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Elastic Waves in Layered Media

W. MAURICE EWING

Director, Lamont Geological Observatory
Professor of Geology, Columbia University

WENCESLAS S. JARDETZKY

Associate Professor of Mechanics, Manhattan College
Research Associate, Lamont Geological Observatory

FRANK PRESS

Professor of Geophysics, California Institute of Technology

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PREFACE

This work is the outgrowth of a plan to make a uniform presentation of the investigations on earthquake seismology, underwater sound, and model seismology carried on by the group connected with Lamont Geological Observatory of Columbia University The scope was subsequently enlarged to cover a particular selection of related problems The methods and results of the theory of wave propagation in layered media are important in seismology, in geophysical prospecting, and in many problems of acoustics and electromagnetism

Although the mathematical discussions of electromagnetic waves, water waves, and shock waves are very close to the methods used in this book, we had to reduce them to a few brief references. Many of the methods which have been used in seismological problems were originally developed in studies on electromagnetic waves It is hoped that a systematic presentation of problems concerning elastic-wave propagation may now be useful in other fields

The experimental viewpoint has, to a large extent, governed the selection of problems For many years, research in seismology has been characterized by separation of the experimental and theoretical methods The interplay of the two methods guided the research program which led to this book, and it has been retained whenever possible Observations of surface waves from explosions and earthquakes, flexural waves in ice, and SOFAR sound propagation are a few examples of topics in which the theoretical and practical investigations benefited each other

An effort was made to compile a comprehensive and systematic bibliography of the world literature for the main topics discussed Few workers in this field could become familiar with all the past investigations, which are scattered in many journals.

We are very grateful to the Air Force Cambridge Research Center, the Bureau of Ships, and the Office of Naval Research for support of the program of research on elastic-wave propagation at the Lamont Geological Observatory. Peter Gottlieb, Dr. Samuel Katz, Dr. A Laughton, Dr. Franklyn Levin, and Stefan Mueller kindly read the manuscript and made helpful suggestions.

MAURICE EWING
WENCESLAS JARDETZKY
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LIST OF SYMBOLS

c_R	Velocity of Rayleigh waves
c	Phase velocity
E	Young's modulus
$e_{xx}, e_{xy}, \cdots, e_{zz}$	Strain components
ϵ	Angle of emergence
f	Frequency or constant of gravitation
$\mathfrak{f}$	Angle of incidence for shear waves
$H_n^{(1)}, H_n^{(2)}$	Hankel functions of the order n
I_n	Modified Bessel function of the first kind of the order n
$\mathfrak{i}$	Angle of incidence
J_n	Bessel functions of the order n
k	Wave number
$\mathfrak{k}$	Coefficient of incompressibility
$\mathcal{K}_n$	Modified Bessel function of the second kind of the order n
l_0, l	Wave length
$p_{xx}, p_{xy}, \cdots, p_{zz}$	Stress components
$\mathcal{P}$	Principal value (of an integral)
p	Hydrostatic pressure
q, w	Displacement components in cylindrical coordinates
$\mathbf{s}(u, v, w)$	Displacement
T	Period
U	Group velocity
$\mathbf{v}(\bar{u}, \bar{v}, i\bar{w})$	Velocity
X, Y, Z	Body forces
α	Compressional-wave velocity
β	Shear-wave velocity
γ	Parameter
ϵ	Phase shift
θ	Cubical dilatation or an angle
θ_{cr}	Critical angle
κ	Root of the Rayleigh equation or parameter
λ, μ	Lamé constants
ρ	Density
σ	Poisson's ratio or an angle
$\varphi, \psi(\psi_1, \psi_2, \psi_3)$	Displacement potentials
$\bar{\varphi}$	Velocity potential
$\boldsymbol{\Omega}(\Omega_x, \Omega_y, \Omega_z)$	Rotation
ω	Angular frequency

CHAPTER 1

FUNDAMENTAL EQUATIONS AND SOLUTIONS

1–1. Equations of Motion. The problems we shall consider concern the propagation of elastic disturbances in layered media, each layer being continuous, isotropic, and of constant thickness We begin with a brief outline of the theory of motion in elastic media and a derivation of the equations of motion. A more detailed treatment may be found in reference books, e.g., Sommerfeld [57] †

When a deformable body undergoes a change in configuration due to the application of a system of forces, the body is said to be strained. Within the body, any point P with space-fixed rectangular coordinates (x, y, z) is then displaced to a new position, the components of displacement being, respectively, u, v, w. If Q is a neighboring point $(x + \Delta x, y + \Delta y, z + \Delta z)$, its displacement components can be given by a Taylor expansion in the form

$$\begin{aligned} &u + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z + \cdots \\ &v + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \frac{\partial v}{\partial z}\Delta z + \cdots \\ &w + \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z + \cdots \end{aligned} \tag{1–1}$$

For the small strains associated with elastic waves, higher-order terms can be neglected. Then, introducing the expressions

$$\Omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{1–2}$$

and others obtained by the cyclic change of letters x, y, z and u, v, w, respectively, we may write the displacement components (1–1) in the form

$$\begin{aligned} &u + (\Omega_y\Delta z - \Omega_z\Delta y) + (e_{xx}\Delta x + e_{xy}\Delta y + e_{xz}\Delta z) \\ &v + (\Omega_z\Delta x - \Omega_x\Delta z) + (e_{yx}\Delta x + e_{yy}\Delta y + e_{yz}\Delta z) \\ &w + (\Omega_x\Delta y - \Omega_y\Delta x) + (e_{zx}\Delta x + e_{zy}\Delta y + e_{zz}\Delta z) \end{aligned} \tag{1–3}$$

†Numerals in brackets in the text correspond to the numbered references at the end of the chapter.

The first terms of these expressions are the components of displacement of the point P. It can be shown that the terms in the first parentheses correspond to a pure rotation of a volume element and that the terms in the second parentheses are associated with deformation or strain of the element. The array

$$\begin{matrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{matrix} \tag{1-4}$$

represents the symmetrical strain tensor at P, since $e_{xy} = e_{yx} \cdots$. The three components

$$e_{xx} = \frac{\partial u}{\partial x} \qquad e_{yy} = \frac{\partial v}{\partial y} \qquad e_{zz} = \frac{\partial w}{\partial z}$$

represent simple extensions parallel to the x, y, z axes, and the other three expressions e_{xy}, e_{yz}, e_{zx} are the shear components of strain, which may be shown to be equal to half the angular changes in the xy, yz, zx planes, respectively, of an originally orthogonal volume element. It is also shown in the theory of elasticity that there is a particular set of orthogonal axes through P for which the shear components of strain vanish. These axes are known as the principal axes of strain. The corresponding values of e_{xx}, e_{yy}, e_{zz} are the principal extensions which completely determine the deformation at P. Thus the deformation at any point may be specified by three mutually perpendicular extensions. It is also known that the sum $e_{xx} + e_{yy} + e_{zz}$ is independent of the choice of the orthogonal coordinate system.

The cubical dilatation θ, defined as the limit approached by the ratio of increase in volume to the initial volume when the dimensions Δx, Δy, Δz approach zero, is

$$\lim \frac{(\Delta x + e_{xx}\,\Delta x)(\Delta y + e_{yy}\,\Delta y)(\Delta z + e_{zz}\,\Delta z) - \Delta x\,\Delta y\,\Delta z}{\Delta x\,\Delta y\,\Delta z}$$

or

$$\theta = e_{xx} + e_{yy} + e_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{1-5}$$

neglecting higher-order terms. Although the principal extensions e_{xx}, e_{yy}, e_{zz} are used in the derivation of (1–5), the result holds for any cartesian system because of the invariance of the sum.

Forces acting on an element of area ΔS separating two small portions of a body are, in general, equivalent to a resultant force or traction **R** upon the element and a couple **C** (Fig. 1–1). As ΔS goes to zero, the limit of the ratio of traction upon ΔS to the area ΔS is finite and defines the

stress. The ratio of the couple to ΔS, involving an additional dimension of length, may be neglected. For a complete specification of the stress at P, it is necessary to give the traction at P acting upon all planes passing through the point. However, all these tractions may be reduced to com-

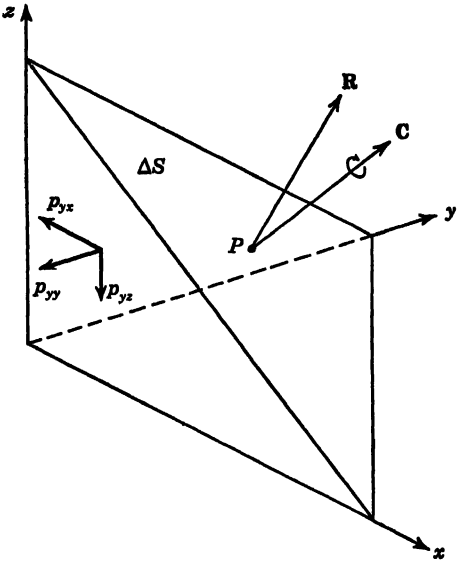


FIG. 1–1. Traction **R** and couple **C** acting on element of area ΔS. Stress components p_{yy}, p_{yz}, and p_{yx} in plane normal to y axis.

ponent tractions across planes parallel to the coordinate planes. Across each of these planes the tractions may be resolved into three components parallel to the axes. This gives nine elements of stress (see Fig. 1–1)

$$\begin{matrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{matrix} \tag{1–6}$$

where the first subscripts represent a coordinate axis normal to a given plane and the second subscripts represent the axis to which the traction is parallel. The array (1–6) is a symmetrical tensor. This may be proved by considering the equilibrium of a small volume element within the medium with sides of length Δx, Δy, Δz, parallel to the x, y, z axes. Moments about axes through the center of mass arise from tractions corresponding to stresses p_{xy}, p_{yx}, $\cdots$. Moments of normal stresses vanish, since the corresponding forces intersect the axes through the center of mass of the

infinitesimal element and moments of body forces are small quantities of higher order than those of stresses. The equilibrium conditions require, therefore, that the shear components of stress be equal in pairs, $p_{xy} = p_{yx}$, etc. As was the case for the shear components of strain, three mutually perpendicular axes of principal stress may be found with respect to which the shear components of stress vanish. Then the stress at a point is completely specified by the principal stresses p_{xx}, p_{yy}, p_{zz} corresponding to these axes.

To derive the equations of motion we consider the tractions across the surfaces of a volume element corresponding to the stress components (1–6) and the body forces X, Y, Z which are proportional to the mass in

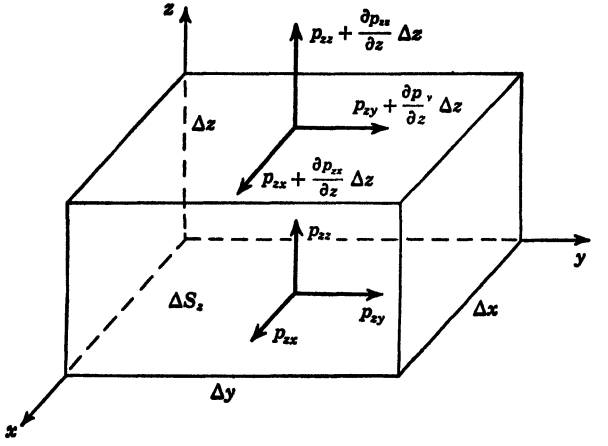


FIG. 1–2. Stress components in the faces ΔS_z of a volume element.

the volume element (Fig. 1–2). When the tractions are considered, the x component of the resultant force acting on an element, e.g., produced by stresses in the faces normal to the x, y, z axes, is (again neglecting higher-order terms)

$$\left(p_{xx} + \frac{\partial p_{xx}}{\partial x} \Delta x - p_{xx}\right) \Delta S_x$$

$$\left(p_{yx} + \frac{\partial p_{yx}}{\partial y} \Delta y - p_{yx}\right) \Delta S_y$$

$$\left(p_{zx} + \frac{\partial p_{zx}}{\partial z} \Delta z - p_{zx}\right) \Delta S_z$$

where ΔS_x, ΔS_y, ΔS_z are the areas of the faces normal to the x, y, z axes, respectively. It follows that the x component of force resulting from all

the tractions is given by the three terms

$$\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}\right) \Delta x\, \Delta y\, \Delta z$$

The equations of motion are obtained by adding all the forces and the inertia terms $-\rho\, d^2u/dt^2\, \Delta x \Delta y \Delta z, \cdots$, for each component:

$$\begin{aligned}
\rho \frac{d^2u}{dt^2} &= \rho X + \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} \\
\rho \frac{d^2v}{dt^2} &= \rho Y + \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z} \qquad (1\text{–}7) \\
\rho \frac{d^2w}{dt^2} &= \rho Z + \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z}
\end{aligned}$$

In these expressions, ρ is the density of the medium.

The Equation of Continuity. This equation expresses the condition that the mass of a given portion of matter is conserved. The total outflow of mass from the elementary volume $\Delta\tau$ during the time Δt is div $\rho\mathbf{v}\, \Delta\tau\, \Delta t$, where $\mathbf{v}$ is the velocity, whose components parallel to the x, y, z axes are $\bar{u}$, $\bar{v}$, $\bar{w}$. The loss of mass during the same time is $-(\partial\rho/\partial t)\, \Delta\tau\, \Delta t$. Equating these last two expressions gives

$$\frac{\partial \rho}{\partial t} + \text{div } \rho\mathbf{v} = 0 \qquad (1\text{–}8)$$

Another form of this equation is

$$\frac{d\rho}{dt} + \rho \text{ div } \mathbf{v} = 0 \qquad (1\text{–}9)$$

where the operation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\text{grad} \qquad (1\text{–}10)$$

represents the "total or material" rate of change following the motion and $\partial/\partial t$ is the local rate of change.

1–2. Elastic Media. In the generalized form of Hooke's law, it is assumed that each of the six components of stress is a linear function of all the components of strain, and in the general case 36 elastic constants appear in the stress-strain relations.

Isotropic Elastic Solid. On account of the symmetry associated with an isotropic body, the number of elastic constants degenerates to two, and the stress-strain relations may be written in the following manner, using

Lamé's constants λ and μ:

$$\begin{aligned} p_{xx} &= \lambda\theta + 2\mu\frac{\partial u}{\partial x} & p_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ p_{yy} &= \lambda\theta + 2\mu\frac{\partial v}{\partial y} & p_{yz} &= \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ p_{zz} &= \lambda\theta + 2\mu\frac{\partial w}{\partial z} & p_{zx} &= \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \end{aligned} \tag{1-11}$$

We also could have written these equations using any two of the constants: Young's modulus E, Poisson's ratio σ, or the coefficient of incompressibility k. The relations between these elastic constants are given by the equations

$$\begin{aligned} \lambda &= \frac{\sigma E}{(1+\sigma)(1-2\sigma)} & \mu &= \frac{E}{2(1+\sigma)} \\ E &= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} & \sigma &= \frac{\lambda}{2(\lambda+\mu)} \\ \mathrm{k} &= \lambda + \tfrac{2}{3}\mu \end{aligned} \tag{1-12}$$

Using Eqs. (1-7) and (1-11), we can write the equations of motion in terms of displacements u, v, w of a point in an elastic solid:

$$\begin{aligned} \rho\frac{\partial^2 u}{\partial t^2} &= (\lambda+\mu)\frac{\partial\theta}{\partial x} + \mu\nabla^2 u + \rho X \\ \rho\frac{\partial^2 v}{\partial t^2} &= (\lambda+\mu)\frac{\partial\theta}{\partial y} + \mu\nabla^2 v + \rho Y \\ \rho\frac{\partial^2 w}{\partial t^2} &= (\lambda+\mu)\frac{\partial\theta}{\partial z} + \mu\nabla^2 w + \rho Z \end{aligned} \tag{1-13}$$

We have replaced d^2/dt^2 by $\partial^2/\partial t^2$, since it follows from (1-10) that the difference between corresponding expressions involves second powers or products of components which are assumed to be small. By neglecting these products, we linearize our differential equations.

For many solids, λ and μ are nearly equal, and we will occasionally use the Poisson relation $\lambda = \mu$ as a simplification. This corresponds to $\mathrm{k} = \frac{5}{3}\mu$ and $\sigma = \frac{1}{4}$.

For an incompressible medium, $\theta = \operatorname{div}\mathbf{v} = 0$ or, by Eq. (1-9), $d\rho/dt = 0$.

Ideal Fluid. If the rigidity μ vanishes, the medium is an ideal fluid. From (1-11) and (1-12) we find $p_{xx} = p_{yy} = p_{zz} = \mathrm{k}\theta = -p$, where $-p$, the value of the remaining independent component of the stress tensor, is the hydrostatic or mean pressure. In liquids the incompressibility k is very large, whereas it has only moderate values for gases. If a liquid is

incompressible, $k = \infty$ and $\sigma = 0.5$. The equations of small motion in an ideal fluid may be obtained from (1–13) with $\mu = 0$.

1–3. Imperfectly Elastic Media. We shall also be concerned with the damping of elastic waves resulting from imperfections in elasticity, particularly from "internal friction." (For a discussion, see Birch [9, pp. 88–91].) The effect of internal friction may be introduced into the equations of motion by replacing an elastic constant such as μ by $\mu + \mu' \partial/\partial t$ in the equations of motion. This is equivalent to stating that stress is a linear function of both the strain and the time rate of change of strain. For simple harmonic motion, the time factor $e^{i\omega t}$ is used, and the effect of internal friction is introduced by replacing μ by the complex rigidity $\mu(1 + i/Q)$, where $1/Q = \omega\mu'/\mu$. In many cases Q may be treated as independent of frequency to a sufficiently good approximation but the more detailed discussion which this case requires is given in Sec. 5–6.

1–4. Boundary Conditions. If the medium to which the equations of motion are applied is bounded, some special conditions must be added. These conditions express the behavior of stresses and displacement at the boundaries. At a free surface of a solid or liquid all stress components vanish. In the problems which follow it will be assumed that solid elastic media are welded together at the surface of contact, implying continuity of all stress and displacement components across the boundary. At a solid-liquid interface slippage can occur, and continuity of normal stresses and displacements alone is required. Since the rigidity vanishes in the liquid, tangential stresses in the solid must vanish at the interface.

1–5. Reduction to Wave Equations. The equations of motion of a fluid [derived from (1–13) with $\mu = 0$ and therefore $\lambda = k$] can be simplified and reduced to one differential equation if a velocity potential $\bar{\varphi}$, defined as follows, exists:

$$\bar{u} = \frac{\partial \bar{\varphi}}{\partial x} \qquad \bar{v} = \frac{\partial \bar{\varphi}}{\partial y} \qquad \bar{w} = \frac{\partial \bar{\varphi}}{\partial z} \tag{1–14}$$

If the body forces are neglected, Eqs. (1–13) reduce to

$$\rho \frac{\partial \bar{u}}{\partial t} = k \frac{\partial \theta}{\partial x} \qquad \rho \frac{\partial \bar{v}}{\partial t} = k \frac{\partial \theta}{\partial y} \qquad \rho \frac{\partial \bar{w}}{\partial t} = k \frac{\partial \theta}{\partial z} \tag{1–15}$$

Now, writing $\alpha^2 = k/\rho$, we easily see from (1–14) and (1–15) that

$$\frac{\partial \bar{\varphi}}{\partial t} = \alpha^2 \theta + F(t) \tag{1–16}$$

and

$$\bar{\varphi} = \alpha^2 \int_0^t \theta \, dt \tag{1–17}$$

where the additive function of t is omitted, being without significance.

From the definition of mean pressure $p = -k\theta$ and (1–16) we have

$$p = -\rho \frac{\partial \bar{\varphi}}{\partial t} \tag{1–18}$$

Then, from (1–16) and (1–5) we obtain

$$\nabla^2 \bar{\varphi} = \frac{1}{\alpha^2} \frac{\partial^2 \bar{\varphi}}{\partial t^2} \tag{1–19}$$

in which small quantities of higher order have been neglected. This wave equation holds for small disturbances propagating in an ideal fluid with velocity α, under the assumptions mentioned above.

For displacements in a solid body, it is convenient to define a scalar potential φ and a vector potential $\psi(\psi_1, \psi_2, \psi_3)$ as follows:

$$\begin{aligned} u &= \frac{\partial \varphi}{\partial x} + \frac{\partial \psi_3}{\partial y} - \frac{\partial \psi_2}{\partial z} \\ v &= \frac{\partial \varphi}{\partial y} + \frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_3}{\partial x} \\ w &= \frac{\partial \varphi}{\partial z} + \frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_1}{\partial y} \end{aligned} \tag{1–20}$$

or, in vector form,

$$\mathbf{s}(u, v, w) = \text{grad } \varphi + \text{curl } \psi(\psi_1, \psi_2, \psi_3) \tag{1–20′}$$

By the definition of θ as given by (1–5), we obtain

$$\theta = \nabla^2 \varphi \tag{1–21}$$

In general, the equations of motion (1–13) represent the propagation of a disturbance which involves both equivoluminal ($\theta = 0$) and irrotational ($\Omega = 0$) motion, where $\theta = \text{div } \mathbf{s}(u, v, w)$ and $\Omega = \frac{1}{2} \text{curl } \mathbf{s}$ [see Eqs. (1–2)]. However, by introduction of the potentials φ and ψ_i, separate wave equations are obtained for these two types of motion. Assuming that the body forces may be neglected, we can write the first of Eqs. (1–13) in the form

$$\frac{\partial}{\partial x}\left(\rho \frac{\partial^2 \varphi}{\partial t^2}\right) + \frac{\partial}{\partial y}\left(\rho \frac{\partial^2 \psi_3}{\partial t^2}\right) - \frac{\partial}{\partial z}\left(\rho \frac{\partial^2 \psi_2}{\partial t^2}\right)$$

$$= (\lambda + \mu) \frac{\partial}{\partial x} \nabla^2 \varphi + \mu \frac{\partial}{\partial x} \nabla^2 \varphi + \mu \frac{\partial}{\partial y} \nabla^2 \psi_3 - \mu \frac{\partial}{\partial z} \nabla^2 \psi_2$$

It is easy to see that this equation and the two others from (1–13) written in a similar form will be satisfied if the functions φ and ψ_i are solutions of the equations

$$\nabla^2 \varphi = \frac{1}{\alpha^2} \frac{\partial^2 \varphi}{\partial t^2} \qquad \nabla^2 \psi_i = \frac{1}{\beta^2} \frac{\partial^2 \psi_i}{\partial t^2} \qquad i = 1, 2, 3 \tag{1–22}$$

where $$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad \beta = \sqrt{\frac{\mu}{\rho}} \tag{1-23}$$

The wave equations (1–22) indicate that two types of disturbances with velocities α and β may be propagated through an elastic solid.

Wave equations involving the dilatation (θ) and rotation (Ω) directly

$$\nabla^2\theta = \frac{1}{\alpha^2}\frac{\partial^2\theta}{\partial t^2} \qquad \nabla^2\Omega_z = \frac{1}{\beta^2}\frac{\partial^2\Omega_z}{\partial t^2} \cdots \tag{1-24}$$

can be readily derived from (1–13) or (1–22). Love [31] discussed expressions which represent solutions satisfying the equations of motion, in the most general case, but it was pointed out by Pendse [42] that the use of scalar and vector potentials is not always free from ambiguity.

In most problems to be considered in subsequent chapters, spherical waves from a point source will be considered. Spherical symmetry does not persist during propagation in a layered medium. However, in many cases axial symmetry will remain, and we shall make use of cylindrical coordinates. If r, z, χ are the cylindrical coordinates (usually taken with the z axis passing through the source and normal to the layering) and q and w are the displacements in the r and the z directions, respectively, the two equations of motion are

$$\begin{aligned}
&(\lambda + 2\mu)\left(\frac{\partial^2 q}{\partial r^2} + \frac{1}{r}\frac{\partial q}{\partial r} - \frac{q}{r^2} + \frac{\partial^2 w}{\partial z\,\partial r}\right) + \mu\left(\frac{\partial^2 q}{\partial z^2} - \frac{\partial^2 w}{\partial z\,\partial r}\right) = \rho\frac{\partial^2 q}{\partial t^2} \\
&(\lambda + 2\mu)\left(\frac{\partial^2 q}{\partial z\,\partial r} + \frac{1}{r}\frac{\partial q}{\partial z} + \frac{\partial^2 w}{\partial z^2}\right) - \frac{\mu}{r}\left(\frac{\partial q}{\partial z} - \frac{\partial w}{\partial r}\right) \\
&\qquad - \mu\left(\frac{\partial^2 q}{\partial z\,\partial r} - \frac{\partial^2 w}{\partial r^2}\right) = \rho\frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{1-25}$$

The angle χ does not appear, because of the axial symmetry. Instead of the potential defined in (1–20), we now use the following:

$$q = \frac{\partial\varphi}{\partial r} - \frac{\partial W}{\partial z} \qquad w = \frac{\partial\varphi}{\partial z} + \frac{\partial(rW)}{r\,\partial r} \tag{1-26}$$

By substitution from (1–26) it may be shown that Eqs. (1–25) are satisfied if the functions φ and W are solutions of equations

$$\nabla^2\varphi = \frac{1}{\alpha^2}\frac{\partial^2\varphi}{\partial t^2} \qquad \nabla^2 W - \frac{1}{r^2}W = \frac{1}{\beta^2}\frac{\partial^2 W}{\partial t^2} \tag{1-27}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{\partial^2}{\partial z^2}$

in cylindrical coordinates. The first equation in (1–27) is a wave equation, and in the second equation the function W may be replaced by another

function ψ defined by

$$W = -\frac{\partial \psi}{\partial r} \tag{1-28}$$

Then, if ψ is a solution of the wave equation

$$\nabla^2 \psi = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2} \tag{1-29}$$

we see immediately, upon differentiating with respect to r, that W satisfies the second equation (1–27).

By introduction of the potentials φ and ψ we have reduced our problem to that of solving the wave equations. For a fluid we obtained a single wave equation representing the propagation of a compressional disturbance. Two wave equations were found for a solid, representing the propagation of compressional and distortional waves with velocities α and β, respectively. The distortional waves (also known as shear, transverse, equivoluminal, or rotational) are represented in general by three functions, ψ_i, which must be solutions of Eqs. (1–22). When there is axial symmetry we have only one function ψ satisfying Eq. (1–29).

It is possible to include additional effects such as those of body forces or finite strains. Stoneley and Scholte (see Chap. 4) applied the classical theory of elasticity in papers concerning gravity waves in compressible media. Stokes' theory concerning waves of finite amplitude is discussed in detail by Lamb [30]. The problem of finite deformations of an elastic body and the effect of high initial stress on wave propagation were discussed in a series of investigations by Hencky [21], Murnaghan [40], Birch [7, 8], Biot [6], and Keller [27]. Several additional examples are given in Chap. 5.

1–6. Solutions of the Wave Equation. The wave equations (1–19) and (1–22) are linear partial differential equations of the second order. The usual method of obtaining a solution for a given problem has been to superpose certain particular integrals, forming a sum which satisfies all conditions of a given problem. A more direct procedure would start with the general solution of the wave equations and adjust this solution to the conditions of the given problem. It is obvious that a general solution must be an expression which holds for all cases of wave propagation in a homogeneous and isotropic medium, i.e., for an arbitrary number of sources with arbitrary positions and for any conditions which can be imposed at the boundaries of the medium. As its form is somewhat complicated, the general solution will be considered after the discussion of simpler cases.

Plane Waves. By making a trial substitution

$$\bar{\varphi} = A \exp\left[i \frac{2\pi}{l} (\nu_1 x + \nu_2 y + \nu_3 z \mp ct) \right] \tag{1-30}$$

we easily see that the wave equation (1–19) is satisfied, if $c = \alpha$ and

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1 \tag{1–31}$$

From condition (1–31) we see that ν_1, ν_2, ν_3 may be considered as the direction cosines of a line L. Equation (1–30) shows that, for a given time $t = t_0$, $\bar{\varphi}$ is constant over any plane (normal to L) determined by the equation $\nu_1 x + \nu_2 y + \nu_3 z \mp ct_0 = P$ and is sinusoidal along L with wave length l. Furthermore, at any given point, $\bar{\varphi}$ is periodic in time with period l/c. Thus Eq. (1–30) represents a system of plane waves of wave length l, advancing along L with velocity c, the direction of advance depending upon the sign chosen.

When we make a similar trial substitution in case of a solid, there are two wave equations (1–22) to be satisfied, and we must obtain two different values of velocity, namely, α and β. It is instructive to substitute as trial expressions for the displacements in the equations of motion (1–13) (body forces being omitted) the values of u, v, w given by the equations

$$u = A_1 \exp\left[i\frac{2\pi}{l}(\nu_1 x + \nu_2 y + \nu_3 z \mp ct)\right]$$
$$v = A_2 \exp \cdots \qquad w = A_3 \exp \cdots \tag{1–32}$$

obtaining the three homogeneous linear equations in A_s

$$\begin{aligned}
-\rho c^2 A_1 + (\lambda + \mu)\nu_1(A_1\nu_1 + A_2\nu_2 + A_3\nu_3) + \mu A_1 &= 0 \\
-\rho c^2 A_2 + (\lambda + \mu)\nu_2(A_1\nu_1 + A_2\nu_2 + A_3\nu_3) + \mu A_2 &= 0 \\
-\rho c^2 A_3 + (\lambda + \mu)\nu_3(A_1\nu_1 + A_2\nu_2 + A_3\nu_3) + \mu A_3 &= 0
\end{aligned} \tag{1–33}$$

The coefficients A_s will determine components of the displacement vector **s**, which may be considered as the resultant of the vectors **B**, **C**, **D**, where **B** is in the direction of L, while **C** and **D** have mutually perpendicular directions. If we take the z axis in the direction of L, the x and y axis in the direction of **C** and **D**, respectively, in order to simplify Eqs. (1–33), we obtain the conditions $\nu_1 = \nu_2 = 0$, $\nu_3 = 1$. For the component **B** we put in (1–33) $A_3 = B$, $A_1 = A_2 = 0$; similarly for **C**, $A_1 = C$, $A_2 = A_3 = 0$; and for **D**, $A_2 = D$, $A_1 = A_3 = 0$. Then (1–33) take the form

$$\begin{aligned}
-\rho c^2 C + \mu C &= 0 \\
-\rho c^2 D + \mu D &= 0 \\
-\rho c^2 B + (\lambda + 2\mu)B &= 0
\end{aligned} \tag{1–33′}$$

Thus we see that the velocity $c = \alpha = \sqrt{(\lambda + 2\mu)/\rho}$ is associated with the component of displacement parallel to the direction of propagation and the velocity $c = \beta = \sqrt{\mu/\rho}$ is associated with the components in any two mutually perpendicular directions normal to it.

Had we solved Eqs. (1–33) directly for c^2 by elimination of A_s, a cubic in c^2 with a single root at α^2 and a double root at β^2 would have resulted.

Thus the system of plane waves traveling along L consists of three independent parts which correspond to the P (compressional), SV (vertically polarized shear), and SH (horizontally polarized shear) waves of seismology.

Spherical Waves. If we write the first equation of (1–22) in spherical coordinates, putting $R^2 = x^2 + y^2 + z^2$ and assuming that $\bar{\varphi} = \varphi(R, t)$, it takes the form

$$\frac{\partial^2 \varphi}{\partial R^2} + \frac{2}{R}\frac{\partial \varphi}{\partial R} = \frac{1}{\alpha^2}\frac{\partial^2 \varphi}{\partial t^2} \tag{1–34}$$

By the substitution

$$\varphi(R, t) = \frac{\vartheta(R, t)}{R} \tag{1–35}$$

we obtain the equation

$$\frac{\partial^2 \vartheta}{\partial R^2} = \frac{1}{\alpha^2}\frac{\partial^2 \vartheta}{\partial t^2} \tag{1–36}$$

and it is evident that its general solution has the form

$$\vartheta = f_1(R - \alpha t) + f_2(R + \alpha t) \tag{1–37}$$

where f_1 and f_2 are arbitrary functions. Thus, by (1–35) and (1–37) the solution of the first equation of (1–22) is, in this case,

$$\varphi = \frac{1}{R}[f_1(R - \alpha t) + f_2(R + \alpha t)] \tag{1–38}$$

Each term in (1–38) has a constant value on a sphere $R = \text{const}$ at $t = t_0$. If f_1 and f_2 are periodic functions, (1–38) will represent infinite trains of spherical waves propagating toward and away from the common center of the spheres $R = \text{const}$.

A similar treatment of the remaining equations in (1–22) yields

$$\psi_s = \frac{1}{R}[g_{s1}(R - \beta t) + g_{s2}(R + \beta t)] \tag{1–38′}$$

The functions f_1 and g_{s1} represent radiations from a point source at the origin. The functions f_2 and g_{s2} correspond to disturbances traveling in the opposite direction and are usually zero.

For spherical waves in a fluid we would start with Eq. (1–19) and obtain for the velocity potential

$$\bar{\varphi} = \frac{1}{R}[\bar{f}_1(R - \alpha t) + \bar{f}_2(R + \alpha t)] \tag{1–39}$$

where $\bar{f}_1$ and $\bar{f}_2$ are again arbitrary functions.

When sources of a disturbance act in a finite domain, a condition is usually imposed that the disturbance vanish at infinity. According to Sommerfeld [56], a further condition is necessary which specifies that no energy may be radiated from infinity into the region of sources. Terms such as $f_2(R + \alpha t)$ in (1–38) are excluded by this "condition of radiation." Bakaliajev [1] extended to three dimensions a similar condition derived by Kupradze for propagation of elastic waves in two dimensions.

Special solutions of the type (1–39) satisfying the radiation condition are the functions

$$\varphi = \frac{A}{R} \exp\left[\pm i(k_\alpha R - \omega t)\right] \tag{1–40}$$

where A = const
$k_\alpha = \omega/\alpha$

A detailed discussion of the mathematical expressions representing radiation from a source, together with proofs of uniqueness, is given, for example, in the work of Stratton [59]. The radiation condition guarantees a unique solution, and it was shown by Haug [20] that this holds even if k_α is complex in expressions such as (1–40).

Very important transformations of expressions representing spherical waves were used in the theory of wave propagation by Lamb, Sommerfeld, and Weyl, and these will be applied in the next chapters. The factor in (1–40) depending only on distance was given by Sommerfeld [55 or 57] in the form

$$\frac{e^{-ik_\alpha R}}{R} = \int_0^\infty J_0(kr)e^{-\nu|z|}\frac{k\,dk}{\nu} \tag{1–41}$$

where J_0 = zero-order Bessel function
z and r = cylindrical coordinates
$\nu^2 = k^2 - k_\alpha^2$, k being a parameter

To prove formula (1–41) Sommerfeld made use of a more general expression for the right-hand member. Since each product $J_0(kr)e^{-\nu|z|}$ is a solution of the wave equation if $\nu^2 = k^2 - k_\alpha^2$, we can attempt to derive a representation of $\exp(-ik_\alpha R)/R$ by superposing such expressions to obtain

$$\int_0^\infty F(k)J_0(kr)e^{-\nu|z|}\,dk$$

In order to determine the coefficient $F(k)$ we can consider the values of both functions at $z = 0$. Thus we put

$$\frac{e^{-ik_\alpha r}}{r} = \int_0^\infty F(k)J_0(kr)\,dk \tag{1–42}$$

The left-hand member can be written in the form of a Fourier-Bessel integral

$$\frac{e^{-ik_\alpha r}}{r} = \int_0^\infty k\,dk\,J_0(kr)\int_0^\infty e^{-ik_\alpha \tau}J_0(k\tau)\,d\tau$$

Hence

$$F(k) = k\int_0^\infty e^{-ik_\alpha \tau}J_0(k\tau)\,d\tau$$

$$= \frac{k}{2\pi}\int_{-\pi}^{\pi} d\sigma \int_0^\infty e^{i\tau(-k_\alpha + k\cos\sigma)}\,d\tau$$

where the latter integral is obtained by substituting the expression (1–69) for the Bessel function and reversing the order of integration. To perform the integration with respect to τ, we apply the Cauchy theorem to transform the path of integration from the real axis to the infinite arc and imaginary axis of the first or fourth quadrant, according as the coefficient of $i\tau$ in the exponent is positive or negative. The contribution from the infinite arc vanishes, and integration along the imaginary axis gives

$$F(k) = -\frac{k}{2\pi i}\int_{-\pi}^{\pi}\frac{d\sigma}{-k_\alpha + k\cos\sigma}$$

Now, calculating the last integral, we find

$$F(k) = \frac{k}{\sqrt{k^2 - k_\alpha^2}} = \frac{k}{\nu} \tag{1–43}$$

thus proving the relationship (1–41).

Instead of using cylindrical coordinates, Weyl [64] considered the expression (1–40) as a result of superposition of plane waves and proved that the first exponential factor in (1–40) can be written in the form of a double integral

$$\frac{e^{-ik_\alpha R}}{-ik_\alpha R} = \frac{1}{2\pi}\int e^{-ik_\alpha(\nu_1 x + \nu_2 y + \nu_3 z)}\,d\Sigma \tag{1–44}$$

where $d\Sigma$ is a surface element. In this expression the integration must be performed over the half sphere $\nu_3 > 0$ of the unit sphere $\nu_1^2 + \nu_2^2 + \nu_3^2 = 1$. If we put $\nu_1 = \sin\vartheta\cos\sigma$, $\nu_2 = \sin\vartheta\sin\sigma$, $\nu_3 = \cos\vartheta$, $d\Sigma = \sin\vartheta\,d\vartheta\,d\sigma$, then $0 \le \sigma \le 2\pi$, while ϑ is to be taken complex and to vary from 0 to $\pi/2$ and subsequently from $\pi/2$ to $\pi/2 + i\infty$.†

As an extension of Weyl's integral, the expression

$$\frac{F(\alpha t - R)}{R} = -\frac{1}{2\pi}\int F'[\alpha t - (\nu_1 x + \nu_2 y + \nu_3 z)]\,d\Sigma \tag{1–45}$$

†Expression (1–41) can be derived from (1–44).

which represents an arbitrary spherical wave, was given by Poritzky [44]. In this formula, F is defined as a real function of its real argument but F' must also range over complex values of its argument.

Solutions in the form (1–40) are used for an infinite medium or for a certain time interval in a finite domain until the effect of boundaries has to be considered. The first representation of the sound field produced by a spherical source of harmonic oscillations was given by Stokes and developed by Rayleigh [45]. The potential of this field was expressed in terms of a set of special functions. Recently Rzhevkin [47] suggested the use of new functions to make the study of the energy transport in a field more convenient. The relationship between these functions and those of Stokes and Rayleigh was also shown.

Some general formulas for the displacements in spherical polar coordinates were recently derived by Homma [22] for cases important in seismology. These formulas represent the displacements at a point produced by a given distribution of initial values Certain applications to elastic waves produced by a source similar to an explosion were studied by Kawasumi and Yosiyama [26], Sezawa [52], Sezawa and Kanai [53], Sharpe [54], Blake [11], and Selberg [51].

If a pressure $p(t)$ is applied to the surface of a spherical cavity of radius R_0 in an elastic medium, the resultant wave can be represented by the equation

$$\varphi(R, t) = \frac{1}{2\pi R}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\omega)p(\tau)\left\{\exp\left[i\omega\left(\tau - t + \frac{R - R_0}{\alpha}\right)\right]\right\} d\omega\, d\tau \quad (1\text{–}46)$$

which is a generalization of (1–40). For $p(t) = p_0$ for $t \geq 0$ and $p(t) = 0$ for $t < 0$, Sharpe obtained the approximate formula for radial displacements

$$u = \frac{R_0^2 p_0}{2\sqrt{2}\,\mu R} e^{-\frac{\delta}{\sqrt{2}}\left(t - \frac{R - R_0}{\alpha}\right)} \sin\delta\left(t - \frac{R - R_0}{\alpha}\right) \quad \text{for } t \geq \frac{R - R_0}{\alpha}$$
$$u = 0 \quad \text{for } t < \frac{R - R_0}{\alpha} \quad (1\text{–}47)$$

where $\delta = 2\sqrt{2}\alpha/3R_0$, in good agreement with several observed characteristics of waves near explosive sources. Another transformation of the integral representing a spherical wave propagating from the wall of a spherical cavity ($R = R_0$) was given by Selberg [51].

Other applications of solutions representing spherical waves were made by Barnes and Anderson [2] to explain the phenomenon of the so-called "tail" in pulse propagation from a spherical source, by Duvall and Atchison [17], who considered displacements in a field due to a pulse at the boundary of a spherical cavity for $\sigma = \frac{1}{4}$, ($\lambda = \mu$), and by Blake [11] for general values of Poisson's constant ($\lambda \neq \mu$).

Poisson and Kirchhoff Solutions. The classic problem of sound propagation was given the following mathematical form by Cauchy: Values of a function $\varphi(x, y, z, t)$ and its derivative with respect to the time are given at an initial instant $t = 0$. These are $\varphi(x, y, z, 0) = \varphi_0(x, y, z)$ and $[\partial\varphi/\partial t]_{t=0} = \varphi_1(x, y, z)$. A function $\varphi(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ is sought which satisfies the wave equation (1–19) and the initial conditions given above. A solution was given by Poisson. Let us take (see, for example, Hadamard [19]) a sphere S having its center at the point $P(\bar{x}, \bar{y}, \bar{z})$ and a variable radius $\sigma = \alpha\bar{t}$. A point of this sphere has coordinates $\bar{x} + \alpha\bar{t} \sin\vartheta \cos\tau$, $\bar{y} + \alpha\bar{t} \sin\vartheta \sin\tau$, $\bar{z} + \alpha\bar{t} \cos\vartheta$, where ϑ and τ are two spherical coordinates. The average value $M_\sigma(\varphi)$ of a function on this sphere is given by the equation

$$M_\sigma = \frac{1}{4\pi} \iint \varphi(\bar{x} + \alpha\bar{t} \sin\vartheta \cos\tau, \cdots) \sin\vartheta \, d\vartheta \, d\tau \qquad (1\text{–}48)$$

Then Poisson's solution of the wave equation may be written in the form

$$\varphi(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = \frac{d}{d\bar{t}} [\bar{t} M_\sigma(\varphi_0)] + \bar{t} M_\sigma(\varphi_1) \qquad (1\text{–}49)$$

It may be proved by substitution that this expression satisfies the wave equation. It gives the value of the function φ at a point P at a moment $\bar{t}$ in terms of initial values of this function and its derivatives with respect to time at the distance $\sigma = \alpha\bar{t}$ from P.

Another form of the solution of the Cauchy problem was obtained by Kirchhoff and is considered as a mathematical form of the Huygens' principle. The Kirchhoff formula can be written even for a more general case than that considered above. Consider the inhomogeneous wave equation

$$\nabla^2\varphi + F(x, y, z, t) = \frac{1}{\alpha^2} \frac{\partial^2\varphi}{\partial t^2} \qquad (1\text{–}50)$$

where F is interpreted as the source-density distribution. (See, for example, Bateman [5].) In order to obtain the value of φ at any given point P in terms of its values in a certain region D we have to assume that, in this region, φ and its first derivatives are continuous and that the second derivatives and F are finite and integrable. Let us now denote by R the distance PQ, Q being any point in the region; by S, the closed boundary of D; and by $\partial/\partial n$, differentiation in the direction of the outer normal. $[F]$ indicates that the value of the function F is to be calculated at time $t - R/\alpha$. Then Kirchhoff's formula, for P inside D, is

$$4\pi\varphi_P = \iiint \frac{[F]}{R} \, dV - \iint \left\{ [\varphi] \frac{\partial}{\partial n}\left(\frac{1}{R}\right) - \frac{1}{R}\left[\frac{\partial\varphi}{\partial n}\right] - \frac{1}{\alpha R} \frac{\partial R}{\partial n}\left[\frac{\partial\varphi}{\partial t}\right] \right\} dS \qquad (1\text{–}51)$$

For P outside D, the value of the integral is zero.

For the homogeneous wave equation, the volume integral representing the so-called retarded potential vanishes. For this case we note that φ_P depends on the value of φ and its derivatives at points Q on the surface at a time preceding the instant t by R/α, which is the time for a disturbance to travel from Q to P with the speed α. This corresponds to Huygens' principle in that Q may be regarded as a secondary source sending disturbances to P.

If we assume that S is a sphere of radius $R = \alpha t$ with its center at P, Kirchhoff's formula may be reduced to Poisson's formula (1–49). We see that $\varphi(t - R/\alpha) = \varphi(0)$, $\partial/\partial n = \partial/\partial R$, and $\partial\varphi/\partial t = -\alpha\, \partial\varphi/\partial R$. For a periodic function of time, a simpler formula of Helmholtz is easily obtained. A general discussion of conditions which hold at the wave fronts as well as an extension of the Kirchhoff formula and of solutions representing the propagation of a disturbance in an infinite elastic medium has been given by Love [32, 33].

General Solution of Wave Equation. A general solution of a partial differential equation may be written in different forms, and it is sometimes impossible to transform one into another. Whittaker's form [66] of the general solution is

$$\varphi = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x \sin u \cos v + y \sin u \sin v + z \cos u + \alpha t, u, v)\, du\, dv \quad (1\text{–}52)$$

Another form of the solution of the wave equation was given by Bateman [3]. Since the integrand in (1–52) is itself an arbitrary function of coordinates and time and a solution of (1–19), the number of parameters can be easily increased if desired. Previously we made use of the plane-wave solution (1–30). We can now take the most general linear expression in x, y, z, t for the argument of the function f. Let

$$\chi = \nu_1(x - \hat{x}) + \nu_2(y - \hat{y}) + \nu_3(z - \hat{z}) + \nu_4(t - \hat{t}) \quad (1\text{–}53)$$

where ν_i, $\hat{x}$, $\hat{y}$, $\hat{z}$, $\hat{t}$ are eight parameters.† An arbitrary function

$$f(\chi; \nu_1, \nu_2, \nu_3, \cdots, \hat{t}) \quad (1\text{–}54)$$

is a solution of (1–19), provided that

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = \frac{\nu_4^2}{\alpha^2} \quad (1\text{–}55)$$

This condition may be written either in the form

$$\nu_3 = \pm\sqrt{\frac{\nu_4^2}{\alpha^2} - \nu_1^2 - \nu_2^2} \quad (1\text{–}56)$$

or

$$\nu_4 = \pm\alpha\sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2} = \pm\nu \quad (1\text{–}57)$$

†We assume here that these parameters are real. The case of complex ν_i will be considered later.

Let us first consider (1–57). Then we can make use of two arguments

$$\begin{aligned} \chi_1 &= \nu_1(x - \hat{x}) + \nu_2(y - \hat{y}) + \nu_3(z - \hat{z}) - \nu(t - \hat{t}) \\ \chi_2 &= \nu_1(x - \hat{x}) + \nu_2(y - \hat{y}) + \nu_3(z - \hat{z}) + \nu(t - \hat{t}) \end{aligned} \tag{1–58}$$

and write, instead of (1–52),

$$\varphi = \int_{-\infty}^{\infty} \cdots \int f_1(\chi_1; \nu_1, \nu_2, \nu_3, \hat{x}, \hat{y}, \hat{z}, \hat{t})\, d\nu_1\, d\nu_2\, d\nu_3\, d\hat{x}\, d\hat{y}\, d\hat{z}\, d\hat{t} + \int_{-\infty}^{\infty} \cdots \int f_2(\chi_2; \nu_1, \cdots, \hat{t})\, d\nu_1 \cdots d\hat{t} \tag{1–59}$$

where f_1 and f_2 are two arbitrary functions and both integrals are sevenfold. The condition (1–56) is used in many investigations, and we obtain

$$\varphi = \int_{-\infty}^{\infty} \cdots \int f_1(\chi_1; \nu_1, \nu_2, \nu_4, \hat{x}, \hat{y}, \hat{z}, \hat{t})\, d\nu_1\, d\nu_2\, d\nu_4\, d\hat{x}\, d\hat{y}\, d\hat{z}\, d\hat{t} + \int_{-\infty}^{\infty} \cdots \int f_2(\chi_2; \nu_1, \nu_2, \nu_4, \hat{x}, \hat{y}, \hat{z}, \hat{t})\, d\nu_1\, d\nu_2\, d\nu_4\, d\hat{x}\, d\hat{y}\, d\hat{z}\, d\hat{t} \tag{1–60}$$

where f_1 and f_2 are again two arbitrary functions but

$$\begin{aligned} \chi_1 &= \nu_1(x - \hat{x}) + \nu_2(y - \hat{y}) + \sqrt{\frac{\nu_4^2}{\alpha^2} - \nu_1^2 - \nu_2^2}\,(z - \hat{z}) + \nu_4(t - \hat{t}) \\ \chi_2 &= \nu_1(x - \hat{x}) + \nu_2(y - \hat{y}) - \sqrt{\frac{\nu_4^2}{\alpha^2} - \nu_1^2 - \nu_2^2}\,(z - \hat{z}) + \nu_4(t - \hat{t}) \end{aligned} \tag{1–61}$$

We can also consider special functions f_ν. For example, writing one term only

$$f = P(\nu_1, \cdots, \hat{t})e^{i\chi} \tag{1–62}$$

we obtain Fourier integrals

$$\varphi = \int_{-\infty}^{\infty} \cdots \int Pe^{i\chi}\, d\nu_1\, d\nu_2\, d\nu_4\, d\hat{x}\, d\hat{y}\, d\hat{z}\, dt \tag{1–63}$$

Both expressions (1–59) and (1–60), given by Jardetzky [23], are general solutions of the wave equation. Taken together, they contain all particular cases considered in the literature. By a specialization of functions involved in this solution we can adjust them to all conditions of a particular problem.

Some special forms used. We shall show first how the well-known expression for the potential φ for a single source may be derived from the general solution. If a source is located at a point $S(0, 0, h)$, we have to put $\hat{x} = \hat{y} = 0$, $\hat{z} = h$. If the source begins to emit a disturbance at $t = 0$, we also put $\hat{t} = 0$ and write, by (1–63) and (1–61),

$$\varphi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\nu_1, \nu_2, \nu_4) e^{i[\nu_1 x + \nu_2 y + \nu_3(z-h) + \nu_4 t]}\, d\nu_1\, d\nu_2\, d\nu_4 \qquad (1\text{–}64)$$

Even though the disturbance has spherical symmetry, we consider now only symmetry with respect to the z axis, having in view subsequent applications. Thus, we put

$$\begin{aligned} x &= r\cos u \qquad \nu_1 = \kappa\cos\tau \qquad 0 \le \kappa \le \infty \\ y &= r\sin u \qquad \nu_2 = \kappa\sin\tau \qquad -\pi \le \tau \le \pi \end{aligned} \qquad (1\text{–}65)$$

Then (1–64) takes the form

$$\varphi = \int_{-\infty}^{\infty} e^{i\nu_4 t}\, d\nu_4 \int_0^{\infty}\int_{-\pi}^{\pi} e^{i\nu_3(z-h)+i\kappa r\cos(\tau-u)} \bar{P}(\kappa, \tau, \nu_4)\, d\kappa\, d\tau \qquad (1\text{–}66)$$

By (1–56) and (1–65),

$$\nu_3 = \pm\sqrt{\frac{\nu_4^2}{\alpha^2} - \kappa^2} \qquad (1\text{–}67)$$

is independent of τ. It is apparent, because of the assumed isotropy of the medium, that the function $\bar{P}$ is also independent of τ. We can then write

$$\varphi = \int_{-\infty}^{\infty} e^{i\nu_4 t}\, d\nu_4 \int_0^{\infty} e^{i\nu_3(z-h)} \bar{P}(\kappa, \nu_4)\, d\kappa \int_{-\pi}^{\pi} e^{i\kappa r\cos(\tau-u)}\, d\tau \qquad (1\text{–}68)$$

The last integral represents the Bessel function (see Watson [61]).

$$J_0(\kappa r) = \frac{1}{\pi}\int_0^{\pi} e^{i\kappa r\cos\sigma}\, d\sigma = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\kappa r\cos(\sigma+\epsilon)}\, d\sigma \qquad (1\text{–}69)$$

where ϵ is arbitrary. Including the factor $1/2\pi$ in the arbitrary function $\bar{P}$, we obtain

$$\varphi = \int_{-\infty}^{\infty} e^{i\nu_4 t}\, d\nu_4 \int_0^{\infty} \bar{P}(\kappa, \nu_4) J_0(\kappa r) e^{i\nu_3(z-h)}\, d\kappa \qquad (1\text{–}70)$$

If we assume that $\bar{P}(\kappa, \nu_4) = P_1(\kappa)P_2(\nu_4)$, it is possible to write

$$\varphi = \int_{-\infty}^{\infty} P_2(\nu_4) e^{i\nu_4 t}\, d\nu_4 \int_0^{\infty} P_1(\kappa) J_0(\kappa r) e^{i\nu_3(z-h)}\, d\kappa \qquad (1\text{–}71)$$

Now we are able to see that, in order to obtain the expression of the form (1–41)

$$\frac{e^{-ikR}}{R} = \int_0^{\infty} e^{-\nu|z-h|} J_0(\kappa r)\, \frac{\kappa\, d\kappa}{\nu} \qquad (1\text{–}72)$$

where $R^2 = r^2 + (z-h)^2$ corresponding to spherical waves emitted by a point source, we have to put

$$\nu = \mp i\nu_3 \qquad P_1(\kappa) = \frac{\kappa}{\nu} \qquad (1\text{–}73)$$

taking the upper sign if $z - h$ is positive. Finally,

$$\varphi = \varphi_0 = \int_0^\infty \frac{P_2(\omega)}{R} e^{i\omega[t-(R/\alpha)]}\, d\omega \tag{1-74}$$

if we write ω instead of ν_4. The function $P_2(\omega)$ is not yet fixed, and one can easily see that the most natural assumption is to connect it with the properties of the source.

If we consider, for example, sources distributed continuously along the z axis, we can easily derive from (1–63) the potential φ due to a line source in the form given by Coulomb [15].

Assuming $\hat{z} = 0$, we can also obtain from (1–63) expressions for the potentials φ and ψ, used by Schermann [49] in the problem of propagation of a disturbance in a half space, when the displacements or applied forces are given at the plane $z = 0$ at some initial moment.

Other expressions for the functions φ and ψ, representing solutions of particular problems will be discussed in the following chapters.

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CHAPTER 2

HOMOGENEOUS AND ISOTROPIC HALF SPACE

The solutions of wave equations considered in the preceding chapter represent disturbances propagating in media of infinite extent in all three dimensions. At the instant the disturbance reaches a boundary, new conditions must be taken into account; these will affect the form of solution.

2–1. Reflection of Plane Waves at a Free Surface. Various types of waves generated in a homogeneous and isotropic half space will be discussed in this chapter, and we shall begin with the simplest problem of this kind. We assume that the free boundary of a homogeneous and isotropic medium is a plane ($z = 0$) and that a train of plane waves propagates

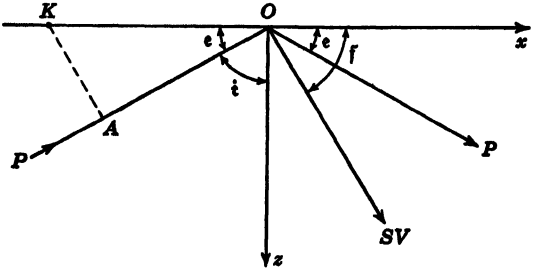


FIG. 2–1. Reflection of P wave at free surface of elastic solid.

in a direction AO in the xz plane which makes an angle e with the boundary or $i = 90° - e$ with the normal to it (see Fig. 2–1).

We shall consider two types of plane incident waves. The case of incident P waves is represented in Fig. 2–1 and that of SV waves in Fig. 2–2.

For the reference axes chosen, the plane P and SV waves do not depend on y, Eqs. (1–20) form two separated groups. The displacements corresponding to these waves

$$u = \frac{\partial \varphi}{\partial x} - \frac{\partial \psi}{\partial z} \qquad w = \frac{\partial \varphi}{\partial z} + \frac{\partial \psi}{\partial x} \tag{2–1}$$

will be considered together; the *SH* component

$$v = \frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_3}{\partial x} \tag{2-2}$$

representing a pure distortion can be treated separately. The functions φ and ψ satisfy the wave equations (1-22)

$$\nabla^2 \varphi = \frac{1}{\alpha^2} \frac{\partial^2 \varphi}{\partial t^2} \qquad \nabla^2 \psi = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2} \tag{2-3}$$

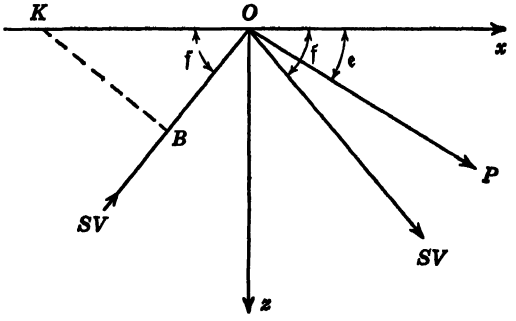


FIG. 2-2. Reflection of *SV* wave at free surface of elastic solid.

It will be shown later that, to satisfy the boundary conditions at $z = 0$, both potentials φ and ψ must be used and, when solutions of (2-3) having the form

$$\varphi = f(z)e^{ik(ct-x)} \qquad \psi = g(z)e^{ik(ct-x)} \tag{2-4}$$

are used, the exponential terms must be identical.

Substituting φ, for example, in (2-3), we obtain

$$\frac{d^2 f}{dz^2} + \left(\frac{c^2}{\alpha^2} - 1\right)k^2 f = 0 \tag{2-5}$$

and the integral of this equation is

$$f(z) = A_1 \exp\left(ik\sqrt{\frac{c^2}{\alpha^2} - 1}\, z\right) + A_2 \exp\left(-ik\sqrt{\frac{c^2}{\alpha^2} - 1}\, z\right) \tag{2-6}$$

From these results we can write the solutions (2-4) in the convenient

form

$$\varphi = A_1 \exp\left[ik\left(ct + z\sqrt{\frac{c^2}{\alpha^2} - 1} - x\right)\right] + A_2 \exp\left[ik\left(ct - z\sqrt{\frac{c^2}{\alpha^2} - 1} - x\right)\right]$$
$$\psi = B_1 \exp\left[ik\left(ct + z\sqrt{\frac{c^2}{\beta^2} - 1} - x\right)\right] + B_2 \exp\left[ik\left(ct - z\sqrt{\frac{c^2}{\beta^2} - 1} - x\right)\right] \tag{2-7}$$

To interpret these expressions we note that c is an apparent velocity along the surface. In Fig. 2–1, $AO = \alpha$ represents the distance traveled by the compressional wave front KA in unit time, $OK = c$ is the corresponding distance traced by the wave front along the free surface, and it follows that

$$\sqrt{\frac{c^2}{\alpha^2} - 1} = \tan \mathrm{e}$$

In a similar way, we obtain for distortional waves

$$\sqrt{\frac{c^2}{\beta^2} - 1} = \tan \mathrm{f}$$

and $c = \alpha \sec \mathrm{e} = \beta \sec \mathrm{f}$, which is similar to Snell's law. Let

$$k = \frac{2\pi \cos \mathrm{e}}{l} = \frac{2\pi \cos \mathrm{f}}{l'}$$

where l and l' are wave lengths for compressional and distortional waves, respectively. Then (2–7) represent compressional waves with emergence and reflection angle e and distortional waves with emergence and reflection angle f.

Incident P Waves. Considering the case where only P waves are incident, we set $B_1 = 0$ in Eqs. (2–7) and determine the relations between the remaining coefficients by use of the boundary conditions

$$[p_{zz}]_{z=0} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \mu\left(2\frac{\partial^2 \varphi}{\partial x\, \partial z} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2}\right) = 0$$
$$[p_{zz}]_{z=0} = \lambda\theta + 2\mu\frac{\partial w}{\partial z} = \lambda\nabla^2\varphi + 2\mu\left(\frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x\, \partial z}\right) = 0 \tag{2-8}$$

Incidentally, it may be seen from Eqs. (2–7) and (2–8) that if $B_2 = 0$, φ exists only if $\mathrm{e} = 0$ or $\mathrm{e} = \pi/2$, and $A_2 = -A_1$. Thus, in general, an incident compressional wave produces both reflected compressional and shear waves.

It follows from the definitions of tan e and tan f given above that

$$\cos^2 e = \frac{\alpha^2}{\beta^2} \cos^2 f \tag{2-9}$$

If $\sigma = \frac{1}{4}$, $\lambda = \mu$, then $\alpha^2 = 3\beta^2$, $\cos^2 e = 3 \cos^2 f$, and the boundary conditions take the form

$$\begin{aligned} 2(A_1 - A_2) \tan e + B_2(\tan^2 f - 1) &= 0 \\ (A_1 + A_2)(1 + 3 \tan^2 e) + 2B_2 \tan f &= 0 \end{aligned} \tag{2-10}$$

From Eqs. (2-9) and (2-10) the ratios A_2/A_1, B_2/A_1, and f can be expressed in terms of e:

$$\begin{aligned} \frac{A_2}{A_1} &= \frac{4 \tan e \tan f - (1 + 3 \tan^2 e)^2}{4 \tan e \tan f + (1 + 3 \tan^2 e)^2} \\ \frac{B_2}{A_1} &= \frac{-4 \tan e (1 + 3 \tan^2 e)}{4 \tan e \tan f + (1 + 3 \tan^2 e)^2} \end{aligned} \tag{2-11}$$

From the second of Eqs. (2-11) we see that B_2 vanishes for two cases. First, for normal incidence, $e = \pi/2$, $f = \pi/2$, the denominator being of a higher order with respect to tan e, and second, for grazing incidence, $e = 0$. In both cases the reflection consists of a P wave only.

Since $\tan^2 f = 3 \tan^2 e + 2$, the coefficient A_2 will vanish if

$$4 \tan e (3 \tan^2 e + 2)^{\frac{1}{2}} = (1 + 3 \tan^2 e)^2$$

This equation has two roots $e = 12°47'$ or $e = 30°$, and, therefore, for these two special directions of an incident wave no reflected P wave exists. These roots correspond to $c/\beta = 1.776$ or 2.000.

In order to measure the angle of emergence e from seismograms, one makes use of the amplitudes A_v and A_h of the vertical and horizontal ground displacements.

The angle $\bar{e} = \tan^{-1} A_v/A_h$ is called the apparent angle of emergence. By Eqs. (2-1), (2-7), and (2-11) we obtain

$$\tan \bar{e} = \frac{1 + 3 \tan^2 e}{2 \tan f} = \frac{\tan^2 f - 1}{2 \tan f} = -\cot 2f \tag{2-12}$$

Walker [54] derived the relation between e and $\bar{e}$ for $\alpha^2 \neq 3\beta^2$ in the form

$$2 \cos^2 e = \frac{\alpha^2}{\beta^2} (1 - \sin \bar{e}) \tag{2-13}$$

which he calls Wiechert's relation and which for Poisson's relation $\sigma = \frac{1}{4}$ takes the form

$$2 \cos^2 e = 3(1 - \sin \bar{e}) \tag{2-14}$$

This equation also follows from (2-12) and (2-9).

If $\alpha^2 \neq 3\beta^2$, the conditions for vanishing of A_2 are changed. Making use of the value $\alpha/\beta = 1.788$, corresponding to $\alpha = 7.17$ km/sec, $\beta = 4.01$ km/sec, Walker found that A_2 will not vanish for any real value of ϵ. It attains a small minimum near $\epsilon = 20°$.

Incident S Waves. Let us now consider an incident SV wave (Fig. 2–2). The boundary conditions are satisfied if the incident transverse wave gives rise to a reflected transverse wave and a reflected longitudinal wave. In (2–7) we put $A_1 = 0$ and substitute the other terms in the boundary conditions (2–8). Again assuming that Poisson's relation holds, we obtain the following equations for the reflection coefficients:

$$\frac{A_2}{B_1} = \frac{4 \tan \phi (1 + 3 \tan^2 \epsilon)}{4 \tan \epsilon \tan \phi + (1 + 3 \tan^2 \epsilon)^2}$$
$$\frac{B_2}{B_1} = \frac{4 \tan \epsilon \tan \phi - (1 + 3 \tan^2 \epsilon)^2}{4 \tan \epsilon \tan \phi + (1 + 3 \tan^2 \epsilon)^2} \qquad (2\text{–}15)$$

The amplitude of the reflected distortional wave B_2 vanishes for $\phi = 55°44'$ and $\phi = 60°$, also corresponding to $c/\beta = 1.776$ and 2.000.

If $\alpha/\beta = 1.788$, B_2 does not vanish but has a small minimum near $\phi = 58°$. One can easily derive the expression for the apparent angle of emergence $\bar{\phi}$ for the case of incident SV when $\sigma = \frac{1}{4}$:

$$\tan \bar{\phi} = \frac{2 \tan \epsilon}{1 + 3 \tan^2 \epsilon} \qquad (2\text{–}16)$$

For an incident SH wave a similar derivation shows that all the energy is reflected as SH, the horizontal displacement of the free surface being twice that of the incident wave.

Partition of Reflected Energy. To derive an expression for the energy partition in the system of incident and reflected waves, Eqs. (2–7), we write the corresponding particle velocities in the form $\bar{u} = \partial^2\varphi/\partial x\, \partial t$, $\bar{w} = \partial^2\varphi/\partial z\, \partial t$ for P waves. Then

$$\bar{u} = A_1 k^2 c \cos k\chi_1 \qquad \bar{w} = -A_1 k^2 c \tan \epsilon \cos k\chi_1$$
$$\bar{u} = A_2 k^2 c \cos k\chi_2 \qquad \bar{w} = A_2 k^2 c \tan \epsilon \cos k\chi_2$$

for incident and reflected P waves. Similarly, for incident and reflected SV waves we can write

$$\bar{u} = B_1 k^2 c \tan \phi \cos k\chi_1' \qquad \bar{w} = B_1 k^2 c \cos k\chi_1'$$
$$\bar{u} = -B_2 k^2 c \tan \phi \cos k\chi_2' \qquad \bar{w} = B_2 k^2 c \cos k\chi_2'$$

In these equations, χ and χ' are the expressions in parentheses in Eqs. (2–7).

Taking the kinetic energy per unit volume as $\frac{1}{2}\rho(\bar{u}^2 + \bar{w}^2)$, we may compute the energy flux for the waves mentioned above by multiplying

the total energy per unit volume (double the mean kinetic-energy density) by the velocity of propagation and the area of the wave front involved. Thus we may write the equality between incident P-wave energy and reflected P- and SV-wave energy for unit area of the free surface as

$$\tfrac{1}{2}\rho A_1^2 k^4 c^2 \alpha \sec^2 \mathfrak{e} \sin \mathfrak{e} = \tfrac{1}{2}\rho A_2^2 k^4 c^2 \alpha \sec^2 \mathfrak{e} \sin \mathfrak{e} + \tfrac{1}{2}\rho B_2^2 k^4 c^2 \beta \sec^2 \mathfrak{f} \sin \mathfrak{f} \qquad (2\text{–}17)$$

For the case of incident SV waves, the corresponding equation is

$$\tfrac{1}{2}\rho B_1^2 k^4 c^2 \beta \sec^2 \mathfrak{f} \sin \mathfrak{f} = \tfrac{1}{2}\rho A_2^2 k^4 c^2 \alpha \sec^2 \mathfrak{e} \sin \mathfrak{e} + \tfrac{1}{2}\rho B_2^2 k^4 c^2 \beta \sec^2 \mathfrak{f} \sin \mathfrak{f} \qquad (2\text{–}18)$$

If the energy flux of the incident waves is taken as unity in both cases, Eqs. (2–17) and (2–18) reduce to the form, useful for computations,

$$1 = a^2 + b^2 \qquad (2\text{–}19)$$

where $$a^2 = \frac{A_2^2}{A_1^2} \quad \text{and} \quad b^2 = \frac{B_2^2 \tan \mathfrak{f}}{A_1^2 \tan \mathfrak{e}}$$

for an incident P wave or

$$a^2 = \frac{A_2^2 \tan \mathfrak{e}}{B_1^2 \tan \mathfrak{f}} \quad \text{and} \quad b^2 = \frac{B_2^2}{B_1^2}$$

for an incident SV wave.

Computations for reflections from a free surface have been given by Jeffreys [15] and Gutenberg [10], among others. Jeffreys presents his results in terms of the reflection coefficients A_2/A_1, B_2/A_1, etc., whereas Gutenberg prefers the square roots of the energy ratios, a, b. In Figs. 2–3 and 2–4 Gutenberg's calculations for various assumed values of the ratio α/β are reproduced.

Reflection at Critical Angles. Since $\alpha \cos \mathfrak{f} = \beta \cos \mathfrak{e}$, no real value of $\mathfrak{e}$ exists until $\mathfrak{f}$ reaches the critical angle $\cos^{-1} \beta/\alpha$, indicating that for an incident SV wave there is no reflected P wave in the range $0 < \mathfrak{f} < \cos^{-1} \beta/\alpha$. Within this range, $\tan \mathfrak{e} = -i\sqrt{1 - c^2/\alpha^2}$, $\tan \mathfrak{f} = \sqrt{c^2/\beta^2 - 1}$, and $\alpha > c > \beta$. It follows from Eqs. (2–15) that the amplitude B_2 becomes complex with $B_2 = -B_1 e^{2i\delta}$, where

$$\tan \delta = \frac{4\sqrt{1 - c^2/\alpha^2}\,\sqrt{c^2/\beta^2 - 1}}{(2 - c^2/\beta^2)^2} \qquad (2\text{–}20)$$

This represents total reflection with no change in amplitude and a phase change in the reflected SV wave. The coefficient A_2 also becomes complex, and the expression $\sqrt{c^2/\alpha^2 - 1}$ becomes negative imaginary. The motion corresponding to the potential φ decreases exponentially with depth from the surface.

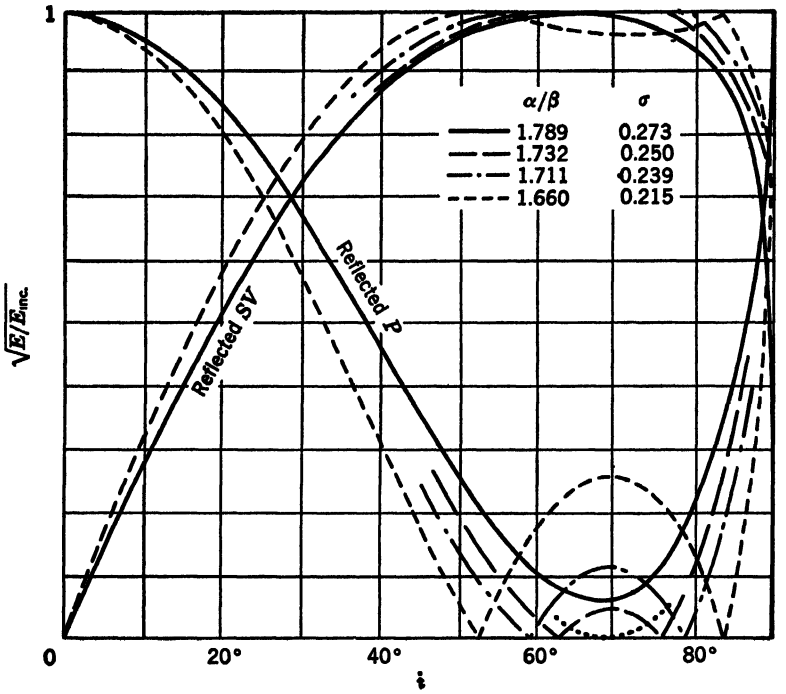


FIG. 2-3. Square root of ratio of reflected to incident energy for P wave incident at free surface for various values of Poisson's constant. (*After Gutenberg.*)

A critical case occurs for the grazing incidence of P waves, $\epsilon = 0$ ($c = \alpha$), where $A_2 = -A_1$, $B_2 = B_1 = 0$, and the expressions (2-7) fail to represent trains of waves. To see the reason and to find another form of the potential, we go back to Eqs. (2-4) and (2-5). If we write the solutions of the wave equations (2-3) in the form (2-4), the function $f(z)$ must satisfy the ordinary differential equation (2-5). Now the roots of the characteristic equation of (2-5) are

$$\tau_{1,2} = \pm ik\left(\frac{c^2}{\alpha^2} - 1\right)^{\frac{1}{2}} = \pm ik \tan \epsilon \tag{2-21}$$

For grazing incidence, $c = \alpha$, $\epsilon = 0$, and the roots $\tau_1 = \tau_2 = 0$. Under these conditions, the general integral of (2-5) is

$$f(z) = C_1 e^{\tau_1 z} + C_2 z e^{\tau_1 z} = C_1 + C_2 z \tag{2-22}$$

(see Jardetzky [14]). This holds for an incident P wave as well as for an

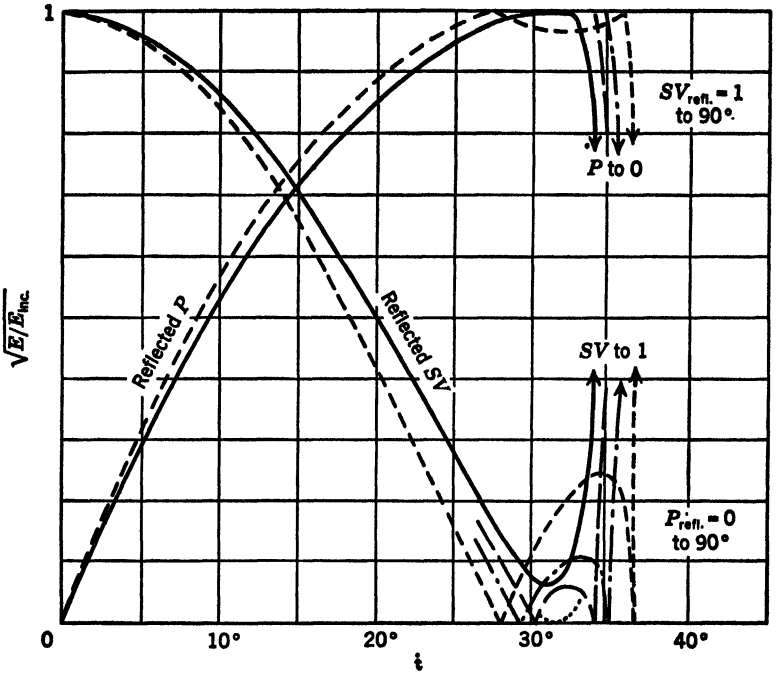


FIG. 2–4. Square root of ratio of reflected to incident energy for SV wave incident at free surface. (*After Gutenberg.*)

incident SV wave. From the equation $\tan^2 f = 3 \tan^2 e + 2$ it follows for $e = 0$ that $\tan f = \pm\sqrt{2}$, and the expressions (2–7) can be written in the form

$$\varphi = (C_1 + C_2 z)e^{ik(at-x)}$$
$$\psi = -Be^{-ikz\sqrt{2}}e^{ik(at-x)} \tag{2–23}$$

derived by Goodier and Bishop [9]. A solution of this form represents a wave with amplitudes increasing with distance from the interface.

2–2. Free Rayleigh Waves. Rayleigh [39] gave the theory for surface waves on the free surface of a semi-infinite elastic solid, showing that the motion became negligible at a distance of a few wavelengths from the free surface. Assume a simple harmonic wave train traveling in the x direction such that (1) the disturbance is independent of the y coordinate and (2) it decreases rapidly with distance z from the free surface. Waves satisfying the second condition are called surface waves. A solution cor-

responding to this definition may be obtained from Eqs. (2–7):

$$\varphi = A \exp\left[ik\left(ct \pm \sqrt{\frac{c^2}{\alpha^2} - 1}\, z - x\right)\right] \qquad \text{for } 0 \leq z < \infty \tag{2–24}$$

$$\psi = B \exp\left[ik\left(ct \pm \sqrt{\frac{c^2}{\beta^2} - 1}\, z - x\right)\right] \tag{2–25}$$

provided that $c < \beta < \alpha$ and the sign is chosen so that the potentials approach zero as z approaches ∞. In a similar way, the *SH* component is given by

$$v = C \exp\left[ik\left(ct \pm \sqrt{\frac{c^2}{\beta^2} - 1}\, z - x\right)\right] \tag{2–26}$$

The arbitrary constants A, B, C in Eqs. (2–24), (2–25), and (2–26) can be determined from the boundary conditions. Since we assume that the plane boundary of a half space is a free surface, the stresses must vanish at $z = 0$. In order to find the expressions for the stresses, we use Eqs. (2–24), (2–25), (2–26), (2–8), and (1–11).

From the conditions $p_{xy} = 0$, $p_{zz} = 0$, $p_{zx} = 0$, it follows that $C = 0$ and

$$\begin{aligned} \left(2 - \frac{c^2}{\beta^2}\right)A \pm 2\sqrt{\frac{c^2}{\beta^2} - 1}\, B &= 0 \\ \mp 2\sqrt{\frac{c^2}{\alpha^2} - 1}\, A + \left(2 - \frac{c^2}{\beta^2}\right)B &= 0 \end{aligned} \tag{2–27}$$

In order to have values A and B different from zero, the parameter c must satisfy an equation obtained by putting the determinant of (2–27) equal to zero. Thus for either upper or lower signs

$$\left(2 - \frac{c^2}{\beta^2}\right)^2 = 4\left(1 - \frac{c^2}{\alpha^2}\right)^{\frac{1}{2}}\left(1 - \frac{c^2}{\beta^2}\right)^{\frac{1}{2}} \tag{2–28}$$

The quantity c^2/β^2 can be factored out after rationalization, and the Rayleigh equation takes the form

$$\frac{c^2}{\beta^2}\left[\frac{c^6}{\beta^6} - 8\frac{c^4}{\beta^4} + c^2\left(\frac{24}{\beta^2} - \frac{16}{\alpha^2}\right) - 16\left(1 - \frac{\beta^2}{\alpha^2}\right)\right] = 0 \tag{2–29}$$

If $c = 0$, Eqs. (2–24) and (2–25) are independent of time, and from (2–27) $A = -iB$ and $u = w = 0$. Hence this solution is not of interest. Now the second factor in (2–29) is negative for $c = 0$, $\beta < \alpha$, and is positive for $c = \beta$. There is always a root of Eq. (2–29) if $0 < c < \beta < \alpha$, and under these conditions surface waves can exist.

For an incompressible solid $\alpha \to \infty$, and (2–29) reduces to

$$\frac{c^6}{\beta^6} - 8\frac{c^4}{\beta^4} + 24\frac{c^2}{\beta^2} - 16 = 0 \tag{2–30}$$

This cubic equation in c^2 has a real root at $c^2 = 0.91275\beta^2$, corresponding to surface waves with velocity $c \cong 0.95\beta$. The other two roots for this case are complex and do not represent surface waves.

If Poisson's relation $\lambda = \mu$ holds, $\alpha = \sqrt{3}\beta$, and (2–29) becomes

$$\frac{c^6}{\beta^6} - 8\frac{c^4}{\beta^4} + \frac{56}{3}\frac{c^2}{\beta^2} - \frac{32}{3} = 0 \qquad (2\text{–}30')$$

This equation yields three real roots, $c^2/\beta^2 = 4, 2 + 2/\sqrt{3}, 2 - 2/\sqrt{3}$. The last root alone can satisfy condition 2 for surface waves, since the radicals in (2–24) and (2–25) become positive imaginary. The last root corresponds to the velocity

$$c_R = 0.9194\beta \qquad (2\text{–}31)$$

The other roots of Eq. (2–30′) correspond to real values of the radicals in (2–24) and (2–35) and therefore do not represent surface waves as mentioned above. These roots arise from squaring Eq. (2–28) and satisfy (2–28) except for a change in sign. The determinant corresponding to this new equation is obtained from the boundary conditions if we consider solutions given by (2–7) for the two cases $A_2 = B_1 = 0$ and $A_1 = B_2 = 0$. The first case represents a compressional wave incident upon the free surface at an angle such that only reflected shear waves occur. The second case represents the reverse situation of an incident shear wave and a reflected compressional wave. Values of c/β computed for these special cases in Sec. 2–1 from the general expressions for reflection coefficients are identical to the values given by the extraneous roots of Eq. (2–30′). For a more detailed discussion, see Fu [7].

For the condition (2–31) one finds for the displacements

$$\begin{aligned} u &= D(e^{-0.8475ks} - 0.5773e^{-0.3933ks}) \sin k(ct - x) \\ w &= D(-0.8475e^{-0.8475ks} + 1.4679e^{-0.3933ks}) \cos k(ct - x) \end{aligned} \qquad (2\text{–}32)$$

where D is a function of k. It may be seen from (2–32) that the particle motion for Rayleigh waves is elliptical retrograde in contrast to the elliptical direct orbit for surface waves on water. The vertical displacement is about one and one-half times the horizontal at the surface. Horizontal motion vanishes at a depth of 0.192 of a wavelength and reverses sign below this.

Dobrin, Simon, and Lawrence [3] experimentally determined the particle-trajectory variation with depth below the earth's surface for Rayleigh waves from small explosions. They found that the motion is retrograde above 40 ft and direct below. The crossover depth was 0.136 of a wavelength. The displacements decrease continuously below 40 ft. These experiments were conducted in a region of layered unconsolidated and semiconsolidated rocks. Despite the departure from homogeneity, the experimental results offer good agreement with the theory.

Knopoff [20] has computed the ratios c_R/β, c_R/α for Rayleigh waves from Eq. (2–29) for all possible values of Poisson's ratio. These are shown in Fig. 2–5.

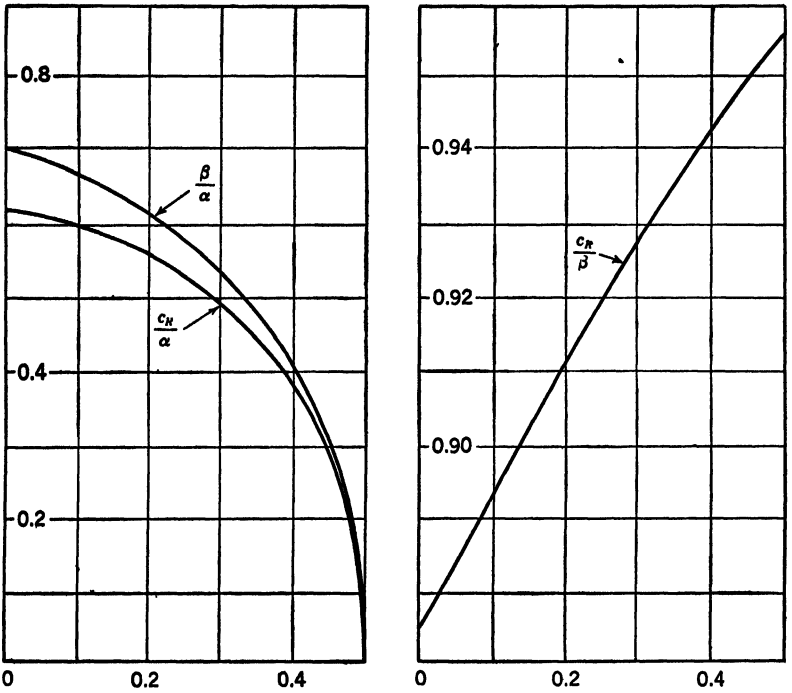


FIG. 2–5. Ratios β/α, c_R/α, c_R/β as functions of Poisson's ratio. (*After Knopoff.*)

Calculations on wave propagation in a half space using plane-wave concepts as illustrated above are inadequate for problems in which characteristics of the wave source must be considered.

2–3. Integral Solutions for a Line Source. In a classic paper, Lamb [22] first considered the disturbance generated in a semi-infinite medium by an impulsive force applied along a line or at a point on the surface. He also wrote the formal solutions for internal sources as integrals which were later studied by Nakano [28], Lapwood [25], and others. Since the methods and solutions in Lamb's paper form the basis of much of the material in later chapters, the more important points of it will now be discussed in some detail.

Surface Source. In this section we consider a two-dimensional problem and derive expressions for surface displacements arising from a force

applied normal to the free surface along a line coincident with the y axis. The displacements u and w are given by Eqs. (2–1), where the functions φ and ψ are solutions of the wave equations (2–3). The effect of a periodic force applied perpendicular to the free surface is expressed by the conditions

$$[p_{zz}]_{z=0} = 0 \qquad [p_{xz}]_{z=0} = Ze^{i(\omega t - kx)} \tag{2–33}$$

where the amplitude Z depends only on k. Henceforth the time factor $e^{i\omega t}$ is omitted to save space. The stresses p_{zz} and p_{xz} are given by (2–8), and in order to find their values we now make use of potentials†

$$\varphi = Ae^{-\nu z - ikx} \qquad \psi = Be^{-\nu' z - ikx} \tag{2–34}$$

which satisfy the wave equations (2–3), provided that

$$\nu^2 = k^2 - k_\alpha^2 \qquad \nu'^2 = k^2 - k_\beta^2 \qquad k_\alpha = \frac{\omega}{\alpha} \qquad k_\beta = \frac{\omega}{\beta} \tag{2–35}$$

A and B are functions of the parameter k as specified by conditions (2–33). On inserting (2–34) in (2–33), using (2–8), we obtain

$$\begin{aligned} 2Ai\nu k - (2k^2 - k_\beta^2)B &= 0 \\ (2k^2 - k_\beta^2)A + 2Bik\nu' &= \frac{Z(k)}{\mu} \end{aligned} \tag{2–36}$$

If the conventions $\nu = (k^2 - k_\alpha^2)^{\frac{1}{2}} = ik(c^2/\alpha^2 - 1)^{\frac{1}{2}}$ and $\operatorname{Re} \nu \geq 0$ with similar ones for ν' are attended to, the signs in these equations correspond to the lower signs in Eqs. (2–24), (2–25), and (2–27).

Solving these equations for A and B, we obtain

$$A = \frac{2k^2 - k_\beta^2}{F(k)} \frac{Z(k)}{\mu} \qquad B = \frac{2ik\nu}{F(k)} \frac{Z(k)}{\mu} \tag{2–37}$$

where

$$F(k) \equiv (2k^2 - k_\beta^2)^2 - 4k^2\nu\nu' \tag{2–38}$$

is Rayleigh's function.

We now wish to superpose an infinite number of stress distributions of the form (2–33) such that the resultant is a concentrated line source. To do this, we put $Z = -Q\,dk/2\pi$ in (2–33) and integrate with respect to k from $-\infty$ to $+\infty$, obtaining

$$[p_{xz}]_{z=0} = f(x) = -\frac{Q}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx}\,dk \tag{2–39}$$

Then if we put in (2–39)

$$\int_{-\infty}^{+\infty} f(\xi) e^{ik\xi}\,d\xi = -Q(k) \tag{2–40}$$

†Because $\exp[i(\omega t - kx)]$ instead of the factor $\exp[i(pt + \xi x)]$ of Lamb [22] is used, some of our expressions differ from his in sign.

we obtain the Fourier integral

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx}\, dk \int_{-\infty}^{+\infty} f(\xi) e^{ik\xi}\, d\xi \tag{2-41}$$

representing the distribution of the normal surficial stresses. In order to obtain a concentrated normal force at $x = 0$, assume that the normal force $f(x)$ along the x axis vanishes everywhere except at $x = 0$, where it approaches infinity in such a way that $\int_{-\infty}^{+\infty} f(\xi)\, d\xi = -Q$ is finite. Then Eq. (2–41) reduces to (2–39), with Q a constant.

With the stress specified by (2–39), the displacement of any point in the surface $z = 0$ may be written, using Eqs. (2–1), (2–34), and (2–37), as

$$\begin{aligned} u_0 &= \frac{iQ}{2\pi\mu} \int_{-\infty}^{+\infty} \frac{k(2k^2 - k_\beta^2 - 2\nu\nu')}{F(k)} e^{-ikx}\, dk \\ w_0 &= -\frac{Q}{2\pi\mu} \int_{-\infty}^{+\infty} \frac{k_\beta^2 \nu}{F(k)} e^{-ikx}\, dk \end{aligned} \tag{2-42}$$

The effect of a concentrated horizontal force P acting parallel to the x axis at the origin can be represented in a similar way. Methods of evaluating integrals such as (2–42) will be discussed later, as well as a generalization for an impulsive source.

Internal Source. Lamb [22] also considered an internal line source of compressional waves. Lapwood [25] has given a more detailed discussion of this problem.

A solution of the wave equation representing compressional waves propagating cylindrically outward from a line source at $x = 0$, $z = h$ may be written as

$$\varphi_0 = i\pi e^{i\omega t} H_0^{(2)}(k_\alpha R) \qquad \psi_0 = 0 \tag{2-43}$$

where $H_0^{(2)}$ = Hankel function of second kind of zero order
$R^2 = x^2 + (z - h)^2$

This particular form is chosen because, where $|k_\alpha R|$ is large,

$$H_0^{(2)}(k_\alpha R) \sim \sqrt{\frac{2i}{\pi k_\alpha R}}\, e^{-ik_\alpha R}$$

(see Watson [56, p. 198]), and Eqs. (2–43) represent cylindrical waves diverging outward with velocity α. Another transformation of the Hankel function may be used to write (2–43) in the form

$$\varphi_0 = i\pi e^{i\omega t} H_0^{(2)}(k_\alpha R) = -2e^{i\omega t} \int_0^\infty e^{-\nu|h-z|} \cos kx\, \frac{dk}{\nu} \tag{2-44}$$

Hereafter the factor $e^{i\omega t}$, although not written, will be understood. This transformation may be proved by a method not unlike that used for the

corresponding three-dimensional transformation (1–41). By a simple transformation,

$$-2\int_0^\infty \cos kx\, e^{-\nu z}\frac{dk}{\nu} = -\int_{-\infty}^{\infty} e^{-\nu z - ikx}\frac{dk}{\nu} \qquad (2\text{–}44')$$

Comparing (2–44) and (2–44′) with (2–34), we can obtain an interpretation of the former in terms of the superposition of plane waves.

We note that the convention $\operatorname{Re}\nu \geq 0$ for ν given by (2–35) must be adhered to in Eq. (2–44) in order to ensure the vanishing of φ_0 as $z \to \infty$. Now, in order to have vanishing normal stress p_{zz} at the free surface, we can make use of the functions

$$\varphi_{0r} = -4\int_0^\infty e^{-\nu h}\frac{\sinh \nu z}{\nu}\cos kx\, dk \qquad \psi_0 = 0 \qquad (2\text{–}45)$$

formed by adding to (2–43) the potential of an equal and opposite image source $(0, -h)$ corresponding to a reflection at the free surface. If the functions (2–45) are inserted into (2–8), it is found that $p_{zz} = 0$ at $z = 0$. In order to make the tangential stress p_{xz} also vanish at $z = 0$ we add to (2–45) the potentials φ and ψ in the form

$$\begin{aligned}\varphi &= 4\int_0^\infty (A\cos kx + B\sin kx)e^{-\nu z}\,dk \\ \psi &= 4\int_0^\infty (C\cos kx + D\sin kx)e^{-\nu' z}dk\end{aligned} \qquad (2\text{–}46)$$

The necessity for superposing functions such as φ and ψ in order to obtain vanishing stress at the free surface is connected with the curvature of the incident wave front corresponding to φ_0. All boundary conditions can be satisfied by incident and reflected P and SV waves alone only when the waves are plane.

The first integrand in (2–46) satisfies the condition of vanishing potential as $z \to \infty$ from our choice of sign for $\operatorname{Re}\nu$. In order to insure that ψ satisfies this condition, we require $\operatorname{Re}\nu' \geq 0$. Now, substitution of the functions $\varphi_{0r} + \varphi$ and ψ in the boundary conditions (2–8) leads to two integrals which must vanish for all values of x. Since the coefficients of $\cos kx$ and $\sin kx$ in these integrals must vanish separately, we obtain four equations for A, B, C, D. One finds first that $B = C = 0$. Then, solving for A and D and substituting in (2–46), we obtain for a compressional source

$$\begin{aligned}\varphi &= 16\int_0^\infty \frac{k^2\nu'}{F(k)}e^{-\nu(h+z)}\cos kx\, dk \\ \psi &= -8\int_0^\infty \frac{k(2k^2 - k_\beta^2)}{F(k)}e^{-\nu h - \nu' z}\sin kx\, dk\end{aligned} \qquad (2\text{–}47)$$

Equations (2–47) together with (2–45) complete the steady-state solution for an internal line source of compressional waves. For an internal source of shear waves the corresponding functions may be found in a similar manner.

We see from Eqs. (2–37) that for $Z = 0$ the coefficients A and B can have finite values only if $F(k) = 0$, that is, for those values of k for which free Rayleigh waves are possible. These values of k are also singularities of the integrands in (2–47).

A method of evaluating integrals of the type (2–47) will be given in Sec. 2–5.

2–4. Integral Solutions for a Point Source. Let us assume that the primary disturbance varies as a simple harmonic function of time.

Surface Source. The case of a point source located at the surface of a semi-infinite medium has more applications than the two-dimensional problems considered in previous sections. Symmetry about the z axis (taken through the source) is usually assumed, and we can put

$$r = \sqrt{x^2 + y^2} \qquad u = \frac{x}{r} q \qquad v = \frac{y}{r} q \tag{2–48}$$

The displacements w and q, parallel and perpendicular to the z axis, are represented in terms of potentials φ and ψ, as in Eqs. (1–26) and (1–28):

$$q = \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z} \qquad w = \frac{\partial \varphi}{\partial z} - \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = \frac{\partial \varphi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{\beta^2}\frac{\partial^2 \psi}{\partial t^2} \tag{2–49}$$

If we again assume a time variation $e^{i\omega t}$, the wave equations (1–27) and (1–29) take the reduced form

$$(\nabla^2 + k_\alpha^2)\varphi = 0 \qquad (\nabla^2 + k_\beta^2)\psi = 0 \tag{2–50}$$

$$k_\alpha^2 = \frac{\omega^2}{\alpha^2} \qquad k_\beta^2 = \frac{\omega^2}{\beta^2} \tag{2–51}$$

For the axial symmetry,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2–52}$$

and particular solutions of (2–50) may be taken in the form

$$\varphi = S(kr)T(z) \qquad \psi = S_1(kr)T_1(z) \tag{2–53}$$

provided that the functions S and T satisfy the equations

$$\begin{gathered} \frac{d^2S}{dr^2} + \frac{1}{r}\frac{dS}{dr} + (k_\alpha^2 + \nu^2)S = 0 \\ \frac{d^2T}{dz^2} - \nu^2 T = 0 \end{gathered} \tag{2–54}$$

with $\nu^2 = k^2 - k_\alpha^2$. Similar equations hold for S_1, T_1, with $\nu'^2 = k^2 - k_\beta^2$. The second of Eqs. (2–54) has a particular solution $T = e^{-\nu' z}$; the first is a form of Bessel's equation which is satisfied by the Bessel function $J_0(kr)$. Thus we have two particular solutions

$$\varphi = Ae^{-\nu z} J_0(kr) \qquad \psi = Be^{-\nu' z} J_0(kr) \tag{2–55}$$

A and B being two constants. If k is real and $k_\alpha < k_\beta < k$, ν and ν' must be positive real. The solutions (2–55) vanish as $z \to \infty$ and also vanish as $r \to \infty$ because of the property of the Bessel function $J_0(kr)$.

By (2–49) we have

$$\begin{aligned} q &= -(kAe^{-\nu z} - k\nu' Be^{-\nu' z}) J_1(kr) \\ w &= (-\nu Ae^{-\nu z} + k^2 Be^{-\nu' z}) J_0(kr) \end{aligned} \tag{2–56}$$

using the relation

$$J_1(kr) = -\frac{dJ_0(kr)}{d(kr)}$$

The stress equations in cylindrical coordinates are

$$p_{zr} = \mu\left(\frac{\partial q}{\partial z} + \frac{\partial w}{\partial r}\right) \qquad p_{zz} = \lambda\theta + 2\mu \frac{\partial w}{\partial z} \tag{2–57}$$

where $\theta = \nabla^2\varphi = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r \dfrac{\partial \varphi}{\partial r}\right) + \dfrac{\partial^2 \varphi}{\partial z^2}$

With Eqs. (2–57) we get for the stresses in the plane $z = 0$, with $\lambda = \mu$,

$$\begin{aligned} [p_{zr}]_{z=0} &= \mu[2k\nu A - k(2k^2 - k_\beta^2)B] J_1(kr) \\ [p_{zz}]_{z=0} &= \mu[(2k^2 - k_\beta^2)A - 2k^2\nu' B] J_0(kr) \end{aligned} \tag{2–58}$$

by Eqs. (2–56). Equations (2–56) and hence (2–58) may be also derived from Eqs. (2–1) and (2–34) for a line source if we consider the effect of an infinite number of line sources in a uniform arrangement about the z axis. To form the resultant displacements we note that the variable x corresponds to $r \cos\theta$ and integrate (2–34) with respect to θ from 0 to π, dividing the integral by the length of the interval π. The displacements in the z direction are obviously directly additive, whereas the sum of displacement components in the r direction must be taken, by multiplying by $\cos\theta$ before the integration is performed. Thus we have

$$\begin{aligned} q &= (-ikAe^{-\nu z} + B\nu' e^{-\nu' z}) \int_0^\pi e^{-ikr\cos\theta} \cos\theta \frac{d\theta}{\pi} \\ w &= (-\nu Ae^{-\nu z} - Bike^{-\nu' z}) \int_0^\pi e^{-ikr\cos\theta} \frac{d\theta}{\pi} \end{aligned} \tag{2–56′}$$

If $\bar{B}$ is replaced by Bik, this result becomes identical with (2–56), as may be seen from the expression for a Bessel function of the nth order

$$J_n(\xi) = \frac{i^{-n}}{\pi} \int_0^\pi e^{i\xi \cos\theta} \cos n\theta \, d\theta$$

$J_0(\xi)$ being an even function of ξ and $J_1(\xi)$ an odd function of ξ.

To obtain the solution for a point source we now consider a force per unit area $ZJ_0(kr)$ acting normal to the free surface. Appropriate boundary conditions for this case are

$$[p_{rr}]_{z=0} = 0 \qquad [p_{zz}]_{z=0} = ZJ_0(kr) \tag{2–59}$$

Then by (2–57) and (2–58) we have

$$\begin{aligned} -2\nu A + (2k^2 - k_\beta^2)B &= 0 \\ (2k^2 - k_\beta^2)A - 2k^2\nu' B &= \frac{Z}{\mu} \end{aligned} \tag{2–60}$$

whence

$$A = \frac{2k^2 - k_\beta^2}{F(k)} \frac{Z}{\mu} \qquad B = \frac{2\nu}{F(k)} \frac{Z}{\mu} \tag{2–61}$$

where $F(k) = (2k^2 - k_\beta^2)^2 - 4k^2\nu\nu'$

The expression (2–61) may be compared to (2–37), since B given by Eqs. (2–37) was replaced by ikB. The function $F(k)$ is identical to (2–38). The displacements (2–56) at the surface $z = 0$ can now be written in the form

$$\begin{aligned} q_0 &= -\frac{k(2k^2 - k_\beta^2 - 2\nu\nu')}{F(k)} J_1(kr) \frac{Z}{\mu} \\ w_0 &= \frac{k_\beta^2 \nu}{F(k)} J_0(kr) \frac{Z}{\mu} \end{aligned} \tag{2–62}$$

FREE ANNULAR RAYLEIGH WAVES. Free surface waves correspond to $Z = 0$. In this case the constants A and B in (2–55) have values different from zero, as was mentioned at the end of Sec. 2–3, if the parameter k is equal to a root κ of the equation $F(k) = 0$. Then, on writing $k = \kappa$, $Z = 0$ in (2–61), we can put

$$A = (2\kappa^2 - k_\beta^2)C \qquad B = 2\nu_R C = 2\sqrt{\kappa^2 - k_\alpha^2}\, C \tag{2–63}$$

where C is a new arbitrary constant. Writing the time factor $e^{i\omega t}$ again, we have by (2–56) the surface displacements ($z = 0$) corresponding to free annular surface waves

$$\begin{aligned} q_0 &= -C\kappa(2\kappa^2 - k_\beta^2 - 2\nu_R\nu_R')J_1(\kappa r)e^{i\omega t} \\ w_0 &= Ck_\beta^2\nu_R J_0(\kappa r)e^{i\omega t} \end{aligned} \tag{2–64}$$

where $\nu_R^2 = \kappa^2 - k_\alpha^2$

$\nu_R'^2 = \kappa^2 - k_\beta^2$

FORCED WAVES. In order to pass to a concentrated force $Le^{i\omega t}$ acting at the origin, we can make use of the Fourier-Bessel integral (see, for example, Ref. 18, p. 559) to represent the stress distribution p_{zz}

$$[p_{zz}]_{z=0} = f(r) = \int_0^\infty J_0(kr)k\,dk \int_0^\infty f(\sigma)J_0(k\sigma)\sigma\,d\sigma \tag{2-65}$$

Now suppose that $f(\sigma)$ vanishes for all but infinitesimal values of σ, where it becomes infinite in such a way that

$$\int_0^\infty f(\sigma)2\pi\sigma\,d\sigma = -L \tag{2-66}$$

is finite. Then (2-65) reduces to

$$[p_{zz}]_{z=0} = -\frac{L}{2\pi}\int_0^\infty J_0(kr)k\,dk \tag{2-67}$$

To obtain the solutions for a surface point source, we can put $Z = -Lk\,dk/2\pi$ in (2-59) and (2-62) and integrate with respect to k from 0 to ∞:

$$\begin{aligned} \hat{q}_0 &= \frac{L}{2\pi\mu}\int_0^\infty \frac{k^2(2k^2 - k_\beta^2 - 2\nu\nu')}{F(k)} J_1(kr)\,dk \\ \hat{w}_0 &= -\frac{L}{2\pi\mu}\int_0^\infty \frac{k_\beta^2 k\nu}{F(k)} J_0(kr)\,dk \end{aligned} \tag{2-68}$$

By finding the surface stresses corresponding to Eqs. (2-68) it may be verified that $[p_{rz}]_{z=0} = 0$ and $[p_{zz}]_{z=0}$ reduces to Eq. (2-67) which, by (2-65) and (2-66), satisfies the condition for a concentrated force at the origin.

In order to compare these results with previous ones for two dimensions we use the relations which hold for Bessel functions of the zero and first order:

$$\begin{aligned} J_0(kr) &= -\frac{i}{\pi}\int_0^\infty (e^{ikr\cosh u} - e^{-ikr\cosh u})\,du \\ J_1(kr) &= -\frac{1}{\pi}\int_0^\infty (e^{ikr\cosh u} + e^{-ikr\cosh u})\cosh u\,du \end{aligned} \tag{2-69}$$

(See, for example, Ref. 56, p. 180.) We may now rewrite Eqs. (2-68) in the form

$$\begin{aligned} \hat{q}_0 &= -\frac{L}{2\pi^2\mu}\int_0^\infty \cosh u\,du \int_{-\infty}^\infty \frac{k^2(2k^2 - k_\beta^2 - 2\nu\nu')}{F(k)} e^{-ikr\cosh u}\,dk \\ \hat{w}_0 &= -\frac{Li}{2\pi^2\mu}\int_0^\infty du \int_{-\infty}^\infty \frac{k_\beta^2 k\nu}{F(k)} e^{-ikr\cosh u}\,dk \end{aligned} \tag{2-70}$$

We see that the definite integrals with respect to k in Eqs. (2–70) are obtained from the integrals (2–42) for the two-dimensional case by performing the operation $-\partial/\pi\partial x$ and substituting

$$x = r \cosh u \qquad Q = L$$

Internal Source. In order to obtain the potentials corresponding to an internal source in a half space, we again follow the method of Lamb.

COMPRESSIONAL-WAVE SOURCE IN AN UNLIMITED SOLID. To represent a point source of compressional waves, let $R = \sqrt{r^2 + z^2}$ be the distance from the source and write the potentials

$$\varphi = \frac{e^{i(\omega t - k_\alpha R)}}{R} = e^{i\omega t} \int_0^\infty J_0(kr) e^{-\nu|z|} \frac{k\,dk}{\nu} \tag{2–71}$$

$$\psi = 0$$

The transformation (1–41) enables us to use the integral form of the potentials in which r and z occur in separate factors. In practical applications this solution represents the effect of an explosive source in a medium, provided that the wave lengths considered are large compared with the diameter of the source and small compared with the distance to the nearest boundary.

CONCENTRATED FORCE. This problem involves a normal periodic force $ZJ_0(kr)$ per unit area acting at the plane $z = 0$. On considering an unlimited solid in addition to expressions (2–55) for $z > 0$, we now need the following potentials for $z < 0$:

$$\varphi = A'e^{\nu' z} J_0(kr) \qquad \psi = B'e^{\nu' z} J_0(kr) \tag{2–72}$$

The conditions for stresses on the plane $z = 0$ are

$$\begin{aligned} [p_{zz}]_{+0} - [p_{zz}]_{-0} &= ZJ_0(kr) \\ [p_{zr}]_{+0} - [p_{zr}]_{-0} &= 0 \end{aligned} \tag{2–73}$$

By Eqs. (2–8), (2–55), and (2–72) we obtain for $\lambda = \mu$

$$\begin{aligned} (2k^2 - k_\beta^2)(A - A') - 2k^2\nu'(B + B') &= \frac{Z}{\mu} \\ -2\nu(A + A') + (2k^2 - k_\beta^2)(B - B') &= 0 \end{aligned} \tag{2–74}$$

Similarly, the condition of continuity of displacements on $z = 0$ leads to

$$\begin{aligned} A - A' - \nu'(B + B') &= 0 \\ \nu(A + A') - k^2(B - B') &= 0 \end{aligned} \tag{2–75}$$

From (2–74) and (2–75) it follows that

$$A = -A' = -\frac{Z}{2k_\beta^2\mu} \qquad B = B' = -\frac{Z}{2k_\beta^2\mu\nu'} \tag{2–76}$$

and Eqs. (2–55) now take the form (for $z > 0$)

$$\varphi = \frac{-Z}{2k_\beta^2\mu} e^{-\nu' z} J_0(kr) \qquad \psi = \frac{-Z}{2\mu\nu' k_\beta^2} e^{-\nu' z} J_0(kr) \tag{2–77}$$

To represent a concentrated force $Le^{i\omega t}$ acting at the origin we follow the procedure of the previous section. Put $Z = -Lk\,dk/2\pi$ in Eqs. (2–77) and integrate with respect to k from 0 to ∞. Using $k_\beta = \omega/\beta$ and $\mu = \beta^2\rho$, we obtain

$$\begin{aligned} \hat{\varphi} &= \frac{L}{4\pi\omega^2\rho} \int_0^\infty e^{-\nu z} J_0(kr) k\,dk \\ \hat{\psi} &= \frac{L}{4\pi\omega^2\rho} \int_0^\infty e^{-\nu' z} J_0(kr) \frac{k\,dk}{\nu'} \end{aligned} \qquad \text{for } z > 0 \tag{2–78}$$

Similarly, by using Eqs. (2–72) and (2–76) we obtain for $z < 0$

$$\hat{\varphi} = -\frac{L}{4\pi\omega^2\rho} \int_0^\infty e^{\nu z} J_0(kr) k\,dk \tag{2–79}$$

$$\hat{\psi} = \frac{L}{4\pi\omega^2\rho} \int_0^\infty e^{\nu' z} J_0(kr) \frac{k\,dk}{\nu'} \tag{2–80}$$

The expression (1–41) may be used to transform the potentials $\hat{\varphi}$ and $\hat{\psi}$ in Eqs. (2–78) into

$$\hat{\varphi} = -\frac{L}{4\pi\omega^2\rho} \frac{\partial}{\partial z} \frac{e^{-ik_\alpha R}}{R} \qquad \hat{\psi} = \frac{L}{4\pi\omega^2\rho} \frac{e^{-ik_\beta R}}{R} \qquad \text{for } z > 0 \tag{2–81}$$

which represent the solution of the problem as given by Stokes [52].

INTERNAL COUPLE. Of particular interest in the study of earthquakes is a source which simulates the failure in shear that is known to be the principal action at the foci of earthquakes. A suitable expression for the potential may be obtained by extending the solution of the previous section to the case of two equal and opposite forces constituting a couple. As might be expected, this potential may be derived by taking suitable derivatives of Eqs. (2–81).

COMPRESSIONAL-WAVE SOURCE IN A HALF SPACE. Next we consider the problem of finding the disturbances produced in a half space by an internal point source of compressional waves. Let us consider a potential φ representing two spherical compressional waves, one originating at a source $S(0, 0, h)$, the other apparently originating at the image point $S'(0, 0, -h)$. Thus from (2–71)

$$\begin{aligned} \varphi &= \int_0^\infty e^{-\nu|z-h|} J_0(kr) \frac{k\,dk}{\nu} + \int_0^\infty e^{-\nu|z+h|} J_0(kr) \frac{k\,dk}{\nu} \\ \psi &= 0 \end{aligned} \tag{2–82}$$

For $0 < z < h$ we put $|z - h| = h - z$ and obtain

$$\varphi = 2\int_0^\infty \cosh \nu z\, e^{-\nu h} J_0(kr)\, \frac{k\, dk}{\nu} \qquad \psi = 0 \tag{2-83}$$

From (2–49) and (2–83) the displacements at $z = 0$ are found to be

$$q_0 = -2\int_0^\infty e^{-\nu h} J_1(kr)\, \frac{k^2\, dk}{\nu} \qquad w_0 = 0 \tag{2-84}$$

The stresses at the free surface determined by the potentials (2–83) are

$$[p_{zr}]_{z=0} = 0 \qquad [p_{zz}]_{z=0} = 2\mu\int_0^\infty \frac{(2k^2 - k_\beta^2)}{\nu} e^{-\nu h} J_0(kr) k\, dk \tag{2-85}$$

Thus an additional system of surface stresses is required to satisfy the condition $p_{zz} = 0$ at $z = 0$. To obtain potentials which will annul the residual stress of (2–85) we put

$$Z = -2\mu\, \frac{2k^2 - k_\beta^2}{\nu} e^{-\nu h} k\, dk$$

in Eqs. (2–59), (2–55), (2–61), and (2–62) and integrate with respect to k from 0 to ∞. By adding the displacements (2–62) with the changes mentioned and (2–84) we obtain the resultant surface displacements

$$\begin{aligned} q_0 &= 4e^{i\omega t}\int_0^\infty \frac{\nu' k_\beta^2 k^2}{F(k)} e^{-\nu h} J_1(kr)\, dk \\ w_0 &= -2e^{i\omega t}\int_0^\infty \frac{k_\beta^2 k(2k^2 - k_\beta^2)}{F(k)} e^{-\nu h} J_0(kr)\, dk \end{aligned} \tag{2-86}$$

2–5. Evaluation of Integral Solutions. The integrals obtained as solutions of the problems treated in this chapter cannot be evaluated by direct integration, and evaluation by numerical integration is exceedingly difficult.

Application of Contour Integration. A useful approach is to replace the variable of integration k by the complex variable ζ and to use contour integration in the ζ plane. We shall evaluate integrals of the form

$$\int \Phi(\zeta)\, d\zeta = \int \frac{\zeta(2\zeta^2 - k_\beta^2 - 2\sqrt{\zeta^2 - k_\alpha^2}\sqrt{\zeta^2 - k_\beta^2})}{(2\zeta^2 - k_\beta^2)^2 - 4\zeta^2\sqrt{\zeta^2 - k_\alpha^2}\sqrt{\zeta^2 - k_\beta^2}} e^{-i\zeta z}\, d\zeta \tag{2-87}$$

$$\int \Psi(\zeta)\, d\zeta = \int \frac{k_\beta^2\sqrt{\zeta^2 - k_\alpha^2}}{(2\zeta^2 - k_\beta^2)^2 - 4\zeta^2\sqrt{\zeta^2 - k_\alpha^2}\sqrt{\zeta^2 - k_\beta^2}} e^{-i\zeta z}\, d\zeta \tag{2-88}$$

which occur in the solutions (2–42) representing the surface displacements for a surface line source. Our results may be also adapted to the integrals with respect to k, which are part of the solutions (2–70) for a surface point source, by performing the operation $-\partial/\pi\partial x$ and substituting $x = r \cosh u$.

We note first that when ζ is a complex variable, $\zeta = k \pm i\tau$, the integrand $\Phi(\zeta)$ or $\Psi(\zeta)$ has real poles $(\pm\kappa, 0)$, determined by the zeros κ of $F(\zeta)$, the denominator of the integrand defined by (2-88). Branch points $(\pm k_\alpha, 0)$, $(\pm k_\beta, 0)$ are introduced by the radicals $\nu = \pm\sqrt{\zeta^2 - k_\alpha^2}$ and $\nu' = \pm\sqrt{\zeta^2 - k_\beta^2}$. The existence of branch points requires that the integrands be made uniform functions before Cauchy's theorem is applied. This is accomplished by introducing cuts in the complex plane. The signs of the radicals ν and ν' will be determined by the conditions $\text{Re } \nu \geq 0$, $\text{Re } \nu' \geq 0$ in agreement with the signs used previously, since with these choices infinite values of φ as $z \to \infty$ are avoided.

In (2-87) and (2-88) the branch points $\pm k_\alpha$ and $\pm k_\beta$ are located on the real axis but when the methods of operational analysis are applied, the complex values of ω also have to be considered. Therefore we will first take the cuts for complex $k_\alpha = \omega/\alpha$ and $k_\beta = \omega/\beta$. The Riemann surface for integrands in (2-87) and (2-88) has four sheets, as there are four combinations of signs of ν and ν'. The permissible sheet must be selected according to the requirements $\text{Re } \nu \geq 0$ and $\text{Re } \nu' \geq 0$. Thus, the cuts will be given by $\text{Re } \nu = 0$, $\text{Re } \nu' = 0$. For a complex $\omega = s - i\sigma$ we have $k_\alpha = (s - i\sigma)/\alpha$, $k_\beta = (s - i\sigma)/\beta$. $\text{Re } \nu = 0$, where $\nu^2 = \zeta^2 - k_\alpha^2$, requires that $k^2 - \tau^2 + 2ik\tau - (s^2 - \sigma^2 - 2is\sigma)/\alpha^2$ be real and negative, or

$$k\tau = -s\sigma/\alpha^2 \qquad \text{and} \qquad k^2 - \tau^2 < (s^2 - \sigma^2)/\alpha^2 \tag{2-89}$$

The first of these conditions shows that for $s > 0$ the branch points and cuts must lie as in Fig. 2-6, these cuts being parts of hyperbolas. The second condition defines the part of the hyperbola to be used as a cut. To simplify the further discussion we shall assume that there is only one pair of branch points $\pm k_\alpha$. For a real ω, that is, $\sigma = 0$, the conditions (2-89) take the form

$$k\tau = 0 \qquad k^2 - \tau^2 < \frac{s^2}{\alpha^2} \tag{2-90}$$

They show that either

$$\tau = 0 \qquad k^2 < \frac{s^2}{\alpha^2} \tag{2-91}$$

or

$$k = 0 \qquad -\tau^2 < \frac{s^2}{\alpha^2} \tag{2-92}$$

Thus, since the condition $\text{Re } \nu \geq 0$ restricts the choice of a cut, we can use, according to (2-91), a part of the real axis between the branch points $-k_\alpha(B)$ and $k_\alpha(A)$ [see Fig. (2-7)]. The imaginary axis determined by the conditions (2-92) is not an independent cut since it does not pass through a branch point. However, it can be used if it is combined with that part of the real axis to form the cuts *AOE* and *BOL*, as shown in Fig. 2-7.

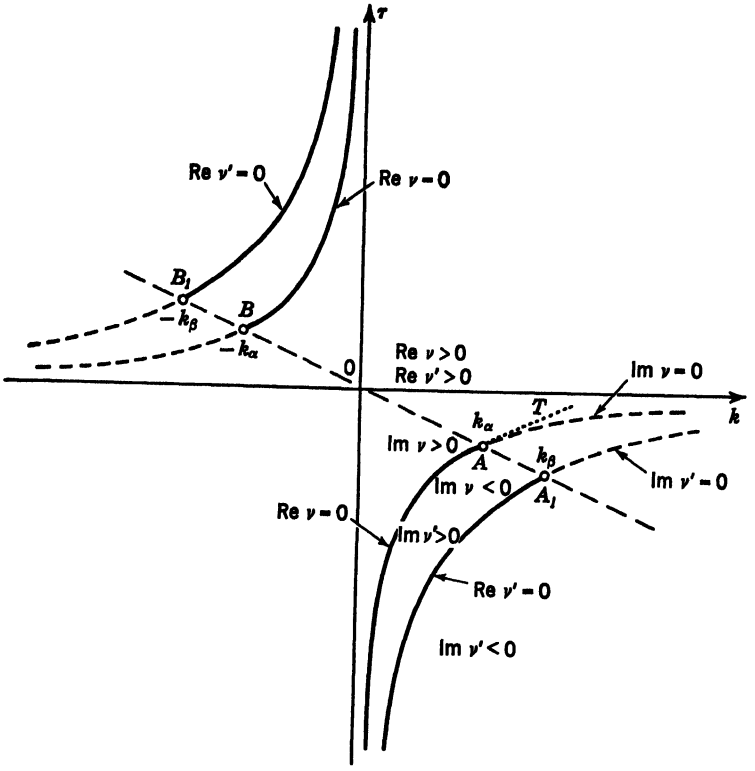


FIG. 2-6. Branch points and cuts in the complex $\zeta = k + i\tau$ plane for Re $\omega > 0$.

The cuts AOE and BOL appear then as limiting cases of those parts of a hyperbola (Fig. 2-6) which are cuts for complex ω.

For ω real, Re ν does not change sign on any permissible path in the right half plane. Only for $\zeta = k \geq k_\alpha$, that is, on Ak in Fig. 2-7, Im $\nu = 0$. Thus, for any permissible path in the right half plane, Im ν can change sign only on crossing Ak.

Now, from the transformation

$$\zeta = k_\alpha + \delta e^{i\theta} \qquad (-\pi < \theta < \pi) \tag{2-93}$$

where θ is measured from the k axis,

$$\nu = +\sqrt{2k_\alpha\delta}\, e^{i\theta/2} = \sqrt{2k_\alpha\delta}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \tag{2-94}$$

for a small region around A, in which δ^2 may be neglected.

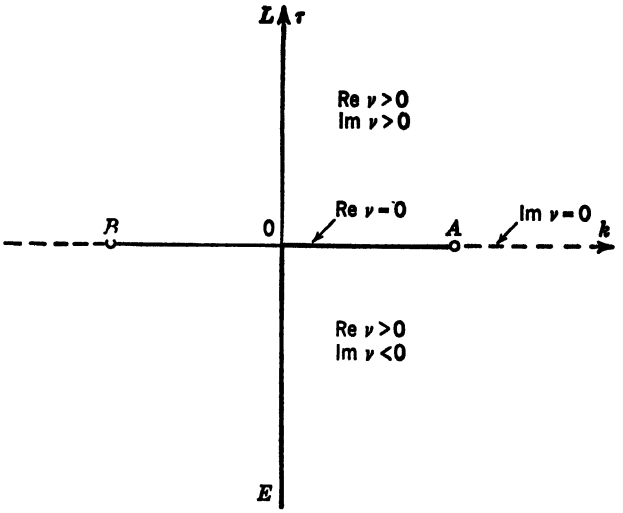


FIG. 2–7. Branch points A and B and cuts AOE and BOL in the complex ζ plane for real ω.

From the foregoing we see that for all permissible paths in the right half plane Re $\nu > 0$ while Im $\nu > 0$ in the first and Im $\nu < 0$ in the fourth quadrant.

For complex ω, Re $\nu = 0$ on the first part of a hyperbola determined by Eqs. (2–89). The Im ν is zero in the second part given by (2–89). Im ν is discontinuous along the branch line, being positive on the left side and negative on the right side of the cut in Fig. 2–6. To determine the signs of Im ν, introduce the angle $-\pi < \theta < \pi$ measured from AT, the tangent to the hyperbolic cut (see Fig. 2–6). Following the same procedure as for real ω, we may determine the signs as shown.

Residues. Lamb [22] demonstrated that Rayleigh waves are the largest disturbance at a surface point far removed from a surface impulse. In addition, he found other terms representing waves which diminish more rapidly with increasing distance from the source. We shall now evaluate the surface-wave terms.

If we consider the integrals (2–42) for a real ω, the singular points of the integrand are on the path of integration, and principal values of integrals must be used, as was done by Lamb. The assumption that ω is complex will displace the points k_α and k_β as well as the roots κ of the equation $F(\zeta) = 0$ from the real k axis to a line whose slope is Im ω/Re ω. Therefore, it seems to be more useful to consider first the contour shown

in Fig. 2–8 and then to take the limiting case ω real. Now taking, for example, the integral (2–87), we see that

$$\oint \Phi(\zeta)d\zeta = \int_M^N + \int_N^H + \int_{L_\alpha} + \int_{L_\beta} + \int_G^M = 2\pi i \sum \text{Res}$$

On the infinite arcs NH and GM, because of our choice of the contour in the lower half plane, the presence of the factor $\exp(-i\zeta x) = \exp(-ikx)$ ·

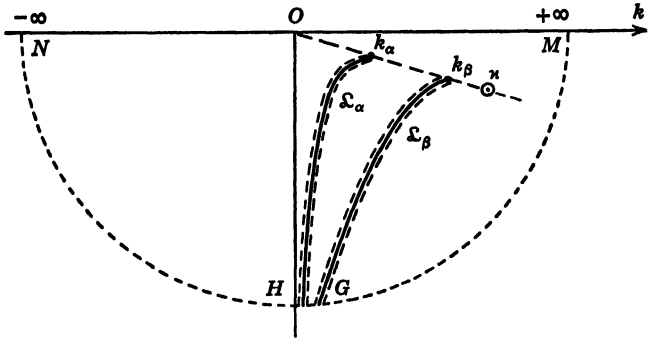


FIG. 2–8. Integration path in the complex ζ plane.

$\exp(-\tau x)$ makes the integrals zero. Therefore the integral along the real axis is

$$\int_{-\infty}^{\infty} \Phi(k)\, dk = -2\pi i \sum \text{Res} + \int_{L_\alpha} + \int_{L_\beta} \tag{2–95}$$

There is only one pole (κ) in this case, and the integrals along the loops L_α and L_β are branch line integrals. Using for the residue of an integrand $M(x)/N(x)$ at a pole $x = a$ the expression $M(a)/N'(a)$, we can easily find that by Eqs. (2–87) and (2–88)

$$\int_{-\infty}^{\infty} \frac{k(2k^2 - k_\beta^2 - 2\nu\nu')}{F(k)} e^{-ikz}\, dk = 2\pi i H e^{-i\kappa z} + \int_{L_\alpha} \Phi(\zeta)\, d\zeta + \int_{L_\beta} \Phi(\zeta)\, d\zeta \tag{2–96}$$

$$\int_{-\infty}^{\infty} \frac{k_\beta^2 \nu}{F(k)} e^{-ikz}\, dk = 2\pi i K e^{-i\kappa z} + \int_{L_\alpha} \Psi(\zeta)\, d\zeta + \int_{L_\beta} \Psi(\zeta)\, d\zeta \tag{2–97}$$

where

$$H = -\frac{\kappa(2\kappa^2 - k_\beta^2) - 2\sqrt{\kappa^2 - k_\alpha^2}\sqrt{\kappa^2 - k_\beta^2}}{F'(\kappa)}$$

$$K = -\frac{k_\beta^2 \sqrt{\kappa^2 - k_\alpha^2}}{F'(\kappa)} \tag{2–98}$$

Applying these results to the integrals for the displacements (2–42) produced by a surface line source, we obtain

$$u_0 = -\frac{QH}{\mu} \exp [i(\omega t - \kappa x)] + \cdots$$
$$w_0 = -i\frac{QK}{\mu} \exp [i(\omega t - \kappa x)] + \cdots \tag{2–99}$$

where terms derived from the branch line integrals have been omitted for the present. The first terms in Eqs. (2–99) result from the contribution of the pole in the integrands (2–42) and represent a train of Rayleigh waves traveling away from the source with an amplitude of displacement independent of x. The velocity is that of Rayleigh waves, since κ is a root of Rayleigh's equation $F(k) = 0$ [see (2–38)]. It is seen that the orbital motion of a surface particle is retrograde elliptical, the ratio of vertical to horizontal amplitudes being K/H. This ratio is the same as that obtained for free waves in Sec. 2–2.

Branch Line Integrals: Line Source. To obtain further information about the waves represented by the branch line integrals, we first examine for real ω the last two terms in Eqs. (2–96) and (2–97). In this case, the two loops L_α and L_β degenerate into one (L), with branch points k_α and k_β on the real axis. This loop is shown in Fig. 2–9. The cut A_1AOH obtained

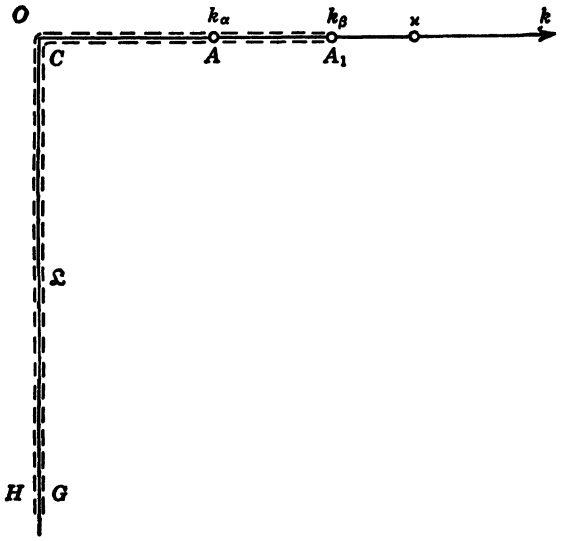


FIG. 2–9. Loop $\mathfrak{L}$ formed by contraction of $\mathfrak{L}_\alpha$ and $\mathfrak{L}_\beta$ for ω real.

by contracting two hyperbolas is characterized by the condition that Re $\nu' = 0$ along the entire cut and Re $\nu = 0$ along its part AOH. Now under the conditions given earlier in this section the imaginary parts of ν and ν' are discontinuous along the corresponding cut. Im ν and Im ν' are positive in the first quadrant and on the left side of the cut OH and are negative in the fourth quadrant. Writing now, for example, for (2–87)

$$\int_L \frac{\zeta(2\zeta^2 - k_\beta^2 - 2\nu\nu')}{(2\zeta^2 - k_\beta^2)^2 - 4\zeta^2\nu\nu'} e^{-i\zeta z}\, d\zeta$$

we see that the integrand is a function of the product $\nu\nu'$. On the part of the loop HOA the product $\nu\nu'$ has the same value as on GCA (Fig. 2–9), and the corresponding parts of the integral along L cancel each other. Thus, the branch line integrals of the type (2–87) are reduced to the term

$$I_1 = \int_{k_\alpha}^{k_\beta} \left\{ \frac{2k^2 - k_\beta^2 - 2\nu\nu'_{,}}{(2k^2 - k_\beta^2)^2 - 4k^4\nu\nu'_{,}} - \frac{2k^2 - k_\beta^2 - 2\nu\nu'_{,,}}{(2k^2 - k_\beta^2)^2 - 4k^4\nu\nu'_{,,}} \right\} e^{-ikz} k\, dk \quad (2\text{–}100)$$

where
$$\begin{aligned} \nu &= (k^2 - k_\alpha^2)^{\frac{1}{2}} \text{ is real positive} \\ \nu'_{,} &= \text{Im } \nu' > 0 \text{ in first quadrant} \\ \nu'_{,,} &= \text{Im } \nu' < 0 \text{ in fourth quadrant} \end{aligned} \quad (2\text{–}101)$$

Since $\nu'_{,,} = -\nu'_{,}$ near OA_1 Eq. (2–100) can be written in the form

$$I_1 = 4k_\beta^2 \int_{k_\alpha}^{k_\beta} \frac{(2k^2 - k_\beta^2)\nu\nu'_{,}k}{(2k^2 - k_\beta^2)^4 - 16k^4\nu^2\nu'^2_{,}} e^{-ikz}\, dk \quad (2\text{–}102)$$

The product of exp $(i\omega t)$ and (2–102) can be interpreted as an aggregate of waves traveling with velocities ranging from α to β. The more rapid fluctuations of the factor exp $(-ikx)$ for increasing x imply diminishing values of the integral (2–102). The integrals of the type (2–88) cannot be reduced similarly because the factor ν stands alone and the parts from HOA and GCA no longer cancel. The resultant value of (2–88) is

$$\begin{aligned} I_2 = 2k_\beta^2 \int_\infty^0 \frac{i\sqrt{\tau^2 + k_\alpha^2}}{F(-i\tau)} e^{-\tau z} d(-i\tau) + 2k_\beta^2 \int_0^{k_\alpha} \frac{\nu_{,}}{F(k)} e^{-ikz}\, dk \\ + 8k_\beta^2 \int_{k_\alpha}^{k_\beta} \frac{k^2\nu^2\nu'_{,}}{(2k^2 - k_\beta^2)^4 - 16k^4\nu^2\nu'^2_{,}} e^{-ikz}\, dk \end{aligned} \quad (2\text{–}103)$$

This expression was given by Lamb [22, p. 17]. In order to obtain expressions which yield more information about the aggregate of waves (2–102), we return to the two loops L_α and L_β (Fig. 2–8). For the case ω real we contract these contours but maintain the separation between the loops and adhere to the convention of values for ν and ν' on the different banks of these loops as given in Fig. 2–10. If we put $\nu = \nu_{,}$ = Im $\nu > 0$, Eq. (2–100), for example, must be replaced by

$$\int_{L_\alpha} = \int_{-i\infty}^{0} \left\{ \frac{2\zeta^2 - k_\beta^2 - 2\nu_\zeta\nu'_\zeta}{(2\zeta^2 - k_\beta^2)^2 - 4\zeta^2\nu_\zeta\nu'_\zeta} - \frac{2\zeta^2 - k_\beta^2 + 2\nu_\zeta\nu'_\zeta}{(2\zeta^2 - k_\beta^2)^2 + 4\zeta^2\nu_\zeta\nu'_\zeta} \right\} e^{-i\zeta z}\zeta \, d\zeta$$
$$+ \int_0^{k_\alpha} \left\{ \frac{2k^2 - k_\beta^2 - 2\nu_\zeta\nu'_\zeta}{(2k^2 - k_\beta^2)^2 - 4k^2\nu_\zeta\nu'_\zeta} - \frac{2k^2 - k_\beta^2 + 2\nu_\zeta\nu'_\zeta}{(2k^2 - k_\beta^2)^2 + 4k^2\nu_\zeta\nu'_\zeta} \right\} e^{-ikz}k \, dk \quad (2\text{–}104)$$

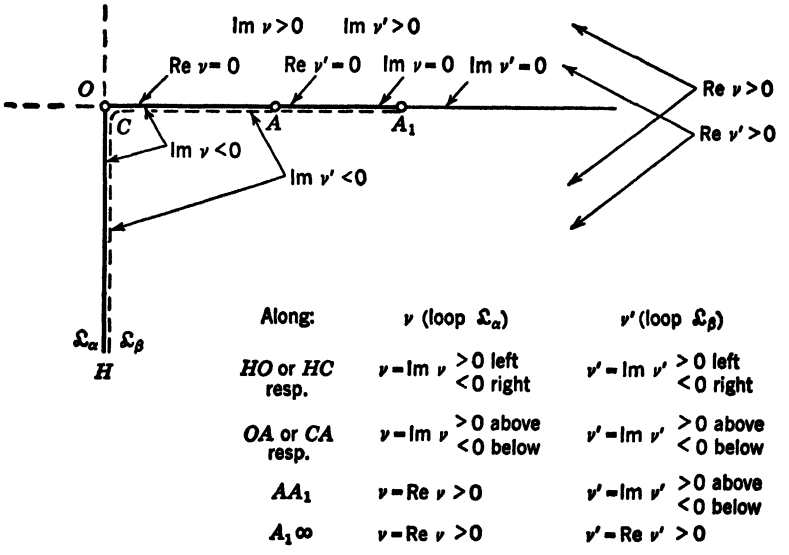
and

$$\int_{L_\beta} = \int_{-i\infty}^{0} \left\{ \frac{2\zeta^2 - k_\beta^2 + 2\nu_\zeta\nu'_\zeta}{(2\zeta^2 - k_\beta^2)^2 + 4\zeta^2\nu_\zeta\nu'_\zeta} - \frac{2\zeta^2 - k_\beta^2 - 2\nu_\zeta\nu'_\zeta}{(2\zeta^2 - k_\beta^2)^2 - 4\zeta^2\nu_\zeta\nu'_\zeta} \right\} e^{-i\zeta z}\zeta \, d\zeta$$
$$+ \int_0^{k_\alpha} \left\{ \frac{2k^2 - k_\beta^2 + 2\nu_\zeta\nu'_\zeta}{(2k^2 - k_\beta^2)^2 + 4k^2\nu_\zeta\nu'_\zeta} - \frac{2k^2 - k_\beta^2 - 2\nu_\zeta\nu'_\zeta}{(2k^2 - k_\beta^2)^2 - 4k^2\nu_\zeta\nu'_\zeta} \right\} e^{-ikz}k \, dk$$
$$+ \int_{k_\alpha}^{k_\beta} \left\{ \frac{2k^2 - k_\beta^2 - 2\nu\nu'_\zeta}{(2k^2 - k_\beta^2)^2 - 4k^2\nu\nu'_\zeta} - \frac{2k^2 - k_\beta^2 - 2\nu\nu'_{\zeta\zeta}}{(2k^2 - k_\beta^2)^2 - 4k^2\nu\nu'_{\zeta\zeta}} \right\} e^{-ikz}k \, dk \quad (2\text{–}105)$$

The integral (2–105) can also be written in the form

$$\int_{L_\beta} = \int_{-i\infty}^{0} \left\{ \frac{2\zeta^2 - k_\beta^2 + 2\nu_\zeta\nu'_\zeta}{(2\zeta^2 - k_\beta^2)^2 + 4\zeta^2\nu_\zeta\nu'_\zeta} - \frac{2\zeta^2 - k_\beta^2 - 2\nu\nu'_\zeta}{(2\zeta^2 - k_\beta^2)^2 - 4\zeta^2\nu_\zeta\nu'_\zeta} \right\} e^{-i\zeta z}\zeta \, d\zeta$$
$$+ \int_0^{k_\beta} \left\{ \frac{2k^2 - k_\beta^2 + 2\bar{\nu}\nu'_\zeta}{(2k^2 - k_\beta^2)^2 + 4k^2\bar{\nu}\nu'_\zeta} - \frac{2k^2 - k_\beta^2 - 2\bar{\nu}\nu'_{\zeta\zeta}}{(2k^2 - k_\beta^2)^2 - 4k^2\bar{\nu}\nu'_{\zeta\zeta}} \right\} e^{-ikz}k \, dk \quad (2\text{–}105')$$

if the convention on radicals ν and ν' is attended to. On the upper bank of the loop L_β, which is on the side of the lower bank of L_α, we have



Along:	ν (loop $\mathfrak{L}_\alpha$)	ν' (loop $\mathfrak{L}_\beta$)
HO or *HC* resp.	$\nu = \text{Im}\,\nu$ >0 left <0 right	$\nu' = \text{Im}\,\nu'$ >0 left <0 right
OA or *CA* resp.	$\nu = \text{Im}\,\nu$ >0 above <0 below	$\nu' = \text{Im}\,\nu'$ >0 above <0 below
AA_1	$\nu = \text{Re}\,\nu > 0$	$\nu' = \text{Im}\,\nu'$ >0 above <0 below
$A_1\infty$	$\nu = \text{Re}\,\nu > 0$	$\nu' = \text{Re}\,\nu' > 0$

FIG. 2–10. Specification of signs of Im ν and Im ν' when loops $\mathfrak{L}_\alpha$ and $\mathfrak{L}_\beta$ are contracted for ω real.

$\tilde{\nu} = -\nu_{,} = \nu_{,,} = \operatorname{Im} \nu < 0$ from 0 to k_α, and $\tilde{\nu} = \nu = \operatorname{Re} \nu$ from k_α to k_β. Figure 2–10 shows also that on the lower bank of L_β from 0 to k_β the second radical ν' has the value $\nu'_{,} = -\nu'_{,} = \operatorname{Im} \nu' < 0$.

Now, since $\zeta = -i\tau$ in the first integral in (2–104), it can be written in the form

$$-4k_\beta^2 \int_0^\infty \frac{\tau(2\tau^2 + k_\beta^2)\nu_{,}\nu'_{,}}{(2\tau^2 + k_\beta^2)^4 - 16\tau^4\nu_{,}^2\nu_{,}'^2} e^{-\tau z}\, d\tau \qquad (2\text{–}106)$$

If in the second integral in (2–104) we substitute $u = k_\alpha - k$, it takes the form

$$4k_\beta^2 e^{-ik_\alpha x} \int_0^{k_\alpha} u^{\frac{1}{2}} G(u) e^{iux}\, du \qquad (2\text{–}107)$$

where

$$G(u) = u^{-\frac{1}{2}} \frac{[2(k_\alpha - u)^2 - k_\beta^2]\nu_{,}\nu'_{,}(k_\alpha - u)}{[2(k_\alpha - u)^2 - k_\beta^2]^4 - 16(k_\alpha - u)^4\nu_{,}^2\nu_{,}'^2} \qquad (2\text{–}108)$$

It can be easily seen that at a large distance x the integral (2–106) becomes small because of the factor τ exp $(-\tau x)$ and that the integral (2–107) becomes small because of very rapid fluctuations of the exponential factor. In order to obtain their approximate values, we can make use of the formula

$$\int_0^\infty \tau^{1/2} G(\tau) e^{-\tau z}\, d\tau = \frac{\Gamma(3/2)}{x^{3/2}} G(0) + \frac{\Gamma(5/2)}{x^{5/2}} \frac{G'(0)}{1!} + \frac{\Gamma(7/2)}{x^{7/2}} \frac{G''(0)}{2!} + \cdots \qquad (2\text{–}109)\dagger$$

Now, in the case of the integral (2–106), $G(0) = 0$, and the contribution of this integral diminishes as $x^{-5/2}$. The integral (2–107), however, yields a term in $x^{-3/2}$. In order to show this, we note that for large x the predominant contribution to the integral (2–107) occurs in the vicinity of the lower limit because of the rapid fluctuations of the last factor.

†Formula (2–109) can be derived as follows: First we observe that the major contribution to this integral is due, in general, to smaller values of the variable τ, since for $x \neq 0$ the integrand decreases rapidly for $\tau x \gg 1$ because of the exponential factor. Substituting, therefore,

$$G(\tau) = G(0) + \tau G'(0) + \frac{\tau^2}{1 \cdot 2} G''(0) + \cdots$$

we can write

$$\int_0^\infty \tau^{1/2} G(\tau) e^{-\tau z}\, d\tau = G(0) \int_0^\infty e^{-\tau z} \tau^{(3/2)-1}\, d\tau + G'(0) \int_0^\infty e^{-\tau z} \tau^{(5/2)-1}\, d\tau + \frac{G''(0)}{1 \cdot 2} \int_0^\infty e^{-\tau z} \tau^{(7/2)-1}\, d\tau + \cdots$$

Now, according to an integral definition of the gamma function in Ref. 27, p. 75,

$$\frac{\Gamma(\sigma)}{x^\sigma} = \int_0^\infty e^{-\tau z} \tau^{\sigma-1}\, d\tau$$

and the formula (2–109) is proved.

We can therefore extend the upper limit to ∞ with negligible error and distort the path of integration to the imaginary axis. Then we can apply formula (2–109), noting that

$$\nu_\alpha = [(k_\alpha - u)^2 - k_\alpha^2]^{\frac{1}{2}} = (-2k_\alpha u + u^2)^{\frac{1}{2}} \cong (-2k_\alpha u)^{\frac{1}{2}}$$

for $u \ll k_\alpha$, and

$$G(0) = \frac{(k_\alpha^2 - k_\beta^2)^{\frac{1}{2}}(-2)^{\frac{1}{2}}k_\alpha^{\frac{1}{2}}}{(2k_\alpha^2 - k_\beta^2)^3} \tag{2–110}$$

Thus we obtain

$$\int_{L_\alpha} = C(k_\alpha x)^{-3/2}e^{-ik_\alpha x} + O(x^{-5/2}) \tag{2–111}$$

where

$$C = -2\sqrt{2\pi}\,\frac{k_\alpha^3 k_\beta^2(k_\beta^2 - k_\alpha^2)^{\frac{1}{2}}}{(k_\beta^2 - 2k_\alpha^2)^3}\exp\left(-i\frac{\pi}{4}\right) \tag{2–112}$$

In a similar manner, the approximate value of the second branch line integral (2–105′) is

$$\int_{L_\beta} = D(k_\beta x)^{-3/2}e^{-ik_\beta x} + O(x^{-5/2}) \tag{2–113}$$

where D depends on k_α and k_β but not on the distance x.

Upon inclusion of the branch line integrals the first expression in Eqs. (2–99) takes the form

$$u_0 = -\frac{QH}{\mu}\exp\left[i(\omega t - \kappa x)\right] + \frac{iQ}{2\pi\mu}\{C(k_\alpha x)^{-\frac{1}{2}}\exp\left[i(\omega t - k_\alpha x)\right] + D(k_\beta x)^{-\frac{1}{2}}\exp\left[i(\omega t - k_\beta x)\right]\} + \cdots \tag{2–114}$$

Following a similar procedure, we find for the second expression in Eqs. (2–99)

$$w_0 = -i\frac{QK}{\mu}\exp\left[i(\omega t - \kappa x)\right] - \frac{Q}{2\pi\mu}\{C_1(k_\alpha x)^{-\frac{1}{2}}\exp\left[i(\omega t - k_\alpha x)\right] + D_1(k_\beta x)^{-\frac{1}{2}}\exp\left[i(\omega t - k_\beta x)\right]\} + \cdots \tag{2–115}$$

where C_1 and D_1 are determined for the function Ψ in Eq. (2–88) and do not depend on the distance x.

The expressions D, C_1, and D_1 are written for reference:

$$\begin{aligned} D &= -2i\sqrt{2\pi}\sqrt{1 - \frac{k_\alpha^2}{k_\beta^2}}\exp\left(-i\frac{\pi}{4}\right) \\ C_1 &= -i\sqrt{2\pi}\,\frac{k_\alpha^2 k_\beta^2}{(k_\beta^2 - 2k_\alpha^2)^2}\exp\left(-i\frac{\pi}{4}\right) \\ D_1 &= -4i\sqrt{2\pi}\left(1 - \frac{k_\alpha^2}{k_\beta^2}\right)\exp\left(-i\frac{\pi}{4}\right) \end{aligned} \tag{2–116}$$

According to the formula (2–109), the next terms decrease as $x^{-5/2}$ The second terms in (2–114) and (2–115) represent compressional waves propagating with velocity $\alpha = \omega/k_\alpha$, and the third terms represent shear waves propagating with velocity $\beta = \omega/k_\beta$. Amplitudes are proportional to $(k_\alpha x)^{-\frac{1}{2}}$ and $(k_\beta x)^{-\frac{1}{2}}$, respectively, whereas in an unlimited solid the amplitudes decrease as x^{-1}. The surface vibrations corresponding to these waves are rectilinear. Lamb [22] gives for the ratio of vertical to horizontal amplitudes of the compressional waves $(k_\beta^2 - 2k_\alpha^2)/2k_\alpha\sqrt{k_\beta^2 - k_\alpha^2}$ or 0.3535 for $\lambda = \mu$. For shear waves the ratio is $2\sqrt{1 - k_\alpha^2/k_\beta^2} = 2\sqrt{2/3} = 1.633$ for $\lambda = \mu$.

SURFACE POINT SOURCE. As we have seen in Sec. 2–4, the factors represented by the definite integrals with respect to k in the point-source problem are obtained from the corresponding two-dimensional equations by performing the operation $-\partial/\pi\partial x$ and substituting $x = r \cosh u$. We can therefore use Eqs. (2–96) and (2–97) to obtain the corresponding expressions (2–70) for a surface point source. Now to obtain the more important terms, where $k_\alpha r$ and $k_\beta r$ are large, we also make use of Eqs. (2–114) and (2–115) with the constant Q replaced by L. We shall derive, for example, the first term of w_0. Performing the operations just indicated upon the first term in Eq. (2–115) and integrating with respect to the variable u, we find

$$\frac{LK}{\pi\mu}\kappa e^{i\omega t}\int_0^\infty e^{-i\kappa r\cosh u}\,du$$

Using the definition of the Hankel function

$$H_0^{(2)}(\kappa r) = \frac{2i}{\pi}\int_0^\infty e^{-i\kappa r\cosh u}\,du$$

we obtain

$$-\frac{i\kappa KL}{2\mu}H_0^{(2)}(\kappa r)e^{i\omega t}$$

Applying the asymptotic expansion for $H_0^{(2)}$ (see Ref. 56, p. 198)

$$H_0^{(2)}(z) = \sqrt{\frac{2}{\pi z}}\exp\left[-i\left(z - \frac{\pi}{4}\right)\right]\left(1 + \frac{i}{8z} + \cdots\right)$$

we can write the first term in the form

$$\frac{\kappa KL}{2\mu}\sqrt{\frac{2}{\pi\kappa r}}\exp\left[i\left(\omega t - \kappa r - \frac{\pi}{4}\right)\right] \tag{2–117}$$

A similar transformation of the first term in Eq. (2–114) leads to $-(iLH/\pi\mu)\,\kappa\exp[i(\omega t - \kappa x)]$ in which, however, the function

$$H_1^{(2)}(\kappa r) = -\frac{2}{\pi}\int_0^\infty \exp(-i\kappa r\cosh u)\cosh u\,du = -[H_0^{(2)}(\kappa r)]'$$

is involved, and it yields the term

$$i\frac{\kappa LH}{2\mu}H_1^{(2)}(\kappa r)\exp(i\omega t) \sim -i\frac{\kappa LH}{2\mu}\sqrt{\frac{2}{\pi\kappa r}}\exp\left[i\left(\omega t - \kappa r - \frac{\pi}{4}\right)\right] \quad (2\text{–}118)$$

Transforming terms due to branch line integrals from Eqs. (2–114) and (2–115), we finally obtain for the displacements

$$q_0 = -\frac{i\kappa LH}{\mu}\sqrt{\frac{1}{2\pi\kappa r}}\exp\left[i\left(\omega t - \kappa r - \frac{\pi}{4}\right)\right] + \frac{M}{(k_\alpha r)^{\frac{1}{2}}}\int_0^\infty \frac{\exp\left[i(\omega t - k_\alpha r\cosh u)\right]}{(\cosh u)^{\frac{1}{2}}}\,du + \frac{N}{(k_\beta r)^{\frac{1}{2}}}\int_0^\infty \frac{\exp\left[i(\omega t - k_\beta r\cosh u)\right]}{(\cosh u)^{\frac{1}{2}}}\,du + \cdots \quad (2\text{–}119)$$

and

$$w_0 = \frac{\kappa KL}{\mu}\sqrt{\frac{1}{2\pi\kappa r}}\exp\left[i\left(\omega t - \kappa r - \frac{\pi}{4}\right)\right] + \frac{M_1}{(k_\alpha r)^{\frac{1}{2}}}\int_0^\infty \frac{\exp\left[i(\omega t - k_\alpha r\cosh u)\right]}{(\cosh u)^{\frac{1}{2}}}\,du + \frac{N_1}{(k_\beta r)^{\frac{1}{2}}}\int_0^\infty \frac{\exp\left[i(\omega t - k_\beta r\cosh u)\right]}{(\cosh u)^{\frac{1}{2}}}\,du + \cdots \quad (2\text{–}120)$$

where the factors M, M_1, N, N_1 can be readily expressed in terms of L, μ, C, C_1, D, and D_1 by Eqs. (2–112) and (2–116). These expressions hold only for large $k_\alpha r$ and $k_\beta r$.

The first terms of Eqs. (2–119) and (2–120) again represent Rayleigh waves having retrograde elliptical vibrations with the same ratio of horizontal to vertical axes as in the solutions for a line source. In this case, however, the amplitudes diminish with distance as $(\kappa r)^{-\frac{1}{2}}$, the familiar law for divergence for annular waves. The remaining terms again represent shear and compressional waves with amplitudes diminishing at least as

$$\frac{1}{(k_\beta r)^{\frac{1}{2}}}\int_0^\infty \frac{\exp(-ik_\beta r\cosh u)}{\pi(\cosh u)^n}\,du \qquad \frac{1}{(k_\alpha r)^{\frac{1}{2}}}\int_0^\infty \frac{\exp(-ik_\alpha r\cosh u)}{\pi(\cosh u)^n}\,du$$

respectively, where $n = \frac{1}{2}$ for q_0 and $n = \frac{3}{2}$ for w_0. The functions of $k_\beta r$ and $k_\alpha r$ represented by second factors in these expressions are integrals of the type considered in Appendix A. This may be easily seen by putting in Eq. (A–1) $x = k_\beta r$ or $k_\alpha r$. Then Eq. (A–13) shows that such an integral can be approximated by an expression with a factor $|x|^{-\frac{1}{2}}$, that is, $|k_\beta r|^{-\frac{1}{2}}$ or $|k_\alpha r|^{-\frac{1}{2}}$, which can be combined with the first factors, namely, $(k_\beta r)^{-\frac{1}{2}}$ and $(k_\alpha r)^{-\frac{1}{2}}$. At large distances these reduce to $(k_\beta r)^{-2}$ and $(k_\alpha r)^{-2}$, and the Rayleigh waves predominate. The amplitude decrease of r^{-2} is also

more rapid than is the case for elastic waves diverging from a point in an unlimited medium.

INTERNAL LINE SOURCE. A detailed discussion was given by Lapwood [25] for integrals of the type (2–47). Another method for evaluation of these integrals was used by Sakai [41] and recently discussed by Honda and Nakamura [12]. Following Lapwood, we note first that the integrals (2–47) have the form

$$I_1 = \frac{1}{2}\int_0^\infty G(k)e^{ihz}\,dk + \frac{1}{2}\int_0^\infty G(k)e^{-ihz}\,dk \tag{2–121}$$

$$I_2 = \frac{1}{2i}\int_0^\infty kG(k)e^{ihz}\,dk - \frac{1}{2i}\int_0^\infty kG(k)e^{-ihz}\,dk \tag{2–122}$$

where $G(k)$ is an even-valued function of k.

Following the procedure given in Sec. 2–5 for ω complex, we distort the path of the first integrals in Eqs. (2–121) and (2–122) into the contour

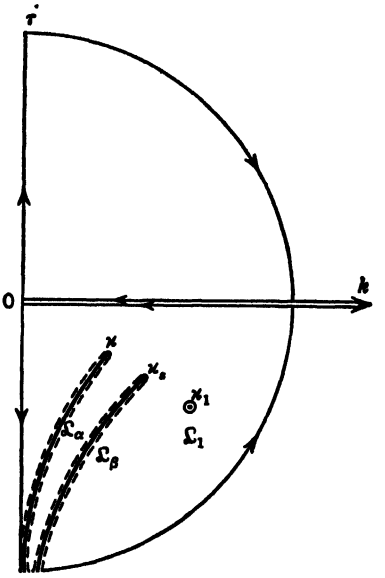


FIG. 2–11. Integration path in the complex ζ plane for internal line source.

shown in the first quadrant of Fig. 2–11, which includes the positive imaginary axis and the infinite arc. The path of the second integral is distorted into the contour, shown in the fourth quadrant, containing the negative imaginary axis, the loops L_α and L_β around the cuts, and a

small circle around each pole. The use of similar contours for the evaluation of integrals like (2–121) and (2–122), which also occur in the theory of electromagnetic waves, was suggested by Sommerfeld (see Chap. 1, Ref. 55). Since the contributions from the infinite arcs are zero, we have

$$I_1 = \frac{1}{2}\int_0^{i\infty} G(\zeta)e^{i\zeta x}\,d\zeta + \frac{1}{2}\int_0^{-i\infty} G(\zeta)e^{-i\zeta x}\,d\zeta + \frac{1}{2}\int_{L_\alpha, L_\beta} G(\zeta)e^{-i\zeta x}\,d\zeta - 2\pi i \sum \text{Res}$$

$$I_2 = \frac{1}{2i}\int_0^{i\infty} \zeta G(\zeta)e^{i\zeta x}\,d\zeta + \frac{1}{2i}\int_0^{-i\infty} \zeta G(\zeta)e^{-i\zeta x}\,d\zeta + \frac{1}{2i}\int_{L_\alpha, L_\beta} \zeta G(\zeta)e^{-i\zeta x}\,d\zeta - 2\pi i \sum \text{Res} \tag{2–123}$$

The integrals along the imaginary axis cancel, provided that the positive and negative parts of this axis are located on a sheet where no change is required in the definition of some factor in the expression $G(\zeta)$ (such as $\nu = \pm\sqrt{k^2 - k_\alpha^2}$ in cases considered before) which would affect the integrands. Then

$$I_1 = \frac{1}{2}\int_{L_\alpha, L_\beta} G(\zeta)e^{-i\zeta x}\,d\zeta - 2\pi i \sum \text{Res}$$

$$I_2 = \frac{1}{2i}\int_{L_\alpha, L_\beta} \zeta G(\zeta)e^{-i\zeta x}\,d\zeta - 2\pi i \sum \text{Res} \tag{2–124}$$

where L_α and L_β indicate that the path consists of loops lying indefinitely close to the two cuts connecting $-i\infty$ with k_α and k_β, respectively. Now, on substituting $\zeta = k - i\tau$, $\tau > 0$, it is easy to see that the major contribution to the integrals comes from the neighborhood of the branch points, since the modulus of $\exp(-i\zeta x)$ decreases rapidly as ζ moves away from these points. This fact suggests that, to a first approximation valid for large distances, factors such as $\exp[i\omega(t - x/\alpha)]$ and $\exp[i\omega(t - x/\beta)]$ will result from the integration along L_α and L_β, respectively, and that we may associate with these loops waves which have traveled most of the distance from source to detector as compressional and shear waves. From the type of potential we surmise the velocity of the wave near the detector. For example, for a compressional source we would have the potentials ${}_p\varphi$ and ${}_p\psi$ and for a shear source, ${}_s\varphi$ and ${}_s\psi$. Integrals such as (2–124) will lead to wave types which can be discussed as follows for the loop L_α:

1. ${}_p\varphi_\alpha$ is a wave beginning and ending as a P wave, having traveled with velocity α. This would represent a contribution to the reflected P wave (PP in seismological nomenclature).

2. ${}_p\psi_\alpha$ starts as a P wave, ends as a shear wave, and traverses most of the path with velocity α. This is the reflected wave PS.

3. ${}_{s}\psi_{\alpha}$ starts and ends as an S wave and travels most of the way with velocity α. This is the surface-generated wave denoted by sPs.

4. ${}_{s}\varphi_{\alpha}$ starts as S, ends as P, travels as P along most of the path. This is the reflected wave SP.

These four waves are depicted in Figs. 2–12 and 2–13. They travel minimum time paths and are predictable from the rules of geometric optics.

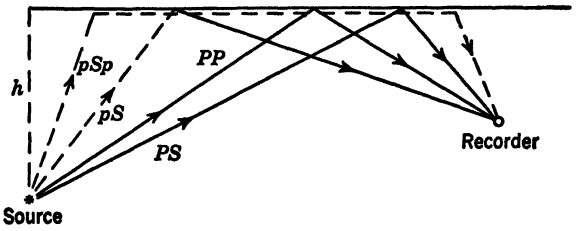


FIG. 2–12. Waves from a P source near a free surface.

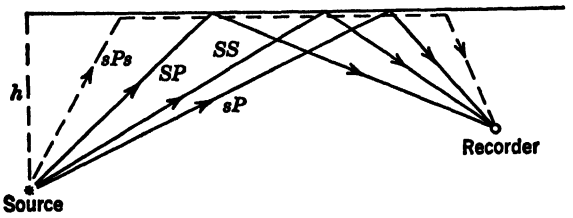


FIG. 2–13. Waves from an SV source near a free surface.

Contributions from the loop L_{β} represent waves traveling over most of the path with velocity β:

5. ${}_{p}\varphi_{\beta}$ is a wave which starts and ends as P and travels most of the path as S. It represents the surface-generated wave pSp.

6. ${}_{p}\psi_{\beta}$ starts as P, finishes as S, and travels primarily as S. This wave apparently is reflected near the source and will be called pS.

7. ${}_{s}\varphi_{\beta}$ is analogous to 6 in that it starts as S, ends as P, and travels most of the way as S. This wave appears to be reflected near the detector and will be called sP.

8. ${}_{s}\psi_{\beta}$ obviously represents the reflected S wave denoted by SS.

These waves are also illustrated in Figs. 2–12 and 2–13. The first three types of the L_{β} group do not satisfy a minimum time condition; hence they are not represented by the laws of geometric optics.

The contribution from the pole κ contains the factor $\exp[i\omega(t - x/c_R)]$ which readily identifies it as the Rayleigh wave discussed earlier. It occurs for both P and S sources.

Application of the Method of Steepest Descent. This method (see Appendix A) has been frequently applied to the evaluations of integrals of the type Eqs. (2–47). It provides more information about the disturbances at short ranges. Replacing sin kx in Eqs. (2–47), for example, by exponential functions, extending the integration from $-\infty$ to ∞, and restoring the time factor, let us consider the integral

$$\psi = -4i \exp(i\omega t) \int_{-\infty}^{\infty} \frac{\zeta(2\zeta^2 - k_\beta^2)}{F(\zeta)} \exp(-\nu h - \nu' z - i\zeta x)\, d\zeta \qquad (2\text{–}125)$$

Referring to Eq. (A–1) we put

$$f(\zeta) = -\frac{\nu h}{x} - \frac{\nu' z}{x} - i\zeta = \rho + i\sigma \qquad (2\text{–}126)$$

The line of steepest descent is given by $\sigma(k, \tau) = \text{const}$. Writing $\zeta = k_\alpha u$, we can examine the variation of σ for real u. In the range $0 \leq u \leq 1$, with $\omega = s - ic$, $(s > 0, c > 0)$, we have

$$-\sigma = \frac{s}{\alpha}\left[u + \frac{h}{x}\sqrt{1 - u^2} + \frac{z}{x}\sqrt{\frac{\alpha^2}{\beta^2} - u^2}\right] \qquad (2\text{–}127)$$

The function $-\sigma$ has a maximum $-\sigma_0$ at the saddle point ζ_0 given by $df(\zeta)/d\zeta = 0$, or

$$x = \frac{hu_0}{\sqrt{1 - u_0^2}} + \frac{zu_0}{\sqrt{\alpha^2/\beta^2 - u_0^2}} \qquad (2\text{–}128)$$

From (2–128) we see that a saddle point exists for u_0 real. It depends on h, z, and x and lies in the range $0 \leq u_0 \leq 1$, that is, on the line between the origin and k_α. The contribution of the saddle point is obtainable from Eq. (A–13). It contains the factor

$$\exp\left[i\omega\left(t - \frac{xu_0}{\alpha} - \frac{h}{\alpha}\sqrt{1 - u_0^2} - \frac{z}{\alpha}\sqrt{\frac{\alpha^2}{\beta^2} - u_0^2}\right)\right] \qquad (2\text{–}129)$$

where u_0 is given by (2–128). Substituting $u_0 = \cos\delta$ in (2–128) and (2–129), we can easily verify that these equations define the phase PS depicted in Fig. 2–14 and that δ is the angle of incidence of the P wave at the free surface.

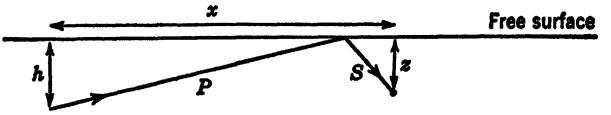


FIG. 2–14. *PS* phase from a *P* source near a free surface.

To illustrate another application of the method of steepest descent we shall follow Newlands [32] and derive an expression for the minimum distance at which the Rayleigh wave appears. The path of steepest descent recuts the line of branch points at or before a point where $-\sigma \geq -\sigma_0$, or

$$\frac{1}{\alpha}\left[su - \frac{ch}{x}\sqrt{u^2 - 1} - \frac{cz}{x}\sqrt{u^2 - \frac{\alpha^2}{\beta^2}}\right] \geq \frac{s}{\alpha}\left[u_0 + \frac{h}{x}\sqrt{1 - u_0^2} + \frac{z}{x}\sqrt{\frac{\alpha^2}{\beta^2} - u_0^2}\right] \tag{2-130}$$

If this occurs for any value of ω, it must occur when $c = 0$, or

$$xu = xu_0 + h\sqrt{1 - u_0^2} + z\sqrt{\frac{\alpha^2}{\beta^2} - u_0^2} \tag{2-131}$$

If the intersection occurs to the left of the pole, then the contribution of the pole must be considered when the path of integration is distorted from the real axis to that of steepest descent. This occurs when $\alpha/c_R > u$. It follows that the Rayleigh pulse appears when

$$\frac{x\alpha}{c_R} > xu_0 + h\sqrt{1 - u_0^2} + z\sqrt{\frac{\alpha^2}{\beta^2} - u_0^2} \tag{2-132}$$

At the free surface $z = 0$ Eqs. (2–132) and (2–128) reduce to the condition

$$x > \frac{c_R h}{\sqrt{\alpha^2 - c_R^2}} \tag{2-133}$$

a result first derived by Nakano (see Ref. 28 and Sec. 2–7).

If we introduce the angle $\theta = \sin^{-1}(c_R/\alpha)$, Eq. (2–133) takes the form $x > h \tan\theta$. Equation (2–133) has usually been interpreted to mean that the Rayleigh wave does not exist in the interval EP (Fig. 2–15). From Eq. (2–133) it may be concluded that the travel times are identical, whether one considers the Rayleigh waves to be excited by arrival of

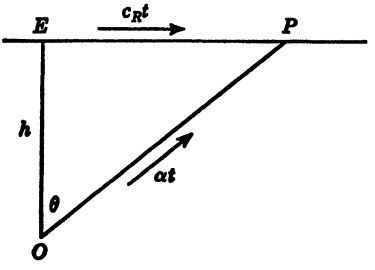


FIG. 2–15. Interpretation of the minimum distance at which the Rayleigh wave appears.

compressional waves at P or excited instantaneously at E. The advantage of the latter point of view is apparent if one considers that the disturbance at O has both shear and compressional components. The travel-time curve of Rayleigh waves is such that extrapolation to zero time gives zero distance from E.

2–6. Generalization for an Arbitrary Time Variation. The steady-state solutions considered in preceding sections are characterized by a time factor of the form $\exp i\omega t$ and represent a primary disturbance varying as a simple harmonic function of the time. For the discussion of different phenomena these solutions can be generalized for an arbitrary law of time variation. The effect of a single impulse of short duration is particularly important. For such generalizations the Fourier transform has been used in most cases. Jeffreys [16] suggested the use of a "pulse" represented by a simple Heaviside unit function $H(t)$ changing from zero for $t < 0$ to 1 for $t > 0$.

In most problems which we shall consider, the solutions are constructed from potentials $\bar{\varphi}_0 e^{i\omega t}$, $\bar{\varphi}_1 e^{i\omega t}$, $\bar{\psi}_1 e^{i\omega t}$, where $\bar{\varphi}_0(r, z, \omega)$ corresponds to the direct compressional wave, $\bar{\varphi}_1$ and $\bar{\psi}_1$ correspond to waves in the first medium resulting from the presence of boundaries, and $\bar{\varphi}_2$, $\bar{\psi}_2$ represent waves transmitted in the second medium, etc. In Eq. (1–40) or (1–41) the angular frequency appears in the exponent only but we can assume that the coefficient A in the former expression is a function of ω. If we put $A = G(\omega)$, each of the terms in a solution is found to vary with this parameter and with time as $f_s(r, z, \omega)G(\omega)e^{i\omega t}$. In many cases the functions f_s are products of the form $f(\omega)F(r, z)$. For example, the first terms in Eqs. (2–119) and (2–120) are of the form

$$F(r, z)\omega^{\frac{1}{2}} \exp\left[i\omega\left(t - \frac{r}{c_R}\right)\right]$$

since $\kappa = \omega/c_R$. If we put $t_* = t - r/c_R$, or $t_* = t - r/\alpha$, or $t - r/\beta$, we shall see that such terms as $F(r, z)\omega^{n-1}e^{i\omega t_*}$ occur frequently.

Now define a function $S_d(t)$ giving the time variation. Then, by the Fourier integral theorem, we can put

$$S_d(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega)e^{i\omega t}\, d\omega \tag{2–134}$$

where the transform $g(\omega)$ is given by

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S_d(\hat{t})e^{-i\omega \hat{t}}\, d\hat{t} \tag{2–135}$$

If we put

$$G(\omega)\varphi_0(0, h, \omega) = g(\omega) \tag{2–136}$$

terms such as

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\omega) f(\omega) F(r, z) e^{i\omega t_*} \, d\omega \tag{2-137}$$

arise by superposition. They represent the waves generated at the boundaries.

Among different functions representing the time variation at the source considered by Lamb [22, pp. 26 and 37] we now choose a pulse defined by (Fig. 2–16)

$$S_d(t) = \frac{pL}{p^2 + t^2} \tag{2-138}$$

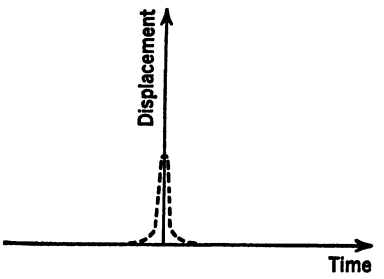


FIG. 2–16. Assumed initial time variation at the source.

where p and L are arbitrary real and positive constants. The Fourier transform of (2–138) can readily be obtained from (2–135) by contour integration in the upper (for $\omega < 0$) or lower half plane (for $\omega > 0$) of the complex variable $\hat{t}$ (Fig. 2–17). The integral along the infinite half circle will vanish because of the factor $\exp(-i\omega t)$.

The residues at the poles $\mp ip$ yield

$$g(\omega) = \sqrt{\frac{\pi}{2}} L e^{\mp \omega p} \qquad \omega \gtrless 0$$

Then expression (2–137) for $f(\omega) = \omega^{n-1}$, for example, takes the form

$$\frac{LF}{2} \int_{-\infty}^{\infty} \omega^{n-1} \exp(\mp \omega p + i\omega t_*) \, d\omega \qquad \text{for } \omega \gtrless 0 \tag{2-139}$$

which converges, provided that $n > 0$. This integral can be evaluated using Euler's formula

$$\int_0^{\infty} \omega^{n-1} e^{-p\omega} \begin{matrix} \cos \\ \sin \end{matrix} \omega t_* \, d\omega = \Gamma(n)(p^2 + t_*^2)^{-n/2} \begin{matrix} \cos \\ \sin \end{matrix} nv \tag{2-140}$$

where $\tan v = t_*/p$ (see, for example, Ref. 57, p. 260). We obtain for (2–139)

$$\frac{LF(r, z)}{2} \Gamma(n)(p^2 + t_*^2)^{-n/2}\{\exp(inv) - \exp[-in(v + \pi)]\} \qquad (2\text{–}141)$$

For $n \leq 0$ the integral (2–139) diverges but certain information concerning the movements may be obtained by working with velocities $\bar{q}$ and $\bar{w}$. The differentiation with respect to t can be performed in convergent integrals of the form (2–139).

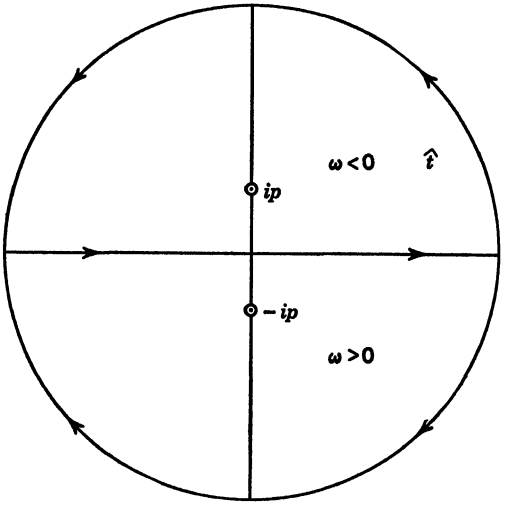


FIG. 2–17. Integration paths in the complex $\hat{t}$ plane for $\omega \gtrless 0$.

Instead of (2–134) and (2–135), Lamb considered a real form of the Fourier transform and a related function, i.e., the equations

$$S_1(t) = \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^\infty S_1(\hat{t}) \cos\omega(t - \hat{t})\, d\hat{t} \qquad (2\text{–}142)$$

$$S_2(t) = \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^\infty S_2(\hat{t}) \sin\omega(t - \hat{t})\, d\hat{t} \qquad (2\text{–}143)$$

where the summation is for positive ω only. If we therefore make use of the exponential form of (2–140) to generalize the terms in Eqs. (2–119) and (2–120) which represent Rayleigh waves, we obtain

$$F = -i\frac{H}{\mu}\sqrt{\frac{1}{2\pi r c_R}} \exp\left(-i\frac{\pi}{4}\right) \quad \text{or} \quad \frac{K}{\mu}\sqrt{\frac{1}{2\pi r c_R}} \exp\left(-i\frac{\pi}{4}\right)$$

$$n = \frac{3}{2} \qquad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi} \qquad \cos v = \frac{p}{\sqrt{p^2 + t_*^2}} \qquad (2\text{–}144)$$

$$q_0 = -\frac{HL}{2\mu(2rc_R)^{\frac{1}{2}}p^{\frac{1}{2}}}\cos^{\frac{1}{2}}\nu \sin\left(\frac{\pi}{4} - \frac{3}{2}\nu\right) + \cdots$$
$$w_0 = \frac{KL}{2\mu(2rc_R)^{\frac{1}{2}}p^{\frac{1}{2}}}\cos^{\frac{1}{2}}\nu \cos\left(\frac{\pi}{4} - \frac{3}{2}\nu\right) + \cdots \qquad (2\text{–}145)\dagger$$

Following a similar procedure, using the transformation indicated in (2–139) and (2–141), we can generalize the other terms in (2–119) and (2–120) (see Ref. 25).

In Figs. 2–16 and 2–18 are plotted the initial pulse $S(t)$ and the hori-

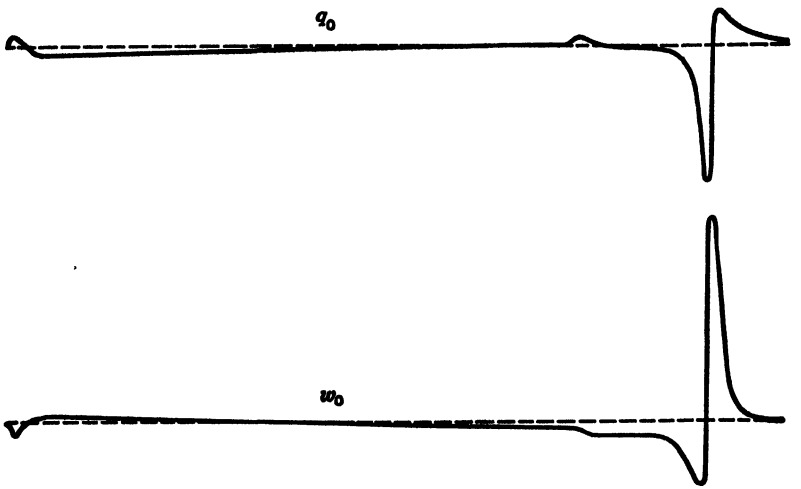


FIG. 2–18. Ground motion, according to Lamb, from a distant impulse. Upper curve is horizontal motion; lower curve is vertical motion.

zontal and vertical surface displacements q_0 and w_0 corresponding to the passage of the Rayleigh waves at a distant point. The surface particle velocities corresponding to passage of the preliminary compressional and shear waves are plotted in Fig. 2–19.

2–7. Other Investigations. The propagation of a disturbance in a half space, sometimes called Lamb's problem, was also investigated under different conditions by Nakano [28, 29], Sobolev [48], Naryškina [30, 31], Hallen [11], Schermann [44], and as a limiting case in several papers dealing with propagation in two semi-infinite media. This last problem is considered in the next chapter.

In the first of the papers quoted above, Nakano assumed a line source

†In the derivation of Eqs. (2–145), note that K is an odd function in κ and that $K \gtrless 0$ as $\mathrm{Re}\,\omega \gtrless 0$.

in the interior of a solid half space producing (1) a longitudinal cylindrical wave and (2) a transverse cylindrical wave. A point source of distortional waves was considered by Sakai [41]. Following Lamb's method, Nakano [28] used contour integration in a complex plane in order to evaluate the displacements at the free surface. Debye's method of the steepest descent

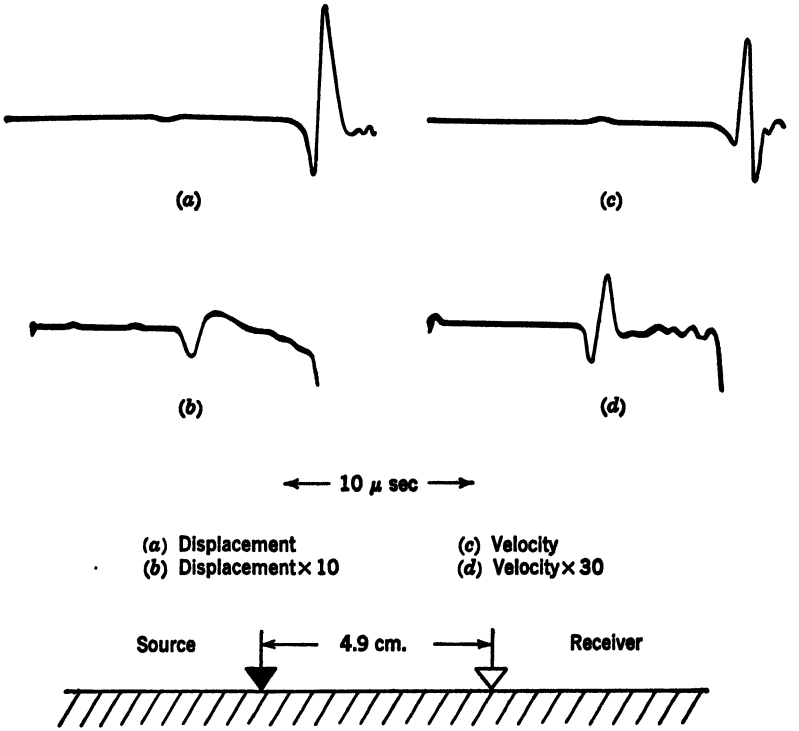


FIG. 2–19. Tatel's model seismograms showing vertical surface motion from a distant surface point impulse. (*a*) Displacement; (*b*) displacement times 10; (*c*) velocity; (*d*) velocity times 30.

(see Appendix A) was applied in the physical interpretation of the special paths required for this case. Nakano showed that the Rayleigh waves do not appear at places near the sources, i.e., where the epicentral distance is smaller than (1)

$$\frac{c_R h}{\sqrt{\alpha^2 - c_R^2}}$$

or (2) $$\frac{c_R h}{\sqrt{\beta^2 - c_R^2}}$$

where h = depth of source
c_R = velocity of Rayleigh waves

Moreover, it was proved that these waves do not have their full amplitude near these limit distances. This fact is due to interferences with other kinds of waves. At large distances from the epicenter the Rayleigh waves appear at about the time required to traverse these distances with velocity c_R. This part of Nakano's investigation was developed in order to explain the difficulties in their identification on seismograms at a station whose epicentral distance is not great compared with the depth of the source. The results obtained for simple harmonic waves were also extended to a more general case where the action at the source may be any function of time.

It was also pointed out by Nakano [28, p. 268] that it is possible to resolve a displacement into longitudinal, transverse, and Rayleigh waves under the assumption of a simple harmonic train but these components cannot be separated in the rigorous sense. To simplify the problem, it is assumed, however, that such separation is possible. This separation is "somewhat arbitrary" (see Ref. 28, p. 274), since on taking different paths when computing the integrals we can obtain different interpretations. However, the interpretation given before seems to be a natural one. In an unlimited homogeneous and isotropic solid there are only compressional and distortional waves but in the case of a half space the Rayleigh wave is produced from body waves of either type when a curved wave front reaches the free surface.

Nakano [29] also considered normal and tangential forces distributed at the surface in an arbitrary way, paying special attention to periodic forces and displacements repeated in n sectors. Then, besides compressional, transverse, and Rayleigh waves, there is a displacement component of a different nature. This wave propagates with the velocity of transverse waves and is the only wave present in case of a transverse force distributed symmetrically about a vertical axis.

Sobolev [48, 49] applied a new method to the solution of Lamb's problem. His solution coincides with that of Lamb in that longitudinal, shear, and Rayleigh waves are found. Wave-front diagrams are given for these waves.

Sobolev [49] also gave the solution of the two-dimensional problem for arbitrary initial conditions and external forces acting on the boundary of a half space. Following his method, Naryškina [30] found the solution of the three-dimensional problem of propagation of oscillations in a half space when there are arbitrary initial conditions given and no external forces. Nonvanishing external forces were later considered by Sobolev as

well as by Schermann [44] in a three-dimensional problem. Hallen [11] also gave a solution of the two-dimensional problem for arbitrary external forces. Schermann's solution is based on the Cauchy-Fourier method, and the expressions for potentials are assumed in the form mentioned at the end of Sec. 1–6.

The two-dimensional problem was recently discussed by Sauter [42, 43] for surface normal and shear stresses which depend on the coordinates x and t.

The method of Smirnov and Sobolev [47] was generalized by Petrashen [37*a*, 37*b*]. Fourier integrals and a special contour in the complex plane are used, and the Rayleigh waves in the solution are separated from terms representing longitudinal and transverse waves. Petrashen applied this method also to the problem of wave propagation in a layer overlying a semi-infinite solid. Some particular cases of the problem of vibrations due to given displacements at the boundary of an elastic half space were treated by Shatashvili [45, 46]. On applying Schermann's method [44] he reduced the problem to a system of integral equations of the Fredholm type.

Assuming a pressure pulse varying like the Heaviside function, Pekeris [36] recalculated the solution of Lamb's problem for a surface source and a buried source. The vertical component could be obtained in a closed form, while the horizontal component was expressed in terms of elliptic integrals. In this solution the arrival of the shear wave is marked by a change in slope of the displacements. Both components become infinite, however, at the time of arrival of the Rayleigh wave.

2–8. Traveling Disturbance. In the investigations discussed in the preceding paragraphs, the sources were assumed to have fixed positions with respect to the half space, and the boundary conditions were usually taken to be independent of time. Lamb [23] has also considered the case of an impulsive disturbance traveling with a constant velocity c_0 in a fixed direction, say, the direction of x negative. The effect of a traveling disturbance can be obtained by the application of a succession of infinitesimal impulses at equal time intervals. Each impulse produces a system of waves which may be represented by an equation having a form similar to (2–42). Assuming in this two-dimensional problem that a concentrated force acts at the origin, we can write

$$w_0 = \frac{1}{2\pi}\int_0^\infty A(k)\exp\left[i(\omega t - kx)\right]dk + \frac{1}{2\pi}\int_0^\infty A(k)\exp\left[i(\omega t + kx)\right]dk \qquad (2\text{–}146)$$

Since these integrals may become indeterminate, Lamb introduces a factor $\exp(-\gamma t)$ which represents the effect of a slight dissipative action.

The final results are the limits approached when the coefficient of absorption γ goes to zero. We take the origin of x at the position of the traveling disturbance at $t = 0$. The result of an impulse delivered at an earlier time t' can be given by (2–146), if x is replaced by $c_0t' - x$, and Eq. (2–146) is multiplied by dt'. Integrating from $t' = 0$ to $t' = \infty$ to obtain the effect of a traveling impulse, Eq. (2–146) takes the form

$$w_0 = \frac{1}{2\pi}\int_0^\infty \left\{\int_0^\infty A(k)\exp\left[i\omega t' - ik(c_0t' - x) - \gamma t'\right]dk + \int_0^\infty A(k)\exp\left[i\omega t' + ik(c_0t' - x) - \gamma t'\right]dk\right\}dt' \qquad (2\text{–}147)$$

or

$$w_0 = \frac{1}{2\pi}\int_0^\infty \frac{A(k)e^{ikx}\,dk}{\gamma - i(\omega - kc_0)} + \frac{1}{2\pi}\int_0^\infty \frac{A(k)e^{-ikx}\,dk}{\gamma - i(\omega + kc_0)} \qquad (2\text{–}148)$$

In these expressions, ω is not assumed to be equal to kc_0 but, γ being very small, the most important part of the first integral will be due to the root κ of the equation

$$\omega = kc_0 \qquad (2\text{–}149)$$

that is, for those waves whose phase velocity $c = \omega/k$ equals the velocity c_0 of the traveling impulse.

Writing $k = \kappa + k_1$ and taking the first term of the expansion,

$$\omega - kc_0 = \left(\frac{d\omega}{dk} - c_0\right)k_1 = (U - c_0)k_1\dagger$$

where $\omega = \omega(k)$

U = group velocity

we obtain the most important part of the first integral in (2–148) in the form

$$w_{01} = \frac{1}{2\pi}A(\kappa)e^{i\kappa x}\int_{-\infty}^{\infty}\frac{e^{ik_1x}\,dk_1}{\gamma - i(U - c_0)k_1} \qquad (2\text{–}150)$$

The extension of the limits of k_1 to $\pm\infty$ will make little difference to the value of w_{01} in Eq. (2–150).

Since this integral can be evaluated as

$$\int_{-\infty}^{\infty}\frac{e^{ims}\,dm}{a + im} = \begin{cases} 2\pi e^{-as} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \quad \text{or} \quad \int_{-\infty}^{\infty}\frac{e^{ims}\,dm}{a - im} = \begin{cases} 0 & \text{for } x > 0 \\ 2\pi e^{as} & \text{for } x < 0 \end{cases} \qquad (2\text{–}151)$$

we have, if $U < c_0$,

$$w_0 = \begin{cases} \dfrac{A(\kappa)}{c_0 - U}\exp(i\kappa x)\exp\left(-\dfrac{\gamma x}{c_0 - U}\right) & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \qquad (2\text{–}152)$$

†The next term is $\frac{1}{2}k_1^2\,dU/dk$.

If $U > c_0$,

$$\begin{aligned} w_0 &= 0 && \text{for } x > 0 \\ w_0 &= \frac{A(\kappa)}{U - c_0} \exp(i\kappa x) \exp\left(\frac{\gamma x}{U - c_0}\right) && \text{for } x < 0 \end{aligned} \tag{2-153}$$

For $\gamma = 0$, the second factor is unity, and

$$w_0 = \frac{A(\kappa)e^{i\kappa x}}{|c_0 - U|} \quad \text{or} \quad w_0 = 0 \tag{2-154}$$

This is Lamb's result representing a wave train which follows the traveling disturbance, depending on whether the group velocity is less or greater than the phase velocity. Gravity waves in water illustrate the former, capillary waves the latter. Lamb has also shown how the procedure must be modified if the root of (2–149) also makes $U = c_0$. The result is in this case

$$w_0 = \pm\frac{A(\kappa)}{\sqrt{2U\,|dU/dk|}} \exp\left[i\left(\kappa x \pm \frac{\pi}{4}\right)\right] \quad \text{for } x \gtrless 0 \tag{2-155}$$

An application of the formula (2–154) is made to the problem of air-coupled surface waves in Chaps. 4 and 5. The case $U = c$ occurs if c is equal to Kelvin's minimum wave velocity, when gravity and capillarity both are taken into account.

Equations (2–154) can be successfully applied to the calculations of wave resistance and to the phenomenon of "dead water." In the latter problem the waves in two superposed liquids are considered (a layer of finite depth on a semi-infinite liquid). In a second paper on the subject Lamb [24] considered the waves generated by a traveling point source.

2–9. Experimental Study of Lamb's Problem. The theoretical studies of Stokes [52] and Rayleigh [39] showed that compressional waves and shear waves may be propagated through a homogeneous, isotropic solid body and that surface waves may be propagated along a free surface of such a body. The first systematic seismographic recordings from distant earthquakes were made in 1889 by von Rebeur-Paschwitz but it was not until 1900 that Oldham [34] recognized the threefold character of the disturbance produced by a distant earthquake and the fact that these three parts correspond to the compressional, shear, and Rayleigh waves predicted theoretically.

Lamb's [22] often neglected solution for the disturbances in a semi-infinite elastic solid resulting from an impulsive disturbance in a limited region is one of the most important papers in the literature of wave propagation. Lamb's calculation (Fig. 2–18) accounted for many of the principal features of seismograms from distant earthquakes for a source not unlike

that at the focus of an earthquake. Actual seismograms are considerably more complicated than the idealized one computed by Lamb but it is gradually being shown that most of the complications result from the fact that the earth is a sphere and is layered. For example, the presence of large transverse motions in the early part of surface-wave trains and also the long duration and oscillatory character of these trains were so much at variance with the results of the Rayleigh-Lamb theory that there were persistent doubts about the applicability of the calculation. Love [26] showed how both of these features resulted from the layering in the earth, which affects surface waves far more strongly than it does body waves. In fact, for experimental verification of the Rayleigh-Lamb theory of surface waves in a homogeneous semi-infinite medium, it is necessary to go to model experiments.

Northwood and Anderson [33], Kaufman and Roever [19], Knopoff [21], Oliver, Press, and Ewing [35], Tatel [53], and others performed experiments on models, using ultrasonic-pulse techniques. These investigators conclude that the theory of Lamb adequately explains their experimental results.

In the model study of Lamb's problem a pressure pulse is applied at a point on the surface or in the interior of an elastic "half space." Actually, the model consists of a block or slab of steel, limestone, or concrete of sufficiently large dimensions to be considered a half space for the distances involved. The pressure pulse is generated either by a spark or by brief voltage pulses applied to a small piezoelectric transducer. Motion generated by the impulse is detected by small piezoelectric transducers, amplified and displayed on a cathode-ray oscilloscope whose sweep is triggered by the initial pulse.

In Fig. 2–18 is presented Lamb's calculation of the horizontal and vertical ground motion from a distant point impulse applied normal to the surface. Figure 2–19 is Tatel's model seismogram taken with transmitter and vertical-component detector spaced 5 cm apart on the surface of a large block of steel. Trace (*a*) is the vertical displacement, trace (*b*) is the same with amplification increased by a factor of 10, trace (*c*) is the vertical velocity of surface particles, and trace (*d*) is this function amplified 30 times. It is seen that the essential details of traces (*a*) and (*b*) agree with Lamb's theoretical description.

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CHAPTER 3

TWO SEMI-INFINITE MEDIA IN CONTACT

It is a well-established fact that a disturbance of any kind propagating in one medium and impinging upon an interface gives rise, in general, to reflected and refracted waves. We shall see in this chapter under what conditions additional disturbances may arise.

3–1. Reflection and Refraction of Plane Waves at an Interface. In this section we shall discuss several problems concerning the propagation of disturbances in two semi-infinite elastic media in contact at a plane interface. However, before considering this problem in all details, an elementary discussion of reflection and refraction of plane elastic waves will be given.

Knott [22] seems to have been the first to derive the general equations for reflection and refraction at plane boundaries. His work was elaborated by other investigators, and in this section we shall make use of Jeffreys' [19, 20] treatment.

Both liquids and solids will be taken into account. In the case of the earth we are concerned with the interfaces between the atmosphere and the land or water, between the water and the ocean bottom, and between the different rock layers. We have seen in Sec. 2–1 that the problem of propagation is essentially simplified by the assumption that all functions involved are independent of one coordinate, say y, whose axis lies in the interface. We can then discuss two separate groups of displacements, one represented by u and w, and the other only by v, this displacement being parallel to the y axis.

Rigid Boundary. Under a simple assumption a first approximation may be made, which can be applied only in a few cases. Assume that the common boundary between two media is a plane and that the physical properties and conditions are such that waves impinging on the boundary from one medium produce no motion of the boundary. No disturbance is transmitted to the second medium.

We represent the disturbances produced by a plane P wave incident on the interface from below by the equations obtained from (2–7):

$$\begin{aligned} \varphi &= A_1 \exp [ik(ct - x + az)] + A_2 \exp [ik(ct - x - az)] \\ \psi &= B_2 \exp [ik(ct - x - bz)] \end{aligned} \tag{3–1}$$

where $a = \tan e$
$b = \tan f$

The directions of waves corresponding to these terms at any point of the interface are depicted in Fig. 3–1. Then, from the condition $u = 0$, $w = 0$ at $z = 0$ we obtain, using (2–1),

$$\frac{A_1}{ab + 1} = \frac{A_2}{ab - 1} = \frac{B_2}{2a} \tag{3–2}$$

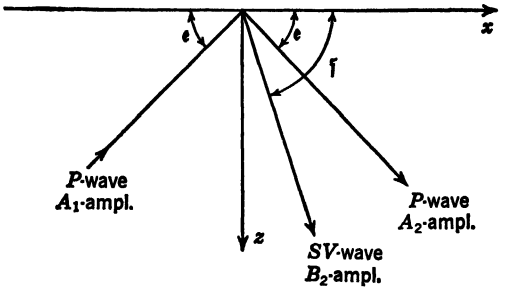


FIG. 3–1. Reflection of P waves at a rigid boundary.

A reflected distortional wave represented by the function ψ exists always, except for grazing and normal incidence, since $e \to 0$, $a \to 0$, and $B_2 \to 0$ in the first case, $B_2/A_1 \to 0$ in the second. There is no reflected P wave if

$$ab = \tan e \tan f = 1 \tag{3–3}$$

If we use the definitions of tan e and tan f given in Sec. 2–1 and Eq. (2–9) for the case $\lambda = \mu$, this condition becomes

$$(3 \tan^2 e - 1)(\tan^2 e + 1) = 0 \tag{3–3′}$$

If we take an incident SV wave (Fig. 3–2), Eqs. (2–7) take the form

$$\varphi = A_2 \exp [ik(ct - x - az)]$$
$$\psi = B_1 \exp [ik(ct - x + bz)] + B_2 \exp [ik(ct - x - bz)] \tag{3–4}$$

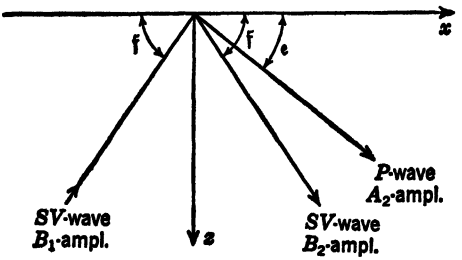


FIG. 3–2. Reflection of SV waves at a rigid boundary.

and the condition of vanishing displacements at $z = 0$ (Eqs. 2–1) leads to

$$\frac{B_1}{ab + 1} = \frac{B_2}{ab - 1} = -\frac{A_2}{2b} \tag{3–5}$$

General Equations. Any incident wave at the interface of two elastic-solid bodies will, in general, produce compressional and distortional waves in both media. Four boundary conditions must be satisfied, requiring continuity of the two components of displacement u, w and the two stresses p_{zz}, p_{xz} across the interface. Indicating by the subscript 1 and 2 quantities referring to incident and reflected waves, respectively, and by accents quantities referring to transmitted waves, we have, in general,

$$\varphi = A_1 \exp [ik(ct - x + az)] + A_2 \exp [ik(ct - x - az)] \tag{3–6}$$

$$\psi = B_1 \exp [ik(ct - x + bz)] + B_2 \exp [ik(ct - x - bz)] \tag{3–7}$$

$$\varphi' = A' \exp [ik(ct - x + a'z)] \quad \psi' = B' \exp [ik(ct - x + b'z)] \tag{3–8}$$

where $a = \tan e$
$b = \tan f$
$a' = \tan e'$
$b' = \tan f'$

and e, f, e′, f′ are defined in Figs. 3–3 and 3–4.

Since we assume that the boundary conditions at the interface $z = 0$ are independent of x and t, the coefficients c and k must be the same in

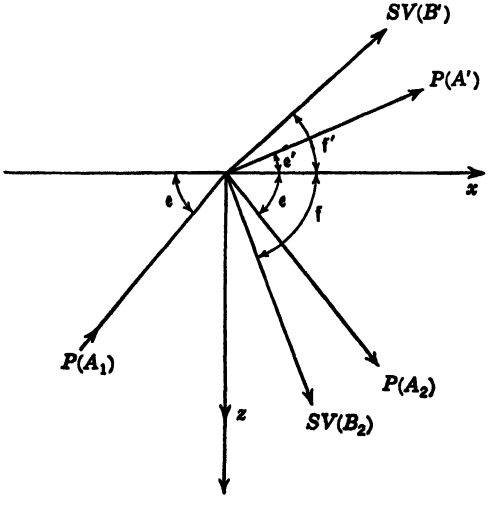


FIG. 3–3. Reflection of P waves at an interface between two elastic solids.

the solutions (3–6), (3–7), and (3–8). The elementary laws of reflection and refraction immediately follow, as in Sec. 2–1, from the fact that these expressions satisfy appropriate wave equations, and we have

$$c = \frac{\alpha}{\cos \mathfrak{e}} = \frac{\beta}{\cos \mathfrak{f}} = \frac{\alpha'}{\cos \mathfrak{e}'} = \frac{\beta'}{\cos \mathfrak{f}'} \tag{3–9}$$

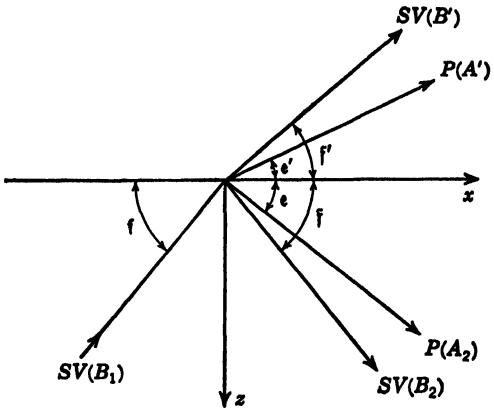


FIG. 3–4. Reflection of SV waves at an interface between two elastic solids.

These conditions imply that for real $\mathfrak{e}$, $\mathfrak{f}$, $\mathfrak{e}'$, $\mathfrak{f}'$ the velocity c must be greater than α, β, α', and β'.

Using the methods of Sec. 2–1, we may write

$$\begin{aligned} a &= \sqrt{\frac{c^2}{\alpha^2} - 1} & c &> \alpha & b &= \sqrt{\frac{c^2}{\beta^2} - 1} & c &> \beta \\ a' &= \sqrt{\frac{c^2}{\alpha'^2} - 1} & c &> \alpha' & b' &= \sqrt{\frac{c^2}{\beta'^2} - 1} & c &> \beta' \\ a' &= -i\sqrt{1 - \frac{c^2}{\alpha'^2}} & c &< \alpha' & b' &= -i\sqrt{1 - \frac{c^2}{\beta'^2}} & c &< \beta' \end{aligned} \tag{3–10}$$

It will be recalled that when any of the coefficients defined in (3–10) are imaginary, complex reflection coefficients will occur, indicating phase changes.

If the displacements and stresses are taken in the form (2–1) and (2–8), the four boundary conditions require $u = u'$, $w = w'$, $p_{zz} = p'_{zz}$, $p_{xz} = p'_{xz}$ at $z = 0$, or

$$A_1 + A_2 + b(B_1 - B_2) = A' + b'B' \tag{3–11}$$

$$a(A_1 - A_2) - (B_1 + B_2) = a'A' - B' \tag{3–12}$$

$$\rho\beta^2\{-(b^2-1)(A_1+A_2)+2b(B_1-B_2)\} = \rho'\beta'^2\{-(b'^2-1)A'+2b'B'\} \tag{3-13}$$

$$\rho\beta^2\{2a(A_1-A_2)+(b^2-1)(B_1+B_2)\} = \rho'\beta'^2\{2a'A'+(b'^2-1)B'\} \tag{3-14}$$

By (1-23), $\mu = \rho\beta^2$, and (3-13) is obtained after a transformation in which use is made of conditions (3-10). An incident wave of a single type usually occurs so that either $A_1 = 0$ or $B_1 = 0$, and the four amplitude coefficients may be expressed in terms of the amplitude of the incident wave.

To examine the case where the medium $z > 0$ is liquid, let $\beta \to 0$ and note that $b \to \infty$ in such a manner that $\beta^2 b^2 \to \alpha^2 \sec^2 \epsilon = c^2$, $\beta^2 b \to 0$, and $B_1 = B_2 = 0$. Slippage occurs at the interface, Eq. (3-11) becomes extraneous, and the corresponding tangential stress in (3-14) is zero.

Liquid-Liquid Interface. For this case Eqs. (3-12) and (3-13) lead to

$$A_1 - A_2 = \frac{a'}{a}A' \tag{3-15}$$

$$A_1 + A_2 = \frac{\rho'}{\rho}A' \tag{3-16}$$

These equations can be derived directly from the boundary conditions for liquids, using (3-6), the first equations (3-8), (3-13), and (1-18). From (3-15) and (3-16) we find the reflection and transmission coefficients

$$\frac{A_2}{A_1} = \frac{\rho'/\rho - a'/a}{\rho'/\rho + a'/a} = \frac{\rho'/\rho - \sqrt{c^2/\alpha'^2 - 1}/\sqrt{c^2/\alpha^2 - 1}}{\rho'/\rho + \sqrt{c^2/\alpha'^2 - 1}/\sqrt{c^2/\alpha^2 - 1}} \tag{3-17}$$

$$\frac{A'}{A_1} = \frac{2}{\rho'/\rho + a'/a} = \frac{2}{\rho'/\rho + \sqrt{c^2/\alpha'^2 - 1}/\sqrt{c^2/\alpha^2 - 1}} \tag{3-18}$$

where $c = \alpha \sec \epsilon = \alpha' \sec \epsilon'$. We have by (3-9)

$$\frac{\alpha}{\alpha'} = \frac{\cos \epsilon}{\cos \epsilon'} = n$$

n being the refraction index. For normal incidence, $\epsilon = \pi/2$, $c = \infty$, and we have

$$\frac{A_2}{A_1} = \frac{\rho'/\rho - \alpha/\alpha'}{\rho'/\rho + \alpha/\alpha'} \qquad \frac{A'}{A_1} = \frac{2}{\rho'/\rho + \alpha/\alpha'} \tag{3-19}$$

For grazing incidence, $\epsilon = 0$, $c = \alpha$, and

$$\frac{A_2}{A_1} = -1 \qquad \frac{A'}{A_1} = 0 \tag{3-20}$$

The reflected wave vanishes when $\rho' \tan \epsilon = \rho\sqrt{\sec^2 \epsilon \cdot \alpha^2/\alpha'^2 - 1}$, and the reflection coefficient becomes unity when $c = \alpha'$ or $\cos \epsilon = \alpha/\alpha'$ and $\epsilon' = 0$.

For the case $\alpha' > c > \alpha$, (3–17) becomes

$$\frac{A_2}{A_1} = \frac{\rho'/\rho + i\sqrt{1 - c^2/\alpha'^2}/\sqrt{c^2/\alpha^2 - 1}}{\rho'/\rho - i\sqrt{1 - c^2/\alpha'^2}/\sqrt{c^2/\alpha^2 - 1}} = e^{i2\epsilon} \tag{3–21}$$

where

$$\tan \epsilon = \frac{\rho\sqrt{1 - c^2/\alpha'^2}}{\rho'\sqrt{c^2/\alpha^2 - 1}} \qquad 0 \leq \epsilon \leq \frac{\pi}{2} \tag{3–22}$$

The effect of the phase shift is to increase the time factor from t to $t + 2\epsilon/\omega$, regardless of choice of axes and direction of propagation. It follows from (3–9) and (3–22) that ϵ is real and, therefore, total reflection occurs without change in amplitude and with a phase change of 2ϵ. The refracted wave for this case is given by the first expression in Eqs. (3–8), with a' (hence ϵ') imaginary. The factor representing the phase of the refracted wave can be written in the form

$$\exp\,[ik(ct - x)] \exp\,(-k\sqrt{1 - c^2/\alpha'^2}\,|z|) \tag{3–23}$$

For $c \leq \alpha'$ no disturbance is transmitted in the interior of the second medium, since by (3–9) ϵ' is imaginary or zero. Nevertheless, for $c < \alpha'$, the formulas represent a disturbance in the second medium which, according to the first factor in (3–23), propagates along the interface, decreasing exponentially with the distance from it. It can be seen from (3–22) and (3–9) that 2ϵ varies from 0 to π as ϵ goes from the critical value, i.e., from $\epsilon_{cr} = \cos^{-1}(\alpha/\alpha')$ to 0. A plot of reflection coefficients for normal incidence given by Eqs. (3–19) appears in Fig. 3–5. The phase change 2ϵ is plotted as a function of angle of incidence in Fig. 3–6.

Liquid-Solid Interface. Following a similar procedure, we may derive for this case the reflection and transmission coefficients for an incident compressional wave in the liquid medium. In addition to the waves discussed in the preceding section, a transmitted shear wave occurs. One finds

$$\frac{A_2}{A_1} = \frac{-\rho a'c^4/\beta'^2 + \mu' a[(c^2/\beta'^2 - 2)^2 + 4a'b']}{\rho a'c^4/\beta'^2 + \mu' a[(c^2/\beta'^2 - 2)^2 + 4a'b']} \tag{3–24}$$

$$\frac{A'}{A_1} = \frac{2\rho ac^2(c^2/\beta'^2 - 2)}{\rho a'c^4/\beta'^2 + \mu' a[(c^2/\beta'^2 - 2)^2 + 4a'b']} \tag{3–25}$$

$$\frac{B'}{A_1} = -\frac{4\rho aa'c^2}{\rho a'c^4/\beta'^2 + \mu' a[(c^2/\beta'^2 - 2)^2 + 4a'b']} \tag{3–26}$$

For the case $\alpha < \beta' < c < \alpha'$, a' is negative imaginary, and φ' decreases exponentially with distance from the interface. The reflected compressional wave undergoes a phase change as does the transmitted shear wave. For the case $\alpha < c < \beta' < \alpha'$ $[\pi/2 > \epsilon > \cos^{-1}(\alpha/\beta')]$ total reflection occurs in the liquid, the attendant phase change being given by 2ϵ, where

$$\cot \epsilon = \frac{\rho'\beta'^4\sqrt{c^2/\alpha^2 - 1}}{\rho c^4\sqrt{1 - c^2/\alpha'^2}}\left[\left(2 - \frac{c^2}{\beta'^2}\right)^2 - 4\sqrt{1 - \frac{c^2}{\alpha'^2}}\sqrt{1 - \frac{c^2}{\beta'^2}}\right] \tag{3–27}$$

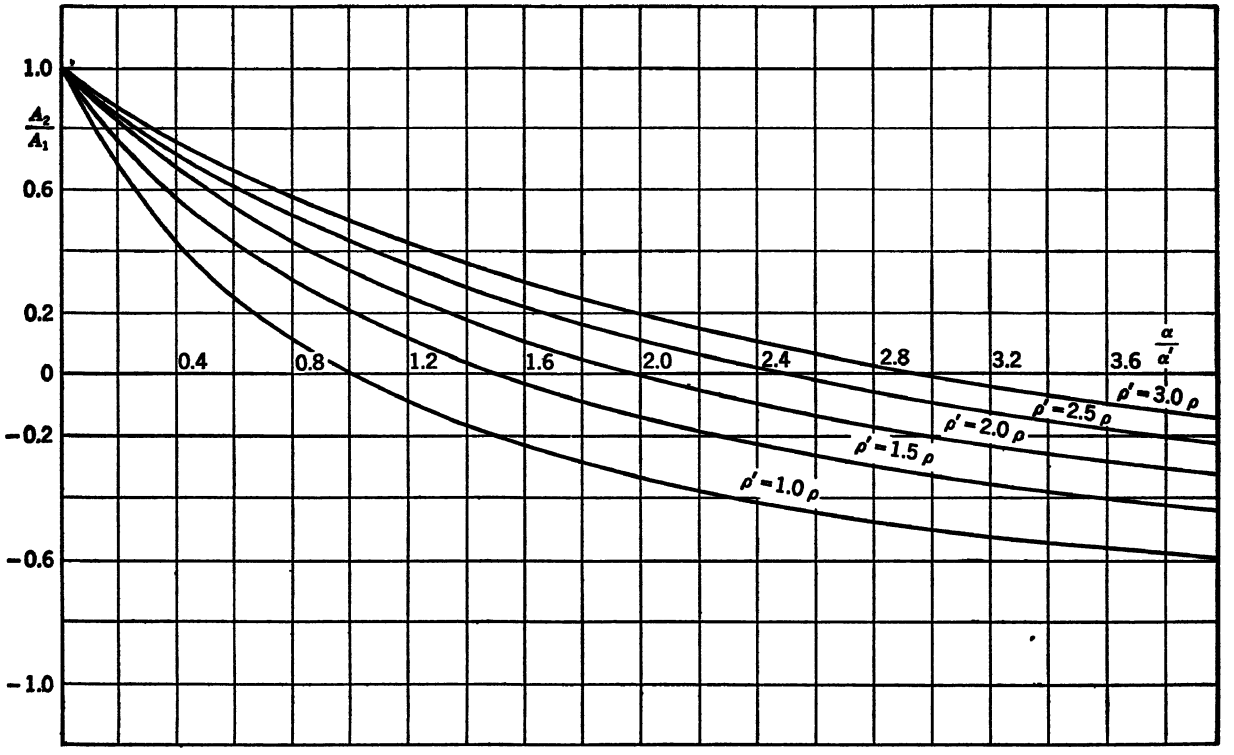


FIG. 3–5. Reflection coefficient for normal incidence on interface between two liquid layers.

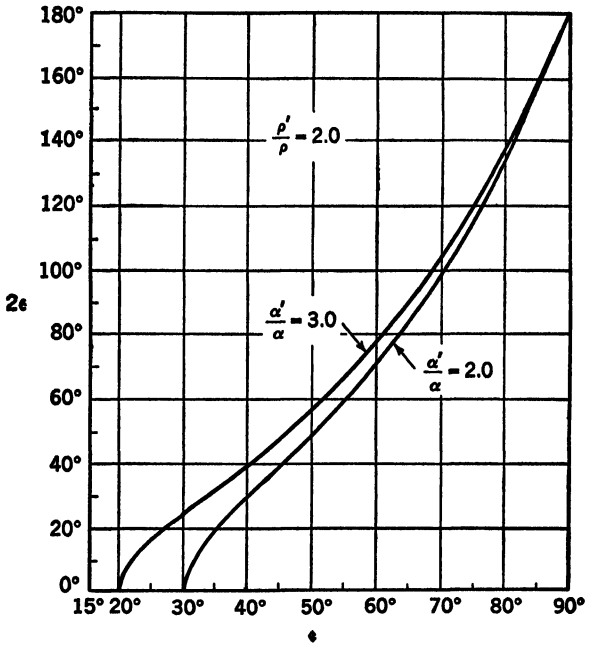


FIG. 3–6. Phase change 2ϵ for various angles of incidence.

Both a' and b' are imaginary in Eqs. (3–8), and φ' and ψ' decrease exponentially with distance from the interface.

Ergin [10] has computed the square roots of the energy ratios ξ_2, ξ', and η' for reflected P, transmitted P, and transmitted S, respectively, where

$$\xi_2 = \frac{A_2}{A_1} \qquad \xi' = \sqrt{\frac{\rho' \tan e'}{\rho \tan e}}\,\frac{A'}{A_1} \qquad \eta' = \sqrt{\frac{\rho' \tan f'}{\rho \tan e}}\,\frac{B'}{A_1}$$

and

$$1 = \xi_2^2 + \xi'^2 + \eta'^2$$

He studied the following three cases:

Case	α'/β'	α'/α	ρ'/ρ
1	1.6	3 0	3.0
2	1.7	3 0	3.0
3	1.8	3.0	3.0

His curves for a P wave in the water incident on the solid are shown in Figs. 3–7 and 3–8. For a P or SV wave in the solid incident on the liquid

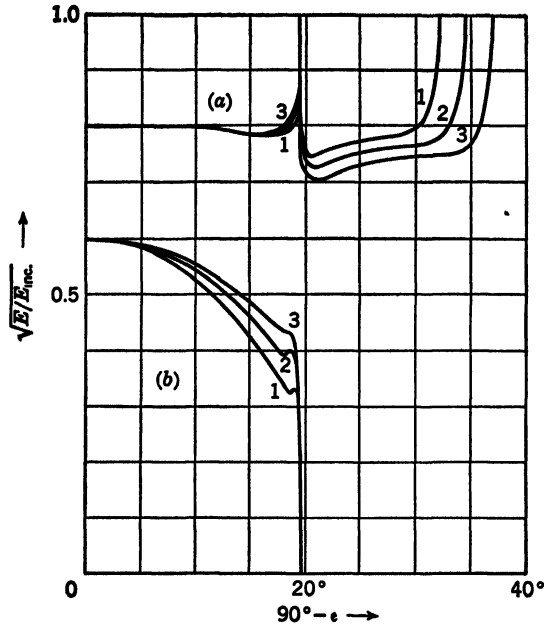


FIG. 3–7. (*a*) Square root of the energy ratio for the reflected *P* wave (*b*) Square root of the energy ratio for the refracted *P* wave for a *P* wave incident in water against a solid. (*After Ergin.*)

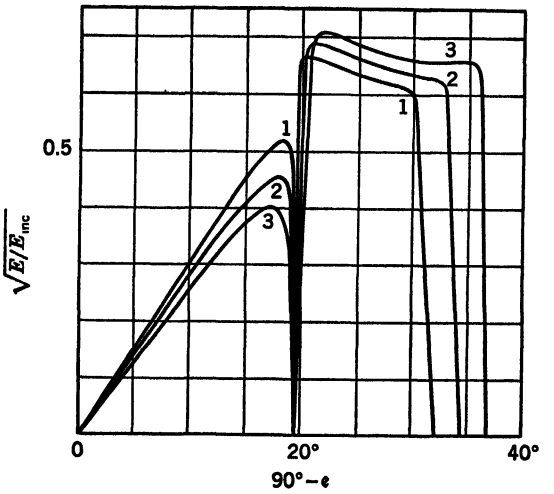


FIG. 3–8. Square root of the energy ratio for the refracted *SV* wave for a *P* wave incident in water against a solid. (*After Ergin.*)

the corresponding quantities for reflected P, reflected SV, and transmitted P are plotted in Figs. 3–9 to 3–14 with $\alpha/\alpha' = 0.2$, $\rho/\rho' = 0.3$, and $\alpha'/\beta' = 1.6, 1.7, 1.75$ in Figs. 3–12 to 3–14.

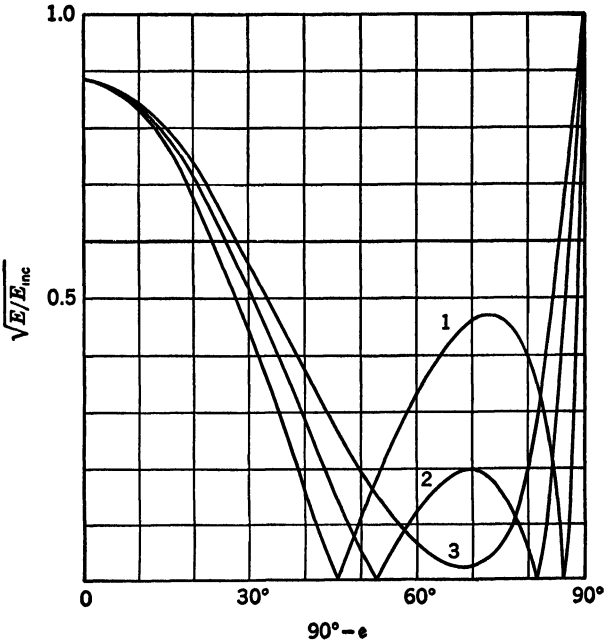


FIG. 3-9. Square root of the energy ratio for the reflected P wave for a P wave incident in a solid against water. (*After Ergin.*)

Two other sets of reflection and refraction coefficients for the case of a liquid-solid interface can be derived for an incident compressional or distortional wave in the solid.

Solid-Solid Interface. We note first that Eqs. (3–11), (3–12), (3–13), and (3–14) form two separate groups, with unknowns $A_1 + A_2 = S$, $B_1 - B_2 = D'$ in the first and $A_1 - A_2 = D$, $B_1 + B_2 = S'$ in the second group respectively. Now, solving Eqs. (3–11) and (3–13), we obtain

$$\Delta = \mu b(b^2 + 1)$$

$$\Delta_S = [2\mu b + \mu' b(b'^2 - 1)]A' + 2bb'(\mu - \mu')B'$$

$$\Delta_{D'} = [\mu(b^2 - 1) - \mu'(b'^2 - 1)]A' + b'[2\mu' + \mu(b^2 - 1)]B'$$

$$S = \frac{\Delta_S}{\Delta} = \frac{[2\mu + \mu'(b'^2 - 1)]A' + 2b'(\mu - \mu')B'}{\mu(b^2 + 1)} \tag{3-28}$$

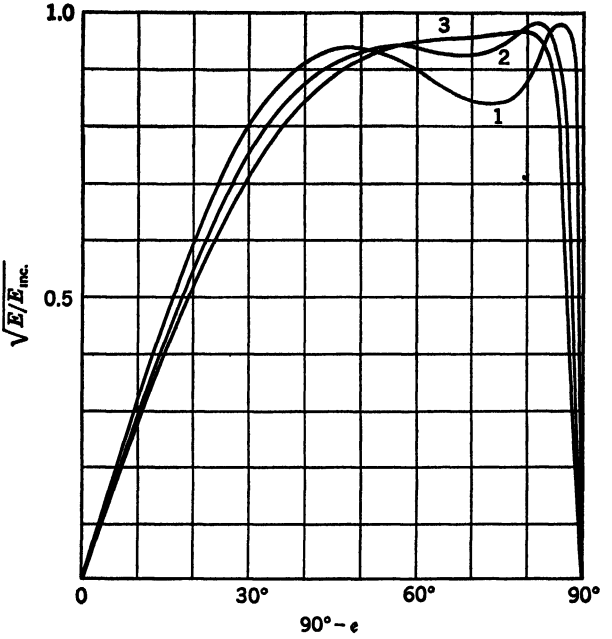


FIG. 3–10. Square root of the energy ratio for the reflected SV wave for a P wave incident in a solid against water. (*After Ergin.*)

$$D' = \frac{\Delta_{D'}}{\Delta} = \frac{[\mu(b^2 - 1) - \mu'(b'^2 - 1)]A' + b'[2\mu' + \mu(b^2 - 1)]B'}{\mu b(b^2 + 1)} \tag{3–29}$$

Solving Eqs. (3–12) and (3–14), we have

$$\Delta' = \mu a(b^2 + 1)$$

$$\Delta_D = a'[2\mu' + \mu(b^2 - 1)]A' - [\mu(b^2 - 1) - \mu'(b'^2 - 1)]B'$$

$$\Delta_{S'} = 2aa'(\mu' - \mu)A' + a[2\mu + \mu'(b'^2 - 1)]B'$$

and

$$D = \frac{\Delta_D}{\Delta'} = \frac{a'[2\mu' + \mu(b^2 - 1)]A' - [\mu(b^2 - 1) - \mu'(b'^2 - 1)]B'}{\mu a(b^2 + 1)} \tag{3–30}$$

$$S' = \frac{\Delta_{S'}}{\Delta'} = \frac{2a'(\mu' - \mu)A' + [2\mu + \mu'(b'^2 - 1)]B'}{\mu(b^2 + 1)} \tag{3–31}$$

It was assumed that the incident waves occur in the first medium. Now we shall be able to find the reflection and refraction coefficients from

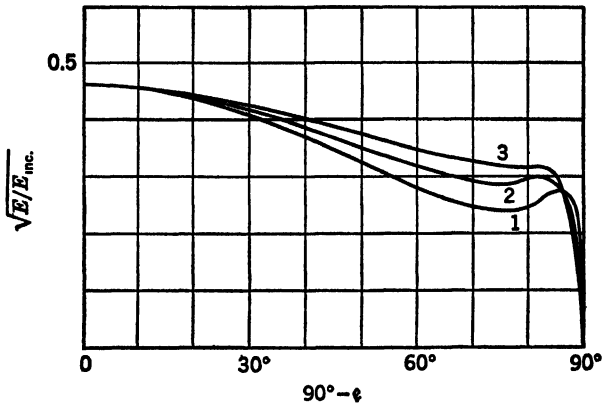


FIG. 3–11. Square root of the energy ratio for the refracted P wave for a P wave incident in a solid against water. (*After Ergin.*)

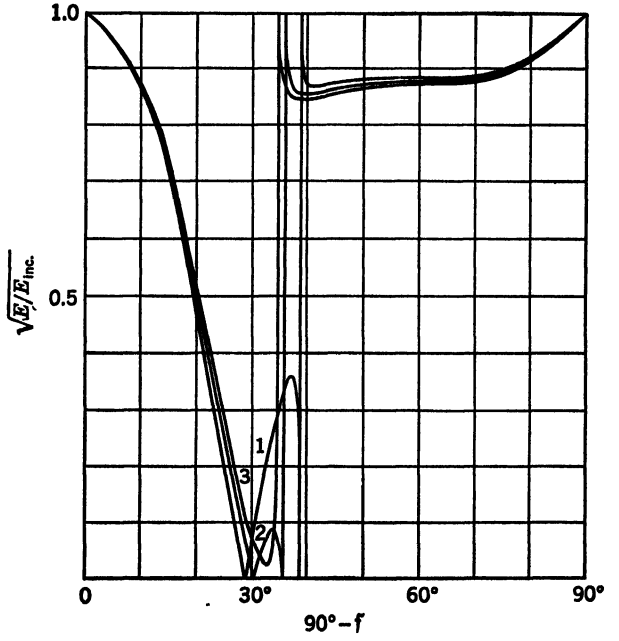


FIG. 3–12. Square root of the energy ratio for the reflected SV wave for an SV wave incident in a solid against water. (*After Ergin.*)

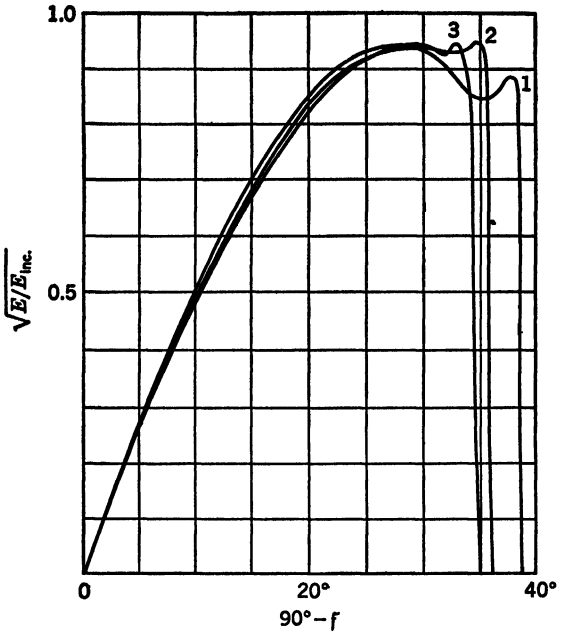


FIG. 3–13. Square root of the energy ratio for the reflected P wave for an SV wave incident in a solid against water. (*After Ergin.*)

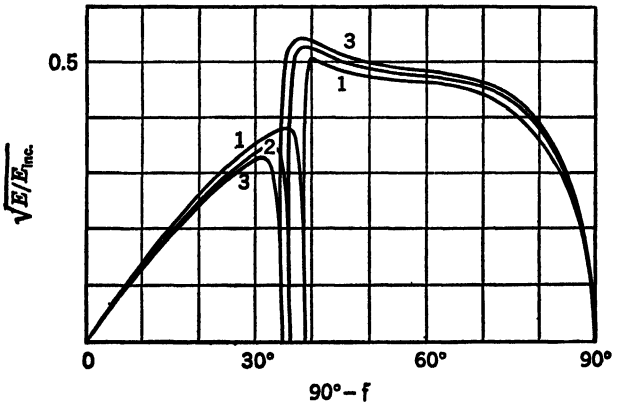


FIG. 3–14. Square root of the energy ratio for the refracted P wave for an SV wave incident in a solid against water. (*After Ergin.*)

(3-28) to (3-31) if there is a single incident wave, e.g., a compressional wave ($B_1 = 0$). Then these equations take the form

$$\begin{aligned} -\frac{A_2}{A_1} + l_1\frac{A'}{A_1} + m_1\frac{B'}{A_1} &= 1 \\ \frac{B_2}{A_1} + l_2\frac{A'}{A_1} + m_2\frac{B'}{A_1} &= 0 \\ \frac{A_2}{A_1} + l_3\frac{A'}{A_1} + m_3\frac{B'}{A_1} &= 1 \\ -\frac{B_2}{A_1} + l_4\frac{A'}{A_1} + m_4\frac{B'}{A_1} &= 0 \end{aligned} \tag{3-32}$$

Hence

$$\begin{aligned} (l_1 + l_3)\frac{A'}{A_1} + (m_1 + m_3)\frac{B'}{A_1} &= 2 \\ (l_2 + l_4)\frac{A'}{A_1} + (m_2 + m_4)\frac{B'}{A_1} &= 0 \end{aligned} \tag{3-33}$$

and

$$\frac{A'}{A_1} = \frac{2(m_2 + m_4)}{(l_1 + l_3)(m_2 + m_4) - (l_2 + l_4)(m_1 + m_3)} \tag{3-34}$$

$$\frac{B'}{A_1} = \frac{-2(l_2 + l_4)}{(l_1 + l_3)(m_2 + m_4) - (l_2 + l_4)(m_1 + m_3)} \tag{3-35}$$

$$\frac{A_2}{A_1} = \frac{(l_1 - l_3)(m_2 + m_4) - (l_2 + l_4)(m_1 - m_3)}{(l_1 + l_3)(m_2 + m_4) - (l_2 + l_4)(m_1 + m_3)} \tag{3-36}$$

$$\frac{B_2}{A_1} = \frac{2(l_4 m_2 - m_4 l_2)}{(l_1 + l_3)(m_2 + m_4) - (l_2 + l_4)(m_1 + m_3)} \tag{3-37}$$

where the expressions for the coefficients l and m must be taken from Eqs. (3-28) to (3-31). Muskat and Meres [29] developed systematic tables of the reflection and transmission coefficients for the various types of interfaces for application in seismic-reflection surveys.

The reflection and refraction of elastic waves at a plane separating two media were also discussed in several other investigations (Schuster [48], Krüger [24], Ott [36, 37], Brekhovskikh [4], Gutenberg [Chap. 2, Ref. 10], Slichter and Gabriel [50, 51]).

Gutenberg gave curves for the square root of the energy ratio of the reflected and transmitted waves for several values of the elastic parameters. The ratios may be obtained from the energy equations (particular cases of which were considered in Sec. 2-1)

$$1 = \frac{A_2^2}{A_1^2} + \frac{\tan \mathfrak{f}\, B_2^2}{\tan \mathfrak{e}\, A_1^2} + \frac{\rho' \tan \mathfrak{e}'\, A'^2}{\rho \tan \mathfrak{e}\, A_1^2} + \frac{\rho' \tan \mathfrak{f}'\, B'^2}{\rho \tan \mathfrak{e}\, A_1^2}$$

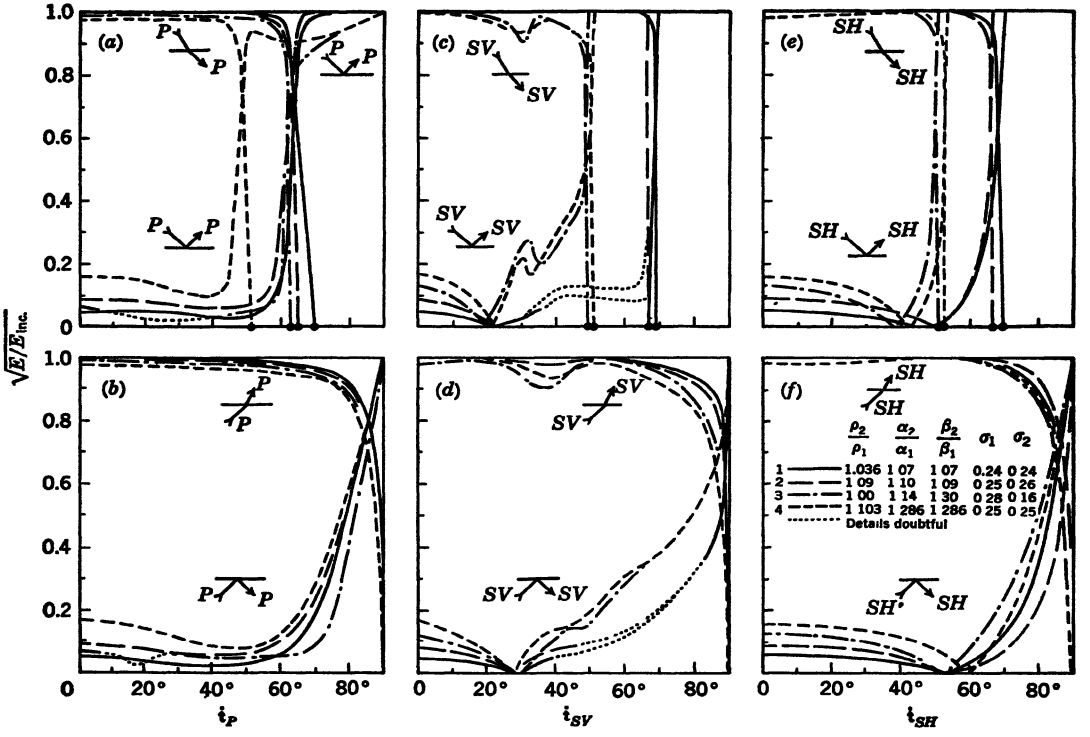


FIG. 3–15. Square roots of ratio of reflected or transmitted to incident energy if no change in wave type occurs. (*After Gutenberg.*) Subscript 1 refers to upper layer, subscript 2 to lower layer.

for an incident P wave and

$$1 = \frac{\tan e\, A_2^2}{\tan f\, B_1^2} + \frac{B_2^2}{B_1^2} + \frac{\rho' \tan e'\, A'^2}{\rho \tan f\, B_1^2} + \frac{\rho' \tan f'\, B'^2}{\rho \tan f\, B_1^2}$$

for an incident SV wave. The terms on the right-hand side are, respectively, the energy ratios for the reflected P, reflected SV, transmitted P, and transmitted SV waves. The square roots of the ratios are plotted versus the angle of incidence in Figs. 3–15 and 3–16 for various cases.

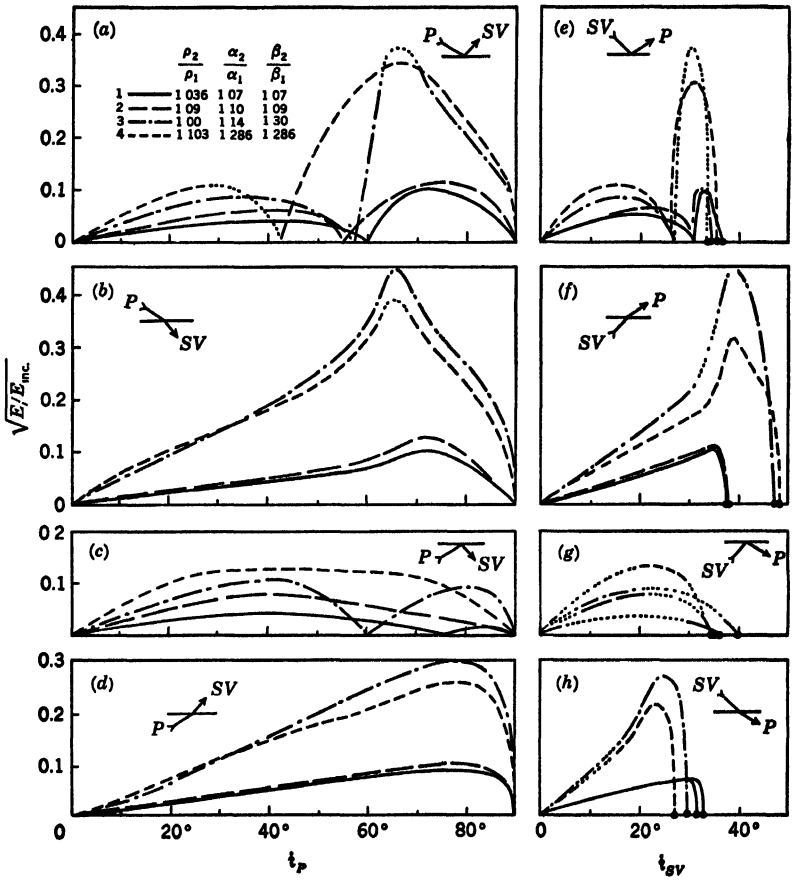


FIG. 3–16. Square roots of ratio of reflected or transmitted to incident energy if incident and reflected or transmitted waves are of different type. (*After Gutenberg.*) Subscript 1 refers to upper layer, subscript 2 to lower layer.

3–2. Reflection of a Pulse Incident beyond the Critical Angle. In Sec. 3–1 we derived expressions for the reflection of simple harmonic plane waves from the interface between two liquid media. For reflection beyond the critical angle only expressions for the incident and reflected compressional waves are involved, and by (3–21) the second wave is subjected to a phase change 2ϵ, equivalent to a time increase of $2\epsilon/|\omega|$. For this case, expression (3–6) may be written in the form

$$\varphi_i = A_1 \exp\left[i\omega(t - \bar{\alpha}x + \bar{a}z)\right] \tag{3–38}$$

$$\varphi_r = A_1 \exp\left[i\omega\left(t - \bar{\alpha}x - \bar{a}z + \frac{2\epsilon}{|\omega|}\right)\right] \tag{3–39}$$

where

$$\bar{\alpha} = \frac{1}{c} = \frac{\cos \mathfrak{e}}{\alpha}$$

$$\bar{a} = \frac{a}{c} = \frac{\sin \mathfrak{e}}{\alpha}, \quad \text{by (3–9)}$$

$$a = \tan \mathfrak{e}$$

The factor ϵ is given by (3–22). This equation shows that the phase shift 2ϵ is determined by the physical constants of the media. Had we started with a negative ω in Eqs. (3–6), (3–7), and (3–8), the sign of the imaginary radicals as defined in (3–10) would be reversed, changing the sign in the exponent of (3–21), giving $A_2 = A_1 \exp(-i2\epsilon)$; hence the factor $2\epsilon/|\omega|$ is to be added to the time for any value of ω. Take the time variation in the incident pulse in the form

$$\hat{\varphi}_i(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t}\, d\omega \tag{3–40}$$

where the Fourier transform $g(\omega)$ is given by

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}_i(\tau) e^{-i\omega\tau}\, d\tau \tag{3–41}$$

Similarly, the time variation in the reflected pulse can be given by a Fourier integral

$$\hat{\varphi}_r(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\omega) e^{i\omega t}\, d\omega \tag{3–42}$$

According to (3–38) and (3–39), we can put

$$G(\omega) = g(\omega) \exp i2\epsilon \frac{\omega}{|\omega|} \tag{3–43}$$

using the transformation of (3–42) given by Arons and Yennie [2]. From (3–42) and (3–43)

$$\hat{\phi}_r = \frac{\cos 2\epsilon}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t}\, d\omega + \frac{i \sin 2\epsilon}{\sqrt{2\pi}} \left[-\int_{-\infty}^{0} g(\omega) e^{i\omega t}\, d\omega + \int_{0}^{\infty} g(\omega) e^{i\omega t}\, d\omega \right] \tag{3–42′}$$

The factor of $\cos 2\epsilon$ is the incident pulse $\hat{\phi}_i$. The last two integrals may be combined into a single one, despite their difference in sign, if we introduce the function

$$\mathcal{P} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{i\omega\xi}}{\xi}\, d\xi = \begin{cases} -1 & \omega < 0 \\ 0 & \text{for } \omega = 0 \\ 1 & \omega > 0 \end{cases} \tag{3–44}$$

where $\mathcal{P}$ denotes the principal value of the integral. We obtain

$$\hat{\phi}_r = \hat{\phi}_i(t) \cos 2\epsilon + F(t) \sin 2\epsilon \tag{3–45}$$

where

$$F(t) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \hat{\phi}_i(\tau)\, d\tau \int_{-\infty}^{\infty} e^{i\omega(t-\tau)}\, d\omega \mathcal{P} \int_{-\infty}^{\infty} \frac{e^{i\omega\xi}}{\xi}\, d\xi \tag{3–46}$$

Take the incident pulse to have the form commonly used to represent an explosion

$$\hat{\phi}_i(\tau) = \begin{cases} 0 & \tau < 0 \\ A_1 e^{-\sigma\tau} & \tau > 0, \sigma > 0 \end{cases} \tag{3–47}$$

and integrate with respect to τ to obtain

$$F(t) = \frac{A_1}{2\pi^2} \mathcal{P} \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \int_{-\infty}^{\infty} \frac{e^{i\omega(t+\xi)}}{\sigma + i\omega}\, d\omega \tag{3–48}$$

To perform the integration with respect to ω we follow in the usual manner a semicircular contour in the lower half of the complex ω plane for $\xi < -t$ and one in the upper half for $\xi > -t$. The integrand has a single pole at $\omega = i\sigma$; hence the integral for $\xi < -t$ is zero. Thus (3–48) becomes

$$F(t) = \frac{A_1}{\pi} e^{-\sigma t} \mathcal{P} \int_{-t}^{\infty} \frac{e^{-\sigma\xi}}{\xi}\, d\xi \tag{3–49}$$

The principal part of this integral is equal to the negative value of the function $Ei(\sigma t)$ (see E. Jahnke and F. Emde, "Tables of Functions," pp. 1–8, Dover Publications, New York, 1943). Thus the expression for the

time variation in the reflected pulse becomes

$$\phi_r(t) = \phi_i(t) \cos 2\epsilon - \frac{A_1}{\pi} e^{-\sigma t} \overline{Ei}(\sigma t) \sin 2\epsilon \qquad t > 0$$
$$\phi_r(t) = -\frac{A_1}{\pi} e^{-\sigma t} Ei(\sigma t) \sin 2\epsilon \qquad t < 0 \tag{3-50}$$

This formula yields the obvious results that for $2\epsilon = 0$, $\phi_r(t) = \phi_i(t)$, and for $2\epsilon = \pi$, $\phi_r(t) = -\phi_i(t)$. Arons and Yennie computed the shape of the reflected pulse for various values of 2ϵ. Their results are presented in Fig. 3–17 and show general agreement with their observations on

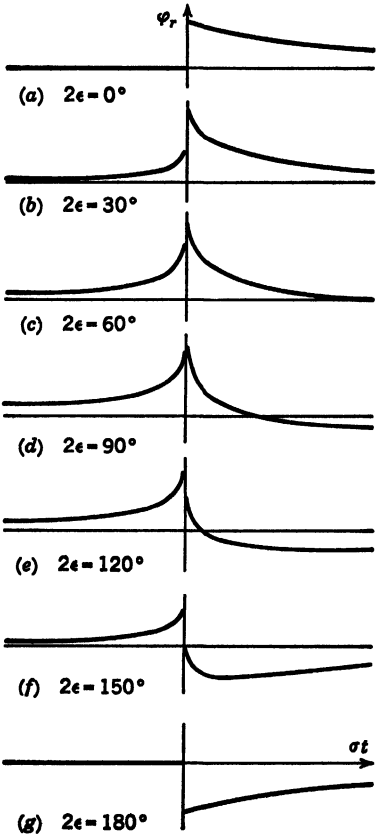


FIG. 3–17. Shape of reflected pulse for several values of phase change 2ϵ, computed from Eqs. 3–50. (*After Arons and Yennie.*)

reflection of explosion sounds in shallow water underlain by unconsolidated sediments.

In many problems on sound transmission in layers multiple reflections are involved. If a pulse undergoes n reflections from the bottom, with phase shift 2ϵ for each reflection, and m reflections from a free surface with phase shift π, the expression for the reflected pulse becomes

$$\phi_{rmn}(t) = (-1)^m \left[\phi_i(t) \cos 2n\epsilon - \frac{A_1 e^{-\sigma t}}{\pi} \overline{Ei}(\sigma t) \sin 2n\epsilon \right] \qquad t > 0 \qquad (3\text{–}51)$$

The theory given by Arons and Yennie seems to be adequate and useful in all cases involving pulses reflected beyond the critical angle, provided that plane-wave approximations are valid and the reflected waves of various orders are added when they overlap. As the distance becomes large compared with the layer thickness, interference between waves of various orders becomes important, and other methods, e.g., normal mode calculations, must be applied (see Chap. 4).

It is interesting to note that the second Eq. (3–50) for the reflected pulse does not exclude a disturbance even for $t < 0$, that is, for times prior to the application of the incident pulse. In this respect our problem is similar to that of the transient response of an idealized electrical network, one with phase distortion but without amplitude distortion (see Ref. 8, chap. 4).

3–3. Propagation in Two Semi-infinite Media: Point Source. The propagation of plane waves in two semi-infinite media separated by a plane interface was discussed in Sec. 3–1. Sommerfeld [55 and Chap. 1, Ref. 55], Jeffreys [20], Muskat [28], and others have discussed wave propagation for the case where the distance of the point source from the plane interface is finite. Their results are directly related to an important practical problem, that of the "refraction arrival" in seismology of near earthquakes and in seismic-refraction investigations. When an impulsive source and a receiver are located in a lower-velocity medium separated by a distance large compared with the distance of either from the plane of contact with the higher-velocity medium, it is observed that the first disturbance arrives at a time corresponding to propagation along the path shown in Fig. 3–18.

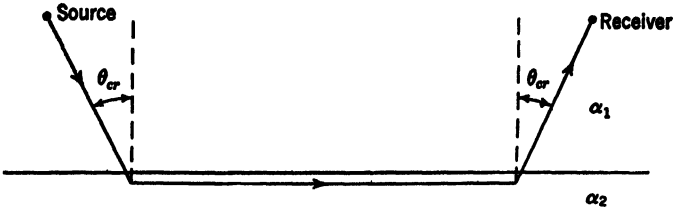


FIG. 3–18. Path of "refraction arrival" when $\alpha_2 > \alpha_1$.

From observations of travel times it can be inferred that the part of the path along the interface is traversed at the higher velocity α_2, the remainder of the path at the lower velocity α_1, and the angle of incidence is equal to the critical angle $\theta_{cr} = \sin^{-1}(\alpha_1/\alpha_2)$. This is the well-known refraction arrival, first used by Mohorovičić [27] in 1909 for deducing continental crustal layering. It is the basis of the seismic-refraction method of exploration. The refraction arrival presented a serious difficulty in that no energy would be expected for this path from the viewpoint of geometric optics. This difficulty was first resolved by Jeffreys [20] who, using wave theory, found terms corresponding to the refraction arrival.

Sommerfeld [Chap. 1, Ref. 55] developed a method similar to that used by Lamb in the case of a half space. He was concerned with the propagation of electromagnetic waves from a source located at the interface. The solution for a dipole located at a certain distance from the interface may be found in his textbook on partial differential equations [55, p. 237]. It was shown by Joos and Teltow [21] that Sommerfeld's formulas can be transformed to represent the propagation of a disturbance in elastic media. It is interesting to note that this problem was studied by several investigators, few of whom took account of the earlier results. We shall begin the discussion with the most simple case of liquid media, which requires only two potentials.

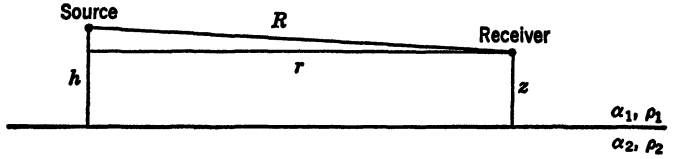


FIG. 3–19. Coordinates for propagation from a point source.

Two Liquids. It was mentioned in Sec. 1–6 that an expression for spherical waves emitted by a point source at $S(0, 0, h)$ (see Fig. 3–19) may be written in the form (1–72). If the time factor $\exp(i\omega t)$ is omitted,

$$\varphi_0 = \frac{e^{-ik_a R}}{R} = \int_0^\infty \frac{e^{-\nu|z-h|}}{\nu} J_0(kr)k\,dk \tag{3–52}$$

where $R = \sqrt{r^2 + (z-h)^2}$

$$\nu = \sqrt{k^2 - k_a^2}$$

$$k_a = \frac{\omega}{\alpha}$$

Now in the problem of wave propagation in two semi-infinite media we have to use two different expressions for the displacement potential. For

the medium ($z > 0$) which contains the source we may write

$$\varphi_1 = \int_0^\infty \frac{e^{-\nu_1|z-h|}}{\nu_1} J_0(kr)k\,dk + \int_0^\infty \frac{Q_1 e^{-\nu_1(z-h)}}{\nu_1} J_0(kr)k\,dk \qquad (3\text{–}53)$$

assuming that besides the primary disturbance φ_0 represented by the first term there is a second one due to the presence of the boundary.

For the second medium ($z < 0$) we may write an expression similar to the second term in (3–53):

$$\varphi_2 = \int_0^\infty \frac{Q_2 e^{\nu_2(z-h)}}{\nu_2} J_0(kr)k\,dk \qquad (3\text{–}54)$$

With the separation of the factors k/ν_1 and k/ν_2, the arbitrary functions Q_1 and Q_2 in Eqs. (3–53) and (3–54) can be related to plane-wave reflection and refraction coefficients. It will be seen later how inclusion of these factors in the arbitrary coefficients leads to simple and symmetrical expressions. These expressions satisfy the wave equations

$$\nabla^2\varphi_i = \frac{1}{\alpha_i^2}\frac{\partial^2\varphi_i}{\partial t^2} \qquad i = 1, 2$$

where α_i is the velocity of propagation of compressional waves in the corresponding medium, provided that

$$\nu_1 = \sqrt{k^2 - k_{\alpha_1}^2} \qquad \nu_2 = \sqrt{k^2 - k_{\alpha_2}^2} \qquad (3\text{–}55)$$

The quantities ν_1 and ν_2 are taken to have positive real parts. This choice is required by the condition of vanishing potential as $|z| \to \infty$, the positive direction of z being taken in the medium containing the source.

The potentials φ_1 and φ_2 must satisfy the boundary conditions at the interface $z = 0$:

$$\frac{\partial\varphi_1}{\partial z} = \frac{\partial\varphi_2}{\partial z} \qquad (3\text{–}56)$$

$$\rho_1\varphi_1 = \rho_2\varphi_2 \qquad (3\text{–}57)$$

These equations express the fact that the normal displacement w and the pressure as defined in (1–26) and (1–18) are continuous across the interface. Substituting (3–53) and (3–54) in (3–56) and (3–57) gives

$$e^{-\nu_1 h} - Q_1 e^{\nu_1 h} = Q_2 e^{-\nu_2 h} \qquad (3\text{–}58)\dagger$$

$$\frac{\rho_1}{\nu_1}(e^{-\nu_1 h} + Q_1 e^{\nu_1 h}) = \frac{\rho_2}{\nu_2} e^{-\nu_2 h} Q_2 \qquad (3\text{–}59)$$

†The exponential in the first integral of (3–53) is taken as $-\nu_1(h - z)$ for $z < h$ to obtain this result.

Solving for Q_1 and Q_2, and writing $\delta = \rho_2/\rho_1$, we find

$$Q_1 = \frac{\delta\nu_1 - \nu_2}{\delta\nu_1 + \nu_2} e^{-2\nu_1 h} \tag{3-60}$$

$$Q_2 = \frac{2\nu_2}{\delta\nu_1 + \nu_2} e^{(\nu_2 - \nu_1)h} \tag{3-61}$$

The solutions (3-53) and (3-54) now become

$$\varphi_1 = \int_0^\infty \frac{e^{-\nu_1|z-h|} J_0(kr)k\,dk}{\nu_1} + \int_0^\infty \left[\frac{\delta\nu_1 - \nu_2}{\delta\nu_1 + \nu_2}\right] \frac{e^{-\nu_1(z+h)} J_0(kr)k\,dk}{\nu_1} \tag{3-62}$$

$$\varphi_2 = \int_0^\infty \left[\frac{2\nu_2}{\delta\nu_1 + \nu_2}\right] \frac{e^{\nu_2 z - \nu_1 h} J_0(kr)k\,dk}{\nu_2} \tag{3-63}$$

where the expressions in brackets have been arranged for interpretation as the reflection and transmission coefficients for plane waves, according to (3-17) and (3-18). Weyl's formula (1-44) for a spherical wave shows that the primary disturbance represented by the first term in (3-62) can be interpreted as a superposition of plane waves. Similarly, the remaining integrals in (3-62) and (3-63) may be interpreted as a superposition of reflected and transmitted plane waves.

To evaluate φ_1, first add the two integrals, using in the exponential function in the first term of (3-62) $|z - h| = h - z$ for $z < h$ and $|z - h| = z - h$ for $z > h$. Two expressions for φ_1 are as follows:

$$\varphi_1 = 2\int_0^\infty \frac{\delta\nu_1 \cosh \nu_1 z + \nu_2 \sinh \nu_1 z}{\nu_1(\delta\nu_1 + \nu_2)} e^{-\nu_1 h} J_0(kr)k\,dk \qquad z < h \tag{3-64}$$

$$\varphi_1 = 2\int_0^\infty \frac{\delta\nu_1 \cosh \nu_1 h + \nu_2 \sinh \nu_1 h}{\nu_1(\delta\nu_1 + \nu_2)} e^{-\nu_1 z} J_0(kr)k\,dk \qquad z > h \tag{3-65}$$

or

$$\varphi_1 = 2\int_0^\infty \Lambda(k) J_0(kr)k\,dk \tag{3-66}$$

For $z < h$ the function $\Lambda(k)$ is given by

$$\Lambda(k) = \frac{\delta\nu_1 \cosh \nu_1 z + \nu_2 \sinh \nu_1 z}{\nu_1(\delta\nu_1 + \nu_2)} e^{-\nu_1 h} \tag{3-67}$$

A similar expression for $z > h$ is obtained by interchanging z and h. It is seen that the integrands do not contain poles since, from the definitions of ν_1 and ν_2, the sum $\delta\nu_1 + \nu_2$ cannot vanish. Addition of the two integrals in (3-62) to obtain (3-64) or (3-65) removed the algebraic singularities at $k = \pm k_{\alpha 1}$ or $\nu_1 = 0$, because the reflection coefficient becomes -1 at this point. Physically this implies cancellation of the direct wave by the reflected wave at grazing incidence, i.e., the limiting case of the Lloyd

mirror effect. In evaluating (3–64) and (3–65) we shall use a method similar to that of Lapwood (1949). (See also Sec. 2–5.) These integrals represent the disturbance corresponding to a source varying with time as exp ($i\omega t$), ω being real. A time variation appropriate for an explosion has been defined in Eq. (3–47). If we use notations of operational analysis, it may also be written in the form

$$S(t) = \frac{1}{2\pi i}\int_{\Omega}\frac{e^{i\omega t}}{\omega - i\sigma}\,d\omega = e^{-\sigma t}H(t) \qquad \sigma \geq 0 \tag{3–68}$$

where $H(t) = 0$ for $t < 0$, $H(t) = 1$ for $t > 0$ is the Heaviside unit function, and the contour Ω runs from $-\infty - ic$ to $\infty - ic$. If $c > 0$ and $t < 0$, Ω is equivalent to the infinite semicircle in the lower half of the complex ω plane along which the integrand vanishes. For $t > 0$ the contour Ω is equivalent to the infinite semicircle in the upper half plane plus a small circle surrounding the pole $\omega = i\sigma$. Thus Eq. (3–68) may be readily verified. When $\sigma = 0$, Eq. (3–68) defines the unit function $H(t)$.

As in (2–137), we may write the solution corresponding to the initial pulse $S(t)$ defined by (3–68) in the form

$$\frac{1}{2\pi i}\int_{\Omega}\frac{f(\omega, r, z)}{\omega - i\sigma}\,e^{i\omega t}\,d\omega \tag{3–69}$$

where $f(\omega, r, z)e^{i\omega t}$ represents the steady-state solution. In applying (3–69) one must be sure that $f(\omega)$ is analytic, that the integral converges, and that approximations used in obtaining $f(\omega)$ are valid over the contour [see Lapwood (Chap. 2, Ref. 25, pp. 66 and 84)].

Now returning to the evaluation of the solutions (3–64) and (3–65), replace k by the complex variable $\zeta = k + i\tau$ and consider complex values of ω.

The signs of ν_1 and ν_2 have already been specified by the requirements Re $\nu_1 > 0$ and Re $\nu_2 > 0$, confining the integrand to a single leaf of the four-leaved Riemann surface. The branch points at which $\nu_1 = 0$, $\nu_2 = 0$ are given by $\zeta = \pm k_{\alpha_1}$, $\zeta = \pm k_{\alpha_2}$, where $k_{\alpha_1} = \omega/\alpha_1$, $k_{\alpha_2} = \omega/\alpha_2$, and the cuts forming the boundaries of the chosen leaf are given by Re $\nu_1 = 0$ and Re $\nu_2 = 0$. The definition of the contour Ω used in (3–68) involves complex values of the variable $\omega = s - ic$ with $c > 0$. Then the last relations (for example, Re $\sqrt{\zeta^2 - \omega^2/\alpha_j^2} = 0$, for $j = 1, 2$) imply that $k^2 - \tau^2 + 2ik\tau - (s^2 - c^2 - 2isc)/\alpha_j^2$ be real and negative or $k^2 - \tau^2 < (s^2 - c^2)/\alpha_j^2$ and $k\tau = -sc/\alpha_j^2$. Under these conditions, the cuts in the complex ζ plane must be parts of hyperbolas defined by the last equation (see Sec. 2–5) and lying as shown in Fig. 3–20. Since $\omega = s - ic$ is now assumed complex, the time factor exp ($i\omega t$) in φ becomes exp $[(s - ic)it] =$ exp ($ct + ist$). For c positive the integrand becomes infinite as $t \to \infty$. However, for the particular time dependence assumed in Eq. (3–68), the

integration with respect to ω performed in the complex plane along the Ω contour results in a solution without a singularity at $t = \infty$. After these preliminary remarks, we make the substitution

$$2J_0(kr) = H_0^{(1)}(kr) + H_0^{(2)}(kr)$$

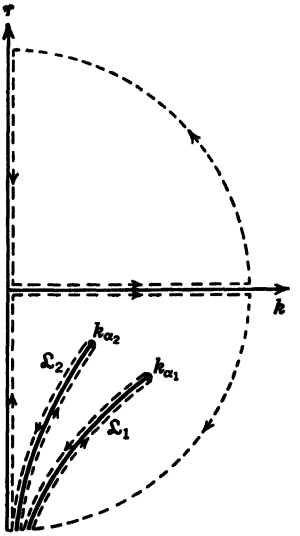


FIG. 3-20. Integration paths in the complex ζ plane for Re $\omega > 0$.

in Eq. (3-66) and write the sum of two integrals

$$\varphi_1 = I_1 + I_2 = \int_0^\infty \Lambda(k)H_0^{(1)}(kr)k\,dk + \int_0^\infty \Lambda(k)H_0^{(2)}(kr)k\,dk \qquad (3\text{-}70)$$

The function $\Lambda(k)$ has no poles, as mentioned above, and for complex ω the branch points are not on the real axis. There are no other singular points on this axis except those of the Hankel functions at the origin. Using the fact that $H_0^{(1)}$ and $H_0^{(2)}$ vanish along the infinite arcs in the first and fourth quadrants, respectively, we can distort the path of integration for the two cases Re $\omega > 0$ and Re $\omega < 0$ as follows (see Figs. 3-20 and 3-21):

Re $\omega > 0$. Distort the contour of the first integral (3-70) to the positive imaginary axis. Distort the contour of the second integral to the negative imaginary axis together with the loops $\mathcal{L}_1$ and $\mathcal{L}_2$ lying close to the cuts given by Re $\nu_1 = 0$ and Re $\nu_2 = 0$, respectively.

Thus

$$I_1 = -\int_0^{\infty} \Lambda(i\tau)H_0^{(1)}(i\tau r)\tau\, d\tau \tag{3-71}$$

$$I_2 = -\int_0^{-\infty} \Lambda(i\tau)H_0^{(2)}(i\tau r)\tau\, d\tau + \int_{\mathcal{L}_1} \Lambda(\zeta)H_0^{(2)}(\zeta r)\zeta\, d\zeta + \int_{\mathcal{L}_2} \Lambda(\zeta)H_0^{(2)}(\zeta r)\zeta\, d\zeta \tag{3-72}$$

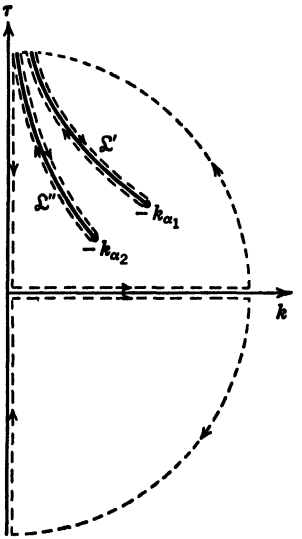


FIG. 3-21. Integration paths in the complex ζ plane for Re $\omega < 0$.

Since $H_0^{(1)}(i\tau r) = -H_0^{(2)}(-i\tau r)$ and $\Lambda(k)$ is a function in which the same value of ν is now used on the positive and negative parts of the imaginary axis, the first integrals in (3-71) and (3-72) cancel, so that

$$\varphi_1 = \int_{\mathcal{L}_1} \Lambda(\zeta)H_0^{(2)}(\zeta r)\zeta\, d\zeta + \int_{\mathcal{L}_2} \Lambda(\zeta)H_0^{(2)}(\zeta r)\zeta\, d\zeta \tag{3-73}$$

Re $\omega < 0$. For this case the branch lines $\mathcal{L}'$ and $\mathcal{L}''$ lie in the first quadrant, and by a similar procedure we find

$$\varphi_1 = \int_{\mathcal{L}'} \Lambda(\zeta)H_0^{(1)}(\zeta r)\zeta\, d\zeta + \int_{\mathcal{L}''} \Lambda(\zeta)H_0^{(1)}(\zeta r)\zeta\, d\zeta \tag{3-74}$$

CONTRIBUTION FROM $\mathcal{L}_2$. The contribution from the loop $\mathcal{L}_2$, for Re $\omega > 0$, can now be approximated. Since this contour lies close to the

cut Re $\nu_2 = 0$, we can put for points on this line

$$\nu_2 = \pm iu \qquad \zeta^2 = k_{\alpha_2}^2 - u^2 \qquad \text{and} \qquad \zeta\, d\zeta = -u\, du$$

To determine the sign of Im ν_2 on different sides of the branch line we have the assumptions discussed in Sec. 2–5. With those assumptions, a positive imaginary component corresponds to the left-hand side of the branch line and a negative imaginary component to its right side. The

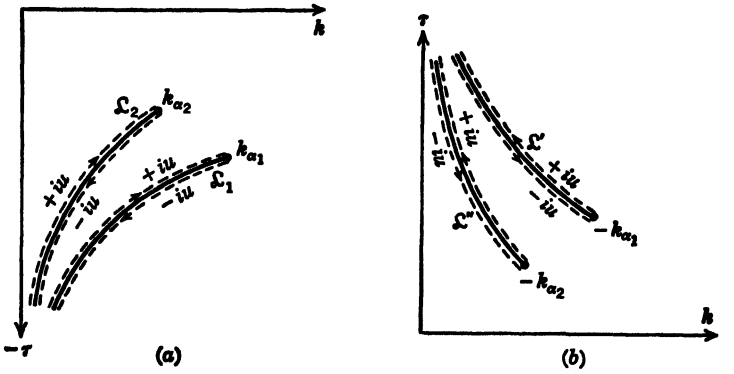


FIG 3–22. (*a*) Loops $\mathcal{L}_1$ and $\mathcal{L}_2$ for Re $\omega > 0$. (*b*) Loops $\mathcal{L}'$ and $\mathcal{L}''$ for Re $\omega < 0$.

signs are indicated in Fig. 3–22*a*. From the asymptotic form for $H_0^{(2)}$† we note that the principal contribution of the second integral (3–73) occurs for ζ close to k_{α_2}, where Im ζ has its smallest value. We can therefore consider u small and make the following approximations when ζr is large:

$$\zeta \cong k_{\alpha_2} - \frac{u^2}{2k_{\alpha_2}} \qquad \nu_1 = (\zeta^2 - k_{\alpha_1}^2)^{\frac{1}{2}} \cong \left(\frac{\omega}{\gamma} + \frac{u^2\gamma}{2\omega}\right) i$$

where $\gamma = (1/\alpha_1^2 - 1/\alpha_2^2)^{-\frac{1}{2}}$. If we use the first term of the asymptotic expansion of $H_0^{(2)}$ and keep second-order approximations in the exponential only, the second integral (3–73) takes the approximate form

$$\varphi_1^{(2)} \cong \sqrt{\frac{2}{\pi k_{\alpha_2} r}} \exp\left[-i\left(k_{\alpha_2} r - \frac{\pi}{4}\right)\right] \int_0^\infty [\Delta(iu) - \Delta(-iu)] \exp \frac{iru^2}{2k_{\alpha_2}} u\, du \tag{3–75}$$

where on $\mathcal{L}_2$ for both (3–64) and (3–65)

$$\Delta(iu) - \Delta(-iu) = \frac{2i\gamma^2 u}{\delta\omega^2} \exp\left[-i\left(\frac{\omega}{\gamma} + \frac{u^2\gamma}{2\omega}\right)(z + h)\right]$$

† $H_0^{(2)}(\zeta r) \sim \sqrt{\dfrac{2}{\pi\zeta r}} \exp\left[-i\left(\zeta r - \dfrac{\pi}{4}\right)\right]$

Using a table of definite integrals, we can evaluate (3–75) exactly, and we find, again including the factor exp ($i\omega t$),

$$\varphi_1^{(2)} = \frac{2i\gamma^2}{\delta\alpha_2\omega r^2} e^{i\omega t_*} \tag{3–76}$$

provided that $(z + h)/r$ is small, $k_{\alpha_2} r$ is large, and

$$t_* = t - \frac{r}{\alpha_2} - \frac{z + h}{\gamma}$$

For Re $\omega < 0$, the potential φ_1'' is given by the second integral of (3–74). For this case the only change is the use of the $\mathfrak{L}''$ contour and the substitution of $H_0^{(1)}$ for $H_0^{(2)}$. Following the same procedure, we may readily verify that the result is identical to (3–76).

We may now generalize these results for an initial impulse having the form of (3–68). Applying (3–69) to the source function (3–52), we find

$$\Phi_0 = \frac{1}{2\pi i R} \int_\Omega \frac{e^{i\omega t_*}}{\omega - i\sigma} d\omega = \frac{1}{R} e^{-\sigma t_*} H(t_0) \tag{3–77}$$

where $t_0 = t - R/\alpha_1$. Similarly, (3–76) leads to

$$\Phi_1^{(2)} = \frac{2i\gamma^2}{\delta\alpha_2 r^2} \cdot \frac{1}{2\pi i} \int_\Omega \frac{e^{i\omega t_*}}{\omega(\omega - i\sigma)} d\omega \tag{3–78}$$

so that the displacements $q_1^{(2)}$ and $w_1^{(2)}$ may be obtained by differentiating with respect to r and z. Neglecting the term containing r^{-3}, we get

$$q_1^{(2)} = -\frac{2\gamma^2}{\delta\alpha_2^2 r^2} \cdot \frac{1}{2\pi i} \int_\Omega \frac{e^{i\omega t_*}}{\omega - i\sigma} d\omega = -\frac{2\gamma^2}{\delta\alpha_2^2 r^2} e^{-\sigma t_*} H(t_*) \tag{3–79}$$

$$w_1^{(2)} = -\frac{2\gamma}{\delta\alpha_2 r^2} e^{-\sigma t_*} H(t_*) \tag{3–80}$$

These displacements are seen to have a definite beginning at the time

$$t = \frac{r}{\alpha_2} + \frac{z + h}{\gamma} = \frac{r}{\alpha_2} + (z + h) \frac{\cos \theta_{cr}}{\alpha_1} \tag{3–81}$$

where $\theta_{cr} = \sin^{-1} (\alpha_1/\alpha_2)$. This is precisely the time required for travel from the source to the receiver along the refraction path shown in Fig. 3–18. The path is one of least time, and the angle of incidence is the critical angle θ_{cr}. Comparing Eqs. (3–79) and (3–80) with the expression for the initial disturbance (3–77), we see that the displacements have the same time variation as the initial potential. Similarly, the particle velocities $\bar{q}_1^{(2)}$ and $\bar{w}_1^{(2)}$ obtained by differentiating (3–79) and (3–80) have the same time variation as the initial displacements if we neglect terms of higher order in r and z. The refraction arrival is seen to decrease with distance

as r^{-2}. The singularity at $t_* = 0$ for $\bar{q}_1^{(2)}$, $\bar{w}_1^{(2)}$ and at $t_0 = 0$ for $\bar{q}_0$, $\bar{w}_0$ follows from the assumption of an instantaneous rise time of $S(t)$. This behavior may be thought of as an abrupt jerk followed by an exponential recovery.

This theory accounts for the existence of a "refraction arrival" but cannot be used for any quantitative description of field data. In the great majority of actual cases the refracting medium is a layer, and the effects of its other boundary must be taken into account, at least over a large part of the frequency spectrum. An experimental study of this problem was made by Press, Oliver, and Ewing [40].

CONTRIBUTION FROM $\mathfrak{L}_1$. For Re $\omega > 0$ we use the loop shown in Fig. 3–22 with $\nu_1 = \mp iu$ on the right and left side of the cut, respectively. In the vicinity of the branch point we can write

$$\zeta = \sqrt{k_{\alpha_1}^2 - u^2} = k_{\alpha_1} - \frac{u^2}{2k_{\alpha_1}} \qquad \nu_2 = \frac{\omega}{\gamma}$$

as required by the condition Re $\nu_2 > 0$, where $\gamma = (1/\alpha_1^2 - 1/\alpha_2^2)^{-\frac{1}{2}}$. As was discussed in Sec. 2–5, the major contribution to the branch line integrals occurs in the vicinity of the branch points.

Then the first term in (3–73) yields the contribution of the contour $\mathfrak{L}_1$ to φ_1:

$$\varphi_1^{(1)} = \int_0^\infty H_0^{(2)}(\zeta r)[\Lambda(iu) - \Lambda(-iu)]u\, du \tag{3–82}$$

where, by (3–67),

$$\Lambda(iu) - \Lambda(-iu) = \frac{\cos u(h-z)}{iu} - \frac{\cos u(h+z)}{iu}$$

$$- \frac{2i\delta\gamma}{\omega} \sin u(h+z) - \frac{\delta^2\gamma^2}{\omega^2} iu[\cos u(h+z) + \cos u(h-z)]$$

Using the asymptotic expansion of $H_0^{(2)}(\zeta r)$, we may write (3–82) as

$$\varphi_1^{(1)} = \sqrt{\frac{2}{\pi k_{\alpha_1} r}} \exp i\left(\omega t - k_{\alpha_1} r + \frac{\pi}{4}\right)$$

$$\cdot \int_0^\infty \exp \frac{iru^2}{2k_{\alpha_1}} [\Lambda(iu) - \Lambda(-iu)]u\, du$$

$$= \varphi_1^{(\mathrm{I})} + \varphi_1^{(\mathrm{II})} + \varphi_1^{(\mathrm{III})} + \varphi_1^{(\mathrm{IV})} \tag{3–83}$$

where, upon integration,

$$\varphi_1^{(\mathrm{I})} = \frac{1}{r} e^{i\omega t_{*0}} \qquad t_{*0} = t - \frac{r}{\alpha_1} - \frac{(h-z)^2}{2r\alpha_1} \tag{3–84}$$

$$\varphi_1^{(\mathrm{II})} = -\frac{1}{r} e^{i\omega t_{*1}} \qquad t_{*1} = t - \frac{r}{\alpha_1} - \frac{(h+z)^2}{2r\alpha_1} \tag{3–85}$$

$$\varphi_1^{(\mathrm{III})} = \frac{2\delta i\gamma}{r^2\alpha_1}(h+z)e^{i\omega t_{*1}} \tag{3-86}$$

$$\varphi_1^{(\mathrm{IV})} = \frac{\delta^2 i\gamma^2}{r^2\alpha_1}\left\{\left[\frac{1}{\omega} - \frac{i(h+z)^2}{\alpha_1 r}\right]e^{i\omega t_{*1}} + \left[\frac{1}{\omega} - \frac{i(h-z)^2}{\alpha_1 r}\right]e^{i\omega t_{*0}}\right\} \tag{3-87}$$

For Re $\omega < 0$ we follow the contour $\mathfrak{L}'$ as depicted in Fig. 3-22. Im ν_1 is positive to the left and negative to the right of the cuts. To determine the contribution of the first term in (3-74) we follow a procedure similar to that used for the contour $\mathfrak{L}_1$. In this case we use the asymptotic expansion for $H_0^{(1)}(\zeta r)$† and substitute for the vicinity of the branch point $-k_{\alpha_1}$ in the first quadrant

$$\zeta = -k_{\alpha_1} + \frac{u^2}{2k_{\alpha_1}} \qquad \nu_2 = -\frac{\omega}{\gamma}$$

We obtain (3-83) except for a change in sign of terms with γ. Solutions (3-84), (3-85), and (3-87) are unaffected but (3-86) becomes

$$\varphi_1^{(\mathrm{III})} = \pm\frac{2\delta i\gamma}{r^2\alpha_1}(h+z)e^{i\omega t_{*1}} \qquad \text{for Re}\,\omega \gtrless 0 \tag{3-88}$$

Again generalizing these results for an initial time variation (3-86) we find, by applying (3-69) to (3-84), (3-85), (3-88), and (3-87),

$$\Phi_1^{(\mathrm{I})} = \frac{1}{r}e^{-\sigma t_{*0}}H(t_{*0}) \tag{3-89}$$

$$\Phi_1^{(\mathrm{II})} = -\frac{1}{r}e^{-\sigma t_{*1}}H(t_{*1}) \tag{3-90}$$

$$\Phi_1^{(\mathrm{III})} = \frac{2\delta i\gamma}{r^2\alpha_1}(h+z)M \tag{3-91}$$

where

$$M = \frac{1}{2\pi i}\left[\int_{-\infty}^{0}\frac{e^{i\omega t_{*1}}}{\omega - i\sigma}\,d\omega - \int_0^{\infty}\frac{e^{i\omega t_{*1}}}{\omega - i\sigma}\,d\omega\right] = e^{-\sigma t_{*1}}\overline{Ei}(\sigma t_{*1})$$

In deriving (3-91) the Ω contour has been distorted to the real axis and the integral evaluated by the method used in Sec. 3-2. Finally, we have

$$\Phi_1^{(\mathrm{IV})} = -\frac{\delta^2\gamma^2}{r^2\alpha_1}\left[\int_0^{t_{*1}} e^{-\sigma t_{*1}}H(t_{*1})\,dt_{*1} - \frac{(h+z)^2}{\alpha_1 r}e^{-\sigma t_{*1}}H(t_{*1}) + \int_0^{t_{*0}} e^{-\sigma t_{*0}}H(t_{*0})\,dt_{*0} - \frac{(h-z)^2}{\alpha_1 r}e^{-\sigma t_{*0}}H(t_{*0})\right] \tag{3-92}$$

†$H_0^{(1)}(\zeta r) \sim \sqrt{\frac{2}{\pi\zeta r}}\exp i\left(\zeta r - \frac{\pi}{4}\right) \qquad -\pi < \arg \zeta r < 2\pi$

It is seen that $\Phi_1^{(I)}$ is a disturbance having the same shape as the initial pulse and beginning abruptly at

$$t = \frac{r}{\alpha_1} + \frac{(h-z)^2}{2r\alpha_1} \tag{3-93}$$

It decreases with distance as r^{-1} and depends only on the properties of the first medium. Since (3-93) represents the first two terms of the expansion of R/α_1 for large r, $\Phi_1^{(I)}$ may be interpreted as the direct spherical pulse from source to receiver. The expression $\Phi_1^{(II)}$ represents a similar pulse beginning abruptly at

$$t = \frac{r}{\alpha_1} + \frac{(h+z)^2}{2r\alpha_1} \tag{3-94}$$

equal in magnitude but 180° out of phase with $\Phi_1^{(I)}$. This corresponds to the grazing reflection from the interface. From a plot of $e^{-\sigma t_{*1}}\overline{Ei}(\sigma t_{*1})$ in Fig. 3-17*d* it may be seen that $\Phi_1^{(III)}$ does not have a definite beginning but reaches a maximum value at the time given in Eq. (3-94) for a reflection from the interface. The amplitude of this wave depends on the properties of both media and decreases as r^{-2}. The expression for $\Phi_1^{(IV)}$ contains four terms, the first two beginning at the time for a reflection from the interface, the next two beginning at a time corresponding to direct travel from source to receiver. All terms in $\Phi_1^{(IV)}$ have amplitudes dependent on the properties of both media. The first and third terms correspond to a pulse decreasing as r^{-2} and having a shape given by the integral of the initial pulse. The second and fourth terms decrease as r^{-3} and have the same shape as the initial pulse. Thus $\Phi_1^{(III)}$ and $\Phi_1^{(IV)}$ include effects arising from the curvature of the incident wave front, the former term showing the "reflection tail" which is characteristic of three-dimensional wave propagation. If we omit $\exp i\omega t$, the superposition of $\varphi_1^{(I)}$ and $\varphi_1^{(II)}$ leads to the Lloyd mirror effect at large distances:

$$\frac{1}{r}e^{i\omega r/\alpha_1}[e^{-i\omega(z-h)^2/2r\alpha_1} - e^{-i\omega(z+h)^2/2r\alpha_1}]$$

$$\cong \frac{2i}{r}\sin\left(\frac{\omega zh}{\alpha_1 r}\right)e^{-i\omega r/\alpha_1} \cong \frac{2i\omega zh}{r^2\alpha_1}e^{-i\omega r/\alpha_1} \tag{3-95}$$

Alternative discussions of the reflection of spherical waves have been given by Sommerfeld [55], Weyl [Chap. 1, Ref. 64], Niessen [33], van der Pol [58], Norton [34], Rudnick [42], Pekeris [39], and Brekhovskikh [4]. Brekhovskikh discussed the problem of reflection of spherical waves, electromagnetic as well as acoustic, from a plane boundary separating two media. He derived expressions for potentials for different values of the refraction index n at various distances from the source. To find

these potentials, Brekhovskikh made use of Weber's functions and of a series the terms of which are inversely proportional to the powers of the product of the distance and frequency. He gave his results in a form suitable when the two media differ only slightly. Brekhovskikh's formulas should have wider application than the solutions of Ott [36] and Krüger [24] which hold only for angles not too near the angle of total reflection.

Fluid and Solid Half Spaces. In the second problem concerning two semi-infinite media we assume that the first medium ($z > 0$) displays the properties of a fluid and the second ($z < 0$) those of an elastic solid. If we assume axial symmetry, the displacements in the fluid are represented by

$$q_1 = \frac{\partial \varphi_1}{\partial r} \qquad w_1 = \frac{\partial \varphi_1}{\partial z} \tag{3-96}$$

and in the solid body by Eqs. (1–26) and (1–28),

$$q_2 = \frac{\partial \varphi_2}{\partial r} + \frac{\partial^2 \psi_2}{\partial r\, \partial z} \qquad w_2 = \frac{\partial \varphi_2}{\partial z} + \frac{\partial}{r\, \partial r}\left(-r \frac{\partial \psi_2}{\partial r}\right) \tag{3-97}$$

The functions φ_1, φ_2, and ψ_2 must be solutions of the wave equations

$$\nabla^2 \varphi_i = \frac{1}{\alpha_i^2} \frac{\partial^2 \varphi_i}{\partial t^2} \qquad i = 1, 2 \qquad \nabla^2 \psi_2 = \frac{1}{\beta_2^2} \frac{\partial^2 \psi_2}{\partial t^2} \tag{3-98}$$

where α_1 and α_2 = velocities of compressional waves
β_2 = velocity of distortional waves in second medium

As to the source, there are more special cases than in the preceding paragraph. A source in a fluid half space can produce only compressional waves. A source located in the elastic solid medium can emit both compressional or distortional waves. For the case of a point source in the fluid half space we can make use of formulas in the preceding problems (3–53) and (3–54), which represent compressional waves propagating in both media, and add a solution of the third equation in (3–98) to represent distortional waves. We can put

$$\psi_2 = \int_0^\infty S_2(k) J_0(kr) e^{\nu_2'(z-h)} dk \tag{3-99}$$

where $S_2(k)$ is a function to be determined from the boundary conditions and the coefficient ν_2' must be chosen as usual to satisfy the wave equation. Like (3–55), we obtain

$$\nu_2' = \pm\sqrt{k^2 - \frac{\omega^2}{\beta_2^2}} = \pm\sqrt{k^2 - k_{\beta_2}^2} \tag{3-100}$$

where the real part of ν_2' must again be positive. The function ψ_2 represents distortional waves below the interface ($z \leq 0$). The formal solutions for

a source of simple harmonic waves are

$$\varphi_1 = \int_0^\infty \frac{k}{\nu_1} J_0(kr)e^{-\nu_1|z-h|}\,dk + \int_0^\infty Q_1(k)J_0(kr)e^{-\nu_1(z-h)}\,dk \qquad (3\text{–}101)\dagger$$

$$\varphi_2 = \int_0^\infty Q_2(k)J_0(kr)e^{\nu_2(z-h)}\,dk \qquad (3\text{–}102)$$

$$\psi_2 = \int_0^\infty S_2(k)J_0(kr)e^{\nu_2'(z-h)}\,dk \qquad (3\text{–}103)$$

The three unknown functions Q_1, Q_2, S_2 will now be determined by three boundary conditions. These conditions correspond to the continuity of displacements in the z direction and of stresses at the interface ($z = 0$):

$$w_1 = w_2 \qquad (p_{zz})_1 = (p_{zz})_2 \qquad (p_{zr})_1 = (p_{zr})_2 \qquad (3\text{–}104)$$

The first equation of (3–104) takes the form

$$\frac{\partial\varphi_2}{\partial z} + \frac{\partial^2\psi_2}{\partial z^2} + k_{\beta_2}^2\psi_2 = \frac{\partial\varphi_1}{\partial z} \qquad \text{at } z = 0 \qquad (3\text{–}105)$$

by (1–26), (1–27), (1–28), and (1–29) when the time factor in (3–101) to (3–103) is taken as exp ($i\omega t$). The second equation of (3–104) leads, by (2–57), to

$$\lambda_2\nabla^2\varphi_2 + 2\mu_2\frac{\partial w_2}{\partial z} = \lambda_1\nabla^2\varphi_1 \qquad \text{at } z = 0 \qquad (3\text{–}106)$$

Since no tangential stresses act in a perfect fluid, the third equation of (3–104) gives

$$0 = \mu\left(\frac{\partial q_2}{\partial z} + \frac{\partial w_2}{\partial r}\right) = (p_{zr})_2 \qquad \text{at } z = 0 \qquad (3\text{–}107)$$

On inserting (3–101) to (3–103) in (3–105) to (3–107) we obtain linear equations with respect to Q_1 exp ($\nu_1 h$), Q_2 exp ($-\nu_2 h$), S_2 exp ($-\nu_2' h$):

$$\nu_1 Q_1 e^{\nu_1 h} + \nu_2 Q_2 e^{-\nu_2 h} + k^2 S_2 e^{-\nu_2' h} = ke^{-\nu_1 h} \qquad (3\text{–}108)$$

$$\rho_1\omega^2 Q_1 e^{\nu_1 h} + (2\mu_2 k^2 - \rho_2\omega^2)Q_2 e^{-\nu_2 h} + 2\mu_2 k^2\nu_2' S_2 e^{-\nu_2' h} = -\rho_1\omega^2\frac{k}{\nu_1}e^{-\nu_1 h} \qquad (3\text{–}109)$$

$$2\nu_2\mu_2 Q_2 e^{-\nu_2 h} + (2\mu_2 k^2 - \rho_2\omega^2)S_2 e^{-\nu_2' h} = 0 \qquad (3\text{–}110)$$

Thus if we put

$$\Delta = \begin{vmatrix} \nu_1 & \nu_2 & k^2 \\ \rho_1\omega^2 & 2\mu_2 k^2 - \rho_2\omega^2 & 2\mu_2 k^2\nu_2' \\ 0 & 2\nu_2\mu_2 & 2\mu_2 k^2 - \rho_2\omega^2 \end{vmatrix}$$

$$= \nu_1[(2\mu_2 k^2 - \rho_2\omega^2)^2 - 4\mu_2^2 k^2\nu_2\nu_2'] + \rho_1\rho_2\omega^4\nu_2 \qquad (3\text{–}111)$$

†See remark following Eq. (3–58).

and if Δ_1, Δ_2, Δ_2' are the other determinants for solving Eqs. (3–108) to (3–110), we obtain

$$Q_1 = \frac{\Delta_1}{\Delta} e^{-2\nu_1 h} \qquad Q_2 = \frac{\Delta_2}{\Delta} e^{-(\nu_1-\nu_2)h} \qquad S_2 = \frac{\Delta_2'}{\Delta} e^{-(\nu_1-\nu_2')h} \tag{3–112}$$

where

$$\frac{\Delta_1}{\Delta} = \frac{k}{\nu_1} \frac{\nu_1[(2\mu_2 k^2 - \rho_2\omega^2)^2 - 4\mu_2^2 k^4 \nu_2 \nu_2'] - \rho_1\rho_2\omega^4\nu_2}{\Delta} \tag{3–113}$$

$$\frac{\Delta_2}{\Delta} = \frac{k}{\nu_1} \frac{-2\nu_1\rho_1\omega^2(2\mu_2 k^2 - \rho_2\omega^2)}{\Delta} \tag{3–114}$$

$$\frac{\Delta_2'}{\Delta} = \frac{k}{\nu_1} \frac{4\nu_1\nu_2\rho_1\omega^2\mu_2}{\Delta} \tag{3–115}$$

By simple transfor iations it may be seen again that (3–113), (3–114), and (3–115) are expressions for reflection and transmission coefficients for plane waves. Equations (3–101) to (3–103) with (3–112) to (3–115) represent the formal steady-state solutions. These integrals may be evaluated by the approximate methods of the preceding sections with several modifications. In addition to the branch points $k_{\alpha 1}$ and $k_{\alpha 2}$, we must consider the branch points $k_{\beta 2} = \pm\omega/\beta_2$ and the possible poles at $\Delta = 0$. The corresponding terms will represent, in addition to those discussed in the preceding section, a transmitted shear wave and waves tied to the interface. Later in this section it will be shown that $\Delta = 0$ corresponds to waves propagating with a velocity less than that of compressional or shear waves in either medium and called Stoneley waves. Biot [3] has called attention to the possible importance of Stoneley waves at a liquid-solid interface in connection with transmission of elastic waves through oceans.

Two Solids. We now discuss the problem of propagation of a disturbance from a compressional-wave source located in a solid half space which is in contact with a second solid half space at $z = 0$.

With the use of definitions similar to those made in the preceding problem, the displacements are represented by the equations

$$q_i = \frac{\partial\varphi_i}{\partial r} + \frac{\partial^2\psi_i}{\partial r\,\partial z} \qquad w_i = \frac{\partial\varphi_i}{\partial z} + \frac{\partial^2\psi_i}{\partial z^2} + k_{\beta i}^2\psi_i \qquad i = 1, 2 \tag{3–116}$$

where the potentials φ_i and ψ_i are solutions of the wave equations

$$\nabla^2\varphi_i = \frac{1}{\alpha_i^2}\frac{\partial^2\varphi_i}{\partial t^2} \qquad \nabla^2\psi_i = \frac{1}{\beta_i^2}\frac{\partial^2\psi_i}{\partial t^2} \qquad i = 1, 2 \tag{3–117}$$

If we put

$$\nu_1' = \sqrt{k^2 - k_{\beta_1}^2} \tag{3–118}$$

and make use of (3–100) and (3–55), the following expressions may be taken as solutions of (3–117), the time factor being omitted:

$$\varphi_1 = \int_0^\infty \frac{k}{\nu_1} J_0(kr) e^{-\nu_1 |z-h|}\, dk + \int_0^\infty Q_1(k) J_0(kr) e^{-\nu_1 (z-h)}\, dk \qquad \text{at } z > 0 \tag{3–119}$$

$$\psi_1 = \int_0^\infty S_1(k) J_0(kr) e^{-\nu_1' (z-h)}\, dk \tag{3–120}$$

$$\varphi_2 = \int_0^\infty Q_2(k) J_0(kr) e^{\nu_2 (z-h)}\, dk \qquad \text{at } z < 0 \tag{3–121}$$

$$\psi_2 = \int_0^\infty S_2(k) J_0(kr) e^{\nu_2' (z-h)}\, dk \tag{3–122}$$

Now the four coefficients Q_i, S_i can be chosen to satisfy the boundary conditions. In those problems which we shall discuss, a "welded contact" is usually assumed, and, therefore, we have four conditions which hold at $z = 0$:

$$q_1 = q_2 \qquad w_1 = w_2 \tag{3–123}$$

$$(p_{zz})_1 = (p_{zz})_2 \qquad (p_{zr})_1 = (p_{zr})_2 \tag{3–124}$$

These equations express the continuity of displacements and stresses at the surface of contact ($z = 0$) of the two solid media and have the form

$$\frac{\partial \varphi_1}{\partial r} + \frac{\partial^2 \psi_1}{\partial z\, \partial r} = \frac{\partial \varphi_2}{\partial r} + \frac{\partial^2 \psi_2}{\partial z\, \partial r} \tag{3–125}$$

$$\frac{\partial \varphi_1}{\partial z} + \frac{\partial^2 \psi_1}{\partial z^2} + k_{\beta_1}^2 \psi_1 = \frac{\partial \varphi_2}{\partial z} + \frac{\partial^2 \psi_2}{\partial z^2} + k_{\beta_2}^2 \psi_2 \tag{3–126}$$

$$\lambda_1 \nabla^2 \varphi_1 + 2\mu_1 \frac{\partial w_1}{\partial z} = \lambda_2 \nabla^2 \varphi_2 + 2\mu_2 \frac{\partial w_2}{\partial z} \tag{3–127}$$

$$\mu_1 \left(\frac{\partial q_1}{\partial z} + \frac{\partial w_1}{\partial r} \right) = \mu_2 \left(\frac{\partial q_2}{\partial z} + \frac{\partial w_2}{\partial r} \right) \tag{3–128}$$

On inserting (3–119) to (3–122) we obtain a system of four linear equations for the functions

$$\hat{Q}_1 = Q_1 e^{\nu_1 h} \qquad \hat{Q}_2 = Q_2 e^{-\nu_2 h} \qquad \hat{S}_1 = S_1 e^{\nu_1' h} \qquad \hat{S}_2 = S_2 e^{-\nu_2' h}$$

This system is

$$\hat{Q}_1 - \hat{Q}_2 - \nu_1'\hat{S}_1 - \nu_2'\hat{S}_2 = -\frac{k}{\nu_1}e^{-\nu_1 h}$$

$$\nu_1\hat{Q}_1 + \nu_2\hat{Q}_2 - k^2\hat{S}_1 + k^2\hat{S}_2 = ke^{-\nu_1 h}$$

$$-(2\mu_1 k^2 - \rho_1\omega^2)\hat{Q}_1 + (2\mu_2 k^2 - \rho_2\omega^2)\hat{Q}_2 + 2\mu_1 k^2\nu_1'\hat{S}_1 \tag{3-129}$$

$$+ 2\mu_2 k^2\nu_2'\hat{S}_2 = \frac{k}{\nu_1}(2\mu_1 k^2 - \rho_1\omega^2)e^{-\nu_1 h}$$

$$2\mu_1\nu_1\hat{Q}_1 + 2\mu_2\nu_2\hat{Q}_2 - (2\mu_1 k^2 - \rho_1\omega^2)\hat{S}_1 + (2\mu_2 k^2 - \rho_2\omega^2)\hat{S}_2 = 2k\mu_1 e^{-\nu_1 h}$$

Thus, on putting

$$2\mu_1 k^2 - \rho_1\omega^2 = a_1 \qquad 2\mu_2 k^2 - \rho_2\omega^2 = a_2 \tag{3-130}$$

the determinant of (3-129) is

$$\Delta(k) = \begin{vmatrix} 1 & -1 & -\nu_1' & -\nu_2' \\ \nu_1 & \nu_2 & -k^2 & k^2 \\ -a_1 & a_2 & 2\mu_1 k^2\nu_1' & 2\mu_2 k^2\nu_2' \\ 2\mu_1\nu_1 & 2\mu_2\nu_2 & -a_1 & a_2 \end{vmatrix} \tag{3-131}$$

Now the coefficients Q_s and S_s can be written in the form

$$Q_1 = \frac{\Delta_1}{\Delta}e^{-2\nu_1 h} \qquad S_1 = \frac{\Delta_1'}{\Delta}e^{-(\nu_1+\nu_1')h}$$

$$Q_2 = \frac{\Delta_2}{\Delta}e^{-(\nu_1-\nu_2)h} \qquad S_2 = \frac{\Delta_2'}{\Delta}e^{-(\nu_1-\nu_2')h} \tag{3-132}$$

where Δ_1, Δ_1', Δ_2, and Δ_2' are the determinants formed according to the well-known rule, the factor $\exp(-\nu_1 h)$ being separated from them. The factors $\Delta_1/\Delta, \cdots, \Delta_2'/\Delta$ have the form of reflection and transmission coefficients for plane simple harmonic waves. By (3-119) to (3-122) and (3-132) the solution of the problem of wave propagation in two solids is represented by the functions

$$\varphi_1 = \int_0^\infty \frac{k}{\nu_1}J_0(kr)e^{-\nu_1|z-h|}\,dk + \int_0^\infty \frac{\Delta_1}{\Delta}J_0(kr)e^{-\nu_1(z+h)}\,dk \tag{3-133}$$

$$\psi_1 = \int_0^\infty \frac{\Delta_1'}{\Delta}J_0(kr)e^{-\nu_1' z-\nu_1 h}\,dk \tag{3-134}$$

$$\varphi_2 = \int_0^\infty \frac{\Delta_2}{\Delta}J_0(kr)e^{\nu_2 z-\nu_1 h}\,dk \tag{3-135}$$

$$\psi_2 = \int_0^\infty \frac{\Delta_2'}{\Delta}J_0(kr)e^{\nu_2' z-\nu_1 h}\,dk \tag{3-136}$$

The first term in (3–133) represents the direct compressional wave. All other terms in (3–133) to (3–136) represent waves generated in both media by it. To investigate these waves one has to insert the time factor again and evaluate the integrals in (3–133) to (3–136) by the methods used earlier in this section or by some other method. Different types of waves are determined by a set of branch line integrals corresponding to $k = k_{\alpha 1}, k_{\beta 1}, k_{\alpha 2}, k_{\beta 2}$ and residues corresponding to roots κ of the equation

$$\Delta(k) = 0 \tag{3–137}$$

where $\Delta(k)$ is given by (3–131), or

$$\begin{aligned}\Delta(k) = 4(\mu_2 - \mu_1)^2\Bigg[& k^2\left(k^2 - \frac{\omega^2(\rho_2 - \rho_1)}{2(\mu_2 - \mu_1)}\right)^2 \\ & - \nu_1\nu_1'\left(k^2 - \frac{\rho_2\omega^2}{2(\mu_2 - \mu_1)}\right)^2 - \nu_2\nu_2'\left(k^2 + \frac{\rho_1\omega^2}{2(\mu_2 - \mu_1)}\right)^2 \\ & - (\nu_1\nu_2' + \nu_2\nu_1')\frac{\omega^4\rho_1\rho_2}{4(\mu_2 - \mu_1)^2} + \nu_1\nu_2\nu_1'\nu_2'k^2\Bigg]\end{aligned} \tag{3–138}$$

Without carrying out the analysis, we may surmise from what has preceded that each of the wave types associated with the branch points of Eqs. (3–133) to (3–136) may be considered to travel along a path composed of three parts: (1) source to interface, (2) along the interface, and (3) interface to receiver. The coefficient of h in the exponential indicates whether the first part of the path is traversed by compressional or shear waves, the coefficient of z gives the same information about the third part, while the value of k at the branch point indicates the mode of travel along the interface. In all cases when the exponents are imaginary, the propagation paths are minimum-time paths which can be represented by rays. Some of these coefficients may assume real values at certain branch points, corresponding to a wave for which energy is propagated parallel to the interface although the ray cannot be drawn. An apparent exception occurs when two or three consecutive parts of a path are traversed with the same velocity, in which case the wave is a reflected type confined to one medium. These results are summarized in Fig. 3–23 for a compressional source when $\alpha_2 > \beta_2 > \alpha_1 > \beta_1$. The heavy lines represent rays, and the velocity along each segment is indicated. Angles of incidence, reflection, and refraction are governed by Snell's law. The wavy lines indicate waves for which the paths cannot be drawn.

The characteristic equation (3–138) in the form given later in this section was investigated for the first time by Stoneley [56]. Under certain conditions a real root of (3–138) exists corresponding to a velocity of propagation less than that of body waves in either medium. All the ν factors are real, and the resultant wave cannot be represented by rays.

Its amplitude decreases exponentially with distance from the interface and can be shown to fall off as $r^{-\frac{1}{2}}$ with distance, as would be expected for an interface wave.

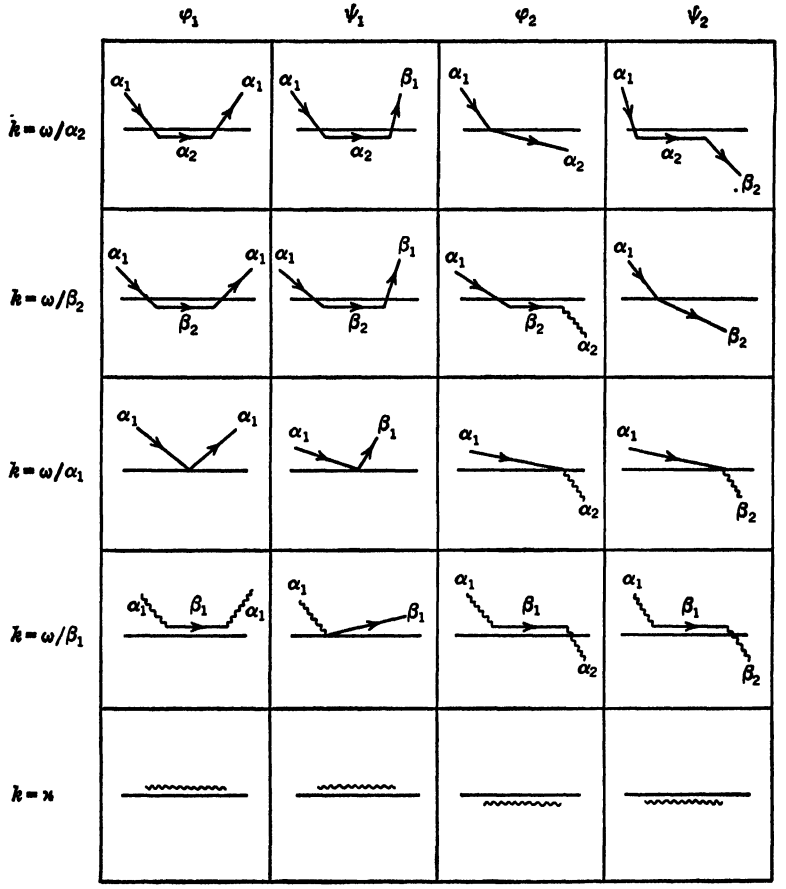


FIG. 3–23. Waves from a compressional source near interface between two solid half spaces, $\alpha_2 > \beta_2 > \alpha_1 > \beta_1$. Source located in lower-velocity medium.

Stoneley Waves. The existence of surface waves in an elastic half space was shown by Rayleigh in a derivation using plane waves, and Lamb extended these results to cylindrical and spherical waves (see Chap. 2). Love [26, pp. 165–177] investigated the effect of a surface layer on the propagation of Rayleigh waves and discovered another wave of

the same type. For wave lengths short compared with the layer thickness Love found, in addition to the ordinary waves with velocity determined only by the properties of the surface layer, that a modified Rayleigh wave with velocity depending on the properties of both media could exist under the stringent condition that shear-wave velocities in the two media were nearly equal. Stoneley later thoroughly investigated the propagation of this generalized Rayleigh wave (now commonly known as the Stoneley wave). Assuming a solution in the form U_1, V_1, $W_1 \exp ik(ct - x)$, where U_1, V_1, and W_1 are functions of z approaching zero at infinite distance from $z = 0$, Stoneley obtained the frequency equation in the form

$$\begin{aligned} c^4\{(\rho_1 - \rho_2)^2 - (\rho_1 A_2 + \rho_2 A_1)(\rho_1 B_2 + \rho_2 B_1)\} \\ + 2Kc^2\{\rho A_2 B_2 - \rho_2 A_1 B_1 - \rho_1 + \rho_2\} \\ + K^2(A_1 B_1 - 1)(A_2 B_2 - 1) = 0 \end{aligned} \tag{3-139}$$

which is equivalent to the characteristic equation (3–138). This is an equation for the phase velocity c, and the following transformation of variables must be taken into account:

$$\begin{aligned} A_1 &= \left(1 - \frac{c^2}{\alpha_1^2}\right)^{\frac{1}{2}} \qquad A_2 = \left(1 - \frac{c^2}{\alpha_2^2}\right)^{\frac{1}{2}} \\ B_1 &= \left(1 - \frac{c^2}{\beta_1^2}\right)^{\frac{1}{2}} \qquad B_2 = \left(1 - \frac{c^2}{\beta_2^2}\right)^{\frac{1}{2}} \\ K &= 2(\rho_1 \beta_1^2 - \rho_2 \beta_2^2) = 2(\mu_1 - \mu_2) \end{aligned} \tag{3-140}$$

with

$$k = \frac{\omega}{c} \tag{3-141}$$

and

$$\nu_1 = \sqrt{k^2 - k_{\alpha_1}^2} = kA_1 \qquad \nu_1' = kB_1 \qquad \nu_2 = kA_2 \qquad \nu_2' = kB_2 \tag{3-142}$$

If $\mu_1 = 0$, the left-hand side of Eq. (3–139) is replaced by Eq. (3–111). If $\rho_2 = 0$, Eq. (3–139) is the ordinary equation for Rayleigh waves. Equation (3–139) was numerically solved by Koppe [23] who concluded that c^2/β_1^2 cannot have smaller values than the root of the Rayleigh equation, i.e., the velocity of surface waves at the interface of two solid media falls between the velocity of Rayleigh waves and that of transverse waves in the medium of greater acoustic density. There is a region in the plane of variables μ and $\beta = \beta_1/\beta_2$ where surface waves are impossible. On the other hand, generalized Rayleigh waves are always possible at the interface of a solid and a fluid medium. Their velocity is smaller than that of regular Rayleigh waves. Other analyses of Eq. (3–139) were given by Sezawa

and Kanai [49], Cagniard [6], and Scholte [47]. Figures 3–24 and 3–25 represent Scholte's principal results for the conditions under which Stoneley waves may exist.

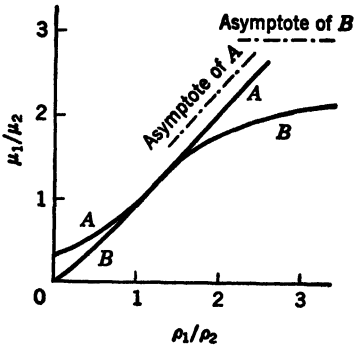


FIG. 3–24. Range of existence of Stoneley waves for $\lambda_1/\mu_1 = \lambda_2/\mu_2 = 1$. Stoneley waves can exist for every value of μ_1/μ_2 and ρ_1/ρ_2 which lies between the curves A and B. (*After Scholte.*)

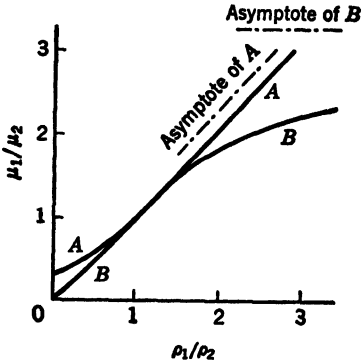


FIG. 3–25. Range of existence of Stoneley waves for $\lambda_1 = \lambda_2 = \infty$. (*After Scholte.*)

3–4. Further Remarks on Waves Generated at an Interface. In the preceding paragraphs our study of the propagation of a disturbance in two semi-infinite media gave terms which represent "refraction arrivals" in addition to the direct, reflected, and refracted waves. "Refraction arrivals" are readily observed in the field in seismic exploration. They have also been observed in laboratory experiments of Schmidt [45], where a spark served as a point source of compressional waves in an acoustic medium consisting of xylol (sound velocity 1,175 m/sec) on NaCl solution (sound velocity 1,600 m/sec). Schlieren photographs revealed all the wave types. Refraction arrivals observed by Press, Oliver, and Ewing [40] in a

laboratory model consisting of a thin plate of Plexiglas (longitudinal velocity 7,350 ft/sec) cemented to a thin plate of aluminum (longitudinal velocity 17,750 ft/sec) are illustrated in Fig. 3–26. Although longitudinal or plate waves (Sec. 6–1) are used here, the results are completely analogous

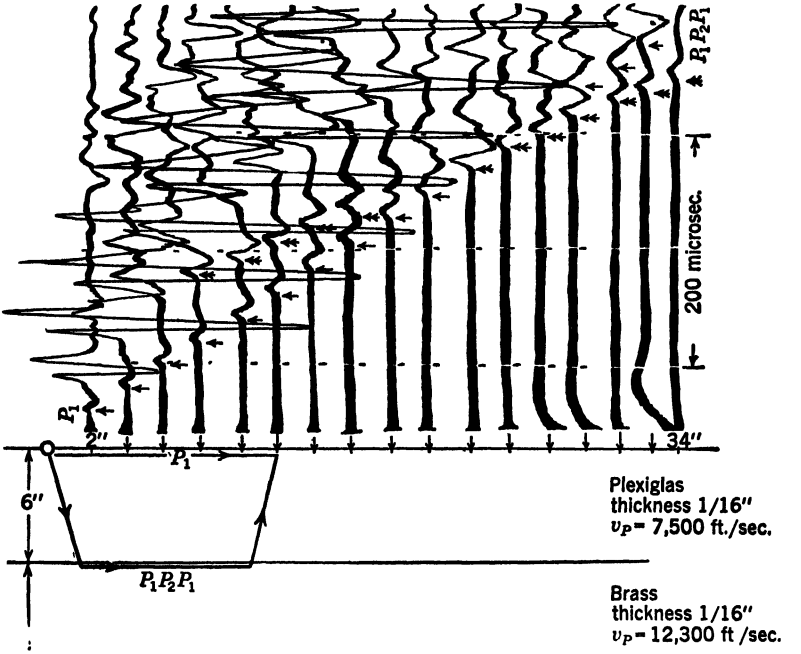


FIG. 3–26. Direct waves P_1 and refracted waves $P_1P_2P_1$ for a two-dimensional model of two layers.

to three-dimensional propagation. Experimentally and theoretically studied "refraction arrivals" have been variously referred to as "refraction waves," "waves corresponding to the fourth ray," "traveling reflections," "head waves" (not the *Kopfwelle* generated by a projectile), "*Flankenwelle*," etc.

A simple picture of these waves may be given by the use of Huygens' principle which was first applied to these problems by Merten in 1927, according to Thornburgh [57]. (See also Dix [9] and Ansel [1].) Jardetzky (see Chap. 1, Ref. 23) elaborated on this approach to show the physical conditions for the generation of different types of waves in a layered medium.

A theory including terms representing waves generated at an interface was given by Ott [36] for electromagnetic as well as acoustical waves.

These terms were obtained by the method of steepest descent applied to the neighborhood of one of the existing saddle points. In the case of electromagnetic waves, Ott also found surface waves in the ground when an antenna emits a disturbance in the air, as was asserted by Sommerfeld (see Chap. 1, Ref. 55). After Weyl (Chap. 1, Ref. 64) could not confirm Sommerfeld's result, the question of the existence of this wave became the object of numerous papers. According to Ott's conclusion, such a wave is similar to Schmidt's head wave. On making use of a complex refraction index, Krüger [24] could derive the existence of the wave in question by Weyl's method.

3–5. Other Investigations. As mentioned above, other methods of solving problems of wave propagation in elastic media have been applied. Cagniard [6] worked with Carson's integral equations [7] and the Laplace transformation. He used the functions φ_i of (3–119) and (3–121) but, instead of the functions ψ_i, he took $U_i = -\partial\psi_i/\partial r$, according to Eq. (1–28). Considering axial symmetry, put

$$\hat{\varphi}_i = e^{pt} X_{pi}(r, z) \qquad \hat{U}_i = e^{pt} Y_{pi}(r, z) \tag{3–143}$$

where, according to Cagniard, p is real and positive. This form of the time factor is a limiting case for the complex exponent $\omega = s - ic$ used in Sec. 3–3, where the real part $s = 0$, $c \equiv p$, and $\exp(i\omega t) = \exp(pt)$. The conditions for cuts in the complex ζ plane derived in Sec. 3–3 show that the corresponding cuts now have to be taken on the imaginary axis.

Since the factor p replaces $i\omega$ used in the case of simple harmonic motion, the expressions (3–55), (3–100), and (3–118) take the form

$$\begin{aligned} \nu_1 &= \sqrt{k^2 + \frac{p^2}{\alpha_1^2}} & \nu_2 &= \sqrt{k^2 + \frac{p^2}{\alpha_2^2}} \\ \nu_1' &= \sqrt{k^2 + \frac{p^2}{\beta_1^2}} & \nu_2' &= \sqrt{k^2 + \frac{p^2}{\beta_2^2}} \end{aligned} \tag{3–144}$$

Again, omit the time factor and write the solutions of the second factors in (3–143) in a form similar to that used in Sec. 3–3 [Eqs. (3–119) to (3–122)]

$$X_{p1} = \int_0^\infty e^{-\nu_1|z-h|} J_0(kr) \frac{k\,dk}{\nu_1} + \int_0^\infty Q_1(k) J_0(kr) e^{-\nu_1 z}\,dk \tag{3–145}$$

for $z > 0$

$$Y_{p1} = \int_0^\infty S_1(k) \frac{d}{dr}[J_0(kr)] e^{-\nu_1' z}\,dk \tag{3–146}$$

$$X_{p2} = \int_0^\infty Q_2(k) J_0(kr) e^{\nu_2 z}\,dk \tag{3–147}$$

for $z < 0$

$$Y_{p2} = \int_0^\infty S_2(k) \frac{d}{dr}[J_0(kr)] e^{\nu_2' z}\,dk \tag{3–148}$$

The functions Q_1, Q_2, S_1, and S_2 can be determined from the boundary conditions in the same way as was done previously. For further transformation we will now choose new variables u and v. The variable u having dimensions of reciprocal velocity is defined and introduced into the expressions (3–144) as follows:

$$k = pu \qquad \nu_1 = p\sqrt{u^2 + \frac{1}{\alpha_1^2}} \qquad \cdots \tag{3–149}$$

We also put

$$\begin{aligned} a_1 &= \sqrt{u^2 + \frac{1}{\alpha_1^2}} \qquad b_1 = \sqrt{u^2 + \frac{1}{\beta_1^2}} \\ a_2 &= \sqrt{u^2 + \frac{1}{\alpha_2^2}} \qquad b_2 = \sqrt{u^2 + \frac{1}{\beta_2^2}} \end{aligned} \tag{3–150}$$

The second variable v will be determined by the conditions that the expressions (3–145) to (3–148) can be written in the form

$$X_{p,}(p, r, z) = p \int_0^\infty e^{-pv} A_{,}(v, r, z)\, dv \tag{3–151}$$

$$Y_{p,}(p, r, z) = p \int_0^\infty e^{-pv} B_{,}(v, r, z)\, dv \tag{3–152}$$

where $A_{,}$, $B_{,}$ are Laplace transforms of $X_{p,}/p$ and $Y_{p,}/p$ and Eqs. (3–151) and (3–152) are the Carson integral equations for $A_{,}$ and $B_{,}$. The fundamental statement in the Cagniard theory is that the functions φ and U giving the displacements in both media can be represented in terms of

$$\phi_1 = \int_0^t F'(t - v)A_{,}(v)\, dv \qquad \hat{U}_{,} = \int_0^t F'(t - v)B_{,}(v)\, dv \tag{3–153}$$

where $F(t)$ represents the action of the source. For a given problem, the procedure is to determine $X_{p,}$ and $Y_{p,}$ by methods similar to those used in Sec. 3–3, to obtain $A_{,}$ and $B_{,}$ from (3–151) and (3–152), and to introduce the initial time variation by (3–153).

In order to see how the variable v can be introduced, let us write the expression for $Y_{p,}$, making use of Eqs. (3–134), (3–146), (3–149), (3–150), and (3–143) and the definition $U_{,} = -\partial\psi_{,}/\partial r$ by (1–28). Then

$$Y_{p1}(p, r, z) = -\int_0^\infty \frac{\Delta_1'}{\Delta}\left[\frac{d}{dr} J_0(pur)\right] e^{-p(a_1 h + b_1 z)} p\, du \tag{3–154}$$

The Bessel function $J_0(pur)$ is the only factor depending on r. Making use of definition (1–69) of the even function $J_0(kr)$, we obtain

$$J_0(kr) = \frac{1}{\pi}\int_0^\pi e^{-ikr\cos\sigma}\, d\sigma \tag{3–155}$$

and, therefore,

$$\frac{d}{dr} J_0(kr) = -\frac{ik}{\pi} \int_0^{\pi} e^{-ikr \cos \sigma} \cos \sigma \, d\sigma$$

$$= \frac{2k}{\pi} \operatorname{Im} \int_0^{\pi/2} e^{-ikr \cos \sigma} \cos \sigma \, d\sigma \qquad (3\text{–}156)$$

Thus (3–154) takes the form

$$\frac{Y_{p1}(p, r, z)}{p} = -\frac{2}{\pi} \operatorname{Im} \int_0^{\pi/2} \cos \sigma \, d\sigma \int_0^{\infty} \frac{p\Delta_1'}{\Delta} u e^{-pv} \, du \qquad (3\text{–}157)$$

where, according to (3–150),

$$v = iur \cos \sigma + h\sqrt{u^2 + \frac{1}{\alpha_1^2}} + z\sqrt{u^2 + \frac{1}{\beta_1^2}} \qquad (3\text{–}158)$$

The variable σ is real but u and v are considered as complex variables. To make the radicals in (3–158) uniform functions, Cagniard [6, p. 55] takes a cut along the imaginary axis in the complex plane of u between the points $-i/\beta_1$ and i/β_1 (when $\beta_1 < \alpha_1$ and $\beta_1 < \beta_2$). We take a_1 and b_1 in Eqs. (3–150) to be given by their arithmetic values when u is real and positive. They are uniform functions in the first and fourth quadrant of the u plane. Since v is determined by (3–158), it is also a complex variable. For $u = 0$, we have $v_0 = h/\alpha_1 + z/\beta_1$, and in general the real values

$$v \geq \frac{h}{\alpha_1} + \frac{z}{\beta_1} = v_0 \qquad (3\text{–}159)$$

play an important part. It may be proved that, in a certain region of the complex variable v, Eq. (3–158) has a unique root u and that this root is a uniform and holomorphic function of v.

If we compare (3–157) and (1–66), we may easily see that Cagniard's expression (3–157) may be readily derived from that form of solution of a wave equation.

Now, if u is expressed in terms of v and σ,

$$u = u(v, \sigma) \qquad (3\text{–}160)$$

[u depends also on other variables in (3–158)], the last integral in (3–157) can be transformed as follows:

$$I = \int_0^{\infty} u \frac{p\Delta_1'}{\Delta} e^{-p[ir u \cos \sigma + a_1 h + b_1 z]} \, du$$

$$= \int_{C_\sigma} u(v, \sigma) K(v, \sigma) \frac{\partial}{\partial v} [u(v, \sigma)] e^{-pv} \, dv \qquad (3\text{–}161)$$

The new integration path C_σ, starting at v_0 (for $u = 0$), depends on the value of $\cos \sigma$. It may be easily seen that the variable v has dimensions of time and that the value $v = v_0$ represents the minimum time required by a disturbance in the form of a compressional wave to reach the interface and, after that moment, in the form of a reflected distortional wave to reach the height z. At $u = \infty$, $v = \infty$ in the first quadrant, since for u real the real part of v satisfies the condition (3–159). Since there are no singularities between C_σ and the real axis, the integration path C_σ may be replaced by the part of the real axis (v_0, ∞). Cagniard proved that the function in (3–161) is an integrable one. Then, we have

$$I = \int_{v_0}^{\infty} Ku \frac{\partial u}{\partial v} e^{-pv} \, dv \tag{3–162}$$

and, therefore, substituting in (3–157), we obtain

$$\begin{aligned} \frac{Y_{p1}}{p} &= -\frac{2}{\pi} \int_0^{\pi/2} \cos \sigma \, d\sigma \int_{v_0}^{\infty} \operatorname{Im} \left[Ku \frac{\partial u}{\partial v} \right] e^{-pv} \, dv \\ &= -\frac{2}{\pi} \int_{v_0}^{\infty} e^{-pv} \, dv \int_0^{\pi/2} \operatorname{Im} \left[Ku \frac{\partial u}{\partial v} \right] \cos \sigma \, d\sigma \end{aligned} \tag{3–163}$$

Now from (3–163) and (3–152) the direct conclusion can be drawn that the solution of Carson's equation for $i = 1$ is given by the expression

$$B_1(v) = \begin{cases} 0 & \text{for } v < v_0 \\ -\dfrac{2}{\pi} \displaystyle\int_0^{\pi/2} \operatorname{Im} \left[Ku \frac{\partial u}{\partial v} \right] \cos \sigma \, d\sigma & \text{for } v \geq v_0 \end{cases} \tag{3–164}$$

Cagniard changed the variable of integration from σ to u, using (3–158). A very long transformation and evaluation of this so-called transmission coefficient $B_1(v, r, z)$ then yielded a set of wave fronts. This set was completed by a similar interpretation of the other three functions B_2, A_1, A_2.

Different distributions of wave fronts are determined, depending on the relative values of α_1 and β_1. For the case, for example,

$$\beta_1 < \alpha_1 < \beta_2 < \alpha_2 \tag{3–165}$$

the wave fronts are represented in Fig. 3–27. This and similar figures were computed by Cagniard [6, p. 122] for certain particular values of parameters.

In this figure W_1 is the direct compressional wave emitted by a source S at a distance h from the interface. W_1' and $\overline{W}_1'$ are reflected compressional and distortional refracted waves. W_2 and $\overline{W}_2$ are compressional and distortional refracted waves. The waves denoted by $''$ are the conical waves, i.e., the waves generated at an interface.

In order to see the way in which these wave fronts are determined from

the expressions just used, we shall give a short discussion of one example. If the integrals (3–145) to (3–148) are evaluated by using contour integration in the complex plane, the major contribution to those integrals is determined by singularities of the integrands, for example, at their

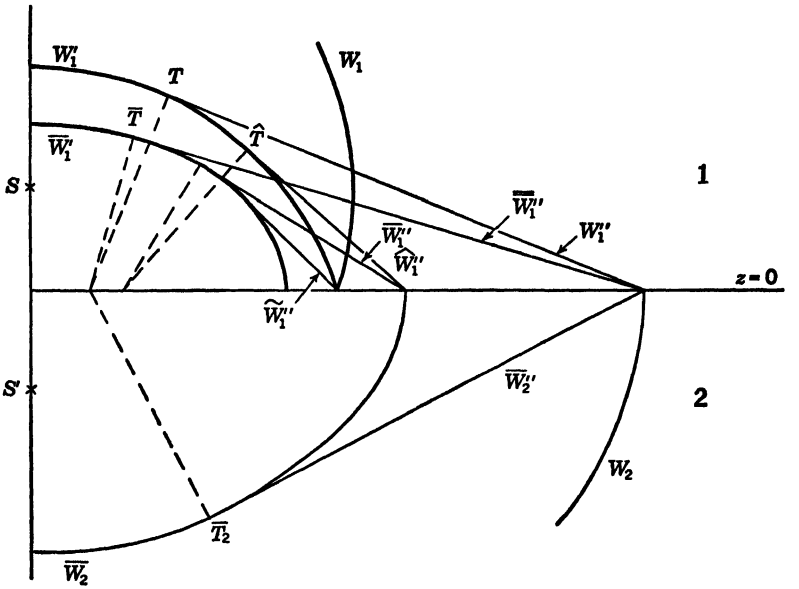


FIG. 3–27. A complete set of waves in two solid media produced by a point source of compressional waves when $\alpha_2 > \beta_2 > \alpha_1 > \beta_1$. (*After Cagniard.*)

poles. In (3–164) the root $u = u_0$ of the derivative $\partial v/\partial u$ is such a pole. If we first put $u = \imath l$ in (3–158), where the real parameter l varies from $-\infty$ to ∞, the root l_0 satisfies the condition

$$-r \cos \sigma - \frac{hl_0}{\sqrt{1/\alpha_1^2 - l_0^2}} - \frac{zl_0}{\sqrt{1/\beta_1^2 - l_0^2}} = 0 \qquad (3\text{–}166)$$

According to Cagniard, Eq. (3–166) can be given the following interpretation: Putting

$$-l_0 = \frac{\sin i_1}{\alpha_1} = \frac{\sin i_2}{\beta_1} \qquad (3\text{–}167)$$

and

$$h \tan i_1 = r_1 \cos \sigma \qquad z \tan i_2 = r_2 \cos \sigma \qquad (3\text{–}168)$$

we obtain $r = r_1 + r_2$ and the relationships represented in Fig. 3–28.

It is easy to see now that for $\sigma = 0$ the condition (3–166) corresponds to the wave front of a disturbance traveling first as a compressional wave from the source S in the direction SP and as a reflected shear wave in

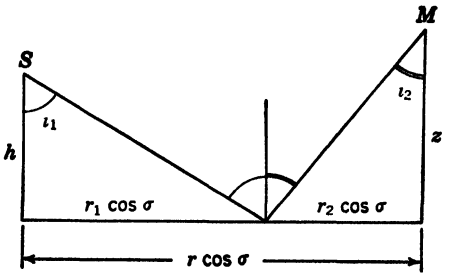


FIG 3–28 Interpretation of Eq. (3–166)

the direction PM. The time when this disturbance reaches an observer at M is found as follows: If $\sigma = 0$, Eq. (3–167) takes the form

$$-l_{00} = \frac{\sin i_{10}}{\alpha_1} = \frac{\sin i_{20}}{\beta_1} \tag{3–169}$$

Substituting $u = il_{00}$ and $r = r_1 + r_2$ in Eq. (3–158), we obtain from Fig. 3–29

$$v = \frac{D_1}{\alpha_1} + \frac{D_2}{\beta_1} \tag{3–170}$$

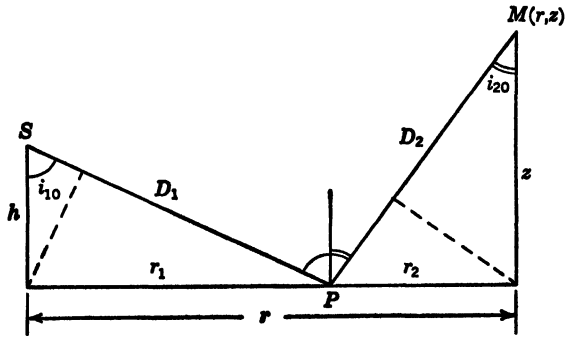


FIG. 3–29. Reduction to the reflected wave PS when $\sigma = 0$.

which is obviously the time of travel for the wave front just mentioned.

In a series of investigations the method of Sobolev [54], mentioned in Sec. 2–7, was applied to problems discussed in this chapter. Kupradze and Sobolev [25] had shown earlier that when an elastic-solid medium is

overlain by a liquid half space the system of oscillations as given by Lamb in his problem is essentially the same but is supplemented by the existence of new compressional waves in the liquid medium. They also studied the influence of the ocean on seismic waves. A similar case of the propagation of a disturbance emitted by a source in the solid or liquid was discussed by Naryškina [31, 32], and the displacements were determined to the first approximation. Smirnov and Sobolev [52] gave a new method of solution of two- and three-dimensional problems. As mentioned earlier, Muskat [28], using the methods of Jeffreys and Sommerfeld, discussed the problem in connection with refraction shooting. Recently Zaicev and Zvolinskii [60, 61] have investigated the problem of a wave generated at the interface of two elastic liquids by Sobolev's method.

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CHAPTER 4

A LAYERED HALF SPACE

It is only the exceptional problem of wave propagation in which a homogeneous layer exists whose thickness is so great, compared with the wavelength, that the theories for half space considered in Chaps. 2 and 3 are applicable. In the present chapter we shall consider cases in which one or more layers are superposed on the half space, obtaining results of wider applicability to practical problems.

Love (Chap. 2, Ref. 26) gave the first comprehensive treatment of the case of an elastic-solid half space covered by a single solid layer. He calculated the dispersion of Rayleigh waves and showed that a new surface wave having particle motion parallel to the surface and perpendicular to the direction of propagation could exist under suitable conditions. Stoneley [193] investigated the effect of the ocean on the transmission of Rayleigh waves, treating the bottom as a solid half space. Problems of this kind dealing with two- and three-layered media will be discussed in the following pages. A general discussion for a multilayered half space will also be given.

4–1. General Equations for an n-layered Elastic Half Space. Many problems of interest in geophysics and acoustics involve propagation of elastic disturbances in a layered half space. In the most usual problem the half space is divided into homogeneous and isotropic layers by the planes $z_j = \sum_{i=1}^{j} H_i$, where $z = 0$ is the free surface, H_j is the thickness of the jth layer, and $H_n = \infty$. By ρ_j, λ_j, and μ_j we denote the densities and elastic constants of the media forming the layers ($j = 1, 2, \cdots, n$).

An important case is again that of a point source, and, because of the axial symmetry in the distribution of all quantities involved, we have to consider only two components of displacement, q_j and w_j. The latter is taken parallel to the z axis; the former is perpendicular to it. The differential equations of motion for the jth layer are

$$\begin{aligned}
&(\lambda_j + 2\mu_j)\left[\frac{\partial^2 q_j}{\partial r^2} + \frac{1}{r}\frac{\partial q_j}{\partial r} - \frac{q_j}{r^2} + \frac{\partial^2 w_j}{\partial z\,\partial r}\right] + \mu_j \frac{\partial^2 q_j}{\partial z^2} - \mu_j \frac{\partial^2 w_j}{\partial z\,\partial r} = \rho_j \frac{\partial^2 q_j}{\partial t^2} \\
&(\lambda_j + 2\mu_j)\left[\frac{\partial^2 q_j}{\partial r\,\partial z} + \frac{1}{r}\frac{\partial q_j}{\partial z} + \frac{\partial^2 w_j}{\partial z^2}\right] - \frac{\mu_j}{r}\left[\frac{\partial q_j}{\partial z} - \frac{\partial w_j}{\partial r}\right] \\
&\qquad - \mu_j\left[\frac{\partial^2 q_j}{\partial z\,\partial r} - \frac{\partial^2 w_j}{\partial r^2}\right] = \rho_j \frac{\partial^2 w_j}{\partial t^2}
\end{aligned} \tag{4–1}$$

If, as before, we assume that the layers are in "welded contact" at each interface, the conditions

$$q_j = q_{j+1} \qquad w_j = w_{j+1} \qquad \text{at } z = z_j \tag{4-2}$$

will hold. The continuity of the normal and tangential stresses

$$(p_{zz})_j = \lambda_j \nabla^2 \varphi_j + 2\mu_j \frac{\partial w_j}{\partial z} \qquad (p_{zr})_j = \mu_j \left(\frac{\partial q_j}{\partial z} + \frac{\partial w_j}{\partial r} \right) \tag{4-3}$$

at the interfaces is also assumed, and we obtain the other set of boundary conditions

$$\begin{aligned} p_{zz} &= 0 & p_{zr} &= 0 & &\text{at } z = 0 \\ (p_{zz})_j &= (p_{zz})_{j+1} & (p_{zr})_j &= (p_{zr})_{j+1} & &\text{at } z = z_j \end{aligned} \tag{4-4}$$

If any layer is a perfect fluid, the tangential stresses at its boundaries disappear, and the equations concerning the tangential displacements at its boundaries are eliminated. As to the conditions at infinity, the velocity or displacement must vanish for all values of time. Thus

$$\left.\begin{matrix} q_j \to 0 \\ w_j \to 0 \end{matrix}\right\} \quad \text{as } r \to \infty \quad j = 1, 2, \cdots, n \quad \left.\begin{matrix} q_n \to 0 \\ w_n \to 0 \end{matrix}\right\} \quad \text{as } z \to \infty \tag{4-5}$$

These conditions are usually replaced by the following:

$$\left.\begin{matrix} \varphi_j \to 0 \\ \psi_j \to 0 \end{matrix}\right\} \quad \text{as } r \to \infty \qquad \left.\begin{matrix} \varphi_n \to 0 \\ \psi_n \to 0 \end{matrix}\right\} \quad \text{as } z \to \infty \tag{4-6}$$

where φ_j and ψ_j are displacement potentials defined below.

In the preceding chapters we have seen that the equations of motion (4–1) are satisfied if we put

$$q_j = \frac{\partial \varphi_j}{\partial r} + \frac{\partial^2 \psi_j}{\partial r \, \partial z} \qquad w_j = \frac{\partial \varphi_j}{\partial z} + \frac{\partial}{r \partial r} \left(-r \frac{\partial \psi_j}{\partial r} \right) \tag{4-7}$$

and φ_j and ψ_j are solutions of the wave equations

$$\nabla^2 \varphi_j = \frac{1}{\alpha_j^2} \frac{\partial^2 \varphi_j}{\partial t^2} \qquad \nabla^2 \psi_j = \frac{1}{\beta_j^2} \frac{\partial^2 \psi_j}{\partial t^2} \tag{4-8}$$

where

$$\alpha_j = \sqrt{\frac{\lambda_j + 2\mu_j}{\rho_j}} \qquad \beta_j = \sqrt{\frac{\mu_j}{\rho_j}} \tag{4-9}$$

The results for two semi-infinite media indicate that for layered media we can use solutions of the type (3–119) to (3–122). In order to adjust the general solution (1–71) of the wave equation to a case involving some

layers of finite thickness, we retain both the positive and the negative values of the coefficient of z in exponent for each layer whose thickness is finite. It will be seen that these terms correspond to upward and downward traveling waves in the layer. For the nth layer of the present problem we retain only factors of the form $\exp(-\nu z)$, $\nu > 0$, as required by Eqs. (4–6). Omitting the time factor and choosing positive real parts for the coefficients

$$\nu_j = \sqrt{k^2 - k_{\alpha_j}^2} \qquad \nu_j' = \sqrt{k^2 - k_{\beta_j}^2} \tag{4–10}$$

where

$$k_{\alpha_j} = \frac{\omega}{\alpha_j} \quad \text{and} \quad k_{\beta_j} = \frac{\omega}{\beta_j}$$

we can write, for $j = 1, 2, \cdots, n-1$,

$$\varphi_j = \int_0^\infty Q_j J_0(kr) e^{-\nu_j(z-h)}\, dk + \int_0^\infty Q_j^* J_0(kr) e^{\nu_j(z-h)}\, dk \tag{4–11}$$

$$\psi_j = \int_0^\infty S_j J_0(kr) e^{-\nu_j'(z-h)}\, dk + \int_0^\infty S_j^* J_0(kr) e^{\nu_j'(z-h)}\, dk \tag{4–12}$$

and

$$\varphi_n = \int_0^\infty Q_n J_0(kr) e^{-\nu_n(z-h)}\, dk \tag{4–13}$$

$$\psi_n = \int_0^\infty S_n J_0(kr) e^{-\nu_n'(z-h)}\, dk \tag{4–14}$$

Written in the form (4–11) to (4–14) the solutions do not yet include the contribution of a point source located in some layer. We will assume now that such a source is at a distance $z = h$ from the free surface. Since this source $S(0, 0, h)$ can be located in any one of these layers, we will introduce the spherical wave emitted by the source by adding its potential to the corresponding φ_j or ψ_j. If, for example, a point source of compressional waves is in the first layer, we put

$$\hat{\varphi}_1 = \varphi_0 + \varphi_1 \tag{4–15}$$

where

$$\varphi_0 = \frac{e^{-ik_{\alpha_1}R}}{R} = \int_0^\infty J_0(kr) e^{-\nu_1|z-h|} \frac{k\, dk}{\nu_1} \tag{4–16}$$

represents the spherical wave propagating from the source with the velocity α_1.

For a shear-wave source, α_1 has to be replaced by β_1 and ν_1 by ν_1' in Eq. (4–16).

4–2. Two-layered Liquid Half Space. The dispersive waves observed by Ewing [35] and Worzel and Ewing [211] in experiments on explosion sounds in shallow water (10 to 20 fathoms) were first interpreted by Pekeris [116]. He considered a problem of propagation of a disturbance in

a two-layered liquid half space. Let the first liquid layer be water (ocean) and the second the "liquid bottom" extending from $z = H$ to $z = \infty$. A point source S is placed at the depth h in the first liquid layer. Let the density and the velocity of sound propagation in each medium be ρ_j

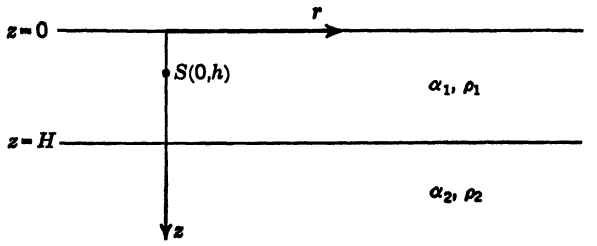


FIG. 4-1. Point source in liquid layer over liquid half space.

and α_j, respectively (Fig. 4-1). The displacements q_j and w_j are expressed in terms of the potentials φ_j:

$$q_j = \frac{\partial \varphi_j}{\partial r} \qquad w_j = \frac{\partial \varphi_j}{\partial z} \qquad j = 1, 2 \tag{4-17}$$†

These potentials are solutions of the wave equations

$$\nabla^2 \varphi_j = \frac{1}{\alpha_j^2} \frac{\partial^2 \varphi_j}{\partial t^2} \qquad j = 1, 2 \tag{4-18}$$

The boundary conditions are as in (3-56) and (3-57):

$$\frac{\partial \varphi_1}{\partial z} = \frac{\partial \varphi_2}{\partial z} \qquad \rho_1 \varphi_1 = \rho_2 \varphi_2 \qquad \text{at } z = H \tag{4-19}$$

and

$$\varphi_1 = 0 \qquad \text{at } z = 0 \tag{4-20}$$

In boundary conditions involving pressure we have replaced the pressure $\rho_j(\partial^2 \varphi_j / \partial t^2)$ by $\rho_j \varphi_j$. In the usual application to simple harmonic waves this involves only the omission of a factor ω^2, which would cancel.

We make use of expressions for potentials obtained earlier. Omitting the time factor, we write

$$\varphi_1 = \int_0^\infty \frac{k}{\nu_1} J_0(kr) e^{-\nu_1 |z-h|}\, dk + \int_0^\infty Q_1(k) J_0(kr) e^{-\nu_1 (z-h)}\, dk + \int_0^\infty Q_1^*(k) J_0(kr) e^{\nu_1 (z-h)}\, dk \tag{4-21}$$

$$\varphi_2 = \int_0^\infty Q_2(k) J_0(kr) e^{-\nu_2 (z-h)}\, dk \tag{4-22}$$

†Since Pekeris used velocity potentials φ_j our results will differ from his in dimensions.

The first term in (4–21) represents the direct wave emitted by the source S. The other two terms correspond to summations of upward and downward traveling waves which have been reflected one or more times from the boundaries, as in the general solution (4–11). In (4–22) the term with the positive sign in the exponent has been omitted because of the condition $\varphi_2 \to 0$ as $z \to \infty$. The potentials φ_1 and φ_2 will satisfy the boundary conditions (4–19) and (4–20) under the following conditions:

$$-ke^{-\nu_1(H-h)} - \nu_1 Q_1 e^{-\nu_1(H-h)} + \nu_1 Q_1^* e^{\nu_1(H-h)} = -\nu_2 Q_2 e^{-\nu_2(H-h)} \tag{4–23}$$

$$\delta_1\left[\frac{k}{\nu_1} e^{-\nu_1(H-h)} + Q_1 e^{-\nu_1(H-h)} + Q_1^* e^{\nu_1(H-h)}\right] = Q_2 e^{-\nu_2(H-h)} \tag{4–24}$$

$$\frac{k}{\nu_1} e^{-\nu_1 h} + Q_1 e^{\nu_1 h} + Q_1^* e^{-\nu_1 h} = 0 \tag{4–25}$$

where

$$\delta_1 = \frac{\rho_1}{\rho_2} \tag{4–26}$$

Thus we have a system of three linear equations (4–23) to (4–25) from which the coefficients Q_1, Q_1^*, Q_2 may be determined. Then Eqs. (4–21) and (4–22) represent a solution of the problem.

The determinant of Eqs. (4–23) to (4–25) is

$$\Delta = e^{\nu_2 h}\begin{vmatrix} 1 & 1 & 0 \\ -\nu_1 e^{-\nu_1 H} & \nu_1 e^{\nu_1 H} & \nu_2 e^{-\nu_2 H} \\ \delta_1 e^{-\nu_1 H} & \delta_1 e^{\nu_1 H} & -e^{-\nu_2 H} \end{vmatrix}$$

$$= -2e^{-\nu_2(H-h)}(\nu_1 \cosh \nu_1 H + \delta_1 \nu_2 \sinh \nu_1 H) \tag{4–27}$$

and we obtain

$$Q_1 = \frac{\Delta_1}{\Delta} = -\frac{k}{\nu_1} e^{-\nu_1 h} \frac{\nu_1 \cosh \nu_1(H-h) + \delta_1 \nu_2 \sinh \nu_1(H-h)}{\nu_1 \cosh \nu_1 H + \delta_1 \nu_2 \sinh \nu_1 H} \tag{4–28}$$

$$Q_1^* = -\frac{k}{\nu_1} \sinh \nu_1 h \, e^{-\nu_1(H-h)} \frac{\delta_1 \nu_2 - \nu_1}{\nu_1 \cosh \nu_1 H + \delta_1 \nu_2 \sinh \nu_1 H} \tag{4–29}$$

$$Q_2 = 2\delta_1 k e^{\nu_2(H-h)} \frac{\sinh \nu_1 h}{\nu_1 \cosh \nu_1 H + \delta_1 \nu_2 \sinh \nu_1 H} \tag{4–30}$$

As a result of the term $|z-h|$ in Eq. (4–21), different expressions for the function φ_1 occur for $z > h$ or $z < h$. For $0 \leq z < h$ all three terms in (4–21) may be written in a single integral with the integrand

$$\left[\frac{k}{\nu_1} e^{\nu_1(z-h)} + Q_1 e^{-\nu_1(z-h)} + Q_1^* e^{\nu_1(z-h)}\right] J_0(kr) \tag{4–31}$$

Because of (4–25), this expression takes the form

$$-2Q_1 e^{\nu_1 h} \sinh \nu_1 z \, J_0(kr) \tag{4–31′}$$

Thus, by (4–21), (4–31′), and (4–28), we obtain for $0 \leq z \leq h$

$$\varphi_1' = 2\int_0^\infty kJ_0(kr)\frac{\sinh \nu_1 z}{\nu_1}\,\frac{\nu_1 \cosh \nu_1(H-h) + \delta_1\nu_2 \sinh \nu_1(H-h)}{\nu_1 \cosh \nu_1 H + \delta_1\nu_2 \sinh \nu_1 H}\,dk \tag{4–32}$$

For $h \leq z < H$ the factor depending on z in the integrand of (4–21) takes the form

$$\frac{k}{\nu_1}e^{-\nu_1(z-h)} + Q_1 e^{-\nu_1(z-h)} + Q_1^* e^{\nu_1(z-h)} \tag{4–33}$$

and the potential φ_1'' for $h \leq z < H$ can be written as

$$\varphi_1'' = 2\int_0^\infty kJ_0(kr)\frac{\sinh \nu_1 h}{\nu_1}\,\frac{\nu_1 \cosh \nu_1(z-H) - \delta_1\nu_2 \sinh \nu_1(z-H)}{\nu_1 \cosh \nu_1 H + \delta_1\nu_2 \sinh \nu_1 H}\,dk \tag{4–34}$$

Inserting (4–30) in (4–22), we obtain the potential φ_2 for $H < z < \infty$ in the form

$$\varphi_2 = 2\delta_1\int_0^\infty kJ_0(kr)\frac{\sinh \nu_1 h}{\nu_1 \cosh \nu_1 H + \delta_1\nu_2 \sinh \nu_1 H}e^{-\nu_2(z-H)}\,dk \tag{4–35}$$

Equations (4–32), (4–34), and (4–35) represent the solution of the problem. We shall see now that these equations are identical with the solution found by Pekeris [116] using another method. To obtain a formal solution it is not necessary to start with the expression (4–21) in which a point source is represented by the first term. Since the exponential factor depending on the variable z can be expressed in terms of trigonometric or hyperbolic functions, we can consider solutions of the wave equation of the form

$$\varphi = \int_0^\infty F(k)J_0(kr)\sin \nu z\,dk \qquad \text{or} \qquad \int_0^\infty F(k)J_0(kr)\cos \nu z\,dk$$

We follow Pekeris in dividing the first layer into two parts by the plane $z = h$. Then the potential φ_1 is represented by two different expressions:

$$\varphi_1' = \int_0^\infty A(k)J_0(kr)\sin \nu_1 z\,dk \qquad 0 \leq z \leq h$$

$$\varphi_1'' = \int_0^\infty B(k)J_0(kr)\sin \nu_1 z\,dk \tag{4–36}$$

$$+ \int_0^\infty C(k)J_0(kr)\cos \nu_1 z\,dk \qquad h \leq z \leq H$$

and

$$\varphi_2 = \int_0^\infty D(k)J_0(kr)e^{-\nu_2 z}\,dk \qquad H \leq z \tag{4–37}$$

where

$$\nu_1 = -i\nu_1 \qquad \nu_2 = -i\nu_2 \tag{4–38}$$

By this choice of φ_1' the boundary condition (4–20) is satisfied. There are two new conditions, namely,

$$\varphi_1' = \varphi_1'' \qquad \text{at } z = h \tag{4–39}$$

and a condition expressing the discontinuity of the vertical component of velocity at the point source. This component is everywhere continuous in the plane $z = h$ except at the point $S(0, 0, h)$. The fact that at this point the liquid above and below moves in opposite directions can be expressed by the equation

$$\frac{\partial \varphi_1'}{\partial z} - \frac{\partial \varphi_1''}{\partial z} = 2 \int_0^\infty J_0(kr)k \, dk \tag{4–40}\dagger$$

The function represented by

$$\frac{1}{2\pi} \int_0^\infty J_0(kr)k \, dk \tag{4–41}$$

vanishes everywhere except at $r = 0$, where it becomes infinite in such a manner that its integral over the plane $z = h$ is unity. This may be seen from the Fourier-Bessel integral

$$f(r) = \int_0^\infty J_0(kr)k \, dk \int_0^\infty f(\lambda) J_0(k\lambda)\lambda \, d\lambda \tag{4–42}$$

Noting that $J_0(k\lambda) \to 1$ as $\lambda \to 0$, we choose $f(\lambda)$ to vanish for all but infinitesimal values of λ in such a manner that the integral over the plane $z = h$ is unity, or $\int f(\lambda) 2\pi\lambda \, d\lambda = 1$. Then $f(r)$ is given by (4–41).

Inserting (4–36) and (4–37) in (4–19), (4–39), and (4–40), we can solve these equations for the four unknown functions $A(k)$, $B(k)$, $C(k)$, $D(k)$:

$$\begin{aligned} A &= \frac{2k}{\nu_1} \frac{\nu_1 \cos \nu_1(H - h) + i\delta_1\nu_2 \sin \nu_1(H - h)}{\nu_1 \cos \nu_1 H + i\delta_1\nu_2 \sin \nu_1 H} \\ B &= \frac{2k \sin \nu_1 h}{\nu_1} \frac{\nu_1 \sin \nu_1 H - i\delta_1\nu_2 \cos \nu_1 H}{\nu_1 \cos \nu_1 H + i\delta_1\nu_2 \sin \nu_1 H} \\ C &= \frac{2k \sin \nu_1 h}{\nu_1} \qquad D = \frac{2\delta_1 k \sin \nu_1 h}{\nu_1 \cos \nu_1 H + i\delta_1\nu_2 \sin \nu_1 H} e^{i\nu_2 H} \end{aligned} \tag{4–43}$$

Thus Eqs. (4–36) take the form, equivalent to (4–32) and (4–34),

$$\varphi_1' = 2 \int_0^\infty J_0(kr) \frac{k \sin \nu_1 z}{\nu_1} \frac{\nu_1 \cos \nu_1(H - h) + i\delta_1\nu_2 \sin \nu_1(H - h)}{\nu_1 \cos \nu_1 H + i\delta_1\nu_2 \sin \nu_1 H} \, dk \tag{4–44}$$

$$\varphi_1'' = 2 \int_0^\infty J_0(kr) \frac{k \sin \nu_1 h}{\nu_1} \frac{\nu_1 \cos \nu_1(H - z) + i\delta_1\nu_2 \sin \nu_1(H - z)}{\nu_1 \cos \nu_1 H + i\delta_1\nu_2 \sin \nu_1 H} \, dk \tag{4–45}$$

†The factor 2 is used in (4–40) instead of −2 in order to conform with the sign of the first integral in (4–21)

Now Eq. (4–37) becomes equivalent to (4–35), or

$$\varphi_2 = 2\delta_1 \int_0^\infty J_0(kr) k e^{-i\bar{\nu}_2(z-H)} \frac{\sin \bar{\nu}_1 h}{\bar{\nu}_1 \cos \bar{\nu}_1 H + i\delta_1 \bar{\nu}_2 \sin \bar{\nu}_1 H} dk \qquad (4\text{–}46)$$

All these integrals reduce to a form representing a spherical wave emitted by the source and another one reflected at the free surface, if we assume that $\delta_1 \to 1$ and $\bar{\nu}_2 \to \bar{\nu}_1$. For these conditions, expressions (4–45) and (4–46) become

$$\varphi_1'' = \int_0^\infty J_0(kr) \frac{k}{i\bar{\nu}_1} e^{-i\bar{\nu}_1(z-h)} dk - \int_0^\infty J_0(kr) \frac{k}{i\bar{\nu}_1} e^{-i\bar{\nu}_1(z+h)} dk = \varphi_2 \qquad (4\text{–}47)$$

Equations (4–44) to (4–46) can be transformed in different ways, and two of the transformations will be discussed here. One corresponds to a representation of wave propagation by rays and the other by normal modes.

The expression (4–45) for φ_1'' can be written in another form. Using K, the reflection coefficient (3–17) for plane waves,

$$K = \frac{\bar{\nu}_1 - \delta_1 \bar{\nu}_2}{\bar{\nu}_1 + \delta_1 \bar{\nu}_2} \qquad (4\text{–}48)$$

and taking exponential functions instead of the trigonometric functions, we can expand the last ratio in Eq. (4–45) in a series as follows:

$$\frac{\bar{\nu}_1 \cos \bar{\nu}_1(H-z) + i\delta_1 \bar{\nu}_2 \sin \bar{\nu}_1(H-z)}{\bar{\nu}_1 \cos \bar{\nu}_1 H + i\delta_1 \bar{\nu}_2 \sin \bar{\nu}_1 H} = e^{-i\bar{\nu}_1 z} \frac{1 + Ke^{-i2\bar{\nu}_1(H-z)}}{1 + Ke^{-i2\bar{\nu}_1 H}}$$

$$= e^{-i\bar{\nu}_1 z}[1 + Ke^{-i2\bar{\nu}_1(H-z)}][1 - Ke^{-i2\bar{\nu}_1 H} + K^2 e^{-i4\bar{\nu}_1 H} + \cdots] \qquad (4\text{–}49)$$

Therefore

$$\varphi_1'' = \int_0^\infty J_0(kr) \frac{k\,dk}{i\bar{\nu}_1} \{[e^{-i\bar{\nu}_1(z-h)} - e^{-i\bar{\nu}_1(z+h)}] + K[e^{-i\bar{\nu}_1(-z-h+2H)}$$

$$- e^{-i\bar{\nu}_1(-z+h+2H)} - e^{-i\bar{\nu}_1(z-h+2H)} + e^{-i\bar{\nu}_1(z+h+2H)}] + K^2[\cdots] + \cdots\} \qquad (4\text{–}50)$$

As we have seen earlier, the first two terms of the right-side member represent the direct wave (or ray) and the ray reflected from the free surface $z = 0$. Now, according to Pekeris, the successive terms in (4–50) can be identified with rays reflected a certain number of times from the bottom. This interpretation had been used by Sezawa [160] for the case of a liquid layer over a rigid bottom. Thus the four terms in the coefficient of K represent four rays reflected once, and those multiplied by K^n correspond to n reflections from the bottom. In support of this identification, Pekeris takes the fact that for an impulsive source the integrals representing the rays vanish until the appropriate arrival times. This expansion of the integrand into a series of terms which can be interpreted as succes-

sively reflected pulses is due to Bromwich [12]. Newlands [105] used this method and gave detailed descriptions of the pulses. In the present work, our principal interest is in cases where the horizontal distance greatly exceeds the thickness of the layer. In such cases pulses become prolonged, and travel times for successive pulses become nearly equal, so that they overlap. As would be expected, a more suitable and direct method for these cases lies in evaluation of the original integrals by methods of contour integration used in the preceding chapters.

Denoting by $F(\nu_1, \nu_2)$ the multiplier of $2J_0(kr)k\,dk$ in any of the integrals (4–32), (4–34), and (4–35), we can write

$$\varphi = 2\int_0^\infty J_0(kr)F(\nu_1, \nu_2)k\,dk \tag{4–51}$$

or

$$\begin{aligned}\varphi &= \int_0^\infty H_0^{(1)}(kr)F(\nu_1, \nu_2)k\,dk + \int_0^\infty H_0^{(2)}(kr)F(\nu_1, \nu_2)k\,dk \\ &= \int_0^\infty I_1\,dk + \int_0^\infty I_2\,dk\end{aligned} \tag{4–52}$$

where $H_0^{(1)}$ and $H_0^{(2)}$ are Hankel functions of the first and second kind, respectively.

Integrals of the form (4–51) and (4–52) are improper. The integrands become infinite at the zeros of the determinant (4–27) or at the roots of the denominator in (4–32), (4–34), (4–35), or (4–44) to (4–46). To evaluate these integrals, contour integration in the complex $\zeta = k + i\tau$ plane is used, as before. The meaning chosen for improper integrals such as (4–51) is the limit of the integral along a path like that indicated by the continuous lines *OM* in Figs. 4–2 and 4–3. This definition has the advantage over the definition of (4–51) as identical with its principal part in that the value of the integral is unchanged if the poles are displaced infinitesimal distances into the fourth quadrant, as was done in Sec. 2–5. The contours and the cuts, shown in Figs. 4–2 and 4–3, are chosen with particular attention to functions involved.

The integrals along the real axis may be replaced, using Cauchy's theorem, by integrals along the imaginary axis, by branch line integrals, and by the residues. Thus, from Eq. (4–52)

$$\varphi = \int_0^N I_1\,d\zeta + \int_E^D I_2\,d\zeta + \int_{OAE} I_2\,d\zeta - 2\pi i \sum \text{Res}\, I_2 \tag{4–53}$$

provided that the complex poles are not located on the permissible sheet of the Riemann surface, as will be shown later.

More precision is required in the definition of the integral (4–51) since

the part of the real axis from the origin to the branch points is a part of the cut corresponding to the condition Re $\nu_1 = 0$, chosen in Sec. 2–5. Moreover, substituting Hankel functions instead of the Bessel function, we note that the origin is a logarithmic singularity for both of those functions. It is a regular point, however, for J_0.

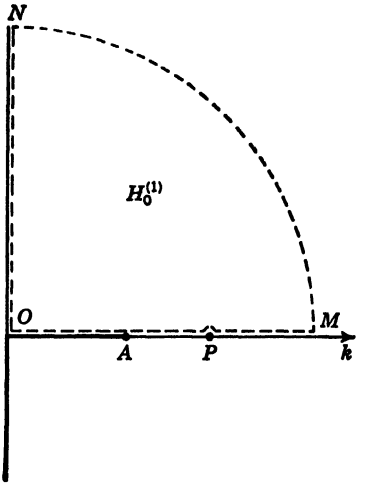


FIG. 4–2. Integration path for integral containing $H_0^{(1)}$.

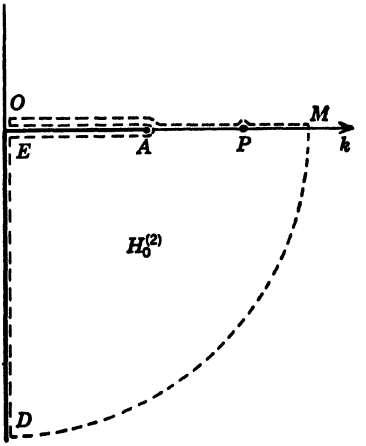


FIG. 4–3. Integration path for integral containing $H_0^{(2)}$.

As to the type of the Riemann surface which can be used to transform the integrals by Cauchy's theorem, it seems at first that there are two radicals introduced and, therefore, the surface should be four-leaved. If the integrals for the first layer are written in the form (4–21), where the direct wave from the source appears as a separate term, both integrands are mixed-valued functions of ν_1 and ν_2. The solution seems to contain terms with contributions from both branch line integrals corresponding to branch points k_{α_1} and k_{α_2}. However, if the direct wave is combined with the reflected wave, as in (4–32), (4–34), and (4–35), then all integrands are even-valued functions of ν_1 and the corresponding branch line integrals vanish. It appears that the disturbances corresponding to the branch point k_{α_1} vanish through destructive interference between the direct wave and waves arising from reflections at the surface and at the interface

It will be proved in Sec. 4–8 that in the general problem of this kind where there are parallel layers overlying an elastic half space only the branch points corresponding to the latter medium determine the Riemann surface. Thus, in the problem of a liquid layer overlying a liquid half space now considered, the Riemann surface is two-leaved, as represented in Figs. 4–2 and 4–3. The branch point A of these figures corresponds to the factor $\nu_2 = \pm\sqrt{\zeta^2 - k_{\alpha_2}^2}$, which makes $F(\nu_1, \nu_2)$ a two-valued function of ν_2.

If we proceed as in Sec. 2–5, the cut given by the condition Re $\nu_2 = 0$ begins at $k_{\alpha 2}$, runs to the origin along the real axis, and then goes to $-i\infty$ along the negative imaginary axis Im ν_2 is positive in the first quadrant and negative in the fourth quadrant. Re ν_2 is positive in both quadrants by the choice of the Riemann surface. We shall denote by the symbol ν_2^+ or ν_2^- that Re $\nu_2 = 0$ and Im $\nu_2 > 0$ or Im $\nu_2 < 0$, respectively.

By use of the relation $H_0^{(1)}(\zeta r) = -H_0^{(2)}(-\zeta r)$, the first integral in (4–53) can now be written as

$$\int_0^\infty H_0^{(1)}(i\tau r)F(\nu_1, \nu_2^+)i\tau i\, d\tau = -\int_0^{-\infty} H_0^{(2)}(i\tau r)F(\nu_1, \nu_2^+)i\tau i\, d\tau$$

which combines readily with the second term of (4–53). Thus

$$\varphi = \int_0^{-\infty} H_0^{(2)}(i\tau r)[F(\nu_1, \nu_2^-) - F(\nu_1, \nu_2^+)]i\tau i\, d\tau$$

$$+ \int_{k_\alpha}^0 H_0^{(2)}(kr)[F(\nu_1, \nu_2^-) - F(\nu_1, \nu_2^+)]k\, dk - 2\pi i \sum \text{Res}\,(I_2)$$

$$= \int_{\mathcal{L}} - 2\pi i \sum \text{Res}\,(I_2) \qquad (4\text{–}54)$$

The solution (4–54) is composed of a sum of residues corresponding to

real roots and a branch line integral along $\mathfrak{L}$. The integrand of the second integral in (4–52) has the form

$$I_2 = H_0^{(2)}(kr)\,\frac{M(k)}{N(k)} \tag{4–55}$$

The residue at a simple pole κ_n for an integrand of this form is given by

$$\text{Res}\,(I_2) = H_0^{(2)}(\kappa_n r)\,\frac{M(\kappa_n)}{N'(\kappa_n)} \tag{4–56}$$

For solutions of the form (4–44) to (4–46) κ_n is the root of the equation

$$N(k) = \nu_1 \cos \nu_1 H + i\delta_1\nu_2 \sin \nu_1 H = 0 \tag{4–57}$$

with this condition

$$M(\kappa_n) = \frac{\sin \nu_1 z \sin \nu_1 h}{\sin \nu_1 H}\,\kappa_n \tag{4–58}$$

$$N'(\kappa_n) = \frac{\kappa_n}{\nu_1 \sin \nu_1 H}\,[\nu_1 H - \sin \nu_1 H \cos \nu_1 H - \delta_1^2 \sin^2 \nu_1 H \tan \nu_1 H] \tag{4–59}$$

By (4–44), (4–45), and (4–54) to (4–59), the potential for the upper layer is

$$\varphi_1' = \varphi_1'' = \int_{\mathfrak{L}} - \frac{2\pi i}{H} \sum_n \frac{H_0^{(2)}(\kappa_n r)\nu_1 H \sin \nu_1 h \sin \nu_1 z}{\nu_1 H - \sin \nu_1 H \cos \nu_1 H - \delta_1^2 \sin^2 \nu_1 H \tan \nu_1 H} \tag{4–60}$$

In treating an improper integral of the type (4–51) Lamb (Chap. 2, Ref. 22) chose to define the integral as identical with its principal value. His final result for the residues represented standing waves. His definition of the improper integrals (4–52) differs from ours by the amounts $-\pi i \sum \text{Res}\,(I_1)$ and $-\pi i \sum \text{Res}\,(I_2)$, that is, by the contribution to the integrals along the lines OM in Figs. 4–2 and 4–3 from the small indentations above the poles. These terms correspond to the free waves added by Lamb to his result to produce outward propagation of the waves [Jardetzky [72]; see also Whittaker and Watson (Chap. 1, Ref. 66, p. 117)].

Returning to a discussion of the roots of the period equation (4–57), it will be recalled that we assumed the absence of complex roots on the permissible sheet of the Riemann surface in writing Eq. (4–53). The equation

$$\tilde{\Delta}_1 \equiv \nu_1 \cosh \nu_1 H + \delta_1\nu_2 \sinh \nu_1 H = 0 \tag{4–61}$$

is identical with (4–57) because of the definitions of $\nu_i = -i\nu_i$ given in (4–38). For $\alpha_1 < \alpha_2$, $k_{\alpha_1} < k_{\alpha_2}$, and for $k > k_{\alpha_1}$ both ν_1 and ν_2 are real and

positive. The left-hand member of (4–61) cannot vanish for $k_{\alpha_1} < k < \infty$ and similarly for $0 < k < k_{\alpha_2}$. Thus the real roots κ_n of (4–61) must lie in the interval $k_{\alpha 2} \leq \kappa_n \leq k_{\alpha_1}$.

To locate complex roots, we put

$$\begin{aligned} H\nu_1 &= H\sqrt{\zeta^2 - k_{\alpha_1}^2} = p_1 + iq_1 \\ H\nu_2 &= H\sqrt{\zeta^2 - k_{\alpha_2}^2} = p_2 + iq_2 \end{aligned} \tag{4–62}$$

with $p_1 \geq 0$ and $p_2 \geq 0$. The coefficients q_1 and q_2 are positive in the first quadrant (Fig. 2–7) and negative in the fourth. For ω real, k_{α_1} and k_{α_2} are also real, and therefore the preceding equations yield the condition

$$p_1 q_1 = p_2 q_2 \tag{4–63}$$

Substituting (4–62) in (4–61) and separating the real and imaginary parts, we obtain two equations

$$\begin{aligned} p_1 + \delta_1 p_2 \tanh p_1 &= \tan q_1 \left(\frac{p_2 q_2}{p_1} \tanh p_1 + \delta_1 q_2 \right) \\ \frac{p_2 q_2}{p_1} + \delta_1 q_2 \tanh p_1 + \tan q_1 (p_1 \tanh p_1 + \delta_1 p_2) &= 0 \end{aligned} \tag{4–64}$$

Eliminating $\tan q_1$, we obtain

$$\frac{p_1(p_1 + \delta_1 p_2 \tanh p_1)}{q_2(p_2 \tanh p_1 + \delta_1 p_1)} = -\frac{q_2(p_2 + \delta_1 p_1 \tanh p_1)}{p_1(p_1 \tanh p_1 + \delta_1 p_2)} \tag{4–65}$$

Because of the convention of signs mentioned above, this condition cannot be satisfied in the first or fourth quadrant. Therefore we must draw the conclusion that the complex roots of the period equation (4–61) cannot be located on the permissible sheet of the Riemann surface as defined in the preceding sections.

The same conclusion can be reached if we write Eq. (4–61) in the form

$$\delta_1 \tanh (p_1 + q_1 i) = -\frac{p_1 + q_1 i}{p_2 + q_2 i} \tag{4–66}$$

and separate the real and imaginary parts. Two equations result:

$$\frac{\delta_1 \tanh p_1 (1 + \tan^2 q_1)}{1 + \tanh^2 p_1 \tan^2 q_1} = -\frac{p_1 p_2 + q_1 q_2}{p_2^2 + q_2^2} \tag{4–67}$$

$$\frac{\delta_1 \tan q_1 (1 - \tanh^2 p_1)}{1 + \tanh^2 p_1 \tan^2 q_1} = \frac{p_1 q_2 - p_2 q_1}{p_2^2 + q_2^2} \tag{4–68}$$

and Eq. (4–67) cannot be satisfied with the convention of signs used earlier.

We can easily see that there are no roots of the period equation (4–61)

on the imaginary axis. If we put $\zeta = -im$, this equation takes the form

$$\sqrt{m^2 + k_{\alpha_1}^2}\cos H\sqrt{m^2 + k_{\alpha_1}^2} + i\delta_1\sqrt{m^2 + k_{\alpha_2}^2}\sin H\sqrt{m^2 + k_{\alpha_1}^2} = 0 \qquad (4\text{–}69)$$

which has no real roots m.

Schermann [151] investigated roots of the period equation of a liquid layer over a semi-infinite solid. In principle, his results for this more complicated case should be the same as those for the present problem. Schermann found a finite number of real roots and an infinite number of complex roots. From the discussion just given, we must conclude that these complex roots do not fall on the permissible Riemann sheet.

Discussion of Solutions. In many investigations involving contour integration, branch line integrals are interpreted as terms which are nonessential at large distances. The asymptotic behavior of such integrals and their physical interpretation can be discussed by different methods (see, for example, Secs. 2–5 and 3–3). The branch line integral in Eq. (4–60) corresponds to a wave traversing the refraction path shown in Fig. 3–18. Schermann has shown that in the two-dimensional case a branch line integral is of the order $r^{-\frac{1}{2}}$. The amplitudes of waves corresponding to these integrals in three dimensions (see Sec. 3–3) diminish as r^{-2}. We shall, therefore, discuss in greater detail the more important part of the solution (4–60) given by the residues, for which the decrease with distance is less rapid.

It may be seen that the expression (4–45) for φ_1'', having the same poles as φ_1' in Eq. (4–44), also takes the form (4–60). Thus these terms represent waves in the whole layer (0, H). Similar considerations prove that the part of φ_2 in Eq. (4–46) due to residues is given by

$$\varphi_{2R} = -\frac{2\pi i\delta_1}{H}\sum \frac{H_0^{(2)}(\kappa_n r)\nu_1 H\sin\nu_1 h\sin\nu_1 H\, e^{-i\nu_2(z-H)}}{\nu_1 H - \sin\nu_1 H\cos\nu_1 H - \delta_1^2\sin^2\nu_1 H\tan\nu_1 H} \qquad \text{for } z > H \qquad (4\text{–}70)$$

An approximation of the residues in Eqs. (4–60) and (4–70) convenient for discussion can be obtained if we make use of the asymptotic expansion of the Hankel function

$$H_0^{(2)}(\kappa_n r) \sim \sqrt{\frac{2}{\pi\kappa_n r}}\exp\left[i\left(\frac{\pi}{4} - \kappa_n r\right)\right] \qquad (4\text{–}71)$$

Let $\nu_1 H = x$

$$\nu_1^{(n)}H = -H\sqrt{k_{\alpha_1}^2 - \kappa_n^2} = x_n \qquad \nu_2^{(n)} = -\sqrt{k_{\alpha_2}^2 - \kappa_n^2} \qquad (4\text{–}72)$$

and

$$V(x_n) = \frac{x_n}{x_n - \sin x_n\cos x_n - \delta_1^2\sin^2 x_n\tan x_n} \qquad (4\text{–}73)$$

Then, if we use $-i = \exp(-i\pi/2)$ and replace the time factor $\exp(i\omega t)$, the residue terms in Eqs. (4–60) and (4–70) yield the terms

$$\varphi_{1R} = \frac{2}{H}\sqrt{\frac{2\pi}{r}} \sum \frac{1}{\sqrt{\kappa_n}} \exp\left[i\left(\omega t - \kappa_n r - \frac{\pi}{4}\right)\right] \cdot V(x_n) \sin\frac{x_n h}{H} \sin\frac{x_n z}{H} \qquad \text{for } 0 \leq z \leq H \tag{4–74}$$

$$\varphi_{2R} = \frac{2\delta_1}{H}\sqrt{\frac{2\pi}{r}} \sum \frac{1}{\sqrt{\kappa_n}} \exp\left[i\left(\omega t - \kappa_n r - \frac{\pi}{4}\right)\right] \cdot V(x_n) \sin\frac{x_n h}{H} \sin x_n \exp\left[-i\nu_2^{(n)}(z - H)\right] \qquad \text{for } z > H \tag{4–75}$$

Each term in Eqs. (4–74) and (4–75) will be called a "normal mode," the mode being characterized by the subscript n. This subscript indicates that the corresponding quantity is to be evaluated at $k = \kappa_n$, where κ_n are the roots of the period equation (4–57) or (4–61) for a given ω. From the exponential factor it can be seen that the phase velocity c_n for the nth root can be expressed by

$$c_n = \frac{\omega}{\kappa_n} \tag{4–76}$$

The period equation (4–57) defines an implicit relationship between any two of the three variables k, ω, and c, as may be seen by writing it in the form

$$\delta_1 \tan H(k_{\alpha_1}^2 - k^2)^{\frac{1}{2}} = -\frac{H(k_{\alpha_1}^2 - k^2)^{\frac{1}{2}}}{H(k^2 - k_{\alpha_2}^2)^{\frac{1}{2}}} \tag{4–77}$$

For $k_{\alpha_2} \leq k \leq k_{\alpha_1}$, all terms are real, and real roots κ_n exist. In Fig. 4–4 are plotted the curves representing the right and left sides of Eq. (4–77) as functions of the parameter $H(k_{\alpha_1}^2 - k^2)^{\frac{1}{2}}$. The intersections shown by circles define the values of κ_n which are real roots of the period equation. It is seen that for any finite value of ω the number of real roots is the nearest integer to the number $H(k_{\alpha_1}^2 - k_{\alpha_2}^2)^{\frac{1}{2}}/\pi + 1$, the trivial root at the origin being included. Thus there is a finite number of poles of the integrands of (4–44), (4–45), and (4–46) on the real axis.

It is particularly instructive to write the period equation in terms of c and k as follows:

$$\tan kH\sqrt{\frac{c^2}{\alpha_1^2} - 1} = -\frac{\rho_2}{\rho_1}\sqrt{\frac{c^2}{\alpha_1^2} - 1} \Big/ \sqrt{1 - \frac{c^2}{\alpha_2^2}} \tag{4–78}$$

When Eq. (4–77) is written in this form, real roots correspond to the case $\alpha_2 \geq c \geq \alpha_1$. This last equation is an implicit relation between the frequency

$$f = \frac{kc}{2\pi} \tag{4–79}$$

and the phase velocity c. For a given value of phase velocity it is seen that the frequency is a multiple-valued function of phase velocity, each

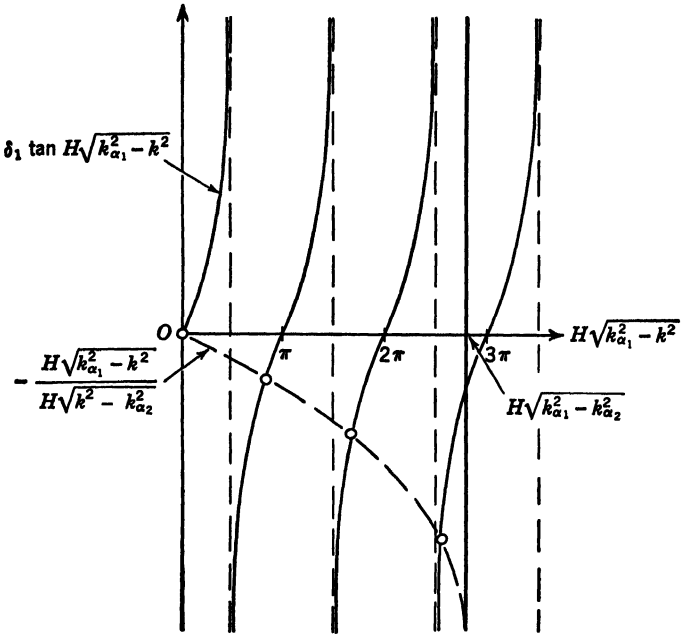


FIG. 4–4. Method of determining real roots of Eq (4–77).

value belonging to a different mode of propagation. This equation is best treated numerically by choosing a value of c and calculating the corresponding value kH. It is evident that as $c/\alpha_1 \to \alpha_2/\alpha_1$

$$kH \to (2n - 1)\frac{\pi}{2} \Big/ \sqrt{\frac{\alpha_2^2}{\alpha_1^2} - 1} \tag{4–80}$$

This corresponds to the cutoff frequency of each mode, i.e., the lower limit of possible frequency for this mode. As $c/\alpha_1 \to 1$,

$$kH \to n\pi \Big/ \sqrt{\frac{c^2}{\alpha_1^2} - 1} \to \infty \tag{4–81}$$

and it is seen that propagation at the phase velocity $c = \alpha_1$ occurs in each mode at the highest frequencies. More precisely, from the definition of the wave number $k = 2\pi/l$, large values of kH correspond to large values of the dimensionless parameter H/l. Thus for $l \ll H$, $c \to \alpha_1$ for all modes. For $\alpha_2/\alpha_1 > c/\alpha_1 > 1$ the range of $\kappa_n H$ is such that

$$\left(n - \frac{1}{2}\right)\pi < \kappa_n H \sqrt{\frac{c^2}{\alpha_1^2} - 1} < n\pi \tag{4-82}$$

As $n \to \infty$, that is, for the higher modes,

$$\kappa_n H \to n\pi \Big/ \sqrt{\frac{c^2}{\alpha_1^2} - 1},$$

and the modes form a harmonic series.

It can be shown that the period equation expresses the condition of constructive interference between plane waves undergoing multiple reflection in the liquid layer at angles of incidence beyond the critical angle $\theta_{cr} = \sin^{-1}(\alpha_1/\alpha_2)$. In Fig. 4–5 $ADEF$ represents a portion of the

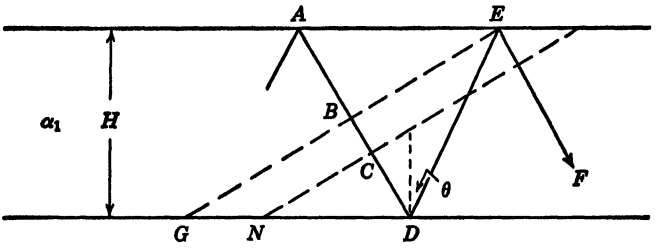


FIG. 4–5. Interpretation of frequency equation $BC/GN = l_0/l$.

path of a plane wave which has been totally reflected at the bottom with a phase change 2ϵ and reflected at the free surface with a phase change of $-\pi$. As the wave front, shown by the dashed line, moves a distance BC in unit time, it traces the distance GN in the horizontal direction, where $GN = c = \alpha_1/\sin\theta$ is the phase velocity defined earlier. For this representation the wave number k may be defined by the equation $k = 2\pi \sin\theta/l_0$, where l_0 is the wavelength measured in the layer along the path BC and $l = l_0/\sin\theta$ is the wavelength measured in the horizontal direction. If the wave front GBE is to interfere constructively with the coincident wave front which has traversed the additional path BDE, it is required that

$$\frac{2\pi}{l_0} BDE - 2\epsilon(\theta) + \pi = 2n\pi \qquad n = 1, 2, 3, \cdots \tag{4-83}$$

From Fig. 4–5, $BDE = 2H \cos \theta$, and from the results of Sec. 3–1

$$\cot \epsilon = \frac{\rho_2 \sqrt{c^2/\alpha_1^2 - 1}}{\rho_1 \sqrt{1 - c^2/\alpha_2^2}}$$

Taking the tangent of both sides of Eq. (4–83) and using these definitions, we obtain the equivalent of Eq. (4–78). From this point of view it is seen that the normal modes are interference phenomena, each successively higher mode representing a higher order of interference, and the disturbance at a distant point may be obtained by the superposition of waves arriving at the point along the oblique rays for which constructive interference occurs.

In Eqs. (4–74) and (4–75) the factors $V(x_n)/\sqrt{\kappa_n}$ give the relative excitation of each mode. The factor $\sin (x_n h/H)$ shows the influence of the depth of the source on the amplitudes of the different modes. The vertical-pressure distribution for each mode is given by the factor $\sin (x_n z/H)$ which is plotted in Fig. 4–6 for the first two modes. It is apparent that,

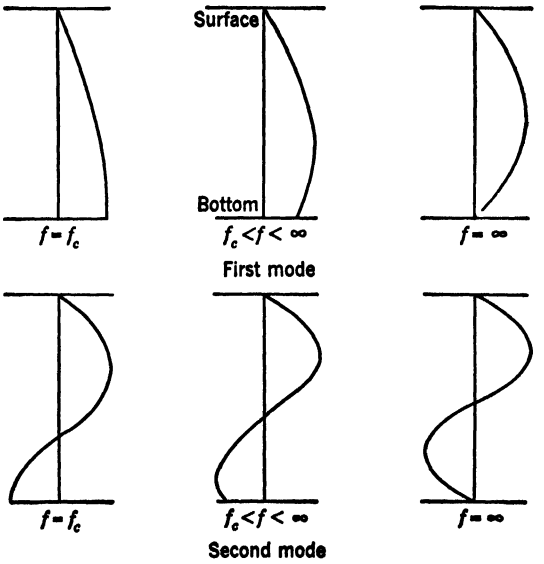


FIG. 4–6. Vertical-pressure variation in liquid layer for cutoff frequency f_c, intermediate frequency f, and infinite frequency in the first two modes.

for all modes and for all frequencies, the free surface is a node in pressure and in horizontal particle velocity and an antinode in vertical particle velocity.

As would be expected, at the great distances for which our approximations hold, the steady-state amplitudes decrease as $r^{-\frac{1}{2}}$ for the normal modes and r^{-2} for the refraction waves from the branch line integrals, which may be compared with the r^{-1} decrease for body waves in a homogeneous medium.

It was pointed out by Satô [145] and Officer [110] that under certain conditions the branch line integral can yield terms as important as the residue. In the branch line integral $\mathfrak{L}$ in Eq. (4–54) the denominator of $F(\nu_1, \nu_2^+)$ is given by Eq. (4–57), using the relations $\nu_1 = i\bar{\nu}_1$, $\nu_2 = i\bar{\nu}_2$. To obtain the common denominator of the factor in brackets of (4–54) we write

$$\frac{1}{\bar{\nu}_1 \cos \bar{\nu}_1 H - i\delta_1 \bar{\nu}_2 \sin \bar{\nu}_1 H} - \frac{1}{\bar{\nu}_1 \cos \bar{\nu}_1 H + i\delta_1 \bar{\nu}_2 \sin \bar{\nu}_1 H} \tag{4-84}$$

Thus, when $h \leq z \leq H$, for example, the branch line integral in (4–54) takes the form

$$\varphi'_{1\mathfrak{L}} = 2i\delta_1 \int_{\mathfrak{L}} H_0^{(2)}(\zeta r) \frac{\bar{\nu}_2 \sin \bar{\nu}_1 h \sin \bar{\nu}_1 z}{\bar{\nu}_1^2 \cos^2 \bar{\nu}_1 H + \delta_1^2 \bar{\nu}_2^2 \sin^2 \bar{\nu}_1 H} \zeta \, d\zeta \tag{4-85}$$

Since $\bar{\nu}_2$ is very small in the vicinity of the branch point, the term containing $\sin^2 \bar{\nu}_1 H$ is usually neglected in evaluating (4–85), and the amplitude is found to decrease as $r^{-\frac{1}{2}}$. However, when $\cos \bar{\nu}_1 H = 0$, this result is invalid, and evaluation of (4–85) yields an amplitude which decreases as r^{-1}. It is interesting to note the physical interpretation of the condition $\cos \bar{\nu}_1 H = 0$. With the substitution $k = 2\pi \sin \theta / l_0$ and $c = \omega/k = \alpha_1/\sin \theta$, this condition can be written in the form

$$\frac{2\pi}{l_0} 2H \cos \theta + \pi = 2n\pi \tag{4-86}$$

Equation (4–86) expresses the condition for constructive interference between waves traversing the paths shown in Fig. 4–7.

Generalization for a Pulse. As stated in Sec. 2–6, to study the propagation of an impulsive disturbance it is convenient to use the Fourier integral to represent the initial disturbance as the summation of a complete spec-

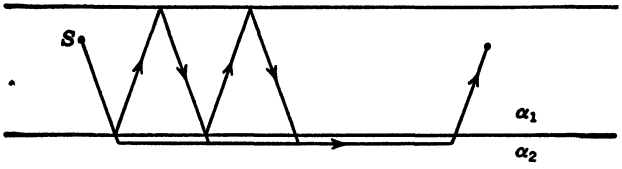


FIG. 4–7. Paths of multiple-reflected and refracted waves which may interfere.

trum of simple harmonic waves by a suitable choice of the initial phases and amplitudes. If the propagation is through a dispersive medium, it is well known that distortion of the pulse will result from variation of phase velocity with frequency. At a sufficiently great distance from the source the initial pulse will be transformed into a train of sinusoidal waves in which the frequency and amplitude vary gradually along the train.

Our derivation of the steady-state solution started with a spherical wave, which in an unlimited homogeneous medium could be expressed as $\varphi_0 = \exp [i(\omega t - kR)]/R$. If, instead of a harmonic function of time, we consider an arbitrary disturbance $S(t)$, we can make use of the Fourier transform

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S(t) e^{-i\omega t}\, dt \tag{4-87}$$

to obtain the initial time variation in the form

$$S(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t}\, d\omega \tag{4-88}$$

In the problem of two liquid layers, whose steady-state solution is given by (4–74) and (4–75), application of the Fourier transform to Eq. (4–74) would give, for an arbitrary initial disturbance $S(t)$,

$$\Phi_{1R} = \frac{2}{H\sqrt{2\pi r}} \sum_n \int_{-\infty}^{\infty} g(\omega) e^{i(\omega t - \kappa_n r - \pi/4)} \frac{V(x_n)}{\sqrt{\kappa_n}} \sin \frac{x_n h}{H} \sin \frac{x_n z}{H}\, d\omega \tag{4-89}$$

for $0 < z < H$ and a similar expression Φ_{2R} for $z > H$.

To represent the initial disturbance due to an explosion we choose $S(t)$ as follows:

$$S(t) = \begin{cases} e^{-\sigma t} & t > 0 \\ 0 & t < 0 \end{cases} \tag{4-90}$$

where σ is a parameter which depends on the energy of the explosive charge, among other factors (see Cole [19]). The corresponding Fourier transform is

$$g(\omega) = \frac{1}{\sqrt{2\pi}\,(\sigma + i\omega)} \tag{4-91}$$

which gives for (4–89)

$$\Phi_{1R} = \frac{1}{H}\sqrt{\frac{2}{\pi r}} \sum_n \int_{-\infty}^{\infty} \frac{e^{i(\omega t - \kappa_n r - \pi/4)}}{\sqrt{\kappa_n}\,(\sigma + i\omega)} V(x_n) \sin \frac{x_n h}{H} \sin \frac{x_n z}{H}\, d\omega \tag{4-92}$$

To obtain an approximate value for this expression we use Kelvin's method of stationary phase. The exponent in Eq. (4–92) will have a stationary value at a frequency ω_0, where ω_0 is the root of the equation $d(\omega t - \kappa_n r - \pi/4)/d\omega = 0$, that is, at $t = r\, d\kappa_n/d\omega$. As shown by Kelvin (see Appendix A), for given values of t and r the integral for Φ_{1R} will be zero, because of the rapid alternations in sign of the exponential factor and the more gradual changes of the remaining factors in the integrand, except for a narrow range of ω near ω_0. By use of Eqs. (A–14) and (A–15) the value of (4–92) is obtained for large r:

$$\Phi_{1R} = \frac{2\alpha_1}{Hr} \sum_n \frac{U_n}{\alpha_1} \left[\gamma \left| \frac{d(U_n/\alpha_1)}{d\gamma} \right| \right]^{1/2} \frac{e^{i(\omega_0 t - \kappa_n r - \pi/4 \pm \pi/4)}}{\sigma + i\omega_0} V(x_n) \sin \frac{x_n h}{H} \sin \frac{x_n z}{H} \qquad \text{for } 0 \leq z \leq H \tag{4–93}$$

Here $\gamma = \omega_0 H/2\pi\alpha_1$, $U = d\omega/dk$, the upper sign in the exponential is taken if $d(U_n/\alpha_1)/d\gamma$ is positive, otherwise the negative sign is used, and x_n and κ_n are to be taken for ω_0.

The quantity $U = d\omega/dk = r/t$, defining the velocity of a group of waves having angular frequency ω_0, is known as the group velocity. It is related to the wave or phase velocity c by the equation

$$U = c + k \frac{dc}{dk} \tag{4–94}$$

At large distances amplitudes can be calculated from Eq. (4–93).

It should be noted that Φ_{1R} and, similarly, Φ_{2R} decrease with distance as r^{-1}, an additional factor of $r^{-\frac{1}{2}}$ having been introduced to Eqs. (4–74) and (4–75) as a result of dispersion. The amplitude of Φ_{1R} varies inversely as the square root of the slope of the group-velocity curve, if the group velocity is considered as a function of the dimensionless parameter

$$\gamma = \frac{H}{l_0} = c_n \kappa_n \frac{H}{2\pi\alpha_1} = \frac{Hf}{\alpha_1} \tag{4–95}$$

For values of ω near a maximum or minimum value of the group velocity Eq. (4–93) is not valid, for it involves the assumption that higher derivatives are negligible in comparison with $dU/d\gamma$. In the neighborhood of a stationary value of group velocity, the next approximation, given in a convenient form by Pekeris, leads to

$$\Phi_{1R} = \frac{4\alpha_1}{3^{2/3} H r^{5/6}} \sum_n \frac{\cos\left[\omega_0 t - \kappa_n r - \tan^{-1}(\omega/\sigma) - \pi/4\right]}{[(\sigma^2 + \omega^2)\kappa_n/2\pi]^{1/2}[(H^2/2\pi)\, dZ_n/d\gamma]^{1/3}} \cdot E(m) V(x_n) \sin \frac{x_n h}{H} \sin \frac{x_n z}{H} \tag{4–96}$$

for $0 \leq z \leq H$, where

$$Z_n = \frac{\alpha_1^2}{H}\frac{d^2\kappa_n}{d\omega^2} = -\frac{\alpha_1}{2\pi U_n^2}\frac{dU_n}{d\gamma} \tag{4-97}$$

$$E(m) = m^{\frac{1}{2}}[J_{-\frac{1}{3}}(m) + J_{\frac{1}{3}}(m)] \quad \text{if } \frac{r}{U_{max}} < t < \frac{r}{U_{min}} \tag{4-98}$$

$$E(m) = m^{\frac{1}{2}}[I_{-\frac{1}{3}}(m) - I_{\frac{1}{3}}(m)] \quad \text{if } t < \frac{r}{U_{max}} \quad \text{or} \quad t > \frac{r}{U_{min}} \tag{4-99}$$

$$m = \frac{4}{3}\pi^{\frac{1}{2}}\left(-\frac{dZ_n}{d\gamma}\right)^{-\frac{1}{2}}\frac{r}{H}\left(\frac{\alpha_1}{U} - \frac{\alpha_1}{U_{min}}\right) \tag{4-100}$$

Pekeris has given the name Airy phase to the waves associated with a stationary value of group velocity. The factor $E(m)$ represents the envelope of the waves in the Airy phase and is plotted in Fig. 4–8 from the data given by Pekeris. It is to be noted that the amplitude in the Airy phase depends on $r^{-5/6}$ in contrast to r^{-1} for other waves in the train. Thus the Airy phase becomes relatively stronger with increasing distance.

The frequency equation (4–78) was used to calculate the ratio c/α_1, as a function of the dimensionless parameter γ (4–95), from which U/α_1

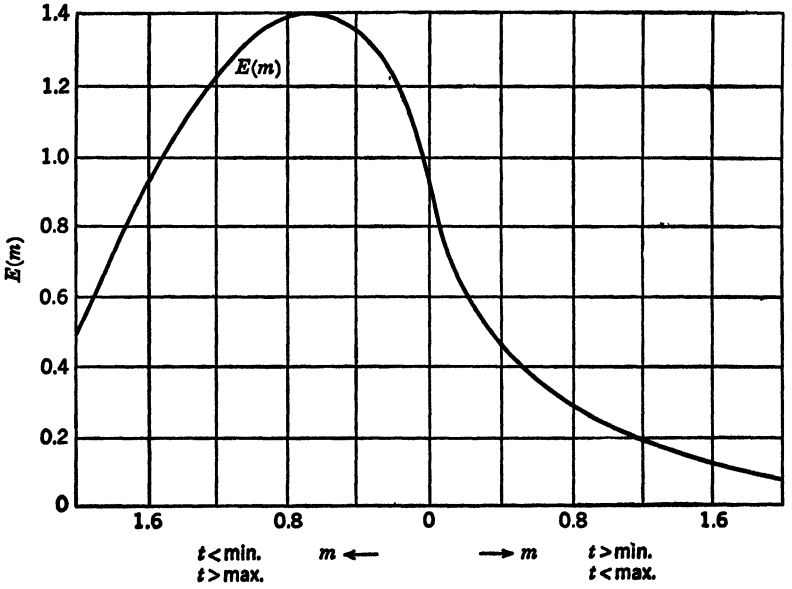


FIG. 4–8. Function $E(m)$ representing the envelope of the waves in the Airy phase. (*After Pekeris.*)

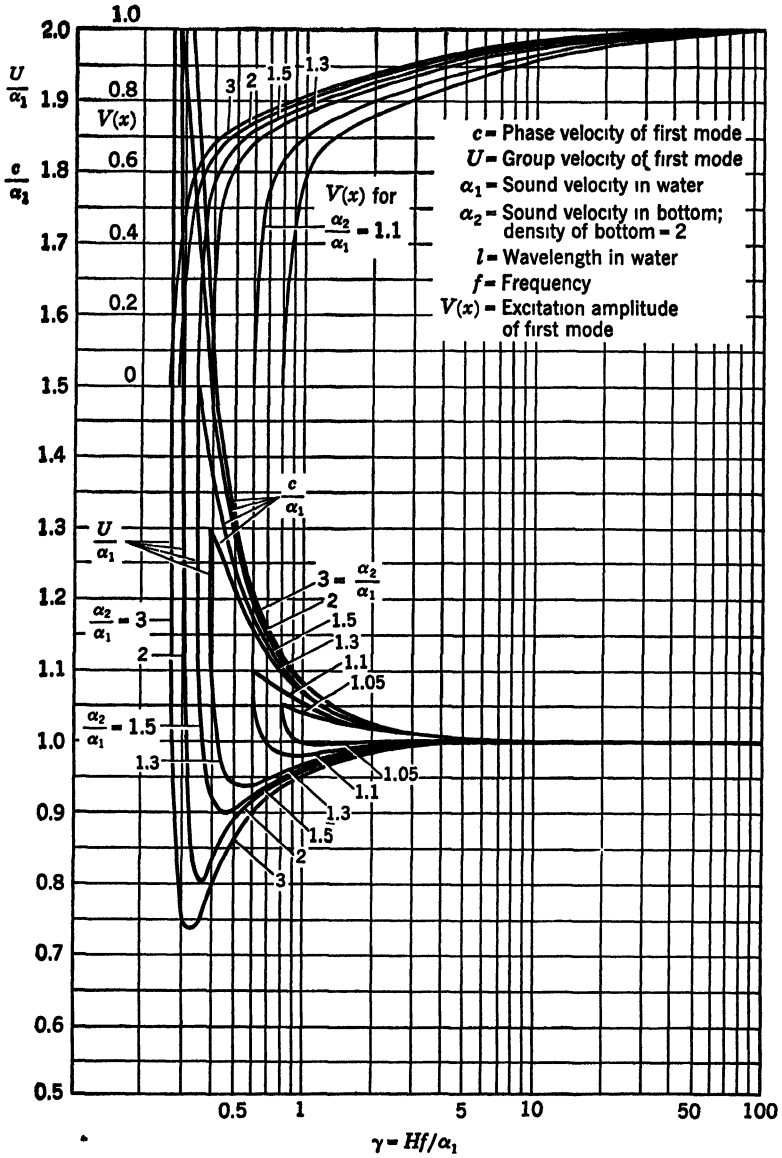


FIG. 4-9. First-mode phase- (c/α_1) and group- (U/α_1) velocity curves and excitation amplitude $V(x)$ for liquid layer over liquid substratum; $\rho_2/\rho_1 = 2.0$, $\alpha_2/\alpha_1 = 3.0$, 2.0, 1.5, 1.3, 1.1, 1.05 as a function of parameter γ. (*After Pekeris.*)

is obtained by the aid of Eq. (4–94). In Fig. 4–9, given by Pekeris, a family of phase- and group-velocity curves for the first mode for various values of α_2/α_1 when $\rho_2/\rho_1 = 2.0$ is presented. It is seen that the phase- and group-velocity curves cut off at a low-frequency limit where $U = c = \alpha_2$ and that they approach α_1 asymptotically for large γ. A striking feature is the occurrence of a minimum on each group-velocity curve, the minimum value being lower and occurring at a lower frequency, the greater the ratio α_2/α_1.

To illustrate features of the higher modes, phase- and group-velocity curves for the first three modes and for the ratio $\alpha_2/\alpha_1 = 1.5$ are shown in

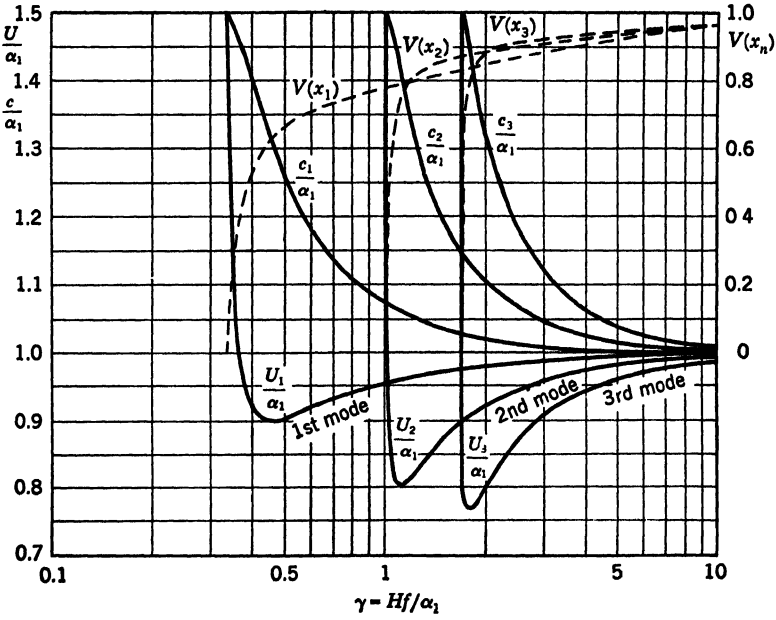


FIG. 4–10. Liquid layer over liquid substratum. Phase- (c/α_1) and group- (U/α_1) velocity curves and excitation amplitudes for $\rho_2/\rho_1 = 2.0$, $\alpha_2/\alpha_1 = 1.5$ in the first three modes. (*After Pekeris.*)

Fig. 4–10. Generally speaking, higher modes involve higher frequencies, and for any given frequency only a finite number of modes is possible.

In preceding sections the steady-state solutions were expressed by a sum of residues of the integrand and a branch line integral corresponding to the branch point $c = \alpha_2$. The residues lead to the normal-mode solutions which predominate at large distances because of the factor r^{-1}. The normal-

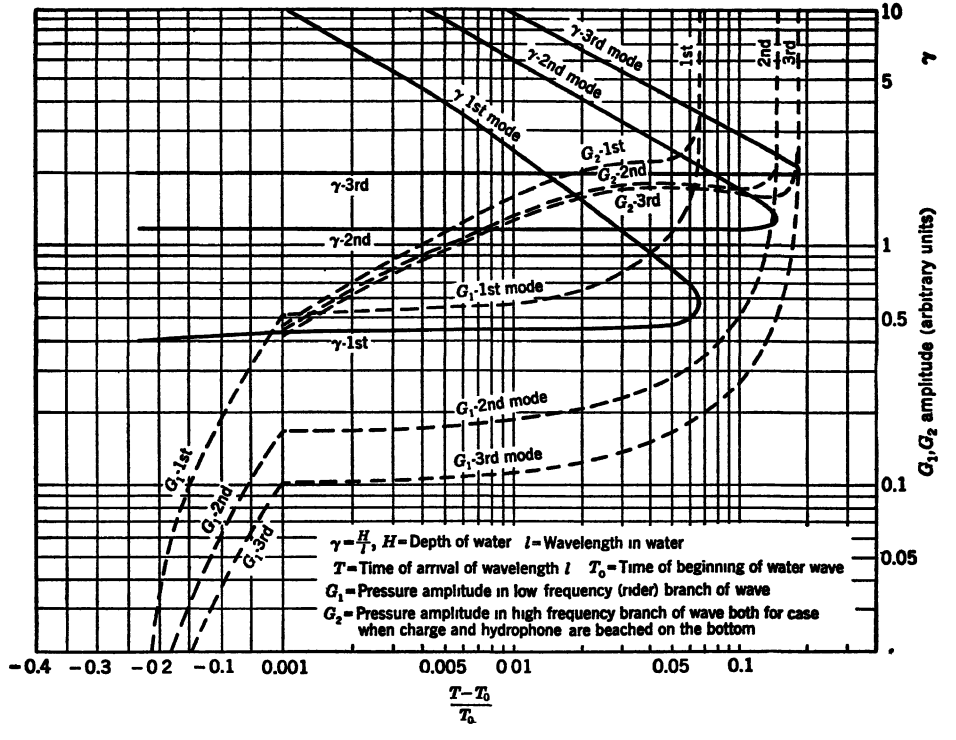


FIG. 4–11. Liquid layer over liquid substratum. Theoretical amplitude, frequency, and arrival time of first three modes for case $\rho_2/\rho_1 = 2.0$ and $\alpha_2/\alpha_1 = 1.3$. (*After Pekeris.*)

mode solutions vanish at $c = U = \alpha_2$. At a time greater than r/α_2 (that is, immediately after the arrival of compressional waves through the lower layer), the normal-mode contributions begin gradually, increasing in amplitude with increasing time. The wave amplitude and frequency, for example, can be obtained from Fig. 4–11 as a function of travel time.

It is seen that the first of the normal-mode waves arrive at $t = r/\alpha_2$ with the cutoff frequency. These waves correspond to the left end of the group-velocity curve (Fig. 4–9), and because of the steepness of the group-velocity curve their frequency increases only very gradually with time. At the time r/α_1, corresponding to travel in the upper layer, high-frequency waves associated with the right-hand end of the group-velocity curve arrive, superposed on the earlier low-frequency oscillations. Their frequency rapidly decreases until a time corresponding to travel at the minimum value of group velocity, at which time the frequencies of the two superimposed wave trains become equal, forming an Airy phase. The branch line integrals contribute waves traversing the refraction paths shown in Fig. 3–18. The waves decrease with distance as r^{-2} except near the cutoff frequency, where the multiple refractions interfere constructively and decrease as r^{-1}.

All modes theoretically contribute to the motion at any point but in many cases recorded wave trains consist almost entirely of contributions from a single mode, usually the fundamental. This situation results generally from the actions of filters in the recording system.

The results of Pekeris for two liquid layers have often been found applicable to studies on explosion sound transmission in shallow water, implying absence of rigidity of ocean-bottom materials. This is not surprising since core samples consist of unconsolidated sediments almost everywhere. An obvious result of the Pekeris theory explains the early experimental observation of Ewing [35] that any accurate determination of the sound velocity for horizontal transmission through the surface layer must be made at the highest frequencies. An example taken from shots recorded at a distance of 17 miles in a water depth of 90 ft is shown in Fig. 4–12. The general similarity of the observed wave train with that

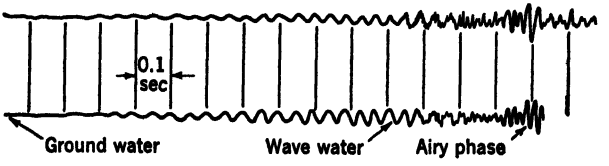


FIG. 4–12. Waves (through two different filters) from an explosive charge of 55 lb observed at a range of 1,030 times water depth, water depth = 90 ft. Bottom sediment thickness = 220 ft, bottom sound velocity = 1.13 times velocity in water. Source and receiver in water. (*Courtesy of C. L. Drake.*)

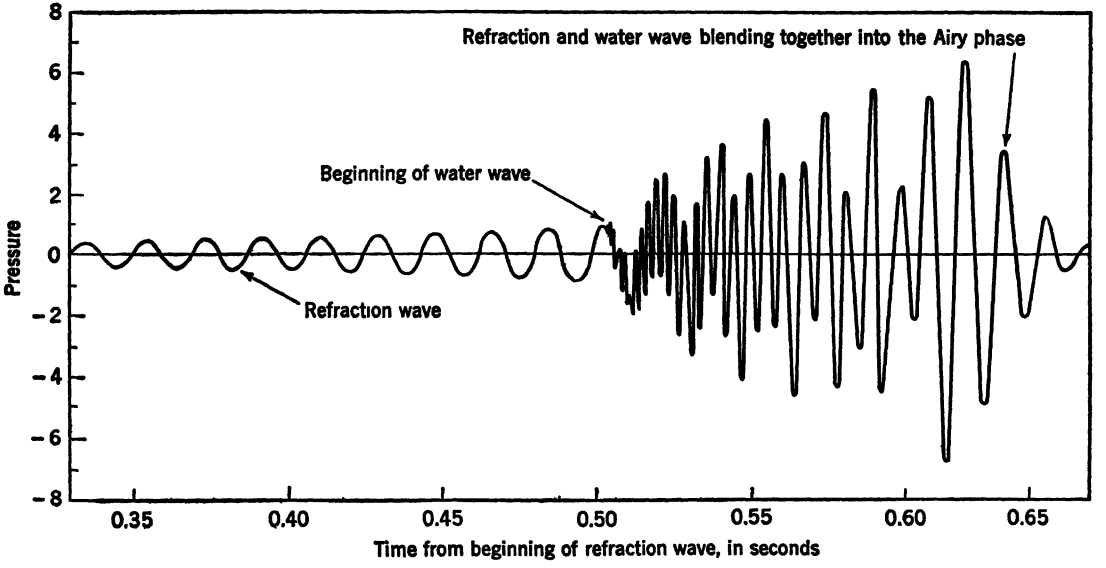


FIG. 4–13. Theoretical wave motion in first mode for range 460 times water depth, water depth 60 ft, bottom velocity 1.1 times velocity in water, density 2.0, charge weight 5 lb. (*After Pekeris.*)

computed for the first mode alone from Eqs. (4–93) and (4–96) is striking (Fig. 4–13).

Dobrin [26] used the observed dispersion to deduce properties of the lagoon bottom on Bikini atoll. He found that when the theory was applicable it could give useful information on the bottom to a depth comparable to water depth. Cases where the simple liquid-layer theory has been found inadequate have been attributed to a layered bottom, a low-velocity bottom, or a solid bottom.

4–3. Three-layered Liquid Half Space. This problem was investigated by Pekeris [116] and Press and Ewing [118]. The method developed in the preceding section can be applied to the new case with slight modifications. Figure 4–14 shows the notations which will be used. To the boundary

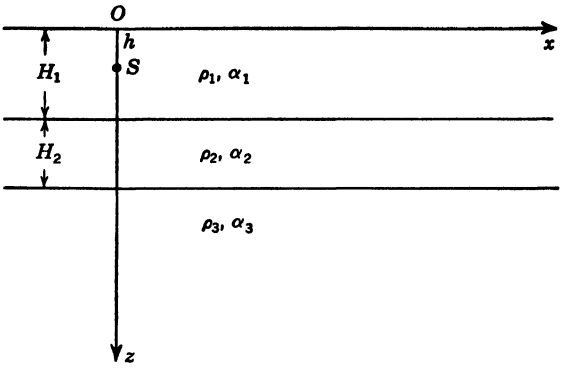


FIG. 4–14. Point source in three-layered liquid half space.

conditions (4–19) and (4–20) we add the two conditions

$$\frac{\partial \varphi_2}{\partial z} = \frac{\partial \varphi_3}{\partial z} \qquad \rho_2\varphi_2 = \rho_3\varphi_3 \qquad \text{at } z = H_1 + H_2 \tag{4–101}$$

The former layer thickness H is now denoted by H_1. In order to satisfy the five boundary conditions, we shall have five arbitrary functions Q in expressions (4–11) and (4–13), and we can introduce the potential (4–16) representing the source. In the preceding section, for a layer of finite thickness, the potential was alternatively represented by expressions containing trigonometric functions of z, such as (4–36) and (4–37). In the first layer, where the source is located, we made use of two different expressions φ_1' and φ_1'' for the potential above and below the source, respectively. These expressions satisfied the conditions of continuity in φ and of discontinuity in displacement; that is, we put [see (4–39) and (4–40)]

$$\varphi_1' = \varphi_1'' \qquad \frac{\partial \varphi_1'}{\partial z} - \frac{\partial \varphi_1''}{\partial z} = 2\int_0^\infty J_0(kr)k\,dk \qquad \text{at } z = h \tag{4–102}$$

We choose φ_1' such that the condition of vanishing pressure at the free surface is satisfied. The solutions may now be written

$$\begin{aligned}
\varphi_1' &= \int_0^\infty A(k)J_0(kr)\sin \nu_1 z\, dk \qquad \text{for } 0 \le z \le h \\
\varphi_1'' &= \int_0^\infty B(k)J_0(kr)\sin \nu_1 z\, dk + \int_0^\infty C(k)J_0(kr)\cos \nu_1 z\, dk \\
&\qquad\qquad \text{for } h \le z \le H_1 \\
\varphi_2 &= \int_0^\infty D(k)J_0(kr)\sin \nu_2 z\, dk + \int_0^\infty E(k)J_0(kr)\cos \nu_2 z\, dk \\
&\qquad\qquad \text{for } H_1 \le z \le H_1 + H_2 \\
\varphi_3 &= \int_0^\infty F(k)e^{-i\nu_3 z}\, dk \qquad \text{for } H_1 + H_2 < z
\end{aligned} \tag{4-103}$$

where

$$\nu_i = -i\sqrt{k^2 - k_{\alpha_i}^2}, \qquad \text{for } k > k_\alpha \tag{4-104}$$

The conditions (4–102) are satisfied if by (4-103)

$$\begin{aligned}
A \sin \nu_1 h &= B \sin \nu_1 h + C \cos \nu_1 h \\
A \cos \nu_1 h - B \cos \nu_1 h + C \sin \nu_1 h &= \frac{2k}{\nu_1}
\end{aligned} \tag{4-105}$$

Hence

$$\begin{aligned}
A - B &= \frac{2k}{\nu_1}\cos \nu_1 h \\
C &= \frac{2k}{\nu_1}\sin \nu_1 h
\end{aligned} \tag{4-106}$$

and the number of unknown factors $A, B, C, D, E, F, \cdots$ is reduced to four. To determine these factors we substitute (4–103) into the remaining boundary conditions (4–19) and (4–101). Putting

$$\frac{\rho_1}{\rho_2} = \delta_1 \qquad \frac{\rho_2}{\rho_3} = \delta_2 \tag{4-107}$$

we obtain four equations

$$\begin{aligned}
&\delta_1 \sin \nu_1 H_1\, B - \sin \nu_2 H_1\, D - \cos \nu_2 H_1\, E = -\frac{2\delta_1 k}{\nu_1}\sin \nu_1 h \cos \nu_1 H \\
&\nu_1 \cos \nu_1 H_1\, B - \nu_2 \cos \nu_2 H_1\, D + \nu_2 \sin \nu_2 H_1\, E = 2k \sin \nu_1 h \sin \nu_1 H_1 \\
&\delta_2 \sin \nu_2(H_1 + H_2)\, D + \delta_2 \cos \nu_2(H_1 + H_2)\, E - e^{-i\nu_3(H_1+H_2)} F = 0 \\
&\nu_2 \cos \nu_2(H_1 + H_2)\, D - \nu_2 \sin \nu_2(H_1 + H_2)\, E + i\nu_3 e^{-i\nu_3(H_1+H_2)} F = 0
\end{aligned} \tag{4-108}$$

Substituting the factors $A, \cdots, F$ into the expressions (4–103) and using the notations

$$S = \frac{\delta_2 \nu_3 \tan \nu_2 H_2 - i\nu_2}{\delta_2 \nu_3 + i\nu_2 \tan \nu_2 H_2} \tag{4–109}$$

and

$$V = \delta_1 \nu_2 \sin \nu_1 H_1 + \nu_1 S \cos \nu_1 H_1 \tag{4–110}$$

we obtain the solutions

$$\varphi_1' = 2 \int_0^\infty J_0(kr) \frac{\sin \nu_1 z}{\nu_1 V} [S\nu_1 \cos \nu_1(H_1 - h) + \delta_1 \nu_2 \sin \nu_1(H_1 - h)]k \, dk \tag{4–111}$$

$$\varphi_1'' = 2 \int_0^\infty J_0(kr) \frac{\sin \nu_1 h}{\nu_1 V} [S\nu_1 \cos \nu_1(H_1 - z) + \delta_1 \nu_2 \sin \nu_1(H_1 - z)]k \, dk \tag{4–112}$$

$$\varphi_2 = 2\delta_1 \int_0^\infty J_0(kr) \frac{\sin \nu_1 h}{V} [S \cos \nu_2(z - H_1) - \sin \nu_2(z - H_1)]k \, dk \tag{4–113}$$

$$\varphi_3 = 2\delta_1 \delta_2 \int_0^\infty J_0(kr) \frac{\sin \nu_1 h}{V} [S \cos \nu_2 H_2 - \sin \nu_2 H_2] e^{-i\nu_3(z - H_1 - H_2)} k \, dk \tag{4–114}$$

The integrands in (4–111) to (4–114) are even functions of ν_1 and ν_2, and therefore the corresponding branch line integrals will vanish.

We now discuss one of these solutions, e.g., (4–112), in more detail. Since there is one branch line integral which corresponds to the branch point $k_{a3} = \omega/\alpha_3$, we can write, as in (4–54),

$$\varphi_1'' = \int_{\mathcal{L}} - 2\pi i \sum \text{Res} \tag{4–115}$$

The residues are determined by the roots of the frequency equation, which is, by (4–110),

$$V = 0 \qquad \text{or} \qquad \tan \nu_1 H_1 = -\nu_1 S/\nu_2 \tag{4–116}$$

Again use has to be made of real roots only.

As before, we replace the Bessel function $J_0(kr)$ by the Hankel function $H_0^{(2)}(kr)$, and, by similar transformations used in Sec. 4–2, we obtain the branch line integral in the form

$$\int_{\mathcal{L}} = \int_0^{-i\infty} H_0^{(2)}(\zeta r) N \zeta \, d\zeta + \int_{k_{a3}}^0 H_0^{(2)}(\zeta r) N \zeta \, d\zeta \tag{4–117}$$

where

$$N = \frac{\sin \nu_1 h}{\nu_1 V} [\nu_2 \delta_1 \sin \nu_1(H_1 - z) + \nu_1 S \cos \nu_1(H_1 - z)] \Big|_{\nu_3}^{-\nu_3} \tag{4–118}$$

As to the wave trains represented by residue terms, we find on applying the same rules we used in Sec. 4–2 that they are given by the sum

$$-2\pi i e^{i\omega t} \sum_n H_0^{(2)}(k_n r) k_n \frac{\sin \bar{\nu}_1 h}{\bar{\nu}_1} \frac{\bar{\nu}_1 S \cos \bar{\nu}_1(H_1 - z) + \bar{\nu}_2 \delta_1 \sin \bar{\nu}_1(H_1 - z)}{(\partial V/\partial k)_n} \tag{4–119}$$

The frequency equation (4–116) takes different forms for $c > \alpha_2$ or $c < \alpha_2$.

Now, if $c = \alpha_3$ or $k = k\alpha_3$, we shall have $\bar{\nu}_3 = 0$ [see Eq. (4–104)]. Then by Eqs. (4–116) and (4–109)

$$\tan \bar{\nu}_1 H_1 \tan \bar{\nu}_2 H_2 = \frac{\bar{\nu}_1}{\delta_1 \bar{\nu}_2} \tag{4–120}$$

from which the cutoff frequencies can be computed.

Since by (4–116) the phase velocity c is a function of frequency or of k, the group velocity of each of the normal modes for a three-layered liquid half space may be computed. Pekeris has shown that under certain conditions the group-velocity curve now has two minima.

Some results of Pekeris' calculations are presented in Fig. 4–15, where group-velocity curves for a three-layered half space are shown for the three cases indicated. In case (*a*), where $H_2/H_1 = 0.1$, $\alpha_2/\alpha_1 = 1.1$, $\alpha_3/\alpha_1 = 3.0$, the second layer is thin, and its properties are nearly the same as those of the surface layer. Comparison with Fig. 4–9 shows that the group-velocity curve could be approximated by treating these two layers as a unit having the same composition as the upper layer. For case (*c*), where $H_2/H_1 = 10$, we may approximate the high-frequency end of the group-velocity curve by considering $H_2 = \infty$ and the low-frequency end by considering $H_1 = 0$. In cases such as (*b*), where the layers are of comparable thickness, the complete three-layer calculation must be used except for very high modes where any problem may be approximated by considering two layers at a time. The case for which the intermediate layer has a lower sound velocity than the first layer was investigated by Press and Ewing [118]. They were interested in the fact that in some areas the "water wave" consisted of a brief burst of high-frequency sound which did not show the dispersion and Airy phase normally found. Considering the possibility that a low-velocity sea bottom could account for this phenomenon, they determined the phase- and group-velocity curves for a three-layered liquid half space with $\alpha_2 < \alpha_1 < \alpha_3$ and $H_2 = 5H_1$.

In the first mode there is found a curve not unlike that which would be obtained if $H_1 = 0$, that is, it has a low-frequency cutoff at $U = c = \alpha_3$ and a minimum group velocity, and it approaches α_2 asymptotically at high frequencies. Only the low-frequency branch of this curve was observed experimentally, absorption of high-frequency sound or possibly sublayering

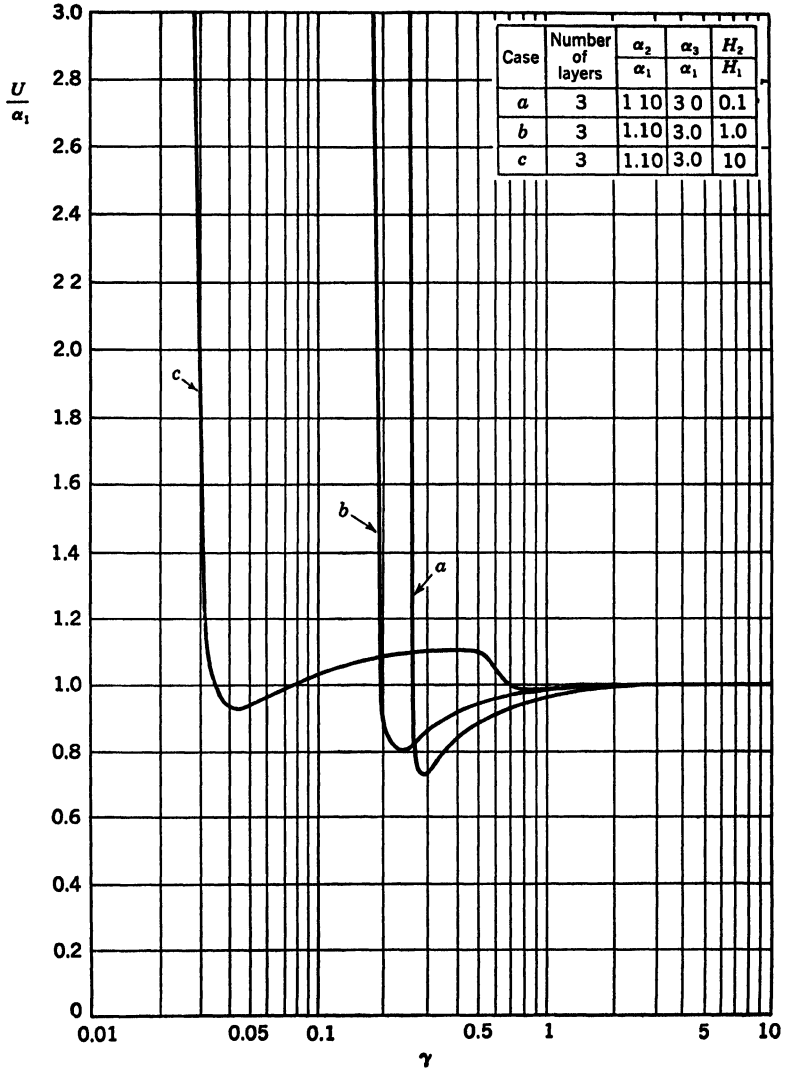


FIG. 4–15. Group-velocity curve, that is, U/α_1, as a function of γ for the first mode in a three-layered half space. (*After Pekeris.*)

in the α_2 layer being suggested as an explanation for the absence of the high-frequency branch. To deduce the properties of the direct waves, which traveled at velocity α_1, the authors proposed that reflections at the H_1H_2 interface at grazing incidence could explain the retention of high-frequency energy in the water layer.

Officer [109] has derived the frequency equation and solutions for the three-liquid half space by the use of rays and plane-wave reflection and transmission coefficients. He found that the frequency equation expressed the condition for constructive interference between the primary wave P_1 in Fig. 4–16 and the sum of the multiply reflected and transmitted waves P_2, P_3, P_4, $\cdots$ in a plane normal to a wave front.

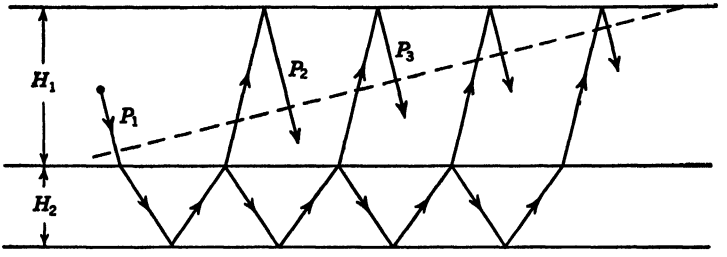


FIG. 4–16. Ray interpretation of period equation for three-layered half space.

4–4. Liquid Layer on a Solid Bottom. As in all other problems treated in this chapter, we assume that the boundaries are parallel planes and consider the problem of propagation of a disturbance in a medium which is composed of a liquid layer and of an underlying solid half space. This problem applies directly to the propagation of earthquake surface waves across the ocean and to the transmission of explosion sound in shallow-water areas when the bottom is solid. Calculations will be given for a source of compressional waves located in either the liquid or the solid.

To investigate the effect of the ocean on transmission of Rayleigh waves, Stoneley [193], using the theory of plane waves, calculated phase and group velocities for a water layer assumed to be 3 km thick over a solid substratum. He confined his attention to the longer-period Rayleigh waves and concluded that the effect of the water layer was unimportant. In a note added to that paper, Jeffreys proved from Stoneley's equation that there exists a minimum of group velocity at some period shorter than thqse investigated by Stoneley. Sezawa [162] obtained an approximate solution for the propagation of cylindrical waves, provided that the wave length was great compared with the water depth, but he also neglected the shorter waves which are prominent on many seismograms. Scholte [152], while attempting to explain microseism generation by transfer of

energy from gravity surface waves to elastic waves in the bottom, considered the combined effects of gravity and compressibility in a layer of water in contact with an elastic-solid bottom. Not being concerned with details of horizontal transmission, he gave no mention of dispersion or group velocity. Press and Ewing [117], in a study of microseisms, presented curves of phase and group velocity for the first and second normal modes for plane waves in a liquid layer superposed on a solid bottom. Later, in a search for Airy phases from submarine earthquakes, they extended the theory in order to include the case of an impulsive point source of compressional waves located within the solid bottom [120]. For application to transmission of explosion sound in shallow water they gave calculations for an impulsive source within the liquid layer. Ewing and Press [39, 43] used this theory to explain some features of the propagation of Rayleigh waves across ocean areas. Longuet-Higgins [90] applied a similar theory to the horizontal transmission of microseismic energy across the oceans.

Compressional-wave Source in the Solid Substratum. Following Press and Ewing [120] we use the notations of Sec. 4–2 and Fig. 4–17. Introduce

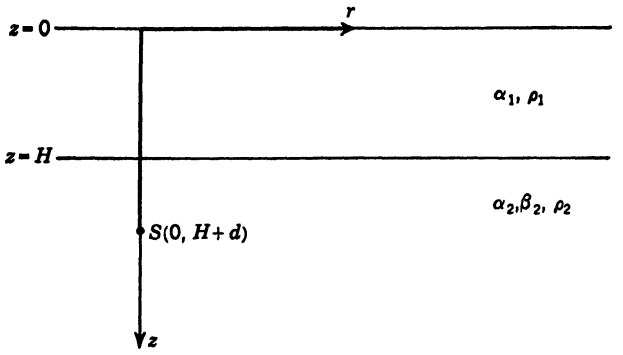


FIG. 4–17. Compressional-wave source in solid substratum.

the two velocities α_2 and β_2 for the propagation of waves in the solid layer. The displacements are expressed in terms of the potentials φ_1, φ_2, and ψ_2. Thus

$$q_1 = \frac{\partial \varphi_1}{\partial r} \qquad w_1 = \frac{\partial \varphi_1}{\partial z}$$

$$q_2 = \frac{\partial \varphi_2}{\partial r} + \frac{\partial^2 \psi_2}{\partial z\, \partial r} \qquad w_2 = \frac{\partial \varphi_2}{\partial z} + \frac{\partial^2 \psi_2}{\partial z^2} + k_{\beta_2}^2 \psi_2 \tag{4–121}$$

with the boundary conditions

$$\varphi_1 = 0 \qquad \text{at } z = 0 \tag{4-122}$$

$$w_1 = w_2 \qquad \text{at } z = H \tag{4-123}$$

$$(p_{zr})_2 = 0 \qquad \text{at } z = H \tag{4-124}$$

$$(p_{zz})_1 = (p_{zz})_2 \qquad \text{at } z = H \tag{4-125}$$

In order to satisfy condition (4–122), we can put, as in Sec. 4–2,

$$\varphi_1 = \int_0^\infty A(k) J_0(kr) \sin \nu_1 z \, dk \qquad \text{for } 0 \leq z \leq H \tag{4-126}$$

where

$$\nu_1^2 = k_{\alpha 1}^2 - k^2 \tag{4-127}$$

A time factor $\exp(i\omega t)$ is understood.

With a slight change of notation we can use the expression (4–16) to represent spherical waves emitted by a point source at $r = 0$, $z = H + d = h$. Thus,

$$\begin{aligned} \varphi_2 &= \int_0^\infty \frac{k}{i\nu_2} e^{-i\nu_2|z-h|} J_0(kr)\, dk + \int_0^\infty Q_2(k) J_0(kr) e^{-i\nu_2 z}\, dk \\ \psi_2 &= \int_0^\infty S_2(k) J_0(kr) e^{-i\nu_2' z}\, dk \end{aligned} \tag{4-128}$$

where

$$\nu_2^2 = k_{\alpha 2}^2 - k^2 \qquad \nu_2'^2 = k_{\beta 2}^2 - k^2 \tag{4-129}$$

If $z < H + d$, we have

$$\begin{aligned} \varphi_2 &= \int_0^\infty \frac{k}{i\nu_2} e^{i\nu_2(z-H-d)} J_0(kr)\, dk + \int_0^\infty Q_2 J_0(kr) e^{-i\nu_2 z}\, dk \\ \psi_2 &= \int_0^\infty S_2 J_0(kr) e^{-i\nu_2' z}\, dk \end{aligned} \tag{4-130}$$

The boundary conditions (4–123) to (4–125) at $z = H$ now take the form

$$\nu_1 A \cos \nu_1 H = k e^{-i\nu_2 d} + k^2 S_2 e^{-i\nu_2' H} - i\nu_2 Q_2 e^{-i\nu_2 H} \tag{4-131}$$

$$2i\nu_2 Q_2 e^{-i\nu_2 H} + (\nu_2'^2 - k^2) S_2 e^{-i\nu_2' H} = 2k e^{-i\nu_2 d} \tag{4-132}$$

$$\frac{\lambda_1}{\alpha_1^2} \frac{\partial^2 \varphi_1}{\partial t^2} = \frac{\lambda_2}{\alpha_2^2} \frac{\partial^2 \varphi_2}{\partial t^2} + 2\mu_2 \left[\frac{\partial^2 \varphi_2}{\partial z^2} + \frac{\partial^3 \psi_2}{\partial z^3} + k_{\beta 2}^2 \frac{\partial \psi_2}{\partial z} \right] \tag{4-133}$$

Equations (4–125) and (3–106) and the wave equation have been used in

deriving Eq. (4–133). After a rearrangement of terms, (4–133) takes the form

$$\rho_1\omega^2 A \sin \nu_1 H + (2\mu_2 k^2 - \rho_2\omega^2)Q_2 e^{-i\nu_2 H} - 2\mu_2 k^2 i\nu_2' S_2 e^{-i\nu_2' H} = -\frac{2\mu_2 k^2 - \rho_2\omega^2}{i\nu_2} k e^{-i\nu_2 d} \tag{4–134}$$

The determinant of the three equations (4–131), (4–132), and (4–134) is

$$\Delta = \begin{vmatrix} \nu_1 \cos \nu_1 H & i\nu_2 & -k^2 \\ 0 & 2i\nu_2 & \nu_2'^2 - k^2 \\ \rho_1\omega^2 \sin \nu_1 H & 2\mu_2 k^2 - \rho_2\omega^2 & -2\mu_2 k^2 i\nu_2' \end{vmatrix} \tag{4–135}$$

and the values of A, Q_2, S_2 in terms of k and other parameters can be found if $\Delta \neq 0$.

Using $\mu_2 = \rho_2\beta_2^2$, we can write for (4–135)

$$\Delta(k) = \frac{\rho_1\omega^4 i\nu_2}{\beta_2^2} \sin \nu_1 H + \nu_1\rho_2\beta_2^2[4k^2\nu_2\nu_2' + (2k^2 - k_{\beta 2}^2)^2] \cos \nu_1 H \tag{4–136}$$

and the coefficients A, Q_2, S_2 are given by the expressions

$$A = -\frac{2\rho_2\omega^2(2k^2 - k_{\beta 2}^2)k}{\Delta} e^{-i\nu_2 d} \tag{4–137}$$

$$Q_2 = \frac{k}{i\nu_2\Delta} e^{i\nu_2(H-d)}\left\{\frac{\rho_1\omega^4 i\nu_2}{\beta_2^2} \sin \nu_1 H + \nu_1\rho_2\beta_2^2[4k^2\nu_2\nu_2' - (2k^2 - k_{\beta 2}^2)^2] \cos \nu_1 H\right\} \tag{4–138}$$

$$S_2 = -\frac{4k(2\mu_2 k^2 - \rho_2\omega^2)\nu_1 \cos \nu_1 H}{\Delta} e^{-i\nu_2 d + i\nu_2' H} \tag{4–139}$$

Now write (4–138) in the form

$$Q_2 = \frac{k}{i\nu_2} e^{i\nu_2(H-d)} - 2\frac{k(2k^2 - k_{\beta 2}^2)^2\nu_1}{i\nu_2\Delta} e^{i\nu_2(H-d)}\rho_2\beta_2^2 \cos \nu_1 H \tag{4–140}$$

and substitute in (4–128) to obtain

$$\varphi_2 = \int_0^\infty \frac{k}{i\nu_2} e^{-i\nu_2|z-H-d|} J_0(kr)\, dk + \int_0^\infty \frac{k}{i\nu_2} e^{-i\nu_2(z-H+d)} J_0(kr)\, dk - 2\int_0^\infty \frac{k}{i\nu_2} \frac{\nu_1\rho_2\beta_2^2 \cos \nu_1 H\,(2k^2 - k_{\beta 2}^2)^2}{\Delta} e^{-i\nu_2 d} e^{-i\nu_2(z-H)} J_0(kr)\, dk \tag{4–141}$$

The second term may be interpreted as a spherical wave emitted by the image of the source in the interface. Its simple form is exp $(-ik\alpha_2 R')/R'$, where $R'^2 = r^2 + (z - H + d)^2$.

The first two terms in Eq. (4–141) may be combined as follows:

$$2\int_0^\infty \frac{k}{i\nu_2}\cos\nu_2(z-H)e^{-i\nu_2 d}J_0(kr)\,dk \qquad \text{for } H \le z \le H+d \tag{4–142}$$

or

$$2\int_0^\infty \frac{k}{i\nu_2}e^{-i\nu_2(z-H)}\cos\nu_2 d\,J_0(kr)\,dk \qquad \text{for } H+d \le z < \infty \tag{4–143}$$

The functions φ_1 and ψ_2 are given by Eqs. (4–126) and (4–128), where the expressions (4–137) and (4–139) have to be inserted for A and S_2.

By Eqs. (4–121), (4–126), and (4–128) with (4–137) to (4–139) we can find the displacements q and w. Thus for the solid bottom $z = H$ we make use of φ_2, given by (4–142) and the third integral in (4–141), and ψ_2 by (4–128) to obtain, on putting

$$\Delta(k) = \Upsilon(k)\rho_2\beta_2^2\nu_1\cos\nu_1 H \tag{4–144}$$

two expressions for the displacement at the interface:

$$q_H = -2k_{\beta 2}^2\int_0^\infty \frac{k^2}{\Upsilon(k)}\left[\frac{\rho_1}{\rho_2}\frac{k_{\beta 2}^2}{\nu_1}\tan\nu_1 H - 2i\nu_2'\right]e^{-i\nu_2 d}J_1(kr)\,dk \tag{4–145}$$

$$w_H = -2k_{\beta 2}^2\int_0^\infty \frac{2k^2-k_{\beta 2}^2}{\Upsilon(k)}e^{-i\nu_2 d}J_0(kr)k\,dk \tag{4–146}$$

As before, the integrals can be represented by the sum of branch line integrals and residues. The residues correspond to the poles $k = \kappa_n$ given by the roots of the equation

$$\Upsilon(k) = 0 \tag{4–147}$$

As we have seen, the amplitudes of waves determined by branch line integrals diminish as r^{-2}. Since we are interested in an approximation which holds for large values of r, the terms corresponding to branch points, which by analogy with earlier results represent waves with phase velocities α_2 and β_2, are left out of consideration. Then only the residues are computed by the methods used earlier, and asymptotic values of displacements are obtained as follows (the time factor is written again):

$$q_H = \frac{2}{H^2}\sqrt{\frac{2\pi}{r}}\sum_n \frac{1}{\sqrt{\kappa_n}}Q(\kappa_n)e^{-i\nu_{2n}d}e^{i(\omega t-\kappa_n r+\pi/4)} \tag{4–148}$$

$$w_H = \frac{2}{H^2}\sqrt{\frac{2\pi}{r}}\sum_n \frac{1}{\sqrt{\kappa_n}}W(\kappa_n)e^{-i\nu_{2n}d}e^{i(\omega t-\kappa_n r-\pi/4)} \tag{4–149}$$

where the phase velocity c_n and the factor ν_{2n} for each mode are given by

$$c_n = \frac{\omega}{\kappa_n} \qquad \nu_{2n}^2 = k_{\alpha 2}^2 - \kappa_n^2 \tag{4–150}$$

and

$$Q(\kappa_n) = \frac{\kappa_n^2 H^2}{R(\kappa_n)} \frac{c_n^2}{\beta_2^2} \left[\frac{\rho_1}{\rho_2} \frac{c_n^2}{\beta_2^2} \frac{\sqrt{1 - c_n^2/\alpha_2^2}}{\sqrt{c_n^2/\alpha_1^2 - 1}} \tan\left(\kappa_n H \sqrt{\frac{c_n^2}{\alpha_1^2} - 1}\right) - 2\sqrt{1 - \frac{c_n^2}{\alpha_2^2}} \sqrt{1 - \frac{c_n^2}{\beta_2^2}} \right] \tag{4-151}$$

$$W(\kappa_n) = \frac{\kappa_n^2 H^2}{R(\kappa_n)} \frac{c_n^2}{\beta_2^2} \left(2 - \frac{c_n^2}{\beta_2^2}\right) \sqrt{1 - \frac{c_n^2}{\alpha_2^2}} \tag{4-152}$$

with

$$\begin{aligned} R(\kappa_n) = \frac{\rho_1}{\rho_2} \frac{c_n^4}{\beta_2^4} \Bigg[& \left(1 + \frac{1 - c_n^2/\alpha_2^2}{c_n^2/\alpha_1^2 - 1}\right) \frac{\tan \kappa_n H \sqrt{c_n^2/\alpha_1^2 - 1}}{\sqrt{c_n^2/\alpha_1^2 - 1}} \\ & - \frac{\kappa_n H (1 - c_n^2/\alpha_2^2)}{c_n^2/\alpha_1^2 - 1} \sec^2 \kappa_n H \sqrt{\frac{c_n^2}{\alpha_1^2} - 1} \Bigg] \\ & - 4 \Bigg[\left(3 - 2\frac{c_n^2}{\alpha_2^2}\right) \sqrt{1 - \frac{c_n^2}{\beta_2^2}} + \frac{1 - c_n^2/\alpha_2^2}{\sqrt{1 - c_n^2/\beta_2^2}} \\ & - 2\left(2 - \frac{c_n^2}{\beta_2^2}\right) \sqrt{1 - \frac{c_n^2}{\alpha_2^2}} \Bigg] \end{aligned} \tag{4-153}$$

The period equation (4–147) can now be written in the dimensionless form

$$\tan\left(\kappa_n H \sqrt{\frac{c_n^2}{\alpha_1^2} - 1}\right) = \frac{\rho_2 \beta_2^4 \sqrt{c_n^2/\alpha_1^2 - 1}}{\rho_1 c_n^4 \sqrt{1 - c_n^2/\alpha_2^2}} \left[4\sqrt{1 - \frac{c_n^2}{\alpha_2^2}} \sqrt{1 - \frac{c_n^2}{\beta_2^2}} - \left(2 - \frac{c_n^2}{\beta_2^2}\right)^2 \right] \tag{4-154}$$

It defines as usual a relationship between the period $T = 2\pi/c_n\kappa_n = 2\pi/\omega$ and the phase velocity, with the elastic constants of the system as parameters. Each of the roots κ_n of (4–154) can be expressed in terms of ω or T.

As mentioned earlier, Schermann [151] proved that there is a finite number of real roots. If $-K$ is the right-hand member of (4–154) and if

$$\begin{aligned} (n - 1)\pi < H\sqrt{k_{\alpha 1}^2 - k^2} \leq n\pi \\ n\pi - \tan^{-1} K \leq H\sqrt{k_{\alpha 1}^2 - k^2} \end{aligned} \tag{4-155}$$

there are $2(n + 1)$ real roots on the k axis symmetrically distributed with respect to the origin. In the case

$$\begin{aligned} (n - 1)\pi < H\sqrt{k_{\alpha 1}^2 - k^2} \leq n\pi \\ n\pi - \tan^{-1} K > H\sqrt{k_{\alpha 1}^2 - k^2} \end{aligned} \tag{4-156}$$

the number of roots reduces to $2n$.

A case commonly encountered in geophysical studies of water-covered areas, $\alpha_2 > \beta_2 \geq c \geq \alpha_1$, will now be discussed in detail.

The period equation (4–154) and Eq. (4–94) were used to obtain the phase velocity c and the group velocity U in terms of kH. Following Pekeris, we express c_n and κ_n in terms of a dimensionless parameter $\gamma = c_n\kappa_n H/2\pi\alpha_1$. In Figs. 4–18 and 4–19 are shown the results of computa-

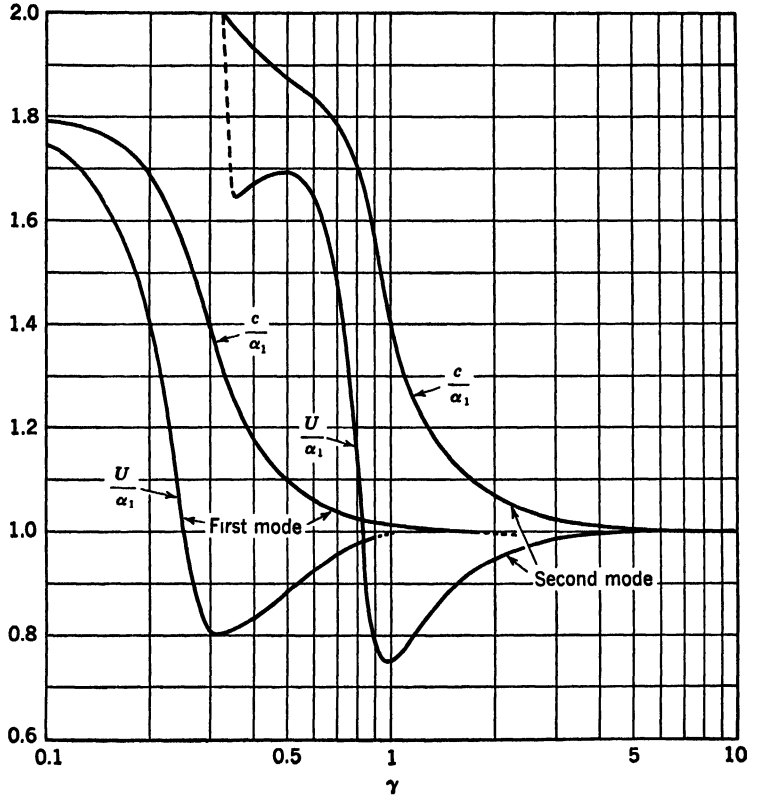


FIG. 4–18. Liquid layer over solid substratum. Phase- and group-velocity curves for first two modes when $\rho_2/\rho_1 = 2.5$, $\alpha_2/\beta_2 = \sqrt{3}$, $\beta_2/\alpha_1 = 2$.

tion for the numerical values: $\rho_2/\rho_1 = 2.5$, $\alpha_2 = \sqrt{3}\beta_2$, $\beta_2 = 2\alpha_1$, and $\rho_2/\rho_1 = 3.0$, $\alpha_2 = \sqrt{3}\beta_2$, $\beta_2 = 3\alpha_1$, which represent the approximate conditions for a granitic and basaltic ocean bottom, respectively.

The phase and group velocities of the first mode ($n = 1$) will approach the velocity c_R of Rayleigh waves in the solid layer as $\gamma \to 0$ or as the

wavelength becomes very long in comparison with the thickness of the first layer. As long as $c > \alpha_1$, the waves corresponding to this mode are termed first-mode generalized Rayleigh waves. For $c < \alpha_1$, ν_1 in Eq. (4–127) is imaginary, the amplitude in the first layer now decreases with distance above the interface, and the waves may appropriately be called Stoneley waves (see Sec. 3–3 and Fig. 4–18 where these waves are repre-

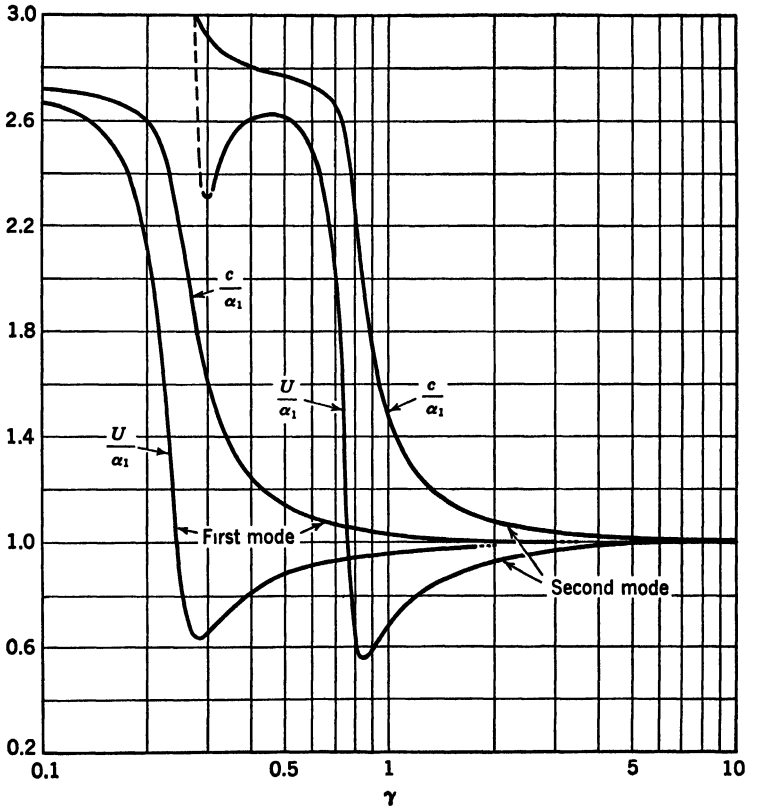


FIG. 4–19. Liquid layer over solid substratum. Phase- and group-velocity curves for first two modes when $\rho_2/\rho_1 = 3.0$, $\alpha_2/\beta_2 = \sqrt{3}$, $\beta_2/\alpha_1 = 3$.

sented by dots to the right of $c/\alpha_1 = 1$ on the first mode). Note that, in contrast with the results for liquid layers, the c/α_1 and U/α_1 curves extend over the range $0 \leq \gamma < \infty$ and that for large γ these ratios approach the Stoneley-wave velocity $0.998\alpha_1$. Coulomb [20] and Biot [6] discussed the

theory of propagation of Stoneley waves along the sea floor in greater detail.

In the second-mode generalized Rayleigh waves (Fig. 4–18) $c = U = \beta_2$ for a value of γ corresponding to a cutoff period. For longer periods, k is complex, and attenuation occurs with increasing distance, corresponding to radiation of energy into the bottom. As c and U approach α_{1_h} the periods become infinitely small. There can exist higher modes of propagation ($n = 3, 4, \cdots$), each having the same cutoff velocity $c = U = \beta_2$ but increasingly shorter cutoff periods. In general, the periods of the higher modes corresponding to a given phase or group velocity become progressively shorter. The group velocities are further characterized by stationary values, as shown in Fig. 4–19 for the first and second modes.

The solution for an arbitrary initial disturbance can now be written, as in previous cases, in the form

$$q_H = \frac{1}{H^2}\sqrt{\frac{2}{\pi r}} \sum_n \int_{-\infty}^{\infty} \frac{1}{\sqrt{\kappa_n}} Q(\kappa_n) g(\omega) e^{-i\gamma_n z_0} e^{i(\omega t - \kappa_n r + \pi/4)}\, d\omega \tag{4–157}$$

$$w_H = \frac{1}{H^2}\sqrt{\frac{2}{\pi r}} \sum_n \int_{-\infty}^{\infty} \frac{1}{\sqrt{\kappa_n}} W(\kappa_n) g(\omega) e^{-i\gamma_n z_0} e^{i(\omega t - \kappa_n r - \pi/4)}\, d\omega \tag{4–158}$$

where the Fourier transform $g(\omega)$ of the initial time variation $S(t)$ is given again by

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S(t) e^{-i\omega t}\, dt \tag{4–159}$$

Assuming that the impulse developed in an earthquake can be represented by the condition that $S(t)$ vanishes for all but infinitesimal values of t in such a manner that $\int_{-\infty}^{\infty} S(t)\, dt = A$, we shall put $g(\omega) = A$. Then Eqs. (4–157) and (4–158) take the form

$$q_H = \frac{A}{H^2}\sqrt{\frac{2}{\pi r}} \sum_n \int_{-\infty}^{\infty} \frac{1}{\sqrt{\kappa_n}} Q(\kappa_n) e^{-i\gamma_n z_0} e^{i(\omega t - \kappa_n r + \pi/4)}\, d\omega \tag{4–160}$$

$$w_H = \frac{A}{H^2}\sqrt{\frac{2}{\pi r}} \sum_n \int_{-\infty}^{\infty} \frac{1}{\sqrt{\kappa_n}} W(\kappa_n) e^{-i\gamma_n z_0} e^{i(\omega t - \kappa_n r - \pi/4)}\, d\omega \tag{4–161}$$

Approximations for these integrals can be obtained by using Kelvin's method of stationary phase, as discussed in Appendix A.

Only the final results are now given. We find for $d(U_n/\alpha_1)/d\gamma > 0$

$$q_H = -\frac{4\alpha_1 A}{H^2 r} \sum_n L_n Q(\kappa_n) e^{-i\gamma_n z_0} \sin(\omega_0 t - \kappa_n r) \tag{4–162}$$

$$w_H = \frac{4\alpha_1 A}{H^2 r} \sum_n L_n W(\kappa_n) e^{-i\gamma_n z_0} \cos(\omega_0 t - \kappa_n r) \tag{4–163}$$

where

$$L_n = \left(\frac{\gamma\alpha_1}{c_n}\frac{\alpha_1^2}{U_n^2}\left|\frac{d(U_n/\alpha_1)}{d\gamma}\right|\right)^{-\frac{1}{2}}$$

and for $d(U_n/\alpha_1)/d\gamma < 0$

$$q_H = \frac{4\alpha_1 A}{H^2 r}\sum_n L_n Q(\kappa_n) e^{-\gamma\nu_{2nd}} \cos(\omega_0 t - \kappa_n r) \tag{4-164}$$

$$w_H = \frac{4\alpha_1 A}{H^2 r}\sum_n L_n W(\kappa_n) e^{-\gamma\nu_{2nd}} \sin(\omega_0 t - \kappa_n r) \tag{4-165}$$

These expressions hold for a large r and for t sufficiently removed from a time corresponding to propagation with a stationary value of group velocity. The train of waves corresponding to this group velocity is called the Airy phase. The final expressions for this phase as given by Press, Ewing, and Tolstoy [119] are

$$q_H = \frac{4\alpha_1 A}{3^{2/3}H^2 r^{5/6}}\sum_n L_n' Q(\kappa_n) E(m) e^{-\gamma\nu_{2nd}} \cos\left(\omega_0 t - \kappa_n r + \frac{\pi}{4}\right) \tag{4-166}$$

$$w_H = \frac{4\alpha_1 A}{3^{2/3}H^2 r^{5/6}}\sum_n L_n' W(\kappa_n) E(m) e^{-\gamma\nu_{2nd}} \cos\left(\omega_0 t - \kappa_n r - \frac{\pi}{4}\right) \tag{4-167}$$

where

$$\begin{aligned} E(m) &= m^{\frac{1}{2}}[J_{-\frac{1}{3}}(m) + J_{\frac{1}{3}}(m)] \quad \text{for } t > \frac{r}{U_{\max}} \quad \text{or} \quad \frac{r}{U_{\min}} > t \\ E(m) &= m^{\frac{1}{2}}[I_{-\frac{1}{3}}(m) - I_{\frac{1}{3}}(m)] \quad \text{for } t < \frac{r}{U_{\max}} \quad \text{or} \quad \frac{r}{U_{\min}} < t \end{aligned} \tag{4-168}$$

with

$$m = \frac{4\sqrt{\pi}}{3\sqrt{|dZ/d\gamma|}}\frac{r}{H}|y - y_M|^{\frac{3}{2}} \qquad Z = -\left[\frac{\alpha_1}{2\pi U^2}\frac{dU}{d\gamma}\right] \tag{4-169}$$

$$y = \frac{t\alpha_1}{r} - 1 \qquad y_M = \frac{\alpha_1}{U_M} - 1 \tag{4-170}$$

The second expression in (4-169) vanishes at a stationary value of group velocity. But only the derivatives of Z are involved in the results. The subscript M denotes that a function is to be evaluated at $U_{\max}$ or $U_{\min}$ and

$$L_n' = \sqrt{\frac{2\pi}{\kappa_n}}\left[\frac{H^2}{2\pi}\left|\left(\frac{dZ}{d\gamma}\right)_M\right|\right]^{-\frac{1}{2}} \tag{4-171}$$

These expressions have been used to compute the amplitudes of the component displacements at the bottom for an impulse produced by a

point source of compressional waves located at a depth $h = H$ or $h = 0$, and at distances of 1,000 km and 10,000 km, respectively. Figures 4–20 and 4–21 were given for $\rho_2/\rho_1 = 2.5$, $\alpha_2 = \sqrt{3}\beta_2$, $\beta_2 = 3\alpha_1$, $H = 5$ km. Discussion of the features of seismograms which correspond to various parts of these curves will be deferred to the section where the case of

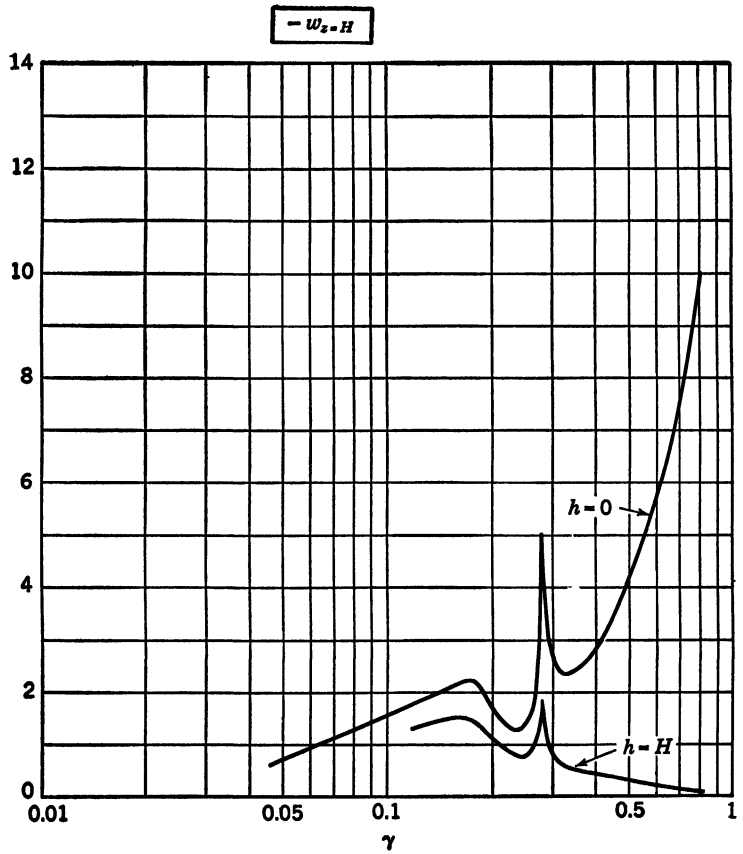


FIG. 4–20. Liquid layer over solid substratum. Vertical displacement $w_{z=H}$ in units $4\alpha_1 AH^{-2}\ 10^{-7}$ for first mode as a function of γ at a range of 1,000 km; $\rho_2 = 2.5\ \rho_1$, $\alpha_2 = \sqrt{3}\ \beta_2$, $\beta_2 = 3\alpha_1$.

a sound source in the liquid layer is treated. The only difference will be in the amplitude functions.

Suboceanic Rayleigh Waves: First Mode. The theory developed in this section can be applied to earthquake Rayleigh waves propagated along a

path that is largely oceanic. It resolves the problem of the "coda," which has long been considered an unsolved problem [e.g., Jeffreys (Chap. 2, Ref. 17, pp. 99–100)].

We reproduce first in Fig. 4–22 the Palisades records from the earthquake of Aug. 12, 1953, in the Tonga Islands. The seismograph has three matched

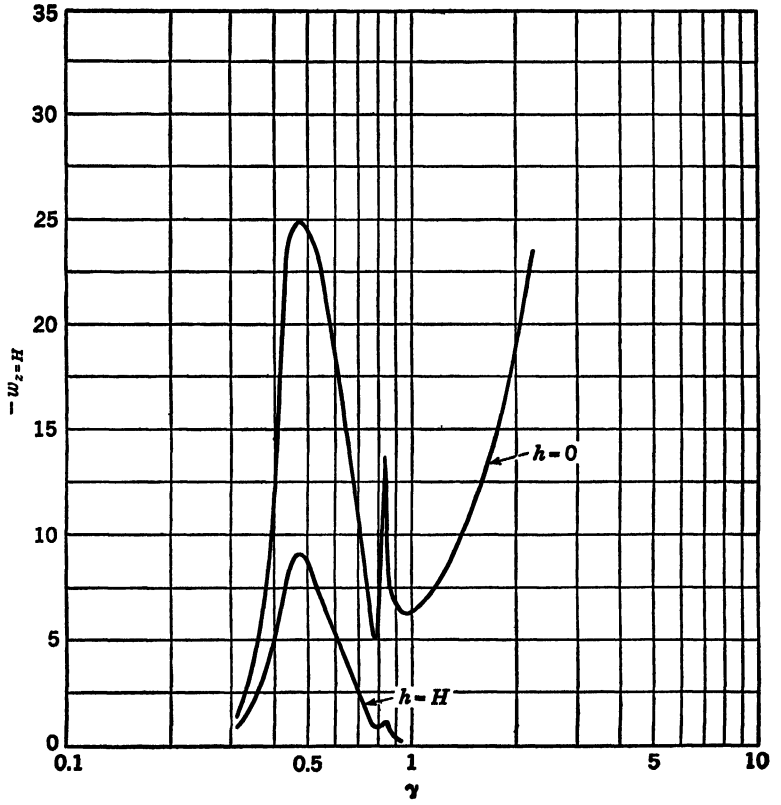


FIG. 4–21. Liquid layer over solid substratum. Vertical displacement for second mode as a function of γ at a range of 1,000 km (see Fig. 4–20).

components each having pendulum period $T_0 = 15$ sec and galvanometer period $T_g = 90$ sec. The path covers about 8,500 km in the Pacific Ocean and about 4,000 km across North America. The azimuth at Palisades is about 260°. The study of these seismograms shows that:

1. The orbital motion is proper for Rayleigh waves arriving from the west.

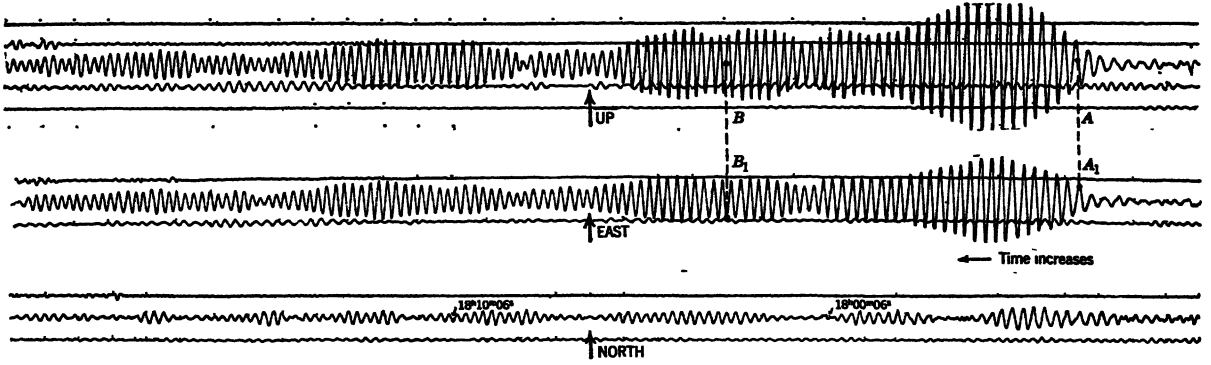


FIG. 4–22. Palisades seismograms for the Tonga earthquake of Aug. 12, 1953, epicentral distance 12,450 km. Note the phase relations at points A, A_1 and B, B_1.

2. The waves are markedly sinusoidal and clearly exhibit dispersion, the period decreasing from about 25 to about 16 sec. The decrease is very rapid at first and so gradual at the end that the period could be judged constant. The part of this wave train in which the period is almost constant has been called the coda.

Since dispersion can modify a pulse into a train of sinusoidal waves whose period varies gradually along the train, it is natural to attempt to explain the coda as a dispersion phenomenon. As will be shown later in this section, layering, such as that present in the crust, can produce dispersion in Rayleigh waves but the difficulty is that the lowest possible group velocity is only slightly less than the lowest shear-wave velocity in any layer. However, no rock layer was available with sufficiently low shear-wave velocity. Guided by experience on dispersion in explosion sound transmitted in shallow water, Ewing and Press [39, 41] reopened the question of the effect of the ocean water on propagation of Rayleigh waves across ocean basins. They found that for paths like those shown in Fig. 4–23, and having a substantial portion across ocean basins the Rayleigh-wave trains could be completely accounted for by the curves of Fig. 4–24 for dispersion over the oceanic and continental parts of the path.

An example of this method of analysis is provided by their study of the surface waves of the Solomon Islands earthquake of July 29, 1950, epicenter 6.8°S 155.1°E, depth 75 km±, origin time $23^h49^m08^s$, and magnitude 7 (Jesuit Seismological Association). Seismograms were studied from Honolulu, Berkeley, Tucson, and Palisades, which lie near a single great-circle path through the epicenter (Fig. 4–23). The direct waves (those along the minor arc of the great circle) were observed at all stations, and the inverse waves (coming along the major arc) were registered at all stations except Honolulu. For Honolulu and Berkeley the path for the direct waves may be considered entirely oceanic. In all cases the seismograms are generally similar to those reproduced in Fig. 4–22. The period of the wave varies gradually within the train, and arrival time can be determined as an empirical function of period. From these data, group velocity has been plotted as a function of period in Fig. 4–24. For the direct paths to Honolulu and Berkeley, group velocity across the ocean was calculated as the ratio of distance to travel time. For all other stations, the total travel time was corrected to represent the oceanic-path segment by subtraction of the time required for travel across the continental segment of the path. This continental travel time was computed by use of a dispersion curve derived by Wilson and Baykal [210] and refined by Brilliant and Ewing [11]. It will be discussed in more detail in Sec. 4–5.

The theoretical dispersion curve in Fig. 4–24 is derived from Eqs. (4–154) and (4–94), with the water depth $H = 5.7$ km, $\alpha_1 = 1.52$ km/sec, $\alpha_2 = 7.95$ km/sec, $\beta_2 = 4.56$ km/sec, and $\rho_2/\rho_1 = 3.0$. The excellent

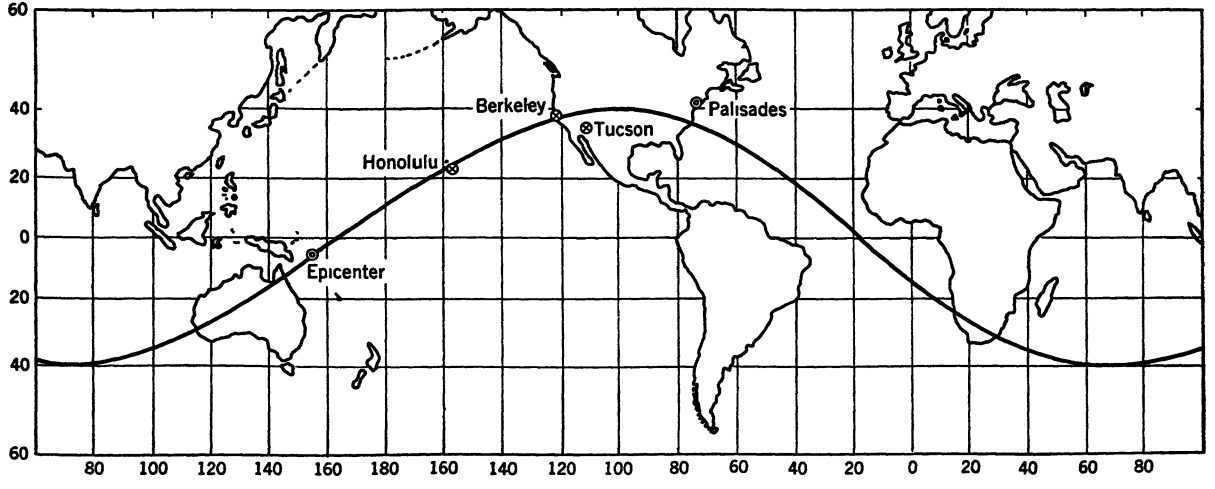


FIG. 4–23. Great-circle path from epicenter in Solomon Islands passing near Honolulu, Berkeley, Tucson, and Palisades stations.

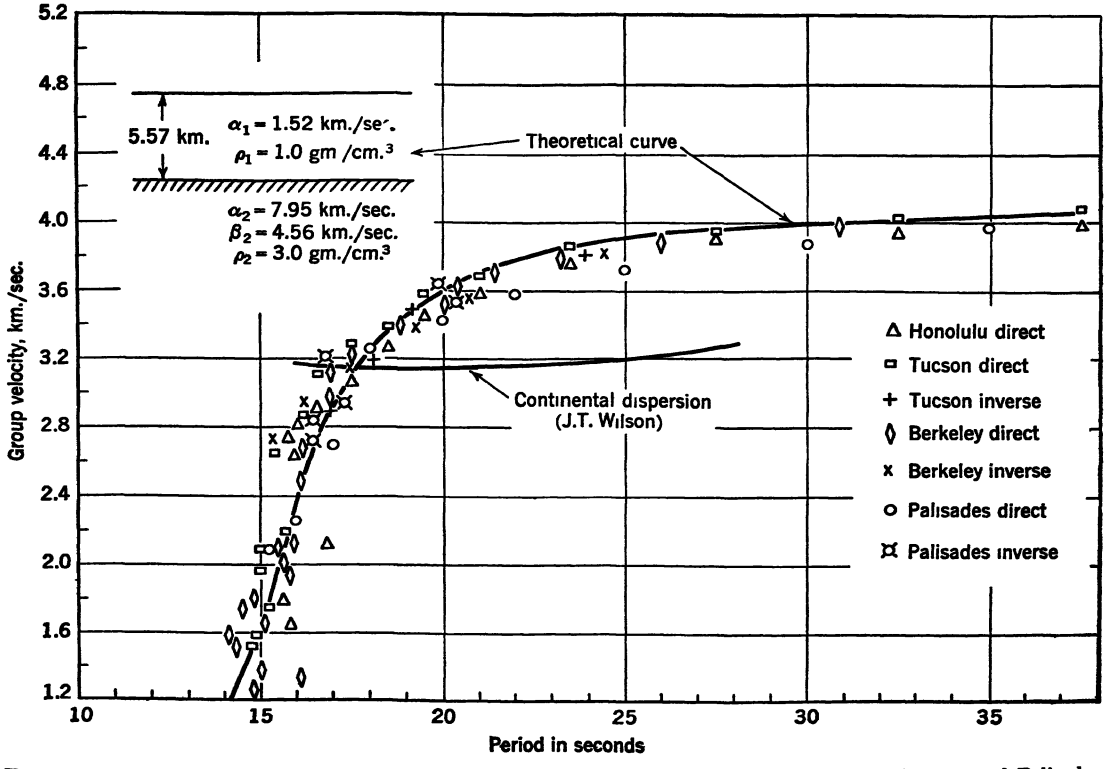


FIG. 4–24. Observed and theoretical dispersion curves for oceanic paths to Honolulu, Berkeley, Tucson, and Palisades.

agreement between this theoretical curve and the observed oceanic dispersion for direct and inverse paths demonstrates the essential identity in crustal structure of the Indian, Atlantic, and Pacific basins along this great-circle path. The liquid layer, depth 5.7 km, which must be used to obtain the best agreement between observation and theory, exceeds the mean depth of water in each basin by roughly 1 km, which may be taken as a measure of the mean thickness of the layer of unconsolidated sediments on the ocean floor along the selected great-circle path.

This representation of oceanic crustal structure involves two simplifications. The sediment of the ocean floor is included in the liquid layer, and the underlying rocks are also represented by a single layer whose properties are very near to those of the ultrabasic rock bounded by the Mohorovičić discontinuity, which lies at 30- to 40-km depth beneath the continents. In a later section, 4–6, it will be shown that the value $\alpha_2 = 7.95$ km/sec represents a sort of average of an actual structure consisting of 5 km of a layer with velocity 6.9 km/sec (basalt) and a very thick underlying layer in which $\alpha_3 = 8.1$ km/sec. The method of deducing arrival time as a function of period from the seismograms consists of numbering the peaks, troughs, and zeros of the wave train and plotting these numbers against travel time. The slope of this curve gives period as a function of travel time, from which the group velocity can be computed.

An objection has sometimes been raised to the idea that long trains of Rayleigh waves, such as those studied here, could occur solely through the effects of dispersion. The view has often been expressed that a long succession of sinusoidal waves of almost constant period was due to some resonant phenomenon. Likewise it has been maintained that the absence of long-period surface waves at the smaller epicentral distances precludes the possibility that dispersion can explain the appearance of the train at great distances. The portion of the coda in which the period seems to remain constant results from the steep portion of the dispersion curve where a large change in group velocity occurs for a very small change in period.

From any seismogram showing a good train of oceanic Rayleigh waves it is possible to estimate the mean thickness of the suboceanic sedimentary layer along the propagation path. Oliver, Ewing, and Press [111] applied this method to the Honolulu seismograms for earthquakes in the circumpacific belt. Their method consists in plotting the dispersion curve for each seismogram on a grid, such as that shown in Fig. 4–25, and subtracting the mean water depth from the liquid-layer thickness read from the grid. They found mean sediment thicknesses ranging from 0.4 km due north of Honolulu to nearly 1.2 km to the southwest and consider the probable error in their results less than 50 per cent.

A puzzling feature of Rayleigh-wave propagation in the oceans is the

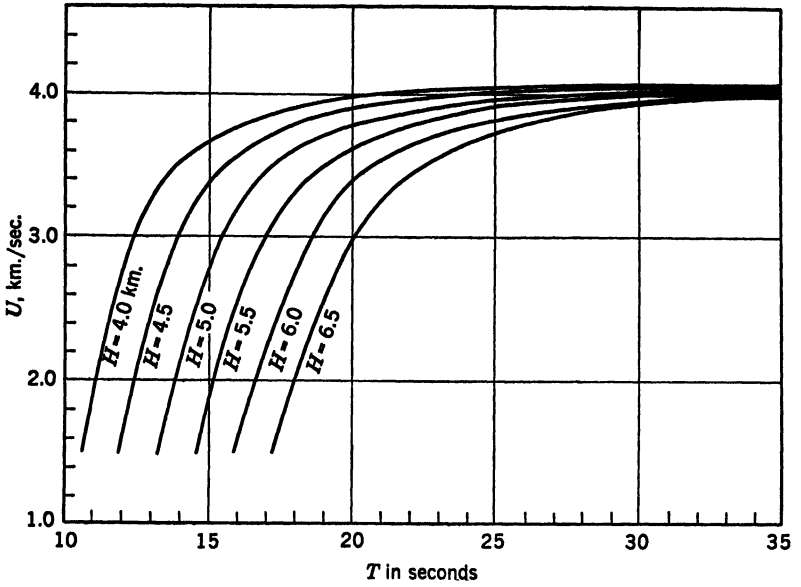


FIG. 4–25. Group velocity of Rayleigh waves as a function of period for various depths H of the liquid layer and for $\alpha_2 = 7.95$ km/sec, $\beta_2 = 4.56$ km/sec, $\alpha_1 = 1.52$ km/sec, $\rho_2 = 2\rho_1$.

absence of first-mode waves in the period range from about 1 to 12 sec. This represents a gap in the spectrum corresponding to the part of the dispersion curve lying to the right of the minimum of group velocity in Fig. 4–19. The important question is whether these waves are not generated by the source or are not transmitted across the ocean because of the effect of some factor not accounted for in the theory. Great depth of focus has been suggested as a possible explanation of the absence of Rayleigh waves with periods shorter than about 12 sec, and it may be seen from Fig. 4–20 that, in an ocean with a uniform depth of 5 km, a depth of focus several times greater than the water depth would reduce very strongly the amplitudes of these waves.

But it seems improbable that depth of focus is the correct explanation because the T phase, i.e., a train of waves with periods less than 1 sec which travels with the speed of sound in water (see Sec. 7–2), is strongly excited by most submarine earthquakes. Thus, even if there are some special conditions which help these short-period waves to enter the water, the 1- to 12-sec gap in the spectrum still remains to be explained. When continental Rayleigh waves, with large amplitudes in the period range 1 to 12 sec, reach the coast they do not continue out to sea as first-mode

Rayleigh waves. Thus it seems clear that first-mode Rayleigh waves in the period range 1 to 12 sec suffer very great attenuation in typical ocean areas. This fact severely limits the usefulness of microseisms for tracking or detecting distant storms, as will be seen later in this section. No gap in the spectrum occurs in propagation of Rayleigh waves in shallow water (see Fig. 4–12). In this seismogram short-period waves between the points marked "water wave" and "Airy phase" (minimum group velocity) are prominent.

It may be inferred from Figs. 4–20 and 4–21 that the Airy phase should contribute to seismograms of oceanic Rayleigh waves a prominent wave train with a period of about 12 sec and a group velocity of about 0.6 to 0.7 times the speed of sound in water. Despite a fairly thorough search, no waves which even roughly fit this description were found except on Bermuda seismograms of West Indian earthquakes (Press, Ewing, and Tolstoy [119]). Since these waves have not been observed elsewhere, an alternative explanation was sought. The observed travel time fits the hypothesis that they are Love waves reflected from the Grand Banks off Newfoundland.

Suboceanic surface waves with periods 6 to 12 sec are observed on seismograms from earthquakes in certain areas. They have the combined characteristics of Love waves and the second-mode Rayleigh waves and will be discussed in Sec. 4–5.

Compressional-wave Source in the Liquid Layer. We shall now consider the second case mentioned above where the point source is in the liquid layer (Press and Ewing [120]). The displacements are represented by Eqs. (4–121), and the boundary conditions are Eqs. (4–122) to (4–125). As in the problem of two liquid layers (Sec. 4–2), we have to make use of two different expressions for the potential φ_1 for liquid layers above and below the source. Thus we write, as before,

$$\varphi_1' = \int_0^\infty A(k)J_0(kr) \sin \nu_1 z \, dk \qquad \text{for } 0 \leq z \leq h \qquad (4\text{–}172)$$

$$\varphi_1'' = \int_0^\infty [B(k) \sin \nu_1 z + C(k) \cos \nu_1 z]J_0(kr) \, dk \qquad \text{for } h \leq z \leq H \qquad (4\text{–}173)$$

$$\varphi_2 = \int_0^\infty Q_2(k)e^{-i\nu_2 z}J_0(kr) \, dk \qquad (4\text{–}174)$$

$$\text{for } H \leq z$$

$$\psi_2 = \int_0^\infty S_2(k)e^{-i\nu_2' z}J_0(kr) \, dk \qquad (4\text{–}175)$$

Condition (4–122) at the free surface $z = 0$ is satisfied by the assumed form of (4–172).

Determining the functions A, B, C, Q_2, S_2 as before and using the determinant $\Delta(k)$ of (4–136), we have in this case

$$\varphi_1' = 2e^{i\omega t}\int_0^\infty J_0(kr)k\frac{\sin \bar{\nu}_1 z}{\bar{\nu}_1\,\Delta(k)}\left\{\frac{\rho_1\omega^4}{\beta_2^2}i\bar{\nu}_2\sin\bar{\nu}_1(H-h)\right.$$
$$\left.+\,\bar{\nu}_1\rho_2\beta_2^2[4k^2\bar{\nu}_2\bar{\nu}_2' + (2k^2 - k_{\beta 2}^2)^2]\cos\bar{\nu}_1(H-h)\right\}dk \qquad (4\text{–}176)$$

$$\varphi_1'' = 2e^{i\omega t}\int_0^\infty J_0(kr)k\frac{\sin \bar{\nu}_1 h}{\bar{\nu}_1\,\Delta(k)}\left\{\frac{\rho_1\omega^4}{\beta_2^2}i\bar{\nu}_2\sin\bar{\nu}_1(H-z)\right.$$
$$\left.+\,\bar{\nu}_1\rho_2\beta_2^2[4k^2\bar{\nu}_2\bar{\nu}_2' + (2k^2 - k_{\beta 2}^2)^2]\cos\bar{\nu}_1(H-z)\right\}dk \qquad (4\text{–}177)$$

$$\varphi_2 = -2e^{i\omega t}\int_0^\infty J_0(kr)k\frac{\sin \bar{\nu}_1 h}{\Delta(k)}\,\rho_1\omega^2(2k^2 - k_{\beta 2}^2)e^{-i\bar{\nu}_2'(z-H)}\,dk \qquad (4\text{–}178)$$

$$\psi_2 = -4e^{i\omega t}\int_0^\infty J_0(kr)k\frac{\sin \bar{\nu}_1 h}{\Delta(k)}\,\rho_1\omega^2 i\bar{\nu}_2 e^{-i\bar{\nu}_2'(z-H)}\,dk \qquad (4\text{–}179)$$

As a check we may use reciprocity considerations upon Eq. (4–178) which represents the compressional disturbance at $z > H$ from a compressional source $z = h < H$. Replacing z by $H + d$, h by z, and ρ_1 by ρ_2 in Eq. (4–178), we obtain the expression given by (4–126) and (4–137) for the disturbance in the first layer from a source in the second.

The integrals (4–176) to (4–179) can be evaluated by the methods applied before, and the solutions can again be expressed as the sum of the residues of the integrands and two integrals along branch lines corresponding to the two branch points $k = k_{\alpha 2} = \omega/\alpha_2$ and $k = k_{\beta 2} = \omega/\beta_2$. The residues which diminish as $r^{-\frac{1}{2}}$ yield the normal-mode solutions, whereas the branch line integrals diminish as r^{-2}. The approximate values of these integrals were evaluated by Honda and Nakamura [68].

The period equation for vanishing $\Delta(k)$ in (4–144) yields the roots κ_n of Eq. (4–147). A discussion of the period equation has been given earlier in this section.

Now, the normal-mode solutions can be written in their final form

$$\varphi_1' = \varphi_1'' = \frac{2}{H}\sqrt{\frac{2\pi}{r}}\sum_n \frac{1}{\sqrt{\kappa_n}}e^{i(\omega t - \kappa_n r - \pi/4)}$$
$$\cdot\,\Theta_1(\kappa_n)\sin\left(\kappa_n h\sqrt{\frac{c^2}{\alpha_1^2} - 1}\right)\sin\left(\kappa_n z\sqrt{\frac{c^2}{\alpha_1^2} - 1}\right) \qquad (4\text{–}180)$$

where

$$\Theta_1(\kappa_n) = -\frac{\rho_1 c^4\kappa_n H\sqrt{1 - c^2/\alpha_2^2}}{\rho_2\beta_2^4(c^2/\alpha_1^2 - 1)M\cos(\kappa_n H\sqrt{c^2/\alpha_1^2 - 1})} \qquad (4\text{–}181)$$

and

$$\varphi_2 = \frac{2}{H}\sqrt{\frac{2\pi}{r}}\sum_n \frac{1}{\sqrt{\kappa_n}} \exp\left[i\left(\omega t - \kappa_n r - \frac{\pi}{4}\right)\right] \cdot \Theta_2(\kappa_n) \sin\left(\kappa_n h\sqrt{\frac{c^2}{\alpha_1^2} - 1}\right) \exp\left[-\kappa_n(z - H)\sqrt{1 - \frac{c^2}{\alpha_2^2}}\right] \quad (4\text{–}182)$$

$$\psi_2 = \frac{2}{H}\sqrt{\frac{2\pi}{r}}\sum_n \frac{1}{\sqrt{\kappa_n}} \exp\left[i\left(\omega t - \kappa_n r - \frac{\pi}{4}\right)\right] \cdot \Xi_2(\kappa_n) \sin\left(\kappa_n h\sqrt{\frac{c^2}{\alpha_1^2} - 1}\right) \exp\left[-\kappa_n(z - H)\sqrt{1 - \frac{c^2}{\beta_2^2}}\right] \quad (4\text{–}183)$$

where

$$\Theta_2(\kappa_n) = -\frac{\rho_1 c^2(2 - c^2/\beta_2^2)\kappa_n H}{\rho_2\beta_2^2\sqrt{c^2/\alpha_1^2 - 1}\, M}$$
$$\Xi_2(\kappa_n) = -\frac{2H\rho_1 c^2\sqrt{1 - c^2/\alpha_2^2}}{\rho_2\beta_2^2\sqrt{c^2/\alpha_1^2 - 1}\, M} \quad (4\text{–}184)$$

with

$$M = \frac{\rho_1}{\rho_2}\frac{c^4}{\beta_2^4}\left[\frac{\sin \kappa_n H\sqrt{c^2/\alpha_1^2 - 1}}{\sqrt{c^2/\alpha_1^2 - 1}\,\sqrt{1 - c^2/\alpha_2^2}}\left(1 + \frac{1 - c^2/\alpha_2^2}{c^2/\alpha_1^2 - 1}\right) - \frac{\kappa_n H\sqrt{1 - c^2/\alpha_2^2}}{c^2/\alpha_1^2 - 1}\sec \kappa_n H\sqrt{\frac{c^2}{\alpha_1^2} - 1}\right] - 4\left[\frac{\sqrt{1 - c^2/\beta_2^2}}{\sqrt{1 - c^2/\alpha_2^2}} + \frac{\sqrt{1 - c^2/\alpha_2^2}}{\sqrt{1 - c^2/\beta_2^2}} + 2\sqrt{1 - \frac{c^2}{\alpha_2^2}}\sqrt{1 - \frac{c^2}{\beta_2^2}} - 2\left(2 - \frac{c^2}{\beta_2^2}\right)\right]\cos \kappa_n H\sqrt{\frac{c^2}{\alpha_1^2} - 1} \quad (4\text{–}185)$$

Written in the form (4–180), (4–182), and (4–183), the expressions for the potentials φ_1, φ_2, and ψ_2 show immediately the influence of each variable of the problem. The changes in these potentials produced by a varying depth z are represented by the last factor, $\sin(\kappa_n z\sqrt{c^2/\alpha_1^2 - 1})$, in (4–180). The factor $\sin(\kappa_n h\sqrt{c^2/\alpha_1^2 - 1})$ depends on the depth of the source. The amplitude factors (4–181) and (4–184) determine the relative strength of the various modes as a function of the frequency $f = c_n\kappa_n/2\pi$.

The generalization for a pulse in this case was also given by Press and Ewing. The phase- and group-velocity curves calculated for a granitic and basaltic ocean bottom earlier in this section are applicable here. Additional numerical values representing the conditions for a sedimentary bottom ($\rho_2/\rho_1 = 2.0$, $\alpha_2 = \sqrt{3}\beta_2$, $\beta_2 = 1.5\alpha_1$) are given in Fig. 4–26. Useful curves computed by Tolstoy [204] are reproduced in Figs. 4–27, 4–28, and 4–29.

We can now describe the sequence of normal-mode waves as they will

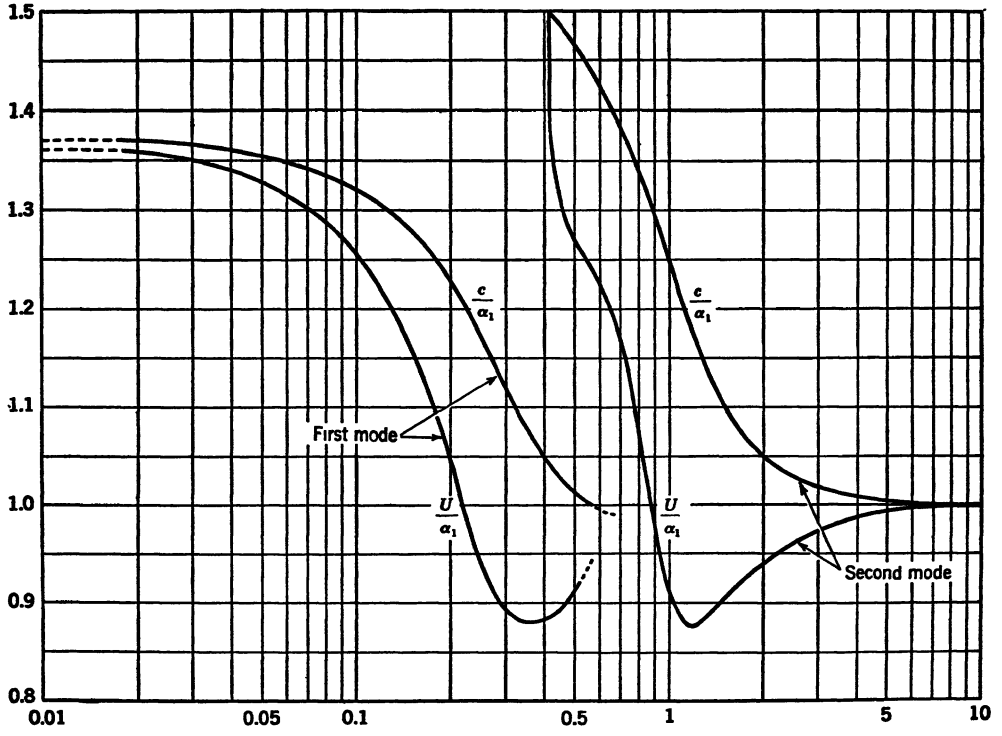


FIG. 4–26. Liquid layer over solid substratum. Phase- and group-velocity curves for the first two modes when $\alpha_2/\beta_2 = \sqrt{3}$, $\beta_2/\alpha_1 = 1.5$, $\rho_2/\rho_1 = 2.0$.

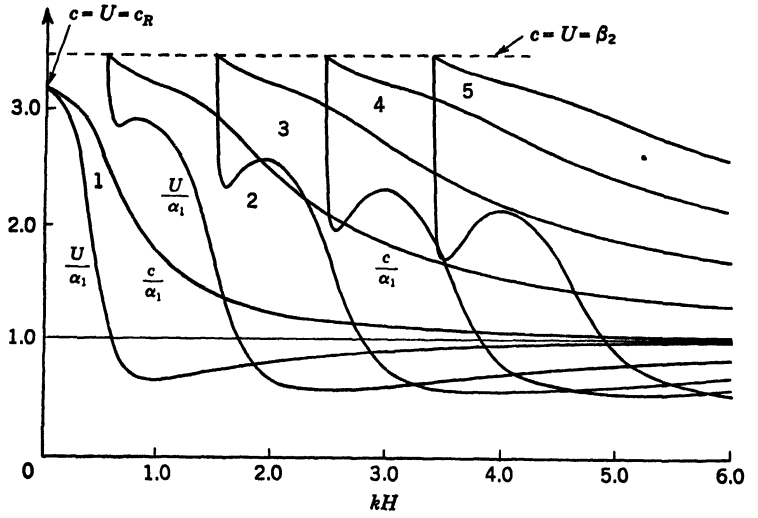


FIG. 4–27. Liquid layer over solid substratum Dispersion curves for first five modes; $\alpha_2/\beta_2 = \sqrt{3}$, $\alpha_2/\alpha_1 = 6$, $\rho_2/\rho_1 = 1.1$. (*After Tolstoy.*)

arrive at a distant point. At a time $t = r/\beta_2$ after the initial impulse at the source (i.e., immediately after the arrival of shear waves), the normal-mode contributions begin, gradually increasing in amplitude to become the predominant waves. The wave amplitudes in the first two modes due to a distant impulsive point source of compressional waves having $g(\omega) =$ const (a flat spectrum), and located within the liquid layer can be taken from Fig. 4–30 as a function of frequency.

In the first mode the first arrivals consist of low-frequency Rayleigh waves with very small amplitudes. With increasing time the frequency increases from zero, rapidly at first and then gradually, and the amplitudes also increase.

At the time $t = r/0.998\alpha_1$ a high-frequency train of Stoneley waves arrives, traveling with a speed slightly less than that of sound in water. According to Fig. 4–30, the amplitudes of these waves are zero at $t = r/\alpha_1(\gamma = 4.36)$ but increase to large amplitudes shortly thereafter.

For $t > r/0.998\alpha_1$ the high-frequency and low-frequency branches of the group-velocity curve contribute waves which arrive simultaneously and approach each other in frequency until they merge to form a conspicuous train of waves, the Airy phase, which terminates at a time corresponding to propagation at the minimum value of group velocity.

The second mode begins with waves arriving with a cutoff frequency at the time $t = r/\beta_2$. The amplitudes are zero at the onset and thereafter

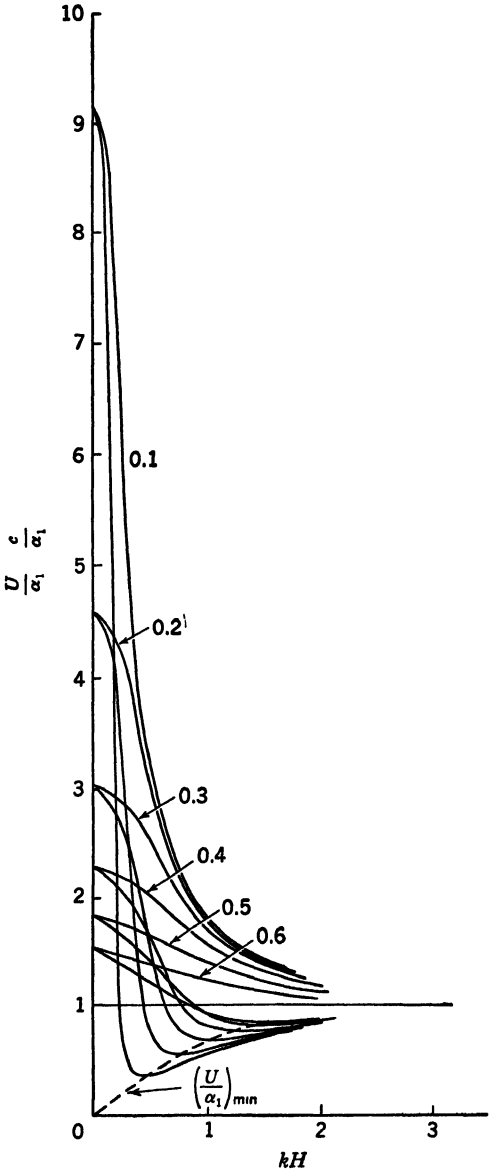


FIG. 4–28. Liquid layer over solid substratum. First-mode dispersion curves for cases $\alpha_2/\beta_2 = \sqrt{3}$, $\rho_2 = \rho_1$, $\alpha_1/\beta_2 = 0$ 1, 0.2, 0.3, $\cdots$, 0.6. (*After Tolstoy.*)

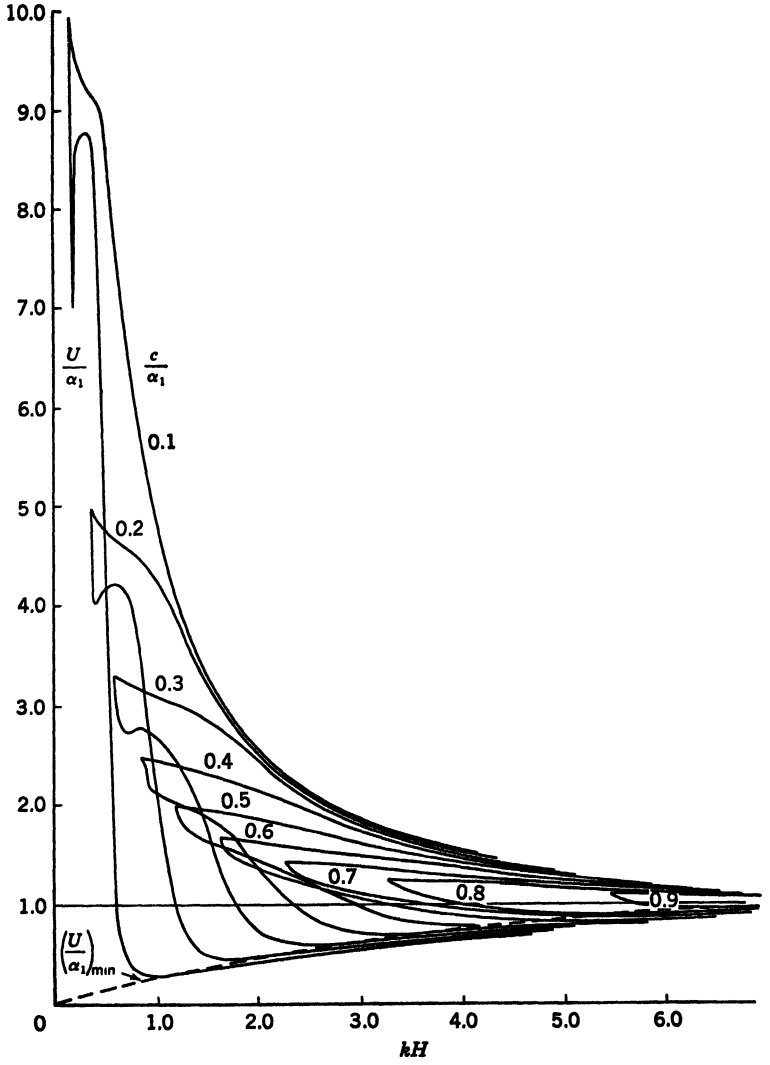


FIG. 4–29. Liquid layer over solid substratum. Second-mode dispersion curves for cases $\alpha_2/\beta_2 = \sqrt{3}$, $\rho_2 = \rho_1$, $\alpha_1/\beta_2 = 0.1, 0.2, 0.3, \cdots, 0.9$. (*After Tolstoy.*)

increase as the frequency increases. At the time $t = r/\alpha_1$ high-frequency waves ($\gamma \to \infty$) arrive, traveling with the speed of sound in water. The amplitude of these waves is zero at the onset but increases rapidly there-

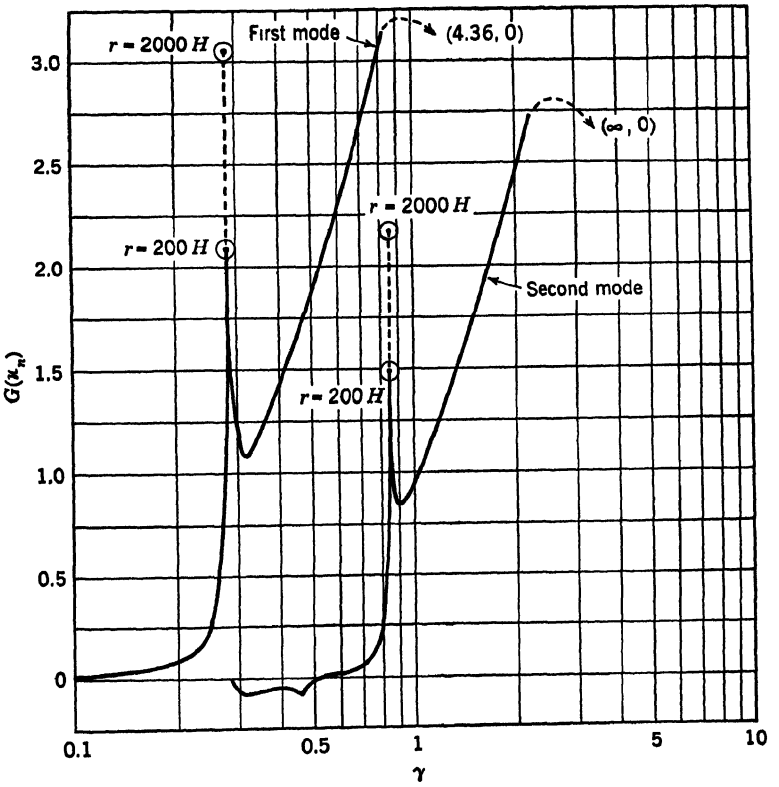


FIG. 4–30 Liquid layer over solid substratum. Amplitude function G for an impulsive source, when $\rho_2/\rho_1 = 2.5$, $\alpha_2/\beta_2 = \sqrt{3}$, $\beta_2/\alpha_1 = 3.0$,

$$G(\kappa_n) = \Theta(\kappa_n) \sin^2 (\kappa_n H \sqrt{c^2/\alpha_1^2 - 1}) \left[\frac{8}{c/\alpha_1} \frac{1}{U^2/\alpha_1^2} \left| \frac{d(U/\alpha_1)}{d\gamma} \right| \right]_n^{-1/2}.$$

after. For $t > r/\alpha_1$ the two arrivals corresponding to the low- and high-frequency branches of the group-velocity curve of this mode approach each other and merge at a minimum value of group velocity, ending the disturbance with the large-amplitude waves of a second-mode Airy phase.

It is to be noted that a maximum value of group velocity is also present in the second mode. Ordinarily one might expect the large-amplitude waves of an Airy phase to begin here but the "excitation" function $\Theta_1(k)$ in Eq. (4–181) almost vanishes for the value of kH corresponding to this stationary value of group velocity, and the resultant amplitudes show only a minor increase.

From Eq. (4–180) it may be seen that the vertical variations of pressure and horizontal displacement in the liquid layer are determined by the factor $\sin(\kappa_n z\sqrt{c^2/\alpha_1^2 - 1})$ and the vertical variation of vertical displacement by $\cos(\kappa_n z\sqrt{c^2/\alpha_1^2 - 1})$. Figures can be easily drawn representing these variations in terms of γ for each mode (Fig. 4–31).

The discussion thus far has been limited to the first two modes. The wave motion at a point is evidently obtained by the superposition of the contributions of all modes.

From the theory just presented a number of important conclusions can

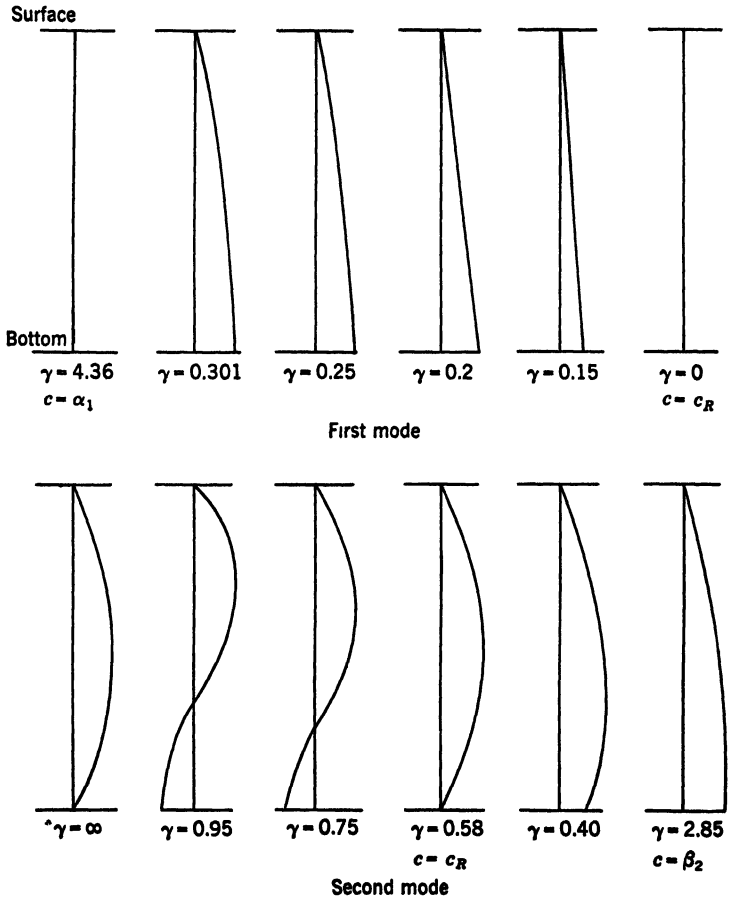


FIG. 4–31. Liquid layer over solid substratum. Vertical-pressure distribution in liquid for the first and second mode when $\alpha_2/\beta_2 = \sqrt{3}$, $\beta_2/\alpha_1 = 3.0$, $\rho_2/\rho_1 = 2.5$.

be drawn concerning the propagation of explosion sound over large ranges in water-covered areas:

1. For a solid bottom, the amplitudes of waves traveling with the speed of compressional waves in the bottom will be relatively small. It is only after the arrival of the first shear waves that large-amplitude waves appear. The shear waves begin with a limiting or cutoff frequency which is characteristic of the depth of water and the elastic constants of the bottom. For a bottom which can be treated as an ideal liquid it was shown in Sec. 4–2 that waves having larger amplitudes appear shortly after the arrival of the bottom compressional or ground waves. These ground waves begin with a cutoff frequency in a manner analogous to the shear waves of the solid-bottom theory.

2. For a solid bottom, a train of waves arrives at a time corresponding to propagation as Rayleigh waves. These waves increase in frequency and amplitude with increasing time.

3. For both the liquid- and solid-bottom theory a high-frequency "water" wave traveling with the speed of sound in water arrives, riding on a low-frequency "rider" wave (see Pekeris [116]). The frequency of the water wave shows a marked decrease with time. An additional feature of first-mode waves over a solid bottom is the higher-frequency Stoneley wave with $c = 0.998\alpha_1$. For the liquid and solid bottom the amplitude of the high-frequency waves increases with time.

4. For both the liquid- and solid-bottom theory the water waves and rider waves merge to form a train of large-amplitude waves which is called the Airy phase. The frequency of the Airy phase is determined by the depth of water and the elastic constants of the bottom. The velocity of the Airy phase is independent of water depth.

5. From the vertical standing wave pattern shown in Fig. 4–31 we see that the response of a hydrophone sensitive to pressure changes or a geophone sensitive to the vertical velocity of a water particle must vary with depth. For any given mode and frequency the best location of a hydrophone is at a pressure antinode, and the ideal location of a geophone is at an antinode of vertical displacement. Antinodes and nodes for pressure correspond to nodes and antinodes for vertical displacement (or velocity), respectively. With the use of curves such as those of Fig. 4–31 the vertical location of a receiver for peak response at a given frequency can readily be obtained.

In most water-covered areas where refraction shooting is undertaken, layering in the bottom occurs, and the assumption of an unstratified bottom made above is indeed an oversimplification. If the thickness of the first bottom layer is several times greater than the water depth, the above theory is applicable for all wavelengths considerably less than this thickness (see Sec. 4–3).

Leaking Modes. In seismic prospecting in shallow water a surface wave has been observed, having the following characteristics, as illustrated in Figs. 4–32 and 4–33:

1. Large amplitudes and long duration
2. Almost constant-frequency train of waves in some cases, fairly simple pattern of beats in others, apparent mixture of several discrete frequencies in others, characterized in all cases by numerous repetitions of a pattern of waves

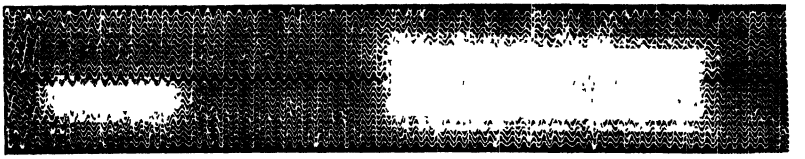


FIG. 4–32. Sixteen- to 40-cycle/sec seismograms showing almost pure sine waves corresponding to leaking-mode propagation in water depth of 132 ft. Seismometers at 8-ft depth; 50 lb of dynamite at 5-ft depth, distance from shot 2,155 to 2,405 ft. (*After Burg, Ewing, Press, and Stulken.*)

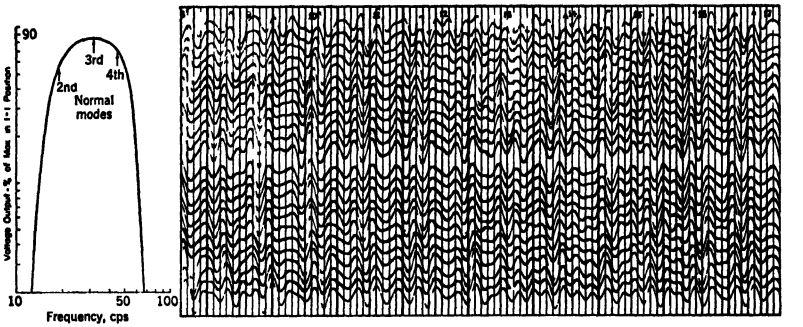


FIG. 4–33 Frequency-response curve on filter setting which on seismogram at right admitted third and fourth leaking modes. Water depth, 192 ft. (*After Burg, Ewing, Press, and Stulken.*)

3. Occurrence usually when a hard stratum is found at or near the sea floor

Burg, Ewing, Press, and Stulken [14] gave a theory for these waves. They stated that waves propagate by multiple reflections at angles of incidence between the normal and the critical angle for total reflection, under the condition of constructive interference. Although a slight leakage of energy occurs with each reflection from the bottom, there is an automatic gain control on the recording apparatus. The attenuation is com-

pensated so that the recorded amplitude remains approximately constant for many seconds after the initial impulse.

The phase velocity and the group velocity can be approximated by assuming infinite density for the bottom. Equation (4–154) then takes the following form:

$$\tan\left(kH\sqrt{\frac{c^2}{\alpha_1^2}-1}\right)\to\infty \tag{4–186}$$

or

$$kH\sqrt{\frac{c^2}{\alpha_1^2}-1}=\frac{2n-1}{2}\pi \qquad n=1,2,\cdots \tag{4–187}$$

If we use the relations $k = 2\pi \sin\theta/l_0$ and $c/\alpha_1 = \csc\theta$, Eq. (4–187) becomes

$$\frac{\alpha_1(2n-1)}{4H\cos\theta}=\frac{\alpha_1}{l_0}=f \tag{4–188}$$

It is easy to prove that the group velocity $U = \alpha_1 \sin\theta = r/t$ approaches zero as the angle of incidence approaches the normal ($\theta = 0$) and the frequency approaches the value

$$f_c=\frac{\alpha_1(2n-1)}{4H} \tag{4–189}$$

For these leaking modes extremely low values of group velocity are significant despite the increased attenuation which accompanies them, because of the automatic gain control mentioned above. It is also seen that many modes may be propagated simultaneously, limited principally by the type of wave filter used in the recording apparatus. Thus for a given water depth, one may observe a single wave train whose frequency approaches that given by Eq. (4–189) if the filter allows only a single mode to pass. If two modes pass the filter, a simple system of beats would be recorded, and several modes together would produce the more complicated patterns mentioned above, as illustrated in Fig. 4–33.

As may be expected, nodes and antinodes occur at various depths in the water, and the contribution of each mode to the seismogram will depend greatly on the depth of the shot and of the detectors.

For a more detailed description of propagation in a leaking mode, one must modify the theory presented earlier in this section.

Some Aspects of Microseisms. The useful sensitivity of most seismographs is limited by background oscillations called microseisms. "Microseism storms" occur in the period range 2 to 10 sec. These storms last from a few hours to a few days, during which time the amplitude of motion gradually rises far above normal and then gradually decays. Although

studies of storm microseisms have resulted in hundreds of papers during the past 50 years, no theory is available for them which can explain all the observations. The sole points on which all agree are that they are generated by the action of storms at sea and that they affect areas of continental dimensions. It is not necessary even to summarize the history of this subject here, as two complete volumes have been devoted to it. See Refs. 100 and 113.

The problem of microseisms may conveniently be divided into four parts: (1) the nature of the source, i.e., the role of the ocean in the transfer of energy from the atmosphere to the earth; (2) the mechanism of transmission over oceanic paths; (3) effects at the continental margin; and (4) the type of propagation over continental paths.

1. NATURE OF SOURCE. After more than 50 years of observations, there is still lack of general agreement about such basic observational data as the role of ocean waves, of storm position, of wind and pressure fluctuations within the storm, of water depth, and of the effect of such parameters on the period and amplitude of the resulting microseisms. Perhaps the major cause of the diversity of opinion is that most observations have been made in latitudes where the general movement of weather is in one direction and the sequence of events during a storm passage from land to sea differs radically from the sequence for a storm passing from sea to land.

An early theory (Wiechert [208]) which still receives some support held that swell breaking on steep coasts introduced microseismic energy into the solid crust. Banerji [2] proposed that gravity waves on the ocean transferred energy to the sea floor, an idea which Scholte [152] tried to revive by including the effect of compressibility of the water. Further studies of the transfer of energy from ocean surface waves were stimulated by the conclusion of Bernard [5] that the period of microseism oscillations on the African coast is half that of the generating sea waves. Deacon [22] noticed the same relationship between microseisms at Kew and sea waves on the north coast of Cornwall. Longuet-Higgins and Ursell [89] and Longuet-Higgins [90] presented a detailed theory for the effect at the ocean bottom of interference between two similar wave trains traveling in opposite directions. In agreement with the finding of Miche [94], they found a pressure fluctuation at the ocean bottom having half the period of the ocean waves and made the first attempt to calculate in detail the amount of energy transmitted to the ocean floor and thence to the point of observation ashore. They concluded that, although it was a second-order effect, it was adequate to produce the observed microseismic disturbances. Gherzi [49] believed barometric pulsations within the storm were able to transmit the necessary energy.

2. MODE OF TRANSMISSION OVER OCEANIC PATHS. It is natural to suggest that the energy of microseisms is transmitted from the source to the

continental margins by the normal modes of wave propagation discussed in this section. Press and Ewing [117] and Longuet-Higgins [90] implicitly assumed that the propagation could be adequately represented by the theory for a homogeneous liquid layer over a homogeneous solid half space. Unfortunately, the gap in Rayleigh-wave spectra for oceanic paths discussed earlier in this section apparently eliminates the possibility of applying this theory to microseisms, despite its utility for longer-period earthquake surface waves. Despite careful search for earthquake surface waves in this period range on seismograms from many coastal stations and from Honolulu and Bermuda (Ewing and Press [43]), no such waves were found where any part of the propagation path crossed an ocean area beyond the continental margin. This is in marked contrast with the abundant energy in the same period range transmitted over continental paths. For example, an earthquake in California of magnitude 5.3 produced clear phases called *Lg* and *Rg* (see Sec. 4–5 and Fig. 4–56) with periods 2 to 8 sec at Palisades. Although the cause of the spectral gap has not yet been found, the gap is clearly of the greatest importance for microseism studies. It strongly supports the conclusions of Ewing and Donn [42], Dinger and Fisher [24], and Carder [18] that ocean areas beyond the continental margins transmit microseisms very poorly and that large microseisms occur only when a portion of the storm (or its swell) reaches shallow water. Several observers report opposite conclusions. Gilmore and Hubert [51] and Whipple and Lee [207] all have stated that storms well beyond the continental margins produce significant microseisms. Perhaps this difference of opinion is due to uncertainty about which portion of the storm produces the microseisms, the radius of the storm area being often comparable with the distance of the center offshore.

3. EFFECTS AT THE CONTINENTAL MARGIN. It was first noted by Gutenberg [58, p. 1308, and 55] that microseisms show marked attenuation in geologically disturbed areas. He has pointed out that microseisms originating off the Atlantic coast of Canada propagate to great distances over the United States and Canada and decrease noticeably only after passing the Rocky Mountain area, and he has called attention to several similar barriers in Europe and in the West Indies. He states that in all instances where the microseisms are propagated over long distances without much loss of energy, stations and source are on the same geological unit. A profound geological change occurs at the continental margin where the crustal thickness increases from about 5 to about 35 km. It is therefore a logical extension of Gutenberg's views about barriers to expect that the transition between ocean and continent will introduce extreme attenuation in microseism waves. The results of Donn [30] and Dinger and Fisher [24] support this idea. They showed that the hurricane microseisms at stations on a continent increase greatly in amplitude as soon as the storm touches the

continental margins. This means that either (1) there is poor generation and transmission of microseisms in deep water or (2) there is extreme attenuation of microseisms at continental borders, or both. The fact (see Sec. 4–5) that the characteristically continental phases *Lg* and *Rg* and 6- to 12-sec surface wave trains from submarine earthquakes do not cross continental borders strongly supports statement 2. The observations of amplitudes smaller than expected which have led to the idea of barriers to microseismic waves are due to the two mechanisms listed above but the relative importance of the two is not clear. Murphy [95] and Gilmore [50] have published additional evidence about the barriers. The details of structure of a continental margin as revealed by seismic refraction and gravity studies are shown schematically in Fig. 4–34. It is seen at once

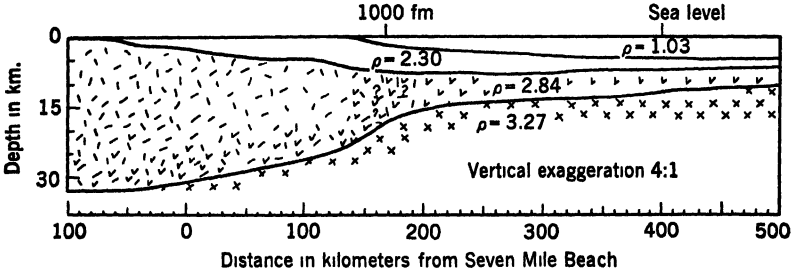


FIG. 4–34. Structure of continental margin deduced from seismic and gravity data. (*After Worzel and Shurbet* [212].)

that transmission across the boundary in either direction is impossible for any wave type in which the crustal layer acts as a wave guide for normal modes. Disturbances from the atmosphere or from the ocean entering this crustal layer in the transition zone will find that the sloping boundary constitutes a Lummer-Gehrcke plate strongly favoring propagation toward the continent rather than toward the ocean. Transformation of surface waves incident upon the continental boundary from one type to another is of greatest importance for microseism studies.

4. MECHANISM OF PROPAGATION OVER CONTINENTAL PATHS. The efficiency of transmission of microseismic waves across continental areas has been considered remarkable since the early days of seismology (see Gutenberg [60]). Carder [18] has made additional studies supporting this point for North America, and Donn [31] has shown that microseisms which are initiated by storms reaching the Pacific coast of Canada may be identified at Palisades by the appropriate Rayleigh-wave particle motion. This is the one aspect of the whole microseism problem which seems to offer no difficulties at the present time. The counterpart of microseismic propaga-

tion across continents is probably represented by the *Lg* and *Rg* earthquake phases described by Press and Ewing [125] (see Sec. 4–5). Both phases meet the requirements of period and efficiency of propagation. The orbital motion of *Lg* is not unlike that found in short-period microseisms, whereas the Rayleigh-wave motion of *Rg* matches that observed for long-period microseisms. The importance of *Lg* and *Rg* for microseism study [125] is that they demonstrate the existence of a very efficient wave guide in the continents. Thus, once microseism energy enters the continent, it can spread over great distances with very little loss.

4–5. Solid Layer over Solid Half Space. Wave propagation in a semi-infinite solid covered by a solid layer of uniform thickness was first studied by Bromwich [12] for steady-state waves of length large compared with the layer thickness. Love (Chap. 3, Ref. 26) extended the work of Bromwich to include waves of length comparable with or small compared with the layer thickness. He was led into this study in an attempt to explain the duration and complexity of earthquake waves. Since that time numerous appeals have been made to imperfections of elasticity, resonance of crustal columns, scattering, even drastic modifications of the fundamentals of the classical theory of wave propagation, in search of a theory of earthquake surface waves. It is now clear that layering is responsible for practically all the observed effects, as Love suspected.

In the preceding sections we considered the solution of problems of wave propagation from an impulsive point source. A similar method is directly applicable to the problems treated in the remainder of the book. The applications have demonstrated, however, that the principal conclusions for surface waves may be obtained directly from the characteristic relation between period and phase velocity which appears in the same form for each physical system, regardless of the source of the wave. From this it follows that in many cases a solution for harmonic plane waves is adequate, and we shall restrict our discussions to this simple type of problem in the following pages.

Rayleigh Waves: General Discussion. The period equation similar to those discussed in previous sections can be readily obtained from its general form derived in Sec. 4–8 for $n - 1$ solid layers overlying a solid half space. It can also be derived by assuming that there are plane waves

$$\begin{aligned}
\varphi_1 &= Ae^{i(\omega t - kx) - \nu_1 z} + Be^{i(\omega t - kx) + \nu_1 z} \\
\psi_1 &= Ce^{i(\omega t - kx) - \nu_1' z} + De^{i(\omega t - kx) + \nu_1' z} \\
\varphi_2 &= Ee^{i(\omega t - kx) - \nu_2 z} \\
\psi_2 &= Fe^{i(\omega t - kx) - \nu_2' z}
\end{aligned} \tag{4–190}$$

propagating in both media.

Substituting these expressions in the boundary conditions (4–4) and (4–2) written for plane waves,

$$\begin{aligned} p_{zz} &= (\lambda_1 + 2\mu_1)\nabla^2\varphi_1 - 2\mu_1\left(\frac{\partial^2\varphi_1}{\partial x^2} - \frac{\partial^2\psi_1}{\partial x\,\partial z}\right) = 0 \\ p_{xz} &= \mu_1\left(2\frac{\partial^2\varphi_1}{\partial x\,\partial z} - \frac{\partial^2\psi_1}{\partial z^2} + \frac{\partial^2\psi_1}{\partial x^2}\right) = 0 \qquad \text{at } z \doteq 0 \end{aligned} \tag{4–191}$$

and

$$\begin{aligned} u_1 &= \frac{\partial\varphi_1}{\partial x} - \frac{\partial\psi_1}{\partial z} = u_2 = \frac{\partial\varphi_2}{\partial x} - \frac{\partial\psi_2}{\partial z} \\ w_1 &= \frac{\partial\varphi_1}{\partial z} + \frac{\partial\psi_1}{\partial x} = w_2 = \frac{\partial\varphi_2}{\partial z} + \frac{\partial\psi_2}{\partial x} \qquad \text{at } z = H \\ (p_{zz})_1 &= (p_{zz})_2 \qquad (p_{xz})_1 = (p_{xz})_2 \end{aligned} \tag{4–192}$$

we obtain six equations:

$$\begin{aligned} (2k^2 - k_{\beta 1}^2)A + (2k^2 - k_{\beta 1}^2)B + 2k\nu_1' iC - 2k\nu_1' iD &= 0 \\ 2k\nu_1 A - 2k\nu_1 B + (2k^2 - k_{\beta 1}^2)iC + (2k^2 - k_{\beta 1}^2)iD &= 0 \\ -kAe^{-\nu_1 H} - kBe^{\nu_1 H} - \nu_1' iCe^{-\nu_1' H} + \nu_1' iDe^{\nu_1' H} &= -kEe^{-\nu_2 H} - \nu_2' iFe^{-\nu_2' H} \\ -\nu_1 Ae^{-\nu_1 H} + \nu_1 Be^{\nu_1 H} - kiCe^{-\nu_1' H} - kiDe^{\nu_1' H} &= -\nu_2 Ee^{-\nu_2 H} - kiFe^{-\nu_2' H} \\ 2k\nu_1 Ae^{-\nu_1 H} - 2k\nu_1 Be^{\nu_1 H} + (2k^2 - k_{\beta 1}^2)iCe^{-\nu_1 H} + (2k^2 - k_{\beta 1}^2)iDe^{\nu_1' H} & \\ = 2\frac{\mu_2}{\mu_1}k\nu_2 Ee^{-\nu_2 H} + \frac{\mu_2}{\mu_1}(2k^2 - k_{\beta 2}^2)iFe^{-\nu_2' H} & \\ (2k^2 - k_{\beta 1}^2)Ae^{-\nu_1 H} + (2k^2 - k_{\beta 1}^2)Be^{\nu_1 H} + 2k\nu_1' iCe^{-\nu_1 H} - 2k\nu_1' iDe^{\nu_1' H} & \\ = \frac{\mu_2}{\mu_1}[(2k^2 - k_{\beta 2}^2)Ee^{-\nu_2 H} + 2k\nu_2' iFe^{-\nu_2' H}] & \end{aligned} \tag{4–193}$$

The six variables $Ae^{-\nu_1 H}$, $iCe^{-\nu_1 H}$, $Be^{\nu_1 H}$, $iDe^{\nu_1' H}$, $Ee^{-\nu_2 H}$, and $iFe^{-\nu_2' H}$ must have values different from zero, and therefore we obtain the period equation

$$\Delta = 0 \tag{4–194}$$

where the determinant can be written in the form shown in (4-195).

$$\Delta = \left|\begin{array}{cc|cccc} (2k^2 - k_{\beta 1}^2)e^{\nu_1 H} & 2k\nu_1' e^{\nu_1' H} & (2k^2 - k_{\beta 1}^2)e^{-\nu_1 H} & -2k\nu_1' e^{-\nu_1' H} & 0 & 0 \\ 2k\nu_1 e^{\nu_1 H} & (2k^2 - k_{\beta 1}^2)e^{\nu_1' H} & -2k\nu_1 e^{-\nu_1 H} & (2k^2 - k_{\beta 1}^2)e^{-\nu_1' H} & 0 & 0 \\ -k & -\nu_1' & -k & \nu_1' & k & \nu_2' \\ -\nu_1 & -k & \nu_1 & -k & \nu_2 & k \\ 2k\nu_1 & 2k^2 - k_{\beta 1}^2 & -2k\nu_1 & 2k - k_{\beta 1} & -2k\nu_2 \frac{\mu_2}{\mu_1} & -\frac{\mu_2}{\mu_1}(2k^2 - k_{\beta 2}^2) \\ 2k^2 - k_{\beta 1}^2 & 2k\nu_1' & 2k^2 - k_{\beta 1}^2 & -2k\nu_1' & -\frac{\mu_2}{\mu_1}(2k^2 - k_{\beta 2}^2) & -2k\nu_2' \frac{\mu_2}{\mu_1} \end{array}\right| \quad (4\text{-}195)$$

If we put

$$F(k) = (2k^2 - k_{\beta 1}^2)^2 - 4k^2\nu_1\nu_1' \qquad (4\text{–}196)$$
$$f(k) = (2k^2 - k_{\beta 1}^2)^2 + 4k^2\nu_1\nu_1'$$

Eq. (4–195) can be represented as the sum

$$\Delta = e^{(\nu_1+\nu_1')H}[D_1 + D_2e^{-2\nu_1'H} + (D_3 + D_4)e^{-(\nu_1+\nu_1')H} + D_5e^{-2\nu_1H} + D_6e^{-2(\nu_1+\nu_1')H}] \qquad (4\text{–}197)$$

where

$$D_1 = F\Delta_{12} \qquad D_2 = f\Delta_{14}$$
$$D_3 = 4k\nu_1(2k^2 - k_{\beta 1}^2)\Delta_{13} \qquad D_4 = -4k\nu_1'(2k^2 - k_{\beta 1}^2)\Delta_{24} \qquad (4\text{–}198)$$
$$D_5 = -f\Delta_{23} \qquad D_6 = F\Delta_{34}$$

and Δ_{12} is the subdeterminant of the fourth order shown in (4–195).

The factors Δ_{ij} are other subdeterminants of the fourth order formed by the elements of the last four lines of Δ, the last two columns being identical for all:

$$\Delta_{14} = \begin{vmatrix} -k & -\nu_1' & \cdot & \cdot \\ \nu_1 & -k & \cdot & \cdot \\ -2k\nu_1 & 2k^2 - k_{\beta 1}^2 & \cdot & \cdot \\ 2k^2 - k_{\beta 1}^2 & 2k\nu_1' & \cdot & \cdot \end{vmatrix} \qquad \Delta_{13} = \begin{vmatrix} -\nu_1' & \nu_1' & \cdot & \cdot \\ -k & -k & \cdot & \cdot \\ 2k^2 - k_{\beta 1}^2 & 2k^2 - k_{\beta 1}^2 & \cdot & \cdot \\ 2k\nu_1' & -2k\nu_1' & \cdot & \cdot \end{vmatrix}$$

$$\Delta_{24} = \begin{vmatrix} -k & -k & \cdot & \cdot \\ -\nu_1 & \nu_1 & \cdot & \cdot \\ 2k\nu_1 & -2k\nu_1 & \cdot & \cdot \\ 2k^2 - k_{\beta 1}^2 & 2k^2 - k_{\beta 1}^2 & \cdot & \cdot \end{vmatrix} \qquad (4\text{–}199)$$

$$\Delta_{23} = \begin{vmatrix} -k & \nu_1' & \cdot & \cdot \\ -\nu_1 & -k & \cdot & \cdot \\ 2k\nu_1 & 2k^2 - k_{\beta 1}^2 & \cdot & \cdot \\ 2k^2 - k_{\beta 1}^2 & -2k\nu_1' & \cdot & \cdot \end{vmatrix} \qquad \Delta_{34} = \begin{vmatrix} -k & -\nu_1' & \cdot & \cdot \\ -\nu_1 & -k & \cdot & \cdot \\ 2k\nu_1 & 2k^2 - k_{\beta 1}^2 & \cdot & \cdot \\ 2k^2 - k_{\beta 1}^2 & 2k\nu_1' & \cdot & \cdot \end{vmatrix}$$

Using the expression (4–197) we have the period equation (4–194) in the form obtained by Newlands [105]. Equation (4–194) will be discussed later in this section. If the columns are combined in another way, the determinant (4–195) yields the period equation in the form discussed by Sezawa and Kanai [169], where exponentials are replaced by hyperbolic and

trigonometric functions. A form often useful for computation is that given by Lee [85]. It is given here for reference.

If X, Y, Z, W are expressions introduced by Love (Chap. 3, Ref. 26),

$$\begin{aligned} X &= \frac{\mu_2}{\mu_1}\frac{k_{\beta 2}^2}{k^2} - 2\left(\frac{\mu_2}{\mu_1} - 1\right) & Y &= \frac{k_{\beta 1}^2}{k^2} + 2\left(\frac{\mu_2}{\mu_1} - 1\right) \\ Z &= \frac{\mu_2}{\mu_1}\frac{k_{\beta 2}^2}{k^2} - \frac{k_{\beta 1}^2}{k^2} - 2\left(\frac{\mu_2}{\mu_1} - 1\right) & W &= 2\left(\frac{\mu_2}{\mu_1} - 1\right) \end{aligned} \tag{4-200}$$

and

$$\begin{aligned} r_1^2 &= (i\nu_1)^2 = k_{\alpha 1}^2 - k^2 & r_2^2 &= k^2 - k_{\alpha 2}^2 = \nu_2^2 \\ s_1^2 &= (i\nu_1')^2 = k_{\beta 1}^2 - k^2 & s_2^2 &= k^2 - k_{\beta 2}^2 = \nu_2'^2 \end{aligned} \tag{4-201}$$

the period equation in the form written by Lee is

$$\xi_1\eta_2 - \xi_2\eta_1 = 0 \tag{4-202}$$

where

$$\begin{aligned} \xi_1 = \left(2 - \frac{k_{\beta 1}^2}{k^2}\right)&\left[X\cos r_1H + \frac{r_2}{r_1}Y\sin r_1H\right] \\ &+ 2\frac{s_1}{k}\left[\frac{r_2}{k}W\sin s_1H - \frac{k}{s_1}Z\cos s_1H\right] \\ \xi_2 = \left(2 - \frac{k_{\beta 1}^2}{k^2}\right)&\left[\frac{s_2}{k}W\cos r_1H + \frac{k}{r_1}Z\sin r_1H\right] \\ &+ 2\frac{s_1}{k}\left[X\sin s_1H - \frac{s_2}{s_1}Y\cos s_1H\right] \end{aligned} \tag{4-203}$$

$$\begin{aligned} \eta_1 = \left(2 - \frac{k_{\beta 1}^2}{k^2}\right)&\left[\frac{r_2}{k}W\cos s_1H + \frac{k}{s_1}Z\sin s_1H\right] \\ &+ 2\frac{r_1}{k}\left[X\sin r_1H - \frac{r_2}{r_1}Y\cos r_1H\right] \\ \eta_2 = \left(2 - \frac{k_{\beta 1}^2}{k^2}\right)&\left[X\cos s_1H + \frac{s_2}{s_1}Y\sin s_1H\right] \\ &+ 2\frac{r_1}{k}\left[\frac{s_2}{k}W\sin r_1H - \frac{k}{r_1}Z\cos r_1H\right] \end{aligned} \tag{4-204}$$

Positive real values of s_i, r_i are obtained when $k^2 < k_{\alpha 1}^2 < k_{\beta 1}^2$ and $k^2 > k_{\beta 2}^2 > k_{\alpha 2}^2$. Equation (4–202) provides an implicit relation between phase velocity c and wave number k, through the dimensionless parameters c/β_1 and kH. Alternatively, c may be obtained as a function of the period T through the relation $T = (2\pi H/\beta_1)/(kH\cdot c/\beta_1)$. Two branches of this function

occur corresponding to the M_1 and M_2 type of propagation first described by Sezawa and Kanai [171] (see also Kanai [81]). Satô [141] and later Tolstoy and Usdin [203] have suggested that these two wave types correspond to the symmetrical and antisymmetrical modes for a free plate (see Sec. 6–1). It may be shown, by substituting in the expressions for displacement the values of ν_1 and ν_1' used in computing roots of the period equation (4–202), that the particle motion at the surface for the M_1 branch is the retrograde elliptical type normal for Rayleigh waves, whereas the M_2 branch leads to the opposite type. For $c < \beta_1$ and $kH \to \infty$ the asymptotic form of Eq. (4–202) becomes factorable, as pointed out by Love. The zero of the first factor represents Rayleigh waves at the upper surface of the layer, while the zero of the second factor represents Stoneley waves at the interface. These results provide a check, giving the expected wave types for wavelengths small compared with the layer thickness. Further discussion of the roots of the period equation may be found in Scholte [156], Sezawa and Kanai [171], Neumark [101–104], and Keilis-Borok [82].

Although the period equation (4–202) is more complicated than that for a liquid layer on a solid (Sec. 4–4), it may be treated by the same method to determine the number and location of the roots. Keilis-Borok found that for a given ω and a given layer thickness the number of real roots of the period equation is limited.

Calculations of dispersion curves for the first two modes ($M_{11}, \cdots, M_{22}$) of the M_1 and M_2 branches were made by Tolstoy and Usdin [203] for the case $\rho_2/\rho_1 = 1.39$, $\beta_2/\beta_1 = 3.147$, and $\alpha_1/\beta_1 = \alpha_2/\beta_2 = \sqrt{3}$. These curves are shown in Fig. 4–35. At the long-wave limit only in M_{11}, the first mode of the M_1 branch, does the phase velocity approach the speed of Rayleigh waves in the substratum. This is the branch which is relevant to the propagation of earthquake Rayleigh waves. In all other modes the velocity approached at the long-wave limit is that of shear waves in the substratum, and for each mode there is a least value of kH, corresponding to a cutoff frequency below which unattenuated propagation does not occur. The cutoff frequency increases with the mode number.

At the short-wave limit ($kH \to \infty$) the phase velocity for the first mode of the M_1 branch approaches that of Rayleigh waves in the layer. Under the very stringent conditions necessary for the existence of Stoneley waves at the interface between two solids, there would be an additional mode in the M_2 branch for which the phase velocity approaches the speed of Stoneley waves as $kH \to \infty$. For all other modes, phase velocity approaches that of shear waves in the layer as $kH \to \infty$. The methods used in previous sections for deducing the group-velocity curves and some characteristics of seismograms from dispersion curves such as those presented in Fig. 4–35 may readily be applied. As in the previous cases,

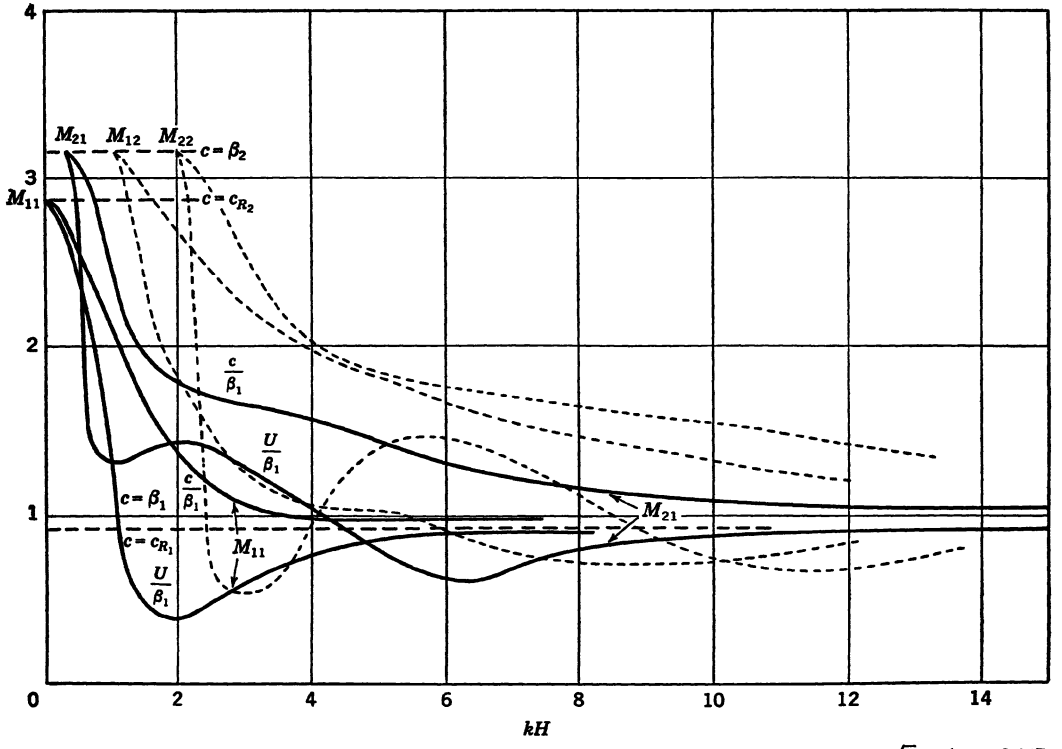


FIG. 4–35. Dispersion curves for elastic layer over semi-infinite elastic solid, for case $\alpha_1/\beta_1 = \alpha_2/\beta_2 = \sqrt{3}$, $\beta_2/\beta_1 = 3.147$, $\rho_2/\rho_1 = 1.39$. (*After Tolstoy and Usdin.*)

the phase-velocity and group-velocity curves meet at both the low-frequency and the high-frequency limits of kH. It is noted that several maximum and minimum values of group velocity occur in the higher modes.

Propagation of Rayleigh Waves across Continents. In accordance with the theory of the propagation of Rayleigh waves in layered media, it is observed that the velocity of Rayleigh waves across a continental area is dependent upon period. Because of a variety of difficulties, attempts to obtain a precise dispersion curve from empirical observations have not been very successful. Ideally, one requires an earthquake of magnitude about 7, which produces Rayleigh waves over a broad range of periods and has a well-determined epicenter situated at one end of a chain of seismograph stations well distributed on a great-circle segment about 70° in length. This large path is required because the dispersion is much weaker across continents than across oceans. It is further desirable that the seismographs have three matched components so that the Rayleigh waves may be positively identified by particle motion, that they have adequate response in the period range 5 to 75 sec, and that the path be free from major crustal irregularities. In no study have these conditions been met adequately.

Observations of Rayleigh-wave dispersion for continental paths have been made by Röhrbach [133], Wilson and Baykal [210], Carder [17], Gutenberg and Richter [56], Sezawa [170], Brilliant and Ewing [11], and Press, Ewing, and Oliver [128], among others. From these studies we have selected the last two to present in Fig. 4–36. These results are consistent with each other, and the conditions of the experiments were the most favorable of any for yielding data about purely continental paths.

Brilliant and Ewing avoided the requirements for long continental path and accurate epicenter location by utilizing waves from a shock in the Southwest Pacific which crossed the west coast of the United States at normal incidence and were recorded at six stations distributed across the United States on a suitable great-circle path. For a number of selected periods the arrival times at each station were read. Figure 4–37 shows arrival time as a function of period for all stations. It is straightforward to determine the group velocity for each selected period across this spread of stations and to demonstrate its constancy across the continent. The results are plotted in Fig. 4–36.

The advantages of this technique are (1) independence from error due to normal inaccuracies in epicenter and origin-time data; (2) improved accuracy in determining periods, resulting from lengthening of the train over the oceanic segment of the path which, in effect, is equivalent to having a pure continental path of great length; and (3) independence from error inherent in other techniques involving mixed paths and from uncertain correction for oceanic and transitional segments.

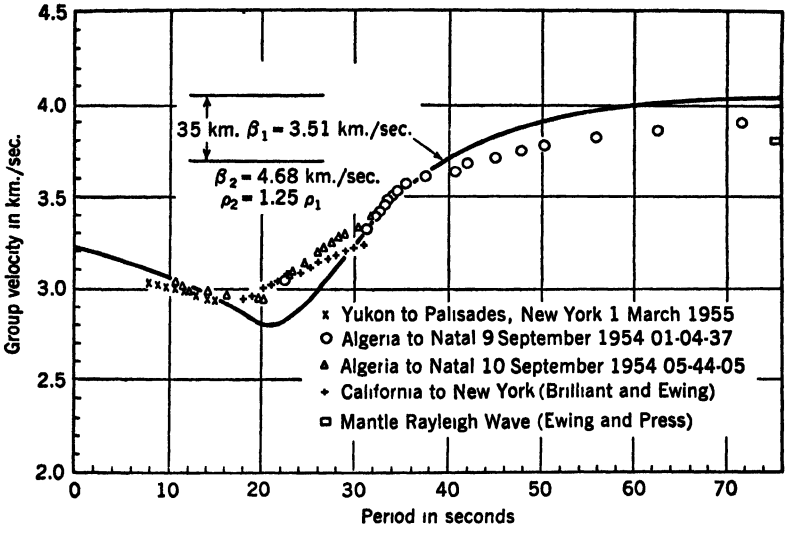


FIG. 4–36. Observed and theoretical dispersion of continental Rayleigh waves.

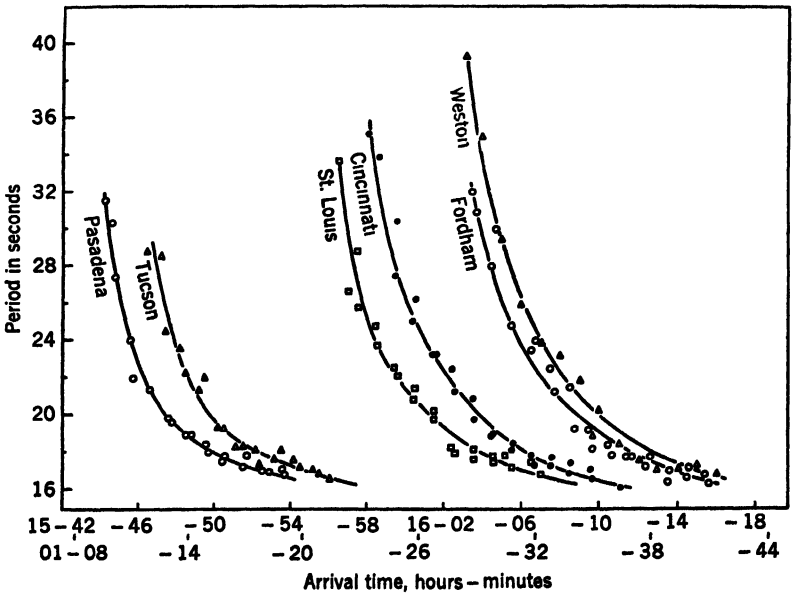


FIG. 4–37. Method of determining dispersion of continental Rayleigh waves. Period is plotted against arrival time for six stations on great-circle path across the United States. (*After Brilliant and Ewing.*)

Press, Ewing, and Oliver studied the Rayleigh waves from the Algerian tremor of Sept. 9, 1954, and the aftershock of Sept. 10, 1954. These were recorded on a long-period Columbia University vertical seismograph installed at Pietermaritzburg, Union of South Africa. The dispersion from these seismograms could be measured with great precision because the path is longer (7,890 km) than any which has been available for a long-period vertical instrument and is free from such obvious crustal anomalies as mountain ranges. The Rayleigh-wave portion of both seismograms is reproduced in Fig. 4–38. Time is marked in minutes after the origin time, and the beginnings of the Rayleigh-wave train, the *Lg* train, and the *Rg* train (to be discussed later in this section) are indicated, along with a group of waves which is interpreted as an *Rg* phase reflected from the continental margin. The Rayleigh-wave train for the main shock clearly shows the normal dispersion, reported in the past, in which the wave period decreases with time. With the arrival of the *Lg* phase, it becomes difficult to read the ordinary Rayleigh waves, but they can be analyzed by shifting from the main shock to the aftershock. *Rg* begins abruptly with great amplitude, as may be seen clearly on the seismogram of the aftershock. The inverse dispersion present in this phase is clearly demonstrated for the first time on this seismogram. The point at 44^m45^s after the origin time, just prior to the arrival of the reflection train, is taken to represent an Airy phase corresponding to the minimum value of group velocity of continental Rayleigh waves. The surface waves arriving during the next 10 or 15 min after the Airy phase are interpreted as due to scattering and reflection from inhomogeneities in the earth's crust.

The method of deducing dispersion from a seismogram (Ewing and Press [41]) involves reading times of zero trace deflection and plotting these against wave number. Periods for a series of arrival times are read as slopes of the resulting curve. The dispersion data thus obtained are plotted in Fig. 4–36 along with the data for the United States path (Brilliant and Ewing [11]). Also plotted is a point indicating the short-period limit of mantle Rayleigh waves (Ewing and Press [46]).

A theoretical curve can be derived using Eqs. (4–202) and (4–94). Jeffreys' calculations [79] were modified so that the elastic constants agree more closely with the most recent determination of crustal structure from explosion and rock-burst data. The curve assumes a homogeneous crust 35 km thick, with shear velocity 3.51 km/sec, overlying a homogeneous mantle with shear velocity 4.68 km/sec and density 1.25 times that of the crustal layer. Poisson's constant σ is taken as 1/4. The following conclusions may be drawn from Fig. 4–36:

1. The remarkable agreement of dispersion data from Africa and North America, in which the discrepancies are less than 0.1 km/sec, indicates an identical crustal structure for the two continents.

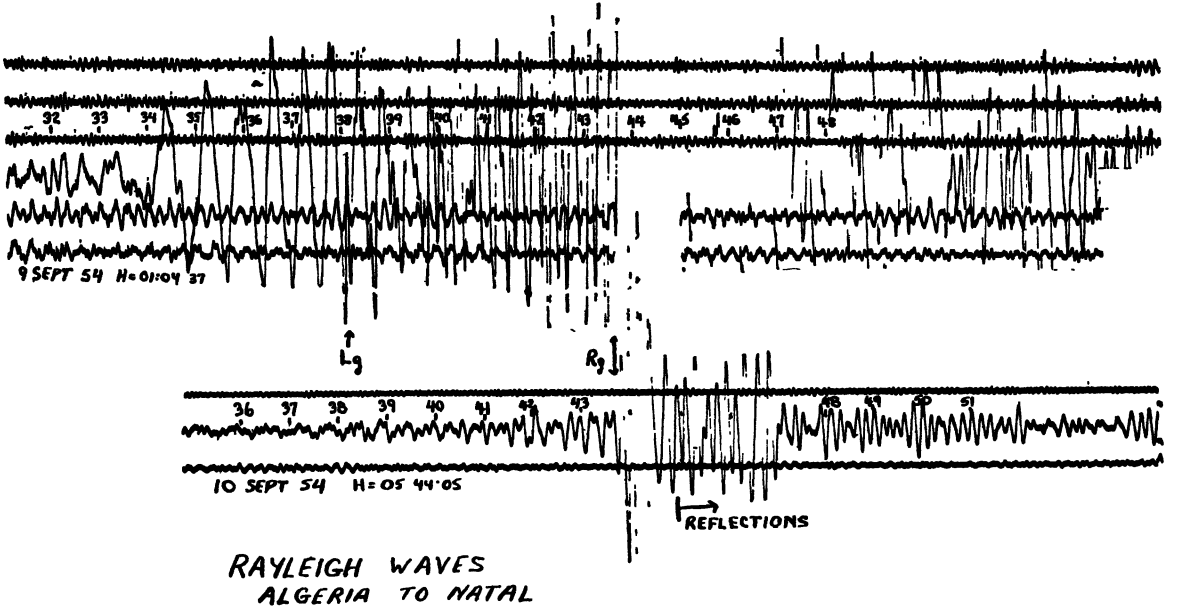


FIG 4–38. Rayleigh waves recorded at Pietermaritzburg, Natal, from the Algerian earthquakes of Sept 9 and 10, 1954. Common time scale refers to minutes after origin time.

2. Although the over-all fit of experimental data with the theoretical curve is reasonably good, there is a tendency for the observed points to fall below the theoretical curve for periods greater than 38 sec. This effect may be explained by the known increase of velocity with depth in the mantle. Theoretical curves including the effect of heterogeneity in the mantle are not readily available. However, it has been experimentally verified in the study of mantle Rayleigh waves (Sec. 7–4) that, because of the velocity gradient in the mantle, the group velocity decreases with increasing period in the range 70 to 225 sec. We therefore conclude that a maximum value of group velocity occurs between 40 and 70 sec and falls below the theoretical curve of Fig. 4–36, as do the experimental points. Previously the adjustment of the theoretical curve was made by decreasing the velocity in the outermost part of the mantle well below the value 4.7 km/sec now fixed by explosion seismology studies.

3. In the period range 18 to 30 sec the experimental points lie above the theoretical curve by amounts ranging up to 0.2 km/sec. The observed minimum group velocity falls at a shorter period than that indicated by the theoretical curve. Both these effects would occur if the average velocity in the crustal layer is higher than that assumed in computing the theoretical curve. Since the velocity near the top of the crust is fairly well established in body-wave studies, we can interpret this discrepancy as an effect introduced by an increase of velocity with depth in the crust.

A new theoretical dispersion curve is needed, one which includes the effect of a velocity gradient in the crust as well as the mantle.

Ground Roll. A small-scale counterpart of the preceding problem is that of the "ground roll" frequently encountered in seismic prospecting for oil. In many regions the near-surface layering consists of a zone of low-velocity, poorly consolidated sediments overlying more competent beds with higher velocity. It is important to note that the discontinuity in compressional-wave velocity at the ground-water table does not enter into this problem, since there is no associated discontinuity in shear velocity, and the shear velocity is the principal factor determining the velocity of Rayleigh waves. A number of papers appeared on this subject, and we cite some of the results of Dobrin et al. [28, 29].

The method of Dobrin et al. consists of detonating explosive charges at the surface and at varying depths beneath it and recording the resulting seismic motion with a spread of detectors spaced at intervals of 50 ft and extending to a distance of several thousand feet from the source. At least at one distance a three-component detector is placed so that the Rayleigh waves may be definitely identified by the orbital motion of the earth particles. In some cases three-component detectors were placed at various depths in a borehole. A sample seismogram is shown in Fig. 4–39. It is seen that the individual crests can be followed across the entire

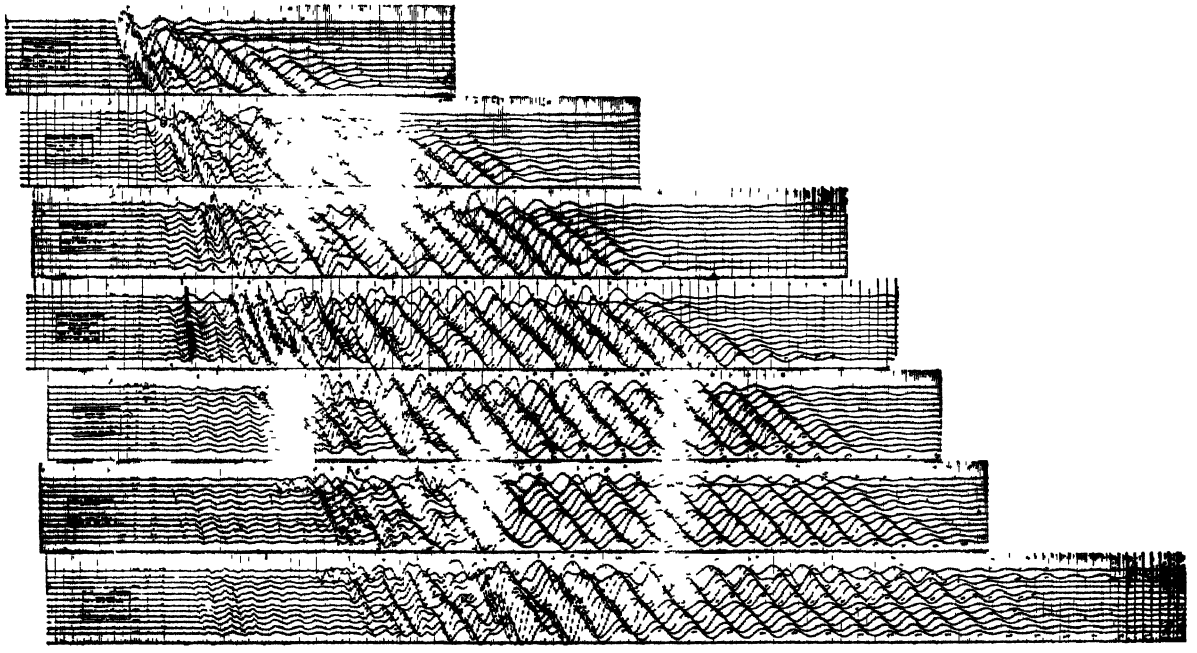


FIG. 4–39. Record array showing Rayleigh waves from explosions recorded at 50-ft intervals over distance range 50 to 3,100 ft from shot. First three traces of each record show, respectively, horizontal radial, horizontal transverse, and vertical motion at stations spaced every 450 ft. The remaining traces show vertical motion every 50 ft, the last trace of each record duplicating the third trace of the following record. (*After Dobrin, Simon, and Lawrence.*)

detector spread, permitting direct determination of phase and group velocities. It is further seen that the dispersion is strong, that the source generates a broad spectrum, and that the data are far more complete than is ever likely to be the case in an earthquake study.

Dobrin et al. applied the theory of Rayleigh-wave dispersion, as given earlier in this section, using values for velocity and layer thickness found in borehole surveys in the area. They obtained the empirical curves represented in Fig. 4–40. A fairly good agreement with the theoretical curve, for periods 0.17 to 0.19 sec, is shown in Fig. 4–41.

Although ground roll consists predominantly of Rayleigh waves, as would be expected from an explosive source, the wave motion is not always

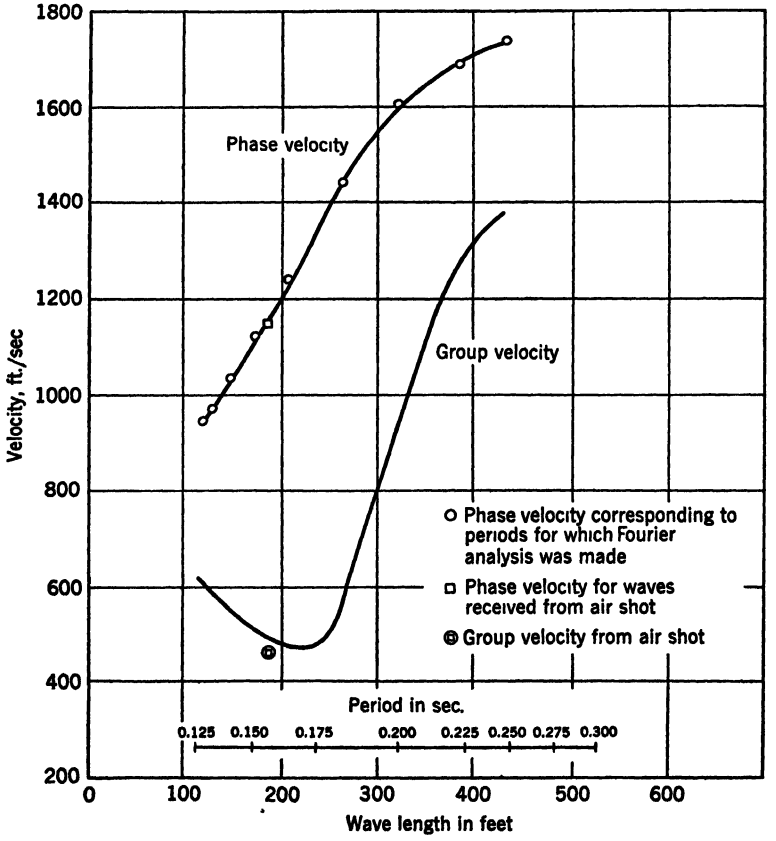


FIG. 4–40. Group and phase velocities of Rayleigh waves obtained from seismograms shown in Fig. 4–39. (*After Dobrin, Simon, and Lawrence.*)

as coherent and regular as that shown in Fig. 4–39 and often contains significant transverse components. It is suggested that heterogeneity and aeolotropy of the upper layers can account for these phenomena.

A low-frequency, large-amplitude arrival, with group velocity as low as a few hundred feet per second, has occasionally been observed on seismograms from explosions in shallow water (Worzel and Ewing [211]). Often dispersion is evident, the lowest frequencies present being 1 to 2 cycles/sec, probably determined by the cutoff frequency of the recording equipment. Although a detailed investigation of these waves has yet to be made, it is probable that they are primarily controlled by the very low rigidity of the bottom sediments. Their frequency is well below any used

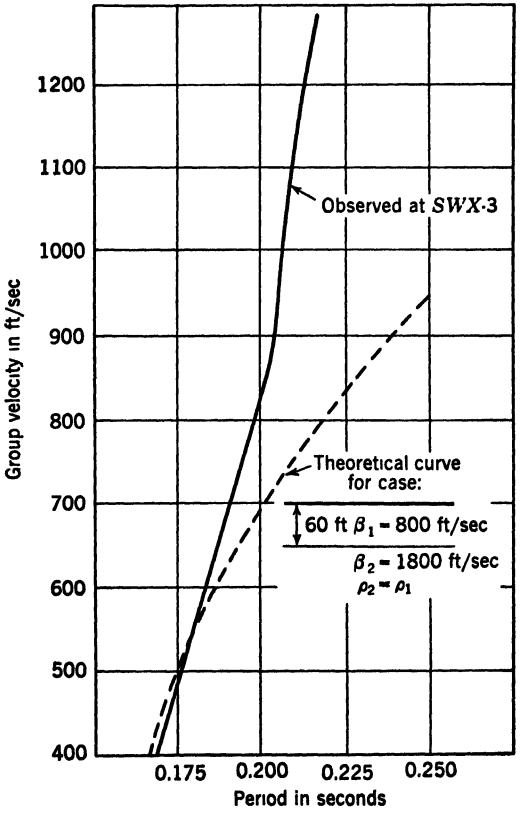


FIG. 4–41. Observed dispersion curve (at the station SWX-3) compared with theoretical curve based on calculations of Sezawa for a layered solid with the indicated characteristics. (*After Dobrin, Simon, and Lawrence.*)

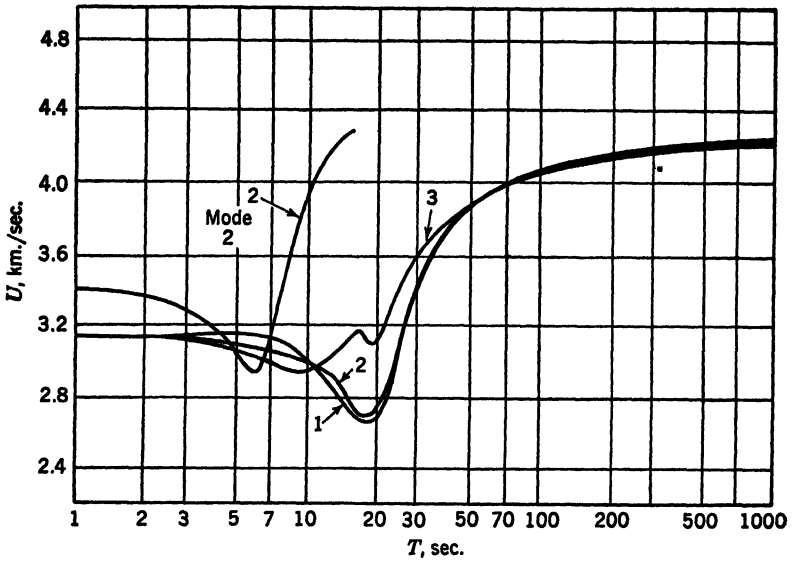


FIG. 4–42. Group velocity of Rayleigh waves for $\beta_1 = 3.39$ km/sec (see Table 4–1). Curve 1, $H_1 = 13.60$ km; curve 2, $H_1 = 28.38$ km; Curve 3, $H_1 = 13.60$ km. (*After Haskell.*)

in seismic-reflection surveys, and their large amplitude can cause serious distortion of other seismic signals if amplifier stages ahead of the filters are allowed to become overloaded.

Theoretical Rayleigh-wave Dispersion Curves. A number of dispersion curves calculated by several investigators for various values of the elastic

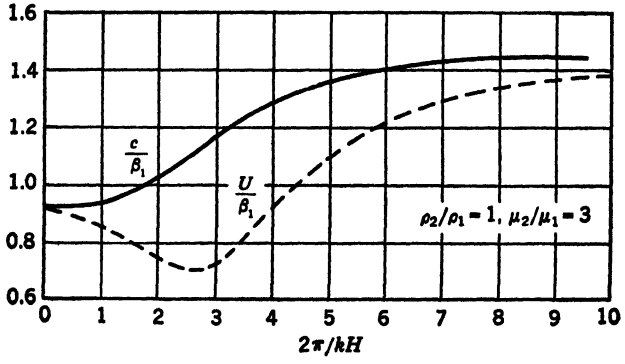


FIG. 4–43. Rayleigh-wave dispersion, case 5, Table 4–1. (*After Kanai.*)

constants of the two media are given here. Table 4–1 identifies each dispersion curve, gives the elastic constants, and cites the references and the figure number.

Love Waves: General Discussion. Since the first long-period seismographs measured horizontal motion only, the presence of large transverse

TABLE 4–1. COMPUTATION OF RAYLEIGH-WAVE DISPERSION IN SOLID LAYERS OVER SOLID SUBSTRATUM

Case	Reference	Layer i	α_i/β_1	β_i/β_1	ρ_i/ρ_1	H_i/H_1	
1.	Haskell [62]	1	1 810	1 000	1 000	1 000	Fig. 4–42,
		2	1 620	0 938	1 000	0 871	curve 1
		3	2 440	1.370	1.110	∞	
2	Haskell [62]	1	1 810	1.000	1 000	1 000	Fig. 4–42,
		2	2 440	1.370	1 110	∞	curve 2
3.	Haskell [62]	1	1 810	1 000	1 000	1.000	Fig. 4–42,
		2	2 060	1 190	1.000	1 560	curve 3
		3	2 440	1 370	1 110	∞	
4.	Jeffreys [79]	1	1 732	1	1	1	Fig. 4–36
		2	2 309	1 333	1 250	∞	
5.	Kanai [81]	1	1 732	1	1	1	Fig. 4–43
		2	3 000	1 732	1	∞	
6.	Kanai [81]	1	1 732	1	1	1	Fig. 4–44
		2	3 873	2 236	1	∞	
7	Kanai [81]	1	1 732	1	1	1	Fig. 4–45
		2	4 898	2 828	1	∞	
8.	Kanai [81]	1	1.732	1	1	1	Fig. 4–46
		2	7.746	4.472	1	∞	
9	Kanai [81]	1	1.732	1	1	1	Fig. 4–47
		2	2 084	1.203	1 140	∞	
10.	Tolstoy and Usdin [203]	1	1 732	1	1	1	Fig. 4–35
		2	5.451	3.147	1 390	∞	
11.	Sezawa [158]	1	1.732	1	1	1	Fig. 4–48
		2	2 449	1.414	1	∞	
12.	Wilson and Baykal [210]	1	1.732	1	1	1	Fig. 4–49
		2	1.916	1.106	1.167	∞	

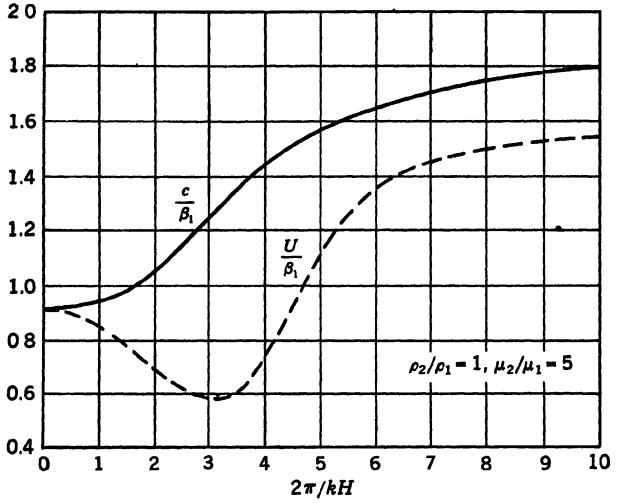


FIG. 4–44. Rayleigh-wave dispersion, case 6, Table 4–1. (*After Kanai.*)

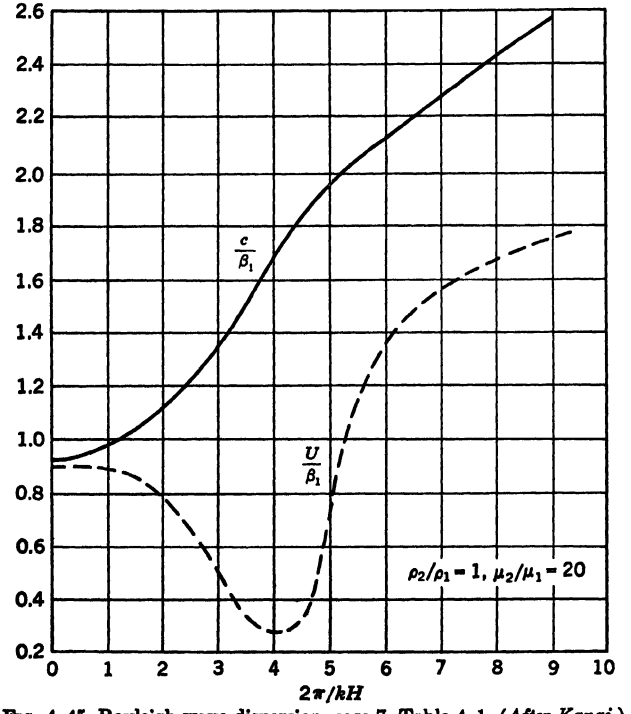


FIG. 4–45. Rayleigh-wave dispersion, case 7, Table 4–1. (*After Kanai.*)

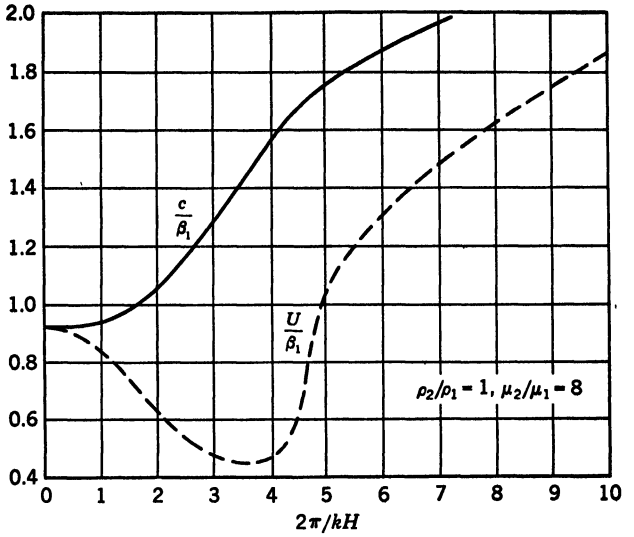


FIG. 4–46. Rayleigh-wave dispersion, case 8, Table 4–1. (*After Kanai.*)

components in the "main tremor" was one of the first established facts of seismology.

It was not until 1911 that an explanation of these waves was provided by Love who showed that they consisted of horizontally polarized shear waves trapped in a superficial layer and propagated by multiple total reflections. To derive the period equation we follow Love's original discussion (Chap. 3, Ref. 26) and consider simple harmonic plane waves. Take the origin of coordinates in the interface, with the x axis in the direction of propagation and the z axis vertically downward. Assume that all dis-

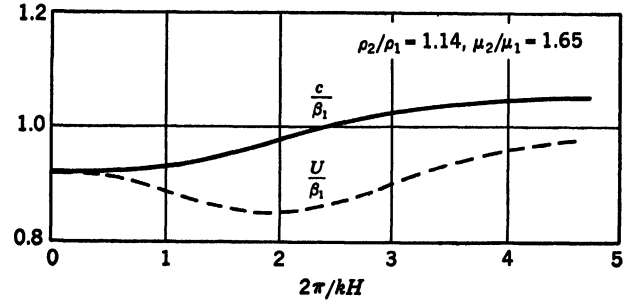


FIG. 4–47. Rayleigh-wave dispersion, case 9, Table 4–1. (*After Kanai.*)

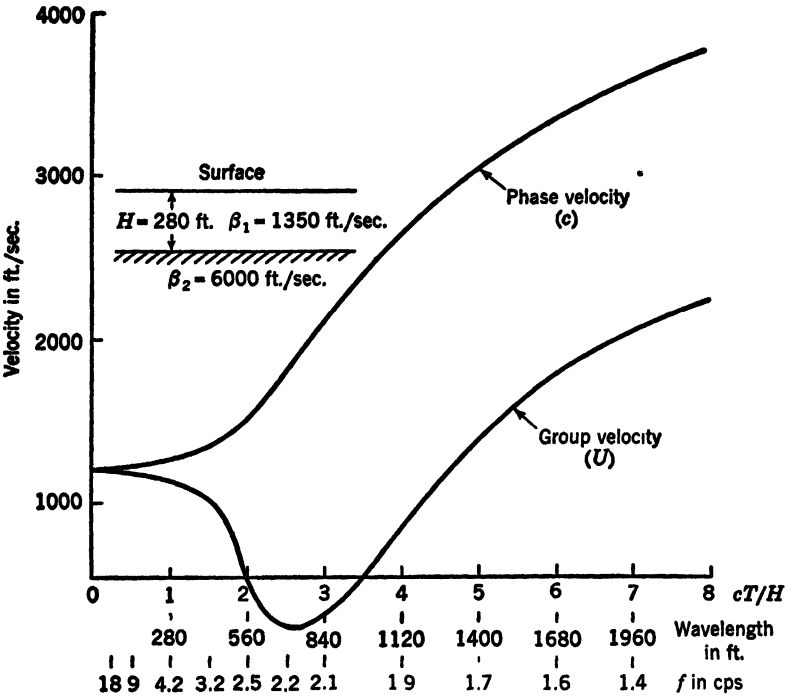


FIG. 4–48. Rayleigh-wave dispersion according to the theory of Sezawa, case 11, Table 4–1. (*After Dobrin et al.* [29].)

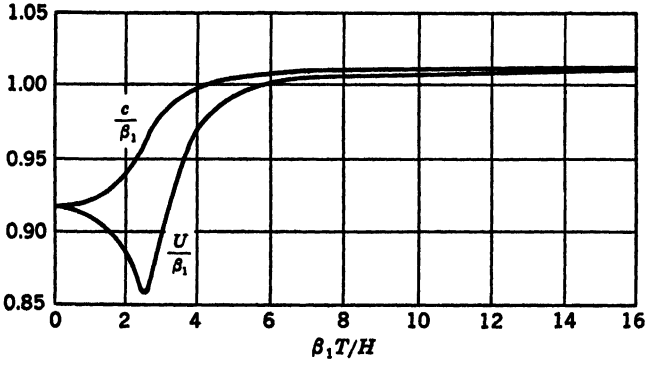


FIG. 4–49. Rayleigh-wave dispersion, case 12, Table 4–1. (*After Wilson and Baykal.*)

placements are independent of the coordinate y and that the time variations are given by the factor $\exp(i\omega t)$. The plane $z = -H$ represents the free surface (Fig. 4–50). The equations of motion (1–13) reduce to $(\nabla^2 + k_{\beta 1}^2)v_1 = 0$ for the layer and to $(\nabla^2 + k_{\beta 2}^2)v_2 = 0$ for the substratum.

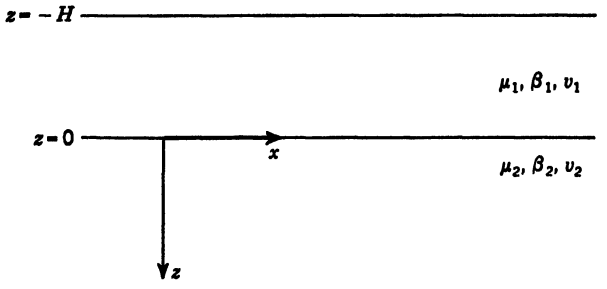


FIG. 4–50. Notations for a layer underlain by a solid half space.

Making use of solutions of the wave equation in the form (2–7), we write for the displacements

$$v_1 = (Ae^{ik\hat{\gamma}_1 z} + Be^{-ik\hat{\gamma}_1 z})e^{-ik(x-ct)} \tag{4–205}$$

$$v_2 = Ce^{ik(-\hat{\gamma}_2 z - x + ct)} \tag{4–206}$$

where

$$\hat{\gamma}_1 = \sqrt{\frac{c^2}{\beta_1^2} - 1} \qquad \hat{\gamma}_2 = \sqrt{\frac{c^2}{\beta_2^2} - 1} \tag{4–207}$$

In order that the energy of these plane waves be confined to the superficial layer, we require that c be less than β_2, that is, that $\hat{\gamma}_2$ be a positive imaginary number. The three constants A, B, and C are determined by the boundary conditions, which require that the stress p_{zv} vanish at the free surface and be continuous, together with the displacement at the interface $z = 0$. Thus by Eqs. (1–11) the condition $p_{zv} = 0$ at $z = -H$ leads to

$$Ae^{-ik\hat{\gamma}_1 H} - Be^{ik\hat{\gamma}_1 H} = 0 \tag{4–208}$$

and $(p_{zv})_1 = (p_{zv})_2$ at $z = 0$ leads to

$$\mu_1\hat{\gamma}_1(A - B) = -\mu_2\hat{\gamma}_2 C \tag{4–209}$$

The continuity of displacement $v_1 = v_2$ at $z = 0$ gives

$$A + B = C \tag{4–210}$$

This system of linear equations has solutions different from zero if

$$\Delta = \begin{vmatrix} e^{-ik\hat{\gamma}_1 H} & -e^{ik\hat{\gamma}_1 H} & 0 \\ \mu_1\hat{\gamma}_1 & -\mu_1\hat{\gamma}_1 & \mu_2\hat{\gamma}_2 \\ 1 & 1 & -1 \end{vmatrix} = 0 \tag{4-211}$$

or

$$\tan k\hat{\gamma}_1 H = i\frac{\mu_2\hat{\gamma}_2}{\mu_1\hat{\gamma}_1} = \frac{\mu_2}{\mu_1}\frac{\sqrt{1 - c^2/\beta_2^2}}{\sqrt{c^2/\beta_1^2 - 1}} \tag{4-212}$$

This period equation may be treated according to methods used in Sec. 4–2, Eq. (4–78). We note that real roots occur when $\beta_1 < c < \beta_2$, and it may be readily shown that when $\beta_2 < \beta_1$ no relevant solutions exist. It is seen by (4–212) when $c \to \beta_2$, $k\hat{\gamma}_1 H \to 0, \pi, 2\pi, \cdots$, and from the last condition the wavelength in the lowest mode becomes infinite compared with H. It is interesting to compare this problem with that of two liquid layers (Sec. 4–2). In both cases a single wave type is involved, and in both cases the phase velocity ranges between the wave velocities in the layer and in the substratum. In the case of liquids, the period approaches a finite limit at the upper limit of phase velocity, and larger periods are excluded (for unattenuated propagation). In the Love-wave case there is no upper limit to the period of waves since, as c approaches β_2, kH approaches zero. This difference in behavior results from the phase change upon reflection at the free surface, which is zero for the Love waves and $\pi/2$ for the liquids. Similarly, when $c \to \beta_1$, $\hat{\gamma}_1 \to 0$, yet the product $kH\hat{\gamma}_1 \to \pi/2, 3\pi/2, \cdots$. The existence of an infinite number of modes follows from the periodicity of the tangent function.

It is easy to show from Eqs. (4–205), (4–206), and (4–212) that the different modes correspond to 0, 1, 2, $\cdots$ nodal planes within the layer. In contrast with the problem for two liquid layers, the free surface here is always an antinode for horizontal displacements. This is one of the few cases where it is more convenient to obtain values of group velocity from an explicit expression than by numerical differentiation of the phase velocity.

Values for phase and group velocity have been computed by Jeffreys [76], Wilson [209], Stoneley [197], and Kanai [81], among others. Kanai's curves appear in Fig. 4–51. Wilson's data are presented in Table 4–2 and Fig. 4–52. Stoneley's results, in addition to a calculation for the second mode, are presented in Fig. 4–53. Again, the group velocity has a minimum value, the Airy phase occurring at a period of about 20 sec when $H = 35$ km. Curves for easy determination of period and velocity for the Airy phase have been given by Satô [140]. Very useful nomograms for determination of phase and group velocity of Love waves for a wide variety of cases have also been prepared by Satô [147, 148].

Generalization of Love-wave theory for an impulsive line or point

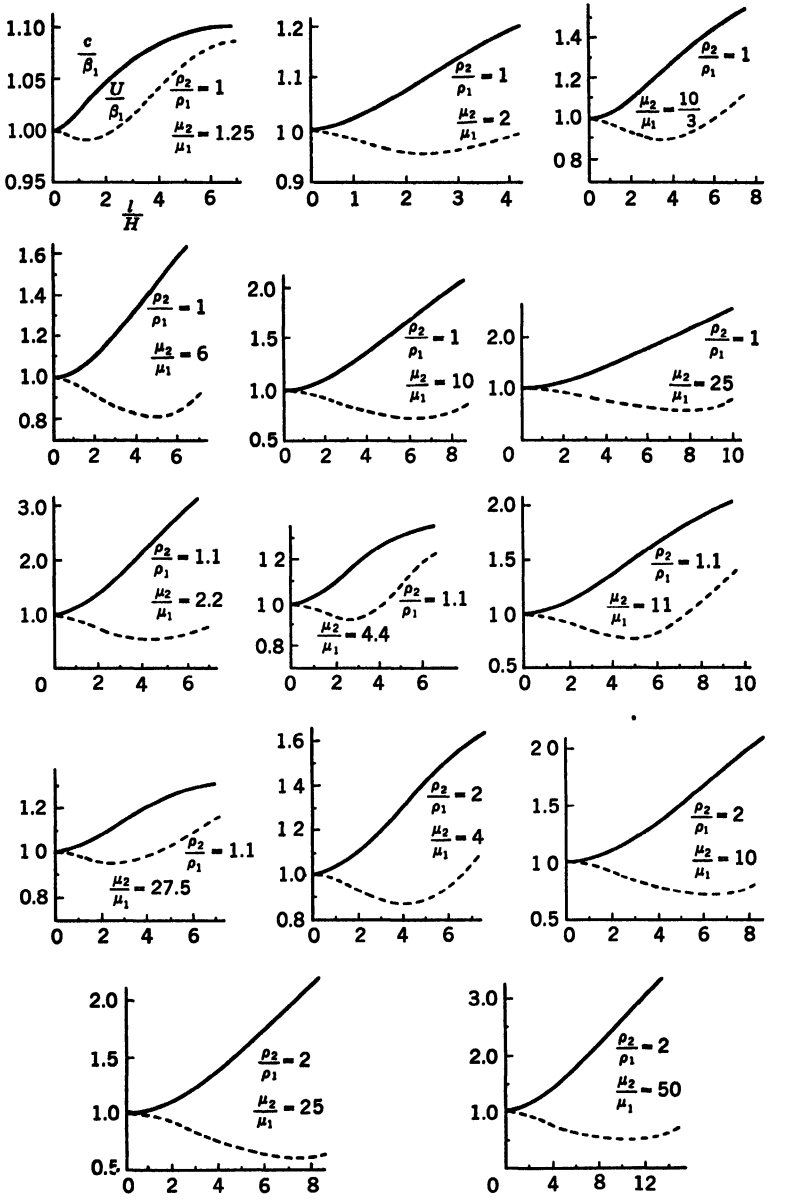


FIG. 4–51. Phase- (c/β_1) and group-velocity (U/β_1) curves of Love waves for various ratios of μ_2/μ_1 and ρ_2/ρ_1. (*After Kanai, with changed notation.*)

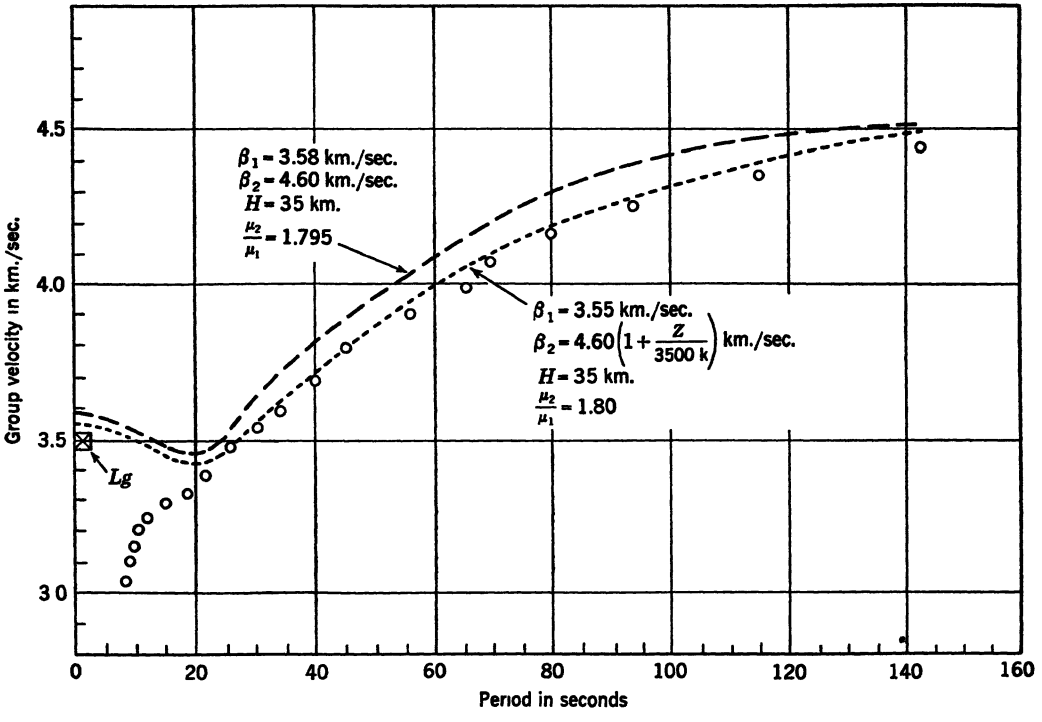


FIG. 4–52. Observed and theoretical (two dashed curves) Love-wave dispersion for continental path. Circles indicate data from Palisades seismogram of Fallon, Nev., earthquake July 6, 1954. Cross corresponds to *Lg* wave. (*Press and Ewing* [125].)

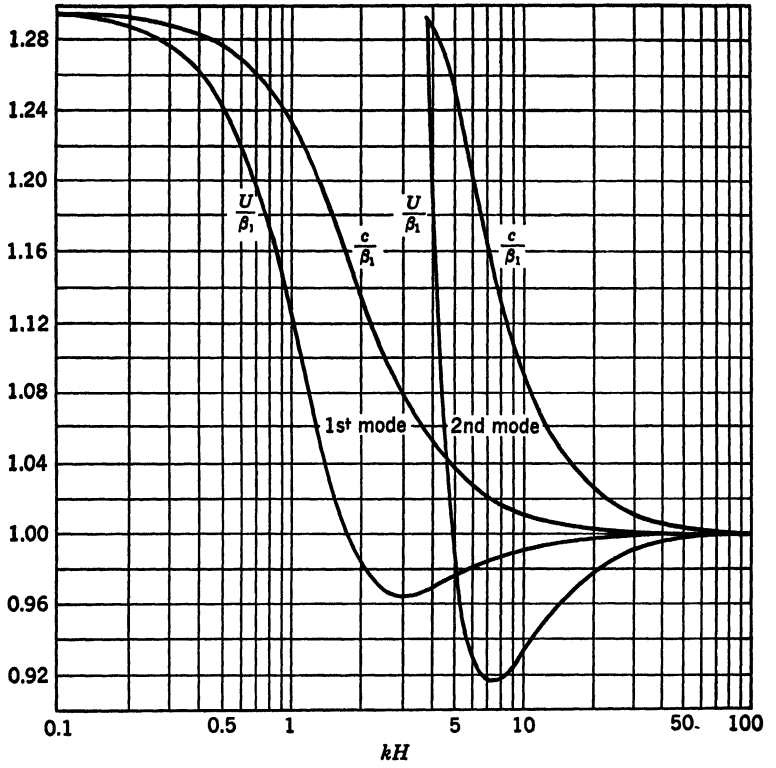


FIG. 4–53. Phase- and group-velocity curves for first- and second-mode Love waves for case $\beta_2/\beta_1 = 1.297$ and $\mu_2/\mu_1 = 2.159$.

source was discussed by Sezawa [167] and further developed by Satô [145]. The methods used are not unlike those given in Sec. 4–2.

Nakano [99*a* and *b*] considered two kinds of solutions representing Love waves. In the case of axial symmetry the motion obtained does not depend on the azimuth and is entirely transverse. If it is assumed that the equal displacements are repeated in n sectors of the free surface, the displacements at each point have both radial and transverse components (and no vertical component). Nakano proved that the amplitude of the radial component must diminish more rapidly than that of the transverse as the product kr increases. Therefore at large distances the latter will predominate.

Love Waves across Continents. Difficulties similar to those described for Rayleigh waves have delayed a complete understanding of Love-wave dis-

TABLE 4–2. THEORETICAL DISPERSION OF LOVE WAVES

μ_2/μ_1	β_1, km/sec	β_2, km/sec	c, km/sec	U, km/sec	$2\pi/kH$
1.80	3.60	4.60	3 60	3 60	0 00
			3 80	3.48	1.72
			4 00	3 51	2.84
			4.20	3.71	4 27
			4 40	4.00	6 86
			4 50	4 31	10 02
			4 56	4.50	16 24
1.795	3 50	4 50	3 50	3.50	0 00
			3 70	3.38	1.72
			3 90	3.43	2.88
			4 00	3 57	3 55
			4.20	3.79	5.37
			4 40	4 23	10 4
			4.48	4.44	23.9
1.80	3 70	4 50	3 70	3.70	0 00
			3 90	3.59	1 71
			4 10	3 67	3 01
			4.30	4 00	5 11
			4.40	4 22	7 66
			4 48	4.44	17.7
1.40	4 00	4 50	4 00	4 00	0 00
			4 10	3 96	1 17
			4 20	3 97	1 97
			4 30	4.10	3 02
			4 40	4 24	5 03
			4 45	4 36	7 19
			4 49	4 47	17 60

persion for continents. Numerical calculation of the dispersion curve by Eq. (4–212) for a single homogeneous layer over a homogeneous substratum is relatively simple, and the many attempts at fitting the observations on the assumptions of this simple type of structure have failed when extended to the entire range of observed periods. In an effort to obtain better agreement with Love-wave observations and to take cognizance of the layering deduced from near-earthquake studies, Stoneley made a series of investigations involving computations for a double surface layer. In his most recent paper [197] on the subject, he offers only tentative support for his preferred scheme of layering and points out that the need for further work is manifest. Jeffreys [76] gave the theory for the effect of a uniform increase in the velocity of shear waves in the substratum.

Wilson [209] used this theory in his study of Love-wave dispersion but concluded that the velocity gradient in the mantle could not be deduced from it. New sources of data from explosion seismology concerning the structure of the upper layers justify reconsideration of this problem.

Wilson and Baykal [210] have discussed the two general methods which have been used in reading data for surface-wave dispersion from seismograms. The older method is to determine period and travel time for the first readable wave or for a conspicuous wave in the train and to construct the dispersion curve by obtaining one or two points from each of many seismograms. The preferred method now involves analysis of the entire train of waves on each seismogram, using only seismograms which provide a clear train of waves in which the period varies gradually with time. By using the method on seismograms from suitably selected earthquakes and propagation paths, the consistency of the experimental data may be greatly increased. It is usually unnecessary to derive true ground displacements from the seismograms prior to dispersion analysis. However, the limitations of all seismographs used must be respected in order to avoid errors from instrumental phase shifts.

Love waves from the Nevada earthquake of July 6, 1954, were recorded for a large range of periods on the Palisades, N. Y., NS seismogram (Fig. 4–54). A dispersion curve covering the period range 8 to 140 sec was

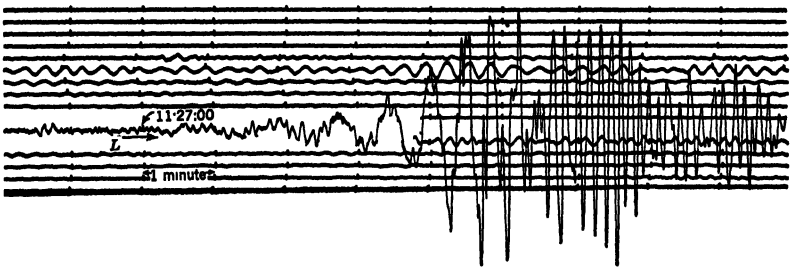


FIG. 4–54. Palisades NS seismograms showing Love waves from the Nevada earthquake of July 6, 1954.

obtained and is plotted in Fig. 4–52. To show the effect of heterogeneity in the mantle, two theoretical curves taken from Wilson have been plotted as dashed lines. These curves are for similar structures except that in one case the mantle velocity gradient is allowed for by Jeffreys' method (Sec. 7–3). It may be seen that the effect of the gradient is to lower the group velocity by about 0.1 km/sec for periods greater than 30 sec. If allowance is made for the effect of the mantle gradient, this theoretical curve would fit the observed data to about 0.1 km/sec for most of the period range 20 to 140 sec. This confirms the seismic-refraction determination of

β_2 to about 0.1 to 0.2 km/sec and suggests that any vertical variation of velocity in the crust must be small. An alternative method of reconciling the theoretical curve with the experimental data would be to assume a shear-wave velocity of about 4.5 km/sec in the substratum. However, this would conflict with observations from explosion and rock-burst seismology and must be rejected.

For periods less than 20 sec the observed points rapidly fall below the theoretical curve. This corresponds to prolongation of the wave train beyond the theoretical limit, an effect not fully understood. Two explanations for the behavior of the short-period waves have been advanced. One involves refraction, reflection, and scattering. The other suggests that it represents an effect of low-velocity sediments.

Another possibility of explaining the low velocities in the period range 8 to 30 sec was eliminated by calculation of the second-mode curve shown in Fig. 4–53. Although the velocities were considerably lowered, the periods involved were too short, the cutoff being at about 13 sec.

A velocity gradient in the crust, as revealed by explosion and near-earthquake investigations, would strongly affect short-period Love-wave propagation, which involves reflections at the top and bottom of the layer at near-grazing angles. A decoupling effect would occur for angles of emergence $\cos^{-1} \beta_1/\beta_{max}$, where β_1 and β_{max} are the shear-wave velocities at the top and bottom of the crust, respectively. Beyond this angle, the energy for the corresponding period (and all shorter periods) may be considered as confined to a "sound channel" bounded by the free surface, and a parallel plane above the interface. Thus reflections from the interface do not occur. No calculation is available in this case.

Love Waves across Oceans. The most recent studies of Love-wave dispersion across ocean basins by Caloi and Marcelli [16] and Wilson [209] reach conclusions which support the result found in seismic-refraction measurements that there is no significant thickness of granite under the ocean basins. However, all the investigators have deduced values for the depth of the Mohorovičić discontinuity which are several times greater than the 5- to 6-km depth commonly found in the refraction work.

For the thin superficial layer indicated by seismic-refraction results, the theoretical group-velocity curve has its minimum at a period between 2 and 4 sec and has essentially reached its constant limiting value of velocity of 4.4 to 4.5 km/sec for periods greater than 20 sec, as may be seen from the curve in Fig. 4–55. It is clearly necessary to investigate periods shorter than 20 sec to study such unexpectedly thin layers. Since the oceanic dispersion is so small compared with the continental dispersion, particularly for periods greater than 20 sec, the correction of the continental part of a mixed path is most important and, in general, cannot be obtained with the accuracy required. It seems that the high values of layer thickness

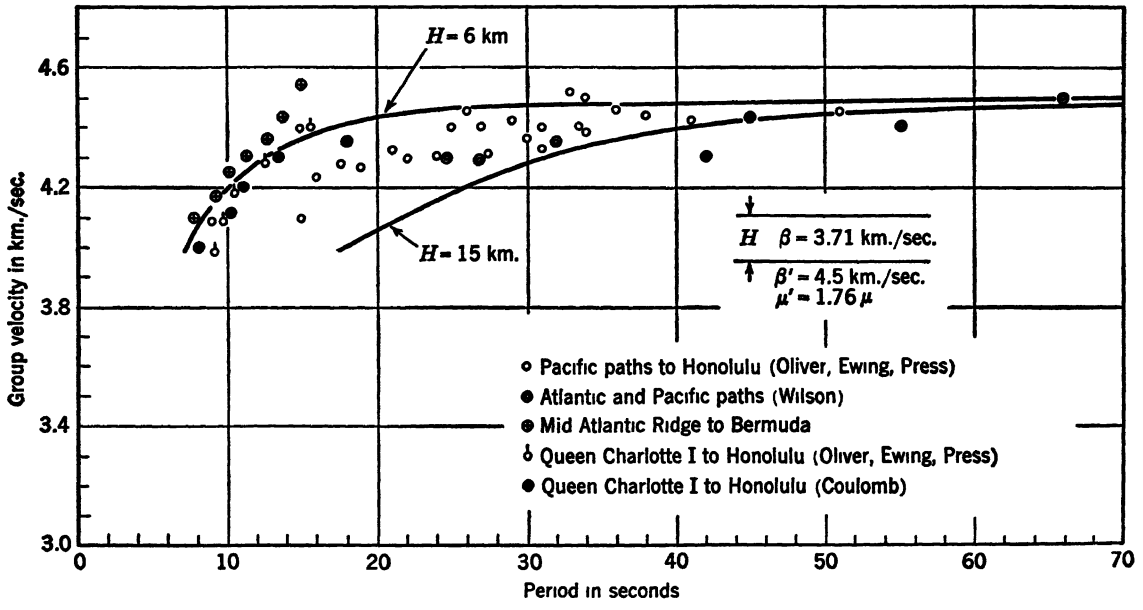


FIG. 4–55. Observed dispersion of Love waves for oceanic path compared with theoretical curves for cases H = 6 km, 15 km.

found by Wilson and Caloi can be attributed to the fact that they confined their attention to waves of period greater than 20 sec and to propagation paths which often included large continental segments.

There are two reasons for the neglect of the shorter-period Love waves. First, they are usually absent, being so severely attenuated at continental boundaries that only seismographs on oceanic islands or extremely near continental boundaries can receive them. Not all oceanic earthquakes excite the short periods, as was shown in a study by Oliver, Ewing, and Press [111] of the Honolulu seismograms. The second reason is that at a period of about 8 sec the short-period Love waves merge with a prolonged train of oscillations which record on all three components with roughly equal amplitudes but without systematic phase relationship. Ignoring this latter part of the surface-wave train, we can extend the observational data on oceanic Love waves to the shorter periods required to determine the crustal structure.

Theoretical curves for the cases $\beta_1 = 3.71$ km/sec, $\beta_2 = 4.50$ km/sec, $\mu_2 = 1.76\mu_1$, $H = 6$ km, and $H = 15$ km are given in Fig. 4–55. Sedimentary and oceanic layers need not be considered for Love waves because of their small or vanishing rigidity.

The curve for $H = 6$ km fits the data reasonably well, a conclusion also reached by Sezawa [170]. For periods greater than 18 sec the observations fall about 0.1 km/sec below the theoretical curve. This discrepancy is of the proper order for the effect of the velocity gradient in the mantle discussed in the previous section. As expected for a thin superficial layer, the observed variations of velocity with period becomes small for periods greater than 20 sec. This accounts for the brief duration of Love waves on seismograms of earthquakes for which the path is principally oceanic.

It is clearly necessary to investigate periods shorter than 20 sec to distinguish between the curves for $H = 6$ km and $H = 15$ km in Fig. 4–55. Since oceanic dispersion for Love waves is small compared with the corresponding continental dispersion, it is important yet difficult to correct for the continental portion of the path. We attribute the excessive values of oceanic crustal thickness deduced in earlier investigations to the fact that few data were available for periods less than 20 sec and that propagation paths contained large continental segments.

Two papers present divergent interpretations of oceanic Love waves. Evernden [33] presents dispersion data for a Pacific path which agrees with our data for periods greater than 20 sec but gives significantly lower group velocities for shorter periods. He infers a crustal structure consisting of 2.5 km with shear velocity 2.31 km/sec, 10 km with 3.87 km/sec, and a mantle with velocity 4.52 km/sec. We prefer to reserve judgment on this result, which proposes a novel crustal structure incompatible with seismic-refraction results, until better-developed short-period Love waves over additional Pacific paths have been examined.

Coulomb [20] studied the Love waves from the Queen Charlotte Islands earthquake of Aug. 22, 1949, on seismograms from Honolulu, Apia, Auckland, and Riverview and presented the first adequate data for the critical period range 8 to 20 sec. Coulomb classified his observations into two groups. The first represents the ordinary long-period Love waves (*G* waves) in the period range 32 to 58 sec. The second group covers the range of periods and velocities from 8 sec and 4.05 km/sec to 34 sec and 4.85 km/sec. No model of crustal layering has been proposed which can transmit surface waves at velocities appreciably greater than 4.5 km/sec, a difficulty that Coulomb pointed out in connection with his tentative suggestion that these were higher-mode Love waves. As an alternative explanation, which is in full accord with the seismic-refraction results, it is here suggested that Coulomb's data be grouped in another way. All velocities less than 4.5 km/sec are taken to represent Love-wave propagation with the single dispersion curve indicated in Fig. 4–55. If the partition of data is allowed, Coulomb's data with that from Honolulu (Oliver, Ewing, and Press [111]) and Bermuda (Ewing and Press, unpublished) give excellent definition of the oceanic Love-wave dispersion curve in the critical short-period range (see Fig. 4–55). The wave with higher velocity may be considered to represent a different kind of phenomenon, related to the sinusoidal-wave trains often observed over continental and mixed paths having velocities between those of the phases *SS* and *Sn* (Caloi [15]).

It is puzzling that no single seismogram has been found which shows an unbroken train of Love waves covering the entire range of periods from 8 to 50 sec.

Lg and Rg Waves. The *Lg* phase is a short-period (1 to 6 sec) large-amplitude arrival in which the motion is predominantly transverse (Figs. 4–56 and 4–57) but accompanied by appreciable vertical components. The phase occurs only when the earthquake epicenter and the seismograph station are so situated as to make the path entirely continental. As little as 2° intervening ocean is sufficient to eliminate the phase entirely (Press and Ewing [125]).

The velocity of *Lg* is 3.51 km/sec, a value essentially equal to the velocity of shear waves in the upper part of the continental crust. Although the precise mechanism of *Lg* propagation is not understood, it is certain that transmission of shear waves through a very efficient wave guide is involved. Short-period transverse (*SH*) waves propagating with this velocity are included in the classical Love-wave theory in the limit $U = c = \beta_1$, as indicated in Fig. 4–53. Similarly, short-period shear waves polarized vertically (*SV*) occur in the second and higher modes of Rayleigh waves propagating in the crustal layer (Fig. 4–35). Both of these mechanisms can explain many of the characteristic features of *Lg*. If velocity gradients occur in the crust, there will be a tendency for the short-period *Lg* phase

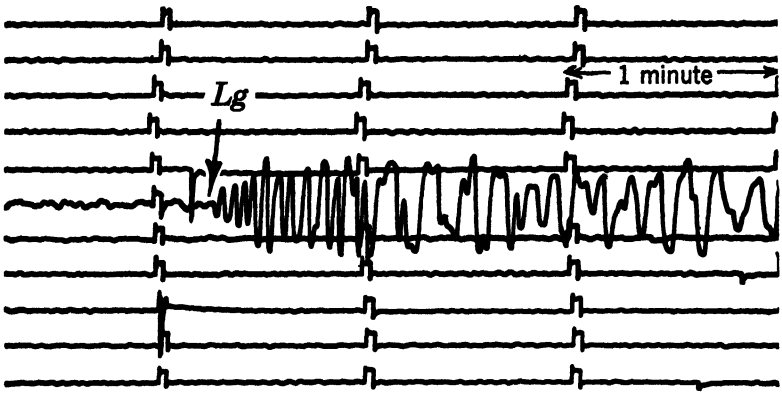


FIG. 4–56. Palisades NS seismogram of *Lg* phase from Southern California earthquake of July 28, 1950, magnitude 5.3.

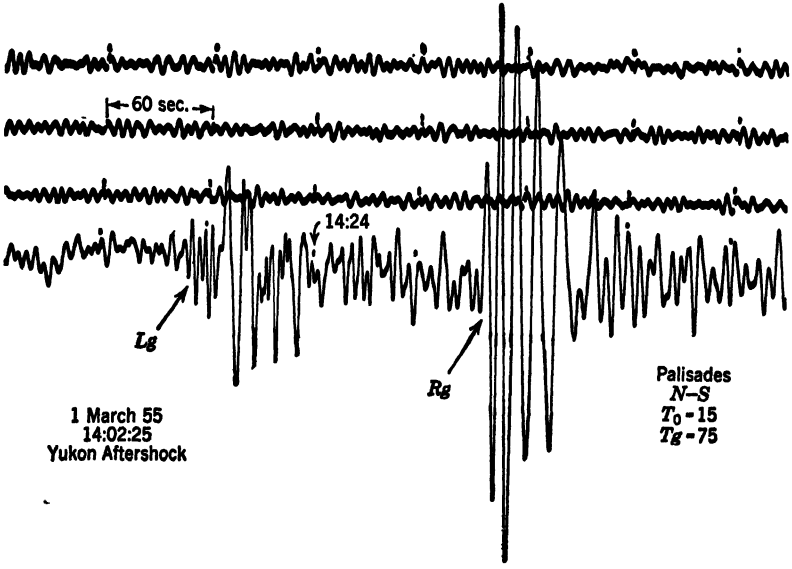


FIG. 4–57. Palisades NS seismogram of *Lg* and *Rg* waves from Yukon aftershock of March 1, 1955.

to concentrate in the zone of lowest velocity. Two such mechanisms are indicated in Fig. 4–58. A principal problem of the *Lg* phase is its long duration, which is probably caused by reflection and scattering rather than low values of group velocity.

Observations of *Rg* (see Figs. 4–38 and 4–57) establish these waves as Rayleigh waves, from their orbital motion and velocity. The phase occurs

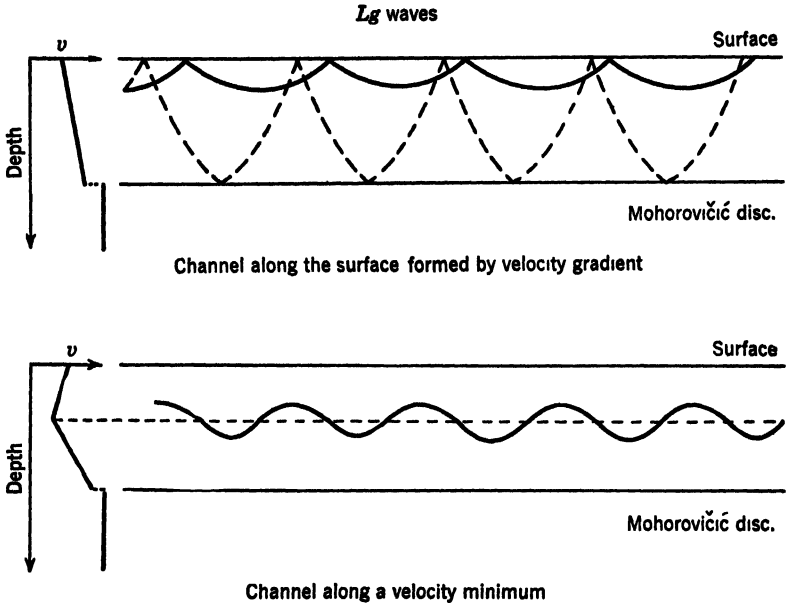


FIG. 4–58. Two possible mechanisms for *Lg* propagation. In the first, *Lg* propagation is confined to the uppermost part of the layer. In the second, *Lg* waves propagate near the depth of minimum velocity.

with periods of 8 to 12 sec and is also restricted to continental paths. Some recordings of *Rg* under particularly favorable conditions (see Fig. 4–57) show inverse dispersion. In all probability, *Rg* waves correspond to propagation according to the portion of the Rayleigh-wave dispersion curve of Fig. 4–36 falling to the left of the minimum value of group velocity.

Lg waves may be used to determine whether the crust beneath a given area is continental or oceanic. The experimental procedure simply is to search for the phase on seismograms, its presence or absence indicating either continental or oceanic plus continental path, respectively. It has been found without exception that the crust is typically continental in any large area where the water depth is less than about 1,000 fathoms

and typically oceanic for water depths greater than about 2,000 fathoms (Press and Ewing [125], and Oliver, Ewing, and Press [112]).

Other Investigations. The important question concerning the existence of a relation between the thickness of a layer and the amplitudes of waves propagating in the system has been discussed by several investigators. Using Sezawa's theory, Sezawa and Kanai [178] expressed the amplitudes of Love waves in terms of the thickness of a surface layer. They discussed also (Sezawa and Kanai [179]) the analogous problem for Rayleigh waves. A conclusion reached by Lee [85, 86] should also be mentioned in this connection. He showed that for Rayleigh waves the effect of a thin layer is more pronounced for horizontal than for vertical motion. Many examples of group-velocity curves for Love waves as well as for Rayleigh waves were computed by Kanai [81].

As mentioned above, the problem of two-dimensional propagation of body waves as well as surface waves in a two-layered solid half space was studied in detail by Newlands [105] by a method different from those discussed in preceding sections. If a line source is at a depth $z = h$ in a layer having the thickness H, the solution for an initial P wave is obtained by combining the direct wave (φ_0) and the wave (φ_r) reflected at the free surface, apparently originating at the image source at $z = -h$. The sum of these is

$$\begin{aligned}\varphi_{0r} = \varphi_0 + \varphi_r &= -4e^{i\omega t}\int_0^\infty e^{-\nu_1 z}\sinh \nu_1 h \cos kx \frac{dk}{\nu_1} \quad \text{for } h \le z \le H \\ &= -4e^{i\omega t}\int_0^\infty e^{-\nu_1 h}\sinh \nu_1 z \cos kx \frac{dk}{\nu_1} \quad \text{for } 0 \le z \le h \end{aligned} \tag{4-213}$$

with the supplementary potentials φ_1 and ψ_1 for the layer ($0 \le z \le H$)

$$\begin{aligned}\varphi_1 &= 4e^{i\omega t}\int_0^\infty [Ae^{-(z-H)\nu_1} + Be^{(z-H)\nu_1}] \cos kx\, dk \\ \psi_1 &= 4e^{i\omega t}\int_0^\infty [Ce^{-(z-H)\nu_1'} + De^{(z-H)\nu_1'}] \sin kx\, dk\end{aligned} \tag{4-214}$$

The potentials for the lower medium can be written in the form

$$\begin{aligned}\varphi_2 &= 4e^{i\omega t}\int_0^\infty Q_2 e^{-(z-H)\nu_2} \cos kx\, dk \\ \psi_2 &= 4e^{i\omega t}\int_0^\infty S_2 e^{-(z-H)\nu_2'} \sin kx\, dk\end{aligned} \tag{4-215}$$

Similar formal solutions can be obtained for an initial S pulse.

Substituting these expressions in the six boundary conditions (4-191) and (4-192), we obtain six linear equations to determine the coefficients A, B, C, D, Q_2, S_2. As usual, each of these coefficients can be written as

a quotient of two determinants. The determinant Δ of this system of six equations may be written in the form

$$\Delta = e^{(\nu_1+\nu_1')H} D_1 \left\{ 1 + \frac{D_2}{D_1} e^{-\nu_1' H} + \frac{D_3 + D_4}{D_1} e^{-(\nu_1+\nu_1')H} + \frac{D_5}{D_1} e^{-2\nu_1 H} + \frac{D_6}{D_1} e^{-2(\nu_1+\nu_1')H} \right\} \tag{4-216}$$

where D_i are represented by expressions (4–198) given earlier in this section.

Now, as in Secs. 2–6 and 3–3, the solutions (4–213) to (4–215) may be generalized for the case of an initial P pulse having the form of a Heaviside unit function. For $0 \leq z \leq h$

$$\Phi_1 = \frac{1}{2\pi i} \int_\Omega (\varphi_{0r} + \varphi_1) e^{i\omega t} \frac{d\omega}{\omega}$$
$$\Psi_1 = \frac{1}{2\pi i} \int_\Omega \psi_1 e^{i\omega t} \frac{d\omega}{\omega} \tag{4-217}$$

where φ_{0r}, φ_1, and ψ_1 are given by (4–213) and (4–214). The coefficients $A, B, \cdots$ must be replaced by the quotients mentioned above, the determinant Δ being written in the form (4–216).

The integrals in (4–213) to (4–215) are of the form $\int_0^\infty G(k) \cos kx\, dk$ or $\int_0^\infty kG(k) \sin kx\, dk$. Newlands obtained their approximate values using the Bromwich [13] expansion method (see also Sec. 2–5). This method yields an important interpretation of the expressions obtained, since each term of the series corresponds to a different kind of pulse. If the complex variable ζ in (4–213) to (4–216) is taken instead of k, it can be easily seen that a general term of the series into which the φ and ψ integrands are expanded contains an exponential of the form

$$\exp [i\omega t - i\zeta x - h_1 \nu_1 - h_2 \nu_1'] \tag{4-218}$$

where h_1 and h_2 are linear forms in h, z, and H. These expansions hold under the condition that the sum of the second and later terms in braces in (4–216) is very small compared with 1.

Now if contour integration is applied to each term in the manner suggested by Lapwood (Chap. 2, Ref. 25), all four branch points $k_{\alpha 1}$, $k_{\alpha 2}$, $k_{\beta 1}$, $k_{\beta 2}$ must be considered since the separate terms are not necessarily even functions of ν_i and ν_i' $(j = 1, 2)$. There are also contributions from the poles due to the existence of roots of the Rayleigh equation (2–28) and of the Stoneley equation (3–139).

The contribution due to each term of the series representation of (4–213) to (4–215) is then composed of several parts. Each of these is generated by the branch line corresponding to Re $\nu_i = 0$, etc. (see Sec. 2–5), or by a pole. The main contribution to the integrand along a loop $\mathfrak{L}_{\alpha 1}$, $\mathfrak{L}_{\beta 1}$, $\cdots$

comes from the neighborhood of the corresponding branch point ($k_{\alpha 1}$, $\cdots$), and appropriate values of 40 "zero-order" and "first-order" terms were computed by Newlands. The zero-order terms represent the generating pulse and those due to the free surface of the layer. The first- and higher-order terms represent the effect due to the finite depth of the layer and to the presence of the substratum.

The terms obtained could be divided into two groups. In the final form obtained by Newlands for a term of the first group we have a factor equal to the Heaviside unit function $H(t_*)$, where the variable t_* is a linear function of t, x, z, and the layer thickness H. Since the corresponding pulse arrives at the instant $t_* = 0$, this condition will determine the time required to travel from the source to the observer along a minimum-time path. Such a path is determined by the form of t_*. For example, the branch point $k_{\alpha 2}$ leads to a term in which

$$t_* = t - \frac{x}{\alpha_2} - (H - h + z)\left(\frac{1}{\alpha_1^2} - \frac{1}{\alpha_2^2}\right)^{\frac{1}{2}} - H\left(\frac{1}{\beta_1^2} - \frac{1}{\alpha_2^2}\right)^{\frac{1}{2}} \qquad (4\text{–}219)$$

When $t_* = 0$, a pulse arrives, with travel time corresponding to the path $SKMNR$ (Fig. 4–59). This represents a compressional wave propagating in the first medium with the velocity α_1 which strikes the second medium

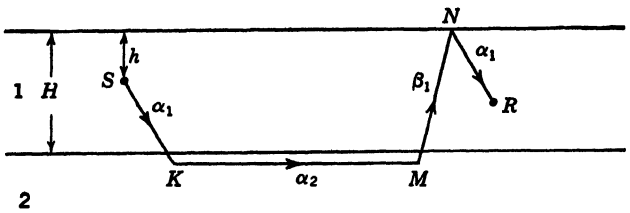


FIG. 4–59. One type of path associated with branch point k_{α_2}.

at the critical angle. It travels with the velocity α_2 in the substratum and emerges as a shear wave traveling in the direction MN. The latter generates a compressional wave at the free surface which reaches the detector at R.

A second group of terms obtained by Newlands cannot be associated with similar paths. They are called "blunt" pulses. In this case the coefficients ν_1 and ν_1' do not appear in the exponent of (4–218) but in the amplitude factor.

4–6. Three-layered Half Space. In many studies of surface-wave propagation it is necessary to consider systems having more than one superficial layer. An important problem is that of Rayleigh-wave propagation along oceanic paths, where a liquid layer and a basaltic layer of approximately

equal thickness are underlain by the mantle. Other examples involving multiple layers arise in Love- or Rayleigh-wave problems where velocity gradients in the crust or the mantle are approximated by several homogeneous layers.

Oceanic Rayleigh Waves with Layered Substratum. Although the theory presented in Sec. 4–5 accounts for the main features of oceanic Rayleigh-wave dispersion, it has been necessary to make calculations on the effect of an interposed solid layer, with thickness approximately equal to the liquid depth†, between the liquid and the mantle. Two cases arise—one in which the layer is assumed to be granite, according to many geological speculations about the Atlantic Ocean, and one in which it is taken to be basalt, according to refraction measurements.

To derive the period equation appropriate for this problem, we assume that a train of plane waves in the given half space may be represented as follows (Jardetzky and Press [70]):

$$\varphi_1 = \{Be^{\nu_1 z} + Ce^{-\nu_1 z}\}e^{-ikx} \qquad \text{for } 0 \le z \le H_1 \tag{4–220}$$

$$\varphi_2 = \{De^{\nu_2 z} + Ee^{-\nu_2 z}\}e^{-ikx} \qquad \text{for } H_1 \le z \le H_1 + H_2 \tag{4–221}$$

$$\psi_2 = \{Me^{\nu_2' z} + Ne^{-\nu_2' z}\}e^{-ikx} \tag{4–222}$$

$$\varphi_3 = Fe^{-\nu_3 z - ikx} \qquad \text{for } H_1 + H_2 \le z \tag{4–223}$$

$$\psi_3 = Pe^{-\nu_3' z - ikx} \tag{4–224}$$

There are now eight boundary conditions. These stipulate vanishing of stress at the free surface of the liquid layer, continuity of normal stress and displacement and vanishing of tangential stress at the liquid-solid interface, and continuity of normal and tangential stress and displacement at the solid-solid interface.

As usual, we substitute expressions (4–220) to (4–224) into the boundary conditions to obtain eight simultaneous linear equations involving the eight coefficients. As before, the condition that the determinant must vanish in order for a solution to exist gives the period equation. The result is

$$\begin{aligned} &l_0 + l_1' \sinh(n_1 kH) \sinh(n_3 kH) + l_2' \sinh(n_1 kH) \cosh(n_3 kH) \\ &\quad + l_3' \cosh(n_1 kH) \sinh(n_3 kH) + l_4' \cosh(n_1 kH) \cosh(n_3 kH) \\ &\quad + [l_1'' \sinh(n_1 kH) \sinh(n_3 kH) + l_2'' \sinh(n_1 kH) \cosh(n_3 kH) \\ &\quad + l_3'' \cosh(n_1 kH) \sinh(n_3 kH) + l_4'' \cosh(n_1 kH) \cosh(n_3 kH)] \\ &\quad \times \tanh(n_0 kH) = 0 \end{aligned} \tag{4–225}$$

†As before, "liquid" here signifies water plus unconsolidated sediments.

where

$$
\begin{aligned}
l_0 &= 4\left(2 - V_1^2\frac{\alpha_1^2}{\beta_2^2}\right)G_1 \\
l_1' &= \left(2 - V_1^2\frac{\alpha_1^2}{\beta_2^2}\right)^2\frac{G_2}{n_1 n_3} - 4n_1 n_3 G_3 \\
l_2' &= -\left(2 - V_1^2\frac{\alpha_1^2}{\beta_2^2}\right)^2\frac{\rho_3}{\rho_2}V_1^4\frac{n_2\alpha_1^4}{n_1\beta_2^4} + 4\frac{\rho_3}{\rho_2}n_1 n_4 V_1^4\frac{\alpha_1^4}{\beta_2^4} \\
l_3' &= -\left(2 - V_1^2\frac{\alpha_1^2}{\beta_2^2}\right)^2\frac{\rho_3}{\rho_1}\frac{n_4}{n_3}V_1^4\frac{\alpha_1^4}{\beta_2^4} + 4\frac{\rho_3}{\rho_2}n_2 n_3 V_1^4\frac{\alpha_1^4}{\beta_2^4} \\
l_4' &= \left(2 - V_1^2\frac{\alpha_1^2}{\beta_2^2}\right)^2 G_3 - 4G_2
\end{aligned}
\tag{4-226}
$$

and

$$
\begin{aligned}
l_1'' &= -\frac{\rho_1\rho_3}{\rho_2^2}\frac{n_1 n_4}{n_0 n_3}V_1^8\frac{\alpha_1^8}{\beta_2^8} \\
l_2'' &= \frac{\rho_1}{\rho_2}\frac{n_1}{n_0}V_1^4\frac{\alpha_1^4}{\beta_1^4}G_3 \\
l_3'' &= \frac{\rho_1}{\rho_2}\frac{1}{n_0 n_3}V_1^4\frac{\alpha_1^4}{\beta_2^4}G_2 \\
l_4'' &= -\frac{\rho_1\rho_3}{\rho_2^2}\frac{n_2}{n_0}V_1^8\frac{\alpha_1^8}{\beta_2^8}
\end{aligned}
\tag{4-227}
$$

with

$$
V_1 = \frac{c}{\alpha_1} \tag{4-228}
$$

$$
G_1 = XZ - n_2 n_4 WY \quad G_2 = Z^2 - n_2 n_4 Y^2 \quad G_3 = n_2 n_4 W^2 - X^2 \tag{4-229}
$$

where

$$
\begin{aligned}
X &= \frac{\rho_3}{\rho_2}V_1^2\frac{\alpha_1^2}{\beta_2^2} - 2\left(\frac{\mu_3}{\mu_2} - 1\right) \\
Y &= V_1^2\frac{\alpha_1^2}{\beta_2^2} + 2\left(\frac{\mu_3}{\mu_2} - 1\right) = V_1\frac{\alpha_1^2}{\beta_2^2} + W \\
Z &= \frac{\rho_3}{\rho_2}V_1^2\frac{\alpha_1^2}{\beta_2^2} - V_1^2\frac{\alpha_1^2}{\beta_2^2} - 2\left(\frac{\mu_3}{\mu_2} - 1\right) = X - V_1^2\frac{\alpha_1^2}{\beta_2^2} \\
W &= 2\left(\frac{\mu_3}{\mu_2} - 1\right)
\end{aligned}
\tag{4-230}
$$

and

$$n_0 = \sqrt{1 - V_1^2} \qquad n_1 = \sqrt{1 - \frac{V_1^2\alpha_1^2}{\alpha_2^2}} \qquad n_2 = \sqrt{1 - \frac{V_1^2\alpha_1^2}{\alpha_3^2}}$$
$$n_3 = \sqrt{1 - \frac{V_1^2\alpha_1^2}{\beta_2^2}} \qquad n_4 = \sqrt{1 - \frac{V_1^2\alpha_1^2}{\beta_3^2}} \tag{4-231}$$

Numerical values, calculated from Eqs. (4–225) to (4–231), are available for three cases, as follows:

	Layer	α, km/sec	β, km/sec	ρ, gm/cm^3	H, km
Case 1	0	1 52	0	1 00	5 57
	1	5.50	3 18	2 67	5 57
	2	8.10	4.68	3.00	∞
Case 2	0	1 52	0	1 00	5 57
	1	7 90	4 56	3 00	∞
Case 3	0	1 52	0	1 00	5 57
	1	6 90	3 98	2 67	5 57
	2	8 10	4 68	3 00	∞

Group-velocity curves for these cases are presented in Fig. 4–60. These show that cases 2 and 3 are experimentally indistinguishable from each other for the periods between 15 and 40 sec, the range covered by observational data. Case 1 differs from these by an amount which might be detected by the study of well-chosen seismograms (see Fig. 4–61).

Love Waves. A necessary condition for the existence of Love waves is the presence of one superficial solid layer. These waves can also occur in more complicated structures. Stoneley [192] gave the theory of a generalized type of Love wave for a three-layered solid medium extended from (1) $z = \infty$ to $z = 0$, (2) $z = 0$ to $z = -H$, and (3) $z = -H$ to $z = -\infty$. Using notations similar to those in Sec. 4–5, we write the corresponding period equation

$$\tan(k\hat{\gamma}_2 H) = \hat{\gamma}_2\mu_2 \frac{\gamma_1\mu_1 + \gamma_3\mu_3}{\hat{\gamma}_2^2\mu_2^2 - \gamma_1\mu_1\gamma_3\mu_3} \tag{4-232}$$

where

$$\gamma_1 = \sqrt{1 - \frac{c^2}{\beta_1^2}} \qquad \hat{\gamma}_2 = \sqrt{\frac{c^2}{\beta_2^2} - 1} \qquad \gamma_3 = \sqrt{1 - \frac{c^2}{\beta_3^2}} \tag{4-233}$$

Real roots of (4–232) occur when $\beta_2 < c < \beta_1$ or β_3.

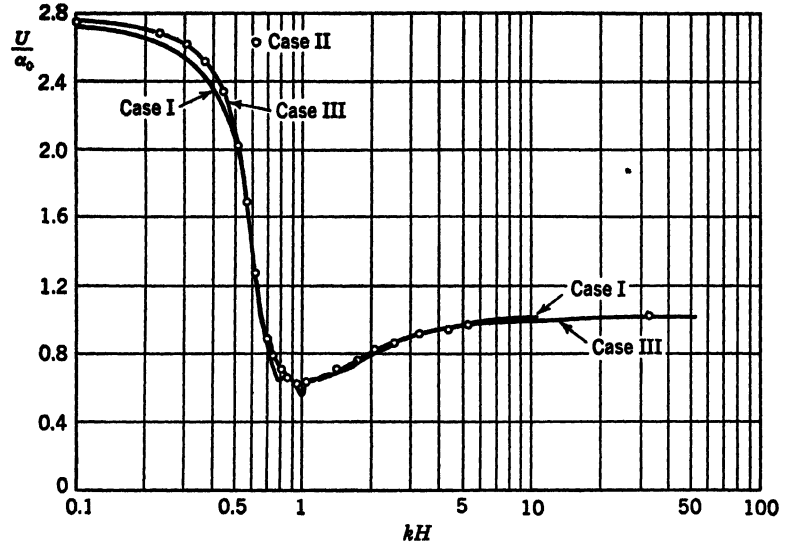


FIG. 4–60. Group-velocity curves for a liquid superposed on two solid layers: cases 1, 2, and 3.

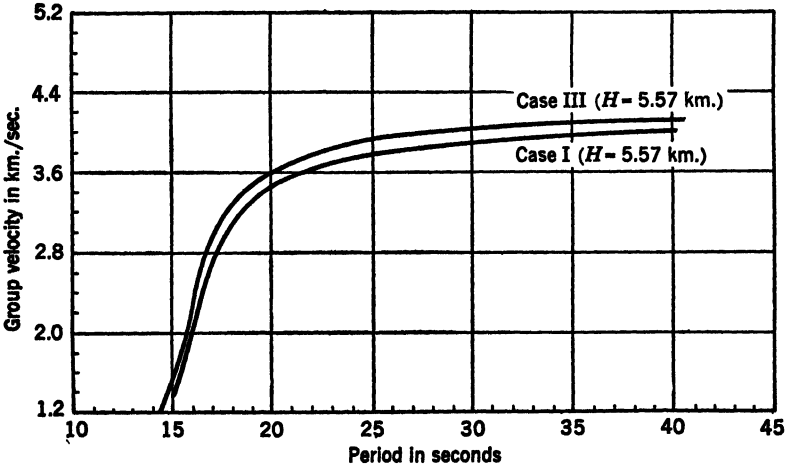


FIG. 4–61. Group-velocity curves for a liquid superposed on two solid layers: cases 1 and 3 with $H = 5.7$ km.

The problem of propagation of Love waves in a double surface layer (three-layered solid half space) was considered by Gutenberg [53], Stoneley and Tillotson [194], and Stoneley [195].

Jeffreys [77] studied the formation of Love waves in two layers in contact, each having finite thickness. In the limit the thickness of the second layer was then taken very large.

To determine the structure of the continental parts of the earth's crust, Stoneley and Tillotson assumed the following layering: (1) granite, from $z = 0$ to $-H_1$; (2) an intermediate basaltic layer, from $z = 0$ to $z = H_2$; and (3) the subjacent material taken as ultrabasic rock, from $z = H_2$ to $z = \infty$. For this case $\beta_3 > \beta_2 > \beta_1$.

Love waves may also exist when $\beta_3 > \beta_1 > \beta_2$. This case was investigated by Stoneley [198]. The effect of the low-velocity layer on Love waves is greater than on Rayleigh waves, since the latter can exist in a homogeneous medium. The low-velocity layer was found to have little influence on the surface amplitudes of Love waves in the range of periods customarily studied. For a high-velocity intermediate layer there is a certain critical wavelength beyond which the Love waves will not exist. Now, under the condition that the velocities β_i increase with the depth, and $\mu_3 > \mu_2 > \mu_1$, $\rho_3 > \rho_2 > \rho_1$, three cases occur: (1) $\beta_3 > c > \beta_2 > \beta_1$, (2) $\beta_3 > \beta_2 > c > \beta_1$, and (3) $\beta_3 > \beta_2 > \beta_1 > c$.

In case (1) we put

$$\hat{\gamma}_1 = \sqrt{\frac{c^2}{\beta_1^2} - 1} \qquad \hat{\gamma}_2 = \sqrt{\frac{c^2}{\beta_2^2} - 1} \qquad \gamma_3 = \sqrt{1 - \frac{c^2}{\beta_3^2}} \tag{4-234}$$

and

$$\begin{aligned} v_1 &= (Ae^{ik\hat{\gamma}_1 z} + Be^{-ik\hat{\gamma}_1 z})e^{ik(ct-x)} \\ v_2 &= (Ce^{ik\hat{\gamma}_2 z} + De^{-ik\hat{\gamma}_2 z})e^{ik(ct-x)} \qquad v_3 = Ee^{-k\gamma_3 z}e^{ik(ct-x)} \end{aligned} \tag{4-235}$$

The boundary conditions such as (4-208) to (4-210) must be supplemented by the equations $v_2 = v_3$ and $(p_{zy})_2 = (p_{zy})_3$ at $z = H_2$. The resulting five homogeneous linear equations with respect to the coefficients $A, B, \cdots, E$ can have solutions different from zero if their determinant is zero. This gives the period equation as follows:

$$\begin{aligned} \mu_2^2\hat{\gamma}_2^2 \tan(k\hat{\gamma}_2 H_2) - \mu_2\hat{\gamma}_2\mu_3\gamma_3 + \mu_1\hat{\gamma}_1\mu_2\hat{\gamma}_2 \tan(k\hat{\gamma}_1 H_1) \\ + \mu_1\hat{\gamma}_1\mu_3\gamma_3 \tan(k\hat{\gamma}_1 H_1) \tan(k\hat{\gamma}_2 H_2) = 0 \end{aligned} \tag{4-236}$$

To pass to case 2, we may observe that $\hat{\gamma}_2$ becomes pure imaginary, and for case 3 $\hat{\gamma}_1$ also does. Putting $\hat{\gamma}_2 = i\gamma_2$ and $\hat{\gamma}_1 = i\gamma_1$, where γ_1 and γ_2 are real, we obtain the corresponding period equations in which tanh will replace the corresponding tan function. Some general conclusions may be drawn from these period equations. The dispersion curve is continuous

through the value $c = \beta_2$. For cases 1 and 2 there exist one or more nodal planes as in the two-layer case. No roots of Eq. (4–236) exist for case 3. Since the phase velocity c is determined in terms of k by this equation, the group velocity can be readily found. Stoneley and Tillotson made numerical calculations for $H_2 = H_1$, and Stoneley [195] has investigated the case $H_2 = 2H_1$.

The extension of these theoretical results to a triple surface layer resting on a uniform substratum was given by Stoneley [196]. The number of boundary conditions is again increased, and we obtain a vanishing determinant of the seventh order. Period equations can be written to conform to the different cases which are determined by values of β, and c, and the range of existence of Love waves can be found. As an application, the thickness of the sedimentary layer of the continents was determined as about 3 km.

Satô [142] discussed the problem of propagation of Love waves in a double superficial layer with special emphasis on the condition of existence of these waves. As a necessary condition, he found that the velocity of distortional waves in one of the layers must be less than that in the semi-infinite substratum. Numerical examples are given for some cases.

Satô [143] also examined the question of using dispersion curves corresponding to an equivalent single layer instead of those for double superficial layers. He concluded that one can find a dispersion curve corresponding to a single layer which fits that of a double-layer structure very well. However, there can be a large discrepancy in the estimated thickness, density, and rigidity of the layers.

4–7. Air-coupled Rayleigh Waves. In most investigations of elastic-wave propagation in solids in contact with the atmosphere, the effect of the atmosphere may be neglected, because of the great density contrast. In some cases, however, resonant coupling may occur for a particular frequency, so that even though the energy flux across the interface is slight, the intensity of the signal transferred may be significant because of constructive interference.

In the first case we shall discuss an impulsive disturbance generated in the air impinging on a plane interface bounding a system of solid or liquid layers in which free waves may travel parallel to the interface with a range of phase velocities including the speed of sound in air.

To a first approximation we may neglect the reaction of the surface wave on the air and follow Lamb's [84, p. 413] (see Sec. 2–8) treatment of the effect of a disturbance produced by a traveling line source. The traveling impulse may be replaced by a succession of infinitesimal impulses placed at equal intervals of time along the path of the disturbance. Each impulse initiates a train of dispersive waves, and constructive interference is possible only for those waves whose phase velocity c equals the speed of the traveling

disturbance c_0. The energy thus transferred will form a train of constant-frequency waves. The duration of the wave train at any distance will be proportional to $|1/c_0 - 1/U_0|$, where U_0 is the group velocity corresponding to c_0. The waves will extend ahead of the air pulse if $U_0 > c_0$; otherwise they will lag behind.

The preceding discussion also applies if the disturbance is in the layered medium and the detector in the air. Here the primary disturbance in the layered medium consists of a train of dispersive waves, and the atmospheric disturbance has the same characteristics as the constant-frequency train mentioned above.

A problem of some practical importance is that of atmospheric coupling to Rayleigh waves generated in the earth by explosions, the "ground roll" of seismic prospecting. Suitable Rayleigh-wave dispersion is introduced by the stratification of sedimentary layers. A quantitative treatment for the case of a homogeneous surface layer over a homogeneous substratum can be given in the following form:

Assume a point source in the air at a distance h above the plane $z = 0$, and denote by ρ_j, α_j and β_j, $j = 0, 1, 2$ the densities and velocities of dilatational and shear waves in the air and in the solid media 1 and 2. Let H be the layer thickness, and use the notations and conditions given in the preceding problems. The potentials which satisfy all boundary conditions can be written in the form

$$\varphi_0' = Ae^{\nu_0 z} J_0(kr) \qquad \text{for } -\infty < z \leq -h \tag{4-237}$$

$$\varphi_0'' = \{Be^{\nu_0 z} + Ce^{-\nu_0 z}\} J_0(kr) \qquad \text{for } -h \leq z \leq 0 \tag{4-238}$$

$$\varphi_1 = \{De^{\nu_1 z} + Ee^{-\nu_1 z}\} J_0(kr) \qquad \text{for } 0 \leq z \leq H \tag{4-239}$$

$$\psi_1 = \{Me^{\nu_1' z} + Ne^{-\nu_1' z}\} J_0(kr) \tag{4-240}$$

$$\varphi_2 = Fe^{-\nu_2 z} J_0(kr) \qquad \text{for } H \leq z < \infty \tag{4-241}$$

$$\psi_2 = Pe^{-\nu_2' z} J_0(kr) \tag{4-242}$$

To determine the nine coefficients $A, B, \cdots, P$, we obtain the following system of equations from the boundary conditions:

$$\rho_0\omega^2(B + C) + (2\mu_1 k^2 - \rho_1\omega^2)(D + E) + 2\mu_1\nu_1' k^2(M - N) = 0 \tag{4-243}$$

$$2\nu_1(D - E) + (\nu_1'^2 + k^2)(M + N) = 0 \tag{4-244}$$

$$(2\mu_1 k^2 - \rho_1\omega^2)(De^{\nu_1 H} + Ee^{-\nu_1 H}) - (2\mu_2 k^2 - \rho_2\omega^2)Fe^{-\nu_2 H} + 2\mu_1\nu_1' k^2(Me^{\nu_1' H} - Ne^{-\nu_1' H}) + 2\mu_2\nu_2' k^2 Pe^{-\nu_2' H} = 0 \tag{4-245}$$

$$2\mu_1\nu_1(De^{\nu_1 H} - Ee^{-\nu_1 H}) + 2\mu_2\nu_2 Fe^{-\nu_2 H} + \mu_1(\nu_1'^2 + k^2)(Me^{\nu_1' H} + Ne^{-\nu_1' H}) - \mu_2(\nu_2'^2 + k^2)Pe^{-\nu_2' H} = 0 \tag{4-246}$$

$$\nu_0(B - C) - \nu_1(D - E) - k^2(M + N) = 0 \tag{4-247}$$

$$\nu_1(De^{\nu_1 H} - Ee^{-\nu_1 H}) + \nu_2 Fe^{-\nu_2 H} + k^2(Me^{\nu_1' H} + Ne^{-\nu_1' H}) - k^2 Pe^{-\nu_0' H} = 0 \tag{4-248}$$

$$De^{\nu_1 H} + Ee^{-\nu_1 H} - Fe^{-\nu_2 H} + \nu_1'(Me^{\nu_1' H} - Ne^{-\nu_1' H}) + \nu_2' Pe^{-\nu_0' H} = 0 \tag{4-249}$$

$$Ae^{\nu_0 h} - Be^{\nu_0 h} - Ce^{-\nu_0 h} = 0 \tag{4-250}$$

$$\nu_0 Ae^{\nu_0 h} + \nu_0(Ce^{-\nu_0 h} - Be^{\nu_0 h}) = 2\hat{Z} \tag{4-251}$$

The last two conditions arise because of the existence of the point source at $z = -h$ [see Eqs. (4–39) and (4–40)], where $\hat{Z}$ represents the strength of the source.

From the last two equations it follows that

$$B = A - \frac{\hat{Z}}{\nu_0} e^{-\nu_0 h} \qquad C = \frac{\hat{Z}}{\nu_0} e^{\nu_0 h} \tag{4-252}$$

These two coefficients can be easily eliminated from the system of Eqs. (4–243) to (4–251). We restrict ourselves to the analysis of the frequency equation.

After certain transformations of the determinant of the preceding system of equations, the frequency equation takes the form (Jardetzky and Press [69]) shown in (4-253), page 233.

In order to develop this determinant, we introduce the notations of Love (Chap. 3, Ref. 26) and Lee [85, 86], which were used by the latter in the investigation of propagation of Rayleigh waves in solid layers. This problem has been discussed in Sec. 4–5. We now put

$$V = \frac{c}{\beta_1} \tag{4-254}$$

$$n_0 = \sqrt{1 - \frac{V^2\beta_1^2}{\alpha_0^2}} \qquad n_1 = \sqrt{1 - \frac{V^2\beta_1^2}{\alpha_1^2}} \qquad n_2 = \sqrt{1 - \frac{V^2\beta_1^2}{\alpha_2^2}}$$
$$n_3 = \sqrt{1 - V^2} \qquad n_4 = \sqrt{1 - \frac{V^2\beta_1^2}{\beta_2^2}} \tag{4-255}$$

$$X = \frac{\rho_2}{\rho_1} V^2 - 2\left(\frac{\mu_2}{\mu_1} - 1\right) \qquad Y = V^2 + 2\left(\frac{\mu_2}{\mu_1} - 1\right) = V^2 + W$$
$$Z = \frac{\rho_2}{\rho_1} V^2 - V^2 - 2\left(\frac{\mu_2}{\mu_1} - 1\right) = X - V^2 \qquad W = 2\left(\frac{\mu_2}{\mu_1} - 1\right) \tag{4-256}$$

$$G_1 = XZ - n_2 n_4 WY \qquad G_2 = Z^2 - n_2 n_4 Y^2 \qquad G_3 = n_2 n_4 W^2 - X^2 \tag{4-257}$$

and $l_0 = 4(2 - V^2)G_1$

$$l_1 = (2 - V^2)^2 \frac{G_2}{n_1 n_3} - 4n_1 n_3 G_3 - \frac{\rho_0 \rho_2}{\rho_1^2} \frac{n_1 n_4}{n_0 n_3} V^8$$

$$
\begin{vmatrix}
1 & \dfrac{2\mu_1 k^2 - \rho_1\omega^2}{\rho_0\omega^2 \cosh \nu_1 H} & 0 & 0 & \dfrac{2\mu_1 \nu_1' k^2}{\rho_0\omega^2 \sinh \nu_1' H} & 0 & 0 \\
0 & 0 & \dfrac{2\nu_1}{\sinh \nu_1 H} & \dfrac{2k^2\beta_1^2 - \omega^2}{\beta_1^2 \cosh \nu_1' H} & 0 & 0 & 0 \\
0 & -\rho_1\omega^2 & -\rho_1\omega^2 & 0 & 0 & \rho_2\omega^2 + 2(\mu_1 - \mu_2)k^2 & 2(\mu_2 - \mu_1)\nu_2' k^2 \\
0 & 0 & 0 & -\rho_1\omega^2 & -\rho_1\omega^2 & 2(\mu_2 - \mu_1)\nu_2 & \rho_2\omega^2 + 2(\mu_1 - \mu_2)k^2 \\
0 & -\dfrac{2\mu_1 k^2 - \rho_1\omega^2}{\rho_0\omega^2 \cosh \nu_1 H} & 0 & \dfrac{-\omega^2}{2\nu_0 \beta_1^2 \cosh \nu_1' H} & \dfrac{-2\mu_1 \nu_1' k^2}{\rho_0\omega^2 \sinh \nu_1' H} & 0 & 0 \\
0 & \nu_1 \tanh \nu_1 H & \nu_1 \coth \nu_1 H & k^2 & k^2 & \nu_2 & -k^2 \\
0 & 1 & 1 & \nu_1' \tanh \nu_1' H & \nu_1' \coth \nu_1' H & -1 & \nu_2'
\end{vmatrix} = 0 \quad (4\text{-}253)
$$

$$l_2 = -(2 - V^2)^2 \frac{\rho_2}{\rho_1}\frac{n_2}{n_1} V^4 + \frac{4\rho_2}{\rho_1} n_1 n_4 V^2 + \frac{\rho_0}{\rho_1}\frac{n_1}{n_0} V^4 G_3 \qquad (4\text{–}258)$$

$$l_3 = -(2 - V^2)^2 \frac{\rho_2}{\rho_1}\frac{n_4}{n_3} V^4 + \frac{4\rho_2}{\rho_1} n_2 n_3 V^4 + \frac{\rho_0}{\rho_1}\frac{1}{n_0 n_3} V^4 G_2$$

$$l_4 = (2 - V^2)^2 G_3 - 4G_2 - \frac{\rho_0 \rho_2}{\rho_1^2}\frac{n_2}{n_0} V^8$$

Then the frequency equation takes the form

$$l_0 + l_1 \sinh(n_1 kH) \sinh(n_3 kH) + l_2 \sinh(n_1 kH) \cosh(n_3 kH)$$
$$+ l_3 \cosh(n_1 kH) \sinh(n_3 kH) + l_4 \cosh(n_1 kH) \cosh(n_3 kH) = 0 \qquad (4\text{–}259)$$

The last terms of the coefficients l, having the factors ρ_0 and $1/n_0$ represent the influence of the air. As a check, we note that for $\rho_0 = 0$ Eq. (4–259) reduces to that given by Lee [85] for the case of a single surface layer (Eq. 4–202). The phase velocity may be calculated from the frequency equation (4–259), and the group velocity by graphical differentiation. The following data were used to compute the phase- and group-velocity curves of Fig. 4–62:

$$\alpha_0 = 1070 \text{ ft/sec} \qquad \beta_1 = 800 \text{ ft/sec} \qquad \frac{\beta_2}{\beta_1} = 3.14$$

$$\frac{\rho_2}{\rho_1} = 1.39 \qquad \frac{\rho_0}{\rho_1} = 0.001 \qquad \frac{\mu_2}{\mu_1} = 13.77 \qquad \frac{\lambda_1}{\mu_1} = \frac{\lambda_2}{\mu_2} = 1$$

These values were chosen as most representative of near-surface conditions often encountered in seismic prospecting. It is obvious that air-coupling effects are negligible when $0.9194\beta_1 > \alpha_0$, since $c > \alpha_1$ for this case, and the air-coupling term of the frequency equation which contains the factor $\rho_0/\rho_1 n_0$ is very small. In the case of air-coupled Rayleigh waves, real values of kH must correspond to the interval $0.9194 \leq V \leq \alpha_0/\beta_1$. The upper limit is equal in this case to 1.3375 (= 1,070/800). For larger values of V, the roots of the frequency equation become complex, corresponding to a radiation of energy from the ground to the air. The roots of Eq. (4–259) were computed by successive approximations, using Lee's computations to obtain the first approximation. Group velocity U was obtained by graphical differentiation of the phase-velocity curve.

The phase-velocity curve in the case of air-coupled Rayleigh waves differs very little from the curve given by Lee until the neighborhood of the critical point $V = 1.3375$ is reached (Fig. 4–62). At this point, it deviates to the left and intersects the V axis at a point $1.3374 < V < 1.3375$.

The real part of complex roots above the critical value is very close to the corresponding roots of the frequency equation without air-connected

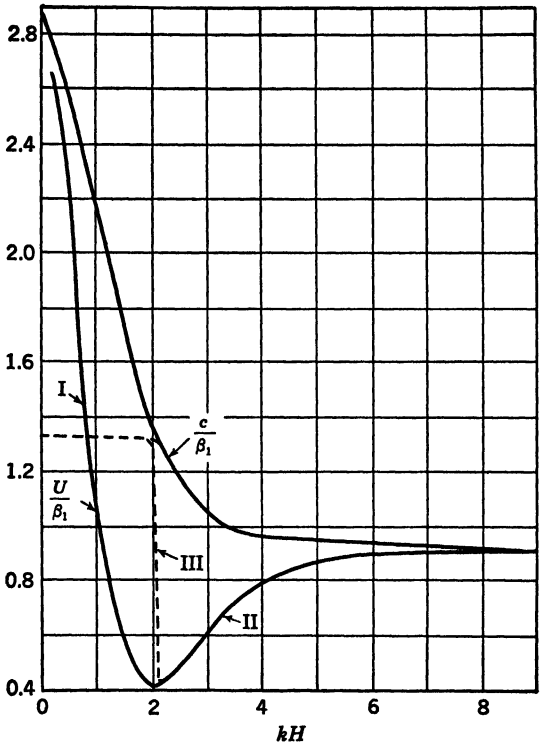


FIG. 4–62. Phase- and group-velocity curves for air-coupled Rayleigh waves.

terms. But it deviates to the right in the neighborhood of the critical value and "ends" at the point $kH = 4.2515$ of the line $V = 1.3375$.

In Fig. 4–62 the heavy lines indicate where atmospheric influence is negligible, and the results are similar to those obtained by Lee. The dashed lines represent new branches introduced by air coupling.

The group-velocity curve in Fig. 4–62 is divided into branches I, II, III, each of which represents a different train of waves. Branch I corresponds to the dispersive train of Rayleigh waves observed on seismograms of earthquakes. Branch I also accounts for the dispersive Rayleigh waves usually associated with ground roll. These waves first appear as long-period arrivals traveling with the speed of Rayleigh waves in the bottom layer, $U = 0.9194\beta_2$. Succeeding waves gradually decrease in period, since kH increases as the group velocity decreases. Waves continue to arrive with decreasing period until a time corresponding to propagation at the minimum value of group velocity. Waves with group-velocity values near the

minimum in branch I have phase velocities approaching the speed of sound in air from the side $c > \alpha_0$. These waves are attenuated, since kH is complex and has an increasingly large imaginary component as $c \to \alpha_0$. Only real parts of kH have been plotted in Fig. 4–62.

Branch II represents a dispersive train of waves beginning as a high-frequency arrival at a time corresponding to propagation with the velocity of Rayleigh waves in the surface layer. The frequency of these waves decreases as time progresses until the time corresponding to propagation at the minimum group velocity, when the waves of branch I and branch II merge to form a single train of waves having a discrete frequency.

Branch III represents an additional train introduced by coupling of Rayleigh waves to atmospheric compressional waves. This train begins at a time corresponding to propagation at the speed of sound in air and continues with almost constant frequency until the time $t = r/0.44\beta_1$. The phase velocity of these waves should be close to the speed of sound in air. From the qualitative discussion at the beginning of this section and from the discussion of air-coupled flexural waves in Sec. 6–3, we might expect these waves to be prominent for a source in the air recorded by a pickup on the ground and for a source within the ground recorded by a microphone in the air. These statements could be established rigorously by a very tedious calculation of amplitude functions. There is an additional branch with dispersion features similar to branch III, corresponding to the complex phase velocities whose real parts are greater than but close to α_0.

Higher modes of propagation exist at correspondingly higher frequencies.

Air-coupled Ground Roll. The introduction of air shooting in seismic prospecting has stimulated studies of ground roll generated by air shots (see also Sec. 4–5). Press and Ewing [122] reported results of a series of field experiments in which the elevation of the shot position was varied from 30 ft above the ground to 40 ft in the ground. The seismograms shown in Fig. 4–63 cover the horizontal distance range from 2,200 to 2,650 feet. In these seismograms the first three traces represent reception at 2,200 feet (1) from a radial horizontal geophone, (2) spurious and (3) from a vertical geophone. The succeeding traces are vertical geophones spaced 50 ft apart out to a distance of 2,650 ft. Ground roll on the air-shot record consists essentially of a constant-frequency train of waves immediately following the air wave. Retrograde elliptical particle motion shown by the first two traces proves these to be Rayleigh waves. The phase velocity of these waves remains close to the speed of sound in air for about 6 cycles. It is interesting to note that the character of the ground roll from an air shot is independent of shot point elevation in the range 0 to 30 ft at horizontal distances which are large in comparison with the shot elevation.

In marked contrast, the ground roll from the buried shots (Fig. 4–63)

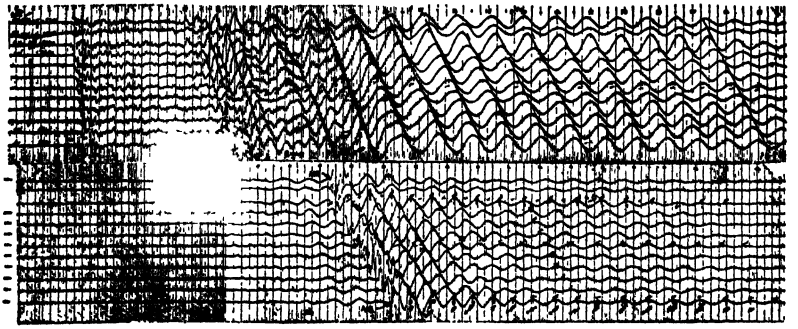


FIG. 4–63. Contrast between dispersive Rayleigh waves from a hole shot at 80-ft depth (upper seismogram) and constant-frequency Rayleigh waves from an air shot 10 ft above the ground (lower seismogram).

consists of a dispersed train of Rayleigh waves. With the use of a Fourier analyzer, phase velocity was obtained as a function of frequency for these waves (Fig. 4–64). It was found that the frequency of waves whose phase velocity equaled the speed of sound in air was identical with the frequency of the waves following the air pulse on the air-shot records. It was also found that a microphone situated a few feet above the ground detected a train of constant-frequency waves from a buried shot. The characteristics of the ground roll for air shots and buried shots are thus seen to confirm the theory developed in the preceding pages.

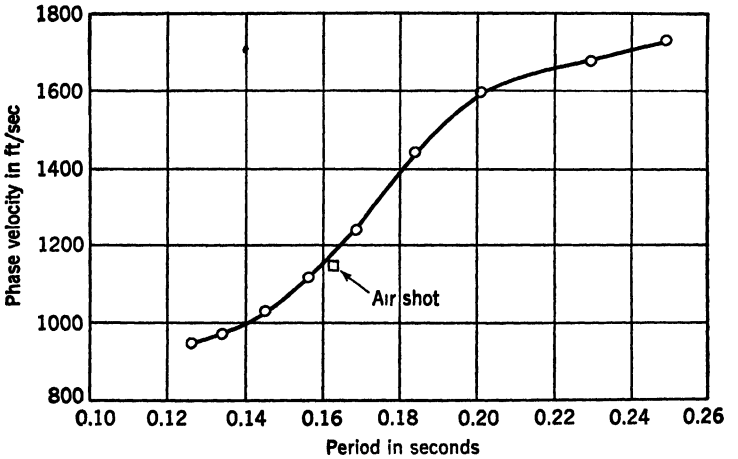


FIG. 4–64. Phase-velocity curve for hole shot and air shot as determined by Fourier analysis of two records at 800 ft and 1,200 ft. (*Courtesy of M. B. Dobrin.*)

Benioff, Ewing, and Press [4] have suggested that the low-frequency sounds often reported as accompanying earthquakes illustrate the same phenomenon and are principally confined to regions in which the superficial strata can propagate surface waves at velocities near the speed of sound in air.

4–8. Remarks concerning the Problem of an n-layered Half Space. The general equations which hold for a stratified medium were given in Sec. 4–1. The equations of motion such as Eqs. (4–1), for example, have to be written separately for each homogeneous part of a medium and the appropriate values given to the elastic constants and the density. The boundary conditions can be formulated in different ways. Assuming a welded contact between the layers, we have conditions (4–2) and the second set of Eqs. (4–4). At the free surface the boundary conditions for the n-layered elastic half space are represented by the first set of Eqs. (4–4). The complete discussion of the problem of wave propagation in an n-layered medium presents very great mathematical difficulties. Therefore this discussion is usually reduced to some special points. Thus the principal aim of Rayleigh's investigation [130] was to determine the reflection coefficient in the first layer of a stratified medium which is composed of a set of equal parallel homogeneous layers. In optics a similar problem has been discussed by several writers (see, for example, Försterling [47] and van Cittert [206]). The transmission of plane compressional waves through a system of alternate layers of two different substances, in connection with the question of acoustic filtering, was investigated by Lindsay [88]. Brekhovskikh [8] suggested a new method of deriving the reflection coefficients by using a certain differential equation of the first order instead of the wave equation. Using the formulas developed by Thomson [202], Haskell [62] reformulated the problem in terms of matrices and suggested a new systematic computational procedure. A derivation of period equations based on the condition of constructive interference is given by Tolstoy and Usdin [203]. Numerous papers have been written about other particular cases of the problem.

The transmission of compressional and distortional waves through layered media in a direction normal to the boundaries was investigated by Sezawa and Nishimura [159] and Sezawa and Kanai [164, 165, 183]. They first considered a single layer embedded in a medium, obtaining a solution in series form, the terms being Fourier integrals. The integrands in these expressions were determined as usual by means of boundary conditions. By contour integration these integrals could be evaluated and some general conclusions drawn.

We have seen the importance of the period equation in every particular problem shown before. Some general properties of solutions for n parallel layers in a half space will now be discussed (see Jardetzky [71]). For each

layer the potentials φ_i and ψ_i, like those in Eqs. (4–11) to (4–14), are

$$\begin{aligned}\varphi_i &= \int_0^\infty Q_i' J_0(kr)e^{-\nu_i z}\,dk + \int_0^\infty Q_i'' J_0(kr)e^{\nu_i z}\,dk \\ \psi_i &= \int_0^\infty S_i' J_0(kr)e^{-\nu_i' z}\,dk + \int_0^\infty S_i'' J_0(kr)e^{\nu_i' z}\,dk\end{aligned} \tag{4–260}$$

with $Q_n'' = S_n'' = 0$. For the layer in which the point source of compressional waves is situated we must add the potential

$$\varphi_0 = \int_0^\infty e^{-\nu_i|z-h|} J_0(kr)\,\frac{k\,dk}{\nu_i} \tag{4–261}$$

There are $4n - 2$ unknown coefficients Q and S in Eqs. (4–260). They can be found as usual by solving the system of linear equations determined by the boundary conditions.

In order to write these conditions, we must use the expressions for q_i, w_i, $(p_{zz})_i$, and $(p_{zr})_i$:

$$q_i = \frac{\partial}{\partial r}\left(\varphi_i + \frac{\partial \psi_i}{\partial z}\right) \qquad w_i = \frac{\partial \varphi_i}{\partial z} + k^2\psi_i \tag{4–262}$$

$$(p_{zz})_i = \mu_i\left(\frac{\partial q_i}{\partial z} + \frac{\partial w_i}{\partial r}\right) = \frac{\partial}{\partial r}\left[2\mu_i \frac{\partial \varphi_i}{\partial z} + \mu_i(\nu_i'^2 + k^2)\psi_i\right] \tag{4–263}$$

$$(p_{zr})_i = \lambda_i \nabla^2 \varphi + 2\mu_i \frac{\partial w_i}{\partial z} = a_i\varphi_i + 2\mu_i k^2 \frac{\partial \psi_i}{\partial z} \tag{4–264}$$

where

$$a_i = 2\mu_i k^2 - \rho_i\omega^2 = 2\mu_i\nu_i^2 - \lambda_i k_{\alpha i}^2 \tag{4–265}$$

Upon substituting (4–260) in (4–262) to (4–264) each boundary condition may be reduced to the vanishing of an integral taken with respect to the parameter k. A sufficient condition for this is that the integrands vanish. Taking the source to be in the first layer at a depth h, we must use for this layer the potential $\phi_1 = \varphi_1 + \varphi_0$. Equating the integrands to zero and canceling common factors such as the Bessel functions, we obtain for the free surface the two equations for the vanishing of stress components at $z = 0$:

$$-2\mu_1\nu_1 Q_1' + 2\mu_1\nu_1 Q_1'' + \mu_1(\nu_1'^2 + k^2)S_1' + \mu_1(\nu_1'^2 + k^2)S_1'' = 2\mu_1 k e^{-\nu_1 h} \tag{4–266}$$

$$a_1 Q_1' + a_1 Q_1'' - 2\mu_1 k^2\nu_1' S_1' + 2\mu_1 k^2 \nu_1' S_1'' = -a_1 \frac{k}{\nu_1} e^{-\nu_1 h} \tag{4–267}$$

Likewise, for the first interface at $z = H_1$ we obtain the four equations:

$$\begin{aligned}&Q_1' e^{-\nu_1 H_1} + Q_1'' e^{\nu_1 H_1} - \nu_1' S_1' e^{-\nu_1' H_1} + \nu_1' S_1'' e^{\nu_1' H_1} - Q_2' e^{-\nu_2 H_1} - Q_2'' e^{\nu_2 H_1} \\ &\quad + \nu_2' S_2' e^{-\nu_2' H_1} - \nu_2' S_2'' e^{\nu_2' H_1} = -\frac{k}{\nu_1} e^{-\nu_1(H_1 - h)}\end{aligned} \tag{4–268}$$

$$-\nu_1 Q_1' e^{-\nu_1 H_1} + \nu_1 Q_1'' e^{\nu_1 H_1} + k^2 S_1' e^{-\nu_1' H_1} + k^2 S_1'' e^{\nu_1' H_1} + \nu_2 Q_2' e^{-\nu_2 H_1} - \nu_2 Q_2'' e^{\nu_2 H_1} - k^2 S_2' e^{-\nu_2' H_1} - k^2 S_2'' e^{\nu_2' H_1} = k e^{-\nu_1 (H_1 - h)} \tag{4-269}$$

$$-2\mu_1 \nu_1 Q_1' e^{-\nu_1 H_1} + 2\mu_1 \nu_1 Q_1'' e^{\nu_1 H_1} + \mu_1 (\nu_1'^2 + k^2) S_1' e^{-\nu_1' H_1} + \mu_1 (\nu_1'^2 + k^2) S_1'' e^{\nu_1' H_1} + 2\mu_2 \nu_2 Q_2' e^{-\nu_2 H_1} - 2\mu_2 \nu_2 Q_2'' e^{\nu_2 H_1} - \mu_2 (\nu_2'^2 + k^2) S_2' e^{-\nu_2' H_1} - \mu_2 (\nu_2'^2 + k^2) S_2'' e^{\nu_2' H_1} = 2\mu_1 k e^{-\nu_1 (H_1 - h)} \tag{4-270}$$

$$a_1 Q_1' e^{-\nu_1 H_1} + a_1 Q_1'' e^{\nu_1 H_1} - 2\mu_1 k^2 \nu_1' S_1' e^{-\nu_1' H_1} + 2\mu_1 k^2 \nu_1' S_1'' e^{\nu_1' H_1} - a_2 Q_2' e^{-\nu_2 H_1} - a_2 Q_2'' e^{\nu_2 H_1} + 2\mu_2 k^2 \nu_2' S_2' e^{-\nu_2' H_1} - 2\mu_2 k^2 \nu_2' S_2'' e^{\nu_2' H_1} = -a_1 \frac{k}{\nu_1} e^{-\nu_1 (H_1 - h)} \tag{4-271}$$

Similar equations may be written for the other interfaces, which will differ from the above only in the absence of terms for the source potential and in the subscripts. For the last interface we finally have

$$Q_{n-1}' e^{-\nu_{n-1} z_{n-1}} + Q_{n-1}'' e^{\nu_{n-1} z_{n-1}} - \nu_{n-1}' S_{n-1}' e^{-\nu_{n-1}' z_{n-1}} + \nu_{n-1}' S_{n-1}'' e^{\nu_{n-1}' z_{n-1}} - Q_n e^{-\nu_n z_{n-1}} + \nu_n' S_n e^{-\nu_n' z_{n-1}} = 0 \tag{4-272}$$†

$$-\nu_{n-1} Q_{n-1}' e^{-\nu_{n-1} z_{n-1}} + \nu_{n-1} Q_{n-1}'' e^{\nu_{n-1} z_{n-1}} + k^2 S_{n-1}' e^{-\nu_{n-1}' z_{n-1}} + k^2 S_{n-1}'' e^{\nu_{n-1}' z_{n-1}} + \nu_n Q_n e^{-\nu_n z_{n-1}} - k^2 S_n e^{-\nu_n' z_{n-1}} = 0 \tag{4-273}$$

$$-2\mu_{n-1} \nu_{n-1} Q_{n-1}' e^{-\nu_{n-1} z_{n-1}} + 2\mu_{n-1} \nu_{n-1} Q_{n-1}'' e^{\nu_{n-1} z_{n-1}} + \mu_{n-1} (\nu_{n-1}'^2 + k^2) S_{n-1}' e^{-\nu_{n-1}' z_{n-1}} + \mu_{n-1} (\nu_{n-1}'^2 + k^2) S_{n-1}'' e^{\nu_{n-1}' z_{n-1}} + 2\mu_n \nu_n Q_n e^{-\nu_n z_{n-1}} - \mu_n (\nu_n'^2 + k^2) S_n e^{-\nu_n' z_{n-1}} = 0 \tag{4-274}$$

$$a_{n-1} Q_{n-1}' e^{-\nu_{n-1} z_{n-1}} + a_{n-1} Q_{n-1}'' e^{\nu_{n-1} z_{n-1}} + 2\mu_{n-1} k^2 \nu_{n-1}' S_{n-1}' e^{-\nu_{n-1}' z_{n-1}} - 2\mu_{n-1} k^2 \nu_{n-1}' S_{n-1}'' e^{\nu_{n-1}' z_{n-1}} - a_n Q_n e^{-\nu_n z_{n-1}} + 2\mu_n k^2 \nu_n' S_n e^{-\nu_n' z_{n-1}} = 0 \tag{4-275}$$

Thus the boundary conditions lead to a system of $4n - 2$ linear equations. The determinant of this system is (4–276), pages 242 and 243.

Then the solutions of Eqs. (4–266) to (4–275) are

$$Q_g' = \frac{\Delta_{gq}'}{\Delta} \qquad S_g' = \frac{\Delta_{gs}'}{\Delta} \tag{4-277}$$

where f denotes $'$ or $''$ (can be omitted for $g = n$), $g = 1, 2, \cdots, n$, and the subscript q or s shows that the corresponding determinant is taken for a coefficient Q or S, respectively, and Δ is given by (4–276).

†The index prime has been omitted for Q_n and S_n.

Now the period equation

$$\Delta = 0 \tag{4-278}$$

expresses the fact that the phase velocity ($c = \omega/k$) depends upon the frequency or wave length in a manner which is determined by the roots of this equation. The period equation (4–278) is too complicated to permit a useful general discussion of its roots. As we have seen before, these roots determine the poles of the integrands in (4–260). However, even in one of the simplest cases, that of a liquid layer overlying a solid half space, Schermann's long proof [151] of the existence of roots shows the difficulties of such a discussion.

We restrict ourselves, therefore, to other considerations concerning the determinant Δ and the solutions (4–260) and (4–261).

This determinant of the $4n - 2$ order is an odd function of each variable ν_j or ν'_j, for $j = 1, 2, \cdots, n - 1$, separately, as may be seen directly in (4–276).

In contrast, Δ is neither an odd nor an even function of ν_n and ν'_n.

Now we can write this determinant in the form

$$\Delta = \sum_{\lambda=1}^{4n-2} (-1)^{\lambda+1} d_{\lambda 1} D_{\lambda 1} = \sum_{\lambda=1}^{4n-2} (-1)^{\lambda+2} d_{\lambda 2} D_{\lambda 2} = \cdots \tag{4-279}$$

where $D_{\lambda\sigma}$ are subdeterminants. We obtain the determinants Δ'_{gq} and Δ'_{g_0} in (4–277) by substituting the right-side members (R_λ) of Eqs. (4–266) to (4–275) in a column determined by the subscripts f, g, q. Then

$$\Delta'_{gq} = \sum_{\lambda=1}^{4n-2} (-1)^{\lambda+\sigma} R_\lambda D'_{\lambda(gq)} \tag{4-280}$$

where σ denotes a column number corresponding to a given combination of f, g, and q and R_λ will replace a column with an odd number for f denoting the index $'$ and a column with an even number for the index $''$.

An important characteristic of the expressions considered above is that we have pairs of coefficients Q or S multiplied by exponential functions in which the exponents differ in sign, for example, $\exp(-\nu_j z)$ and $\exp(\nu_j z)$. We shall consider, therefore, the sums of the corresponding terms, e.g., the sum

$$Q'_g e^{-\nu_g z} + Q''_g e^{\nu_g z}$$

$$= \frac{1}{\Delta} \sum_\lambda R_\lambda [(-1)^{\lambda+\sigma} D'_{\lambda(gq)} e^{-\nu_g z} + (-1)^{\lambda+\sigma+1} D''_{\lambda(gq)} e^{\nu_g z}] \tag{4-281}$$

The exponent $\lambda + \sigma + 1$ in the last term is obviously increased by 1 because the index $'$ in the first term was changed to $''$ in the last one. In order to see whether this expression is an odd or an even function of ν_g, $g = 1, 2, \cdots, n - 1$, let us form the sums in brackets.

$$
\Delta = \begin{vmatrix}
Q_1' & Q_1'' & S_1' & S_1'' & Q_2' & Q_2'' & S_2' & S_2'' & \cdots \\
-2\mu_1\nu_1 & 2\mu_1\nu_1 & \mu_1(\nu_1'^2+k^2) & \mu_1(\nu_1'^2+k^2) & 0 & 0 & 0 & 0 & \cdots \\
a_1 & a_1 & -2\mu_1k^2\nu_1' & 2\mu_1k^2\nu_1' & 0 & 0 & 0 & 0 & \cdots \\
e^{-\nu_1H_1} & e^{\nu_1H_1} & -\nu_1'e^{-\nu_1'H_1} & \nu_1'e^{\nu_1'H_1} & -e^{-\nu_2H_1} & -e^{\nu_2H_1} & \nu_2'e^{-\nu_2'H_1} & -\nu_2'e^{\nu_2'H_1} & \cdots \\
-\nu_1e^{-\nu_1H_1} & \nu_1e^{\nu_1H_1} & k^2e^{-\nu_1'H_1} & k^2e^{\nu_1'H_1} & \nu_2e^{-\nu_2H_1} & -\nu_2e^{\nu_2H_1} & -k^2e^{-\nu_2'H_1} & -k^2e^{\nu_2'H_1} & \cdots \\
-2\mu_1\nu_1e^{-\nu_1H_1} & 2\mu_1\nu_1e^{\nu_1H_1} & \mu_1(\nu_1'^2+k^2)e^{-\nu_1'H_1} & \mu_1(\nu_1'^2+k^2)e^{\nu_1'H_1} & 2\mu_2\nu_2e^{-\nu_2H_1} & -2\mu_2\nu_2e^{\nu_2H_1} & -\mu_2(\nu_2'^2+k^2)e^{-\nu_2'H_1} & -\mu_2(\nu_2'^2+k^2)e^{\nu_2'H_1} & \cdots \\
a_1e^{-\nu_1H_1} & a_1e^{\nu_1H_1} & -2\mu_1k^2\nu_1'e^{-\nu_1'H_1} & 2\mu_1k^2\nu_1'e^{\nu_1'H_1} & -a_2e^{-\nu_2H_1} & -a_2e^{\nu_2H_1} & 2\mu_2k^2\nu_2'e^{-\nu_2'H_1} & -2\mu_2k^2\nu_2'e^{\nu_2'H_1} & \cdots \\
. & . & . & . & . & . & . & . & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{vmatrix}
$$

Q'_{n-1}	Q''_{n-1}	S'_{n-1}	S''_{n-1}	Q_n	S_n
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
.	.	.	.	.	.
$-e^{-\nu_{n-1}z_{n-1}}$	$-e^{\nu_{n-1}z_{n-1}}$	$\nu'_{n-1}e^{-\nu'_{n-1}z_{n-1}}$	$-\nu'_{n-1}e^{\nu'_{n-1}z_{n-1}}$	$-e^{-\nu_n z_{n-1}}$	$\nu'_n e^{-\nu_n' z_{n-1}}$
$\nu_{n-1}e^{-\nu_{n-1}z_{n-1}}$	$-\nu_{n-1}e^{\nu_{n-1}z_{n-1}}$	$-k^2e^{-\nu'_{n-1}z_{n-1}}$	$-k^2e^{\nu'_{n-1}z_{n-1}}$	$\nu_n e^{-\nu_n z_{n-1}}$	$-k^2e^{-\nu_n' z_{n-1}}$
$2\mu_{n-1}\nu_{n-1}e^{-\nu_{n-1}z_{n-1}}$	$-2\mu_{n-1}\nu_{n-1}e^{\nu_{n-1}z_{n-1}}$	$-\mu_{n-1}(\nu'^2_{n-1}+k^2)e^{-\nu'_{n-1}z_{n-1}}$	$-\mu_{n-1}(\nu'^2_{n-1}+k^2)e^{\nu'_{n-1}z_{n-1}}$	$2\mu_n\nu_n e^{-\nu_n z_{n-1}}$	$-\mu_n(\nu'^2_n+k^2)e^{-\nu_n' z_{n-1}}$
$-a_{n-1}e^{-\nu_{n-1}z_{n-1}}$	$-a_{n-1}e^{\nu_{n-1}z_{n-1}}$	$2\mu_{n-1}k^2\nu'_{n-1}e^{-\nu'_{n-1}z_{n-1}}$	$-2\mu_{n-1}k^2\nu'_{n-1}e^{\nu'_{n-1}z_{n-1}}$	$-a_n e^{-\nu_n z_{n-1}}$	$2\mu_n k^2\nu'_n e^{-\nu_n' z_{n-1}}$

(4-276)

By (4–276) it is easy to see that

$$D'_{\lambda(gq)} = \overline{D}'_{\lambda(gq)}e^{-\nu_g s_g} \qquad \overline{D}''_{\lambda(gq)}e^{\nu_g s_g} = D''_{\lambda(gq)} \tag{4–282}$$

where

$$\overline{D}'_{\lambda(gq)} = \sum_{\sigma\neq\lambda}{}' (-1)^{\sigma+\tau} d'_\sigma C'_{\sigma\lambda(gq)} \quad \overline{D}''_{\lambda(gq)} = \sum_{\sigma\neq\lambda}{}' (-1)^{\sigma+\tau+1} d''_\sigma C''_{\sigma\lambda(gq)} \tag{4–283}$$

and the second set of subdeterminants $C''_\sigma = C'_\sigma$ does not depend on ν_g. The elements of the determinant Δ (4–276) denoted by d'_σ and d''_σ and belonging to two neighboring columns for Q or for S coefficients display certain properties. Either

$$d''_\sigma = d'_\sigma = d_\sigma \qquad \text{or} \qquad d''_\sigma = -d'_\sigma \tag{4–284}$$

if there is a factor ν_g present in the coefficient itself, i.e., not in the exponent. We will now denote these elements by $\nu_g d^*_\sigma$ or $-\nu_g d^*_\sigma$.

Thus by (4–281) we consider pairs of terms such as

$$D'_{\lambda(gq)}e^{-\nu_g s} - D''_{\lambda(gq)}e^{\nu_g s} = \overline{D}'_{\lambda(gq)}e^{-\nu_g(s_g+s)} - \overline{D}''_{\lambda(gq)}e^{\nu_g(s_g+s)}$$

Now inserting (4–283) and taking account of the signs, we obtain the sum of two functions

$$\sum_1 d_\sigma C_{\sigma\lambda(gq)}[e^{-\nu_g(s_g+s)} - e^{\nu_g(s_g+s)}] + \sum_2 d^*_\sigma C_{\sigma\lambda(gq)}[e^{-\nu_g(s_g+s)} + e^{\nu_g(s_g+s)}]\nu_g \tag{4–285}$$

It is evident that both functions are odd in ν_g. This final conclusion is reached for $g = 2, \cdots, n-1$, but it holds also for $g = 1$. On comparing the first four columns of Δ and the next set of four we see that for $g = 1$ there are two additional lines but the elements in these lines display the same properties as all others with respect to their composition and signs. Therefore, noting that by (4–260) the potentials φ and ψ depend on ν_g because of the factors Q and S and that these factors are determined by the ratios (4–277), we see that φ and ψ are even functions of ν_g, $g = 1, 2, \cdots, n-1$. The numerators and the denominators in (4–277) are odd functions of these variables.

From this property of the functions φ and ψ an important conclusion about the existence of the branch line integrals can be drawn. The coefficients Q and S in (4–260) are given in terms of the parameter k. When evaluating the integrals (4–260) in the complex k plane it seems that each radical ν requires the consideration of a branch line integral.

On considering wave propagation in a three-layered liquid half space, Pekeris [116] found that two branch line integrals vanish. We now are able to make a general conclusion that, in all cases of wave propagation from a point source in a half space formed by parallel layers displaying

different elastic properties, all expected branch line integrals vanish except the last one corresponding to ν_n. Thus, as was pointed out by Jardetzky [71], in problems dealing with wave propagation in a layered half space, each potential φ and ψ is necessarily obtained in the form of a sum, of which the first part represents a discrete spectrum of modes determined by the residues and the second a continuous spectrum given in the form of a branch line integral.

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CHAPTER 5

THE EFFECTS OF GRAVITY, CURVATURE, AND VISCOSITY

5–1. Gravity Terms in General Equations. The problems discussed in the preceding chapters form the basic part of the theory of wave propagation in stratified media. Despite the fact that the conditions of propagation of a disturbance were simplified, these problems presented great mathematical difficulties. Nevertheless, it was possible to find solutions in several cases, and the approximations obtained have proved adequate for the explanation of many observed phenomena of wave propagation. Three factors, which were not taken into account, may be of importance in some applications. In Eqs. (1–13) we omitted the body forces $\rho X, \cdots$. Moreover, we considered wave propagation only in those cases where all boundaries and interfaces are parallel planes, but in some cases of importance the interfaces are curved, usually being cylinders or spheres. The third effect which has been neglected thus far is that of viscosity or other deviations from ideal elasticity.

We first consider the gravity terms in the equations of motion. Usually we can assume that there is a constant field of forces (X, Y, Z) acting. Since equations for a fluid must hold for small motions starting from an undisturbed state, we can consider the initial conditions $u = v = w = 0$, $\rho = \rho_0$, and $p = p_0$. It may be proved (Lamb [26, p. 556]) that in this case the velocity potential must satisfy the equation

$$\frac{\partial^2 \tilde{\varphi}}{\partial t^2} = \alpha^2 \nabla^2 \tilde{\varphi} + \left(X \frac{\partial \tilde{\varphi}}{\partial x} + Y \frac{\partial \tilde{\varphi}}{\partial y} + Z \frac{\partial \tilde{\varphi}}{\partial z} \right) \tag{5–1}$$

Gravity is the principal force which concerns us in problems of wave propagation, and we can put $X = Y = 0$, $Z = g$ in all cases where the boundaries between homogeneous layers form a set of horizontal parallel planes. It is, of course, assumed that the z axis is perpendicular to these planes and is taken as positive in the direction of the gravity acceleration. Thus, the velocity components of a fluid medium being expressed in terms of $\tilde{\varphi}$ [Eqs. (1–14)], this function must satisfy the equation

$$\frac{\partial^2 \tilde{\varphi}}{\partial t^2} = \alpha^2 \nabla^2 \tilde{\varphi} + g \frac{\partial \tilde{\varphi}}{\partial z} \tag{5–2}$$

instead of the ordinary wave equation formed by the first two terms. For the potential motion of a fluid the equations of motion admit the integral

$$\frac{\partial \bar{\varphi}}{\partial t} + \frac{\mathbf{V}^2}{2} - U + \int \frac{dp}{\rho} = F(t) \tag{5-3}$$

where $\mathbf{V} = \text{grad}\ \bar{\varphi}$

U = potential of body forces

An arbitrary function $F(t)$ is usually included in the potential $\bar{\varphi}$. If the fluid is incompressible, $\rho = \rho_0 = \text{const}$, and if we can neglect the square of the velocity this equation takes the form

$$p = -\rho_0 \frac{\partial \bar{\varphi}}{\partial t} + \rho_0 U + \text{const} \tag{5-4}$$

We can now write

$$p = -\rho_0 \frac{\partial \bar{\varphi}}{\partial t} + \rho_0 g(z + \text{const}) \tag{5-5}$$

and take that value of the arbitrary constant which corresponds to the position of the origin of coordinates.

As to the equations of motion for solid media, they also will be changed by the addition of a term representing the body forces. We write only the first equation of (1–13):

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu) \frac{\partial \theta}{\partial x} + \mu \nabla u + \rho X \tag{5-6}$$

To derive the wave equations (1–22), we assumed that the displacement is represented by a sum of two vectors:

$$\mathbf{s}(u, v, w) = \text{grad}\ \varphi + \text{curl}\ \psi(\psi_1, \psi_2, \psi_3) \tag{5-7}$$

φ and ψ being displacement potentials [Eqs. (1–20)]. We can write, in general, a similar condition for body forces:

$$\mathbf{F}(X, Y, Z) = \text{grad}\ U + \text{curl}\ \mathbf{L}(L_1, L_2, L_3) \tag{5-8}$$

Then instead of Eqs. (1–22) we obtain the equations

$$\frac{\partial^2 \varphi}{\partial t^2} - \alpha^2 \nabla^2 \varphi = U \qquad \frac{\partial^2 \psi_i}{\partial t^2} - \beta^2 \nabla^2 \psi_i = L_i \tag{5-9}$$

under the assumption that ρ is constant. Equations (5–9) have particular solutions (Love [30, p. 304]):

$$\begin{aligned} \varphi &= \frac{1}{4\pi\alpha^2} \iiint \frac{1}{R} U'\left(t - \frac{R}{\alpha}\right) dx'\, dy'\, dz' \\ \psi_i &= \frac{1}{4\pi\beta^2} \iiint \frac{1}{R} L_i'\left(t - \frac{R}{\beta}\right) dx'\, dy'\, dz' \end{aligned} \tag{5-10}$$

If we again consider gravitational forces only, condition (5–8) reduces to $Z = gz + \text{const.}$ Differentiating Eq. (5–6) and two similar equations for the displacements v and w with respect to x, y, z and adding, we obtain

$$\rho \frac{\partial^2 \theta}{\partial t^2} = (\lambda + 2\mu)\nabla^2\theta + \rho\nabla^2 U \tag{5–11}$$

If Poisson's equation is used for the potential U, Eq. (5–11) can be expressed in terms of θ alone.

To evaluate the correction due to fluctuations of the body forces the last term in Eq. (5–6) will be written in the form

$$\rho\left(\frac{\partial U}{\partial x} - \frac{\partial U_0}{\partial x}\right) \tag{5–12}$$

where U_0 corresponds to a certain undisturbed state. Then the last term in Eq. (5–11) becomes

$$\rho(\nabla^2 U - \nabla^2 U_0) \tag{5–13}$$

By Poisson's equation we have, therefore,

$$\nabla^2(U - U_0) = -4\pi f(\rho - \rho_0) = 4\pi f \rho\theta \tag{5–14}$$

approximately, where f is the constant of gravitation and θ is set equal to $-(\rho - \rho_0)/\rho$. Then the equation for θ will have the form

$$\rho \frac{\partial^2 \theta}{\partial t^2} = (\lambda + 2\mu)\nabla^2\theta + 4\pi f \rho^2 \theta \tag{5–15}$$

Jeffreys [19] solved this equation and showed that the correction due to the gravity term is insignificant for compressional waves in the earth. Since the curl of the gravity force vanishes, this force does not affect the propagation of S waves determined by the functions ψ_i.

5–2. Effect of Gravity on Surface Waves. Gravity terms in the equations of motion (5–2) and (5–6) produce modifications in the solutions for surface-wave propagation.

Rayleigh Waves: Incompressible Half Space. In an early paper Bromwich [3] considered the effect of gravity on Rayleigh waves in a solid half space. This effect was introduced in the boundary conditions, omitting the mass terms in the equations of motion (5–6). Moreover, to simplify the problem, Bromwich considered an incompressible solid for which $\lambda \to \infty$ as $\theta \to 0$ in such a manner that $\lambda\theta = \Pi$ remains finite. For the two-dimensional case, Eq. (5–6) and the corresponding equation for w take the form

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \Pi}{\partial x} + \mu\nabla^2 u \qquad \rho \frac{\partial^2 w}{\partial t^2} = \frac{\partial \Pi}{\partial z} + \mu\nabla w \tag{5–16}$$

We also have

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{5-17}$$

Assume u, w, and Π to be proportional to $\exp [i(\omega t - kx)]$; then Eqs. (5–16) and (5–17) become

$$(\nabla^2 + k_\beta^2)u = -\frac{1}{\mu}\frac{\partial \Pi}{\partial x} \qquad (\nabla^2 + k_\beta^2)w = -\frac{1}{\mu}\frac{\partial \Pi}{\partial z} \qquad \nabla^2 \Pi = 0 \tag{5-18}$$

In order to satisfy the last equation, put

$$\Pi = \mu k_\beta^2 P e^{i(\omega t - kx) - kz} \qquad \text{for } 0 \leq z < \infty \tag{5-19}$$

where P is an arbitrary constant. We can define u and w by

$$\begin{aligned} u &= -\frac{1}{\mu k_\beta^2}\frac{\partial \Pi}{\partial x} + Ae^{i(\omega t - kx) - \nu' z} \\ w &= -\frac{1}{\mu k_\beta^2}\frac{\partial \Pi}{\partial z} + Be^{i(\omega t - kx) - \nu' z} \end{aligned} \tag{5-20}$$

which will satisfy Eqs. (5–18), provided that the second terms in (5–20) satisfy the conditions

$$(\nabla^2 + k_\beta^2)u = 0 \qquad (\nabla^2 + k_\beta^2)w = 0 \tag{5-21}$$

or if

$$\nu'^2 = k^2 - k_\beta^2 \tag{5-22}$$

The requirement $\theta = 0$ applied to (5–20) leads to

$$ikA + \nu' B = 0 \tag{5-23}$$

Now, assuming that the normal stress at $z = 0$ is equal to the sum of p_{zz}, by (1–11), and the weight per unit area of an element of height w, we write the boundary conditions in the form

$$\Pi + 2\mu \frac{\partial w}{\partial z} + g\rho w = 0 \qquad \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0 \qquad \text{at } z = 0 \tag{5-24}$$

Substituting (5–19) and (5–20) in (5–24) and taking the determinant of the two equations (5–24) and Eq. (5–23), we obtain the period equation

$$\left(\frac{k_\beta^2}{k^2} - 2\right)^2 - \frac{4\nu'}{k} + \frac{gk}{\omega^2}\frac{k_\beta^4}{k^4} = 0 \tag{5-25}$$

We obtain the reversed sign as compared with the Bromwich period equation since we make use of waves propagating in the positive x direction.

To study the effect of gravity on surface waves in a compressible solid

half space, Love [29] derived the period equation

$$\left(\frac{k_\beta^2}{k^2} - 2\right)^2 - \frac{4\nu\nu'}{k^2} + \frac{4gk}{\omega^2}\left\{\frac{\nu'}{k}\left[\frac{k_\beta^2 k_\alpha^2}{(k_\beta^2 - k_\alpha^2)k^2} - \frac{k_\beta^2 + k_\alpha^2}{k_\beta^2 - k_\alpha^2}\right] - \frac{\nu}{k}\left[\frac{k_\beta^4}{(k_\beta^2 - k_\alpha^2)k^2} - \frac{k_\beta^2 + k_\alpha^2}{k_\beta^2 - k_\alpha^2}\right]\right\} = 0 \qquad (5\text{–}26)$$

where, again,

$$\nu = \sqrt{k^2 - k_\alpha^2} \qquad \nu' = \sqrt{k^2 - k_\beta^2} \qquad k_\alpha = \frac{\omega}{\alpha} \qquad k_\beta = \frac{\omega}{\beta} \qquad (5\text{–}27)$$

In deriving this equation the square of gk/ω^2 was neglected.

If gk/ω^2 is taken to vanish, Eq. (5–26) is readily seen to reduce to Rayleigh's equation for the velocity of surface waves in a solid half space. For an incompressible body, $k_\alpha = 0$, and Eq. (5–26) takes the form (5–25) derived by Bromwich, provided that $(\nu'/k)^2$ is approximated by $\frac{1}{4}(k_\beta^2/k^2 - 2)^2$.

To find the effect of gravity it is convenient to transform Eq. (5–26) to the form, valid for first-order terms in the small quantity $g/k\beta^2$,

$$c = c_R\left(1 + \frac{\delta g}{k\beta^2}\right) \qquad (5\text{–}28)$$

where $\beta = \sqrt{\mu/\rho}$

c_R = velocity of Rayleigh waves in absence of gravity

δ = number which depends upon ratio μ/λ

Table 5–1 gives approximate values for δ and c_R^2/β^2 for several values of μ/λ and Poisson's ratio σ (Love [29, p. 160]).

TABLE 5–1. EFFECT OF GRAVITY ON RAYLEIGH WAVES

σ	μ/λ	c_R^2/β^2	δ
1/2	0	0 9126	0.1089
1/3	1/2	0 8696	0.0462
1/4	1	0 8453	0
1/5	3/2	0.8299	−0 0309

It is seen that for $\sigma = \frac{1}{4}$, $\delta = 0$, and that $\delta \gtrless 0$ for $\sigma \gtrless \frac{1}{4}$. Since for most crystalline rocks $\sigma > \frac{1}{4}$, we may conclude that the velocity of Rayleigh waves is, on the whole, likely to be increased by gravity, the increment being proportional to the wavelength. For the case $\sigma = \frac{1}{3}$, $\beta = 4$ km/sec, we find, for example, that the velocity of Rayleigh waves is increased by about 0.2 per cent when the wavelength is about 500 km.

Gravitating and Compressible Liquid Layer over Solid Half Space. In an attempt to find a theory for the generation of microseisms (Sec. 4–4), Scholte [49] considered this problem in two dimensions. For the velocity potential $\bar{\varphi}_1$ in the water layer, Eq. (5–2) now takes the form

$$\frac{\partial^2 \bar{\varphi}_1}{\partial t^2} = \alpha_1^2 \nabla^2 \bar{\varphi}_1 + g \frac{\partial \bar{\varphi}_1}{\partial z} \tag{5–29}$$

For a train of plane waves propagating in the water we can put

$$\bar{\varphi}_1 = e^{i(\omega t - kx - \vartheta z)} \tag{5–30}$$

By (5–29) and (5–30) we obtain

$$\omega^2 = \alpha_1^2(\vartheta^2 + k^2) + i\vartheta g \tag{5–31}$$

or

$$\vartheta = -i\frac{g}{2\alpha_1^2} \pm \sqrt{k_{\alpha_1}^2 - k^2 - \frac{g^2}{4\alpha_1^4}} = -\frac{ig}{2\alpha_1^2} \pm \eta \tag{5–32}$$

and

$$\bar{\varphi}_1 = \exp\left[-\frac{gz}{2\alpha_1^2} + i(\omega t - kx)\right](Ae^{-i\eta z} + Be^{i\eta z}) \tag{5–33}$$

The velocity components of a water particle are

$$\bar{u}_1 = \frac{\partial u_1}{\partial t} \qquad \bar{w}_1 = \frac{\partial w_1}{\partial t} \tag{5–34}$$

or by Eq. (5–33)

$$\begin{aligned} \bar{u}_1 &= \frac{\partial \bar{\varphi}_1}{\partial x} = -ik \exp\left[i(\omega t - kx) - \frac{gz}{2\alpha_1^2}\right](Ae^{-i\eta z} + Be^{i\eta z}) \\ \bar{w}_1 &= \frac{\partial \bar{\varphi}_1}{\partial z} = -\exp\left[i(\omega t - kx) - \frac{gz}{2\alpha_1^2}\right] \\ &\quad \cdot\left[\left(i\eta + \frac{g}{2\alpha_1^2}\right)Ae^{-i\eta z} - \left(i\eta - \frac{g}{2\alpha_1^2}\right)Be^{i\eta z}\right] \end{aligned} \tag{5–35}$$

Now we can write $\bar{\varphi}_1 = \varphi_1 \cdot i\omega$, where φ_1 is a displacement potential [see Eqs. (5–34)]. The gravity terms in the equations for the solid part of the system are omitted by Scholte.† For the displacement in the solid we have

$$\rho_2 \frac{\partial^2 u_2}{\partial t^2} = (\lambda_2 + \mu_2)\frac{\partial \theta_2}{\partial x} + \mu_2 \nabla^2 u_2 \qquad \rho_2 \frac{\partial^2 w_2}{\partial t^2} = \cdots \tag{5–36}$$

where

$$u_2 = \frac{\partial \varphi_2}{\partial x} - \frac{\partial \psi_2}{\partial z} \qquad w_2 = \frac{\partial \varphi_2}{\partial z} + \frac{\partial \psi_2}{\partial x} \tag{5–37}$$

†It was seen in the preceding pages that this approximation is valid unless the waves are considerably longer than ordinary earthquake Rayleigh waves.

Potentials which vanish as $z \to \infty$ are taken as usual in the form

$$\varphi_2 = \frac{C}{i\omega} e^{i(\omega t - kx - \nu_2 z)} \qquad \psi_2 = \frac{D}{i\omega} e^{i(\omega t - kx - \nu_2' z)} \tag{5-38}$$

where ν_2 and ν_2' must be negative imaginary numbers determined by the conditions

$$\nu_2^2 = k_{\alpha_2}^2 - k^2 \qquad \nu_2'^2 = k_{\beta_2}^2 - k^2 \tag{5-39}$$

As usual, α_2 and β_2 are the velocities of compressional and distortional waves in the second layer. Now we obtain, by Eqs. (5–38) and (5–37),

$$\begin{aligned} \bar{u}_2 &= \frac{\partial u_2}{\partial t} = -(ikCe^{-i\nu_2 z} - i\nu_2' De^{-i\nu_2' z})e^{i(\omega t - kx)} \\ \bar{w}_2 &= \frac{\partial w_2}{\partial t} = -(i\nu_2 Ce^{-i\nu_2 z} + ikDe^{-i\nu_2' z})e^{i(\omega t - kx)} \end{aligned} \tag{5-40}$$

The four coefficients A, B, C, D can be determined from the boundary conditions. The first of these conditions is that the pressure is zero at the free surface of water. In all problems considered before we have assumed that this free surface is a plane ($z = -H$). If we also consider its deformation and denote by $\hat{w}_1$ the vertical displacement, we can easily see that the constant in Eq. (5–5) is equal to H, and on the deformed surface $z = -H + \hat{w}_1$ we have

$$-\frac{\partial \bar{\varphi}_1}{\partial t} + g\hat{w}_1 = 0 \tag{5-41}$$

By (5–33), (5–35), $\bar{w}_1 = \hat{w}_1 \cdot i\omega$ at $z = -H$, and (5–41) we obtain

$$A\left(-\omega^2 + i\eta g + \frac{g^2}{2\alpha_1^2}\right)e^{i\eta H} + B\left(-\omega^2 - i\eta g + \frac{g^2}{2\alpha_1^2}\right)e^{-i\eta H} = 0 \tag{5-42}$$

The other boundary conditions are the continuity of the vertical displacement (or velocity) and of the normal and tangential stresses at the interface $z = 0$. The first of these conditions is, by (5–35) and (5–40),

$$-\left(i\eta + \frac{g}{2\alpha_1^2}\right)A + \left(i\eta - \frac{g}{2\alpha_1^2}\right)B = -i\nu_2 C - ikD \tag{5-43}$$

Now the tangential stress

$$p_{zx} = \mu_2\left(\frac{\partial u_2}{\partial z} + \frac{\partial w_2}{\partial x}\right) = 0$$

or, by (5–37) and (5–40),

$$-2ik\nu_2 C + (k_{\beta 2}^2 - 2k^2)D = 0 \tag{5-44}$$

By (5–5) we have the stress on the side of water

$$p = -\rho_1 \frac{\partial \tilde{\varphi}_1}{\partial t} + \rho_1 g(w_1)_{z=0}$$

$$\cong -\rho_1 A\left(i\omega + \frac{\eta g}{\omega} + \frac{g^2}{2\alpha_1^2 i\omega}\right) - \rho_1 B\left(i\omega - \frac{\eta g}{\omega} + \frac{g^2}{2\alpha_1^2 i\omega}\right) \quad (5\text{–}45)$$

and on the side of the solid medium

$$p_{zz} = \lambda_2 \theta_2 + 2\mu_2 \frac{\partial w_2}{\partial z} - g\rho_2 (w_2)_{z=0} \quad (5\text{–}46)$$

if, according to Scholte, the additional stress is put equal to the weight of the small column due to the deformation.

Noting that $\lambda_2 = \rho_2 (\alpha_2^2 - 2\beta_2^2)$, we obtain by (5–37), (5–38), and (1–5)

$$\lambda_2 \theta_2 + 2\mu_2 \frac{\partial w_2}{\partial z} = i\rho_2 \omega \left[\left(1 - \frac{2k^2}{k_{\beta 2}^2}\right) C e^{-i\nu_{2z}} + \frac{2k\nu_2'}{k_{\beta 2}^2} D e^{-i\nu_2' z}\right] e^{i(\omega t - kx)} \quad (5\text{–}47)$$

By (5–37) and (5–38),

$$(w_2)_{z=0} = -\frac{\nu_2}{\omega} C - \frac{k}{\omega} D \quad (5\text{–}48)$$

Thus the boundary condition for the normal stress may be written in the form ($z = 0$)

$$\rho_1 A\left(i\omega + \frac{\eta g}{\omega} + \frac{g^2}{2\alpha_1^2 i\omega}\right) + \rho_1 B\left(i\omega - \frac{\eta g}{\omega} + \frac{g^2}{2\alpha_1^2 i\omega}\right)$$

$$= \rho_2 C\left[i\omega\left(1 - \frac{2k^2}{k_{\beta 2}^2}\right) + \frac{g\nu_2}{\omega}\right] + \rho_2 D\left[\frac{2ik\nu_2'\omega}{k_{\beta 2}^2} + \frac{gk}{\omega}\right] \quad (5\text{–}49)$$

Thus the boundary conditions yield a system of four homogeneous equations (5–42), (5–43), (5–44), and (5–49) which will have nontrivial solutions if their determinant Δ vanishes.

Expanding the determinant, we find for the period equation

$$\left\{(2s^2 - 1)^2 - 4mns^2 - \frac{gk}{\omega^2}\frac{m}{s}\right\}\left\{1 - \frac{gk}{\omega^2}\cdot\frac{k \tan \eta H}{\eta} + \frac{g}{2\alpha_1^2}\cdot\frac{\tan \eta H}{\eta}\right\}$$

$$+ \frac{\rho_1}{\rho_2} k_{\beta 2} m\left\{1 - \left(\frac{gk}{\omega^2}\right)^2\right\}\frac{\tan \eta H}{\eta} = 0 \quad (5\text{–}50)$$

where

$$s = \frac{k}{k_{\beta 2}} \qquad m = \frac{i\nu_2}{k_{\beta 2}} \qquad n = \frac{i\nu_2'}{k_{\beta 2}}$$

Since $g/k\alpha_1^2 \ll 1$ for naturally occurring waves, Eq. (5–50) can be approximated by

$$\left\{(2s^2 - 1)^2 - 4mns^2 - \frac{gk}{\omega^2}\frac{m}{s}\right\}\left\{1 - \frac{gk}{\omega^2}\cdot\frac{k \tan \nu_1 H}{\nu_1}\right\} + \frac{\rho_1}{\rho_2}\frac{\sqrt{k^2 - k_{\alpha_2}^2}}{\sqrt{k_{\alpha_1}^2 - k^2}} \tan \nu_1 H = 0 \qquad (5\text{–}51)$$

where $\nu_1^2 = k_{\alpha_1}^2 - k^2$

Scholte pointed out that two different types of wave propagation are involved. When $c \ll \beta_2 < \alpha_2$, Eq. (5–51) reduces to that for gravity waves in an incompressible fluid underlain by an immovable bottom:

$$1 - \frac{gk}{\omega^2} \tanh kH = 0$$

For $g/k \ll c^2$, Eq. (5–51) takes the form of (4–154), giving the dispersion of Rayleigh waves in the system formed by a liquid layer underlain by a solid substratum (suboceanic Rayleigh waves). The separation into two types of propagation arises from the great disparity in phase velocities of gravity waves and Rayleigh waves. Although both types of propagation are dispersive, there is no wavelength for which the phase velocities come within an order of magnitude of each other for actual conditions in the ocean.

Another example of superposition of two types of waves will be found (Sec. 6–3) in the problem of flexural waves in floating ice (Ewing and Crary [11]). For long waves the gravity term in the period equation predominates, and the solution is simply that for gravity waves on water. For short waves the solution reduces to that for flexural waves in a thin plate modified slightly by the presence of the water. In this system, for a given period, only one type of propagation occurs, as we might expect. The phase velocities for either system taken separately exist over period ranges which overlap, and the curves cross.

5–3. Effect of Curvature on Surface Waves. In the following sections the theory of wave propagation in a sphere will be used to show the effect of spherical curvature. It is useful to consider first the simpler problem of Rayleigh-wave propagation along the circumferential direction of a cylindrical surface.

Cylindrical Curvature. This problem was solved by Sezawa [50]. If the axial component of motion is omitted, Eqs. (1–24) take the form

$$\rho \frac{\partial^2 \theta}{\partial t^2} = (\lambda + 2\mu)\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \theta}{\partial \sigma^2}\right)$$
$$\rho \frac{\partial^2 \Omega_z}{\partial t^2} = \mu\left(\frac{\partial^2 \Omega_z}{\partial r^2} + \frac{1}{r}\frac{\partial \Omega_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Omega_z}{\partial \sigma^2}\right) \qquad (5\text{–}52)$$

where r, σ, z are cylindrical coordinates. Radial and azimuthal displacement components are q and v, respectively.

The expressions for θ and Ω_z can be obtained for cylindrical coordinates from the general formulas of Sec. 5–4. They are

$$\theta = \frac{1}{r}\frac{\partial(rq)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \sigma} \qquad 2\Omega_z = \frac{1}{r}\frac{\partial(rv)}{\partial r} - \frac{1}{r}\frac{\partial q}{\partial \sigma} \tag{5–53}$$

Now, particular solutions of Eqs. (5–52) are

$$\begin{aligned} \theta &= AJ_{ka}(k_\alpha r)e^{i(\omega t + ka\sigma)} \\ \Omega_z &= BJ_{ka}(k_\beta r)e^{i(\omega t + ka\sigma)} \end{aligned} \tag{5–54}$$

where ω, k, k_α, k_β have the usual definitions, a is the radius of a circular cylinder, and J_{ka} is the Bessel function.

Displacements q and v consistent with the solution for θ given in (5–54) and satisfying the condition $\Omega_z = 0$ are

$$\begin{aligned} q_1 &= -\frac{A}{k_\alpha^2}\frac{dJ_{ka}(k_\alpha r)}{dr}e^{i(\omega t + ka\sigma)} \\ v_1 &= -\frac{A}{k_\alpha^2}\frac{ika}{r}J_{ka}(k_\alpha r)e^{i(\omega t + ka\sigma)} \end{aligned} \tag{5–55}$$

Displacements q_2 and v_2 consistent with the solution for Ω_z given in (5–54) and satisfying the condition $\theta = 0$ are

$$\begin{aligned} q_2 &= \frac{2B}{k_\beta^2}\frac{ika}{r}J_{ka}(k_\beta r)e^{i(\omega t + ka\sigma)} \\ v_2 &= -\frac{2B}{k_\beta^2}\frac{dJ_{ka}(k_\beta r)}{dr}e^{i(\omega t + ka\sigma)} \end{aligned} \tag{5–56}$$

Since the cylindrical surface is free of traction, the boundary conditions are

$$\lambda\theta + 2\mu\frac{\partial q}{\partial r} = 0 \qquad \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial q}{\partial \sigma} = 0 \qquad \text{at } r = a \tag{5–57}$$

where $q = q_1 + q_2$
$v = v_1 + v_2$

Inserting in (5–57) the values for θ, q, v from (5–54) to (5–56), we obtain the period equation which differs in two terms from that given by Sezawa†

$$\left[(k_\beta^2 - 2k_\alpha^2)J_{ka}(k_\alpha a) - 2\frac{d^2J_{ka}(k_\alpha a)}{da^2}\right]\left[(k_\beta^2 - 2k^2)J_{ka}(k_\beta a) + \frac{2}{a}\frac{dJ_{ka}(k_\beta a)}{da}\right]$$
$$- 4k\left[\frac{dJ_{ka}(k_\alpha a)}{da} - \frac{J_{ka}(k_\alpha a)}{a}\right]\left[\frac{dJ_{ka}(k_\beta a)}{da} - \frac{1}{a}J_{ka}(k_\beta a)\right] = 0 \tag{5–58}$$

† The subscripts ka are considered constant when the derivatives with respect to a are computed.

In contrast to the nondispersive Rayleigh waves associated with a plane surface, Eq. (5–58) shows through the parameters k_α, k_β, and k. that the velocity of propagation around a cylindrical surface depends upon the wavelength. To show the magnitude of this effect we have plotted in Fig. 5–1 a corrected Sezawa's curve computed from Eq. (5–58)

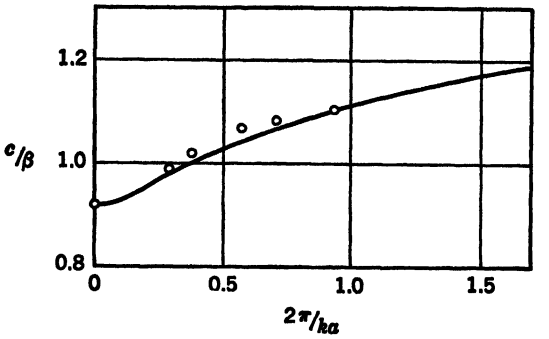


FIG. 5–1. Corrected phase-velocity curve of Rayleigh waves propagating around circumference of a homogeneous cylinder with Poisson's constant of 0.25, and experimental points for a cylinder with Poisson's constant 0.28 (*Calculated by Oliver.*)

for the case $\lambda = \mu$. It is seen that the curvature produces an increase of phase velocity with wavelength. For a wavelength equal to half the radius of curvature the increase is about 10 per cent.

The effect of cylindrical curvature on Rayleigh-wave propagation was obtained using model seismology techniques by Oliver [41]. The computation of the corrected curve was carried out by Oliver, and his experimental results are plotted in Fig. 5–1 for a solid with a Poisson's constant of 0.28. It is seen that the theoretical and experimental results are in agreement.

In Sezawa's paper the corresponding derivation for Love waves is given without numerical calculations.

Spherical Curvature. As in the preceding chapters, the period equation is derived from the condition that the determinant formed by the boundary conditions vanishes. However, the difficulty of the problem is increased greatly for a sphere in that the solutions take the form of spherical harmonics. Love [29] investigated this problem in detail and found that when applied to the earth the solution indicated "quick" waves, controlled primarily by elasticity, and "slow" waves controlled primarily by gravity.

Since Rayleigh waves are affected only slightly by compressibility, a good approximation may be obtained by considering an incompressible medium. We give only the final result, first derived by Bromwich [3] and later obtained by Love [29]:

$$\frac{2}{k_\beta a}\frac{\psi_n'(k_\beta a)}{\psi_n(k_\beta a)} + \frac{(2n+1) + ng\rho a/(2n+1)\mu - k_\beta^2 a^2/2(n-1)}{n(n+2) + ng\rho a/(2n+1)\mu - k_\beta^2 a^2/2(n-1)} = 0 \quad (5\text{–}59)$$

where a is the radius of the sphere and

$$\psi_n(k_\beta a) = (-1)^n\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(k_\beta a)^{-(n+\frac{1}{2})}J_{n+\frac{1}{2}}(k_\beta a) \quad (5\text{–}60)$$

$J_{n+\frac{1}{2}}$ being the Bessel function of the order $n + \frac{1}{2}$.

From Eq. (5–59) one can obtain the dispersion of Rayleigh waves introduced by spherical curvature and gravity. The index n is arbitrary and determines the mode of vibration. Since only sectorial harmonics are involved in this solution, the ratio $2\pi a/n$ represents the wavelength $2\pi/k$, and we can put $n = ka$.

For wavelengths small compared with the radius of curvature, n is very large. For this case Bromwich derived an approximate formula

$$\frac{\psi_n'(k_\beta a)}{\psi_n(k_\beta a)} = \frac{\nu' - k}{k_\beta} \quad (5\text{–}61)$$

which may be used to consider the effect of gravity only and obtain Eq. (5–25).

With the use of model seismology techniques the effect of spherical curvature for compressible media may readily be obtained as in the case of cylindrical curvature.

The propagation of seismic waves in the earth when the effect of curvature is also included was further studied by several investigators. Sezawa [51, 56] formulated the problem of transmission of Rayleigh and Love waves on a spherical surface, neglecting the effect of gravity. In the second of these papers the propagation of Love waves in a surface layer overlying a spherical core was considered for the case of an asymmetrical source. Sezawa found, among other conclusions, that the velocity of propagation of Love waves on a spherical surface is approximately equal to that on a plane surface. Rayleigh waves from an asymmetrical source were also investigated by Sezawa and Nishimura [55]. Recently Jobert [21] reconsidered the Love-wave problem for a layered sphere and found that the velocity of the longer waves is significantly increased as a result of curvature. Matumoto and Satô [32] recently discussed the transverse vibrations of a layered earth for two extreme cases, solid mantle and rigid or liquid core.

5–4. General Solutions for a Spherical Body. Satô [48] considered the boundary conditions at the surface of the sphere in their most general form. Denote by R the radius, δ the colatitude, ϵ the azimuth, u_1, u_2, u_3 the radial, colatitudinal, and azimuthal components of displacement, and by ω_1, ω_2, ω_3 the components of rotation, and take the polar axis to

pass through the epicenter. The equations of motion then take the form

$$\rho \frac{\partial^2 u_1}{\partial t^2} = (\lambda + 2\mu) \frac{\partial \theta}{\partial R} - \frac{2\mu}{R \sin \delta} \frac{\partial(\omega_3 \sin \delta)}{\partial \delta} + \frac{2\mu}{R \sin \delta} \frac{\partial \omega_2}{\partial \epsilon}$$

$$\rho \frac{\partial^2 u_2}{\partial t^2} = (\lambda + 2\mu) \frac{1}{R} \frac{\partial \theta}{\partial \delta} - \frac{2\mu}{R \sin \delta} \frac{\partial \omega_1}{\partial \epsilon} + \frac{2\mu}{R} \frac{\partial(R\omega_3)}{\partial R} \tag{5-62}$$

$$\rho \frac{\partial^2 u_3}{\partial t^2} = \frac{(\lambda + 2\mu)}{R \sin \delta} \frac{\partial \theta}{\partial \epsilon} - \frac{2\mu}{R} \frac{\partial(R\omega_2)}{\partial R} + \frac{2\mu}{R} \frac{\partial \omega_1}{\partial \delta}$$

The equations (5–62) may be easily obtained from the vector form of Eqs. (1–13). If $\mathbf{s}(u, v, w)$ is the displacement and $\theta = \operatorname{div} \mathbf{s}$, these equations are equivalent to

$$\rho \frac{\partial^2 \mathbf{s}}{\partial t^2} = (\lambda + 2\mu) \operatorname{grad} \operatorname{div} \mathbf{s} + \mu(\nabla^2 \mathbf{s} - \operatorname{grad} \operatorname{div} \mathbf{s}) + \rho \mathbf{F} \tag{5-63}$$

Now
$$\operatorname{curl} \operatorname{curl} \mathbf{s} = \operatorname{grad} \operatorname{div} \mathbf{s} - \nabla^2 \mathbf{s} \tag{5-64}$$

and
$$\operatorname{curl} \mathbf{s} = 2\boldsymbol{\omega} \tag{5-65}$$

where $\boldsymbol{\omega}$ is the rotation. Thus Eq. (5–63) may be written in the form

$$\rho \frac{\partial^2 \mathbf{s}}{\partial t^2} = (\lambda + 2\mu) \operatorname{grad} \theta - 2\mu \operatorname{curl} \boldsymbol{\omega} + \rho \mathbf{F} \tag{5-66}$$

The external force $\rho\mathbf{F}$ will now be omitted. In order to obtain Eqs. (5–62) from (5–66), use is made of two formulas for generalized orthogonal coordinates q_1, q_2, q_3. If the line element is given in the form

$$ds^2 = S_1^2\, dq_1^2 + S_2^2\, dq_2^2 + S_3^2\, dq_3^2 \tag{5-67}$$

and $\mathbf{q}_i$ are unit vectors along the axes of curvilinear coordinates, the curl of $\mathbf{s}$ can be written in the form

$$\operatorname{curl} \mathbf{s} = \frac{\mathbf{q}_1}{S_2 S_3} \left[\frac{\partial(S_3 u_3)}{\partial q_2} - \frac{\partial(S_2 u_2)}{\partial q_3} \right] + \frac{\mathbf{q}_2}{S_3 S_1} \left[\frac{\partial(S_1 u_1)}{\partial q_3} - \frac{\partial(S_3 u_3)}{\partial q_1} \right] + \frac{\mathbf{q}_3}{S_1 S_2} \left[\frac{\partial(S_2 u_2)}{\partial q_1} - \frac{\partial(S_1 u_1)}{\partial q_2} \right] \tag{5-68}$$

For spherical coordinates, $ds^2 = dR^2 + R^2\, d\delta^2 + R^2 \sin^2 \delta\, d\epsilon^2$, and we obtain $S_1 = 1$, $S_2 = R$, $S_3 = R \sin \delta$. Furthermore,

$$\theta = \operatorname{div} \mathbf{s} = \frac{1}{S_1 S_2 S_3} \left\{ \frac{\partial(S_2 S_3 u_1)}{\partial q_1} + \frac{\partial(S_3 S_1 u_2)}{\partial q_2} + \frac{\partial(S_1 S_2 u_3)}{\partial q_3} \right\} \tag{5-69}$$

For spherical coordinates, Eq. (5–69) becomes

$$\theta = \frac{1}{R^2 \sin \delta} \left[\frac{\partial(u_1 R^2 \sin \delta)}{\partial R} + \frac{\partial(u_2 R \sin \delta)}{\partial \delta} + \frac{\partial(u_3 R)}{\partial \epsilon} \right] \tag{5-70}$$

and

$$2\omega_1 = \frac{1}{R^2 \sin\delta}\left[\frac{\partial(u_3 R \sin\delta)}{\partial\delta} - \frac{\partial(u_2 R)}{\partial\epsilon}\right]$$

$$2\omega_2 = \frac{1}{R\sin\delta}\left[\frac{\partial u_1}{\partial\epsilon} - \frac{\partial(u_3 R\sin\theta)}{\partial R}\right] \tag{5-71}$$

$$2\omega_3 = \frac{1}{R}\left[\frac{\partial(u_2 R)}{\partial R} - \frac{\partial u_1}{\partial\delta}\right]$$

We could eliminate u_1, u_2, u_3 in Eqs. (5–62) by using Eqs. (5–70) and (5–71) and obtain four partial differential equations of the second order for the variables θ, ω_1, ω_2, ω_3. Jeans [17] assumed that the earth is formed of concentric layers of varying elastic constants λ and μ. He replaced the three equations of motion, in which the gravity terms were retained, by the three equations for the variables θ, u_3, and ω_3. We can also find the solutions of Eqs. (5–62) in a direct way, as was shown by Satô [48].

Omitting the time factor $\exp(i\omega t)$, we can write the solutions of (5–62) in the form

$$\begin{aligned}
u_1 &= \hat{A}P_n^m \cos m\epsilon + \hat{A}'P_n^m \sin m\epsilon \\
u_2 &= \left[\hat{C}\frac{d}{d\delta}P_n^m + \hat{B}'\frac{m}{\sin\delta}P_n^m\right]\cos m\epsilon \\
&\qquad + \left[\hat{C}'\frac{d}{d\delta}P_n^m - \hat{B}\frac{m}{\sin\delta}P_n^m\right]\sin m\epsilon \\
u_3 &= \left[\hat{C}'\frac{m}{\sin\delta}P_n^m - \hat{B}\frac{d}{d\delta}P_n^m\right]\cos m\epsilon \\
&\qquad - \left[\hat{C}\frac{m}{\sin\delta}P_n^m + \hat{B}'\frac{d}{d\delta}P_n^m\right]\sin m\epsilon
\end{aligned} \tag{5-72}$$

where

$$P_n^m = \bar{P}_n^m(\cos\delta) \tag{5-73}$$

is an associated Legendre function. The six expressions $\hat{A}, \cdots, \hat{C}'$ are functions of R as given by the equations

$$\begin{aligned}
\hat{A} &= A\frac{dF}{dR} + C\frac{n(n+1)}{R}G & \hat{A}' &= A'\frac{dF}{dR} + C'\frac{n(n+1)}{R}G \\
\hat{B} &= BG & \hat{B}' &= B'G \\
\hat{C} &= A\frac{F}{R} + \frac{C}{R}\frac{d(RG)}{dR} & \hat{C}' &= A'\frac{F}{R} + \frac{C'}{R}\frac{d(RG)}{dR}
\end{aligned} \tag{5-74}$$

and

$$\begin{aligned}
F &= F(k_\alpha R) = (k_\alpha R)^{-\frac{1}{2}}H_{n+\frac{1}{2}}^{(2)}(k_\alpha R) \\
G &= F(k_\beta R) = (k_\beta R)^{-\frac{1}{2}}H_{n+\frac{1}{2}}^{(2)}(k_\beta R)
\end{aligned} \tag{5-75}$$

The Hankel function $H_{n+\frac{1}{2}}^{(2)}$ is taken in order to have an outgoing wave.

These solutions are expressed in terms of six arbitrary constants $A, B, \cdots, C'$. Now we can assume that on the surface of the sphere $(R = a)$ the displacements have given values

$$u_1 = U_1(\delta, \epsilon) \qquad u_2 = U_2(\delta, \epsilon) \qquad u_3 = U_3(\delta, \epsilon) \tag{5–76}$$

U_1, U_2, and U_3 are arbitrary functions and can be expanded into Fourier series, as follows:

$$\begin{aligned} U_1(\delta, \epsilon) &= \sum_m U_1^m(\cos\delta) \cos m\epsilon + \sum_m U_1^{m\prime}(\cos\delta) \sin m\epsilon \\ U_2(\delta, \epsilon) &= \sum_m U_2^m(\cos\delta) \cos m\epsilon + \sum_m U_2^{m\prime}(\cos\delta) \sin m\epsilon \\ U_3(\delta, \epsilon) &= \sum_m U_3^m(\cos\delta) \cos m\epsilon + \sum_m U_3^{m\prime}(\cos\delta) \sin m\epsilon \end{aligned} \tag{5–77}$$

Then on writing solutions represented by (5–72) for all subscripts m and n and taking the sums, for example,

$$u = \sum_m \sum_n \hat{A}_n^m P_n^m \cos m\epsilon + \sum_m \sum_n \hat{A}_n^{\prime m} P_n^m \sin m\epsilon \tag{5–78}$$

we can compare (5–78) with the first expression in Eqs. (5–76). A set of equations will result by means of which the expressions $\hat{A}_n^m, \cdots, \hat{C}_n^{\prime m}$ may be determined. The arbitrary constants are then found from (5–74). It was pointed out by Satô that the initial-value-problem studies by Homma (Chap. 1, Ref. 22) can be discussed in a similar way, as well as the case where the stress, instead of the displacement components, is known at the surface.

Wave Propagation in a Gravitating Compressible Planet. It is more natural to consider the effects of gravity and curvature of the earth together than to consider them separately. This was done by Love [29], who investigated the laws of wave propagation in the interior of a gravitating compressible planet.

We shall assume that an undisturbed planet is a homogeneous body having a free spherical surface but for the sake of generality the density ρ_0 will at first be taken as a function of the distance R of a point from the center. Since it is assumed that a planet is formed by concentric layers, its gravitational potential V_0 will be a function of R. In a compressible planet which is in a state of equilibrium the "initial" stress can be assumed as an initial pressure determined by the condition of hydrostatic equilibrium. Then, according to Eqs. (1–7) written for the equilibrium of a perfect fluid (see Sec. 1–2),

$$\rho_0 \operatorname{grad} V_0 = \operatorname{grad} p_0 \tag{5–79}$$

Now if we assume that a disturbance occurs, the new potential V will be equal to the sum $V_0 + V_1$, where V_1 is due to the disturbing forces as

well as to the change in the distribution of density. Let ρ be the density at a point $M(x, y, z)$ in the strained state, and s the radial displacement. Then

$$s = u\frac{x}{R} + v\frac{y}{R} + w\frac{z}{R} \tag{5-80}$$

Let ρ' be the density in the deformed state but at the initial position. Since the cubical dilatation $\theta = -(\rho' - \rho_0)/\rho_0$ at the same place, the density ρ of a particle when it is displaced will be

$$\rho = \rho' - s\frac{\partial \rho_0}{\partial R} = \rho_0 - \rho_0\theta - s\frac{\partial \rho_0}{\partial R} \tag{5-81}$$

if ρ_0 is taken at $M_0(x - u, y - v, z - w)$. To simplify the theory, it will be assumed that λ, μ, and ρ_0 have everywhere the same value. By (5–81) we have

$$\rho = \rho_0(1 - \theta) \tag{5-82}$$

The equation of the surface of a planet in the disturbed state can be written in the form

$$R = R_0 + s_0 \tag{5-83}$$

where R_0 = radius of undisturbed surface
s_0 = superficial displacement

Finally, we assume that the stress components p_{zz}, p_{yy}, and p_{xx} at $M(x, y, z)$ at an instant t are equal to the sum of the equilibrium stresses and additional stresses in the disturbed state. Then, by (1–11),

$$p_{zz} = -\left(p_0 - s\frac{\partial p_0}{\partial R}\right) + \lambda\theta + 2\mu\frac{\partial u}{\partial x} \qquad p_{yy} = \cdots \qquad p_{xx} = \cdots \tag{5-84}$$

There are no changes in the shear components p_{yz}, p_{zx}, p_{xy}, and Eqs. (1–7) take the form

$$\rho_0\frac{\partial^2 u}{\partial t^2} = (\lambda + \mu)\frac{\partial \theta}{\partial x} + \mu\nabla^2 u - \frac{\partial}{\partial x}\left(p_0 - s\frac{\partial p_0}{\partial R}\right) + \rho_0(1 - \theta)\frac{\partial V_0}{\partial x} + \rho_0\frac{\partial V_1}{\partial x} \tag{5-85}$$

$$\rho_0\frac{\partial^2 v}{\partial t^2} = \cdots \qquad \rho_0\frac{\partial^2 w}{\partial t^2} = \cdots$$

given by Love [29], if we neglect the products of small quantities of the first order u, θ, V_1, and their derivatives.

By (5–82) and Poisson's equation $\nabla^2 V = -4\pi f\rho$, where f is the constant

of gravitation, we obtain

$$\nabla^2 V_1 = 4\pi f \rho_0 \theta \tag{5-86}$$

If we consider wave propagation and neglect external forces, the term V_1 is due to changes in density only. The potential of a homogeneous sphere is $V_0 = 2\pi f \rho_0 (R_0^2 - \frac{1}{3}R^2)$. By (5-79) and $R^2 = x^2 + y^2 + z^2$, the gravity and pressure terms in Eqs. (5-85) can be transformed as follows:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu) \frac{\partial \theta}{\partial x} + \mu \nabla^2 u$$

$$- \frac{4}{3} \pi f \rho_0^2 \frac{\partial}{\partial x}(Rs) + \frac{4}{3} \pi f \rho_0^2 x \theta + \rho_0 \frac{\partial V_1}{\partial x} \tag{5-87}$$

$$\rho_0 \frac{\partial^2 v}{\partial t^2} = \cdots \qquad \rho_0 \frac{\partial^2 w}{\partial t^2} = \cdots$$

Now

$$g = \left[-\frac{\partial V_0}{\partial R} \right]_{R=R_0} = \frac{4}{3} \pi f \rho_0 R_0 \tag{5-88}$$

and the effect of gravity is represented by two terms in Eqs. (5-87). On operating with curl on Eqs. (5-87) and taking into account that curl grad $\varphi = 0$, we obtain three equations for the components of rotation:

$$\Omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \Omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad \Omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{5-89}$$

We write the first of these equations in the form

$$\rho_0 \frac{\partial^2 \Omega_x}{\partial t^2} = \mu \nabla^2 \Omega_x + \frac{2}{3} \pi f \rho_0^2 \left\{ \frac{\partial (z\theta)}{\partial y} - \frac{\partial (y\theta)}{\partial z} \right\} \tag{5-90}$$

Thus the propagation of rotation will be determined by a wave equation, if the dilatation θ vanishes. On the other hand, by differentiating Eqs. (5-87) with respect to the coordinates and adding, we obtain by using (5-86)

$$\rho_0 \frac{\partial^2 \theta}{\partial t^2} = (\lambda + 2\mu) \nabla^2 \theta - \frac{g \rho_0}{R_0} \nabla^2 (Rs) + \frac{g \rho_0}{R_0} R \frac{\partial \theta}{\partial R} + \frac{6 g \rho_0 \theta}{R_0} \tag{5-91}$$

In order to obtain an equation for the dilatation, we have to eliminate the second term of the right side. Multiplying both sides of the three equations (5-87) by x, y, z in turn, adding the results, and applying the operation ∇^2, we obtain an equation which, together with (5-91), leads to

$$\left(\mu \nabla^2 - \rho_0 \frac{\partial^2}{\partial t^2}\right)\left[(\lambda + 2\mu)\nabla^2 - \rho_0 \frac{\partial^2}{\partial t^2} + \frac{16}{3} \pi f \rho_0^2\right]\theta$$

$$+ \left(\frac{4}{3} \pi f \rho_0^2\right)^2 R^2 \left(\nabla^2 - \frac{\partial^2}{\partial R^2} - \frac{2}{R} \frac{\partial}{\partial R}\right)\theta = 0 \tag{5-92}$$

From Eqs. (5–90) and (5–92) we surmise that when the effect of gravity is included the separate existence of distortional and compressional waves does not occur. When constants corresponding to the earth are inserted in these equations, Love showed that propagation is essentially the same as would occur if gravity and initial stress were neglected.

Substituting $\theta = A \cos k(x - \alpha t)$, where A, k, and α are constants, in (5–92), Love found that, when $2\pi/k$ is small and quantities of the second order in $(g\rho_0 R_0/\mu)(R_0^2 k^2)^{-1}$ are neglected, the velocity of compressional waves is

$$\alpha = \alpha_0 + \delta\alpha \qquad \alpha_0 = \sqrt{(\lambda + 2\mu)/\rho_0} \tag{5–93}$$

where the correction factor $\delta\alpha$ is obtained from

$$2\rho_0\alpha_0\delta\alpha = -\frac{g\rho_0 R_0}{k^2 R_0^2}\left(4 - \frac{g\rho_0 R_0}{\lambda + \mu}\cdot\frac{R^2 - x^2}{R_0^2}\right) \tag{5–94}$$

It is seen that the effect of gravity is to introduce a very slight dispersion depending on locality.

5–5. The Effect of Internal Friction. In all problems considered previously, perfectly elastic media were assumed. It is well known, however, that dissipation accompanies vibrations in solid media, because of the conversion of elastic energy to heat. Several mechanisms have been proposed for energy dissipation in vibrating solids, and these may be grouped collectively under internal friction. For a discussion of internal friction the reader is referred to the work of Kolsky [25].

In general, the effect of internal friction is to produce attenuation and dispersion of elastic waves. In practice, however, the attenuation is slight, and the dispersion is negligible for earthquake waves. The effect of internal friction is more pronounced for higher-frequency explosion-generated elastic waves where it may influence the shape of the elastic pulse.

There is no satisfactory theory of internal friction. Several mathematically convenient mechanisms have been suggested, however, which occasionally fit experimental data over a limited range of frequencies.

Voigt Solid. According to Voigt's definition [64], the stress-strain relations of Sec. 1–2 take the form

$$\begin{aligned} p_{zz} &= \lambda\theta + 2\mu\frac{\partial w}{\partial z} + \lambda'\frac{\partial\theta}{\partial t} + 2\mu'\frac{\partial^2 w}{\partial z\,\partial t} \\ p_{xz} &= \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \mu'\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \end{aligned} \tag{5–95}$$

These expressions, similar to Eqs. (1–11) for an elastic solid, may be obtained by using in (1–11) the operator $\lambda + \lambda'\,\partial/\partial t$ for λ and $\mu + \mu'\,\partial/\partial t$ for μ.

Using these operators in Eqs. (1–13), we obtain the equations of motion in the form

$$\rho \frac{\partial^2 u}{\partial t^2} = \left\{(\lambda + \mu) + (\lambda' + \mu') \frac{\partial}{\partial t}\right\} \frac{\partial \theta}{\partial x} + \left(\mu + \mu' \frac{\partial}{\partial t}\right) \nabla^2 u \qquad (5\text{–}96)$$

$$\rho \frac{\partial^2 v}{\partial t^2} = \cdots \qquad \rho \frac{\partial^2 w}{\partial t^2} = \cdots$$

Sezawa [52] gave these equations in a somewhat different form. Similarly, the wave equations take the form

$$\rho \frac{\partial^2 \varphi}{\partial t^2} = (\lambda + 2\mu) \nabla^2 \varphi + (\lambda' + 2\mu') \nabla^2 \frac{\partial \varphi}{\partial t} \qquad (5\text{–}97)$$

$$\rho \frac{\partial^2 \psi}{\partial t^2} = \mu \nabla^2 \psi + \mu' \nabla^2 \frac{\partial \psi}{\partial t}$$

In general, the four constants λ, λ', μ, μ' must be used to specify a Voigt solid, but simplifying assumptions can also be made. For example, the "dilatational viscosity" $(\lambda' + \frac{2}{3}\mu')$ which corresponds to the bulk modulus $k = \lambda + \frac{2}{3}\mu$ will vanish by taking $\lambda' = -\frac{2}{3}\mu'$, leaving only a single constant μ' for the effect of viscosity.

To consider the effect of viscoelasticity of the Voigt type on a plane wave, we can use the solution

$$\psi = Ae^{i(\omega t - \zeta x)} \qquad (5\text{–}98)$$

for a plane shear wave again propagating in the positive x direction. Substitution in the second of Eqs. (5–97) leads to the expression

$$\omega^2 = \frac{\mu}{\rho} \zeta^2 + \frac{i\mu'}{\rho} \zeta^2 \omega \qquad (5\text{–}99)$$

This equation can be satisfied for complex ζ. Inserting $\zeta = k + i\tau$ in Eq. (5–99) and equating real and imaginary terms, we find

$$k^2 = \frac{\mu\rho\omega^2}{2(\mu^2 + \mu'^2\omega^2)} \left\{\left[1 + \frac{(\mu'\omega)^2}{\mu^2}\right]^{\frac{1}{2}} + 1\right\} \qquad (5\text{–}100)$$

$$\tau^2 = \frac{\mu\rho\omega^2}{2(\mu^2 + \mu'^2\omega^2)} \left\{\left[1 + \frac{(\mu'\omega)^2}{\mu^2}\right]^{\frac{1}{2}} - 1\right\} \qquad (5\text{–}101)$$

From Eqs. (5–101) and (5–98) we find (admitting only the negative root for τ) that the attenuation of the wave, as given by the factor $\exp(-\tau x)$, increases with frequency. The phase velocity $c = \omega/k$ obtained from (5–100) has the value $(\mu/\rho)^{\frac{1}{2}}$, appropriate for an elastic body when $\omega = 0$. For increasing frequency, phase velocity increases, becoming infinite with ω, the attenuation then being complete.

The period equation for Love waves in a Voigt solid may be derived following the procedure of Sec. 4–5.

If the subscripts 1 and 2 refer to a layer and an infinite substratum, respectively, the wave equations are

$$\left(\mu_1 + \mu_1' \frac{\partial}{\partial t}\right)\nabla^2 v_1 = \rho_1 \frac{\partial^2 v_1}{\partial t^2}$$
$$\left(\mu_2 + \mu_2' \frac{\partial}{\partial t}\right)\nabla^2 v_2 = \rho_2 \frac{\partial^2 v_2}{\partial t^2} \tag{5–102}$$

Using the boundary conditions in the form

$$\left(\mu_1 + \mu_1' \frac{\partial}{\partial t}\right)\frac{\partial v_1}{\partial z} = 0 \qquad \text{at } z = 0$$
$$v_1 = v_2 \tag{5–103}$$
$$\left(\mu_1 + \mu_1' \frac{\partial}{\partial t}\right)\frac{\partial v_1}{\partial z} = \left(\mu_2 + \mu_2' \frac{\partial}{\partial t}\right)\frac{\partial v_2}{\partial z} = 0 \qquad \text{at } z = H$$

Sezawa and Kanai [58] derived a generalized Love-wave period equation

$$\tan s_1 H = \frac{(\mu_2 + i\omega\mu_2')s_2}{(\mu_1 + i\omega\mu_1')s_1} \tag{5–104}$$

where

$$s_1^2 = \frac{\rho_1\omega^2}{\mu_1 + i\mu_1'} - k^2$$
$$s_2^2 = k^2 - \frac{\rho_2\omega^2}{\mu_2 + i\mu_2'} \tag{5–105}$$

s_1 and s_2 being coefficients of z in expressions for v_1 and v_2, respectively, with trigonometric functions for the layer and exponential functions for the substratum. When the real and imaginary parts in Eqs. (5–105) are separated, two equations are obtained to determine a complex k.

The propagation of plane Rayleigh waves in a Voigt-solid half space was also discussed by Caloi [6], who gave a generalization of Rayleigh's equation (2–28). For a time factor of the form $\exp(i\omega t)$, where ω can be complex in the general case, Eqs. (1–24) take the form

$$(\nabla^2 + k_\alpha^2)\theta = 0 \tag{5–106}$$
$$(\nabla^2 + k_\beta^2)\Omega_z = 0, \cdots \tag{5–107}$$

where

$$k_\alpha^2 = \frac{\rho\omega^2}{\lambda + 2\mu + i\omega(\lambda' + 2\mu')} \tag{5–108}$$

and

$$k_\beta^2 = \frac{\rho\omega^2}{\mu + i\omega\mu'} \tag{5–109}$$

Equations (5–108) and (5–109) are obviously a generalization of the factors $k_\alpha = \omega/\alpha$ and $k_\beta = \omega/\beta$ introduced in Lamb's problem (Chap. 2). The wave number k now also becomes complex but the formal derivation of Rayleigh's equation follows the usual pattern. We can, therefore, obtain this equation for a Voigt-solid half space from Eq. (2–29), using the transformation

$$\frac{\beta}{\alpha} = \frac{k_\alpha}{k_\beta} \qquad c = \frac{\omega}{k}$$

We obtain

$$1 - 8\frac{k^2}{k_\beta^2} + \left(24 - 16\frac{k_\alpha^2}{k_\beta^2}\right)\frac{k^4}{k_\beta^4} - 16\left(1 - \frac{k_\alpha^2}{k_\beta^2}\right)\frac{k^6}{k_\beta^6} = 0 \qquad (5\text{–}110)$$

where k_α and k_β are now given by Eqs. (5–108) and (5–109). For his numerical calculations Caloi [6] solved this equation assuming that $\lambda = \mu$ and $\lambda' = -\frac{2}{3}\mu'$. The substitution $k_\beta^2/k^2 = B + 8/3$, $k_\alpha^2/k_\beta^2 = 1 - E/16$ reduces (5–110) to the form

$$B^3 + \left(E - \frac{40}{3}\right)B + \left(\frac{5}{3}E - \frac{448}{27}\right) = 0 \qquad (5\text{–}111)$$

The real and imaginary parts of the roots of Rayleigh's equation were calculated for the values of $\mu/\mu' = 30, 50, 100$ sec^{-1}. Caloi's curves show the dispersion and absorption of Rayleigh waves due to viscoelasticity of the Voigt type (Figs. 5–2 and 5–3).

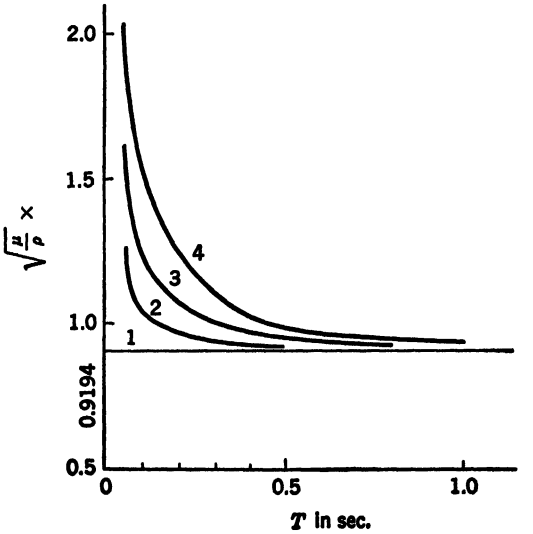


FIG. 5–2. Phase velocity of Rayleigh waves in (1) elastic half space with $\mu' = 0, \sigma = 1/4$; (2) viscoelastic half space with $\mu/\mu' = 100$; (3) $\mu/\mu' = 50$; (4) $\mu/\mu' = 30$. (*After Caloi.*)

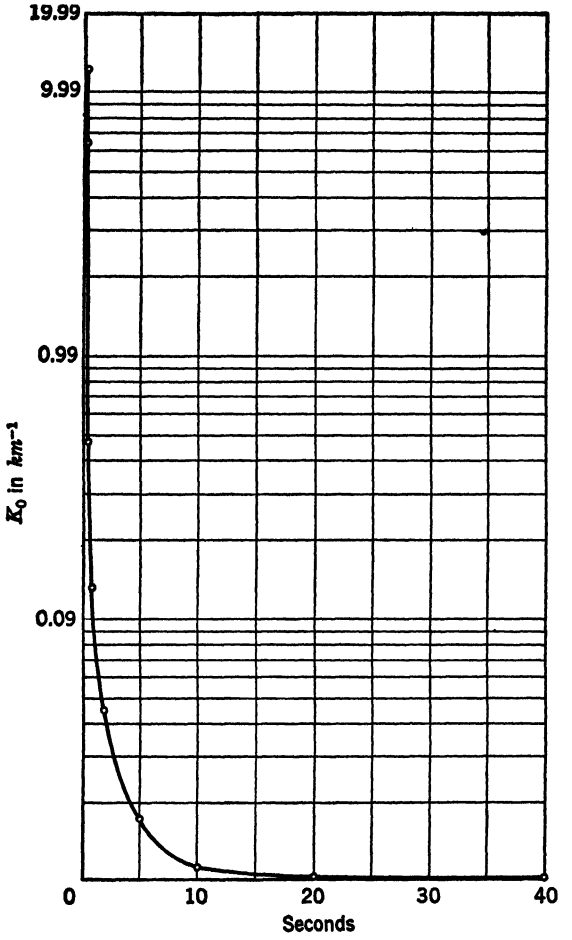


FIG. 5–3. Absorption coefficient K_0 of Rayleigh waves in a viscoelastic half space with $\beta = 3.3$ km/sec and $\mu/\mu' = 50$. (*After Caloi.*)

Newlands [37] generalized Lamb's problem of Sec. 2–3 for a Voigt-type solid in which λ' and μ' vary as $|\omega|^n$. As would be expected, both dispersion and absorption occur for this case. Using the methods of Chaps. 2, 3, and 4, she obtained the characteristics of P, S, and Rayleigh waves.

Maxwell Solid. According to Maxwell's definition [35], the stress-strain relations take forms such as

$$\frac{dp_{xy}}{dt} = 2\mu \frac{de_{xy}}{dt} - \frac{p_{xy}}{\delta} \tag{5–112}$$

where δ is the relaxation time of the solid. For long-continued stresses, such a substance will flow indefinitely, approximating a viscous liquid, the deformation being irrecoverable. For short-period stress variations the Maxwell solid behaves as if perfectly elastic. From the form of (5–112) we note that, with the use of an operator such as $\mu/[1 + 1/\delta(d/dt)]$ for μ in (1–11), the behavior of a Maxwell solid can be found approximately from expressions derived under the assumption of perfect elasticity. As an example, consider the propagation of a plane shear wave given by $\exp[i(\omega t - \zeta x)]$, where $\omega^2/\zeta^2 = \mu/\rho$. For propagation of a disturbance in a Maxwell solid, replace μ by $\mu/(1 + 1/i\omega\delta)$, which gives the equation

$$\zeta^2 = \frac{\rho\omega^2}{\mu}\left(1 + \frac{1}{i\omega\delta}\right) \tag{5–113}$$

This equation is satisfied for complex $\zeta = k + i\tau$ if

$$\begin{aligned} k^2 &= \frac{\rho\omega^2}{2\mu}\left[1 + \sqrt{1 + \frac{1}{\omega^2\delta^2}}\right] \\ \tau^2 &= \frac{\rho\omega^2}{2\mu}\left[\sqrt{1 + \frac{1}{\omega^2\delta^2}} - 1\right] \end{aligned} \tag{5–114}$$

From Eqs. (5–114) we conclude that the effect of viscoelasticity of the Maxwell type on plane shear waves is to introduce an attenuation given by $\exp(-\tau x)$, where τ increases as ω^2 for $\omega\delta \ll 1$ and approaches zero as $\omega\delta$ becomes large. The velocity $\omega/k \to \sqrt{\mu/\rho}$ as $\omega\delta \to \infty$. On the other hand, $\omega/k \to \sqrt{2\mu\omega\delta/\rho}$ as $\omega\delta \to 0$.

Internal Friction in Earth Materials. In the foregoing sections it was shown that the effect of internal friction on steady-state plane waves was to introduce frequency selective absorption and dispersion. Transient seismic pulses generated by earthquakes or explosions will be altered by this effect. Ricker [43–45] discussed this problem in a series of papers for internal friction of the type described by Stokes' differential equation.

Dispersion introduced by internal friction of consolidated rock seems to be negligible for frequencies under 100 cycles/sec. Attenuation is difficult to evaluate because allowance must be made for energy loss upon transmission across interfaces. Ricker [45] used a homogeneous section of shale to study the alteration of a seismic pulse.

There is increasing evidence that for a wide range of frequencies internal friction in crystalline rock is principally of the Voigt type, with quantities such as $\mu'\omega/\mu$ $(= 1/Q)$ mentioned earlier in this section being surprisingly independent of frequency, pressure, and temperature. The dissipation function $1/Q$ is related to the logarithmic decrement Δ of free vibrations by the relation $\Delta = \pi/Q$. For wave propagation it is related to the absorption coefficient by $\tau = \pi f/Qc$. Birch [2] reports values of $1/Q = 170 \times 10^{-5}$

at 1 atm and 280 $\times$ 10^{-5} at 4,000 atm for diabase. These are audio-frequency determinations on laboratory samples using longitudinal and torsional-free vibrations, respectively.

Ewing and Press [12] used the attenuation of mantle Rayleigh waves having periods of several hundred seconds to deduce the value of $1/Q$ for the upper mantle. Since the corresponding wavelengths are large compared with any of the discontinuities encountered by these waves, the effects of scattering and refraction are minimized. It is surprising that values of $1/Q$ determined in this way for the upper mantle are of the same order of magnitude as those found in audio-frequency vibration measurements on crystalline rock despite the difference in physical conditions.

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CHAPTER 6

PLATES AND CYLINDERS

In many respects, wave propagation in elastic plates and cylinders is analogous to propagation in layered spaces. Procedures for developing solutions can be similar, and several of the wave types encountered in the layered-space problems of preceding chapters will again appear. As before, most of the discussion will center on the period equations from which we can infer the main characteristics of wave propagation.

6–1. Plate in a Vacuum. The simplest case is obviously that of a homogeneous plate bounded by two parallel planes. The plate can have either finite or infinite dimensions, and we now restrict ourselves to the latter case. Oscillations of an elastic plate, the surfaces of which are free of stresses, were investigated by Rayleigh [71], Lamb [41], and others, and more recently by Prescott [63], Gogoladze [24], and Satô [73]. To derive the period equation in this problem it is sufficient to consider the propagation of plane waves. The displacements are written in the form

$$u = \frac{\partial \varphi}{\partial x} - \frac{\partial \psi}{\partial z} \qquad w = \frac{\partial \varphi}{\partial z} + \frac{\partial \psi}{\partial x} \tag{6–1}$$

The potentials φ and ψ are solutions of the wave equations

$$\nabla^2 \varphi = \frac{1}{\alpha^2} \frac{\partial^2 \varphi}{\partial t^2} \qquad \nabla^2 \psi = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2} \tag{6–2}$$

which satisfy four boundary conditions at the upper and lower surfaces of the plate. These conditions express the fact that the stresses vanish at the faces $z = -H$ and $z = H$ (the thickness of the plate is denoted by $2H$ and the median plane by $z = 0$). Then we have

$$p_{zz} = \lambda\theta + 2\mu \frac{\partial w}{\partial z} = 0 \qquad p_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0 \qquad \text{at } z = \mp H \tag{6–3}$$

assuming, as usual, solutions of the form

$$\begin{aligned} \varphi &= (A \sinh \nu z + B \cosh \nu z) e^{i(\omega t - kx)} \\ \psi &= (C \sinh \nu' z + D \cosh \nu' z) e^{i(\omega t - kx)} \end{aligned} \tag{6–4}$$

where, from the wave equations,

$$\nu = \sqrt{k^2 - k_\alpha^2} \qquad \nu' = \sqrt{k^2 - k_\beta^2} \qquad k_\alpha = \frac{\omega}{\alpha} \qquad k_\beta = \frac{\omega}{\beta} \tag{6-5}$$

Noting that $\alpha^2 = (\lambda + 2\mu)/\rho$, $\beta^2 = \mu/\rho$ and inserting expressions (6-4) into Eqs. (6-1) and (6-3), we obtain the four boundary conditions in the form

$$\begin{aligned}
&(\rho\omega^2 - 2\mu k^2)(A \sinh \nu H - B \cosh \nu H) \\
&\qquad - 2i\mu k\nu'(C \cosh \nu' H - D \sinh \nu' H) = 0 \\
&2ik\nu(A \cosh \nu H - B \sinh \nu H) \\
&\qquad - (\nu'^2 + k^2)(C \sinh \nu' H - D \cosh \nu' H) = 0 \\
&-(\rho\omega^2 - 2\mu k^2)(A \sinh \nu H + B \cosh \nu H) \\
&\qquad - 2i\mu k\nu'(C \cosh \nu' H + D \sinh \nu' H) = 0 \\
&2ik\nu(A \cosh \nu H + B \sinh \nu H) \\
&\qquad + (\nu'^2 + k^2)(C \sinh \nu' H + D \cosh \nu' H) = 0
\end{aligned} \tag{6-6}$$

The period equation is obtained in a simpler form if we write the columns corresponding to the coefficients A, D, B, C, add the first line of the determinant to the third, and subtract the second from the fourth. Then, on putting

$$\begin{aligned}
a &= 2\mu k^2 - \rho\omega^2 = \mu(2k^2 - k_\beta^2) = \mu(\nu'^2 + k^2) \\
b &= 2\mu k\nu' \cosh \nu' H \qquad d = 2\mu k\nu' \sinh \nu' H
\end{aligned} \tag{6-7}$$

we have

$$\Delta = \begin{vmatrix}
-a \sinh \nu H & id & a \cosh \nu H & -ib \\
-2ik\nu \cosh \nu H & -(\nu'^2 + k^2) \cosh \nu' H & 2ik\nu \sinh \nu H & (\nu'^2 + k^2) \sinh \nu' H \\
0 & 0 & 2a \cosh \nu H & -2ib \\
0 & 0 & -4ik\nu \sinh \nu H & -2(\nu'^2 + k^2) \sinh \nu' H
\end{vmatrix} \tag{6-8}$$

Obviously this equation can be split into two. They are

$$(\rho\omega^2 - 2\mu k^2)(\nu'^2 + k^2) \sinh \nu H \cosh \nu' H + 4\mu k^2 \nu\nu' \cosh \nu H \sinh \nu' H = 0 \tag{6-9}$$

and

$$(\rho\omega^2 - 2\mu k^2)(\nu'^2 + k^2) \cosh \nu H \sinh \nu' H + 4\mu k^2 \nu\nu' \sinh \nu H \cosh \nu' H = 0 \tag{6-10}$$

By Eqs. (6–5), these equations take the form

$$\frac{\tanh \nu H}{\tanh \nu' H} = \frac{4k^2\nu\nu'}{(\nu'^2 + k^2)^2} = \frac{4\sqrt{1 - c^2/\alpha^2}\sqrt{1 - c^2/\beta^2}}{(2 - c^2/\beta^2)^2} \tag{6–11}$$

and

$$\frac{\tanh \nu H}{\tanh \nu' H} = \frac{(\nu'^2 + k^2)^2}{4k^2\nu\nu'} = \frac{(2 - c^2/\beta^2)^2}{4\sqrt{1 - c^2/\alpha^2}\sqrt{1 - c^2/\beta^2}} \tag{6–12}$$

Now the transformation of the determinant Δ which preceded its representation in the form of a product is equivalent to the splitting of Eqs. (6–6) into two separate systems. It is easy to see that the coefficients A and D can be separated from B and C. Thus we can consider a motion symmetric with respect to the plane $z = 0$ which is given by

$$\varphi = B \cosh \nu z\, e^{i(\omega t - kx)} \qquad \psi = C \sinh \nu' z\, e^{i(\omega t - kx)} \tag{6–13}$$

and the antisymmetric motion represented by functions

$$\varphi = A \sinh \nu z\, e^{i(\omega t - kx)} \qquad \psi = D \cosh \nu' z\, e^{i(\omega t - kx)} \tag{6–14}$$

In both cases the nature of the vibrations is determined by the corresponding period equation, i.e., by (6–12) for the symmetric and by (6–11) for the antisymmetric case. The discussion of these transcendental equations in the general form presents certain difficulties, and, therefore, the asymptotic limits for long and short waves are first considered.

Symmetric Vibrations (M_1). For waves long compared with the thickness $2H$ the products kH, νH, $\nu' H$ may be taken as small when $c = \omega/k$ is finite. Then, if the hyperbolic functions are replaced by their arguments, (6–12) takes the form

$$(\nu'^2 + k^2)^2 - 4k^2\nu^2 = 0 \tag{6–15}$$

By (6–5) we obtain

$$\frac{c^2}{\beta^2} = 4\left(1 - \frac{\beta^2}{\alpha^2}\right) = \frac{4(\lambda + \mu)}{\lambda + 2\mu} = \frac{c_p^2}{\beta^2} \tag{6–16}$$

where c_p is the phase velocity of long longitudinal or plate waves. When $\sigma = \frac{1}{4}$, $3\beta^2 = \alpha^2$, we have $c_p = 2\sqrt{2}\,\alpha/3 = 2\sqrt{2/3}\beta$.

For very short waves and $c < \beta < \alpha$, the quantities kH, νH, $\nu' H$ are large, and the left side of (6–12) becomes unity, giving

$$(2k^2 - k_\beta^2)^2 - 4k^2\nu\nu' = 0 \tag{6–17}$$

This equation is recognized as the characteristic equation (2–28) for Rayleigh waves in an elastic half space discussed in Sec. 2–2. For $\sigma = \frac{1}{4}$,

it was found that $c_R = 0.9194\beta$. For $c > \beta$, it may be verified from Eq. (6–12) that $c \to \beta$ as $kH \to \infty$.

In general, the waves described by (6–12) are dispersive. To determine the manner in which the long- and short-wavelength limits are connected one must use the complete equations. The lowest mode M_{11} exists for $c_R \leq c \leq c_p$. An infinite number of higher modes $M_{12}, \cdots$ exists for which $c > \beta$ because of the periodic nature of the functions $\tanh \nu' H$. Phase and group velocities obtained from Eq. (6–12) and $U = c + k\, dc/dk$ are presented for the first two modes M_{11} and M_{12} in Fig. 6–1 (see Chap. 4,

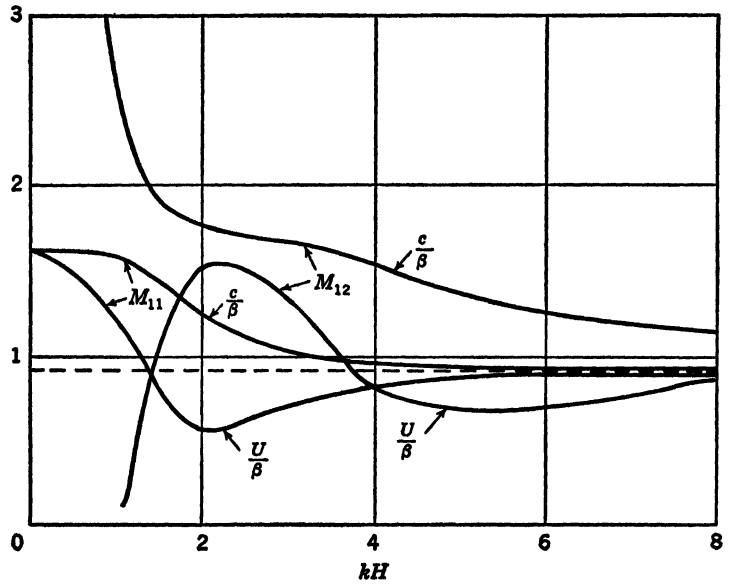


FIG. 6–1. First and second symmetric modes M_{11} and M_{12} of a free elastic plate with Poisson's constant of 1/4. (*After Tolstoy and Usdin.*)

Ref. 203). In the first mode the phase velocity decreases monotonically with increasing values of kH from $c = c_p$ at $kH = 0$ to $c = c_R$ at $kH = \infty$. The group velocity has the same asymptotic limits but exhibits a minimum value when $kH \cong 4$. The second mode M_{12} is typical of all higher modes in that $c > \beta$. For $kH \to 0$, $c \to \infty$ and $U \to 0$, and for $kH \to \infty$, $c \to U \to \beta$. For intermediate values of kH we can find a maximum and a minimum value of group velocity.

A general analysis for arbitrary λ, μ, and ρ was given by Gogoladze [24]. Taking the variable $\vartheta = 1/c = k/\omega$, we can write the frequency equation

(6–12) for symmetric vibrations in the form

$$\left(2\vartheta^2 - \frac{1}{\beta^2}\right)^2 \tan\left(H\omega\sqrt{\frac{1}{\beta^2} - \vartheta^2}\right)$$

$$= -4\vartheta^2\sqrt{\frac{1}{\alpha^2} - \vartheta^2}\sqrt{\frac{1}{\beta^2} - \vartheta^2}\tan\left(H\omega\sqrt{\frac{1}{\alpha^2} - \vartheta^2}\right) \qquad (6\text{–}18)$$

Then, for the interval $0 < \vartheta < 1/\alpha < 1/\beta$ all factors are real. One can now consider in the plane (σ, ϑ) the curves determined by the equations

$$\sigma = \frac{1}{4\vartheta^2}\left(2\vartheta^2 - \frac{1}{\beta^2}\right)^2 \tan\left(H\omega\sqrt{\frac{1}{\beta^2} - \vartheta^2}\right) \qquad (6\text{–}19)$$

and

$$\sigma = -\sqrt{\frac{1}{\alpha^2} - \vartheta^2}\sqrt{\frac{1}{\beta^2} - \vartheta^2}\tan\left(H\omega\sqrt{\frac{1}{\alpha^2} - \vartheta^2}\right) \qquad (6\text{–}20)$$

The expression (6–19) decreases in the interval $(0, 1/\alpha)$, while (6–20) increases. Thus the number of roots of (6–18) is determined by the number of points of intersection or by the number of asymptotes of (6–19) and (6–20). The number of modes corresponding to the phase velocities for $\vartheta < 1/\beta$ increases with increasing frequency. When $\omega H \to \infty$, a real root of the period equation approaches $\vartheta_R = 1/c_R > 1/\beta$. Gogoladze proved that for each ϑ in $(1/\beta, 1/c_R)$ there is a single value of ωH which satisfies the period equation. There are no such roots in $(1/c_R, \infty)$, since $\vartheta_R = 1/c_R = \vartheta_{\max}$ (at $\omega H = \infty$). There is also a minimum value $\vartheta = 1/2\beta\sqrt{1 - \beta^2/\alpha^2}$ which is obtained for $\omega H = 0$.

Antisymmetric Vibrations (M_2). For waves long compared with the thickness $2H$ and $c < \beta < \alpha$ Eq. (6–11) reduces, after some algebraic transformations, to

$$\frac{c^2}{\beta^2} = \frac{4}{3}(kH)^2\left(1 - \frac{\beta^2}{\alpha^2}\right) \qquad (6\text{–}21)$$

In deriving (6–21), the third terms in the expansion of the hyperbolic functions must be retained. This is the period equation for long flexural waves. Dispersion occurs for these waves, with phase velocity decreasing to zero with increasing wave length.

For $kH \to \infty$ and $c < \beta < \alpha$, Eq. (6–11) reduces to Rayleigh's equation (6–17), and the propagation degenerates to Rayleigh waves associated with both free surfaces. For $c > \beta$ and $kH \to \infty$, $c \to \beta$.

To discuss an entire dispersion curve, computations based on Eq. (6–11) must be made. These appear for the first two modes in Fig. 6–2 given by Tolstoy and Usdin. Again $\sigma = \frac{1}{4}$ is assumed. It is seen that for the lowest mode M_{21}, $0 < c < c_R$. For $kH \to 0$, $U \to 0$ (flexural waves), and for $kH \to \infty$, $c \to U \to c_R$. For intermediate kH a maximum value of group velocity occurs at $kH \cong 3.6$, $U \cong \beta$.

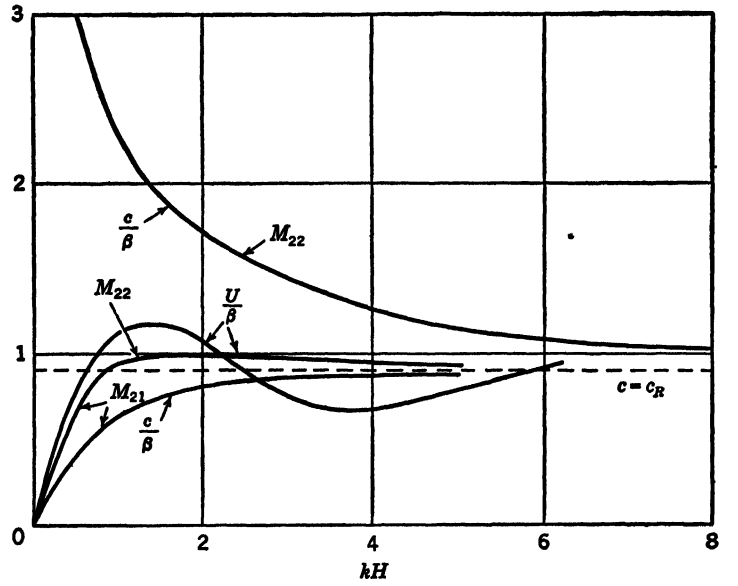


FIG. 6–2. First and second antisymmetric modes M_{21} and M_{22} of a free elastic plate with Poisson's constant of 1/4. (*After Tolstoy and Usdin.*)

The higher modes $M_{22}, M_{23}, \cdots$ are all characterized by $c > \beta, c \to U \to \beta$ as $kH \to \infty$, and $c \to \infty$ as $kH \to 0$. Both maximum and minimum values of group velocity are associated with the higher modes at intermediate wave lengths.

Interpretation in Terms of P and SV Waves. As in the preceding chapters, the period equations (6–11) and (6–12) may be interpreted in terms of multiple-reflected, reinforcing SV waves for $\alpha > c > \beta$ and SV and P waves for $c > \alpha > \beta$. Tolstoy and Usdin discussed this in some detail. We mention a few special cases only. The conditions $kH \to \infty$, $c \to \beta$ correspond to SV waves which are multiple-reflected near grazing incidence, and $c \to \infty$, $kH \to 0$ is the limiting case of SV or P waves normally incident upon the boundaries. For the latter case, the condition $U \to 0$ corresponds to the condition of zero energy transmission in the horizontal direction.

Impulsive Sources. Although a full discussion for this case requires a treatment analogous to those given in Chap. 4, some information can be obtained from the group-velocity curves of Figs. 6–1 and 6–2. In general, a source will stimulate waves associated with the various modes, and the sequence of arrivals at a distant point will be determined by the group-velocity curves. The degree of excitation of waves in any given mode depends on the spectrum of the source and the excitation functions

for the plate. In Fig. 6–3 the model seismogram from an impulsive spark source at the surface of a thin aluminum plate is presented (Press and Oliver [66]). The excitation and the frequency response of the detectors were such that only the lower frequencies of the M_{21} mode, i.e., the flexural waves, were excited. That the dispersion is proper may be seen from a

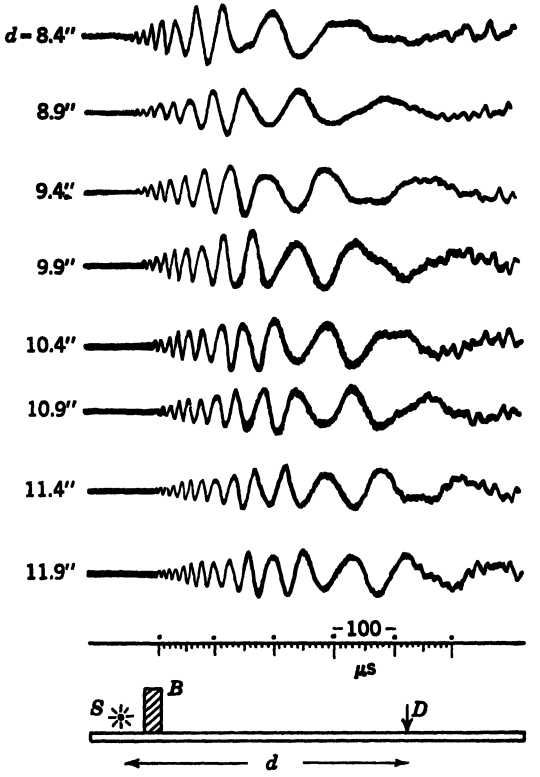


FIG. 6–3. Flexural waves excited in a plate of 24S-T aluminum, 1/32 in. thick. Spark source S is at distance d from detector D. (*After Press and Oliver.*)

comparison of observed phase and group velocities as determined from the seismograms with the theoretical curves computed from Eqs. (6–12) and (4–94), using the elastic constants and thickness appropriate for the experimental plate. The agreement is quite satisfactory, as may be seen in Fig. 6–4.

The discussion of *SH*-wave propagation in a plate will be included in the more general case when the plate is located in a liquid (see Sec. 6–3).

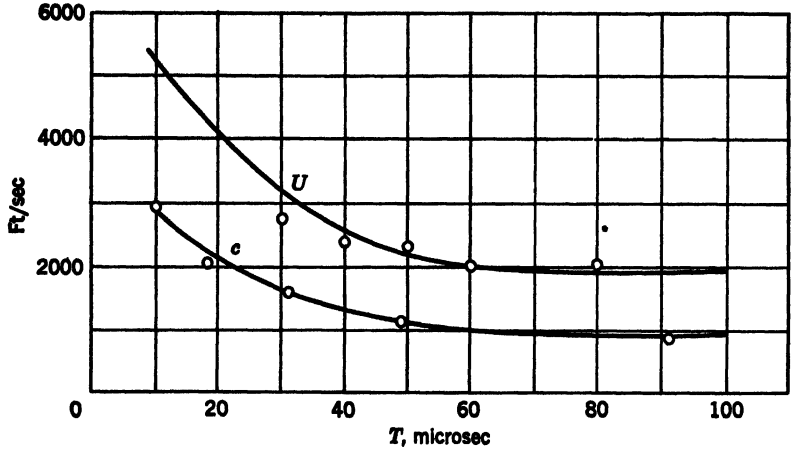


FIG. 6–4. Observed and theoretical phase and group velocities in a plate of 24S-T aluminum 1/32 in. thick. (*After Press and Oliver.*)

Other Investigations. The vibrations of a thin elastic plate were recently discussed by Sauter [74]. Stenzel [80] investigated the acoustical field of a point source in a layer with an acoustically "soft" or "hard" boundary. The analogous problem of transmission and reflection of electromagnetic waves in plates was investigated by Heins and Carlson [28] and Heins [29].

The investigation of flexural motion of plates led Uflyand [87] to a system of equations analogous to those used by Timoshenko [85] in the problem of vibrations of a rod. Mindlin [53] showed how to reduce these equations to three wave equations under certain conditions. These results were applied to the problem of reflection of flexural waves at the edge of a plate by Kane [36].

6–2. Plate in a Liquid. The propagation of elastic waves in a system composed of an infinite plate bounded by two parallel planes and immersed in an infinite liquid was investigated by Reissner [72], Osborne and Hart [59], Fay and Fortier [18], and Fay [20], where other references are given. Sezawa and Nishimura [77] considered a plate in an infinite solid medium.

Osborne and Hart were principally concerned with the so-called waveguide problem (homogeneous case), whereas the other investigators studied the problem of reflection and transmission of waves incident on the plate (inhomogeneous case). We restrict ourselves to the former case, treating only steady-state solutions, but the results could be generalized for an impulsive source, using the methods of Chaps. 2, 3, and 4. Osborne and Hart generalized the solution of Lamb [41] for a plate in a vacuum.

To provide for continuity of normal stresses and displacements and vanishing tangential stresses at the boundaries of the plate, additional potentials are introduced to describe the participation of the liquid. The liquid below and above the plate will now be denoted by the subscripts 2 and 0, respectively (see Fig. 6–5). The z axis is directed downward. Then the

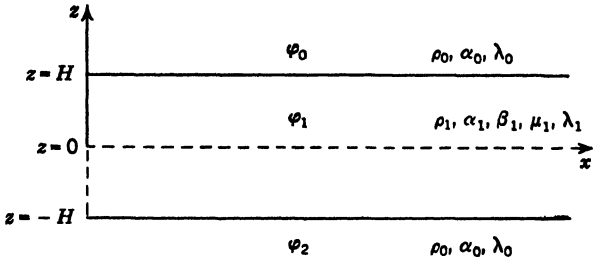


FIG. 6–5. Plate in a liquid.

displacements are

$$u_0 = \frac{\partial \varphi_0}{\partial x} \qquad w_0 = \frac{\partial \varphi_0}{\partial z} \qquad u_2 = \frac{\partial \varphi_2}{\partial x} \qquad w_2 = \frac{\partial \varphi_2}{\partial z} \tag{6–22}$$

and the appropriate solutions for the liquid are

$$\varphi_0 = A_0 e^{\nu_0 z} e^{i(\omega t - kx)} \qquad \varphi_2 = A_2 e^{-\nu_0 z} e^{i(\omega t - kx)} \tag{6–23}$$

where

$$\nu_0 = \sqrt{k^2 - k_{\alpha 0}^2} \tag{6–24}$$

The two equations corresponding to continuity of the normal displacement components w are

$$-\nu_0 e^{-\nu_0 H} A_0 + \nu(A \cosh \nu H - B \sinh \nu H) + ik(C \sinh \nu' H - D \cosh \nu' H) = 0 \tag{6–25}$$

$$\nu(A \cosh \nu H + B \sinh \nu H) - ik(C \sinh \nu' H + D \cosh \nu' H) + \nu_0 e^{-\nu_0 H} A_2 = 0 \tag{6–26}$$

The normal stresses for the liquid ($\mu = 0$) are

$$p_{zz} = \lambda_0 \nabla^2 \varphi_0 = -\frac{\lambda_0}{\alpha_0^2} \omega^2 \varphi_0 = -\rho_0 \omega^2 \varphi_0 \qquad \text{for } z = -H \tag{6–27}$$

$$p_{zz} = -\rho_0 \omega^2 \varphi_2 \qquad \text{for } z = H$$

Denoting the left-side members of Eqs. (6–6) by L_1, L_2, L_3, L_4, we have four other boundary conditions in the form

$$L_1 + \rho_0 \omega^2 e^{-\nu_0 H} A_0 = 0 \qquad L_2 = 0 \tag{6–28}$$

$$\rho_0 \omega^2 e^{-\nu_0 H} A_2 + L_3 = 0 \qquad L_4 = 0 \tag{6–29}$$

$$\begin{vmatrix}
\nu_0 & \nu \cosh \nu H & ik \cosh \nu' H & 0 & 0 & 0 \\
\rho_0\omega^2 & \mu(\nu'^2 + k^2) \sinh \nu H & 2i\mu k\nu' \sinh \nu' H & 0 & 0 & 0 \\
0 & -2ik\nu \cosh \nu H & (\nu'^2 + k^2) \cosh \nu' H & 0 & 0 & 0 \\
0 & 0 & 0 & \nu \sinh \nu H & ik \sinh \nu' H & \nu_0 \\
0 & 0 & 0 & \mu(\nu'^2 + k^2) \cosh \nu H & 2i\mu k\nu' \cosh \nu' H & \rho_0\omega^2 \\
0 & 0 & 0 & -2ik\nu \sinh \nu H & (\nu'^2 + k^2) \sinh \nu' H & 0
\end{vmatrix} = 0 \qquad (6\text{–}30)$$

*

On rearrangement of the determinant of the system of Eqs. (6–25), (6–26), (6–28), and (6–29), the period equation takes the form (if we assume $\lambda = \mu$ for the plate) shown in Eq. (6–30).

Hence

$$\begin{vmatrix} \nu_0 & \nu \cosh \nu H & ik \cosh \nu' H \\ \rho_0 \omega^2 & \mu(\nu'^2 + k^2) \sinh \nu H & 2i\mu k \nu' \sinh \nu' H \\ 0 & -2ik\nu \cosh \nu H & (\nu'^2 + k^2) \cosh \nu' H \end{vmatrix} = 0 \qquad (6\text{–}31)$$

and

$$\begin{vmatrix} \nu \sinh \nu H & ik \sinh \nu' H & \nu_0 \\ \mu(\nu'^2 + k^2) \cosh \nu H & 2i\mu k \nu' \cosh \nu' H & \rho_0 \omega^2 \\ -2ik\nu \sinh \nu H & (\nu'^2 + k^2) \sinh \nu' H & 0 \end{vmatrix} = 0 \qquad (6\text{–}32)$$

Thus the terms in Eqs. (6–4) can be divided into two groups;

$$\begin{aligned} \varphi_0 &= A_0 e^{\nu_0 z + i(\omega t - kx)} & \varphi &= B \cosh \nu z \, e^{i(\omega t - kx)} \\ \varphi_2 &= A_2 e^{-\nu_0 z + i(\omega t - kx)} & \psi &= C \sinh \nu' z \, e^{i(\omega t - kx)} \end{aligned} \qquad (6\text{–}33)$$

is a solution with $A_0 = A_2$, and

$$\begin{aligned} \varphi_0 &= A_0 e^{\nu_0 z + i(\omega t - kx)} & \varphi &= A \sinh \nu z \, e^{i(\omega t - kx)} \\ \varphi_2 &= A_2 e^{-\nu_0 z + i(\omega t - kx)} & \psi &= D \cosh \nu' z \, e^{i(\omega t - kx)} \end{aligned} \qquad (6\text{–}34)$$

with $A_0 = -A_2$. Again these represent symmetric and antisymmetric vibrations of the plate.

The period equation corresponding to the symmetric case [Eqs. (6–32) and (6–33)] takes the form

$$(\nu'^2 + k^2)^2 \cosh \nu H \sinh \nu' H - 4k^3 \nu \nu' \sinh \nu H \cosh \nu' H + \frac{\rho_0 \alpha_0^2 \nu}{\rho \beta^2 \nu_0} (k^2 - \nu_0^2)(k^2 - \nu'^2) \sinh \nu H \sinh \nu' H = 0 \qquad (6\text{–}35)$$

For the antisymmetric case [Eqs. (6–31) and (6–34)] we have

$$(\nu'^2 + k^2)^2 \sinh \nu H \cosh \nu' H - 4k^3 \nu \nu' \cosh \nu H \sinh \nu' H + \frac{\rho_0 \alpha_0^2 \nu}{\rho \beta^2 \nu_0} (k^2 - \nu_0^2)(k^2 - \nu'^2) \cosh \nu H \cosh \nu' H = 0 \qquad (6\text{–}36)$$

The sum of the first two terms in each equation is the expression derived for the case of a free plate [Eqs. (6–12) and (6–11)]. The last terms in (6–35) and (6–36) represent a modification due to the presence of the liquid.

Because of $k = \omega/c$, Eqs. (6–35) and (6–36) define relations between

any two of the three variables c, k, and ω. These equations can yield pairs of real roots for k and c provided that ω is real and $c < \alpha_0$. When Re $c > \alpha_0$, ν_0 becomes complex, and the last terms in (6–35) and (6–36) show that each equation can be split into two. This system of four equations can be satisfied if we assume both c and k are complex, containing four unknowns. Complex c and k indicate attenuation of the waves, the attenuation increasing with the magnitude of the imaginary component. This attenuation is due to the leakage of energy from the plate to the liquid. We can again obtain expressions for phase velocity from (6–35) and (6–36) valid for very large and very small wavelengths before proceeding to the more difficult computations for intermediate wavelengths.

Symmetric Vibrations. As in the case of the plate in a vacuum, in the lowest mode $c_R < c < c_p$. For wavelengths large compared with the plate thickness, or kH small, the hyperbolic functions in (6–35) can be replaced by unity or linear terms, and we obtain the approximation

$$c = 2\beta\left(1 - \frac{\beta^2}{\alpha^2}\right)^{\frac{1}{2}}\left[1 + \frac{i}{2}\frac{\rho_0\alpha_0\omega H}{\rho\beta^2}\left(\frac{1}{4(1 - \beta^2/\alpha^2)} - \frac{\beta^2}{\alpha^2}\right)\right] \qquad (6\text{–}37)$$

The real part of this equation is identical with the velocity of long longitudinal waves of a plate in a vacuum [Eq. (6–16)]. The imaginary component represents the attenuating effect of the liquid which for low frequencies increases as the frequency and vanishes as $\omega \to 0$.

For short wavelengths, or kH large, we use $c < \beta$ and obtain the approximation

$$(k^2 + \nu'^2)^2 - 4k^2\nu\nu' + \frac{\rho_0\alpha_0^2\nu}{\rho\beta^2\nu_0}(k^2 - \nu_0^2)(k^2 - \nu'^2) = 0 \qquad (6\text{–}38)$$

The first two terms may be recognized as the expression whose root gives the velocity of Rayleigh waves in a half space with the elastic properties of the plate. The last term represents the effect of the liquid on the propagation of Rayleigh waves.

The low- and high-frequency limits of the lowest symmetric mode can be found from Eqs. (6–37) and (6–38). Considering steel plates in water, Osborne and Hart computed values of phase velocity for this mode corresponding to intermediate wavelengths and showed that the effect of the water is to add an attenuation, leaving the real part of the phase velocity unchanged. This effect is small for a steel plate in water.

Other modes are introduced, however, by the presence of the water. For example, a mode exists such that when kH is large and $c < \alpha_0$ both (6–35) and (6–36) reduce to

$$4\left(1 - \frac{c^2}{\alpha_0^2}\right)^{\frac{1}{2}}\left[\left(1 - \frac{1}{2}\frac{c^2}{\beta^2}\right)^2 - \left(1 - \frac{c^2}{\alpha^2}\right)^{\frac{1}{2}}\left(1 - \frac{c^2}{\beta^2}\right)^{\frac{1}{2}}\right] + \frac{\rho_0 c^4}{\rho\beta^4}\left(1 - \frac{c^2}{\alpha^2}\right)^{\frac{1}{2}} = 0 \qquad (6\text{–}39)$$

A root exists for c close to but less than α_0, giving the velocity of Stoneley waves (Sec. 3–3) at a liquid-solid interface.

According to Osborne and Hart, the attenuation is great for most of the higher symmetric modes ($c > \beta$).

Antisymmetric Vibrations. For long wavelengths, or kH small, Eq. (6–36) reduces to

$$\frac{c^2}{\beta^2} = \left[\frac{4}{3} k^2 H^2 \left(1 - \frac{\beta^2}{\alpha^2}\right) \frac{\rho}{\rho_0}\right] \Big/ \left(1 + \frac{kH\rho}{\rho_0}\right) \tag{6–40}$$

This equation may be verified by setting $\rho_0 = 0$ and comparing it with Eq. (6–21), which gives the phase velocity for long flexural waves in a plate in vacuum.

For short wavelengths Eq. (6–36) reduces to an expression for Rayleigh waves (6–38) or for Stoneley waves (6–39), as might be expected.

For intermediate wavelengths the first mode is again similar to that found for a plate in a vacuum, with the addition of an imaginary component (small for a steel plate in water).

Other Investigations. The interesting phenomena which occur in the inhomogeneous problem of supersonic-wave transmission through plates immersed in liquids gave rise to the investigations of Cremer, Götz, and Schoch. Maxima of transmission which depend on frequency, thickness of a plate, and the angle of incidence of waves were experimentally observed. Cremer [9] has shown that for thin plates total transmission occurs at an angle of incidence such that the trace velocity (phase velocity) is equal to that of flexural waves. Götz [25] generalized this result for any maximum transmission and simplified Reissner's formulas. Schoch [75, 76] confirmed these theoretical and experimental results.

6–3. Floating Ice Sheet. A discussion of the propagation of elastic waves in a floating ice sheet was given independently by Press and Ewing [64] and by Satô [73] in which an infinite plate underlain by infinitely deep water was used to depict a floating ice sheet. Press and Ewing [65] extended their results to include the effect of the air.

SH Waves. These waves represent a special case of Love-wave propagation in which the solution is unchanged by the presence or absence of liquid media above and below the plate.

The period equation for this case can be obtained from Eq. (4–212) by allowing μ_2 to vanish. Thus we find

$$\tan 2kH\sqrt{\frac{c^2}{\beta_1^2} - 1} = 0 \tag{6–41}$$

where $2H$ is the thickness of the plate. Equation (6–41) is satisfied if $2kH\sqrt{c^2/\beta_1^2 - 1} = n\pi$, where $n = 0, 1, \cdots$. With the substitution $\beta_1/c = \sin\theta$ and $k = 2\pi \sin\theta/l_0$ we can write for (6–41)

$$4H \cos\theta = nl_0 \tag{6–42}$$

which is the condition for constructive interference between multiple-reflected SH waves with incident angle θ and wavelength l_0 in the direction of propagation. The derivation of a similar condition for a liquid layer was given in Eq. (4–83). From (6–41) or (6–42) we may derive for the group velocity

$$U = \beta \sin \theta = \frac{\beta_1^2}{c} \tag{6–43}$$

From Eqs. (6–41) or (6–42) and (6–43) we may compute dispersion curves given in Fig. 6–6. It is seen that $c \to \infty$ (and $U \to 0$) at discrete wave-

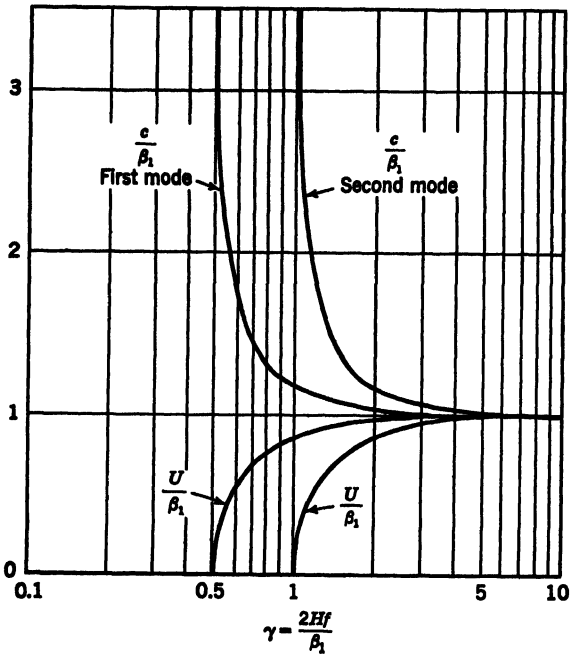


FIG. 6–6. Phase- and group-velocity curves for SH waves in a plate.

lengths given by $2H/l = \frac{1}{2}, 1, \frac{3}{2}, \cdots$. For each of the modes, $U \to c \to \beta_1$ as $2H/l \to \infty$. The sequence of arrivals at a given point corresponding to a given mode can be deduced from group-velocity curves such as those in Fig. 6–6. The first arrivals are high-frequency waves which travel with the velocity β_1. As time progresses, the frequency of the arrivals decreases. The wave train is infinitely long, the slowest waves having the lowest of cutoff frequencies given by $f = n\beta_1/4H$, $n = 1, 2, \cdots$. This case is almost

identical to that of electromagnetic waves in rectangular wave guides (Terman [82]).

SV and P Waves. Let ρ_1 and ρ_2 be the densities of the ice and water, respectively, α, the velocities of compressional waves, and β_1 the velocity of distortional waves in ice. We have for SV and P motion

$$u_1 = \frac{\partial \varphi_1}{\partial x} - \frac{\partial \psi_1}{\partial z} \qquad u_2 = \frac{\partial \varphi_2}{\partial x}$$
$$w_1 = \frac{\partial \varphi_1}{\partial z} + \frac{\partial \psi_1}{\partial x} \qquad w_2 = \frac{\partial \varphi_2}{\partial z} \tag{6-44}$$

The potentials must be solutions of the wave equations (4–8), and the boundary conditions are of the type (4–4). The median plane of the ice sheet is the plane $z = 0$, the free surface is then given by $z = -H$, and the interface by $z = H$. Thus we have the boundary conditions in the form

$$(p_{zz})_1 = 0 \qquad (p_{xz})_1 = 0 \qquad \text{at } z = -H$$
$$w_1 = w_2 \qquad (p_{xz})_1 = 0 \qquad (p_{zz})_1 = (p_{zz})_2 \qquad \text{at } z = H \tag{6-45}$$

By (4–10) we put

$$\nu_1 = \sqrt{k^2 - k_{\alpha_1}^2} \qquad \nu_1' = \sqrt{k^2 - k_{\beta_1}^2} \qquad \nu_2 = \sqrt{k^2 - k_{\alpha_2}^2} \tag{6-46}$$

Taking the solutions of the wave equations (4–8), we have

$$\varphi_1 = (A \sinh \nu_1 z + B \cosh \nu_1 z) e^{i(\omega t - kx)}$$
$$\psi_1 = (C \sinh \nu_1' z + D \cosh \nu_1' z) e^{i(\omega t - kx)} \tag{6-47}$$
$$\varphi_2 = E e^{-\nu_2 z + i(\omega t - kx)}$$

The arbitrary coefficients $A, B, \cdots, E$ are determined by the boundary conditions (6–45). A homogeneous system of five equations is obtained upon substituting the expressions (6–47) in (6–45), and the resultant determinant must vanish. To simplify the results, the assumption $\lambda_1 = \mu_1$ or $\sigma = \frac{1}{4}$ can be made for ice but the observed values of Poisson's constant for lake ice are actually higher. Under these conditions, the period equation can be written in the form

$$P(2Q + \delta \cosh \nu_1 H \cosh \nu_1' H) + Q\delta \sinh \nu_1 H \sinh \nu_1' H = 0 \tag{6-48}$$

where

$$\delta = \rho_2 \alpha_2^2 (\nu_1'^2 - k^2)(\nu_2^2 - k^2) \frac{\nu_1}{\rho_1 \beta_1^2 \nu_2}$$
$$P = (\nu_1'^2 + k^2)^2 \cosh \nu_1 H \sinh \nu_1' H - 4\nu_1 \nu_1' k^2 \sinh \nu_1 H \cosh \nu_1' H \tag{6-49}$$
$$Q = (\nu_1'^2 + k^2)^2 \sinh \nu_1 H \cosh \nu_1' H - 4\nu_1 \nu_1' k^2 \cosh \nu_1 H \sinh \nu_1' H$$

If these equations are compared with those given by Lamb [Eqs. (6–11) and (6–12)], it will be seen that $P = 0$ and $Q = 0$ represent the symmetric and antisymmetric solutions, respectively, for a plate in a vacuum. Osborne and Hart obtained Eqs. (6–35) and (6–36) for the case of a plate in a liquid, or $P + \delta \sinh \nu_1 H \sinh \nu_1' H = 0$ and $Q + \delta \cosh \nu_1 H \cosh \nu_1' H = 0$, which also represent the symmetric and antisymmetric solutions, respectively. It is evident that, unlike these cases, the motion in a floating ice sheet cannot be reduced to purely symmetric and antisymmetric modes. It can also be seen that δ is a modification term, introduced by the presence of the liquid, which vanishes as the density of the liquid approaches zero. The values $c = \alpha_1$ and $c = \beta_1$ are solutions of Eq. (6–48).

The evaluation of phase velocity from (6–48) can be simplified in the limiting cases of very small wavelengths compared with the thickness of the ice sheet ($kH = 2\pi H/l$ very large) and of very large wavelengths compared with H (kH small). In the first case we have $P \cong Q$, and (6–48) reduces to

$$(\nu_1'^2 + k^2)^2 - 4\nu_1\nu_1' k^2 = 0 \tag{6–50}$$

and

$$(\nu_1'^2 + k^2)^2 - 4\nu_1\nu_1' k^2 + \delta = 0 \tag{6–51}$$

The products of hyperbolic functions having large and approximately equal values were canceled.

Equation (6–50) corresponds to Rayleigh waves propagated along the free surface of a semi-infinite solid medium [Eq. (2–28)]. The root of this equation is $c_R = 0.9194\beta_1$. Equation (6–51) is identical with Eq. (6–38) derived by Osborne and Hart for a plate in a liquid and corresponds to Rayleigh waves propagated along the interface between a liquid and solid, both of semi-infinite extent. Since their velocity exceeds that of sound in water, some energy is radiated from the solid. By following Osborne and Hart, an approximation for the phase velocity can be deduced from (6–51) by substituting $c = c_R\ (1 + \epsilon)$, $\epsilon < 1$. If δ is small, we may write the approximation

$$\epsilon = -i\left[\frac{\frac{\rho_2}{\rho_1}\left(1 - \frac{c_R^2}{\alpha_1^2}\right)^{\frac{1}{2}}\frac{c_R^4}{\beta_1^4}}{4\left(\frac{c_R^2}{\alpha_2^2} - 1\right)^{\frac{1}{2}}}\right]\left\{2\left(1 - \frac{c_R^2}{2\beta_1^2}\right)\frac{c_R^2}{\beta_1^2} - \left[\frac{1 - \frac{c_R^2}{\beta_1^2}}{1 - \frac{c_R^2}{\alpha_1^2}}\right]^{\frac{1}{2}}\frac{c_R^2}{\alpha_1^2} - \left[\frac{1 - \frac{c_R^2}{\alpha_1^2}}{1 - \frac{c_R^2}{\beta_1^2}}\right]^{\frac{1}{2}}\frac{c_R^2}{\beta_1^2}\right\}^{-1} \tag{6–52}$$

Using $\rho_1 = 0.917\rho_2$, $\beta_1 = 6{,}300$ ft/sec, $\alpha_2 = 4{,}800$ ft/sec, $\alpha_1 = \sqrt{3}\beta_1$, we

obtain approximately $\epsilon = i/4$. For a steel half space and water, the approximation is much better with $\epsilon = i/100$.

Using (6–46) and $\omega = ck$, we may write Eq. (6–51) in the form

$$\left(1 - \frac{c^2}{\alpha_2^2}\right)^{\frac{1}{2}}\left[\left(2 - \frac{c^2}{\beta_1^2}\right)^2 - 4\left(1 - \frac{c^2}{\alpha_1^2}\right)^{\frac{1}{2}}\left(1 - \frac{c^2}{\beta_1^2}\right)^{\frac{1}{2}}\right] + \left(1 - \frac{c^2}{\alpha_1^2}\right)^{\frac{1}{2}} \frac{\rho_2 c^4}{\rho_1 \beta_1^4} = 0 \qquad (6\text{–}53)$$

Then solving for c with the constants for ice listed above, we obtain $c = 0.87\alpha_2$. This is the speed of Stoneley waves (see Sec. 3–3) traveling along the ice-water interface. Their amplitudes decrease with distance from the interface.

In order to study the period equation (6–48) when kH is small, we write it in the form

$$P(2Q + \delta \cosh \nu_1 H \cosh \nu_1' H) = -Q\delta \sinh \nu_1 H \sinh \nu_1' H \qquad (6\text{–}54)$$

Expanding the hyperbolic functions to terms of the third power in kH, we use the fact that in Eq. (6–40) c^2/β^2 for flexural waves was found to be proportional to $(kH)^3$. With this substitution we find that the lowest order of the right-hand member is $(kH)^{15}$ and the lowest order of the left-hand member is $(kH)^{10}$. We therefore may take the left-hand member to approach zero, provided that we use no terms of higher power than $(kH)^{14}$. Recalling from the previous section that the vanishing of the second factor in the period equation gives flexural waves, we find that the approximate form of Eq. (6–54) for flexural waves is

$$2kH\left\{\frac{4}{3}(kH)^2 \frac{c^2}{\beta_1^2}\left(\frac{\beta_1^2}{\alpha_1^2} - 1\right) + \frac{c^4}{\beta_1^4}\right\} + \frac{\rho_2}{\rho_1}\frac{c^4}{\beta_1^4} = 0 \qquad (6\text{–}55)$$

by retaining terms to the order $(kH)^7$ in this factor. Solving for c^2/β_1^2, we find

$$\frac{c^2}{\beta_1^2} = \frac{8\rho_1(kH)^3}{3\rho_2} \frac{1 - \frac{\beta_1^2}{\alpha_1^2}}{1 + 2\frac{\rho_1(..H)}{\rho_2}} \qquad (6\text{–}56)$$

For $\rho_2 = 0$ this reduces to the corresponding Eq. (6–21) for a plate in a vacuum. It is interesting to note that (6–56) can be reduced to the corresponding expression for a plate in a liquid simply by replacing ρ_2 by $2\rho_2$. The steady-state plane-wave theory of long flexural waves in an ice sheet on water of either finite or infinite depth was given by Ewing and Crary (Chap. 4, Ref. 36) and extended by Press and Ewing [64]. In general, the observed dispersion of flexural waves generated by explosions in floating ice is shown to be in reasonable agreement with the theory in these papers.

Crary [8] computed phase- and group-velocity curves from Eq. (6–54) for the mode which is analogous to the flexural mode for a plate in a vacuum. These curves, shown in Fig. 6–7, are based on the elastic constants for ice and sea water.

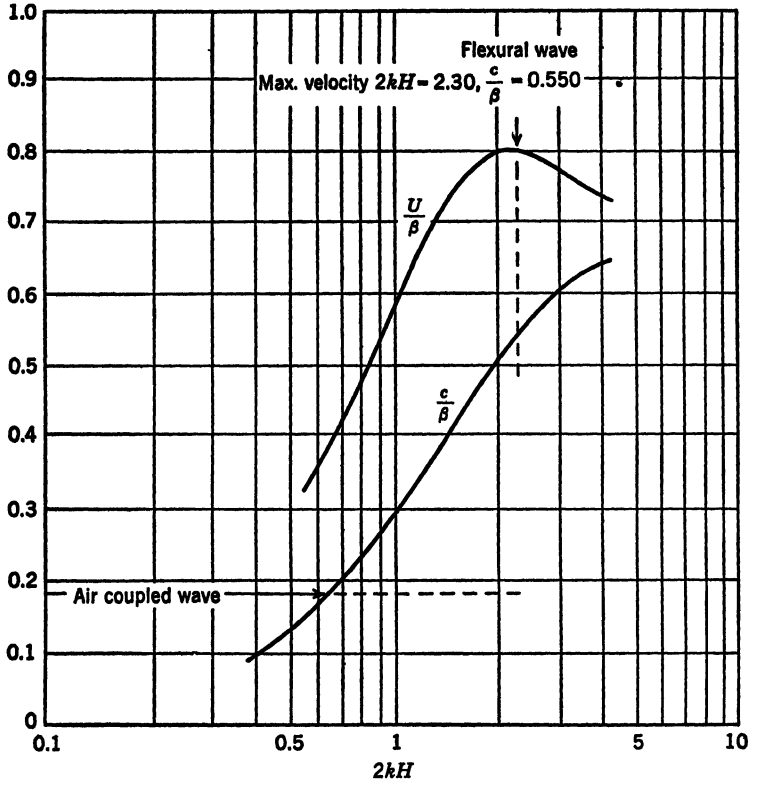


FIG. 6–7. Theoretical phase- and group-velocity curves for a floating ice sheet; α = 12,400 ft/sec, β = 6,040 ft/sec, σ = 0.345 for ice, α = 4,800 ft/sec for water. (*After Crary.*)

For plate waves, where c^2/β_1^2 is not a small quantity, we write Eq. (6–48) in the form

$$P + \frac{Q\delta \sinh \nu_1 H \sinh \nu_1' H}{2Q + \delta \cosh \nu_1 H \cosh \nu_1' H} = 0 \tag{6–57}$$

expand the functions in powers of kH, and divide by $\nu_1' kH$. The first term is of the form $1 + (kH)^2 + \cdots$, and the second, which is complex for

$c > \alpha_2$, is of the order

$$(kH)^2 + i(kH)^3$$

The terms containing $(kH)^2$ introduce a slight dispersive correction to the velocity c_p, and the term $i(kH)^3$ represents attenuation due to loss of energy from the plate to the liquid. Neglecting the slight change in velocity, we find the approximation

$$\frac{c}{\beta_1} = 2\left(1 - \frac{\beta_1^2}{\alpha_1^2}\right)^{\frac{1}{2}}\left\{1 + 2i(kH)^3 \frac{\beta_1\rho_1}{\alpha_2\rho_2}\left(1 - \frac{\beta_1^2}{\alpha_1^2}\right)^{\frac{1}{2}}\left[1 - \frac{4\beta_1^2}{\alpha_1^2}\left(1 - \frac{\beta_1^2}{\alpha_1^2}\right)\right]\right\} \quad (6\text{–}58)$$

In deriving this equation, we have replaced c^2 by c_p^2 in the imaginary term. The solution for a plate in a vacuum is given by Eq. (6–16). If we compare these two expressions, we can easily see that the presence of the liquid hardly affects the real part of the phase velocity but adds a small attenuation, which for large wavelengths increases as the inverse cube of wavelength. This is in agreement with the experimental work of Ewing, Crary, and Thorne (Chap. 4, Ref. 34), in which it was established that the velocity of long longitudinal waves in floating lake ice was given by the real part of Eq. (6–58).

For intermediate wave lengths the evaluation of phase velocity from Eq. (6–48) is very difficult. In general, for $c > \alpha_1$, the phase velocity has an imaginary component, indicating attenuation due to radiation from the ice sheet into water. Both real and imaginary components of phase velocity depend on frequency, the waves being dispersive as well as selectively attenuated. For $c < \alpha_2$, no energy losses due to radiation into the water occur, and the waves are propagated as a dispersive unattenuated train.

Crary Waves. An unusual type of *SV* wave was discovered by Crary [8] in seismic experiments on the floating ice island *T*-3. This wave has the following characteristics (Fig. 6–8):

1. Phase velocity is near the speed of compressional waves in the plate.
2. Travel time is intermediate between that for *P* waves and that for *SH* waves.
3. Principal recording is on longitudinal horizontal seismograph; amplitudes are much smaller on vertical and practically absent on transverse horizontal.
4. Frequency is almost constant, increasing very slightly with time.

These waves are propagated by multiple reflection of *SV* waves arriving at the ice boundaries with an angle of incidence $\theta_{sr} = \sin^{-1} \beta_1/\alpha_1$. At this angle, the *SV* wave is totally reflected in its original form, and the vertical displacement at the surface is zero, as may be seen from Fig. 6–9.

The frequency is determined by the requirement for constructive

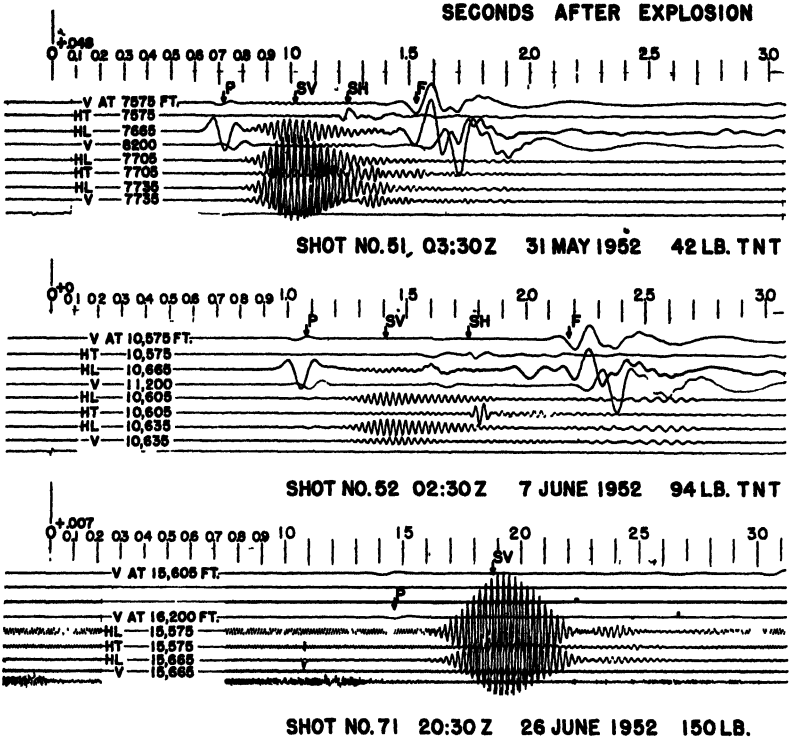


FIG. 6–8. Seismogram obtained on floating ice island, showing *P*, *SV*, Crary, and *SH* waves. The Crary waves are the large trains of nearly sinusoidal waves. (*After Crary.*)

interference

$$4H \cos \theta_{cr} - l = nl \qquad n = 1, 2, \cdots \tag{6–59}$$

where $l = \beta_1/f$ is the wavelength along the path, f being the frequency. The second term in Eq. (6–59) arises from the reversal in phase on reflection at the air-ice and the water-ice interfaces. For $n = 1$, Eq. (6–59) reduces to

$$2H = \frac{\beta_1}{f \cos \theta_{cr}} \tag{6–60}$$

On the ice island Crary found

$$\alpha_1 = 12{,}400 \text{ ft/sec}$$

$$\beta_1 = 6{,}040 \text{ ft/sec}$$

$$f = 40 \text{ cycles/sec}$$

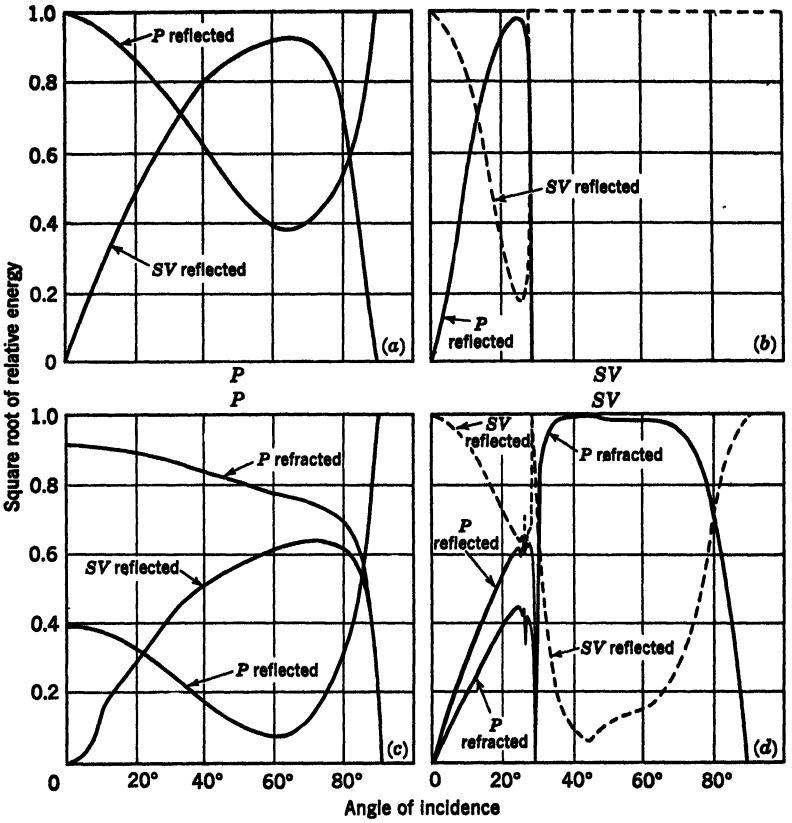


FIG. 6–9. Reflections and refractions at interfaces of a floating ice sheet: (*a*) *P* incident at free surface; (*b*) *SV* incident at free surface; (*c*) *P* incident at ice-water interface; (*d*) *SV* incident at ice-water interface. (*After Crary.*)

Solving for the ice thickness $2H$, Crary found the value 173 ft, which compares well with the thickness obtained by other methods. Equation (6–60) can also be derived from Eq. (6–54) by setting the phase velocity $c = \alpha_1$. The travel time of this phase can be deduced from the corresponding group velocity.

Flexural Waves from an Impulsive Source. A theory for the propagation of flexural waves for the case of an impulsive point source located either in air, or in water beneath a floating ice sheet, was given by Press and Ewing [65]. A partial derivation and some results are given here.

Consider a plate of infinite extent floating on deep water, the thickness of the plate H being small compared with the wavelengths considered.

Overlying the plate is an infinite atmosphere having density ρ_0, and sound velocity α_0. The plate has density ρ_1 and longitudinal-wave velocity† c_p; the water has density ρ_2 and sound velocity α_2. Cylindrical coordinates r, z are used, with z axis positive upward. Assuming simple harmonic motion (exp $(i\omega t)$), we introduce the velocity potentials $\bar{\varphi}_0$ and $\bar{\varphi}_2$ from which the velocity components $\bar{q}$ and $\bar{w}$ and the pressure p can be obtained as follows:

$$p_i = -\rho_i \frac{\partial \bar{\varphi}_i}{\partial t} \qquad \bar{q}_i = \frac{\partial \bar{\varphi}_i}{\partial r} \qquad \bar{w}_i = \frac{\partial \bar{\varphi}_i}{\partial z} \qquad i = 0, 2 \tag{6-61}$$

The functions $\bar{\varphi}_i$ are solutions of the wave equations

$$\alpha_i^2 \nabla^2 \bar{\varphi}_i = \frac{\partial^2 \bar{\varphi}_i}{\partial t^2} \qquad i = 0, 2 \tag{6-62}$$

where $\nabla^2 = \partial^2/\partial r^2 + (1/r)\,\partial/\partial r + \partial^2/\partial t^2$. We assume now that the solutions of these equations satisfy the boundary conditions for a thin plate,

$$\frac{\partial \bar{\varphi}_0}{\partial z} = \frac{\partial w_1}{\partial t} = \frac{\partial \bar{\varphi}_2}{\partial z} \qquad \text{at } z = 0 \tag{6-63}$$

The vertical displacement w_1 of the ice sheet satisfies the equation for flexural vibrations of a thin plate:

$$H\rho_1 \frac{\partial^2 w_1}{\partial t^2} = -\frac{H^3 \rho_1 c_p^2}{12} \nabla^4 w_1 - \rho_2 g w_1 - \rho_2 \frac{\partial \bar{\varphi}_2}{\partial t} + \rho_0 \frac{\partial \bar{\varphi}_0}{\partial t} \tag{6-64}$$

where g is the gravitational acceleration,

$$\nabla^4 = \frac{1}{r}\frac{\partial}{\partial r}\left\{r \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right)\right]\right\} \tag{6-65}$$

and the relation $\lambda = \mu$ is assumed.

Following the procedure in Sec. 4–1, we can write as formal solutions of (6–62) and (6–64) for a source at $z = h$ (in the air)

$$\bar{\varphi}_0 = \int_0^\infty J_0(kr) e^{-\nu_0 |z-h|} \frac{k\,dk}{\nu_0} + \int_0^\infty Q_0 e^{-\nu_0 z} J_0(kr) \frac{k\,dk}{\nu_0} \tag{6-66}$$

$$\bar{\varphi}_2 = \int_0^\infty Q_2 e^{\nu_2 z} J_0(kr) k\,dk \tag{6-67}$$

$$w_1 = \int_0^\infty Q_1 J_0(kr) k\,dk \tag{6-68}$$

where

$$\nu_i = \sqrt{k^2 - k_{\alpha_i}^2} \qquad i = 0, 2 \tag{6-69}$$

†Not to be confused with the compressional-wave velocity α_1.

the factor exp $i\omega t$ being omitted. The first term in (6–66) represents the direct compressional wave emitted by the source; the remaining terms in Eqs. (6–66) to (6–68) represent compressional waves in the air and in the liquid and flexural waves in the plate, respectively, resulting from the action of the direct wave.

Three simultaneous linear equations result when the solutions (6–66) to (6–68) are substituted in Eqs. (6–63) and (6–64). Solving for Q_1, we can write an expression for the displacement of the plate due to a periodic point source in the air:

$$w_1 = 2\frac{\rho_0}{\rho_1} i\omega e^{i\omega t} \int_0^\infty \frac{\nu_2 e^{-\nu_0 h}}{G(k)} J_0(kr)k\, dk \tag{6–70}$$

where

$$G(k) = \frac{\rho_2}{\rho_1}\omega^2\nu_0 + \nu_0\nu_2\left(H\omega^2 - \frac{H^3c_p^2k^4}{12} - g\frac{\rho_2}{\rho_1}\right) + \frac{\rho_0}{\rho_1}\omega^2\nu_2 \tag{6–71}$$

The solutions for a point source in the water can be obtained from (6–66) to (6–68) by interchanging the subscripts 0 and 2.

Integrals of the type (6–70) have been evaluated in Sec. 4–2. The procedure was to transform the path of integration to the complex k plane. The solution was then expressed as the sum of the residues of the integrands corresponding to the poles given by $G(k) = 0$ and two integrals along branch lines corresponding to the branch points $k = \omega/\alpha_2$ and $k = \omega/\alpha_0$. The residues, which diminish as $r^{-\frac{1}{2}}$, give the flexural waves, whereas the branch line integrals represent compressional waves in the media above and below the plate, diminishing as r^{-2}. For large values of r, the contribution of the residues to the value of the integral (6–70) is approximately

$$\begin{aligned} w_1 &= \frac{2\pi\omega\rho_0\kappa\nu_2}{\rho_1(\pi\kappa r/2)^{\frac{1}{2}}\,\partial G(\kappa)/\partial\kappa} \exp(-\nu_0 h) \exp\left[i\left(\omega t - \kappa r + \frac{\pi}{4}\right)\right] \\ &= \frac{W(\kappa)}{c_p H^{\frac{1}{2}} r^{\frac{1}{2}}} \exp(-\nu_0 h) \exp\left[i\left(\omega t - \kappa r + \frac{\pi}{4}\right)\right] \end{aligned} \tag{6–72}$$

where κ is a root of the equation

$$G(k) = 0 \tag{6–73}$$

with $G(k)$ defined by (6–71).

The steady-state solution represented by Eq. (6–72) may be generalized for the case of an arbitrary initial disturbance by the Fourier-integral method of Sec. 4–2. As before, this procedure leads to the conclusion that the predominant disturbance at a distant point satisfies the relation $r/t = U$.

Phase velocity may be obtained as a function of the parameter $\gamma = Hf/\alpha_0$

from Eq. (6–73), and the corresponding group-velocity curve may be obtained using $U = c + k\,dc/dk$. Phase- and group-velocity curves are plotted in Fig. 6–10 for the case $\alpha_0 = 1{,}070$ ft/sec, $c_p = 11{,}500$ ft/sec, $\alpha_2 = 4{,}650$ ft/sec, $\rho_0/\rho_1 = 0.00141$, $\rho_2/\rho_1 = 1.090$. Let γ_a and f_a correspond to $c = \alpha_0$. The portion of the phase-velocity curve for $c < \alpha_0$, that is,

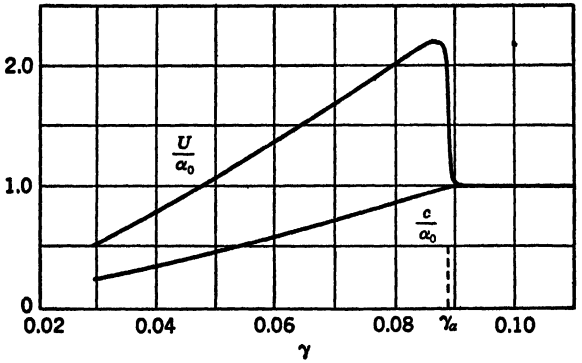


FIG. 6–10. Phase- and group-velocity curves for air-coupled flexural waves in floating ice.

for $\gamma < \gamma_a$, and the corresponding part of the group-velocity curve to the left of the maximum would have been obtained had we neglected the effect of the air. However, the maximum value, the steep limb of the group-velocity curve occurring near $\gamma = \gamma_a$, and the values $U = c = \alpha_0$ for $\gamma > \gamma_a$ all represent effects of the air.

A graph of the steady-state amplitude $W(\kappa)$ is presented in Fig. 6–11 for a source in the air. Interchanging the subscripts 0 and 2 in (6–72) enables one to compute the corresponding amplitudes for a source in the water. Study of these curves reveals that a peak amplitude occurs for an air source at a frequency $f_a = \alpha_0\gamma_a/H$, corresponding to the phase velocity $c = \alpha_0$. For a point source in the water, largest amplitudes are associated with low-frequency waves. As the frequency (and phase velocity) increases, wave amplitudes decrease and abruptly drop to zero as the frequency f_a is approached.

The sequence and character of arrivals at a distant point can be deduced from Figs. 6–10 and 6–11, since the arrival time of waves of a given frequency corresponds to propagation at the associated group velocity and the wave amplitudes are proportional to $W(\kappa)$. For an air shot the first waves to arrive appear at the time $t = r/2.2\alpha_0$, corresponding to propagation at the maximum value of group velocity. These waves appear with large

amplitudes with a frequency close to f_a. Following this, two wave trains arrive simultaneously, corresponding to the two branches of the group-velocity curve on either side of the maximum. Waves propagated according to the left branch are dispersive and rapidly decrease in amplitude as the frequency decreases from the peak value f_a. Waves corresponding to the

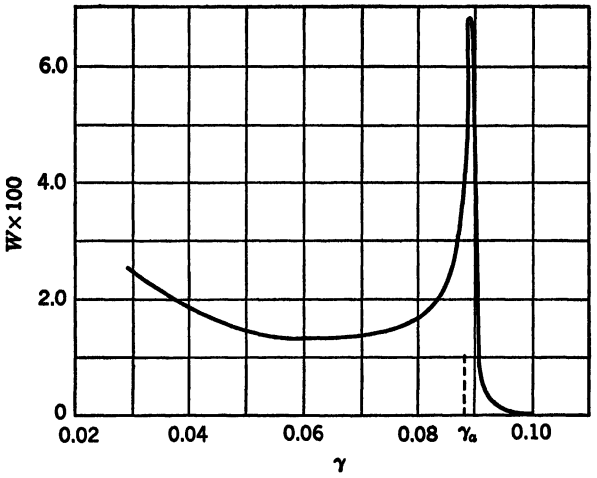


FIG. 6–11. Steady-state amplitude function for an air source.

steep right branch appear as a constant-frequency train continuing to the time $t = r/\alpha_0$. The constant-frequency waves form the predominant disturbance from an air shot since their frequency f_a lies close to the peak frequency.

For a water shot the sequence of arrivals is the same, since the group-velocity curve of Fig. 6–10 is still applicable. However, the amplitudes now follow the curve in Fig. 6–12, where the dispersive waves are predominant. A water shot is therefore characterized by a train of dispersive waves beginning at the time $t = r/2.2\alpha_0$, with a frequency close to f_a. As time increases, frequency decreases and amplitude increases.

In Fig. 6–13 seismograms from an air shot and water shot for lake ice 1.1 ft thick are presented. The constant-frequency train for the air shot and the dispersive train for the water shot are immediately apparent.

6–4. Cylindrical Rod in a Vacuum. In the problems of wave propagation in layered media considered previously, plane or spherical surfaces and interfaces were assumed. A special case having some practical interest is that in which the boundaries are cylinders, the cylindrical rod in a vacuum

being the simplest example. More complicated problems concern a cylindrical rod immersed in a liquid or embedded in another solid.

Three types of vibrations in cylindrical rods are considered here: longitudinal, lateral (flexural), and torsional. These may be treated in various ways, as is well known, but we continue as before to use the general equations of the theory of elasticity. For references on this problem see also Kolsky (Chap. 5, Ref. 25).

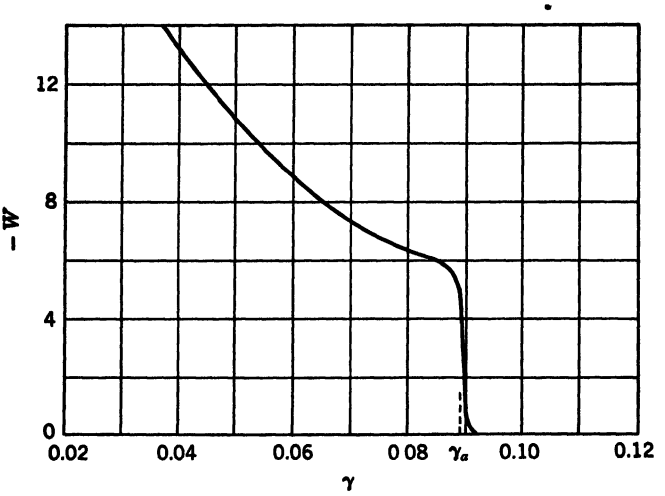


FIG. 6–12. Steady-state amplitude function for a water source.

Longitudinal Vibrations. We assume axial symmetry and take the equations of motion in cylindrical coordinates (1–25). The displacements q and w of a particle perpendicular (r direction) or parallel (z direction) to the axis of the cylinder are expressed in terms of the two functions φ and ψ. By (1–26) to (1–29) we have

$$q = \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \psi}{\partial r\, \partial z} \qquad w = \frac{\partial \varphi}{\partial z} - \frac{\partial}{r\, \partial r}\left(r \frac{\partial \psi}{\partial r}\right) = \frac{\partial \varphi}{\partial z} - \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{6–74}$$

$$\nabla^2 \varphi = \frac{1}{\alpha^2}\frac{\partial^2 \varphi}{\partial t^2} \qquad \nabla^2 \psi \equiv \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2 \psi}{\partial t^2} \tag{6–75}$$

For a rod of infinite length and radius a, the boundary conditions are that the normal and tangential stresses must vanish on the surface of the cylinder:

$$p_{rr} = 0 \qquad p_{sr} = 0 \qquad \text{at } r = a \tag{6–76}$$

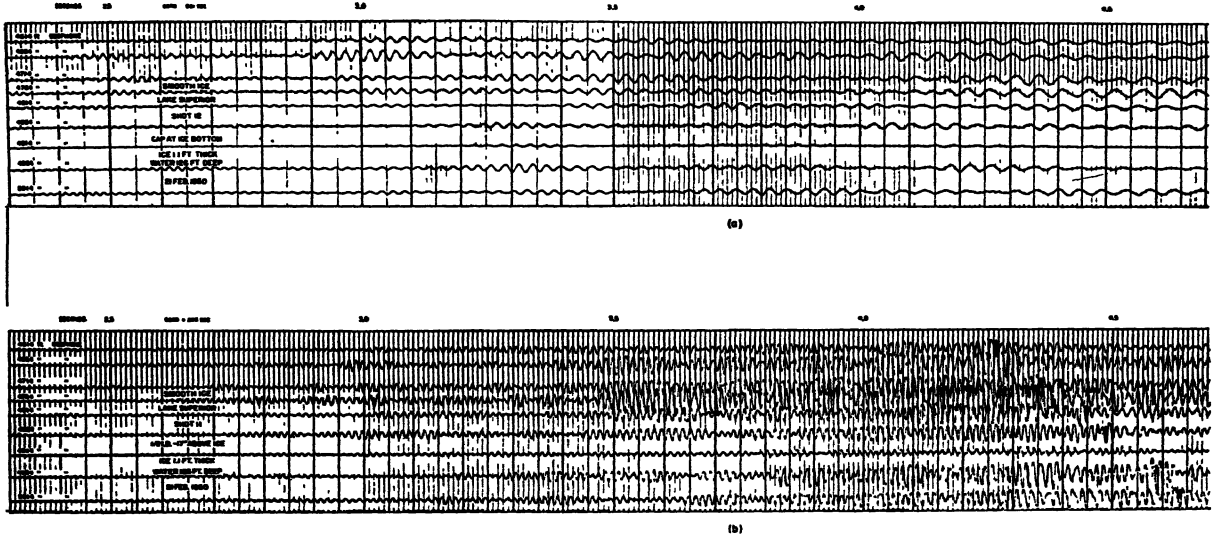


FIG. 6–13. (*a*) Dispersive flexural waves in a floating ice sheet excited by an underwater shot. (*b*) Constant-frequency flexural waves from an air shot. Ice 1.1 ft thick, shot-detector distance 4,614 to 5,014 ft.

where the stresses are given by

$$p_{rr} = \lambda\left(\frac{q}{r} + \frac{\partial q}{\partial r} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial q}{\partial r} \qquad p_{zz} = \mu\left(\frac{\partial q}{\partial z} + \frac{\partial w}{\partial r}\right) \tag{6-77}$$†

Since wave propagation along the axis of the rod is considered, we take a particular solution of Eqs. (6–75) in the form

$$\varphi = AF(r)e^{i(\omega t - \nu z)} \qquad \psi = CG(r)e^{i(\omega t - \nu' z)} \tag{6-78}$$

Then by (6–75) we obtain

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} + (k_\alpha^2 - \nu^2)F = 0 \tag{6-79}$$

$$\frac{d^2G}{dr^2} + \frac{1}{r}\frac{dG}{dr} + (k_\beta^2 - \nu'^2)G = 0 \tag{6-80}$$

If we assume that $\nu' = \nu$, then the boundary conditions will be independent of z, and

$$k_\alpha^2 - \nu^2 = k^2 \qquad k_\beta^2 - \nu^2 = k_1^2 \tag{6-81}$$

Equations (6–79) and (6–80) are satisfied by the Bessel functions $J_0(kr)$ and $J_0(k_1 r)$, respectively. Therefore,

$$\varphi = Ae^{i(\omega t - \nu z)}J_0(kr) \qquad \psi = Ce^{i(\omega t - \nu z)}J_0(k_1 r) \tag{6-82}$$‡

Now, by (6–74) and (6–82), we can take

$$\begin{aligned} q &= \left[A\frac{d}{dr}J_0(kr) - iC\nu\frac{d}{dr}J_0(k_1 r)\right]e^{i(\omega t - \nu z)} \\ w &= \left[-iA\nu J_0(kr) - \frac{C}{r}\frac{d}{dr}\left(r\frac{dJ_0(k_1 r)}{dr}\right)\right]e^{i(\omega t - \nu z)} \end{aligned} \tag{6-83}$$

for waves propagating in the positive z direction. On inserting the expressions (6–83) in Eqs. (6–76) and (6–77), we obtain the boundary conditions

†When the cylindrical coordinates r, χ, z are used, a volume element is built up on the linear elements dr, $rd\chi$, dz. The stress acting at a free face of such an element forming a part of the cylinder surface can have components p_{rr}, p_{rz}, $p_{r\chi}$. Since we consider the case of axial symmetry for longitudinal waves propagating in the z direction, tensions in the direction of the coordinate χ are not produced, and only the first two stress components have to be considered. Azimuthal vibrations were considered in Sec. 5–3.

‡Sometimes a function $W = -\partial\psi/\partial r$ is used in the expressions mentioned above. In this case one can write

$$W = C'e^{i(\omega t - \nu z)}J_1(k_1 r)$$

where $J_1(\xi) = -dJ_0(\xi)/d\xi$ is the Bessel function of the first order.

in the form

$$A\left[2\mu\frac{d^2}{dr^2}J_0(kr) - k_\alpha^2\lambda J_0(kr)\right] - 2iC\mu\nu\frac{d^2}{dr^2}J_0(k_1r) = 0 \quad \text{at } r = a$$
$$2i\nu A\frac{d}{dr}J_0(kr) + C(2\nu^2 - k_\beta^2)\frac{d}{dr}J_0(k_1r) = 0 \tag{6-84}$$

The period equation follows upon elimination of A and C from Eqs. (6–84) and can be given in the form

$$\begin{vmatrix} 2\mu\dfrac{d^2J_0(ka)}{da^2} - k_\alpha^2\lambda J_0(ka) & -2i\mu\nu\dfrac{d^2J_0(k_1a)}{da^2} \\ 2i\nu\dfrac{dJ_0(ka)}{da} & (2\nu^2 - k_\beta^2)\dfrac{dJ_0(k_1a)}{da} \end{vmatrix} = 0 \tag{6-85}$$

It is difficult to discuss this equation in its general form. If the radius a of the cylinder is very small (a thin rod), we may use the expansion of the Bessel function

$$J_0(ka) = 1 - \frac{1}{4}k^2a^2 + \frac{1}{64}k^4a^4 + \cdots \tag{6-86}$$

By (6–86) the period equation (6–85) takes the form

$$(k_\beta^2 - 2\nu^2)k_1a\left(1 - \frac{k_1^2a^2}{8}\right)\left[k^2\left(1 - \frac{3}{8}k^2a^2\right) + \frac{\lambda}{\mu}k_\alpha^2\left(1 - \frac{1}{4}k^2a^2\right)\right]$$
$$+ 2\nu^2k_1\left(1 - \frac{3}{8}k_1^2a^2\right)ak^2\left(1 - \frac{1}{8}k^2a^2\right) = 0 \tag{6-87}$$

If we omit the factor k_1a and neglect all terms of the order a^2, the longitudinal waves are found to propagate along the cylinder with the phase velocity

$$c_0 = \frac{\omega}{\nu} \cong \sqrt{\frac{\mu}{\rho}\frac{(3\lambda + 2\mu)}{\lambda + \mu}} = \sqrt{\frac{E}{\rho}} \tag{6-88}$$

where E is Young's modulus. The second approximation, due to Pochhammer [62], gives

$$c = \frac{\omega}{\nu} \cong \sqrt{\frac{E}{\rho}\left(1 - \frac{1}{4}\sigma^2\nu^2a^2\right)} \tag{6-89}$$

where $\sigma = \lambda/2(\lambda + \mu)$ is Poisson's ratio.

For νa large, Bancroft [1] showed that Eq. (6–87) reduces to Rayleigh's equation (2–28), the root of which gives the velocity of Rayleigh waves at the surface of a solid half space.

Values of the ratio of phase velocities c/c_0 as a function of νa were

computed from Eq. (6–87) by Davies [12] for $\sigma = 0.29$. An infinite number of modes occur, the first three of which are shown in Figs. 6–14 and 6–15. In the first mode the long-wave limit of phase velocity is determined by Young's modulus, $c_0 = \sqrt{E/\rho}$, and the short-wave limit is the Rayleigh-wave velocity, as discussed before. In the higher modes $c \to \infty$ at low

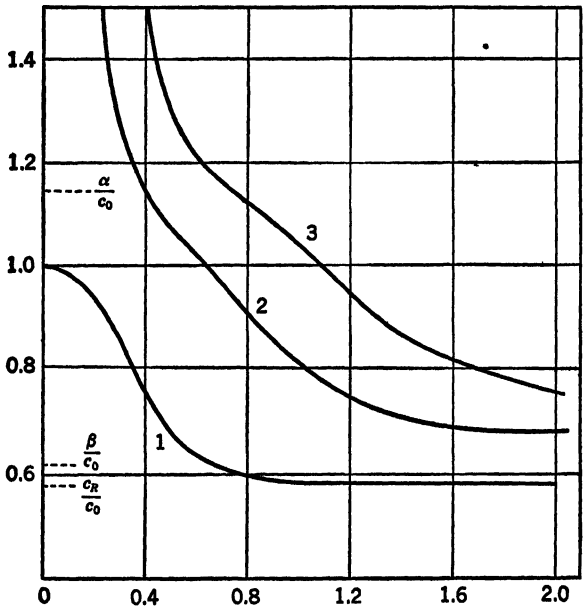


FIG 6–14. Phase velocity (i.e., c/c_0) of longitudinal waves in a cylindrical rod with Poisson's constant 0.29 as a function a/l. First three modes. (*After Davies.*)

frequencies, and $c \to \beta$ at high frequencies. It is interesting to note how closely these results parallel those for the plate (Sec. 6–1).

Oliver [58] has used pulse techniques to study the propagation of waves in long cylindrical rods. The oscillogram for the first longitudinal mode is shown in Fig. 6–16. The first four traces show a dispersive train of waves in which the period decreases with time. In the middle of the fourth trace a train is initiated in which the period increases with time. Group velocity obtained from this oscillogram is plotted as a function of period in Fig. 6–17. The experimental points are seen to fit the theoretical curve quite accurately on both sides of the minimum value of group velocity. The Airy phase corresponding to the minimum group velocity appears as the prominent train on the last trace of Fig. 6–16.

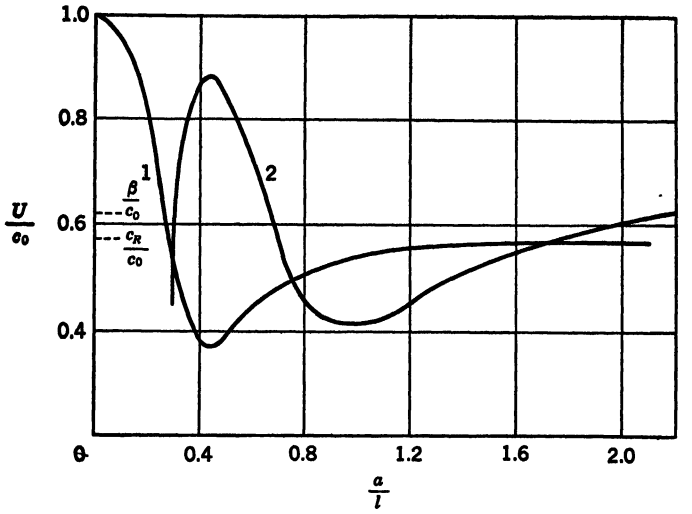


FIG. 6–15. Group velocity corresponding to phase velocity in Fig. 6–14. (*After Davies.*)

Torsional and Flexural Vibrations. To discuss the latter type of wave propagation in cylindrical rods, all three equations of motion in cylindrical coordinates must be taken into account. Torsional vibrations are characterized by the conditions that q and w vanish and that the displacement v_χ corresponding to the third cylindrical coordinate (χ) is independent of χ. Then the propagation is determined by only one differential equation. For harmonic waves along the rod axis the displacement v_χ becomes proportional to the Bessel function of the first order, and the frequency equation is obtained from one boundary condition $p_{r\chi} = 0$. In the first mode the motion is a rotation of each circular section about its center, and the phase velocity is equal to the shear velocity β. Dispersion occurs in higher modes.

It is obvious that, in general, bending of a rod or bar can occur in any direction. For flexural waves propagating in a circular rod an axial section can exist such that points vibrating in this plane remain in it during lateral or flexural motion. All three components of the displacement (q, v_χ, w) are involved in this type of vibration, and we refer to the work of Love (Chap. 1, Ref. 34) where it is shown that for a harmonic oscillation, propagating along the z axis, these components can be expressed in terms of $\sin \chi$, $\cos \chi$ and the Bessel function J_1 and its derivative. The three arbitrary constants in these equations again are determined by the boundary conditions ($p_{rr} = 0$, $p_{r\chi} = 0$, $p_{rz} = 0$), and the frequency equation is thereby obtained (Bancroft [1]). This time the frequency equation is remarkable

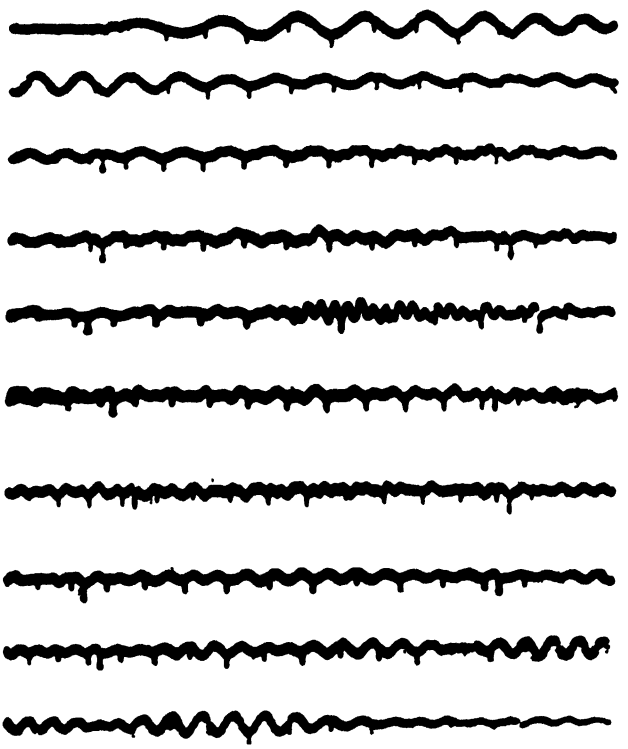


FIG. 6–16. Oscillogram of longitudinal vibrations of a cylindrical rod of hot-rolled steel; thickness 1 in., $\sigma = 0.30$, $\beta_1 = 10{,}400$ ft/sec. Detector 10 ft from impulsive source. Time marks are 10 μsec apart. (*After Oliver.*)

in that only one root occurs, as was proved by Hudson [32]. The phase velocity depends on the ratio of the wavelength to the radius of the cylinder. Phase- and group-velocity curves for the flexural mode are reproduced in Fig. 6–18.

Oliver [58] also studied the flexural waves in rods. An oscillogram of these waves is given in Fig. 6–19, and the observed and the theoretical group velocity in Fig. 6–17.

The problem of vibrations of a cylindrical rod as well as rods or bars having other cross sections has been treated in different ways {see Love (Chap. 1, Ref. 34), Timoshenko [85]}. An approximate equation for flexural waves was derived and solved by Timoshenko [84], and his results agree well with those obtained from the general theory of wave propagation.

To extend the results mentioned here to vibrations of a finite rod, additional boundary conditions at the ends must be taken into account.

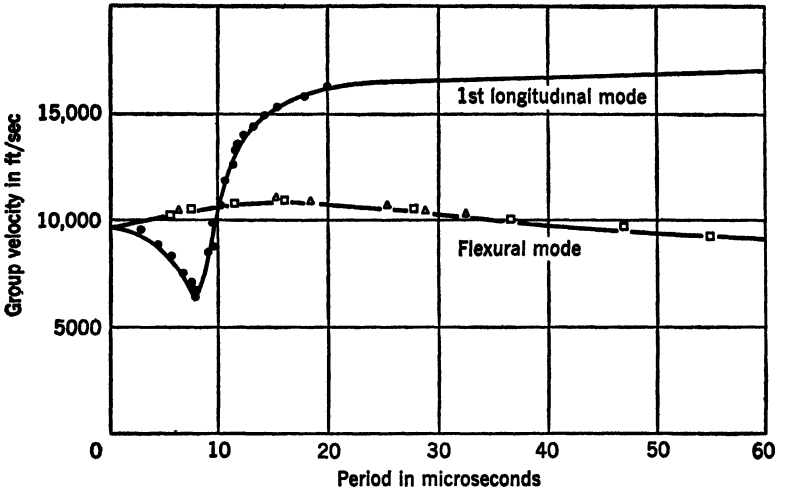


FIG. 6–17. Elastic-wave dispersion in a long cylindrical rod of hot-rolled steel; $\beta_1 = 10{,}400$ ft/sec, $\sigma = 0.30$, diameter 1 in. (*Experimental points by Oliver, theoretical curves by Bancroft and Hudson* [1].)

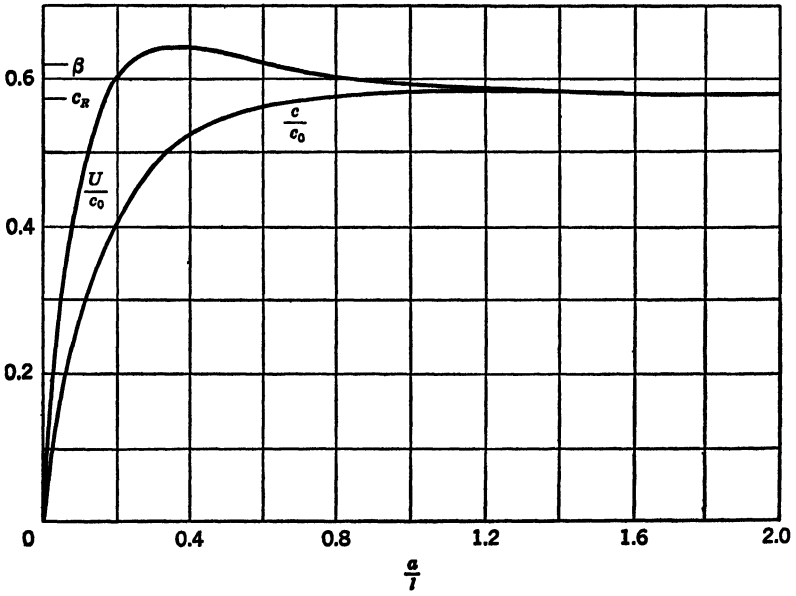


FIG. 6–18. Phase- and group-velocity curves for flexural waves in cylindrical rods for $\sigma = 0.29$. (*After Davies.*)



FIG. 6–19. Oscillogram of flexural vibrations in a cylindrical rod with same parameters as for Fig. 6–16. (*After Oliver.*)

6–5. Cylindrical Rod in a Liquid. Torsional motion of the rod will be unaffected by the liquid. Only longitudinal and flexural waves propagating along the axis need to be considered. In this problem one uses Eqs. (6–74) and (6–75) and the boundary conditions

$$(p_{rr})_1 = (p_{rr})_0 \qquad (p_{sr})_1 = 0 \qquad q_1 = q_0 \qquad \text{at } r = a \tag{6–90}$$

Here the subscripts 1 and 0 refer to the rod and liquid, respectively. For wave propagation in the direction of the z axis the rod potentials φ and ψ of the preceding section are applicable. In addition, a potential φ_0 representing the disturbance in the liquid is needed. The period equation can be derived in the usual manner, and a discussion along the lines of that for a plate in a liquid (Sec. 6–2) can be given. Several investigators have studied this problem, for example, Tamarkin [81] and Faran [16].

The effect of the liquid should be similar to that found in the problem of the plate. Modes with phase velocity exceeding the speed of sound in the liquid will be damped because of radiation of energy from the rod. Additional modes will be introduced owing to the presence of the liquid.

6–6. Cylindrical Hole in an Infinite Solid. The solution of this problem was investigated by Biot [2]. Denote all the quantities referring to the medium ($r > 0$) by the subscript $_1$. Then, by Eqs. (1–26), (1–27), (6–76), (6–77), and $W \equiv \psi_1$ we have

$$p_{rr} = \frac{\lambda_1}{\alpha_1^2}\frac{\partial^2\varphi_1}{\partial t^2} + 2\mu_1\left(\frac{\partial^2\varphi_1}{\partial r^2} - \frac{\partial^2\psi_1}{\partial r\,\partial z}\right) = 0 \qquad \text{at } r = a \tag{6–91}$$

$$p_{sr} = \rho_1\frac{\partial^2\psi_1}{\partial t^2} + 2\mu_1\left(\frac{\partial^2\varphi_1}{\partial r\,\partial z} - \frac{\partial^2\psi_1}{\partial z^2}\right) = 0 \tag{6–92}$$

Instead of expressions (6–78) Biot made use of trigonometric functions in order to represent unattenuated waves propagating in the z direction. Changing his notations, we put†

$$\varphi_1 = A\mathcal{K}_0(\hat{\imath}\hat{m}r)\cos(\nu z - \omega t) \tag{6–93}$$

$$\psi_1 = B\mathcal{K}_1(\hat{\imath}\hat{k}r)\sin(\nu z - \omega t) \tag{6–94}$$

† Since the z axis is usually taken along the rod or hole, the notations in this section differ from those used in earlier chapters, when the wave propagation was considered along the x axis.

with A and B constant, and

$$\hat{m} = \nu\sqrt{1 - \xi_2^2} \qquad \xi_2 = \frac{k_{\alpha 1}}{\nu} = \frac{c}{\alpha_1} \tag{6-95}$$

$$\hat{k} = \nu\sqrt{1 - \xi_1^2} \qquad \xi_1 = \frac{k_{\beta 1}}{\nu} = \frac{c}{\beta_1} \tag{6-96}$$

The phase velocity of waves propagating in the z direction is c. The modified Bessel functions of the second kind of zero and first order $\mathcal{K}_0$ and $\mathcal{K}_1$ can be approximated by the asymptotic expansion

$$\mathcal{K}_n(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \cos n\pi[1 + \cdots] \tag{6-97}$$

for large values of z [see Whittaker and Watson (Chap. 1, Ref. 66, p. 374)]. It follows from (6–97) that (6–93) and (6–94) represent solutions vanishing at infinite distance from the hole.

Inserting (6–93) and (6–94) into Eqs. (6–91) and (6–92) and using the equations satisfied by the Bessel functions $\mathcal{K}_0$ and $\mathcal{K}_1$,

$$\begin{aligned} \frac{d\mathcal{K}_0(z)}{dz} &= -\mathcal{K}_1(z) \qquad \frac{d^2\mathcal{K}_0}{dz^2} + \frac{1}{z}\frac{d\mathcal{K}_0}{dz} = \mathcal{K}_0 \\ \frac{d^2\mathcal{K}_0}{dz^2} &= -\frac{d\mathcal{K}_1}{dz} = \frac{1}{z}\mathcal{K}_1 + \mathcal{K}_0 \end{aligned} \tag{6-98}$$

we can eliminate A and B and obtain the period equation in the form

$$4\sqrt{1 - \xi_1^2}\left[\frac{1}{\hat{k}a} + \frac{\mathcal{K}_0(\hat{k}a)}{\mathcal{K}_1(\hat{k}a)}\right] - \frac{2(2 - \xi_1^2)\sqrt{1 - \xi_2^2}}{\hat{m}a} - \frac{(2 - \xi_1^2)^2\mathcal{K}_0(\hat{m}a)}{\sqrt{1 - \xi_2^2}\,\mathcal{K}_1(\hat{m}a)} = 0 \tag{6-99}$$

If Eqs. (6–95) and (6–96) are taken into account, the phase velocity c of the axial symmetric surface waves becomes a function of the variable $\nu a = 2\pi a/l = \pi D/l$, where l is the wavelength and D the diameter of the hole. If we assume that $\nu a \to \infty$, that is, the waves are very short compared with the diameter D, and if the asymptotic expansion (6–97) of the Bessel functions $\mathcal{K}_n$ is used, Eq. (6–99) reduces to

$$4\sqrt{1 - \xi_1^2} - \frac{(2 - \xi_1^2)^2}{\sqrt{1 - \xi_2^2}} = 0 \tag{6-100}$$

which is the well-known form of Rayleigh's equation (2–28) for surface waves at a plane boundary. Equation (6–99) was solved numerically by Biot, and his phase- and group-velocity curves are shown for various values of Poisson's constant in Figs. 6–20 and 6–21. With increasing

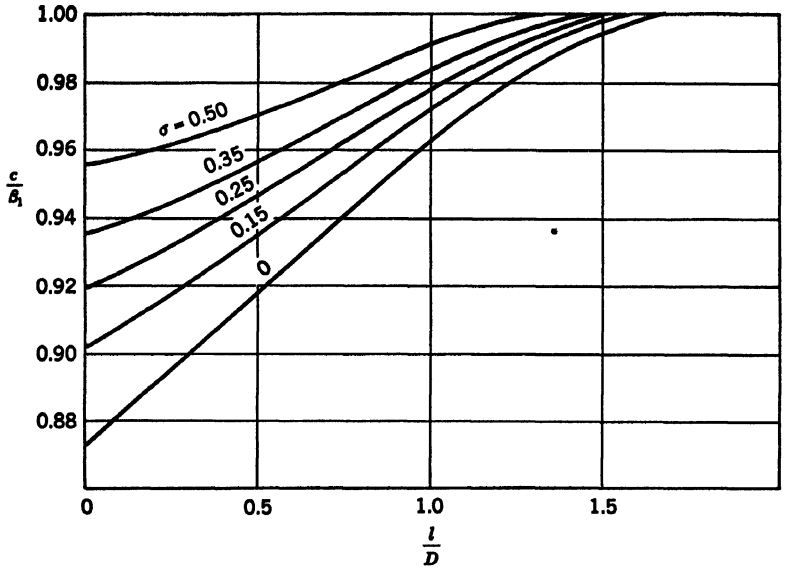


FIG. 6–20. Phase-velocity curves for an empty cylindrical hole for various values of Poisson's constant σ. (*After Biot.*)

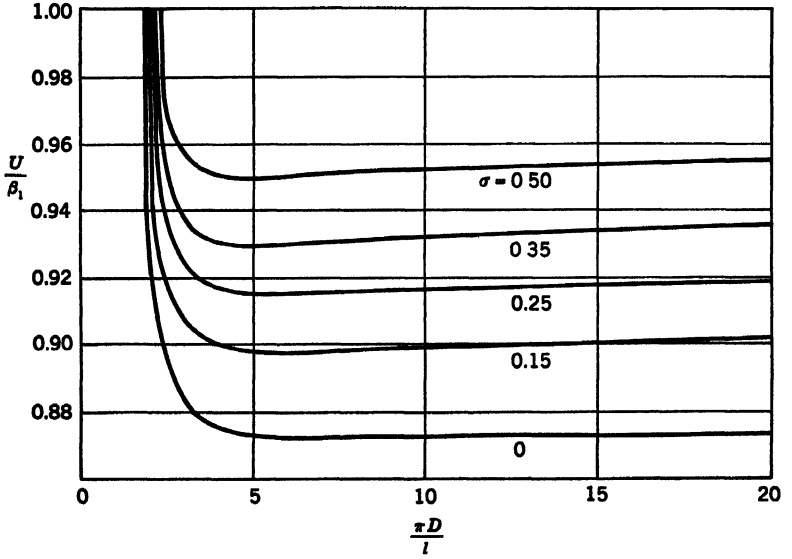


FIG. 6–21. Group-velocity curves for an empty cylindrical hole. (*After Biot.*)

wavelength the phase velocity increases from the velocity of Rayleigh waves to the velocity of shear waves in the solid. The curves terminate at the latter point, which corresponds to a cutoff wavelength l_{∞}, beyond which propagation cannot occur without attenuation.

6–7. Liquid Cylinder in an Elastic Medium. As an extension of the problem considered in the preceding section, the case of a cylindrical hole filled with a fluid was also investigated by Biot [2]. The radial displacement q_0 in the fluid is now given by

$$q_0 = \frac{\partial \varphi_0}{\partial r} \tag{6–101}$$

and the potential φ_0 must satisfy the equation

$$\frac{\partial^2 \varphi_0}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi_0}{\partial r} + \frac{\partial^2 \varphi_0}{\partial z^2} = \frac{1}{\alpha_0^2}\frac{\partial^2 \varphi_0}{\partial t^2} \tag{6–102}$$

The solution of this equation may be written in the form

$$\varphi_0 = J_0[r(k_\alpha^2 - \nu^2)^{\frac{1}{2}}]e^{i(\nu z - \omega t)} \qquad \text{for } k_\alpha^2 > \nu^2 \tag{6–103}$$

where $k_\alpha = \omega/\alpha$ and

$$\varphi_0 = I_0[r(\nu^2 - k_\alpha^2)^{\frac{1}{2}}]e^{i(\nu z - \omega t)} \qquad \text{for } k_\alpha^2 < \nu^2 \tag{6–104}$$

since

$$J_0(iz) = I_0(z) \tag{6–105}$$

where $I_0(z)$ is the modified Bessel function of the first kind of zero order. Putting

$$\xi = \frac{c}{\alpha_0} \tag{6–106}$$

we obtain the fluid pressure in the form

$$(p_{rr})_0 = -\rho_0 \frac{\partial^2 \varphi_0}{\partial t^2} = \rho_0 \omega^2 J_0[r\nu(\xi^2 - 1)^{\frac{1}{2}}]e^{i(\nu z - \omega t)} \qquad \text{for } \xi > 1 \tag{6–107}$$

and

$$(p_{rr})_0 = \rho_0 \omega^2 I_0[r\nu(1 - \xi^2)^{\frac{1}{2}}]e^{i(\nu z - \omega t)} \qquad \text{for } \xi < 1 \tag{6–108}$$

At the surface of the cylinder the radial components of the displacements and the normal stresses are continuous, that is,

$$q_1 = q_0 \qquad (p_{rr})_1 = (p_{rr})_0 \qquad \text{at } r = a \tag{6–109}$$

or

$$\left(\frac{p_{rr}}{q}\right)_0 = \left(\frac{p_{rr}}{q}\right)_1 \qquad \text{at } r = a \tag{6–110}$$

is a composite boundary condition corresponding to the matching of mechanical impedances. The first ratio is determined by Eqs. (6–101),

(6–103), (6–104), (6–107), and (6–108), and if we take into account that

$$J_1(u) = -\frac{d}{du} J_0(u) \qquad I_1(u) = \frac{d}{du} I_0(u) \tag{6–111}$$

we obtain

$$\begin{aligned} q &= -\nu(\xi^2 - 1)^{\frac{1}{2}} J_1[r\nu(\xi^2 - 1)^{\frac{1}{2}}] e^{i(\nu z - \omega t)} && \text{for } \xi > 1 \\ q &= \nu(1 - \xi^2)^{\frac{1}{2}} I_1[r\nu(1 - \xi^2)^{\frac{1}{2}}] e^{i(\nu z - \omega t)} && \text{for } \xi < 1 \end{aligned} \tag{6–112}$$

Thus

$$\left(\frac{p_{rr}}{q}\right)_0 = -\frac{\rho_0 \omega^2}{\nu} \frac{J_0[a\nu(\xi^2 - 1)^{\frac{1}{2}}]}{(\xi^2 - 1)^{\frac{1}{2}} J_1[a\nu(\xi^2 - 1)^{\frac{1}{2}}]} \qquad \text{for } \xi > 1 \tag{6–113}$$

$$\left(\frac{p_{rr}}{q}\right)_0 = \frac{\rho_0 \omega^2}{\nu} \frac{I_0[a\nu(1 - \xi^2)^{\frac{1}{2}}]}{(1 - \xi^2)^{\frac{1}{2}} I_1[a\nu(1 - \xi^2)^{\frac{1}{2}}]} \qquad \text{for } \xi < 1 \tag{6–114}$$

In deriving these formulas it was assumed that $q \neq 0$ at $r = a$. The condition $q = 0$ would mean that the liquid is contained in a rigid wall By Eqs. (6–112), then, the phase velocity would satisfy the equation

$$(\xi^2 - 1)^{\frac{1}{2}} J_1[a\nu(\xi^2 - 1)^{\frac{1}{2}}] = 0 \tag{6–115}$$

Hence

$$\xi = 1 \qquad \text{or} \qquad a\nu(\xi^2 - 1)^{\frac{1}{2}} = u_n \tag{6–116}$$

where u_n are the roots of the Bessel function J_1. The root $\xi = c/\alpha_0 = 1$ corresponds to waves with their plane perpendicular to the axis, while the roots u_n give the dispersion of multiply-reflected conical waves.

One can see from Eq. (6–113) that if a liquid cylinder is free at the boundary, that is, $(p_{rr})_0 = 0$ at $r = a$, the multiply-reflected conical waves will be determined by the roots of the equation

$$J_0[\nu a(\xi^2 - 1)^{\frac{1}{2}}] = 0 \tag{6–117}$$

For the case of a liquid cylinder in an elastic medium we can compute the second ratio in (6–110) from (6–91), (6–93), and (6–94), taking into account that

$$q_1 = \frac{\partial \varphi_1}{\partial r} - \frac{\partial \psi_1}{\partial z} \tag{6–118}$$

The third boundary condition is

$$(p_{rr})_1 = 0 \qquad \text{at } r = a \tag{6–119}$$

It is easy to see that Eq. (6–119) yields the ratio A/B, which can therefore be eliminated from the right-hand member of Eq. (6–110). This results in the period equation, which after certain transformations takes the form given by Biot:

$$L = -\frac{\rho_0}{\rho_1} \frac{\xi_1^4}{(\xi^2 - 1)^{\frac{1}{2}}} \frac{J_0[\nu a(\xi^2 - 1)^{\frac{1}{2}}]}{J_1[\nu a(\xi^2 - 1)^{\frac{1}{2}}]} \tag{6–120}$$

where L denotes an expression equal to the left-hand member of (6–99). Equation (6–120) holds for the reflected waves ($\xi > 1$). For $\xi < 1$ the propagation reduces to Stoneley waves at the liquid-solid interface given by

$$L = \frac{\rho_0}{\rho_1} \frac{\xi_1^4}{(1 - \xi^2)^{\frac{1}{2}}} \frac{I_0[\nu a(1 - \xi^2)^{\frac{1}{2}}]}{I_1[\nu a(1 - \xi^2)^{\frac{1}{2}}]} \tag{6–121}$$

By (6–95), (6–96), and (6–106), it may be seen that the parameters ξ_1 and ξ_2 can be expressed in terms of $\xi = c/\alpha_0$:

$$\xi_1 = \frac{c}{\beta_1} = \xi \frac{\alpha_0}{\beta_1} \qquad \xi_2 = \xi \frac{\alpha_0}{\alpha_1} \tag{6–122}$$

Since

$$\frac{\alpha_1^2}{\beta_1^2} = \frac{\lambda_1 + 2\mu_1}{\mu_1} = \frac{2(1 - \sigma)}{1 - 2\sigma} \tag{6–123}$$

where σ is Poisson's ratio, only three parameters are involved in Eqs. (6–120) and (6–121), β_1/α_0, ρ_0/ρ_1, and σ. Then the variable ξ becomes a function of $\nu a = \pi D/l$, where $\nu = 2\pi/l$ and $2a = D$. Using these relations and Eq. (6–120), Biot computed the phase- and group-velocity curves for multiply-reflected conical waves for the case $\beta_1/\alpha_0 = 1.5$, $\sigma = \frac{1}{4}$, and various values of ρ_0/ρ_1. The curves for the first three modes are presented in Figs. 6–22 and 6–23. The short-wavelength limit of phase and group velocity is the speed of sound in the liquid. The upper value of phase and group velocity is the speed of shear waves in the solid, the cutoff wavelength decreasing with increasing mode number. The group-velocity curves exhibit a minimum value. In the Stoneley-wave branch (Figs. 6–24 and 6–25) computed from Eq. (6–121), the short-wave limit of phase velocity coincides with the Stoneley-wave velocity at the interface between two half spaces, one fluid and the other solid. With increasing wavelength the phase velocity decreases and becomes practically independent of wavelength ($l/D > 5$). In this region the waves correspond to those studied in the classical theory of the "water hammer."

Somers [78] studied this problem for the case of an impulsive ring-shaped line source. The propagation of sound waves along liquid cylinders was also recently considered by Jacobi [35]. The phase-velocity curves for the first two modes were plotted for a liquid cylinder with rigid walls, or with pressure-release walls, and for a liquid cylinder embedded in an infinite liquid. Two other problems concerning cylindrical tubes, which will be discussed in the next section, were considered in the paper of Jacobi, where references on previous work can also be found.

6–8. Cylindrical Tube. The various modes of free vibrations of an infinitely long cylindrical shell were investigated by Lamb [39], and the effect of free edges was considered by Love [47]. Period equations for torsional and longitudinal oscillations of a cylindrical tube were given by

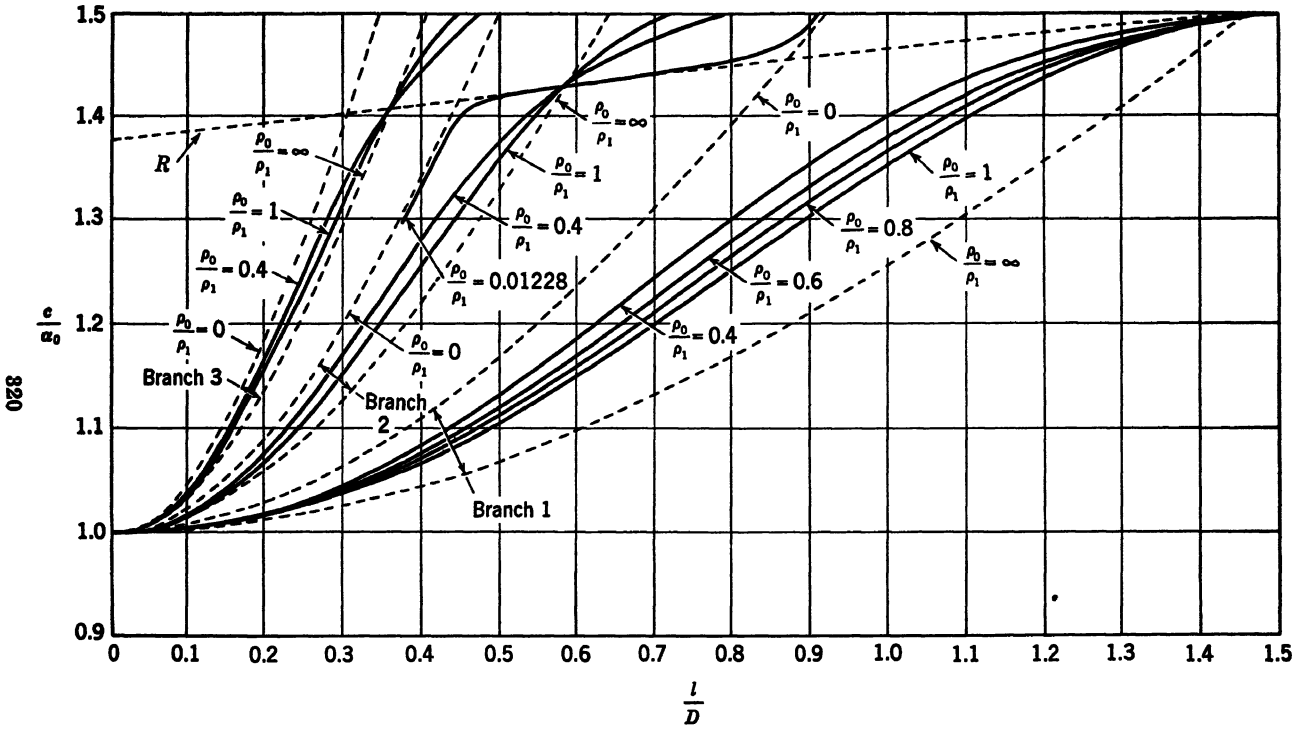


FIG. 6–22. Phase-velocity curves of first three modes ($c > \alpha_0$) of a liquid-filled cylindrical hole; $\beta/\alpha_0 = 1.5$, $\sigma = 1/4$, several values of ρ_0/ρ_1. (*After Biot.*)

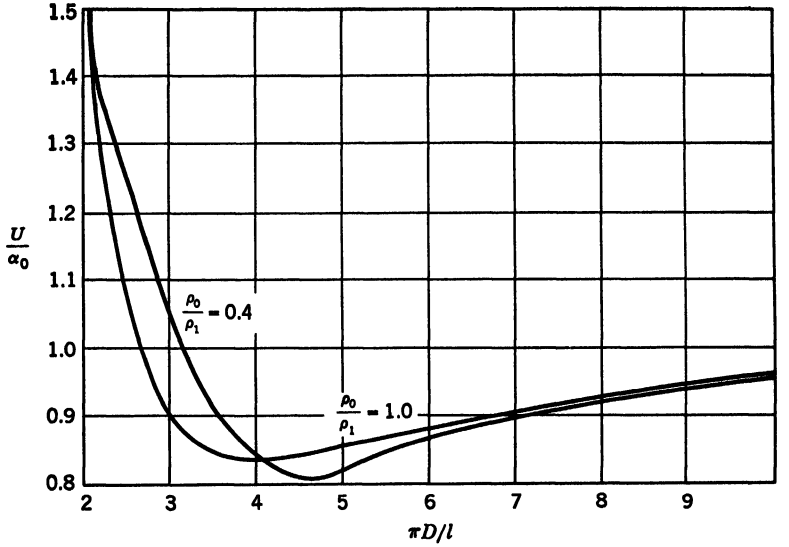


FIG. 6–23. Group velocity for first-mode phase-velocity curve of Fig. 6–22. (*After Biot.*)

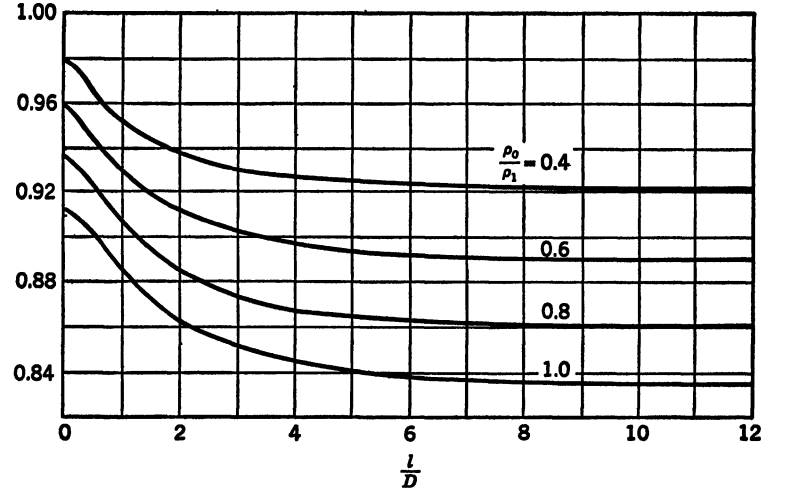


FIG. 6–24. Phase-velocity (i. e., c/α_0) curves for liquid-filled cylindrical hole, Stoneley mode with $\beta_1/\alpha_0 = 1.5$, $\sigma = 1/4$. (*After Biot.*)

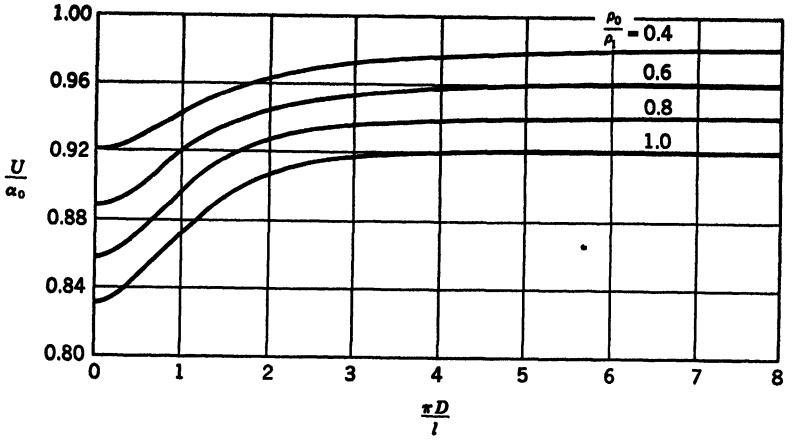


FIG. 6–25. Group velocity for phase-velocity curves of Fig. 6–24. (*After Biot.*)

Ghosh [22]. A case with practical interest, the propagation of waves in a liquid-filled tube, was discussed in several papers {see Rayleigh (Chap. 1, Ref. 45, vol. II, pp. 158 and 323) and Lamb [40] for early investigations of this problem}. In this problem the yielding of the walls cannot be neglected, and therefore two approximations are made. We can consider either a liquid cylinder with a liquid wall or a liquid cylinder with a thin solid wall.

Following Jacobi [35], we assume for the first of these problems that the density and sound velocity are ρ_1, α_1 for the wall and ρ_0, α_0 for the cylinder. Again considering solutions with axial symmetry, we can make use of velocity potentials similar to Eq. (6–103). A solution for this problem can be written in the form

$$\varphi_0 = \sum_{n=-\infty}^{\infty} A_n J_n(mr) e^{i(n\chi + \gamma z - \omega t)} \tag{6–124}$$

$$\varphi_1 = \sum_{n=-\infty}^{\infty} [A_{1n} J_n(m_1 r) + B_{1n} N_n(m_1 r)] e^{i(n\chi + \gamma z - \omega t)} \tag{6–125}$$

where A_n, A_{1n}, B_{1n} are arbitrary constants and

$$m = \sqrt{k_{\alpha 0}^2 - \bar{\nu}^2} \qquad m_1 = \sqrt{k_{\alpha 1}^2 - \bar{\nu}^2} \tag{6–126}$$

The symbol N_n denotes either a Bessel function of the second kind or a Hankel function of the first kind, $H_n^{(1)}$. For convenience in computation, it is best to use $H_0^{(1)}$ where the argument is imaginary. A relatively simple form can be obtained for modes independent of the angle χ, that is, for $n = 0$. Three boundary conditions must be satisfied, expressing continuity

of pressure and of radial components of velocity at the inner boundary, $r = a$, and vanishing of the pressure at the outer boundary of the tube, $r = a_1$. Since only the terms $n = 0$ are used in (6–124) and (6–125), we obtain

$$\begin{aligned} \rho_0 A_0 J_0(ma) - \rho_1 A_{10} J_0(m_1 a) - \rho_1 B_{10} N_0(m_1 a) &= 0 \\ m A_0 J_0'(ma) - m_1 A_{10} J_0'(m_1 a) - m_1 B_{10} N_0'(m_1 a) &= 0 \\ A_{10} J_0(m_1 a_1) + B_{10} N_0(m_1 a_1) &= 0 \end{aligned} \tag{6–127}$$

The period equation may now be written in the form

$$\frac{m J_0'(ma)}{J_0(ma)} = \frac{\rho_0 m_1}{\rho_1} \frac{J_0'(m_1 a) N_0(m_1 a_1) - J_0(m_1 a_1) N_0'(m_1 a)}{J_0(m_1 a) N_0(m_1 a_1) - J_0(m_1 a_1) N_0(m_1 a)} \tag{6–128}$$

The derivatives of the Bessel functions of zero order can be replaced by functions of the first order, using the formulas $J_0' = -J_1$, $N_0' = -N_1$.

The graphical solution of this equation is obtained by plotting the curves of each member of Eq. (6–128) separately and locating intersection points. An approximate expression for the right-hand member was used by Jacobi, and two different cases $\alpha_0 \gtrless \alpha_1$ were discussed. In the first case the phase velocity approaches the sound velocity of the inner medium; in the second case it approaches the sound velocity of the outer medium as the frequency increases. Wave propagation in a liquid cylinder enclosed by a metal-walled tube was investigated by Lamb [40], Gronwall [26], Fay, Brown, and Fortier [17], Jacobi [35], and Fay [19].

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CHAPTER 7

WAVE PROPAGATION IN MEDIA WITH VARIABLE VELOCITY

7–1. Wave Propagation in Heterogeneous Isotropic Media. Problems of this kind are mathematically more difficult than those considered earlier, since each investigation must begin with the most general equations of motion (1–7) and (1–11) in which λ, μ, and ρ are now functions of x, y, z. Thus we have

$$\rho \frac{\partial^2 u}{\partial t^2} = \rho X + \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} \tag{7–1}$$

and two similar equations for the other components, where the stresses are given by

$$p_{xx} = \lambda\theta + 2\mu \frac{\partial u}{\partial x} \qquad p_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad p_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \tag{7–2}$$

and

$$\theta = \operatorname{div} \mathbf{s} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Inserting (7–2) in (7–1) we obtain

$$\rho \frac{\partial^2 u}{\partial t^2} = \rho X + \frac{\partial}{\partial x}(\lambda\theta) + \mu\nabla^2 u + \mu \frac{\partial \theta}{\partial x} + 2\frac{\partial \mu}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \mu}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial \mu}{\partial z}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \tag{7–3}$$

After a simple transformation, the equations of motion take the form

$$\begin{aligned}
\rho \frac{\partial^2 u}{\partial t^2} &= \rho X + \frac{\partial}{\partial x}[(\lambda + 2\mu)\theta] + \mu\nabla^2 u - \mu \frac{\partial \theta}{\partial x} - 2\frac{\partial \mu}{\partial x}\theta \\
&\quad + 2\frac{\partial \mu}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \mu}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial \mu}{\partial z}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\
\rho \frac{\partial^2 v}{\partial t^2} &= \rho Y + \frac{\partial}{\partial y}[(\lambda + 2\mu)\theta] + \mu\nabla^2 v - \mu \frac{\partial \theta}{\partial y} - 2\frac{\partial \mu}{\partial y}\theta \\
&\quad + \frac{\partial \mu}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + 2\frac{\partial \mu}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial \mu}{\partial z}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)
\end{aligned} \tag{7–4}$$

$$\rho \frac{\partial^2 w}{\partial t^2} = \rho Z + \frac{\partial}{\partial z}[(\lambda + 2\mu)\theta] + \mu \nabla^2 w - \mu \frac{\partial \theta}{\partial z} - 2 \frac{\partial \mu}{\partial z} \theta + \frac{\partial \mu}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \frac{\partial \mu}{\partial y}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + 2 \frac{\partial \mu}{\partial z} \frac{\partial w}{\partial z} \quad (7\text{–}4)\ (\text{cont.})$$

It is easy to see that these equations can be written in vector form, if the sum of the three last terms in each of Eqs. (7–4) is understood to represent the double scalar product of the gradient of rigidity $\nabla\mu$ and the symmetrical strain tensor Φ, that is,

$$\nabla\mu\left(\frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial y}, \frac{\partial \mu}{\partial z}\right)$$

and

$$\Phi \begin{Bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) & \dfrac{\partial w}{\partial z} \end{Bmatrix} \quad (7\text{–}5)$$

Then the vector equation for wave propagation in a heterogeneous medium is

$$\rho \frac{\partial^2 \mathbf{s}}{\partial t^2} = \rho \mathbf{F} + \nabla[(\lambda + 2\mu)\theta] + \mu \nabla^2 \mathbf{s} - \mu \nabla \theta - 2\theta \nabla \mu + 2(\nabla \mu \cdot \Phi) \quad (7\text{–}6)$$

The equation of wave propagation in a form equivalent to (7–6) was apparently given first by Uller [75] and independently by Yosiyama [81]. Obviously, the equations needed for various heterogeneous media would follow from Eq. (7–6) through a specialization of conditions. Thus, for example, for fluid media we put $\mu = 0$. Assuming also that there are no external forces acting, $\mathbf{F} = 0$, we have

$$\rho \frac{\partial^2 \mathbf{s}}{pt^2} = \nabla(\lambda\theta) = \theta \nabla \lambda + \lambda \nabla \theta \quad (7\text{–}7)$$

Now we introduce the potential φ defined by

$$u = \frac{\partial \varphi}{\partial x} \qquad v = \frac{\partial \varphi}{\partial y} \qquad w = \frac{\partial \varphi}{\partial z} \quad (7\text{–}8)$$

From Eqs. (7–7) and (7–8) we see that u, v, w must satisfy equations such as

$$\rho \frac{\partial^2 u}{\partial t^2} = \theta \frac{\partial \lambda}{\partial x} + \lambda \nabla^2 u \quad (7\text{–}9)$$

where $\theta = \nabla^2\varphi$, which differ from the usual wave equation by terms representing variations of λ. Equation (7–9) can be written in the form

$$\frac{\partial}{\partial x}\left[\rho\frac{\partial^2\varphi}{\partial t^2} - \lambda\nabla^2\varphi\right] - \frac{\partial\rho}{\partial x}\frac{\partial^2\varphi}{\partial t^2} = 0 \tag{7–10}$$

It is seen, then, that if $\rho(x, y, z)$ is a constant the displacement potential for a heterogeneous isotropic fluid satisfies the wave equation

$$\frac{\partial^2\varphi}{\partial t^2} = \alpha^2(x, y, z)\nabla^2\varphi \tag{7–11}$$

where $\alpha = \sqrt{\lambda/\rho}$.

The effect of variations in density was investigated by Bergmann [9]. Yosiyama [81] used an equation of the type

$$\frac{\partial^2\varphi}{\partial t^2} = \alpha^2\nabla^2\varphi - b\varphi \tag{7–12}$$

in which the last term represents the effect of heterogeneity of the medium.

The small motion of a gas about a state of equilibrium is determined in general by the following equations (see Lamb [44, p. 555])

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}(c^2\theta + Xu + Yv + Zw) + \frac{1}{\rho_0}\frac{\partial\rho_0}{\partial x}\{c^2 - f'(\rho_0)\}\theta, \cdots \tag{7–13}$$

if the forces X, Y, Z are constant and $c^2 = \gamma p_0/\rho_0$, $p_0 = f(\rho_0)$. The motion is irrotational in the case of "convective" equilibrium, i.e., if $f'(\rho_0) = \gamma p_0/\rho_0 = c^2$. Then the potential satisfies the equation

$$\frac{\partial^2\varphi}{\partial t^2} = c^2\nabla^2\varphi + \left(X\frac{\partial\varphi}{\partial x} + Y\frac{\partial\varphi}{\partial y} + Z\frac{\partial\varphi}{\partial z}\right) \tag{7–14}$$

It was shown by Sobolev [69–71] that it is possible to generalize Kirchhoff's formula Eq. (1–51) for wave propagation in heterogeneous media.

If the relative variation of ρ, μ, and λ is very small over a wavelength, Eqs. (7–4) may be approximated by two wave equations for compressional and distortional waves with $\alpha = \alpha(x, y, z)$ and $\beta = \beta(x, y, z)$. Otherwise, coupling between these two wave types occurs at every point of the medium, and it is not possible to make a clear distinction between them (see also Chap. 1).

7–2. Sound Propagation in a Fluid Half Space. Pekeris [58] considered the problem in which the velocity in a fluid depends on the depth z, either as

$$\alpha = az \tag{7–15}$$

or

$$\alpha = \frac{\alpha_0}{(1 + az + bz^2)^{\frac{1}{2}}} \tag{7–16}$$

In the first case the velocity decreases with depth above the plane $z = 0$ shown in Fig. 7–1. It is assumed that the density variation is sufficiently small to permit the use of Eq. (7–11):

$$\nabla^2\varphi = \frac{1}{a^2}\frac{\partial^2\varphi}{\partial t^2} \tag{7–17}$$

At the free surface $z = z_0$, the pressure, hence the potential φ, vanishes as before. φ represents a wave propagating from the source located at

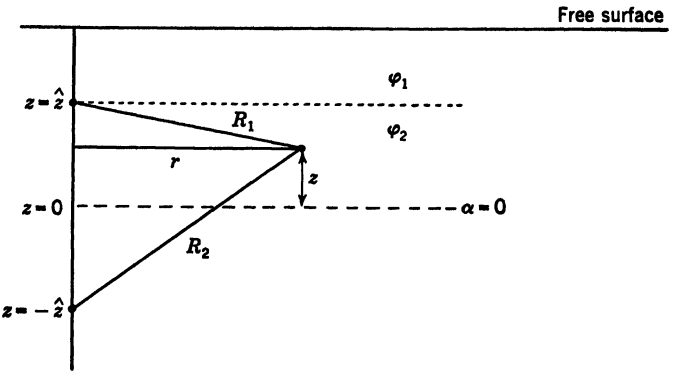


FIG. 7–1. Notations for sound propagation from a source at $(0, \hat{z})$ to a receiver at (r, z).

$z = \hat{z}$, and, as was assumed in Sec. 4–2, both components of velocity are continuous at the level $z = \hat{z}$, except at the source.

If the time factor $\exp(i\omega t)$ is omitted, an expression

$$\varphi = A(k)J_0(kr)F(z) \tag{7–18}$$

will satisfy the equation

$$\nabla^2\varphi + k_\alpha^2\varphi = 0 \tag{7–19}$$

provided that the factor $F(z)$ is an integral of

$$\frac{d^2F}{dz^2} + F\left(\frac{\omega^2}{a^2z^2} - k^2\right) = 0 \tag{7–20}$$

where the velocity distribution given by Eq. (7–15) is already taken into account. If we write $2zk = x$ and $F(z) = F(x/2k) = W(x)$, Eq. (7–20) is reduced to the form

$$\frac{d^2W}{dx^2} + \left(\frac{\omega^2}{a^2x^2} - \frac{1}{4}\right)W = 0 \tag{7–21}$$

Whittaker's function $W_{0,n}(x)$ satisfies this equation if we put [Whittaker and Watson (Chap. 1, Ref. 66, p. 360)]

$$\frac{1}{4} - n^2 = \frac{\omega^2}{a^2} \tag{7-22}$$

The second solution which, together with the first, forms the fundamental system of Eq. (7–21) is $W_{0,n}(-x)$.

On the other hand, the substitution

$$F(z) = z^{\frac{1}{2}}G(z) \tag{7-23}$$

yields the equation

$$\frac{d^2G}{dz^2} + \frac{1}{z}\frac{dG}{dz} + \left[\left(\frac{\omega^2}{a^2} - \frac{1}{4}\right)\frac{1}{z^2} - k^2\right]G = 0 \tag{7-24}$$

If we change the notations as follows and put $kz = \imath\chi$, Eq. (7–24) takes the usual form of a Bessel equation, where $\bar{G}(\chi) = G(z)$:

$$\frac{d^2\bar{G}}{d\chi^2} + \frac{1}{\chi}\frac{d\bar{G}}{d\chi} + \left[1 - \frac{1}{\chi^2}\left(\frac{1}{4} - \frac{\omega^2}{a^2}\right)\right]\bar{G} = 0 \qquad n = \left(\frac{\omega^2}{a^2} - \frac{1}{4}\right)^{\frac{1}{2}} \cong \frac{\omega}{a} \tag{7-25}$$

Now the solution of Eq. (7–20) can be written as follows:

$$F = Az^{\frac{1}{2}}I_{\imath n}(kz) + Bz^{\frac{1}{2}}\mathcal{K}_{\imath n}(kz) \tag{7-26}$$

where I is the Bessel function of the first kind with imaginary argument and $\mathcal{K}$ is the modified Bessel function

$$\mathcal{K}_n(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \cos n\pi W_{0,n}(2z) \tag{7-27}$$

Both functions in Eq. (7–26) are of the order *in* Since $\mathcal{K}_{\imath n}$ corresponds to a wave propagating upward, we write for the medium below the source

$$F_2 = Cz^{\frac{1}{2}}I_{\imath n}(kz) \tag{7-28}$$

The potential φ and therefore F in Eq. (7–26) vanish at $z = z_0$, and we obtain

$$0 = AI_{\imath n}(kz_0) + B\mathcal{K}_{\imath n}(kz_0) \tag{7-29}$$

Thus we can put for the medium above the source

$$F_1 = Bz^{\frac{1}{2}}\left[\mathcal{K}_{\imath n}(kz) - I_{\imath n}(kz)\frac{\mathcal{K}_{\imath n}(kz_0)}{I_{\imath n}(kz_0)}\right] \tag{7-30}$$

Now the constants B and C should be expressed in terms of the intensity of the point source. This result will be obtained as in Sec. 4–1 by imposing the conditions

$$F_1 = F_2 \qquad \frac{\partial F_1}{\partial z} - \frac{\partial F_2}{\partial z} = -Dk \qquad \text{at } z = \hat{z} \tag{7-31}$$

Then φ is continuous across the plane $z = \hat{z}$, and, if we integrate with respect to k between the limits 0 and ∞, $\partial\varphi_1/\partial z - \partial\varphi_2/\partial z$ becomes proportional to $\int_0^\infty J_0(kr)k\,dk$ which vanishes everywhere except at the source, where it becomes infinite in such a way that its integral over the plane is finite. This satisfies the condition for a point source. Thus we have

$$B\left[\mathcal{K}_{\nu n}(k\hat{z}) - I_{\nu n}(k\hat{z})\frac{\mathcal{K}_{\nu n}(kz_0)}{I_{\nu n}(kz_0)}\right] = CI_{\nu n}(k\hat{z}) \tag{7-32}$$

$$B\hat{z}^{\frac{1}{2}}\left[\mathcal{K}'_{\nu n}(k\hat{z}) - I'_{\nu n}(k\hat{z})\frac{\mathcal{K}_{\nu n}(kz_0)}{I_{\nu n}(kz_0)}\right] = -D + C\hat{z}^{\frac{1}{2}}I'_{\nu n}(k\hat{z}) \tag{7-33}$$

From these two equations we obtain

$$B = D\hat{z}^{-\frac{1}{2}}\left[\frac{I'_{\nu n}(k\hat{z})}{I_{\nu n}(k\hat{z})}\mathcal{K}_{\nu n}(k\hat{z}) - \mathcal{K}'_{\nu n}(k\hat{z})\right]^{-1} \tag{7-34}$$

and

$$C = D\hat{z}^{-\frac{1}{2}}\left[\frac{I'_{\nu n}(k\hat{z})}{I_{\nu n}(k\hat{z})}\mathcal{K}_{\nu n}(k\hat{z}) - \mathcal{K}'_{\nu n}(k\hat{z})\right]^{-1}\left[\frac{\mathcal{K}_{\nu n}(k\hat{z})}{I_{\nu n}(k\hat{z})} - \frac{\mathcal{K}_{\nu n}(kz_0)}{I_{\nu n}(kz_0)}\right] \tag{7-35}$$

When we make use of the relationship

$$I'_n(x)\mathcal{K}_n(x) - I_n(x)\mathcal{K}'_n(x) = \frac{1}{x} \tag{7-36}$$

these expressions take the form

$$B = D\hat{z}^{-\frac{1}{2}}I_{\nu n}(k\hat{z})k\hat{z} = Dk\hat{z}^{\frac{1}{2}}I_{\nu n}(k\hat{z}) \tag{7-37}$$

$$C = Dk\hat{z}^{\frac{1}{2}}\left[\mathcal{K}_{\nu n}(k\hat{z}) - \frac{\mathcal{K}_{\nu n}(kz_0)}{I_{\nu n}(kz_0)}I_{\nu n}(k\hat{z})\right] \tag{7-38}$$

Thus the potential φ is determined for both regions above and below the source if we integrate with respect to k from 0 to ∞:

$$\varphi_1 = \int_0^\infty J_0(kr)B(k)z^{\frac{1}{2}}\left[\mathcal{K}_{\nu n}(kz) - I_{\nu n}(kz)\frac{\mathcal{K}_{\nu n}(kz_0)}{I_{\nu n}(kz_0)}\right]dk \tag{7-39}$$

and

$$\varphi_2 = \int_0^\infty J_0(kr)C(k)z^{\frac{1}{2}}I_{\nu n}(kz)\,dk \tag{7-40}$$

where $B(k)$ and $C(k)$ are given by Eqs. (7-37) and (7-38).

Pekeris also derived an expression for the potential for a whole space with the same velocity gradient. This expression is readily obtained by assuming $z_0 \to \infty$. Taking the asymptotic values of Bessel functions

$$\mathcal{K}_{\nu n}(x) \sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}}e^{-x} \qquad I_{\nu n}(x) \sim (2\pi x)^{-\frac{1}{2}}e^{x} \tag{7-41}$$

we obtain

$$\lim_{z_0 \to \infty} \left[\frac{\mathcal{K}_{in}(kz_0)}{I_{in}(kz_0)} \right] = 0 \tag{7-42}$$

and, therefore, we have for $0 < z < \hat{z}$

$$\varphi_2 = D\hat{z}^{\frac{1}{2}} z^{\frac{1}{2}} \int_0^\infty J_0(kr) I_{in}(kz) \mathcal{K}_{in}(k\hat{z}) k \, dk \tag{7-43}$$

The last integral can be replaced by a simple expression (see Pekeris [58], p. 297). If we put (Fig. 7–1)

$$R_1^2 = r^2 + (\hat{z} - z)^2 \qquad R_2^2 = r^2 + (\hat{z} + z)^2 \tag{7-44}$$

we obtain

$$\int_0^\infty J_0(kr) I_n(kz) \mathcal{K}_n(k\hat{z}) k \, dk = \frac{1}{R_1 R_2} \left(\frac{R_2 + R_1}{R_2 - R_1} \right)^{-n} \tag{7-45}$$

Thus the solution (7–43) for a point source in an infinite medium with constant-velocity gradient can be written in the form

$$\varphi_2 = \frac{D\hat{z}^{\frac{1}{2}} z^{\frac{1}{2}}}{R_1 R_2} \left(\frac{R_2 + R_1}{R_2 - R_1} \right)^{-in} = \frac{D\hat{z}^{\frac{1}{2}} z^{\frac{1}{2}}}{R_1 R_2} e^{-2i n \nu} \tag{7-46}$$

where $\tanh \nu = R_1/R_2$. For the vicinity of the source, if we put $z = \hat{z}$, $R_2 = 2\hat{z}$, Eq. (7–46) becomes

$$\varphi_2 \cong \frac{D}{2R_1} \exp \left[i \left(\omega t - \frac{nR_1}{\hat{z}} \right) \right] \tag{7-47}$$

for a point source of unit strength $D = 2$. For vanishing-velocity gradient, $n/\hat{z} \to k_a$, and Eq. (7–47) goes into the familiar form for a spherical wave in a homogeneous infinite medium where the time factor is again introduced.

Pekeris evaluated the integral (7–40) by transforming the path of integration into the complex k plane, the procedure used in Chaps. 3 and 4. He obtained the solution as a sum of residues in the form of a Fourier-Bessel series in which Bessel functions of imaginary order and complex argument are involved.

From the physical viewpoint, this problem is of interest because of the occurrence of a shadow zone into which no rays penetrate. Diffracted radiation, however, can penetrate into this zone, and the intensity can be derived from the residues of Eq. (7–40). Pekeris also outlined the theory for the case of a medium with variable-velocity gradient.

Blokhintzev [10] established the equations of wave propagation in an inhomogeneous and moving medium from a more general viewpoint. They are derived from the equations of hydrodynamics, disregarding only the viscosity and the heat conduction of the medium. Even in the case

of constant entropy and no motion in the medium other than waves, the equation satisfied by the sound potential is complicated. However, it was shown by Blokhintzev that for liquids this more general equation can be approximately replaced by the usual wave equation.

It was pointed out by Morse [54] that, since the case of the linear decrease of sound velocity with depth discussed by Pekeris does not agree with actual measurement of underwater sound propagation near the sea surface, another law should be used. He expressed the sound velocity in terms of depth as follows:

$$\alpha = \alpha_s(1 - az^2) \cong \alpha_s(1 + 2az^2)^{-\frac{1}{2}} \qquad \text{for } H' \geq z \geq 0 \qquad (7\text{–}48)$$

$$\alpha = \alpha_0(1 - bz) \cong \alpha_0(1 + 2bz)^{-\frac{1}{2}} \qquad \text{for } z > H' \qquad (7\text{–}49)$$

where $$\alpha_0^2 = \frac{b\alpha_s^2}{2aH'} \qquad 2a = \frac{b}{H'(1 + bH')} \qquad b = 10^{-4}\ \text{ft}^{-1} \qquad (7\text{–}50)$$

and H' is the depth at which the law of velocity variation is changed.

In Morse's opinion, the assumption that the sound-velocity gradient approaches zero in the uppermost layer of ocean can then explain many features of the shadow zone. The problem similar to that treated in this section but concerning the propagation of electromagnetic waves was considered by Krasnushkin [41]. In earlier papers as well as in the paper cited above he applied the method of normal modes and was able to present the solution in the form of a discrete spectrum combined with a continuous one. He assumed that a vertical electrical current produces electromagnetic waves in a nonmagnetic medium having its dielectric constant a function of z and obtained, instead of Eq. (7–20), a more general equation. This equation is of the Schrödinger type, and the solution can be expressed in terms of Hermite polynomials.

SOFAR Propagation. Special cases of wave propagation in heterogeneous media can have important practical applications. In particular, long-distance propagation of sound waves in the atmosphere and in the oceans is possible because of the favorable variation of sound velocity with depth in these layers. In most of the cases considered in this chapter simple distributions of velocity are assumed, the velocity either increasing or decreasing with depth. In many parts of the deep ocean, however, the sound velocity decreases to a minimum at 400 to 700 fathoms but increases from that depth to the bottom achieving a slightly higher velocity at the bottom than at the surface. The mean-velocity-depth curve in the Atlantic Ocean is presented in Fig. 7–2 (Ewing and Worzel [25]). For related studies in the Pacific Ocean, see Dyk and Swainson [21] and Anderson [5]. This variation in sound velocity is due to two factors. First, the temperature in the oceans decreases rapidly with depth to a little above 0°C at about 700 fathoms in the Atlantic and 500 fathoms

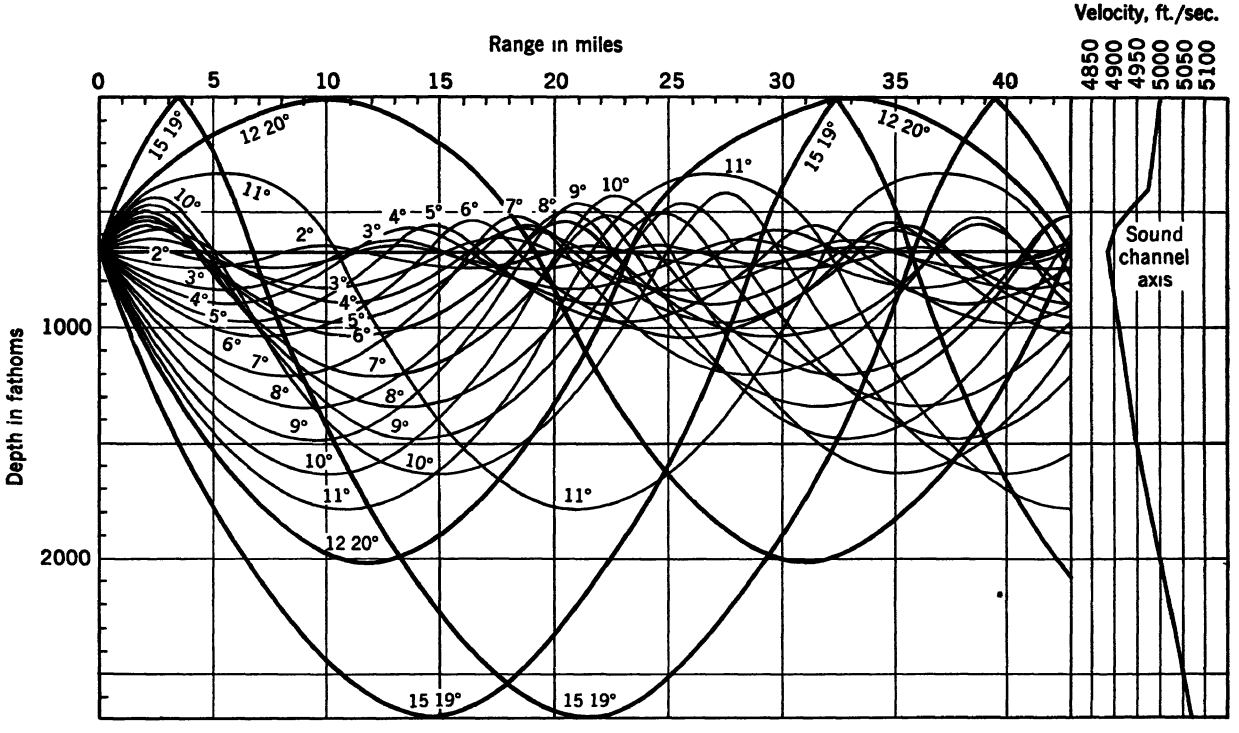


FIG. 7–2. Ray diagrams for typical Atlantic Ocean sound channel. Curve at the right gives sound-velocity variation with depth. (*After Ewing and Worzel.*)

in the Pacific, and the velocity decreases in this range as a consequence. Below these depths the temperature decreases slowly to the bottom, and the increase in velocity in this part of the ocean is caused by the effect of increasing hydrostatic pressure on the compressibility.

Propagation of the sound initiated by a source of spherical waves located at the depth of minimum velocity has characteristics which may be explained by Fig. 7–2. The rays in this figure were calculated from the mean-velocity-depth curve under the assumption that the water column is composed of six layers in each of which the velocity is a linear function of depth (see, for example, Ewing and Leet [23]). If we assume that the velocity of propagation varies continuously in a medium, the sound waves follow curved ray paths because of refraction. In a case such as this, a ray starting with a small angle above the horizontal is bent downward, and a ray which starts at an angle below the horizontal is bent upward. In this way, rays repeatedly return to the depth of minimum velocity and cross it and are bent in the opposite direction. Two-dimensional spreading, absorption, or interception by an obstacle alone limits the horizontal range of propagation in such a case. For oceans the rays behave in this manner if the initial angle lies between 0 and 12°. Part of the initial sound waves are therefore confined to a channel. The name "sound channel" has been used for this type of velocity structure.

Rays with an inclination between 12 and 15° to the axis of the velocity minimum are also curved but return to the axis after reflection at the sea surface. They have been called RSR (refracted and surface-reflected) sounds. If the inclination exceeds 15° at the axis the rays must be reflected at both the surface and the bottom and are called reflected sounds.

SOFAR, or long-distance, signaling in the oceans displays the following characteristics: (1) the extremely long-range transmission of sounds (probably 10,000 miles from small bombs); (2) the unique character of the SOFAR signal, especially the abrupt termination which allows the arrival time to be read with an accuracy better than 0.05 sec; and (3) sound duration depending upon distance in such a way that the distance may be estimated from the signal duration at a single station with a precision of about 3 per cent. With the high-frequency-sensitive detectors used in underwater acoustics, the representation by rays is sufficient to account for much of the character of SOFAR signals. However, a complete wave theory of propagation in the SOFAR channel must consider the elasticity of the bottom as well as the heterogeneity of the water. Such a theory is necessary for waves with periods between 1 and 15 sec. When such a theory becomes available it may explain the puzzling features in this period range on seismograms of earthquakes with large oceanic paths (see Sec. 4–4).

A limited development of the wave theory was attempted in different

ways. Brekhovskikh [13] pointed out that the effect of the velocity gradient due to hydrostatic pressure is quite analogous to the effect of the whispering galleries explained by Rayleigh (Chap. 1, Ref. 45, vol. 2, p. 127). He applied Rayleigh's consideration of rectilinear waves incident on a curved boundary to the case of a plane boundary and curved rays, and in an elementary way he derived some formulas for the effect of this velocity gradient. In a second paper [12] on the subject Brekhovskikh suggested an approximation of the velocity-depth curve ADE by a broken line $ABCDE$ (Fig. 7–3*a*). For the duration T of a recorded signal at a distance r from an instantaneous source he gives the approximate formula

$$T = \frac{ar}{6\alpha_s}(2H + z) \cong \frac{arH}{3\alpha_s} \tag{7–51}$$

The factor α_s is the sound velocity at the ocean surface. H is the ocean depth, and a is the coefficient in the expression for the velocity $\alpha_s(1 + az)$, z being the variable depth. At large distances the maximum of sound inten-

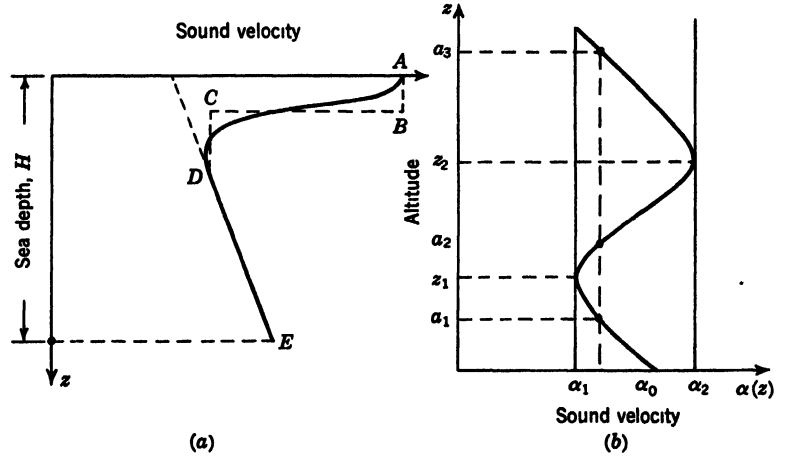


FIG. 7–3. (*a*) Assumed velocity-depth relationship for ocean. (*After Brekhovskikh.*) (*b*) Assumed velocity-altitude relationship for atmosphere. (*After Haskell.*)

sity at the end of a signal decreases as $r^{-\frac{1}{2}}$. This part of the signal is the principal part, confined to the sound channel. Following the abrupt end of this part, a weak disturbance is arriving, which is due to reflections from the bottom. These waves are called by Ewing and Worzel the *RSR* waves (see Ref. 25, Fig. 7). A more extended discussion of the problem of a sound channel was given by the same investigator in other papers [14–16]. (See also the discussion at the end of Ref. 16, p. 546.)

Haskell [36] studied sound propagation in the atmosphere but his results may be extended for the case of the ocean. Thus the considerations which follow must be regarded only as an illustration of the method. The general character of the assumed velocity-altitude relationship shown in Fig. 7–3*b* differs now from those considered before. We assume that (1) the vertical component of velocity vanishes at $z = 0$ (the rigid bottom of the atmosphere); (2) the velocity of a particle

$$\mathbf{v} = \operatorname{grad} \tilde{\varphi} \tag{7–52}$$

vanishes as $z \to \infty$; and (3) in the neighborhood of the source ($r = 0$, $z = h$) the potential $\tilde{\varphi}$ must reduce to

$$\tilde{\varphi} \to \frac{A}{R} \exp\left\{i\omega\left[t - \frac{R}{\alpha(h)}\right]\right\} \qquad R^2 = r^2 + (z - h)^2 \tag{7–53}$$

$\alpha(h)$ being the local velocity. Making use of the equation

$$\nabla^2\tilde{\varphi} = \frac{1}{\alpha^2(z)} \frac{\partial^2\tilde{\varphi}}{\partial t^2} \tag{7–54}$$

which holds approximately in this case, we also put

$$p = -\rho(z) \frac{\partial\tilde{\varphi}}{\partial t} \tag{7–55}$$

Omitting the time factor and changing notation slightly, we make use of Eq. (7–18) again:

$$\tilde{\varphi} = J_0(kr)F(z) \tag{7–56}$$

This function must satisfy conditions (7–31):

$$\frac{\partial F_1}{\partial z} - \frac{\partial F_2}{\partial z} = -2Ak \qquad \text{at } z = h \tag{7–57}$$

and

$$F_1 = F_2 \qquad \text{at } z = h \tag{7–58}$$

The factor $F(z)$ is in this case a solution of a more general equation than Eq. (7–20), namely,

$$\frac{d^2F}{dz^2} + F\left(\frac{\omega^2}{\alpha^2(z)} - k^2\right) = 0 \tag{7–59}$$

and, according to the first boundary condition,

$$\frac{dF_1}{dz} = 0 \qquad \text{at } z = 0 \tag{7–60}$$

Taking two linearly independent solutions of Eq. (7–59), $M(z, k)$ and $N(z, k)$ that behave asymptotically for large values of z (with the factor

$e^{i\omega t}$) like down and upward traveling waves, respectively, we now put $b = MN' - M'N$, where M' and N' are the derivatives with respect to z. Haskell showed that the boundary conditions are satisfied by

$$F_1(z, k) = \left[\frac{2AkN(h, k)}{bN'(0, k)}\right][N(z, k)M'(0, k) - M(z, k)N'(0, k)]$$
$$F_2(z, k) = \left[\frac{2AkN(z, k)}{bN'(0, k)}\right][N(h, k)M'(0, k) - M(h, k)N'(0, k)] \tag{7-61}$$

Moreover, if κ_n are roots of the equation

$$N'(0, k) = 0 \tag{7-62}$$

the solution of Eq. (7–54), which is obtained, as was shown in Chap. 4, by integration with respect to the parameter k, will be represented by a sum of residues corresponding to κ_n and some branch line integrals. If the Bessel function $J_0(kr)$ in (7–56) is replaced by the Hankel function, the sum of residues ($\bar{\varphi}_R$) takes the form

$$\bar{\varphi}_R = -2\pi i A e^{i\omega t} \sum_n \kappa_n H_0^{(2)}(\kappa_n r) N(z, \kappa_n) \frac{N(h, \kappa_n)}{N(0, \kappa_n)} \left[\frac{\partial N'(0, k)}{\partial k}\right]_{k=\kappa_n} \tag{7-63}$$

The factor b now equals $-M'(0, \kappa_n)\, N(0, \kappa_n)$ because of Eq. (7–62). Haskell's expression holds for the region above and below the level of the source. This is a formal solution, and the obvious question now is to choose the appropriate function $N(z, k)$. Let $c(k) = \omega/k$ and

$$Q(z, k) = \begin{cases} k\left\{\left[\dfrac{c(k)}{\alpha(z)}\right]^2 - 1\right\}^{\frac{1}{2}} & \text{for } c(k) > \alpha(z) \\ -ik\left\{1 - \left[\dfrac{c(k)}{\alpha(z)}\right]^2\right\}^{\frac{1}{2}} & \text{for } c(k) < \alpha(z) \end{cases} \tag{7-64}$$

Referring to Fig. 7–3*b*, let a_1, a_2, a_3 be the values of z at which $Q(z, k) = 0$ in the ranges $0 < z < z_1$, $z_1 < z < z_2$, and $z_2 < z < \infty$, respectively. Using a solution of Eq. (7–59), which is an asymptotic approximation for high frequencies, Haskell let

$$F_m = \left(\frac{u_m}{Q}\right)^{\frac{1}{2}} C_m(u_m) \tag{7-65}$$

where

$$u_m(z, k) = \int_{a_m}^{z} Q\, dz \tag{7-66}$$

and C_m is a linear combination of Bessel functions of order $\frac{1}{3}$. The solution with $m = 1$ is valid in the range $0 < z < a_2$, except near $z = a_1$. For $m = 2$, the solution is valid for $0 < z < a_3$ except near $z = a_2$; with $m = 3$ it is valid for $z > a_3$. Substituting the expression (7–65) in Eq. (7–62) leads to an estimation of some of the roots of the frequency equation.

Using asymptotic approximation for the residues in Eq. (7–63) and replacing the sums by equivalent integrals, Haskell evaluated these integrals by the method of stationary phase.

The T Phase. A short period phase, $T < 1$ sec, traveling through the ocean with the velocity of sound in sea water, is frequently found on seismograms of island or coastal observatories for earthquakes in which the path of propagation is mostly oceanic. A typical occurrence is shown in Fig. 7–4. This phase was first noted by Linehan [46], and the mechanism

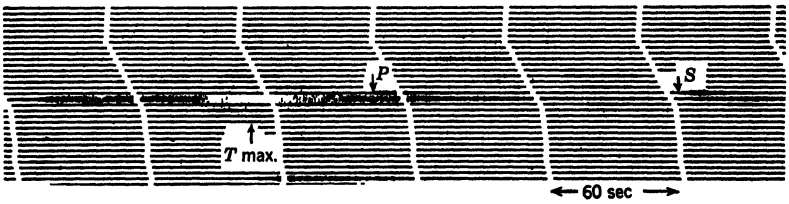


FIG. 7–4. *T* phase from West Indian earthquake of Sept. 21, 1951, recorded by underwater seismograph at Bermuda. (*Courtesy of M. Landisman.*)

of propagation was established in the work of Tolstoy and Ewing [74] and Ewing, Press, and Worzel [26*b*]. In the latter investigation, hydrophones in the SOFAR channel were used to detect earthquake-generated *T* phases.

There is little doubt that the energy crosses the deep ocean as sound waves, probably in the SOFAR channel. However, compressional, shear, and possibly surface waves may be involved in propagation across the land segment of the paths. Recent work with Pacific *T* phases by Wadati and Inouye [76] and Byerly and Herrick [17] and with Atlantic *T* phases by Båth [8] supports these conclusions. Shurbet [68] reported observation of *T* phases recorded at Bermuda from South American shocks in which the energy was transmitted over as much as 51° as *P*, before entering the ocean at the scarp north of Puerto Rico.

From the known great frequency of occurrence of small earthquakes and the efficiency of propagation in the SOFAR channel it may be expected that *T* waves contribute significantly to the acoustic "noise level" in the SOFAR channel. Recently Dietz and Sheehy [20] used *T* waves crossing the Pacific Ocean to study submarine volcanic eruptions off Japan.

7–3. Love Waves in Heterogeneous Isotropic Media. Since velocity gradients are known to exist in the earth's crust and mantle, it is very important to have a theory of Love-wave propagation in a medium where the velocity, rigidity, and density are functions of depth. The simplest case was investigated by Meissner [51], who considered a half space in which the density follows the law $\rho = \rho'(1 + \delta z)$, where z is the depth,

δ a parameter, and the rigidity is $\mu = \mu'(1 + \delta z)^2$. Wilson [79] studied the case $\mu = \mu' \exp(\gamma z)$, $\beta = \beta' \exp(\delta z)$. Meissner [52] added a discussion for media in which one layer is homogeneous, the other heterogeneous. This case (heterogeneity in the lower layer) was investigated in more detail by Jeffreys [43]. He showed that for a mantle given by $\rho = \text{const}$, $\mu = \mu'(1 + \delta z)^2$, the heterogeneity increases the phase velocity and decreases the group velocity of the longer-period waves. A solution for a heterogeneous medium of such kind that the functions involved are of an elementary nature was given by Bateman [6]. One simple case of this problem was also considered by Aichi [4]. Sezawa and Kanai [66] found certain correspondence between the propagation of Love waves through some heterogeneous media and the problem of varying water depth, and Matuzawa [49] gave a solution of the problem on assuming linear changes in the z direction in both rigidity and in the velocity of propagation. Recently the case of a homogeneous layer over a substratum in which $\mu = \mu' - bz$, $\rho = \text{const}$, was discussed by Satô [63], and solutions for the two cases in which variations in the substratum are given by $\mu = \mu' \exp(\delta z)$, $\rho = \rho' \exp(\delta z)$, and $\mu = \mu'(1 + \delta z)^2$, $\rho = \rho'(1 + \delta z)$, were given by Das Gupta [19].

For Love waves, $u = w = 0$, $v = v(x, z)$, and in the absence of body forces the equation of motion for the y component [compare (7–1)] is

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yz}}{\partial z} \tag{7–67}$$

By Eqs. (1–11) we obtain

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(\mu \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right) \tag{7–68}$$

If μ is a function of z only, Eq. (7–68) becomes

$$\rho \frac{\partial^2 v}{\partial t^2} = \mu \nabla^2 v + \frac{\partial \mu}{\partial z} \frac{\partial v}{\partial z} \tag{7–69}$$

If we put $V = \sqrt{\mu}\, v$, this equation takes the form

$$\rho \frac{\partial^2 V}{\partial t^2} = \mu \nabla^2 V + \left[\frac{1}{4\mu}\left(\frac{\partial \mu}{\partial z}\right)^2 - \frac{1}{2}\frac{\partial^2 \mu}{\partial z^2}\right] V \tag{7–70}$$

If we assume that $\mu = \mu(z)$ and

$$V = Z(z) e^{i(\omega t - kx)} \tag{7–71}$$

the first factor will be the solution of the equation

$$\frac{d^2 Z}{dz^2} + \left[k_\beta^2 - k^2 + \frac{1}{4\mu^2}\left(\frac{d\mu}{dz}\right)^2 - \frac{1}{2\mu}\frac{d^2 \mu}{dz^2}\right] Z = 0 \tag{7–72}$$

since $\beta = (\mu/\rho)^{\frac{1}{2}}$.

Homogeneous Layer over Heterogeneous Half Space: Matuzawa's Case. (Fig. 7–5). For $\mu_2 = \mu' - bz$, $\beta_2 = \beta' - \delta z$, Matuzawa [49] wrote Eq. (7–72) in a form valid for small values of z

$$\frac{d^2Z}{dz^2} + (A + Bz)Z = 0 \tag{7–73}$$

where, after minor corrections,

$$A = k_{\beta'}^2 + \frac{b^2}{4\mu'^2} - k^2 \qquad B = 2\left(k_{\beta'}^2 \frac{\delta}{\beta'} + \frac{b^3}{4\mu'^3}\right) \tag{7–74}$$

and obtained the solution

$$Z = \xi^{\frac{1}{2}}\left[C_1 H_{\frac{1}{3}}^{(1)}\left(\frac{2}{3B}\xi^{\frac{3}{2}}\right) + C_2 H_{\frac{1}{3}}^{(2)}\left(\frac{2}{3B}\xi^{\frac{3}{2}}\right)\right] \tag{7–75}$$

where

$$\xi = A + Bz \tag{7–76}$$

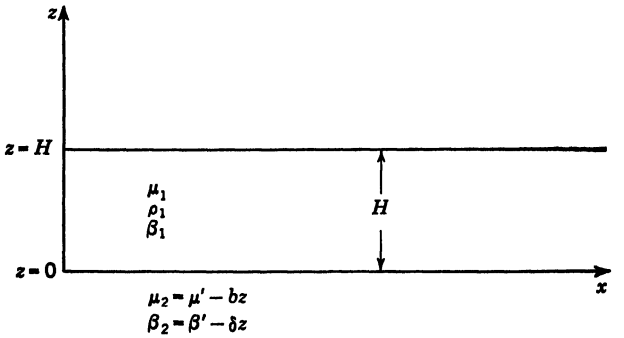


FIG. 7–5. Notations for Love waves in a homogeneous layer over a heterogeneous half space. (Matuzawa's case.)

If a half space displaying such properties is overlain by a homogeneous layer of the thickness H, the solution for this layer can be taken in the usual form (see Sec. 4–5):

$$v_1 = (A_1 e^{ikH\gamma_{1s}} + B_1 e^{-ikH\gamma_{1s}})e^{i(\omega t - kx)} \tag{7–77}$$

with $\gamma_1 = \sqrt{c^2/\beta_1^2 - 1}$, and the boundary conditions for Love waves can easily be written. The period equation given by Matuzawa for this case is

$$\tan kH\gamma_1 = \frac{1}{2}\frac{\mu' B}{\mu_1 k\gamma_1 A}\left(1 + \frac{bA}{B\mu'}\right) + \frac{1}{2}\frac{\mu'\sqrt{A}}{\mu_1 k\gamma_1}\frac{H_{-2/3}^{(1)} - H_{4/3}^{(1)}}{H_{1/3}^{(1)}} \tag{7–78}$$

where argument of the Hankel functions is $2A^{\frac{3}{2}}/3B$. If the asymptotic

expansions of these functions for large values of the argument are used, the approximate form of the period equation is

$$\tan kH\hat{\gamma}_1 = \frac{\mu' s_2}{\mu_1 k\hat{\gamma}_1} - \frac{\mu' B}{2\mu_1 k\hat{\gamma}_1 s_2^2} + \frac{b}{2\mu_1 k\hat{\gamma}_1} \tag{7-79}$$

where $s_2^2 = -A$.

Satô's Case (Fig. 7–6). In Satô's [63] problem the velocity of propagation in the substratum is $\beta_3 = [(\mu' - bz)/\rho]^{\frac{1}{2}}$, and Eq. (7–72) takes the form

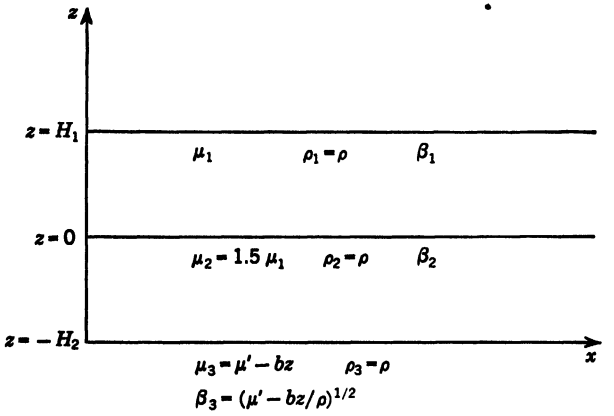


FIG. 7–6. Notations for Love waves in a double homogeneous layer over a heterogeneous half space. (Satô's case.)

$$\frac{d^2Z}{dz^2} + \left[k_{\beta 3}^2 - k^2 + \frac{b^2}{4\mu_3^2}\right]Z = 0 \tag{7-80}$$

By a change of variables

$$y = \frac{\rho\omega^2}{2kb} \qquad \xi = \frac{2k\mu_3}{b} \qquad W = \frac{4k^2}{b^2}Z \tag{7-81}$$

the equation

$$\frac{d^2W}{d\xi^2} + \left[\frac{1}{4\xi^2} + \frac{y}{\xi} - \frac{1}{4}\right]W = 0 \tag{7-82}$$

is obtained. It is satisfied by Whittaker's function

$$W_{y,0}(\xi) = e^{-\xi/2}\xi^y S = e^{-\xi/2}\xi^y\left\{1 + \sum_{n=1}(-1)^n \prod_{m=1}^{n} \frac{(y - m + \frac{1}{2})^2}{m\xi}\right\} \tag{7-83}$$

or

$$M_{y,0}(\xi) = \xi^{\frac{1}{2}}e^{-\xi/2}\,{}_1F_1(\tfrac{1}{2} - y, 1, \xi) \tag{7-84}$$

where ${}_1F_1$ is the confluent hypergeometric function. By Eqs. (7–71) and (7–83) and $V = \sqrt{\mu_3 v_1}$ we have

$$v_1 = A\left(\frac{b}{2k}\right)^{\frac{1}{4}} \Phi(\xi) e^{i(\omega t - kx)} \qquad \Phi(\xi) = \xi^{\nu - \frac{1}{2}} e^{-\xi/2} S \tag{7–85}$$

Considering two homogeneous layers overlying the half space having elastic properties just defined, Satô makes use of the expression (7–77) for each layer. If quantities referred to the layers and half space are denoted by the subscripts 1, 2, and 3, respectively, the boundary conditions may be written as

$$\frac{\partial v_1}{\partial z} = 0 \quad \text{at } z = H_1 \qquad v_1 = v_2 \quad \text{and} \quad \mu_1 \frac{\partial v_1}{\partial z} = \mu_2 \frac{\partial v_2}{\partial z} \quad \text{at } z = 0 \tag{7–86}$$

$$v_2 = v_3 \quad \text{and} \quad \mu_2 \frac{\partial v_2}{\partial z} = \mu_3 \frac{\partial v_3}{\partial z} \quad \text{at } z = -H_2 \tag{7–87}$$

If we put $\hat{\gamma}_i^2 = c^2/\beta_i^2 - 1$, the period equation is obtained in the form

$$\left\{1 - \frac{\mu_1 \hat{\gamma}_1}{\mu_2 \hat{\gamma}_2} \tan kH_1\hat{\gamma}_1 \tan kH_2\hat{\gamma}_2\right\} \frac{2\Phi'(\xi_0)}{\hat{\gamma}_2 \Phi(\xi_0)} + \left\{\frac{\mu_2}{\mu'} \tan kH_2\hat{\gamma}_2 + \frac{\mu_1 \hat{\gamma}_1}{\mu' \hat{\gamma}_2} \tan kH_1\hat{\gamma}_1\right\} = 0 \tag{7–88}$$

where $$\mu_3 = \mu' - b(z + H_2) \quad \text{and} \quad \xi_0 = \frac{2k\mu'}{b} \tag{7–89}$$

The phase and group velocities were determined by Satô for the conditions

$$H_1 = H_2 = H \qquad \mu_2 = 1.5\mu_1 = \frac{\mu'}{1.5} \qquad \rho_1 = \rho_2 = \rho \qquad \frac{\mu'}{bH} = 40$$

His phase- and group-velocity curves are presented in Fig. 7–7.

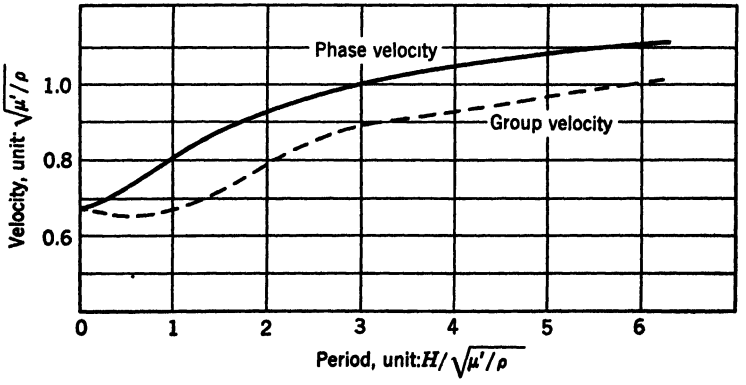


FIG. 7–7. Phase- and group-velocity curves for Satô's case.

Jeffreys' Case (Fig. 7–8). Jeffreys [43] considered the system composed of a homogeneous layer of density ρ_1, and thickness H, and the half-space having a uniform density ρ_2 and the rigidity $\mu_2 = \mu'(1 + z/q)^2$, μ' and q being constants. If we put

$$\xi = 1 + \frac{z}{q} \qquad \gamma_1^2 = \frac{c^2}{\beta_1^2} - 1 \qquad \gamma_2^2 = 1 - \frac{c^2}{\beta_2^2} \tag{7–90}$$

and $v = (A \cos k\gamma_1 z + B \sin k\gamma_1 z)e^{ik(x-ct)}$ for $-H_0 < z < 0$ (7–91)

$$v = Ve^{ik(x-ct)} \qquad \text{for } 0 < z \tag{7–92}$$

Equation (7–68) for the second medium takes the form

$$\frac{d^2}{d\xi^2}(\xi V) - k^2q^2\left(1 - \frac{c^2}{\beta_2^2\xi^2}\right)\xi V = 0 \tag{7–93}$$

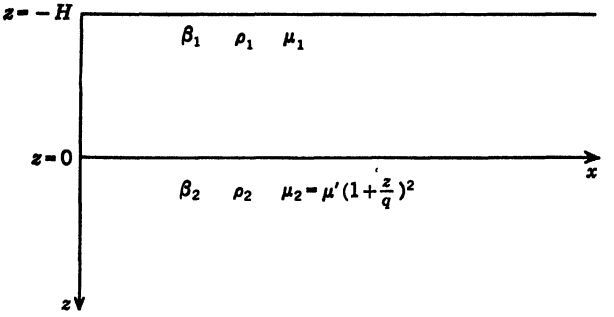


FIG. 7–8. Notations for Love waves in a homogeneous layer over a heterogeneous half space. (Jeffreys' case.)

Two cases have to be considered according as $c \lessgtr \beta_2$, the boundary conditions being of the type used before. The final form of the period equation obtained by Jeffreys is as follows: (1) $c < \beta_2$ [see also Eq. (B–22)];

$$\tan kH\gamma_1 = \frac{\mu_2\gamma_2}{\mu_1\gamma_1}\left[1 + \frac{1}{2kq\gamma_2}\left(1 + \frac{1}{\gamma_2^2}\right) - \frac{1}{8k^2q^2\gamma_2^6}(1 - \gamma_2^2)(5 + \gamma_2^2) + O(kq)^{-3}\right] \tag{7–94}$$

where $\mu_2 = \mu'$ is evaluated at $z = 0$, that is, $\xi = 1$. Since asymptotic solutions of Eq. (7–93) are used, the approximation depends upon the largeness of $kq\gamma_2^3$. (2) $c > \beta_2$. If we use the notations

$$\cosh\theta = \frac{c}{\beta_2} \qquad \sinh\theta = -\gamma_2 = \left(\frac{c^2}{\beta_2^2} - 1\right)^{\frac{1}{2}} \tag{7–95}$$

the period equation is (for large values of $kq\gamma_2^3$)

$$\tan kH\hat{\gamma}_1 = \frac{\mu_2}{\mu_1\hat{\gamma}_1}\left[\hat{\gamma}_2 \cot\left\{kq(\theta\cosh\theta - \sinh\theta) - \frac{3\pi}{4}\right\} - \frac{1-\hat{\gamma}_2^2}{2kq\hat{\gamma}_2^2}\right] \quad (7\text{–}96)$$

Since moderate values $kq\gamma_2^3$ or $kq\hat{\gamma}_2^3$ are also important, Jeffreys derived a third form of the period equation. Using the new variables ξ' and ξ'' given by the equations

$$\xi = \frac{c}{\beta_2}(1 + \xi') \quad (7\text{–}97)$$

$$\xi'' = \left(\frac{2k^2q^2c^2}{\beta_2^2}\right)^{\frac{1}{3}}\xi' \quad (7\text{–}98)$$

he found that Eq. (7–96) becomes

$$\frac{\mu_1\hat{\gamma}_1}{\mu_2}\left(\frac{kqc}{2\beta_2}\right)^{\frac{1}{3}} \tan kH\hat{\gamma}_1 = \frac{Ai'(-\xi'')}{Ai(-\xi'')} \quad (7\text{–}99)$$

where

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(kx - \frac{1}{3}k^3\right)dk \quad (7\text{–}100)$$

is the Airy integral.

Calculations based on Jeffreys' theory have led to dispersion curves which have been compared with experimental observations of Love waves (sec Fig. 4–52).

In the problem investigated by Jeffreys the densities are assumed to be constant. However, variations of densities must also be considered in some applications.

Meissner's Case (Fig. 7–9). Meissner [51] considered a half space in

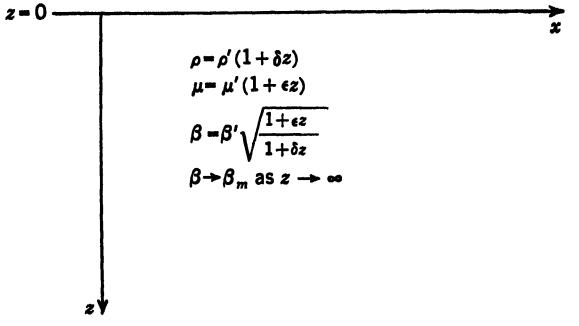


FIG. 7–9. Notations for Love waves in a heterogeneous half space. (Meissner's case.)

which

$$\rho = \rho'(1 + \delta z) \qquad \mu = \mu'(1 + \epsilon z) \tag{7-101}$$

and, therefore,

$$\beta = \beta(z) = \beta'\sqrt{\frac{1 + \epsilon z}{1 + \delta z}} \tag{7-102}$$

The limit value of the velocity at infinity has the finite value

$$\beta_m = \lim_{z \to \infty} \beta(z) = \beta'\sqrt{\frac{\epsilon}{\delta}} \tag{7-103}$$

If we put

$$\eta = 2\gamma \frac{k}{\epsilon}(1 + \epsilon z) \qquad \gamma = \sqrt{1 - \frac{c^2}{\beta_m^2}} \qquad \omega = kc \tag{7-104}$$

Eq. (7-72) takes the form

$$\frac{d^2Z}{d\eta^2} + \left(\frac{1}{4\eta^2} + \frac{p}{2\eta} - \frac{1}{4}\right)Z = 0 \tag{7-105}$$

where

$$p = \frac{c^2 k}{\beta'^2 \epsilon}\left(1 - \frac{\delta}{\epsilon}\right)\left(1 - \frac{c^2}{\beta_m^2}\right)^{-\frac{1}{2}} \tag{7-106}$$

Now, if we put $\delta = 0$, we obtain the case considered by Satô, and Eq. (7-105) is of the type (7-82). We can consider the solution of Eq. (7-105) in terms of Whittaker's functions,

$$Z = AW_{p/2,0}(\eta) + BW_{-p/2,0}(-\eta) \tag{7-107}$$

and find the exact form of the period equation (indicated by Jardetzky). The boundary condition for this case is

$$p_{yz} = \frac{dv}{dz} = 0 \qquad \text{at } z = 0 \tag{7-108}$$

Since surface waves are involved, we have as $z \to \infty$

$$\lim_{\eta \to \infty} Z(\eta) = 0 \tag{7-109}$$

This last condition requires that we put $B = 0$. The solution which tends to 0 as $z \to \infty$ is, then,

$$Z = AW_{p/2,0}(\eta) \tag{7-110}$$

where A is a constant.

Now $v = V/\sqrt{\mu}$, and by Eq. (7-71), if we omit factors which do not depend on z, the condition (7-108) is

$$\frac{d}{dz}\left(\frac{Z}{\sqrt{\mu}}\right) = 0 \tag{7-111}$$

or
$$2\mu \frac{dZ}{dz} - Z \frac{d\mu}{dz} = 0 \tag{7-112}$$

By Eqs. (7–104) and (7–110) we express the required condition (7–112) as the period equation in the form

$$2\eta \frac{d}{d\eta} W_{p/2,0}(\eta) - W_{p/2,0}(\eta) = 0 \qquad \text{at } z = 0,\ \eta = 2\gamma \frac{k}{\epsilon} \tag{7-113}$$

To relate phase velocity to wavelength we first note that Eqs. (7–113) relate the variables p and $\eta_0 = 2(1 - c^2/\beta_m^2)^{\frac{1}{2}}\, k/\epsilon$. On the other hand, we have from (7–106)

$$p = \frac{c^2}{\beta'^2}\left(1 - \frac{\beta'^2}{\beta_m^2}\right)\left(1 - \frac{c^2}{\beta_m^2}\right)^{-1} \frac{\eta_0}{2} \tag{7-114}$$

Finally, if $l = 2\pi/k$ is the wavelength, then

$$\epsilon l = \frac{4\pi}{\eta_0} \sqrt{1 - \frac{c^2}{\beta_m^2}} \tag{7-115}$$

We can now eliminate the two variables p and η_0 from Eqs. (7–113) by using (7–114) and (7–115) to obtain an equation in which c is expressed in terms of l.

Since Whittaker's function $W_{y,m}$ is given by the formula

$$W_{y,m}(\eta) = \frac{\eta^{-m+\frac{1}{2}} e^{-\eta/2}}{(m - y - \frac{1}{2})!} \int_0^\infty e^{-t} t^{m-y-\frac{1}{2}} (t + \eta)^{m+y-\frac{1}{2}}\, dt \tag{7-116}$$

the exact form of the period equation in this case is

$$\int_0^\infty e^{-t} t^{-(p+1)/2} (t + \eta_0)^{(p-3)/2} [p - 1 - (t + \eta_0)]\, dt = 0 \tag{7-117}$$

As an approximation, one can use the asymptotic expansion of the Whittaker function.

For the case of a half space in which

$$\rho = \rho'(1 + \delta z) \qquad \mu = \mu'(1 + \delta z)^2 \tag{7-118}$$

Meissner obtained a solution by means of the Laplace transformation. Das Gupta [19] solved this problem for a homogeneous layer underlain by a half space of the type (7–118), using Whittaker functions.

Observations of Love-wave dispersion from earthquake seismograms were discussed in Sec. 4–5, where it was shown that inclusion of the effect of the velocity gradient in the mantle was necessary to reconcile crustal structure determined by explosion seismology with that determined by surface-wave studies.

7–4. Rayleigh Waves in Heterogeneous Isotropic Media. The first discussion of Rayleigh-wave propagation in a heterogeneous medium seems

to have been given by Honda [40]. Sezawa [65] investigated general questions of wave propagation in a semi-infinite solid body of varying elasticity. He assumed incompressibility, taking $\theta = 0$, the component w equal to zero, and $\mu = c\,(d + z)$, where z is the depth. With external forces omitted, all members of Eqs. (7–4) were expressed in terms of cylindrical components of rotation $\tilde{\omega}_r$, $\tilde{\omega}_\chi$, $\tilde{\omega}_z$. Three equations for these functions could be satisfied by expressions of the form

$$\begin{aligned}
\tilde{\omega}_z &= B_m \Phi(z) H_m^{(2)}(kr) \quad \left.\begin{matrix}\sin\\ -\cos\end{matrix}\right\} m\chi \cdot e^{i\omega t} \\
\tilde{\omega}_r &= \frac{B_m}{k^2}\,\Phi'(z)\,\frac{dH_m^{(2)}(kr)}{dr} \quad \left.\begin{matrix}\sin\\ -\cos\end{matrix}\right\} m\chi \cdot e^{i\omega t} \\
\tilde{\omega}_\chi &= \frac{B_m m}{k^2 r}\,\Phi'(z) H_m^{(2)}(kr) \quad \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} m\chi \cdot e^{i\omega t}
\end{aligned} \tag{7–119}$$

where $\Phi(z)$ is the integral of the differential equation

$$\Phi'' + \frac{1}{\mu}\frac{d\mu}{dz}\,\Phi' + \left(\frac{\rho\omega^2}{\mu} - k^2\right)\Phi = 0 \tag{7–120}$$

This equation, in cases of the simplest laws for rigidity distribution, reduces to well-known types. Thus, for $\mu = cz$ the solution could be expressed in terms of the confluent hypergeometric function.

Stoneley [72] took the equations for two-dimensional motion which represent a particular case of Eqs. (7–4)

$$\begin{aligned}
\rho\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial x}\left\{\lambda\theta + 2\mu\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right\} \\
\rho\frac{\partial^2 w}{\partial t^2} &= \frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right\} + \frac{\partial}{\partial z}\left\{\lambda\theta + 2\mu\frac{\partial w}{\partial z}\right\}
\end{aligned} \tag{7–121}$$

and also assumed that the medium is incompressible. Suppose that u and w are functions of z multiplied by the factor $\exp[ik(ct - x)]$ and put

$$u = \frac{\partial\varphi}{\partial x} - \frac{\partial\psi}{\partial z} \qquad w = \frac{\partial\varphi}{\partial z} + \frac{\partial\psi}{\partial x} \tag{7–122}$$

Now, following Love (Chap. 3, Ref. 26), assume that $-\Pi = \lim \lambda\theta$ as $\lambda \to \infty$ and $\theta \to 0$. Considering the case $\mu = \mu_0 + \mu_1 z$, we now have the equations of motion in the form

$$\begin{aligned}
-\frac{\partial\Pi}{\partial x} + \mu\nabla^2 u + \mu_1\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \rho k^2 c^2 u = 0 \\
-\frac{\partial\Pi}{\partial z} + \mu\nabla^2 w + 2\mu_1\frac{\partial w}{\partial z} + \rho k^2 c^2 w = 0
\end{aligned} \tag{7–123}$$

and
$$\Pi = 2\mu_1 w + \rho k^2 c^2 \varphi \tag{7-124}$$
If we put
$$F = \mu \nabla^2 \psi + \rho k^2 c^2 \psi = 0 \tag{7-125}$$
Eqs. (7–123) take the form
$$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial z} = 0 \tag{7-126}$$
and if we write
$$\varphi = \varphi_0 e^{ik(ct-x)} \qquad \psi = \psi_0 e^{ik(ct-x)} \tag{7-127}$$
we obtain (on omitting the subscript 0)
$$\frac{d^2\varphi}{dz^2} - k^2\varphi = 0 \qquad \frac{d^2\psi}{dz^2} + k^2\left(\frac{\rho c^2}{\mu_0 + \mu_1 z} - 1\right)\psi = 0 \tag{7-128}$$
Since the free surface is $z = 0$ and the z axis is directed into the interior of the body, we have for the velocity at the surface $\beta_0 = (\mu_0/\rho)^{\frac{1}{2}}$. The substitutions $\xi = 2(kz + b)$ and $a = bc^2/2\beta_0^2$, where $b = k\mu_0/\mu_1$, then reduce the second of Eqs. (7–128) to the form
$$\frac{d^2\psi}{d\xi^2} + \left(\frac{a}{\xi} - \frac{1}{4}\right)\psi = 0 \tag{7-129}$$
which is satisfied by Whittaker's function $W_{a,\frac{1}{2}}(\xi)$. For a surface wave, only the negative exponent in the solution of Eqs. (7–128) will be used, and the two functions φ and ψ are taken in the form
$$\varphi = Ae^{-kz}\cos k(ct - x) \qquad \psi = BW_{a,\frac{1}{2}}(2kz + 2b)\sin k(ct - x) \tag{7-130}$$
Then the boundary conditions yield the period equation
$$\left(2 - \frac{c^2}{\beta_0^2}\right)\left(\frac{2}{b} + 2 - \frac{c^2}{\beta_0^2}\right) = \frac{4}{b} - \frac{8W'(2b)}{W(2b)} \tag{7-131}$$
Unfortunately, asymptotic expansions for these Whittaker functions cannot be applied to this case, and Stoneley investigated by other methods the two limiting cases valid for short or long waves. He also considered the problem of a uniform surface layer overlying a heterogeneous half space.

Rayleigh waves in a semi-infinite incompressible medium where a layer in which rigidity varies linearly with depth is underlain by a uniform elastic substratum were investigated by Newlands [55], who also extended the investigation to compressible media.

For a heterogeneous layer Newlands solved the second of Eqs. (7–128) directly instead of using Whittaker's function. Thus, if we put $\vartheta = 1 + \mu_1 z/\mu_0$, this equation takes the form
$$\frac{d^2\psi}{d\vartheta^2} - \left(\frac{k\mu_0}{\mu_1}\right)^2\left(1 - \frac{\rho c^2}{\mu_0 \vartheta}\right)\psi = 0 \tag{7-132}$$

Expanding the integral in a series of powers of $(k\,\mu_0/\mu_1)^2$, namely,

$$\psi(\vartheta) = \psi_0(\vartheta) + \left(\frac{k\mu_0}{\mu_1}\right)^2 \psi_1(\vartheta) + \cdots + \left(\frac{k\mu_0}{\mu_1}\right)^{2n} \psi_n(\vartheta) + \cdots \tag{7–133}$$

we obtain from Eq. (7–132)

$$\psi_0'' = 0 \qquad \psi_1'' = \left(1 - \frac{\rho c^2}{\mu_0 \vartheta}\right)\psi_0, \cdots, \psi_{n+1}'' = \left(1 - \frac{\rho c^2}{\mu_0 \vartheta}\right)\psi_n \tag{7–134}$$

By (7–134) one can take

$$\psi_0(\vartheta) = A_1 + A_2 z \tag{7–135}$$

Then, from the linear differential equation (7–132),

$$\psi = A_1 \psi^{(1)} + A_2 \psi^{(2)} \tag{7–136}$$

where

$$\psi^{(1)} = 1 + \left(\frac{k\mu_0}{\mu_1}\right)^2 \psi_1^{(1)} + \cdots \qquad \psi^{(2)} = z + \left(\frac{k\mu_0}{\mu_1}\right)^2 \psi_1^{(2)} + \cdots \tag{7–137}$$

By the second equation in (7–134) we get

$$\psi_1(\vartheta) = \int_1^{\vartheta} d\xi \int_1^{\xi} \left(1 - \frac{\rho c^2}{\mu_0 \sigma}\right)\psi_0(\sigma)\, d\sigma \tag{7–138}$$

and if we use the absolute value of this expression as well as the other ψ_n, it may be proved that the series converges and that $\psi^{(1)}(\vartheta)$ and $\psi^{(2)}(\vartheta)$ can be written in terms of certain polynomials in ϑ with logarithmic factors.

Now, if two layers are considered, we can assume that for the upper layer the solutions of Eqs. (7–128) hold:

$$\begin{aligned} \varphi_1 &= (P \cosh kz + Q \sinh kz) \cos k(ct - x) \\ \psi_1 &= [A_1 \psi^{(1)}(\vartheta) + A_2 \psi^{(2)}(\vartheta)] \sin k(ct - x) \end{aligned} \tag{7–139}$$

and for the homogeneous substratum

$$\begin{aligned} \varphi_2 &= Re^{-kz} \cos k(ct - x) \\ \psi_2 &= Se^{-k\gamma_2 z} \sin k(ct - x) \end{aligned} \tag{7–140}$$

where

$$\gamma_2^2 = 1 - \frac{c^2}{\beta_2^2} \qquad \beta_2 = \frac{\mu_2}{\rho_2} \tag{7 141}$$

and ρ_2 and μ_2 are the density and the rigidity of the substratum.

The continuity of displacements and stresses at the interface $z = 0$ (the axis is taken positive downward) and vanishing of stresses at the free surface $z = -H$ yield six conditions for the constants A_1, A_2, P, Q, R, and S. Since this system is a homogeneous one, its determinant must

vanish. We thus have the period equation for the medium determined above. If the medium is compressible, λ also becomes a function of coordinates, and, therefore, we have to make use of Eqs. (7–121). These equations take a simpler form for $\mu = \mu_0 + \mu_1 z$ and for a time factor $\exp(\pm i\omega t)$, namely,

$$\begin{aligned}\frac{\partial}{\partial x}\left[(\lambda + 2\mu)\nabla^2\varphi + \rho k^2 c^2 \varphi - 2\mu_1 \frac{\partial \psi}{\partial x}\right] \\ + \frac{\partial}{\partial z}\left[\mu\nabla^2\psi + \rho k^2 c^2 \psi + 2\mu_1 \frac{\partial \varphi}{\partial x}\right] = 0 \\ \frac{\partial}{\partial z}\left[(\lambda + 2\mu)\nabla^2\varphi + \rho k^2 c^2 \varphi - 2\mu_1 \frac{\partial \psi}{\partial x}\right] \\ - \frac{\partial}{\partial x}\left[\mu\nabla^2\psi + \rho k^2 c^2 \psi + 2\mu_1 \frac{\partial \varphi}{\partial x}\right] = 0\end{aligned} \tag{7–142}$$

When the additional constants are neglected, Eqs. (7–142) are satisfied if

$$\begin{aligned} f_1 &= (\lambda + 2\mu)\nabla^2\varphi + \rho k^2 c^2 \varphi - 2\mu_1 \frac{\partial \psi}{\partial x} = 0 \\ f_2 &= \mu\nabla^2\psi + \rho k^2 c^2 \psi + 2\mu_1 \frac{\partial \varphi}{\partial x} = 0 \end{aligned} \tag{7–143}$$

where the factor μ_1 is $\partial\mu/\partial z$ and the last terms are due to heterogeneity of the medium. Following Newlands' method, one expands the functions φ and ψ into series

$$\begin{aligned} \varphi \equiv \varphi(\vartheta) &= F_0 + \frac{k\mu_0}{\mu_1} F_1 + \left(\frac{k\mu_0}{\mu_1}\right)^2 F_2 + \cdots \\ \psi \equiv \psi(\vartheta) &= G_0 + \frac{k\mu_0}{\mu_1} G_1 + \left(\frac{k\mu_0}{\mu_1}\right)^2 G_2 + \cdots \end{aligned} \tag{7–144}$$

where the variable ϑ is $1 + \mu_1 z/\mu_0$, and, as before, makes use of Eqs. (7–143) for successive computation of functions F_s, G_s. The results of this transformation were applied to the problem of Rayleigh waves in a crust where both μ and λ vary linearly with depth. Comparison of the computed group-velocity dispersion curve for Rayleigh-wave observations of Röhrbach [59] did not show any agreement, and Newlands suggested that an adjustment of the assumed constants was needed.

Mantle Rayleigh Waves. Trains of long-period Rayleigh waves which have circled the earth several times are often observed on seismograms of the greatest earthquakes. With the aid of modern long-period seismographs, the dispersion of these waves has been studied in sufficient detail to extend the Rayleigh-wave dispersion curve to the range 70 to 480 sec.

Remarkable seismograms (Fig. 7–10) showing the long waves were

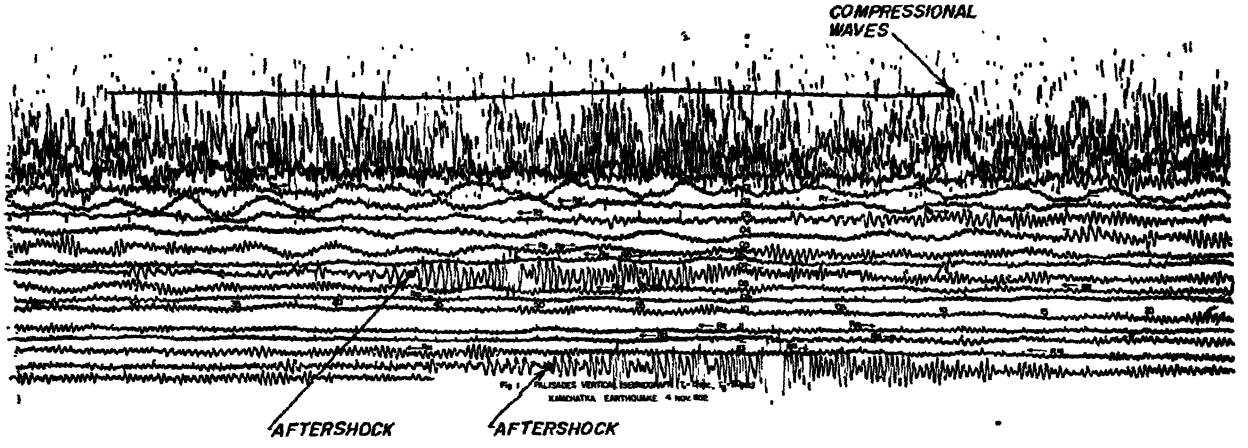


FIG. 7–10. Mantle Rayleigh waves from the Kamchatka earthquake of Nov. 4, 1952. Orders R_6 to R_{15} are indicated.

obtained from the Kamchatka earthquake of Nov. 4, 1952, on the Benioff linear-strain seismograph at Pasadena and on the long-period vertical-pendulum seismograph at Palisades (Ewing and Press, Chap. 5, Ref. 12). Rayleigh-wave trains of orders R_6 to R_{15} are indicated on the seismogram, corresponding to epicentral distance (in degrees) $\Delta_n = (n - 1)180 + \Delta$ for n odd, and $\Delta_n = n \cdot 180 - \Delta$ for n even, where n is the order and Δ the least distance between station and epicenter.

Comparison of the strain and pendulum seismograms from Pasadena demonstrated that the orbital motion was retrograde elliptical, proper for Rayleigh waves. Period and arrival time were read from the record in the usual manner (Ewing and Press [27]), and from these data the group velocity was calculated, using the epicentral distance appropriate for the number of circuits of the earth. Observed group velocity for the long Rayleigh waves is plotted as a function of period in Fig. 7–11 for the various orders. Striking features are the minimum value of group velocity of 3.54 km/sec at a period of 225 sec, a short-period limit of 3.8 km/sec at 70 sec, and the flattening of the curve for periods greater than 400 sec.

In view of the great length of these waves, there can be no doubt that the dispersion is the result of the known increase of shear velocity with depth in the mantle, hence the use of the name "mantle Rayleigh waves." As a rough approximation, the mantle velocity gradient may be replaced by two homogeneous layers and the theory of Sec. 4–5 used to compute the theoretical curve shown in Fig. 7–11. A theoretical curve (Haskell, Chap. 4, Ref. 62) was used with the constants $\beta_2 = 6.15$ km/sec, $\beta_1 = 4.48$ km/sec, and $H = 516$ km, the constants being chosen to fit the observed minimum group velocity. Better agreement with observation could be obtained by computing theoretical curves for a two-layer heterogeneous medium, using the methods of the preceding section, but these calculations are very lengthy.

Several important results emerge from the study of mantle Rayleigh waves:

1. For periods greater than 75 sec, oceanic and continental paths cannot be distinguished. The short-period limit $T \cong 75$ sec corresponds to the least wavelength for which the continent-ocean margin is a negligible barrier.

2. The dispersion of crustal Rayleigh waves for $T < 50$ sec is normal (velocity increasing with period) and that of mantle Rayleigh waves for 75 sec $< T <$ 225 sec is reverse. Since the two dispersion curves must merge, a maximum value of group velocity must exist between periods 50 and 75 sec. A complete Rayleigh-wave dispersion curve including that for oceanic and continental crustal Rayleigh waves and also that for the mantle is shown in Fig. 7–12.

3. The flattening of the dispersion curve for $T > 400$ sec is interpreted as an effect of the vanishing rigidity of the earth's core.

4. The great length of mantle Rayleigh waves frees them from the effects of crustal irregularities, and amplitude measurements can be made with

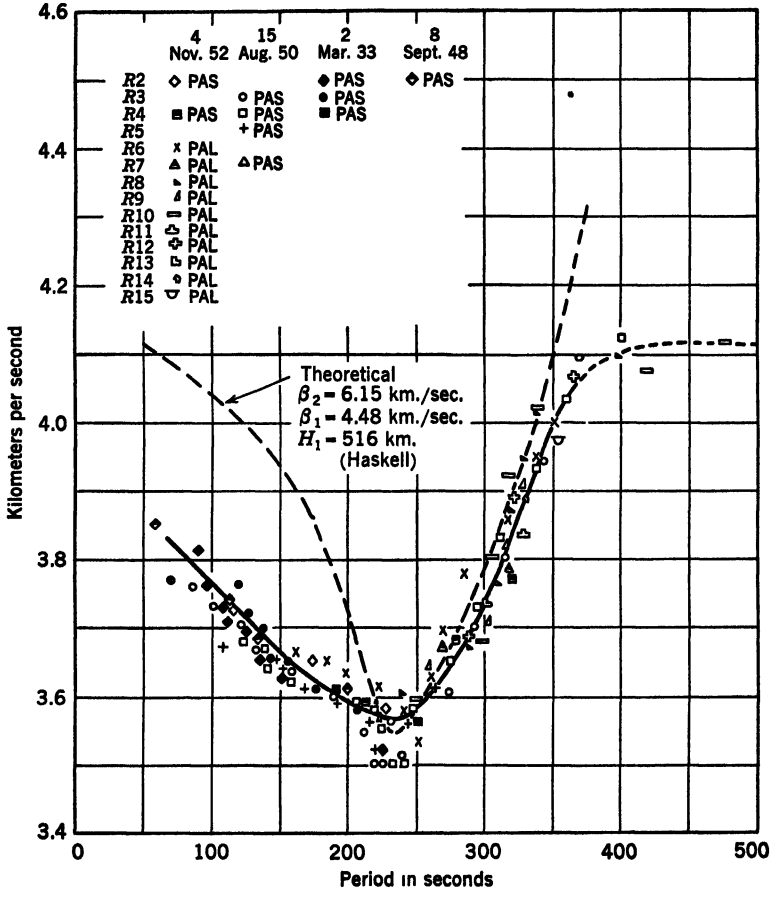


FIG. 7-11. Group-velocity curve for mantle Rayleigh waves obtained from Pasadena and Palisades seismograms.

sufficient precision for studying decrement. For an amplitude decrement given by $\exp(-\delta\Delta_n)$, where $\delta = \pi/QcT$, it was found that $1/Q = 665 \times 10^{-6}$ at $T = 215$ sec, after allowance for geometric spreading by an amplitude factor $(\sin \Delta_n)^{-\frac{1}{2}}$ and dispersion by a factor $\Delta^{-\frac{1}{2}}$. The amplitude A

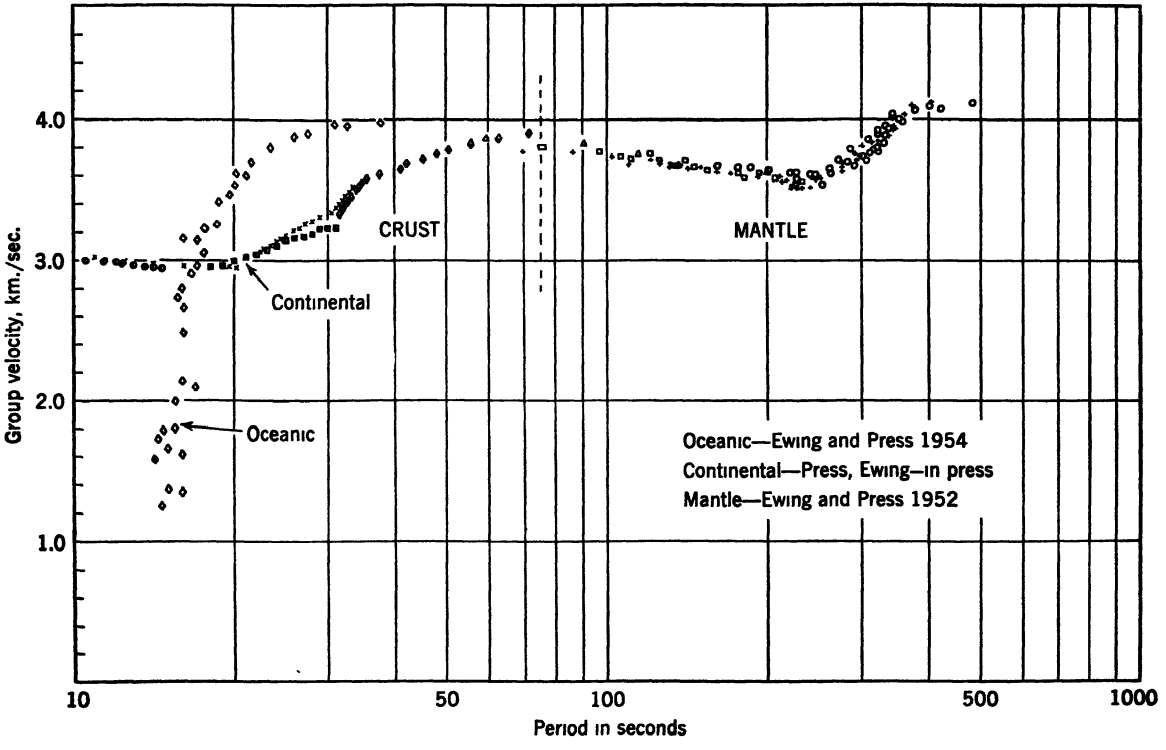


FIG. 7–12. Composite dispersion curve for crustal and mantle Rayleigh waves in the period range 10 to 500 sec.

is given by

$$A = \frac{\overline{A}_0 e^{-\delta\Delta}}{|R_0 \sin \Delta|^{\frac{1}{2}} \Delta^{\frac{1}{2}}} \tag{7-145}$$

for the Airy phase, and

$$A = \frac{A_0 e^{-\delta\Delta}}{|R_0 \sin \Delta|^{\frac{1}{2}} \Delta^{\frac{1}{2}}} \tag{7-146}$$

for other periods in the wave train, R_0 being the earth's radius, A_0 and $\overline{A}_0$ constants.

7-5. Aeolotropic and Other Media. In the investigations discussed in the preceding sections, isotropic elastic media were assumed. Since most igneous rocks are crystalline and the crystals commonly have random orientation, the assumption that the media are isotropic is a reasonable one. Some attempts have been made to explain some discrepancies between observation and theory by assuming aeolotropy in rocks (White and Sengbush [78]). It is well known (see, for example, Love, Chap. 1, Ref. 34) that there are, in general, three principal velocities of wave propagation in an aeolotropic medium. For electromagnetic waves, for example, the so-called Fresnel law determines the velocity in any given direction in terms of these three velocities. For elastic waves in aeolotropic media, one has to make further assumptions about the nature of the medium. If, as usual, the stress components $p_{xx}, \cdots, p_{zz}$ are considered as linear functions of strain components $e_{xx}, \cdots, e_{zz}$, we can write

$$p_{xx} = \sum c_{ij} e_{ij}, \cdots \qquad i, j = x, y, \text{ or } z \tag{7-147}$$

The 36 coefficients c_{ij} reduce to a smaller number for different types of crystals.

According to Satô [62], the first attempt to investigate wave propagation in such media seems to have been made by Homma, who published his results in Japanese in 1942. Satô, making use of these results, discussed a two-dimensional problem of wave propagation in a medium which is horizontally isotropic and vertically aeolotropic, its boundary being a plane surface. He assumed that

$$\begin{aligned} p_{xx} &= M_1(e_{xx} + e_{yy}) - 2N_2 e_{yy} + (M_3 - 2N_1)e_{zz} & p_{yz} &= N_1 e_{yz} \\ p_{yy} &= M_1(e_{xx} + e_{yy}) - 2N_2 e_{xx} + (M_3 - 2N_1)e_{zz} & p_{zx} &= N_1 e_{zx} \\ p_{zz} &= (M_3 - 2N_1)(e_{xx} + e_{yy}) + M_2 e_{zz} & N_2 &= N_1 \end{aligned} \tag{7-148}$$

and, on taking the usual form of solutions, u and w expressed in terms of exponential functions, he obtained the period equation. Then he could show the existence of waves of the Rayleigh type. Matuzawa [50] made use of results of Sakadi [61] concerning wave propagation in crystalline

media. The stresses were taken in the form corresponding to hexagonal crystals, and solutions of equations of motion were given for two kinds of waves. The first, determined by the conditions

$$u = f(my + nz - \omega t) \qquad v = 0 \qquad w = 0 \tag{7-149}$$

displays the character of a distortional wave. The second kind, for which we assume that

$$u = 0 \qquad v = f_1(my + nz - \omega t) \qquad w = f_2(my + nz - \omega t) \tag{7-150}$$

where f_1 and f_2 are arbitrary functions, is similar to a dilatational wave for the first root of the velocity equation and is more like a distortional wave for the second root.

There is no sharp distinction between the dilatational and distortional waves if a disturbance is propagated in an aeolotropic medium, as was pointed out by Stoneley [73]. An explosion in such a medium will produce both P and S waves. Rayleigh and Love waves can be propagated over the surface of a "transversely isotropic" body, as was shown by Stoneley. This body or medium is determined by the condition (Love, Chap. 1, Ref. 34, p. 160) that the strain-energy function has the form

$$\begin{aligned} 2W = {} & A(e_{xx}^2 + e_{yy}^2) + Ce_{zz}^2 + 2F(e_{xx} + e_{yy})e_{zz} \\ & + 2(A - 2N)e_{xx}e_{yy} + L(e_{yz}^2 + e_{zx}^2) + Ne_{xy}^2 \end{aligned} \tag{7-151}$$

If we assume that the coefficients $A, C, \cdots, N$ are constant and if body forces are omitted, the equations of motion (1–7) take the form ($p_{xx} = \partial W/\partial e_{xx}, \cdots$)

$$\begin{aligned} \rho \frac{\partial^2 u}{\partial t^2} &= A \frac{\partial^2 u}{\partial x^2} + L \frac{\partial^2 u}{\partial z^2} + (F + L) \frac{\partial^2 w}{\partial x\, \partial z} \\ \rho \frac{\partial^2 w}{\partial t^2} &= L \frac{\partial^2 w}{\partial x^2} + C \frac{\partial^2 w}{\partial z^2} + (F + L) \frac{\partial^2 u}{\partial x\, \partial z} \end{aligned} \tag{7-152}$$

Investigations of media with other physical properties have been published recently. We shall mention here only some of the definitions of such media and the results concerning the propagation of disturbances in them. A theory of wave propagation in the so-called orthotropic media was given by Carrier [18].

Since there are layers of porous material through which seismic waves are in many cases propagated, attempts were made to account for the existence of empty holes or holes filled with liquid in a solid medium. The first case was considered by Fröhlich and Sack [32], and Mackenzie [47] expressed the elastic constants in terms of the relative density $\delta = \rho/\rho_0$, on assuming that the holes are spherical, ρ_0 being the density of the material without holes, and ρ that of an actual material, i.e., when the holes are

empty. On making use of these results, Satô [64] computed the velocities of P and S waves in terms of the porosity $(1 - \delta)$ for empty holes as well as for holes filled with liquid. These velocities have smaller values than those for the corresponding media.

Wave propagation in granular media is now the subject of numerous experimental as well as theoretical studies.

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APPENDIX A

METHOD OF STEEPEST DESCENT

This method, due to Debye, may be used to advantage in the evaluation of the formal integral solutions obtained in Chaps. 2, 3, and 4, particularly for determining critical distances where certain types of waves first appear. For discussions of the method see Eckart [2], Jeffreys and Jeffreys [4], and Sommerfeld (Chap. 1, Ref. 56, p. 99). Of interest is a paper by Honda and Nakamura (Chap. 4, Ref. 68) in which solutions obtained by the method of steepest descent and by use of the Sommerfeld contour are compared. According to the latter, the name "method of saddle points" or "pass method" is more appropriate.

The method of steepest descent is applied to the evaluation of integrals of the general form

$$I(x) = \int F(\zeta) e^{xf(\zeta)}\, d\zeta \tag{A-1}$$

where x is large, positive, and real, and $f(\zeta)$ is an analytic function. $F(\zeta)$ varies slowly compared with the exponential factor, and the integration follows a path in the complex ζ plane. We separate $f(\zeta)$ into real and imaginary parts

$$f(\zeta) = \rho + i\sigma \tag{A-2}$$

and note that, because of the nonperiodic factor exp $(x\rho)$, the largest values of the integrand occur where ρ is large. The basic idea of the method of steepest descent is to deform the path of integration, if possible, in such a way as to concentrate the large values of ρ in the shortest possible interval, in order that the remainder of the path may be neglected in an approximate evaluation. If we put $\zeta = k + i\tau$, both ρ and σ are functions of the variables k and τ. We can visualize the connection between these variables by representing ρ as an elevation over the ζ plane. Since $f(\zeta)$ is analytic,

$$\frac{\partial \rho}{\partial k} = \frac{\partial \sigma}{\partial \tau} \qquad \frac{\partial \rho}{\partial \tau} = -\frac{\partial \sigma}{\partial k} \tag{A-3}$$

An extremum or a saddle point occurs at points where

$$\frac{\partial \rho}{\partial k} = \frac{\partial \rho}{\partial \tau} = 0 \qquad \text{or} \qquad \frac{\partial \rho}{\partial s} = \frac{\partial \rho}{\partial n} = 0 \tag{A-4}$$

that is, where the derivatives in the tangential (s) and normal (n) directions to the curve $\rho(k, \tau) = \text{const}$ vanish. However, only saddle points are defined by Eqs. (A–4), since $\nabla^2\rho = 0$. Saddle points are also stationary points of σ and zeros of $f'(\zeta)$. There are at least two curves $\rho(k, \tau) = \text{const}$, and in this case the four sectors between the curves are alternately hills and valleys of the surface $\rho = \rho(k, \tau)$. In order to concentrate large values of ρ in short intervals and keep ρ as small as possible elsewhere, the path of integration must avoid the hills and keep to the valleys. If the contour crosses from one valley to another, it must do so through a saddle point, along a path of steepest descent where $|d\rho/ds|$ is as great as possible. At any point of the curve $\rho(k, \tau) = \text{const}$, the maximum variation of ρ occurs along the normal. Since by Eqs. (A–3) we have

$$\frac{\partial\rho}{\partial s} = \frac{\partial\sigma}{\partial n} \qquad \frac{\partial\rho}{\partial n} = -\frac{\partial\sigma}{\partial s} \tag{A–5}$$

$\sigma(k, \tau) = \text{const}$ represents a family of curves, orthogonal to $\rho(k, \tau) = \text{const}$, along which the largest changes in ρ occur. Thus a curve of the family $\sigma(k, \tau) = \text{const}$ represents a line of steepest descent, one such line occurring in each valley and terminating either at infinity or at a singular point.

Suppose now that it is possible to modify the path of integration in (A–1), without altering the value of the integral, in such a way as to pass from one valley to another through a saddle point ζ_0 and coincide with $\sigma(k, \tau) = \text{const}$, at least in the vicinity of ζ_0. It is obvious that the neighborhood of ζ_0 yields the largest contribution to the integral. Near ζ_0, $f(\zeta)$ can be expanded in the form

$$f(\zeta) = f(\zeta_0) + \tfrac{1}{2}(\zeta - \zeta_0)^2 f''(\zeta_0) + \cdots \tag{A 6}$$

where the second term is negative and real along the path. Put

$$\tfrac{1}{2}\kappa^2 = -\tfrac{1}{2}(\zeta - \zeta_0)^2 f''(\zeta_0) \tag{A–7}$$

and change the variable of integration in (A–1) to κ to obtain approximately

$$I = e^{xf(\zeta_0)} \int F(\zeta) e^{-\frac{1}{2}\kappa^2 x} \frac{d\zeta}{d\kappa}\, d\kappa \tag{A–8}$$

To find the factor $d\zeta/d\kappa$ we can write $\zeta - \zeta_0 = r \exp(i\chi)$, where r is small and real and χ is the angle which $\zeta - \zeta_0$ forms with the k axis. For ζ on the path in the neighborhood of ζ_0 we have, by Eq. (A–7), $\kappa^2 = -r^2 f''(\zeta_0) \exp(2i\chi)$. Since κ^2 is real and positive, the coefficient of $-r^2$ must be real and negative. Since $\arg [f''(\zeta_0) \exp(2i\chi)]$ must be $\pm\pi$ and $dr/d\zeta = \exp(-i\chi)$, we obtain

$$\kappa = \pm r\, |f''(\zeta_0)|^{\frac{1}{2}} = \pm(\zeta - \zeta_0) e^{-i\chi}\, |f''(\zeta_0)|^{\frac{1}{2}} \tag{A–9}$$

and

$$\frac{d\kappa}{d\zeta} = \pm e^{-i\chi}\, |f''(\zeta_0)|^{\frac{1}{2}} \tag{A–10}$$

To select the proper value of χ, in the range $(-\pi, \pi)$ two values of χ differing by π are possible. The condition $f'(\zeta_0) = 0$ is used to determine $\zeta_0 = k_0 + i\tau_0$; then $\tan \chi$ is the slope of the path $\sigma(k, \tau) = \sigma(k_0, \tau_0)$. If that value of χ is selected which makes r positive after passing through ζ_0, then the plus sign in Eq. (A–10) is taken. The first term in the asymptotic expansion of (A–8) can now be obtained (see, for example, Jeffreys and Jeffreys [4, p. 473]) by writing Eq. (A–8) in the form

$$I = \frac{e^{xf(\zeta_0)}}{|f''(\zeta_0)|^{\frac{1}{2}}} \int e^{-\frac{1}{2}\kappa^2 x} F(\zeta) e^{i\chi} \, d\kappa \tag{A–11}$$

The larger the value of x, the more closely the higher values of the integrand concentrate about the saddle point. For large values of x, the limits of κ may be extended to the range $\pm\infty$, and use can be made of Watson's lemma

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x\kappa^2} \varphi(\kappa) \, d\kappa \sim \sqrt{2\pi} \left(\frac{\varphi_0}{\sqrt{x}} + \cdots \right) \tag{A–12}$$

where φ_0 is the first term of the series expansion of $\varphi(\kappa)$. Thus

$$I \sim \frac{\sqrt{2\pi}\, e^{xf(\zeta_0)} F(\zeta_0) e^{i\chi}}{|xf''(\zeta_0)|^{\frac{1}{2}}} \tag{A–13}$$

If it is necessary to pass through two or more saddle points in the path of integration of (A–8), then each saddle point gives a contribution to the integral.

In Kelvin's method of stationary phase, integrals of the type (A–1) are evaluated by using paths through saddle points such that ρ, rather that σ, is constant. For this path the modulus of $\exp[xf(\zeta)]$ is constant while the phase varies. Since $f'(\zeta) = 0$ for both cases, we can write as before

$$\int F(\zeta) e^{xf(\zeta)} \, d\zeta \cong F(\zeta_0) e^{xf(\zeta_0)} \int e^{\frac{1}{2}xf''(\zeta_0)(\zeta-\zeta_0)^2} \, d\zeta \tag{A–14}$$

where the path is chosen such that the exponent in the integrand is purely imaginary and the limits are extended to $\pm\infty$. The integral on the right-hand side then reduces to a form which can be evaluated, giving

$$\frac{\sqrt{2\pi}\, F(\zeta_0) e^{xf(\zeta_0)}}{\sqrt{xf''(\zeta_0)/i}} e^{\pm i\pi/4} \qquad \text{according as } f'' \gtrless 0 \tag{A–15}$$

Although a mathematical proof of the method of stationary phase is available (Watson [7]), it is usually justified in terms of interferences of wave motion. In the vicinity of stationary values of $f(\zeta)$, phases are nearly the same, and the contributions are additive. At places where $f'(\zeta)$ does not vanish and is imaginary, the factor $\exp[xf'(\zeta)(\zeta - \zeta_0)]$ in the inte-

grand oscillates rapidly for large values of x, and the resultant contribution to the integral is small.

The two methods are nearly equivalent since the paths pass through the saddle points at an angle $\pi/4$ to each other and can be deformed each into the other, provided that contributions from any singularities crossed are taken into account. For a further discussion of the connection between the methods see Eckart [2].

If a pole of $F(\zeta)$ occurs indefinitely near a saddle point or lies on a path of steepest descent, special methods discussed by Ott [6] are necessary. Emde [3] discussed the case where $f''(\zeta_0) = 0$, $f'''(\zeta_0) \neq 0$, that is, where three valleys meet at the saddle point.

Application of the method of steepest descent to the approximate evaluation of integrals met in problems on wave propagation was discussed by Nakano (Chap. 2, Ref. 28), by Newlands (Chap. 2, Ref. 32), and Honda and Nakamura (Chap. 4, Ref. 68). Writing such an integral in a form slightly different from (A-1), namely,

$$I(x) = \int G(\zeta) e^{f(\zeta)}\, d\zeta \tag{A-16}$$

we have, for example,

$$f(\zeta) = i\omega t - i\zeta x - \nu_1 m - \nu_1' n \tag{A-17}$$

where $$\omega = s - ic \qquad \nu_1 = (\zeta^2 - k_{\alpha 1}^2)^{\frac{1}{2}} \qquad \nu_1' = (\zeta^2 - k_{\beta 1}^2)^{\frac{1}{2}} \tag{A-18}$$

and m and n are linear functions of z, h, and possibly of the depth H of a layer. The saddle point is at ζ_0, where

$$f'(\zeta) = -ix - \frac{m\zeta}{\sqrt{\zeta^2 - k_{\alpha 1}^2}} - \frac{n\zeta}{\sqrt{\zeta^2 - k_{\beta 1}^2}} = 0 \tag{A-19}$$

that is,

$$x = \frac{m\zeta_0}{\sqrt{k_{\alpha 1}^2 - \zeta_0^2}} + \frac{n\zeta_0}{\sqrt{k_{\beta 1}^2 - \zeta_0^2}} \tag{A-20}$$

It can be proved, if we put $\zeta = uk_{\alpha 1}$, that Eq. (A-20) has a single real root for the typical case $\alpha_1^2/\beta_1^2 = 3$. The saddle point ζ_0 is therefore on the line of branch points. Now $\sigma(k, \tau) = \text{const}$ represents a line of steepest descent, and by Eqs. (A-2) and (A-17), taking into account the change in notations, we have

$$\sigma(k, \tau) = \text{const} = \text{Im}\{i\zeta x + im(k_{\alpha 1}^2 - \zeta^2)^{\frac{1}{2}} + in(k_{\beta 1}^2 - \zeta^2)^{\frac{1}{2}}\} \tag{A-21}$$

The time factor being omitted again, this constant σ is determined by Eq. (A-20), and $\zeta = \zeta_0 = u_0(s - ic)/\alpha_1$. For $\alpha_1^2/\beta_1^2 = 3$ we obtain

$$\text{const} = \frac{s}{\alpha_1}[u_0 x + m(1 - u_0^2)^{\frac{1}{2}} + n(3 - u_0^2)^{\frac{1}{2}}] \tag{A-22}$$

Substituting $\zeta = k + i\tau$ in Eq. (A–21) and neglecting $k_{\alpha 1}^2/\zeta^2$ and $k_{\beta 1}^2/\zeta^2$, respectively, we obtain

$$kx \pm (m + n)\tau = 0 \tag{A–23}$$

Thus the portions of the curve (A–21) which correspond to large $|\zeta|$ are straight lines in the fourth and third quadrants, respectively.

By Eq. (A–6), if ζ varies on the curve $\sigma = \text{const}$ in the neighborhood of ζ_0, where ρ has its largest value, we obtain

$$0 > \rho - \rho_0 = \frac{1}{2!}(\zeta - \zeta_0)^2 f''(\zeta_0) + \frac{1}{3!}(\zeta - \zeta_0)^3 f'''(\zeta_0) + \cdots \tag{A–24}$$

From this condition it may be derived [see Newlands (Chap. 2, Ref. 32), for example] that close to the saddle point the curve $\sigma(k, \tau) = \text{const}$ coincides with a parabola. This approximation suggests that the curve of steepest descent is of roughly parabolic shape.

If the original contour, usually the real axis, can be distorted to the path of steepest descent by arcs which contribute nothing, then the major contribution occurs in the vicinity of the saddle point and is given by (A–13). It may be necessary to distort the line of steepest descent to avoid crossing a branch line or a pole by inserting loops. As was the case with the Sommerfeld contour in Sec. 2–5, these loops correspond to definite wave types.

It is not necessary to know the detailed form of the line of steepest descent to determine which loops occur; only the number and location of intersections with the line of branch points are needed. For given values of m, n, α, and β, the numbers and location of intersections depend on x, and the critical distances may be derived where loops (and the corresponding pulses) first appear.

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APPENDIX B

RAYLEIGH'S PRINCIPLE

Rayleigh's principle has been used for approximate computation of frequencies of vibrating systems with much success. Its fundamental idea is derived from the theory of small oscillations of a conservative system [Rayleigh (Chap. 1, Ref. 45)].

If the Lagrangian coordinates q_i of a system with a finite number of degrees of freedom are chosen in such a way that the values $q_s = 0$ determine a configuration of stable equilibrium, the kinetic and potential energies of small oscillations about it can be expressed in the form

$$\begin{aligned} T &= \tfrac{1}{2}a_{11}q_1'^2 + \tfrac{1}{2}a_{22}q_2'^2 + \cdots + a_{12}q_1'q_2' + \cdots \\ W &= \tfrac{1}{2}b_{11}q_1^2 + \tfrac{1}{2}b_{22}q_2^2 + \cdots + b_{12}q_1q_2 + \cdots \end{aligned} \tag{B-1}$$

where a_{rs}, b_{rs} are constants. Since a homogeneous quadratic function can be reduced by a linear transformation to the sum of squares, new variables $\hat{q}_s$ may be introduced to yield the equations

$$\begin{aligned} T &= \tfrac{1}{2}a_1\hat{q}_1'^2 + \tfrac{1}{2}a_2\hat{q}_2'^2 + \cdots \\ W &= \tfrac{1}{2}b_1\hat{q}_1^2 + \tfrac{1}{2}b_2\hat{q}_2^2 + \cdots \end{aligned} \tag{B-2}$$

where a_s and b_s are positive constants.† The variables $\hat{q}_s$ are the normal coordinates, and the equations of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \hat{q}_s'}\right) - \frac{\partial T}{\partial \hat{q}_s} = -\frac{\partial W}{\partial \hat{q}_s} \tag{B-3}$$

take the form

$$a_s\hat{q}_s'' + b_s\hat{q}_s = 0 \tag{B-4}$$

The solutions of Eq. (B-4)

$$\hat{q}_s = A_s \cos(\omega_s t - \epsilon_s) \tag{B-5}$$

show that the periods of these "natural vibrations" or "normal modes of vibration" are given by

$$\omega_s = \frac{2\pi}{\tau_s} = \sqrt{\frac{b_s}{a_s}} \tag{B-6}$$

† A negative value of a_i, b_i corresponds to instability with respect to the coordinate $\hat{q}_s$.

where τ_i is the period. Thus the amplitudes of the normal modes, as well as their frequencies, are independent of one another. Assume that by a suitable constraint the system has only one degree of freedom. Then, if we put

$$q_i = B_i q \tag{B-7}$$

the expressions for T and W become

$$\begin{aligned} T &= \tfrac{1}{2}[a_1 B_1^2 + a_2 B_2^2 + \cdots]q'^2 \\ W &= \tfrac{1}{2}[b_1 B_1^2 + b_2 B_2^2 + \cdots]q^2 \end{aligned} \tag{B-8}$$

Let q vary as $\cos \omega t$ or $\sin \omega t$. If we now take into account that the mean values of the kinetic and potential energies in a simple harmonic motion are equal and that they are

$$T_m = \omega^2 T_0 \cdot \frac{1}{\tau} \int_0^\tau \sin^2 \omega t \, dt \qquad W_m = W_0 \cdot \frac{1}{\tau} \int_0^\tau \cos^2 \omega t \, dt \tag{B-9}$$

where T_0 and W_0 are the factors in brackets in Eqs. (B–8), the last factors being equal to $\frac{1}{2}$ each, we obtain

$$\omega^2 = \frac{b_1 B_1^2 + b_2 B_2^2 + \cdots}{a_1 B_1^2 + a_2 B_2^2 + \cdots} \tag{B-10}$$

On assuming that $b_l/a_l = k_l$ is the smallest or the greatest ratio and using the inequalities $b_i \lessgtr a_i k_l$, respectively, for $i \neq l$, one can immediately see that the limits $\omega_{\min}^2$ and $\omega_{\max}^2$ exist. However, there will be no upper boundary for frequencies in continuous systems (e.g., plates or strings).

By Eq. (B–10) the frequency of a constrained vibration depends on amplitudes of all normal modes, and it cannot be less than the fundamental frequency (or larger than the greatest, in case of a finite number of degrees of freedom). To find an approximate value, say of the least frequency, one has to compute, therefore, the frequency for vibrations of the system constrained to have certain amplitudes B_i, and this is done by equating the mean values of the kinetic and potential energies. The more accurately are given B_i "by observation or intuition," the closer will ω be to the true minimum. Temple [5] has justified the application of this method to continuous systems. For these systems the theory requires the use of integral equations.

The application of the Rayleigh principle to surface-wave propagation was made by Jeffreys [2], and his example will now be discussed. It concerns the approximate determination of the dispersion of Love waves in a system composed of a homogeneous layer ($-H \leq z < 0$) overlying a heterogeneous medium ($0 \leq z < \infty$) (see Sec. 7–3). It is assumed that the transverse wave is given by

$$v = V \cos kx \sin \omega t \qquad \omega = kc \tag{B-11}$$

where the amplitude V is a function of z to be found. If we write the mean value of kinetic energy with regard to x and t,

$$T_m = \frac{1}{2}\int_{-H}^{\infty}\frac{1}{\tau_x}\int_0^{\tau_x}\frac{1}{\tau_t}\int_0^{\tau_t}\rho\left(\frac{\partial v}{\partial t}\right)^2 dz\,dx\,dt \tag{B-12}$$

it is easy to see that it becomes

$$8T_m = \omega^2\int_{-H}^{\infty}\rho V^2\,dz \tag{B-13}$$

since, for example, $\tau_x = \pi/k$ and

$$\frac{1}{\tau_x}\int_0^{\tau_x}\cos^2 kx\,dx = \frac{1}{\pi}\int_0^{\pi}\cos^2\xi\,d\xi = \frac{1}{2} \tag{B-14}$$

The strain energy per unit volume [Love (Chap. 1, Ref. 34, p. 102)] is reduced to a simple form for Love waves, since only two components of strain determined by Eq. (B-11) differ from zero. They are (see Sec. 1–1)

$$e_{yz} = \frac{\partial v}{\partial z} \qquad e_{xy} = \frac{\partial v}{\partial x}$$

and, therefore,

$$\begin{aligned} 2W &= \mu(e_{yz}^2 + e_{xy}^2) \\ &= \mu\left[\left(\frac{dV}{dz}\right)^2\cos^2 kx\sin^2\omega t + k^2V^2\sin^2 kx\sin^2\omega t\right] \end{aligned} \tag{B-15}$$

Hence the mean value with regard to x and t is given by

$$8W_m = \int_{-H}^{\infty}\mu\left[\left(\frac{dV}{dz}\right)^2 + k^2V^2\right]dz \tag{B-16}$$

As a trial form for V, Jeffreys takes that corresponding to two uniform media (see Sec. 4–5)

$$V = A\sec(kH\hat{\gamma})\cos[k(H+z)\hat{\gamma}] \qquad \text{for } -H \le z < 0 \tag{B-17}$$

$$V = Ae^{-k\gamma' z} \qquad \text{for } 0 < z < \infty \tag{B-18}$$

where

$$\hat{\gamma}^2 = \frac{c^2}{\beta^2} - 1 \qquad \gamma'^2 = 1 - \frac{c^2}{\beta'^2}$$

These trial forms satisfy the boundary conditions but (B-18) does not satisfy the equation of motion for the bottom layer. Assume that the rigidity in the lower layer varies as $\mu'(1 + z/s)^2$, μ' and s being constants.

Then the integrals (B–12) and (B–16) take the form

$$8T_m = \frac{1}{2}\omega^2 A^2\left\{\rho \sec^2(kH\hat{\gamma})\left[H + \frac{\sin(2kH\hat{\gamma})}{2k\hat{\gamma}}\right] + \frac{\rho'}{k\gamma'}\right\} \tag{B–19}$$

$$8W_m = \frac{1}{2}\mu A^2 k^2 \sec^2(kH\hat{\gamma})\left\{(1+\hat{\gamma}^2)H + \frac{1-\hat{\gamma}^2}{2k\hat{\gamma}}\sin(2kH\hat{\gamma})\right\}$$
$$+ \frac{1}{2}\mu' A^2 k^2(1+\gamma'^2)\left\{\frac{1}{k\gamma'} + \frac{1}{sk^2\gamma'^2} + \frac{1}{4s^2k^3\gamma'^3}\right\} \tag{B–20}$$

As in the case of Eq. (B–10), we obtain an equation for ω or c by equating the mean values of the kinetic and potential energies. If we use the velocities of shear waves given by $\beta^2 = \mu/\rho$ and $\beta'^2 = \mu'/\rho'$, this equation becomes

$$\mu(1+\hat{\gamma}^2)\sec^2(kH\hat{\gamma})\left[H + \frac{\sin(2kH\hat{\gamma})}{2k\hat{\gamma}}\right] + \frac{\mu'}{k\gamma'}(1-\gamma'^2)$$
$$= \mu\sec^2(kH\hat{\gamma})\left\{(1+\hat{\gamma}^2)H + \frac{1-\hat{\gamma}^2}{2k\hat{\gamma}}\sin(2kH\hat{\gamma})\right\}$$
$$+ \mu'\frac{1+\gamma'^2}{k\gamma'}\left(1 + \frac{1}{sk\gamma'} + \frac{1}{4s^2k^2\gamma'^2}\right) \tag{B–21}$$

In this equation the variables $\hat{\gamma}$ and γ' involve c. After a simplification of Eq. (B–21) we obtain the approximate form of the period equation as follows:

$$\tan(kH\hat{\gamma}) = \frac{\mu'\gamma'}{\mu\hat{\gamma}}\left\{1 + \frac{1+\gamma'^2}{2sk\gamma'^3}\left(1 + \frac{1}{4sk\gamma'}\right)\right\} \tag{B–22}$$

The exact period equation in this problem was also obtained by Jeffreys [1]. An asymptotic expansion of the exact equation agrees with Eq. (B–22) as far as the terms in $1/s$.

Jeffreys also applied this method to the problem of Rayleigh waves in layered media. In this case an equation for c is obtained as a function of the wave number k, and it may be proved that the value of c is unaltered to the first order of small increments in the expressions for displacements, if the latter satisfy the equations of motion and the condition of continuity of stress at the interfaces.

A further development of Rayleigh's method was given by Ritz [4]. The Ritz method yields a better approximation for the first mode as well as for the frequencies of higher modes of vibrations. A simple presentation of this method is given by Timoshenko (Chap. 6, Ref. 85).

As is well known, an upper bound for a normal mode of an oscillating system can be determined by the Ritz-Rayleigh method. It is important, however, to have a method to estimate the error and also a lower bound. Such a method was developed by Temple [7] and Katô [3].

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