UNIVERSAL
LIBRARY

OU_150930

UNIVERSAL
LIBRARY

OSMANIA UNIVERSITY LIBRARY

Call No. 510 | R49 T Accession No. 43596.

Author Rice, Harold S. & Knight, Raymond H

Title Technical mathematics with calculu

This book should be returned on or before the date last marked below.

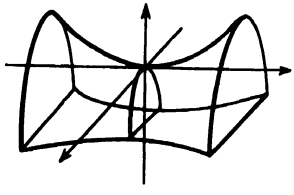
TECHNICAL MATHEMATICS WITH CALCULUS

Technical Mathematics

WITH CALCULUS

HAROLD S. RICE

RAYMOND M. KNIGHT



McGRAW-HILL BOOK COMPANY, INC. *1957*

NEW YORK TORONTO LONDON

TECHNICAL MATHEMATICS WITH CALCULUS

Copyright © 1957 by McGraw-Hill Book Company, Inc.
Copyright, 1954, by the McGraw-Hill Book Company, Inc. Printed in the United States of America. All rights reserved. This book, or parts thereof, may not be reproduced in any form without permission of the publishers.

Library of Congress Catalog Card Number: 57-10918

VIII

52189

THE MAPLE PRESS COMPANY, YORK, PA.

Preface

Every textbook should have a definable purpose and a definable point of view. This particular text is intended to help students of engineering learn how to use the techniques of mathematics as a professional tool. The point of view adopted in this text, therefore, is that mathematics is an engineer's tool. The authors are quite well aware that there are other values of mathematics. They are aware that mathematics is a philosophy; it is a metaphysic; it is an art; and it is logic. However, these values of mathematics, important and worthwhile as they may be, are definitely subordinate to the central theme of this text, namely, that mathematics is an engineer's tool.

The authors are also perfectly well aware that there are many functions and many techniques which are not even hinted at in this volume. They have been omitted intentionally because experience has shown that these functions and techniques are not ordinarily encountered by the beginning engineer and engineering technician. The authors feel very strongly that the classroom time and textbook space can much more properly be devoted to the building up of skills in the use of more elementary techniques rather than to discussing the more theoretical aspects of mathematics, which will probably not be used until the student has developed considerable experience in his chosen field.

The text is the outgrowth of years of classwork with students at Wentworth Institute. These men are high school graduates who have had at least a year or two of algebra. While they expect to enter industry and work in intimate accord with artisans, they must also be at ease in the more precise atmosphere of engineering. Therefore, the authors have given a traditional and thorough treatment of basic algebra and trigonometry and have stressed the applications of these principles to a wide range of specific engineering situations. The calculus section is limited to those techniques of calculus with which the engineering student and the beginning young engineer are likely to be concerned. The more profound and subtle theory and mathematical

thinking involved in the subject of limits, for example, have been intentionally left out.

The topics have been broken down into eight teaching units and three appendixes, as follows:

* Unit I	Slide Rule and Review of Arithmetic and Geometry
* Unit II	Basic Algebra
Unit III	Advanced Algebra and Logarithms
Unit IV	Introduction to Analytical Trigonometry
* Unit V	Numerical Trigonometry of the Right Triangle
Unit VI	Oblique Triangles and Applications of Numerical Trigonometry
Unit VII	Analytical Trigonometry
Unit VIII	Introduction to Calculus
* Appendix A	Computation Aids and Approximations
* Appendix B	Formulas of Geometry and Mensuration
* Appendix C	Solution of Higher-degree Equations

The starred units can be used in any order. Unit I is a mature review of arithmetic and geometry built around a study of the slide rule.

Unit II is a rather thorough review of basic algebra. As a review, it has been handled somewhat differently from a unit intended to introduce algebra.

Unit III continues where Unit II leaves off. This section is not a review and assumes that the student is studying these topics for the first time.

Unit IV introduces analytical trigonometry from the point of view of the electricity student and the more analytically inclined mechanics student.

Unit V is an elementary treatment of numerical trigonometry designed for the beginning student. This unit may be assigned on the opening day of school if desired.

Unit VI continues the study of numerical trigonometry and emphasizes the oblique triangle and trigonometric applications. The section of Chapter 14 on the use of logarithms in computation may be assigned in preparation for Unit VI without the earlier part of Unit V.

Unit VII is intended for electricity students, but Chapter 21 would apply as well for mechanics and building construction students.

Unit VIII is an introduction to the techniques of elementary calculus. The intention of the authors is here to introduce those techniques of calculus which will be useful to the engineering student in the pursuit of his major subjects. It is written for the student engineer and the student engineering technician. It definitely is not written for the physics major.

The Appendixes are used at appropriate times throughout the entire mathematics course.

The authors, in their courses, devote about a year and a half to the material in this book; but in courses that give a greater proportion of time to shopwork, the material may occupy a full two years.

The authors have been assisted in the preparation of the manuscript for this book by many persons.

C. W. Tudbury, now retired, was in charge of the mathematics department at Wentworth Institute when the manuscript was started. His successor, Clarence Paddock, maintained an equal interest until his retirement. Our present colleagues, J. A. Macdonald, A. M. Huyck, and R. C. Wheeler, have been particularly helpful. The following men have reviewed the entire manuscript except that portion on calculus and offered extremely valuable suggestions: Gordon Duvall of Ohio Mechanics Institute; D. H. Craighead of Ryerson Institute of Technology; Chauncey R. Kay of The Wyomissing Polytechnic Institute; LeRoy N. Young of Long Island Agricultural and Technical Institute; Professor T. J. Higgins of the University of Wisconsin.

To R. C. Pickett, president of Pickett and Eckel, Inc., the authors owe permission to photograph one of that company's slide rules. Lastly the authors wish to acknowledge the fine work of Janet Fiske, Mildred Meyer, George Morton, and James Nasson, who typed the manuscript.

HAROLD S. RICE
RAYMOND M. KNIGHT

Contents

Preface v

UNIT ONE **SLIDE RULE AND REVIEW OF ARITHMETIC AND GEOMETRY**

Chapter 1 TREATMENT OF MEASURED DATA 1

Measured data · Rounding off numbers · Scientific notation

Chapter 2 THE SLIDE RULE 5

Scope of the slide rule · Description of the slide rule · Location of numbers on the scales; accuracy of the slide rule · Multiplication · Division · Location of decimal point in scientific notation · Combined multiplication and division · Proportion · Squares and square roots · Cubes and cube roots · Circumference and area of a circle · General suggestions for slide-rule operation

Chapter 3 TABLES AND INTERPOLATION 24

Extending the range of the tables · Interpolation between two points · Other helpful devices

Chapter 4 APPLICATIONS IN GEOMETRY 32

UNIT TWO **BASIC ALGEBRA**

Chapter 5 INTRODUCTION TO ALGEBRA 51

Explicit and literal numbers Absolute values · Addition · Subtraction · Multiplication · Division · Signs of operation · Algebraic expression · Some fundamental rules of exponents · Signs of grouping · Similar terms · Multiplying by a monomial · Multiplying one polynomial by another polynomial · Removing parentheses used to indicate multiplication · Dividing one monomial by

another monomial · Division of a polynomial by a polynomial · Special products or expansions · Factoring · Properties of fractions · Least common multiple · Addition and subtraction of fractions · Multiplying two or more fractions · Dividing one fraction by another · Complex fractions

Chapter 6 LINEAR EQUATIONS IN ONE UNKNOWN 81

Equations · Operations on equations · Degree of an equation · To solve a linear equation · Formulas · Mathematical operations with dimensional units · Suggestions for the solution of applied problems involving linear equations

UNIT THREE **ANALYTIC GEOMETRY, MORE ADVANCED ALGEBRA, AND LOGARITHMS**

Chapter 7 FUNCTIONS AND GRAPHS 103

Constants and variables · Functions · Functional notation · Rectangular coordinates · Graph of a function · Functions not defined by equations

Chapter 8 SYSTEMS OF LINEAR EQUATIONS 127

Graphical solution of a system of two equations · Algebraic methods of solution · Elimination by addition or subtraction · Elimination by substitution · Solution by elimination by comparison · Literal linear equations · Equations linear in the reciprocals of the unknowns · Linear equations in more than two unknowns · Graphical solution of simultaneous linear equations ·

Chapter 9 EXPONENTS AND RADICALS 155

Laws of positive integral exponents · Roots and radicals · Principal roots · Rational and irrational numbers · Fractional exponents · Zero exponent · Negative exponents · Laws of radicals · Rationalizing the denominator in a radical · Reduction of the order of a radical · Addition and subtraction of radicals · Multiplication and division of radicals · Division by an irrational polynomial · Imaginary numbers · Operations with imaginary numbers · Complex numbers

Chapter 10 QUADRATIC EQUATIONS IN ONE UNKNOWN 180

Quadratic equations · Solution of incomplete quadratics · Solution of complete or affected quadratics · Solution by plotting · Solution by factoring · Solution by completing the square · Solution by the formula · Short graphical method · Literal quadratic equations ·

Precautions regarding roots · Vanishing roots · Extraneous roots · Checking · Summary of procedure for solving quadratic equations · Discriminant · Axis of symmetry—extreme value · Equations of the quadratic type · Applications leading to quadratic equations containing one unknown · Equations involving radicals · Applied problems in maxima and minima

Chapter 11 SIMULTANEOUS QUADRATIC EQUATIONS 210

Characteristic types of quadratic curves · Graphical solution · Algebraic solutions of quadratic systems · Solution of systems consisting of one linear and one quadratic equation · Equations of the form $ax^2 + by^2 = c$ · Elimination of constants · Reduction to simpler systems

Chapter 12 RATIO, PROPORTION, VARIATION 227

Quantities of the same kind · Proportions · Variation · Dimensionality of the proportionality constant · Applications

Chapter 13 THE BINOMIAL THEOREM—PROGRESSIONS 240

The binomial theorem · Expansion of a positive integral power of a binomial · The general term of the binomial expansion · Computation of powers and roots · Arithmetic progression · The nth term of an AP · The sum of the first n terms of an AP · Arithmetic means · Applications of arithmetic progressions · Geometric progression · The nth term of a GP · The sum of the first n terms of a GP · Geometric means · Infinite geometric progressions · Applications of geometric progressions

Chapter 14 LOGARITHMS 261

Definition of a logarithm · Graphical representation · Formation of a table of logarithms · Use of exponents in determining logarithms · Properties of logarithms · Multiplication · Division · Powers · Roots · Logarithm of unity · Reciprocal · Change of base · Systems of logarithms · Graphical derivation of table · Characteristics and mantissa · The antilogarithm of an expression · Role of logarithms in rough approximation · Using a table of logarithms · Interpolation · Computation using logarithms · Multiplication by the use of logarithms · Division by the use of logarithms · Combined multiplication and division · Cologarithms · Raising to a power by the use of logarithms · Extracting roots by the use of logarithms · Logarithmic computation of expressions involving addition and subtraction · Accuracy in logarithmic computations · Exponential equations · Aids to logarithmic computation · Relation of logarithms to the slide rule · Logarithmic computations on the slide rule

Chapter 15 MATHEMATICS OF INVESTMENT 303

Principal and interest · Simple interest · Compound interest · General formula · Present value · Nominal vs. effective rate · Use of binomial theorem · Annuities · Amount of annuity · Present value of an annuity · Time payments · Amortization and sinking funds · Perpetuities · Capitalized cost · Depreciation

Chapter 16 EXPONENTIAL FUNCTIONS—RATE OF GROWTH 326

Power function vs. exponential function · Compound-interest law · Significance of e · General formula · Natural logarithms · Use of the table · Relation between common and natural logarithms · Equivalent expressions

UNIT FOUR **INTRODUCTION TO ANALYTICAL TRIGONOMETRY**

Chapter 17 VECTORS AND TRIGONOMETRY 339

Vector and scalar quantities · Vector representation · Vectors and rectangular coordinates · Positive and negative angles · Units of angular measure · Angles of any magnitude · Addition and subtraction of angles · The quadrants · The coordinates of P in the four quadrants · Polar and rectangular coordinates · Components and projections · The trigonometric functions of first-quadrant angles · Functions of complementary angles in the first quadrant · The functions of 30, 45, and 60° · The zero divisor · Signed and unsigned numbers · The signs of the functions · The functions of zero degrees · The functions of 90° · The functions of 180° · The functions of 270 and 360° · Reducing the functions of any angle to functions of a first-quadrant angle · Variations in the functions · Inverse trigonometric functions · Some basic relations between the functions · Trigonometric identities

UNIT FIVE **NUMERICAL TRIGONOMETRY OF THE RIGHT TRIANGLE**

Chapter 18 THE RIGHT TRIANGLE 369

Angles · Arithmetical operations with degrees, minutes, and seconds · Unique determination of triangles · Solution of triangles · Trigonometric functions · Trigonometric tables · Use of table 18·1 · Trigonometric functions defined · The cofunctions · Reciprocal functions · Use of five-place trigonometric tables · Suggestions for solving right triangles · Interpolation in the five-place tables · Isosceles triangles · Regular polygons · Logarithmic solution of right triangles · Solution of oblique triangles without special formulas · Solution of an

oblique triangle when three sides are given · Solution of an oblique triangle when two angles and a side are given · Solution of an oblique triangle when two sides and an angle are given · The slide-rule solution of right triangles · The slide-rule solution of oblique triangles · The solution of right triangles involving small angles

UNIT SIX **OBLIQUE TRIANGLES AND APPLICATIONS OF NUMERICAL TRIGONOMETRY**

Chapter 19 OBLIQUE TRIANGLES 415

The sine law · Derivation of the sine law · The cosine law · Derivation of the cosine law · The tangent law · Derivation of the tangent law · The half-angle laws · Derivation of the half-angle laws · Area of triangles · The ratio of sin ϕ to ϕ

Chapter 20 TRIGONOMETRIC APPLICATIONS AND REVIEW 444

Plane surveying · Measurement of angles · Mapping · Latitudes and departures · Azimuth angles · Interior angles · Calculation of areas · Measurement of distances in a vertical plane · Surveying around obstructions · Applications to machine- and tool-design problems · Solid figures

UNIT SEVEN **ANALYTICAL TRIGONOMETRY**

Chapter 21 VECTORS 471

Traverse courses as vectors · Vector addition · Polar and rectangular coordinates · Topographic and polar diagrams · Vector subtraction · The product of a vector and a scalar quantity · Velocity as a vector · Forces as vectors · Resolution of forces · Resolution of general vectors · The product of two vectors · Reciprocal relations

Chapter 22 TRIGONOMETRIC FORMULAS, IDENTITIES, AND EQUATIONS 501

Functions of the sum of two angles · Functions of the difference between two angles · Functions of twice an angle · Functions of half an angle · Trigonometric identities · Trigonometric equations

Chapter 23 GRAPHS OF THE TRIGONOMETRIC FUNCTIONS 514

Some periodic functions · Sine and cosine graphs · Average ordinates of the curves of sin ϕ and $\sin^2 \phi$ · Plotting the sine and cosine functions by geometric methods · Rotating vectors · Adding sine waves of the same

frequency · Amplitude and phase relationships · The equivalence of graphical and algebraic methods · Adding harmonically related sine functions · Simple harmonic motion · Simple harmonic motion defined · Equations for coordinates of P · Phase · Parametric equations · Parametric equations for a sine curve · Lissajous figures

Chapter 24 COMPLEX NOTATION AND VECTOR ALGEBRA 542

The reference vectors · Addition of the reference vectors · Vector addition · Vector subtraction · Vector multiplication · Vector division · An alternative vector expression · The nth power of a vector · The nth roots of a vector · Demoivre's theorem · Euler's equation · The sine and cosine series · Rotating vectors · Some interesting side lights on Euler's equation · The hyperbolic sine and cosine

UNIT EIGHT **AN INTRODUCTION TO CALCULUS**

Chapter 25 GRAPHICAL METHODS 575

Functions · Dependence · Rates of change · Area under a straight-line graph · Calculation of work from a force-distance graph · Calculation of distance from a speed-time graph · Slope of nonlinear functions · Tangents defined · Average and instantaneous slopes and rates · The derivative · Approximate increments · Differential notation · Maxima and minima · Points of inflection · Derived curves · The area under a nonlinear curve · Summary

Chapter 26 DIFFERENTIATION 621

Average slopes and rates found by algebraic methods · Limits · General rule for differentiating a power function · The derivative of a sum or difference · The derivative of $y = u^n$ when u is a function of x · The derivative of the function of a function · Maxima and minima · The derivative of $\sin \theta$ and $\cos \phi$ · The derivative of $\sin u$ where u is a function of θ · The derivative of $\cos u$ where u is a function of θ · Differentiating the function $y = e^x$ · The derivative of e^u where u is a function of x · The derivative of $\ln x$ · The derivative of $\ln u$ where u is a function of x · The derivative of ka^u where a is any positive constant and u is a function of x · General proof for $dy/dx = knx^{n-1}$ · The derivative of a product and a quotient · The derivative of $\tan u$ · Flexion · Repeated differentiation · Derived curves

Chapter 27 INTEGRATION 674

Introduction to integration · The indefinite integral · The rule for $\int kx^n$ · The multiplied constant in integration · Evaluating the constant of integration · The importance of the differential du · Uniformly

number is revealed by a glance at the exponent (see Table 14·4, page 271).

Several other advantages in this notation will become apparent. Space is saved, a particularly important point in tabulating data. The labor of counting figures to the right or left of the decimal point —a labor attended by risk of error—is eliminated. The accuracy with which a quantity is known is indicated by the number of figures to the right of the decimal point. For example, when we consider the number 72,000, we cannot tell whether there are two, three, four, or five significant figures. No uncertainty exists when we write 7.2×10^4, 7.20×10^4, 7.200×10^4, or 7.2000×10^4.

The ease of dealing with large and small quantities in this manner is illustrated by the following problem.

Simplify the expression

$$\frac{400{,}000 \times 8{,}000{,}000 \times 0.0045}{60{,}000 \times 0.025 \times 100} = \frac{4 \times 10^5 \times 8 \times 10^6 \times 4.5 \times 10^{-3}}{6 \times 10^4 \times 2.5 \times 10^{-2} \times 10^2}$$

$$= \frac{4 \times 8 \times 4.5}{6 \times 2.5} \times 10^{(5+6-3)-(4-2+2)} = 9.6 \times 10^4 = 96{,}000$$

There are two instances in which we depart from the rule of expressing a quantity as a number between 1 and 10 times a suitable power of 10. If we were to extract the square root of 2.5×10^{-7}, we should write this as 25×10^{-8} in order to make the exponent of 10 divisible by the index of the root. The square root is readily seen to be 5×10^{-4}. Also, when quantities are to be added and subtracted, they must have the same exponents. Thus $4 \times 10^{-7} + 7 \times 10^{-5} = 4 \times 10^{-7} + 700 \times 10^{-7} = 704 \times 10^{-7} = 7.04 \times 10^{-5}$.

EXERCISE

1. Translate into ordinary notation: 5.18×10^6; 3.76×10^{-4}; 7.5×10^{-8}; 4.375×10^2.
2. Translate into scientific notation:

   1 year = 31,500,000 sec (approx)
   1 light-second = 186,000 miles
   Wavelength of blue light = 0.000047 cm

3. $\dfrac{5 \times 10^7 \times 9 \times 10^3 \times 400}{4.8 \times 10^4} = ?$

4. $\dfrac{6 \times 10^2 \times 15 \times 10^4 \times 4 \times 10^{-1}}{8 \times 10^3 \times 25 \times 10^5} = ?$

5. $\dfrac{280{,}000 \times 16{,}000 \times 0.009}{2{,}100 \times 2{,}400{,}000 \times 0.04} = ?$

6. $\dfrac{0.12 \times 5{,}000 \times 33{,}000}{550 \times 18 \times 0.0002} = ?$

7. $\dfrac{1.2 \times 10^{5} \times 9 \times 10^{-2}}{16 \times 10^{3} \times 1.5 \times 10^{-4}} - 20{,}000 + \dfrac{7 \times 10^{6}}{2 \times 10^{2}} = ?$

Chapter 2 THE SLIDE RULE

In the last chapter we mentioned four of the most important aids to computation—slide rule, formulas, tables, and graph paper. In this chapter we shall obtain practice in the use of the first of these.

2·1 Scope of the Slide Rule. The student should have a clear idea of what can and what cannot be done on a slide rule. The slide rule is an instrument used for processes of multiplication, division, proportion, and calculation of simple powers and roots. Further discussion of the slide rule will be found in Secs. 14·31, 14·32, and 18·21 to 18·23, but this chapter will deal only with the processes named above. Addition and subtraction cannot be performed on the slide rule. One of the most frequently asked questions is, "Is slide-rule computation good enough?" The answer depends upon the nature of the problem. Many experimental data are accurate to no more than three significant figures. In such cases slide-rule computation is sufficiently accurate. Even in problems requiring more precise methods, the slide rule is accurate enough to detect gross errors and to estimate the order of magnitude of the answer.

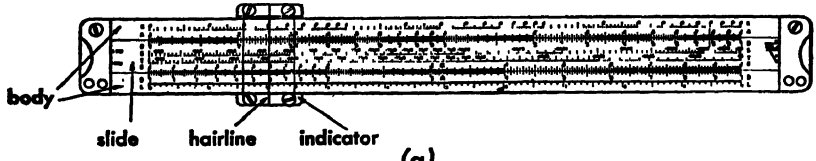
The slide rule will not think for you. Like an automobile, it will take you to your destination if you provide the proper direction.

There are many types of slide rule, but our discussion will be limited to the general technique of operation common to most 10-in. slide rules.

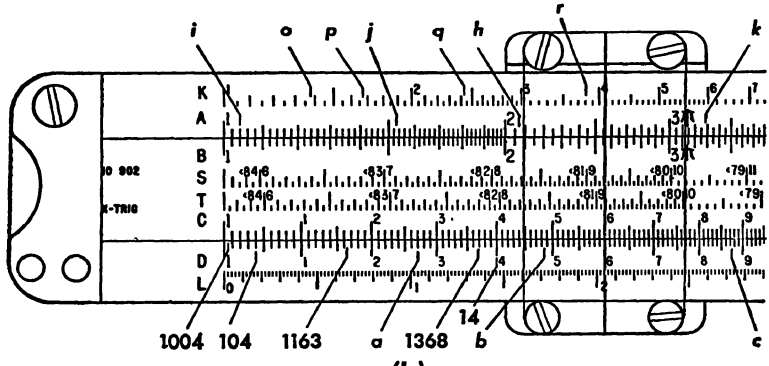
2·2 Description of the Slide Rule. The slide rule consists of three parts (see Fig. 2·1): the *slide*, or central sliding part; the *body*, or the upper and lower bars between which the slide operates; and the *indicator*, which is the movable glass plate marked with a hairline.

The mark associated with the numeral 1 at an end of a scale is called the *index* of the scale. Two positions on two different scales are said to be *opposite* if the hairline can be made to cover both simultaneously without moving the slide.

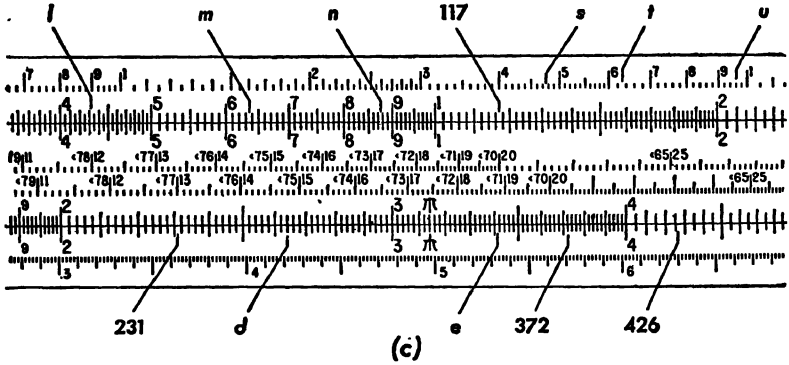
In order to perform the operations mentioned in Sec. 2·1, we need use only the C, D, A, B, and K scales. The C and D scales are identical. The A and B scales are identical with each other, and the distance


body
slide
hairline
indicator

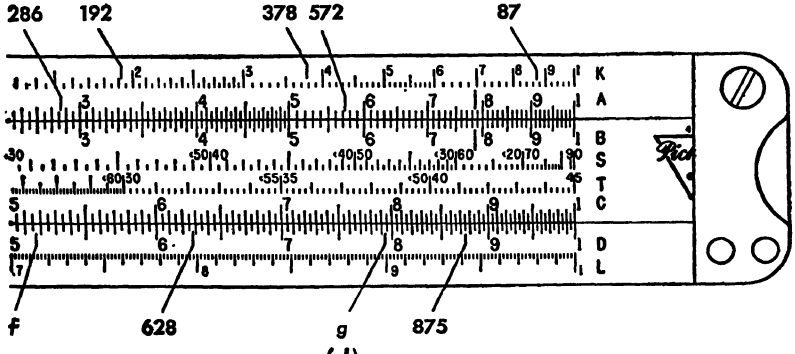
(a)


i
o
p
j
q
h
r
k
K
A
B
S
T
C
D
L
1004
104
1163
a
1368
b
14
c

(b)


l
m
n
117
s
t
u
231
d
e
372
426

(c)


286
192
378
572
87
K
A
B
S
T
C
D
L
f
628
g
875

(d)

Fig. 2·1

between two successive integers is one-half the distance between the same two integers on the C and D scales. The K scale is compressed even further, so that the distance between any two integers is one-third the distance between the same two integers on scales C and D.

The C and D scales are divided into nine principal divisions by *primary marks* bearing the large numbers 1, 2, 3, . . . , 8, 9, 1. The space between any two primary marks is divided into ten parts by *secondary marks*. These are not numbered, except between the primary marks 1 and 2, where they bear the small numerals 1 through 9. The space between two successive secondary marks is divided into two, five, or ten spaces by unnumbered *tertiary marks*.

The A and B scales each have two identical portions. Each portion is divided into nine principal divisions by numbered primary marks. Unnumbered secondary marks further divide each principal division, and still finer division is supplied by tertiary marks between the primary marks numbered 1 and 5.

The K scale is divided into three identical portions. They are divided much as the A and B scales are, but more coarsely.

2·3 Location of Numbers on the Scales. Accuracy of the Slide Rule. It must be remembered that the decimal point plays no part in locating a number on the C and D scales. The first significant digit is located by reference to the primary marks, the second by reference to the secondary marks, and the third by reference to the tertiary marks, or some point between tertiary marks, according to the portion of the scale being used. By estimating fractions of a graduation, a fourth digit may be located for numbers occurring between primary marks numbered 1 and 2 on the C and D scales. This means that the maximum accuracy ordinarily available with a 10-in. slide rule is one part in 1,000. Location of actual numbers will be best illustrated by referring to the examples in Fig. 2·1(*b–d*).

EXERCISE 1

Read as closely as possible the points indicated in Fig. 2·1(*b–d*).

D scale	*a b c d e f g*
A scale	*h i j k l m n*
K scale	*o p q r s t u*

2·4 Multiplication. Either the C and D scales or the A and B scales may be used. Ordinarily the C and D scales are preferable, since their larger-scale divisions make for greater accuracy. Their use is illustrated in Fig. 2·2 and in the following example.

EXAMPLE 1. Multiply 8 × 5.

Set the right index of C opposite 8 on the D scale. Move the indicator so the hairline covers 5 on the C scale. Directly below this 5 will be found the "slide-rule product" 4 on the D scale. The student must locate the decimal place himself, and common sense indicates that the answer is 40, not 4.

It will be appreciated that the same setting would be used for multiplying 80 by 50, 8,000 by 0.5, 0.08 by 500, etc.

If we were to try to use the above setting of the right index to multiply 8 by 12, a reading would be impossible, as the 12 on the C scale is located beyond the end of the D scale. In this case we would

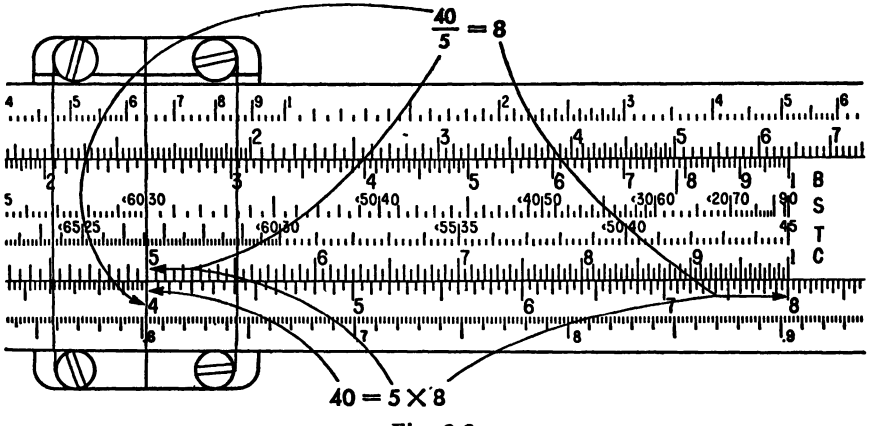


Fig. 2-2

set the *left* index of the C scale over 8 on the D scale and locate on the D scale our answer 96, immediately under 12 on the C scale. It will be useful to remember that when the product of the first significant figures in multiplicand and multiplier is less than 10, the student should use the left index; when it is greater than 10, he should use the right index.

When multiplying more than two numbers, use the hairline to mark the position of the product of the first two. Without reading this product, use it as the multiplicand for the next multiplication.

If the magnitude of the product is not immediately obvious, a rough mental check will suffice.

EXAMPLE 2. The slide-rule product of 1440 × 37.5 × 0.08125 is 439. It is evident that our product is approximately 1,500 × 40 × 0.08, or 4,800. Hence the answer must be 4,390.

To multiply two numbers:

1. *Set the proper index of C opposite the multiplicand on the D scale.*

2. *Place the hairline over the multiplier on the C scale, and read the significant digits of the product under the hairline on the D scale.*

3. *Determine the position of the decimal point by a rough mental estimate.*

4. *In the above steps, C and D may be replaced by B and A, respectively.*

EXERCISE 2

Perform the following multiplications, retaining in the product as many significant figures as you think are justified:

1. 4.00×17.00
2. 7.50×1.20
3. 3.30×9.0
4. 64.0×0.375
5. 288×382
6. $321 \times 1{,}069$
7. $728 \times 1{,}218$
8. $617 \times 1{,}645$
9. $1{,}006 \times 902$
10. $1{,}258 \times 1{,}562$
11. 862×482
12. 66.2×10.3
13. $1.475 \times 1{,}520$
14. 0.981×0.693
15. 329×0.00352
16. 0.342×1.306
17. 8.14×0.0309
18. $2.46 \times 330{,}000 \times 3.14$
19. $3.1 \times 920 \times 0.486 \times 1{,}520$
20. $0.1038 \times 0.0063 \times 28 \times 9.82$
21. $512 \times 62.5 \times 0.0027 \times 87$
22. $0.1047 \times 0.00774 \times 0.349 \times 0.0562$

2·5 Division. Since division is the inverse of multiplication, Fig. 2·2 may be used to illustrate division as well as multiplication. In this example we have the setting for $40 \div 5 = 8$.

To divide one number by another:

1. *Bring the dividend on the D scale opposite the divisor on the C scale by means of the hairline.*

2. *Opposite an index of the C scale, read the significant figures of the quotient on the D scale. If desired, the indicator may be used to aid in this reading.*

3. *Determine the position of the decimal point by a rough mental estimate.*

4. *C and D may be replaced by B and A, respectively, as in multiplication.*

EXERCISE 3

Perform the following divisions, retaining in the quotient as many significant figures as you believe justified:

1. 18.00 ÷ 50.0	**2.** 25.0 ÷ 3.00	**3.** 750 ÷ 5.50
4. 12.8 ÷ 72	**5.** 69.8 ÷ 4.78	**6.** 197.2 ÷ 858
7. 0.924 ÷ 21.0	**8.** 17.5 ÷ 1,646	**9.** 1 ÷ 37.5
10. 0.0752 ÷ 0.000718	**11.** 0.1804 ÷ 363	**12.** 1 ÷ 2.73
13. 0.1875 ÷ 0.078125	**14.** 0.005632 ÷ 18.432	

2·6 Location of Decimal Point in Scientific Notation. Use of scientific notation makes work with very large or very small numbers easier and reduces the likelihood of error.

EXAMPLE 3. Multiply 538,000 × 0.00377.

In scientific notation, we have $5.38 \times 10^5 \times 3.77 \times 10^{-3}$. This product may be written $5.38 \times 3.77 \times 10^2$. Since the product of 5.38 and 3.77 on the slide rule is 20.3, our answer is 20.3×10^2, or 2,030.

EXAMPLE 4. Divide 538,000 by 0.00377.

In scientific notation, we have

$$\frac{5.38 \times 10^5}{3.77 \times 10^{-3}} = 1.427 \times 10^{5-(-3)} = 1.427 \times 10^8 = 142{,}700{,}000$$

2·7 Combined Multiplication and Division. The easiest method of computing the quotient of two products is to alternate between multiplication and division. If we do all the multiplying first, then all the dividing, more moves are required, with greater chance of error.

EXAMPLE 5. Evaluate $\frac{825 \times 184}{227 \times 316}$.

Divide 825 by 227 in the usual way. The quotient will be found on the D scale under the C index. However, we do not read the value, since it serves merely as the multiplicand for 184. Without moving the slide, we move the indicator so that the hairline covers 184 on the C scale. The product will be found on the D scale under the hairline. Since this product is to be the dividend for the division by 316, we do not read the value, nor do we move the hairline, but we move the slide so that 316 is under the hairline. Directly beneath the C index read approximately 2115. The answer is 2.115, with the last figure in doubt.

Sometimes the slide must be moved so that one C index is moved to the spot formerly occupied by the other C index (as marked with the hairline). This situation might have been avoided, with some sacrifice of accuracy, by using the A and B scales.

EXERCISE 4

Perform the following computations, expressing the answer in scientific notation. Retain in the answer as many significant figures as conditions justify.

1. $2.4 \times 6.5 \times 10.37$
2. $1476 \times 37.8 \times 54.0$
3. $0.00842 \times 0.295 \times 6.1875$
4. $67.1 \times 0.000418 \times 3.0$
5. $32.00 \times 5.000 \times 1.900 \times 0.4000$
6. $\dfrac{1}{0.00532 \times 0.0612}$
7. $\dfrac{1.28 \times 3.56}{74.4}$
8. $\dfrac{15.8 \times 1.35}{0.031}$
9. $\dfrac{21.3 \times 0.054}{97.4 \times 3.80}$
10. $\dfrac{1{,}927}{412 \times 0.00592 \times 483}$
11. $\dfrac{24.6 \times 0.359}{296 \times 4.61 \times 98.7}$
12. $\dfrac{560{,}000 \times 0.0045 \times 12{,}500}{1{,}050{,}000 \times 0.072}$

2·8 Proportion. Proportions may usually be solved with only one setting of the slide. Observe in Fig. 2·3 that when 8 on the C scale

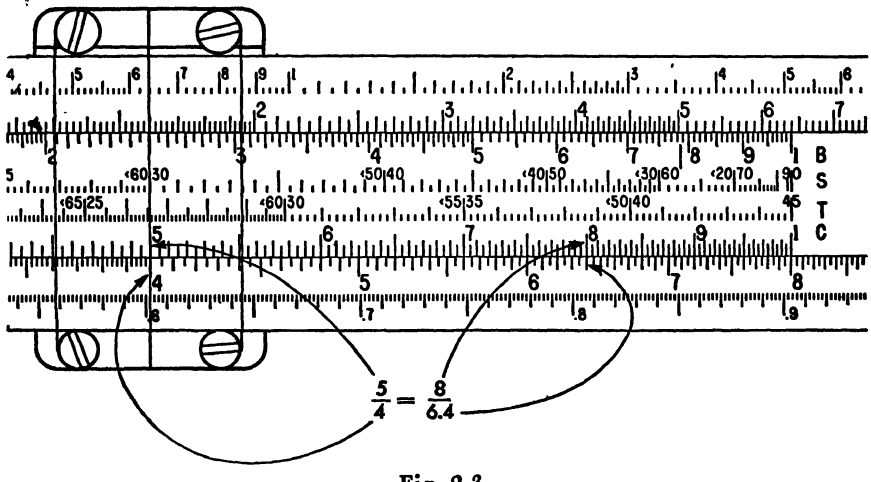


Fig. 2·3

is opposite 64 on the D scale, we find 5 opposite 4. This setting could therefore illustrate the solution of the proportion $5/x = 8/6.4$, in which we set 8 opposite 64 and opposite 5 read $x = 4$. Note that all other pairs of numbers that are opposite each other have the same ratio (*e.g.*, $^{60}/_{48}$, $^{70}/_{56}$, $^{75}/_{60}$, $^{90}/_{72}$).

To solve a proportion, locate the numbers on the C and D scales in the same relative position as in the proportion $a/b = c/d$ (or $c/d =$

a/b if the setting for a falls to the right of the setting for c). When the C and D scales cannot accommodate the simultaneous settings of a over b and c over d, the A and B scales may be used.

EXERCISE 5

In the following proportions, calculate x to three significant figures:

1. $\frac{x}{8.5} = \frac{32}{28.9}$ $x = ?$
2. $\frac{x}{21.5} = \frac{89}{79}$ $x = ?$
3. $\frac{372}{x} = \frac{637}{9.31}$ $x = ?$
4. $\frac{8.2}{377} = \frac{0.323}{x}$ $x = ?$
5. $\frac{18.3}{63.6} = \frac{x}{29}$ $x = ?$
6. $\frac{267}{8.75} = \frac{x}{192}$ $x = ?$
7. $\frac{0.716}{x} = \frac{10.1}{168}$ $x = ?$
8. $\frac{795}{0.109} = \frac{42.3}{x}$ $x = ?$

2·9 Squares and Square Roots. To find the square of a number, set the hairline of the indicator over that number on the D Scale, and

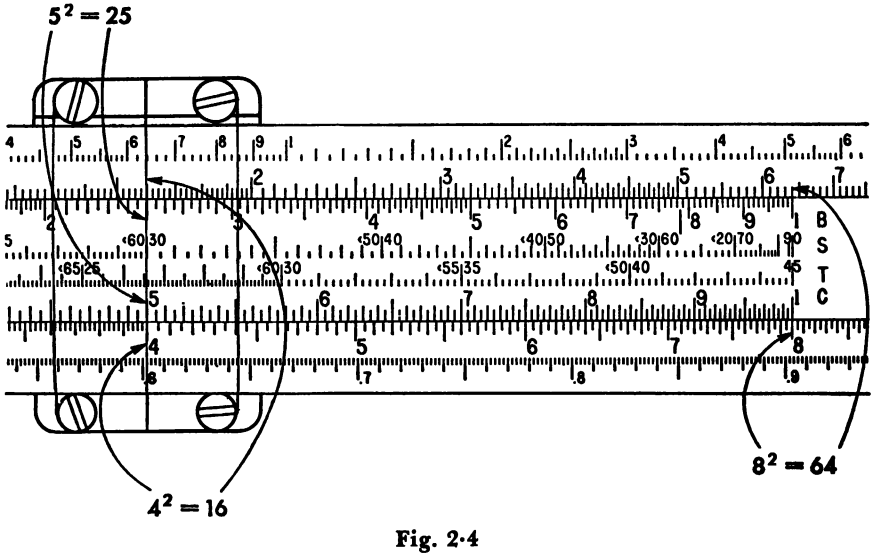


Fig. 2·4

under the hairline read the square of that number on the A scale. We can also read from the C to the B scale in the same way. Figure 2·4 shows that $4^2 = 16$ (D to A scale), that $5^2 = 25$ (C to B scale), and that $8^2 = 64$ (right C index reading to right B index setting).

Figure 2·5 illustrates that $(1.428)^2 = 2.04$, also that $(2.53)^2 = 6.4$.

EXAMPLE 6. Find $(6)^2$. Set the hairline over 6 on the D scale. Read 36 under the hairline on the A scale.

EXAMPLE 7. Find $(8.62)^2$. Set the hairline over 8.62 on the D scale. Read 74.3 under the hairline on the A scale.

EXAMPLE 8. Find $(71{,}700)^2$. On the A scale read 514 directly above the 717 on the D scale. Since $(71{,}700)^2 = (7.17 \times 10^4)^2$, the answer must be $51.4 \times 10^{4\times2} = 51.4 \times 10^8 = 5{,}140{,}000{,}000$.

EXAMPLE 9. Find $(0.00386)^2$. On the A scale read 149 directly above the 386 on the D scale. Since $(0.00386)^2 = (3.86 \times 10^{-3})^2$, the answer must be $14.9 \times 10^{-3\times2} = 14.9 \times 10^{-6} = 0.0000149$.

The process of obtaining square roots is the reverse of that used in calculating squares. Therefore Figs. 2·4 and 2·5 may be used to show

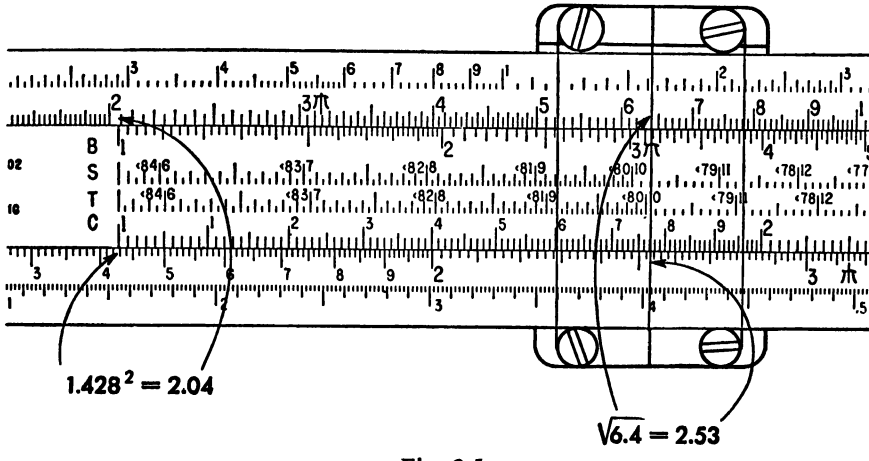


Fig. 2·5

that $\sqrt{16} = 4$, $\sqrt{25} = 5$, $\sqrt{64} = 8$, $\sqrt{2.04} = 1.428$, and $\sqrt{6.4} = 2.53$. We may indeed use Fig. 2·4 to read $\sqrt{64} = 8$; but if we had not had the operation of squaring 8 to guide us, we might have made an unfortunate choice in working from the 64 on the left half of the A scale; *i.e.*, we might have read 253 directly below on the D scale. A quick check will show that 2.53 is the square root of 6.4. (This procedure is opposite to that of squaring 2.53, an operation originally illustrated in Fig. 2·5.)

It is evident that in reversing the procedure for squaring we must first determine which half of the A scale to choose as a starting point. There are a number of ways of making this choice. Perhaps the simplest is that used in the examples on page 25. The procedure is as follows: Write down the number whose square root is to be found. Indicate the grouping of the digits as for longhand extraction of square

root. Write in their proper positions the decimal point and the first significant digit in the square root. Select that half of the A scale which lies above the first significant figure (just determined) on the D scale.

EXAMPLE 10. Find $\sqrt{6{,}870{,}000}$. Indicating the grouping, the decimal point, and the first significant figure, we write $\sqrt{6\ 87\ 00\ 00.}$ (2). It is apparent that we read from 687 on the *left* half of the A scale, since we find 2 on the D scale under that half. Accordingly we read from 687 on the left half of the A scale directly below to 262 on the D scale. This indicates that the square root of 6,870,000 will be written $\sqrt{6\ 87\ 00\ 00.}$ (2 6 2 0.).

EXAMPLE 11. Find $\sqrt{0.0000687}$. A rough indication of the answer is found by proceeding as before and writing $\sqrt{0.\ 00\ 00\ 68\ 7}$ (0. 0 0 8). In this case we evidently read down from the 687 on the *right* half of the A scale. We read 829 on the D scale, and the answer is clearly 0.00829.

2.10 Cubes and Cube Roots. To find the cube of a number, set the hairline over that number on the D scale and read its cube on the K scale under the hairline.

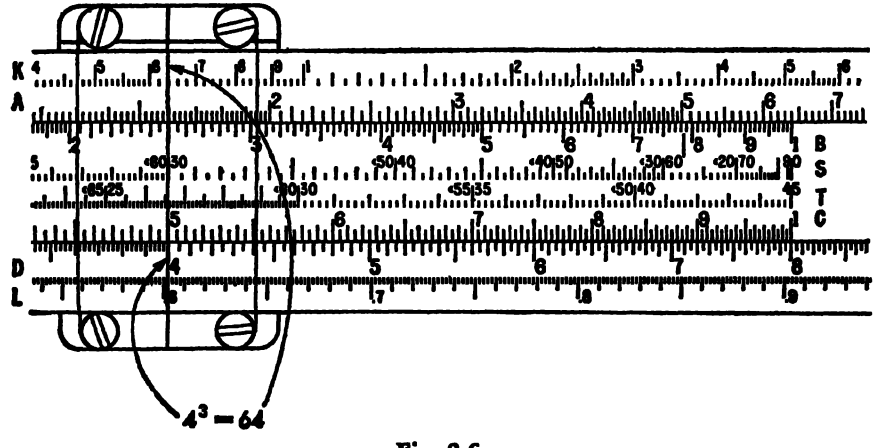


Fig. 2-6

EXAMPLE 12. Find $(6.35)^3$. Set the hairline over 6.35 on the D scale. Read 256 under the hairline on the K scale.

EXAMPLE 13. Find $(0.0439)^3$. Read 846 on the K scale directly opposite 439 on the D scale. Since $(0.0439)^3 = (4.39 \times 10^{-2})^3$, the answer must be $84.6 \times 10^{-2\times3} = 84.6 \times 10^{-6} = 0.0000846$.

To find cube root we reverse the procedure for cubing and work from the K scale to the D scale. When working from the A scale to

the D scale, we had to choose between two sections. On the K scale we must choose the proper section out of three. The process is exactly comparable to that used in finding square root. Figures 2·6, 2·7, and

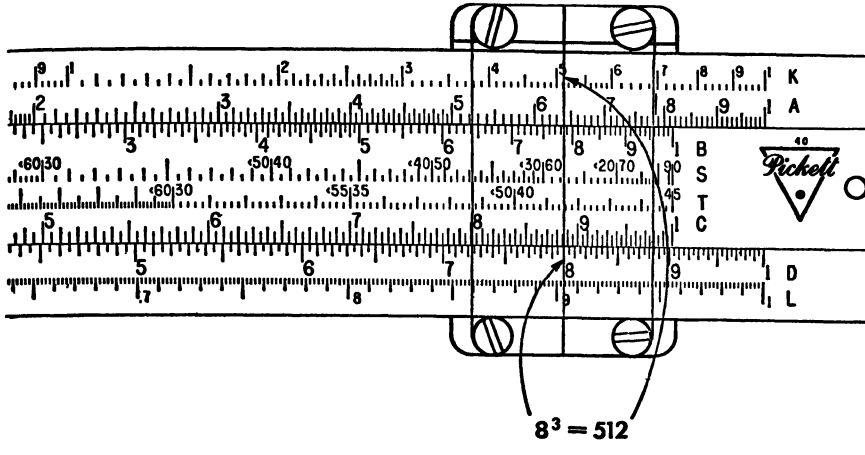


Fig. 2·7

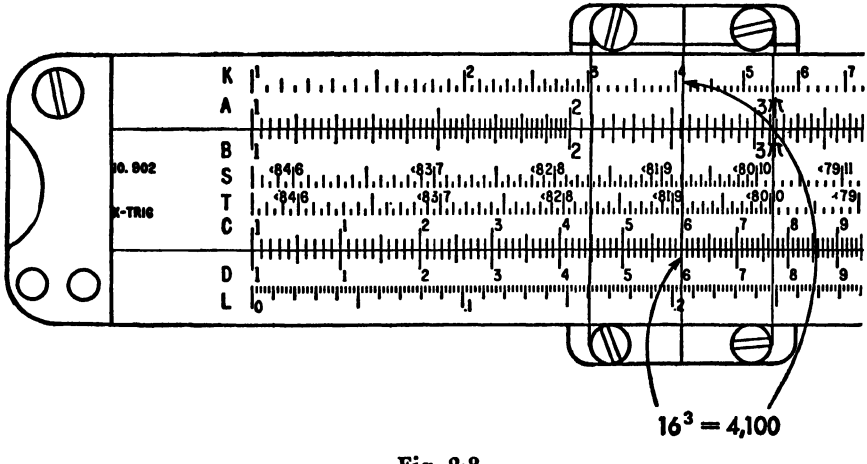


Fig. 2·8

2·8 show the settings for 4^3, 8^3, and 16^3 and therefore also the settings for $\sqrt[3]{64}$, $\sqrt[3]{512}$, and $\sqrt[3]{4{,}100}$.

EXAMPLE 14. Find $\sqrt[3]{0.0000048}$. Indicating the grouping, the decimal point, and the first significant figure, we write $\sqrt[3]{0.000\ 004\ 8}$. Evidently we work from the *left* third of the K scale, since it lies opposite the 1+ on the D scale. Therefore, we read from 48 on the left third of the K scale directly to

1,687 on the D scale. This indicates that the cube root of 0.0000048 will be written $\sqrt[3]{0.000\,004\,8}$ $\overset{0.\ 0\quad 1\ 6\ 8\ 7}{}$.

EXAMPLE 15. Find $\sqrt[3]{58{,}500{,}000}$. Proceeding as before, we write

$$\overset{3\qquad\quad}{\sqrt[3]{58\ 500\ 000.}}$$

and find that the *middle* third of the K scale lies opposite the 3 on the D scale. Reading, therefore, from 585 in the middle section of the K scale, we find 388 on the D scale directly opposite. Hence the answer must be 388.

EXERCISE 6

Evaluate the following as accurately as the slide rule will allow:

1. $(18.00)^2$
2. $(2{,}730)^2$
3. $(6.05)^2$
4. $(34.5)^2$
5. $(167.8)^2$
6. $(0.854)^2$
7. $(0.1054)^2$
8. $(0.00782)^2$
9. $(517.23)^2$
10. $(0.030448)^2$
11. $(63 \times 426)^2$
12. $(9.1 \times 0.0119)^2$
13. $(0.027 \times 1.72 \times 7.95)^2$
14. $(5.1 \times 0.438 \times 14.12)^2$
15. $\left(\frac{1{,}647}{658}\right)^2$
16. $\left(\frac{69.8}{858}\right)^2$
17. $\left(\frac{852}{658}\right)^2$
18. $\left(\frac{61.4 \times 0.673}{2.16}\right)^2$
19. $\left(\frac{582 \times 1.104}{37.2 \times 8}\right)^2$
20. $\sqrt{49.2}$
21. $\sqrt{0.00702}$
22. $\sqrt{2{,}980}$
23. $\sqrt{0.837}$
24. $\sqrt{17{,}840}$
25. $\sqrt{0.0542}$
26. $\sqrt{0.000347}$
27. $\sqrt{45{,}897}$
28. $\sqrt{103.6}$
29. $\sqrt{7{,}080{,}000}$
30. $\sqrt{\frac{1.316}{0.016}}$
31. $\sqrt{\frac{1{,}127 \times 5.47}{21 \times 0.0025}}$
32. $\sqrt{\frac{11.26}{0.787 \times 24.8}}$
33. $(11.9)^3$
34. $(1.33)^3$
35. $(0.157)^3$
36. $(118.5)^3$
37. $(23.19)^3$
38. $(0.0342)^3$
39. $(478)^3$
40. $(6.375)^3$

41. $(783)^3$
42. $(0.0876)^3$
43. $(15.45 \times 0.132)^3$
44. $(8.75 \times 0.037)^3$
45. $\left(\frac{10.12}{17.58}\right)^3$
46. $\left(\frac{48.8}{4.21}\right)^3$
47. $\sqrt[3]{13.5}$
48. $\sqrt[3]{0.0474}$
49. $\sqrt[3]{76{,}215}$
50. $\sqrt[3]{2.85}$
51. $\sqrt[3]{9{,}384{,}260}$
52. $\sqrt[3]{0.00625}$
53. $\sqrt[3]{652 \times 0.725}$
54. $\sqrt[3]{1.447 \times 0.0298}$
55. $\sqrt[3]{\frac{8{,}747}{0.212}}$
56. $\sqrt[3]{\frac{158.5}{255}}$

2·11 Circumference and Area of a Circle. Since the problem of finding the circumference of a circle is simply one of finding the

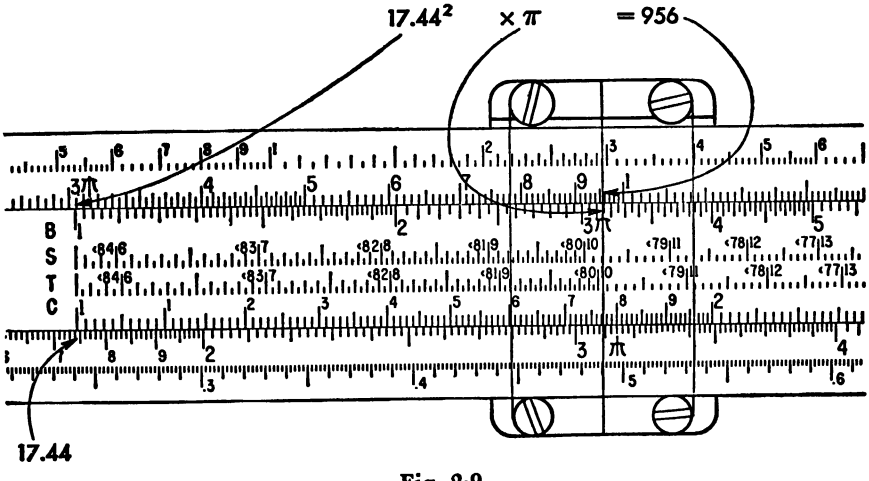


Fig. 2·9

product of a given diameter and 3.14, we follow the usual procedure for multiplication.

To find the area of a circle when the radius is given, set the index of C opposite the radius on the D scale. Place the hairline to cover π on the B scale. The answer is found under the hairline on the A scale. Fig. 2·9 shows that the area of a circle of radius 17.44 in. is about 956 sq in.

To find the area of a circle when the diameter is given, set the index of C opposite the diameter on the D scale. Place the hairline to cover $\pi/4$ or 0.785 on the B scale. The area is found under the hairline on the A scale. Figure 2·10 shows that the area of a 9.03-in.-diameter circle is 64.0 sq in.

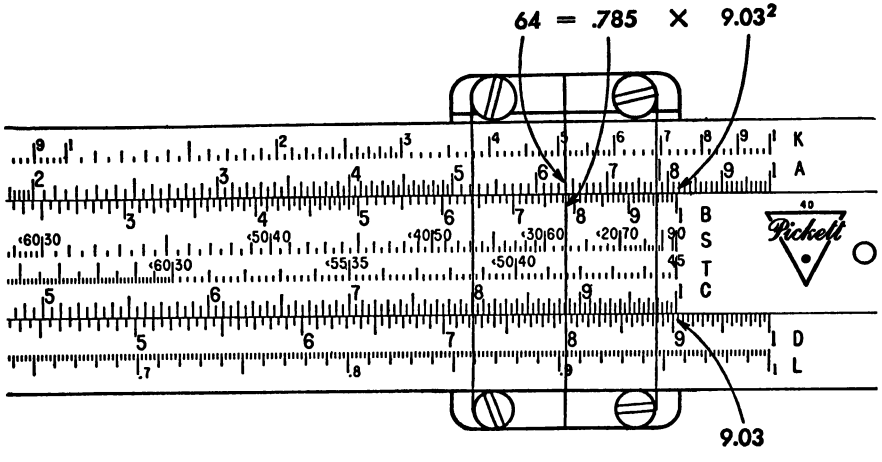


Fig. 2·10

EXERCISE 7

Make a copy of the following table of circles and fill in the blanks.

	Radius	*Diameter*	*Circumference*	*Area*
1.	6.3 miles			
2.	5¼ in.			
3.	78 ft			
4.	9/16 in.			
5.		11.2 yd		
6.		31.6 ft		
7.		⅞ in.		
8.		14½ in.		
9.			8.43 in.	
10.			26.7 ft	
11.				10.00 sq in.
12.				63.8 sq ft

2·12 General Suggestions for Slide-rule Operation. As the student becomes skillful in the use of the slide rule, he will acquire various tricks of the trade which will make this tool even more effective. A few suggestions are outlined in the following paragraphs.

Before setting numbers on the slide rule, cancel or combine simple numbers to reduce the number of moves required.

EXAMPLE 16. $\dfrac{2 \times 43 \times 3}{17 \times 61} = \dfrac{6 \times 43}{17 \times 61}$. A move is saved.

EXAMPLE 17. $\dfrac{60 \times 105}{5 \times 2 \times 97} = \dfrac{6 \times 105}{97}$. Two moves are saved.

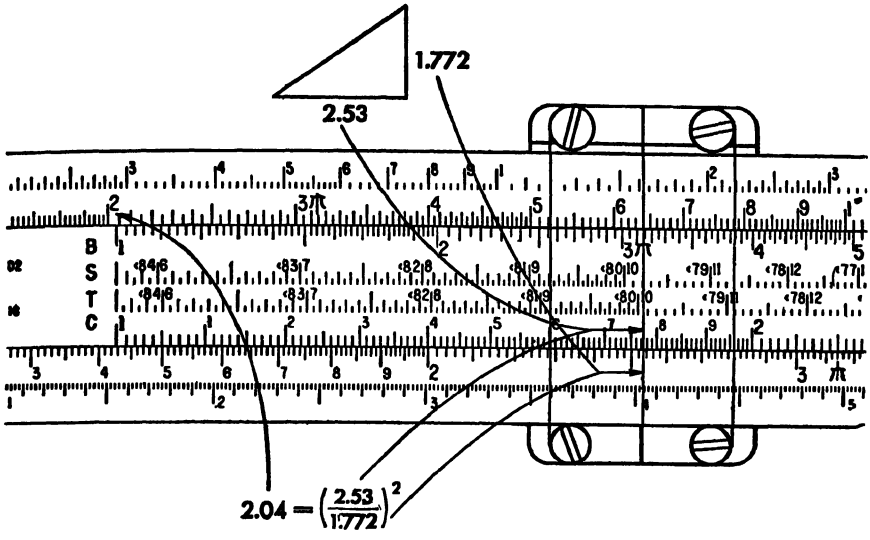


Fig. 2·11

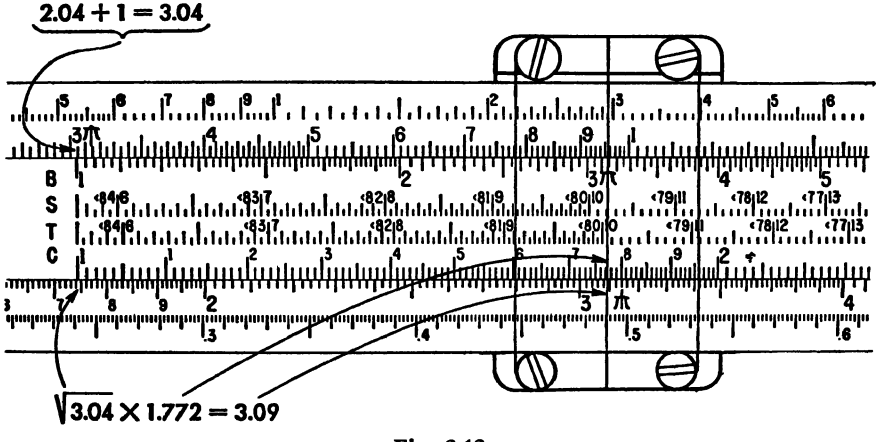


Fig. 2·12

Preliminary rearrangement of formulas is often helpful, as in logarithmic computation (see page 289).

As an alternative to the factoring method, it is suggested that the familiar formulas of the Pythagorean theorem be modified before using on the slide rule, as indicated below:

$a = \sqrt{c^2 - b^2}$	becomes	$a = b\sqrt{(c/b)^2 - 1}$	
$b = \sqrt{c^2 - a^2}$	becomes	$b = a\sqrt{(c/a)^2 - 1}$	
$c = \sqrt{a^2 + b^2}$	becomes	$c = b\sqrt{(a/b)^2 + 1}$	$(a > b)$
		$c = a\sqrt{(b/a)^2 + 1}$	$(b > a)$

With the subtraction or addition of 1 done mentally, a minimum of moves is required.

EXAMPLE 18. Find the hypotenuse of a right triangle whose sides are 2.53 and 1.772. Letting 2.53, the larger side, be represented by a and 1.772 by b and referring to Fig. 2·11, we have the setting for obtaining $(2.53/1.772)^2$, which is seen to be 2.04. Adding 1 mentally to obtain 3.04, or $(a/b)^2 + 1$, we then refer to Fig. 2·12, which shows the setting for $1.772\sqrt{3.04}$, the value of which we read on the D scale as approximately 3.09.

EXERCISE 8

Compute the missing side in the following right triangles where c is the hypotenuse:

	a	b	c
1.	38	47	
2.	7.52	10.36	
3.	523	408	
4.	1⅞ in.	2¾ in.	
5.	9.85	25.6	
6.		0.0645	0.0892
7.		12.33	20.7
8.		3⅛ in.	4¼ in.
9.	5,670		7,030
10.	0.1525		0.291
11.	4⅞ in.		5½ in.

In finding the hypotenuse of right triangles containing a small acute angle, the approximation $c \approx a + (b^2/2a)$ is useful where b is the smallest side (see Appendix A, Table A·2). The smaller b becomes, in relation to a, the closer the approximation. Where b is as large as $0.2a$, c, as calculated by the approximation, will be about 0.02 per cent too large.

EXERCISE 9

Find c, evaluating $b^2/2a$ to three significant figures by slide rule.

	a	b
1.	52	3
2.	109	4
3.	15 in.	⅜ in.
4.	213	7
5.	1.75	0.08

An interesting application of proportion to gear ratios is as follows: Suppose we must have a gear ratio of approximately 1/0.7933, the number of teeth on each gear not to exceed 100. Referring to Fig. 2·13, in which 1 is set opposite 0.7933, we read (in addition to the pair 92 over 73 indicated by the hairline) the following pairs of integers which are approximately matched: 97 over 77; 87 over 69; 63 over 50; 58 over 46 (58 over 46 is a variation of 87 over 69 and would not be considered an independent pair). These results indicate in a rough

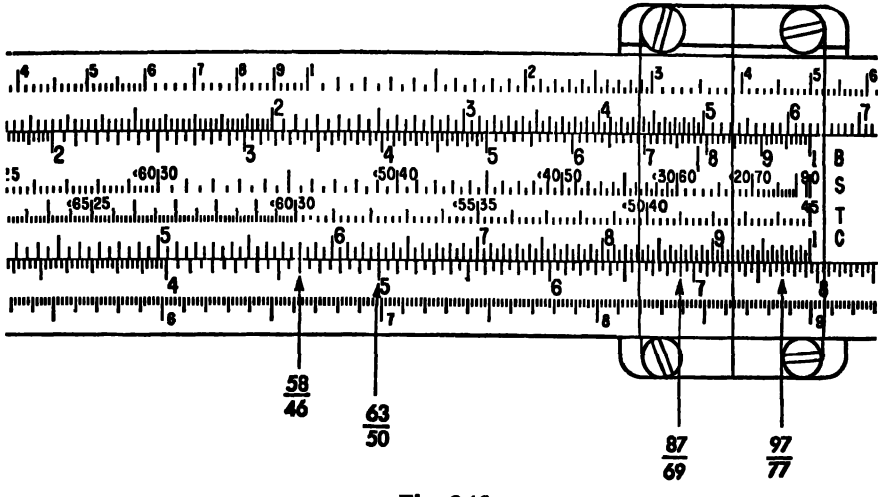


Fig. 2·13

way the number of teeth in gears, giving us the ratio 1:0.7933. The decimal values and accuracy of these ratios are tabulated below.

$$\frac{77}{97} = 0.7938, \text{ or } 0.06 \text{ per cent too large}$$

$$\frac{73}{92} = 0.7935, \text{ or } 0.02 \text{ per cent too large}$$

$$\frac{69}{87} = 0.7931, \text{ or } 0.02 \text{ per cent too small}$$

$$\frac{50}{63} = 0.7936, \text{ or } 0.04 \text{ per cent too large}$$

More precise results can be obtained by using logarithms or continued fractions.

EXERCISE 10

Determine for each of the following ratios all independent pairs of integers, less than 100, whose ratio does not differ from the given value by more than 0.05 per cent:

1. 1.369:1	**2.** 1.841:1	**3.** 2.073:1	**4.** 1.5847:1
5. 0.881:1	**6.** 0.82367:1	**7.** 0.763:1	**8.** 0.6833:1

In the following exercise a set of miscellaneous practice problems involving slide-rule operation is given. If further drill seems advisable, mensuration problems may be selected from Chap. 4 for computation by slide rule.

EXERCISE 11

1. Find the surface area of 1 oz of glass wool whose threads are 0.0008 in. in diameter. Neglect the area of the ends. (One cubic foot of glass weighs 160 lb.)
2. If the national budget is \$71.5 billion, for how many miles would a line of dollar bills represented in this sum extend if laid end to end? Each bill is 6.1 in. long.
3. The product of the wavelength of light and its frequency gives its velocity. Find the wavelength (in centimeters) of red light if its frequency $= 4.00 \times 10^{14}$ vibrations per second. The velocity of light is 186,000 miles per sec.
4. How many miles away is the nearest star whose light takes 4 years to reach the earth? (Refer to velocity of light given in Prob. 3.)
5. How far away from the earth would such a star be in a small-scale model of the solar system? In this model the earth, which has an approximate diameter of 8,000 miles, is represented by a spherical pellet $\frac{1}{4}$ in. in diameter.
6. During a storm 1.5 in. of rain fell. How many gallons of water fell on a square mile of land? 1 gal = 231 cu in.
7. A reservoir having 5 sq miles of surface dropped 2 in. in a week. If it supplies a city of 800,000 inhabitants, how many gallons per week per inhabitant does the consumption amount to?
8. The Mississippi River carries about 3×10^7 cu ft of sediment to the Gulf of Mexico daily. What average yearly erosion (in inches) does this represent if the area of the Mississippi watershed is taken as 1,300,000 sq mile?
9. The earth receives energy from a beam of sunlight 1 sq mile in cross section at the rate of 4.9×10^6 hp approximately. At what rate does a solar-heated house receive energy if the heat-collecting

roof surface is 18 by 25 ft? The sun's rays are assumed to be perpendicular to the roof surface.

10. The polar regions have been estimated to be covered with a sheet of ice averaging 1,500 ft thick over an area of 4,500,000 sq miles. The return of a "tropical age" would melt the ice and raise the level of the oceans. How many feet of rise would this amount to if the water surface is taken as 150,000,000 sq miles? The specific gravity of ice = 0.9.

11. Assuming an average rate of 72 beats per minute, about how many times will the heart beat during a lifetime of 70 years?

12. A manufacturer's advertisement states that a cubic mile of sea water contains 5,000,000 tons of magnesium. If the density of sea water is 64.0 lb per cu ft, find the percentage by weight of magnesium in sea water.

13. It has been stated that a teaspoon of fuel oil will furnish enough energy to draw a 1-ton load for 1 mile along a level track. What over-all coefficient of friction is assumed if a teaspoon contains 4 g of oil? Heat value of oil = 18,000 Btu per lb (1 Btu = 778 ft-lb). (Divide foot-pounds of energy equivalent in the oil by the foot-pounds in a ton-mile.)

Chapter 3 TABLES AND INTERPOLATION

To the average student the utility of the numerical tables is limited to direct readings. In this chapter we shall show how the usefulness of the tables can be extended by appropriate shifting of decimal points, and by interpolation which need not be a hit-or-miss affair. Suggestions for its orderly application will be made.

3·1 Extending the Range of the Tables. The table of squares may be extended beyond the given range by shifting the decimal point twice as many places in the square of the number as in the number itself, for $[x(10)^n]^2 = x^2(10)^{2n}$.

EXAMPLE 1. Find $(7{,}650)^2$. In most tables we shall have to use 765, since 7,650 is beyond the scope of the table. Looking up $(765)^2$, we find 585,225. Therefore, $(7{,}650)^2 = 58{,}522{,}500$.

EXAMPLE 2. Find $(1.38)^2$. Looking up $(138)^2$, we find 19,044. Therefore, $(1.38)^2 = 1.9044$.

NOTE: *The same method applies to finding the areas of circles, since the area varies as the square of the diameter.*

EXAMPLE 3. Find the area of a $2\frac{3}{4}$-in. circle. Looking up the area of a circle of diameter 275, we find 59,396. Therefore, the area of a 2.75-in. circle is 5.9396 sq in.

The table of cubes may be extended by shifting the decimal point 3 times as many places in the cube of the number as in the number itself, for $[x(10)^n]^3 = x^3[10]^{3n}$.

EXAMPLE 4. Find $(3.17)^3$. Looking up $(317)^3$, we find 31,855,013. Therefore, $(3.17)^3 = 31.855013$.

EXAMPLE 5. Find $(0.431)^3$. Looking up $(431)^3$, we find 80,062,991. Therefore, $(0.431)^3 = 0.080062991$.

The use of the square-root table is governed by a rule similar to those just given. That is, we move the decimal place half as many

places in the square root of the number as in the number itself. This is shown by the identity $\sqrt{x(10)^{2n}} = \sqrt{x}\,(10)^n$.

The operation of this rule will be better understood by the following procedure.

Indicate the grouping of the digits of the number as for longhand extraction of square root. Write in their proper positions the decimal point and the first significant digit in the square root. Determine the subsequent digits as found in the square root of the number having the same grouping as the original number. Fill in these digits in proper sequence after the first significant digit already located.

EXAMPLE 6. Find $\sqrt{5{,}090{,}000}$. Indicating the grouping and the decimal point and first significant digit in the answer, we write $\sqrt{5\ 09\ 00\ 00.}$ $\overset{2\quad\quad\quad

EXAMPLE 10. Find $\sqrt[3]{0.00000356}$. Grouping, we get $\sqrt[3]{0.000\ 003\ 560}$. Since the corresponding tabulated number is 3,560, of which the cube root is 15.26921, the desired cube root is $\sqrt[3]{0.000\ 003\ 560}$.

3·2 Interpolation between Two Points. Interpolation between two points assumes a linear relationship over the range between the points. Graphically, a chord replaces the curve connecting the points, and the interpolation error increases with the departure of the chord from the curve. The departure of the chord increases with the curvature of the graph and the distance between the points (see Fig. 3·1).

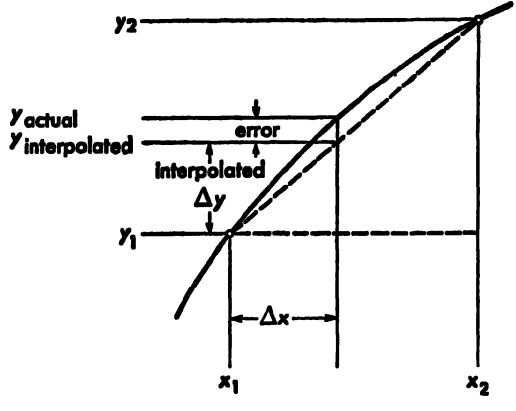


Fig. 3·1

Since Δy is a relatively small (compared to y) additive factor and rarely exceeds three significant figures, slide-rule computation of interpolation is accurate enough for usual cases. (Δx and Δy, read "delta x" and "delta y," represent *changes* in x and y.)

Figure 3·1 shows that linear interpolation is based on similar triangles and that we may write

$$\frac{\Delta y}{y_2 - y_1} = \frac{\Delta x}{x_2 - x_1}$$

or

$$\Delta y = \left(\frac{\Delta x}{x_2 - x_1}\right)(y_2 - y_1) \tag{1}$$

A linear equation is characterized by a constant ratio between the changes in y, (Δy), and the corresponding changes in x, (Δx).

Let us take, for example, the equation $y = 0.8x + 2$ and list some of the pairs of values of x and y satisfying this equation. We note that the ratio

between the values of Δy and the corresponding values of Δx is constant. This is shown graphically in Fig. 3·2.

Δx	x	y	Δy	$\Delta y/\Delta x$
	0	2		
1			0.8	0.8
	1	2.8		
1			0.8	0.8
	2	3.6		
3			2.4	0.8
	5	6.0		
2			1.6	0.8
	7	7.6		
4			3.2	0.8
	11	10.8		

The graph in Fig. 3·3 for a nonlinear equation shows that the ratio between Δy and Δx is not constant.

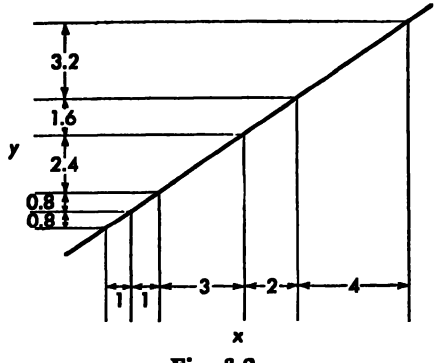


Fig. 3·2

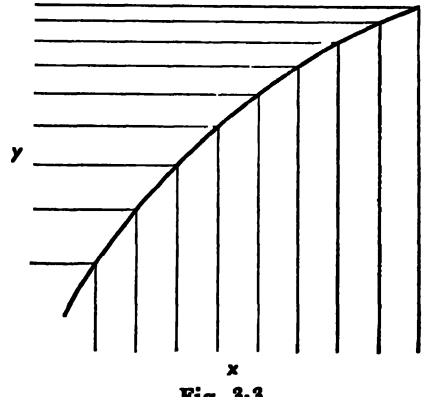


Fig. 3·3

A common type of table lists values of y corresponding to given values of x, y being a function of x. Reliability of an interpolated value of y may be limited by any of the following:

1. Number of significant figures in x. This is frequently unlimited, any limit being imposed by an accuracy of measurement.
2. Precision of tabulated values of y.
3. Departure from linearity of the relationship between x and y.
4. Size of the interval within which interpolation is being calculated.

Examples of these factors are given below:

1. If values of y are *exact* and *linear*, the interpolated value may be calculated to as fine a degree as desired, subject only to the accuracy of x if x is a measured quantity.

EXAMPLE 11

Miles	*Feet* (= 5,280 × *miles*)
3	15,840
3.4	?
4	21,120

Referring to Eq. (1), $\Delta x = 0.4$, $x_1 = 3$, $x_2 = 4$, $y_1 = 15{,}840$, and $y_2 = 21{,}120$. Therefore,

$$\Delta y = \left(\frac{0.4}{4 - 3}\right)(21{,}120 - 15{,}840) = 2{,}112$$

and our interpolated value of $y = 15{,}840 + 2{,}112$, or 17,952. However, this must be rounded off to 18,000 ft. The answer is limited to an accuracy of two significant figures solely because 3.4 miles, a measured quantity, is thus limited.

2. If values of y are *rounded* and *linear*, interpolation may be carried out either (*a*) to as many significant figures as there are in x if x is a measured quantity or (*b*) to the precision of the tabulated values of y, whichever is less reliable.

EXAMPLE 12

Diameter	*Circumference* (= 3.1416 × *diameter*)
4	12.5664
4.74	?
5	15.7080

$$\Delta y = \left(\frac{0.74}{5 - 4}\right)(15.7080 - 12.5664) = 2.3248$$

Therefore, $y = 14.8912$, but we say that the circumference of a 4.74-in. circle is 14.9 in. The accuracy is limited by the three figure accuracy in the diameter.

EXAMPLE 13

Diameter	*Circumference*
9	28.27
9.0852	?
10	31.42

The circumference of a 9.0852-in. circle is 28.54 in. Limit is set by the precision of 0.01 in circumferences.

Note that interpolation does not improve the precision of a value; *i.e.*, 28.27 + 0.0852 (3.15) cannot be written 28.27 + 0.268, or 28.538 (see Example 2, page 2).

3. If values of y are *exact* and *nonlinear*, the extent of interpolation is limited to the less reliable of either (*a*) the accuracy of a measured value of x or (*b*) the precision corresponding to the position of the first digit in Δy showing variation. (Attainable precision may occasionally be one place better than this.)

EXAMPLE 14

x	$y(=x^2)$	Δy
71	5,041	143
72	5,184	
72.2	?	145
73	5,329	

The interpolated value of $(72.2)^2$ is 5,210, a limit of three significant figures of accuracy being set by the measured value 72.2.

EXAMPLE 15

x	$y(=x^3)$	Δy
11	1,331	
11.875	?	397
12	1,728	
13	2,197	469

The interpolated value of $(11.875)^3$ is 1,678, but since the calculated value is 1,674.56+, we are justified in writing only 1,670 or 1,680 as the answer. Δy shows a variation in the hundreds digit; accordingly a conservative calculation would be limited to the nearest 100, or 1,700.

4. If values of y are *rounded* and *nonlinear*, the controlling factor is the least reliable of the following: (*a*) accuracy of measurement of x; (*b*) precision of tabulated values of y; (*c*) precision corresponding to the position of the first digit in Δy showing variation.

EXAMPLE 16

x	$y(=\sqrt{x})$	Δy
37	6.0828	
37.6	?	0.0816
38	6.1644	
39	6.2450	0.0806

The interpolated value of $\sqrt{37.6} = 6.13$. Note that if x had been given as 37.600, then, making the change in Δy the controlling factor (shows change in thousandths place), we might have written $\sqrt{37.600} = 6.132$.

EXAMPLE 17

x	$y(=\sqrt[3]{x})$	Δy
55	3.803	0.023
56	3.826	
56.314	?	0.023
57	3.849	

The controlling factor here is the precision of y (nearest thousandth); therefore we write $\sqrt[3]{56.314} = 3.833$.

EXAMPLE 18

x	$y\left(=\frac{1}{x}\right)$	Δy
15	0.06667	0.00417
16	0.06250	
16.375	?	0.00368
17	0.05882	

Interpolated value of $1/16.375 = 0.061$ (precision limited by change in Δy occurring in thousandths place). Note that an extended interpolation yields 0.06112, compared to a calculated value of 0.0610687. Therefore, we might have written 1/16.375 as 0.0611.

As an example of the effect of the width of the interval on the accuracy of interpolation, find the square of 5.283, the actual value of which is 27.910089.

Interpolation between 5 and 6 gives us $(5.283)^2 = 28$. Interpolation between 5.2 and 5.3 [looking up $(52)^2$ and $(53)^2$ and shifting the decimal point two places to the left] gives us 27.91.

Finally, by interpolation between 5.28 and 5.29 [looking up $(528)^2$ and $(529)^2$ and shifting the decimal point four places to the left], we obtain $(5.283)^2 = 27.9101$.

3·3 Other Helpful Devices. Frequently interpolation may be avoided by judicious use of factoring; *e.g.*, $\sqrt{1{,}328} = \sqrt{4(332)} = 2\sqrt{332} = 2(18.2209) = 36.4418$.

The student should not overlook the advantages of rationalization in extracting roots of fractions. For example, $\sqrt{\frac{2}{3}}$ is much more conveniently found by evaluating $\frac{1}{3}\sqrt{6} = \frac{1}{3}(2.4495) = 0.8165$ than by finding $\sqrt{0.666667}$.

Square roots, for instance, may at times be found with increased accuracy by working backward from the table of squares. This is true if the table of square roots is given only to four or five significant figures or so.

EXAMPLE 19. Find the square root of 174,126.

x	$y(=\sqrt{x})$
174,000	417.13
174,126	?
175,000	418.33

By interpolation, $\sqrt{174{,}126} = 417.28$. Working backward from the table of squares,

Δx	x	$y(=x^2)$	Δy	
1	417	173,889	237	835
	?	174,126		
1	418	174,724		837
	419	175,561		

Interpolating by either of two similar processes, we obtain

$$417 + \frac{237}{835}(1) = 417.284 \qquad \text{or} \qquad 417 + \frac{237}{2(417)} = 417.284$$

(See page 589.)

It should be observed that the nearly constant ratio $\Delta y/\Delta x$, together with the exact values of both x and y, makes it possible to obtain by interpolation three extra digits instead of the usual one.

Circumferences by Addition of Parts.

Since $\pi(a + b) = \pi a + \pi b$, we may consider that the sum of the circumferences of two circles is equal to the circumference of a circle whose diameter is equal to the sum of the diameters of the first two circles.

EXAMPLE 20. Find the circumference of a $5\frac{7}{16}$-in. circle. Divide 5.4375 into groups of not more than three significant figures each to avoid interpolation.

Diameter	*Circumference*
5.4	16.96460
0.0375	0.11781
5.4375	17.08241

In most cases two or more superfluous digits will have to be rounded off to be consistent with the accuracy of the data of the problem, but interpolation will have been avoided.

It should be noted in passing that the method of "addition of parts" is applicable to any linear relation between two variables having a common zero, *e.g.*, degrees to radians and inches to centimeters.

The use of the circumference table is by no means so limited as the name might suggest. For example, the area of an ellipse whose semi-axes are 6 and 8 is found by looking up the circumference of a circle of diameter 48. The circumference table is also useful in the calculation of surface speeds of drills, grinding wheels, pulleys, etc. In electricial engineering, multiples of π are involved in calculations of electromagnetic effects, capacitance, and inductance.

The reciprocal table may be used to evaluate fractions.

EXAMPLE 21. What decimal part of a foot is $\frac{7}{16}$ in.?

$$\frac{7}{16} \div 12 = \frac{7}{16} \times \frac{1}{12} = \frac{7}{192} = 7\left(\frac{1}{192}\right) = 7(0.0052083) = 0.03646$$

Chapter 4 APPLICATIONS IN GEOMETRY

A working knowledge of geometry is essential for solving many practical problems. In many cases the engineer, technician, or machinist is compelled to obtain his information by indirect means when the usual approach is not possible because data are missing or unavailable. The word *geometry* means earth measurement—a field in which many dimensions are either inaccessible or too large for direct measurement.

It is assumed that the student already has some knowledge of geometry; hence in this chapter we are dealing chiefly with applications. However, a list of some of the more important facts of geometry is given in Appendix B. For more detailed information the student is referred to any standard geometry text.[1]

The student is urged to examine the problem critically. Are the data given of the type considered reliable? Are any assumptions (unreasonable or unwarranted) involved? Answers should be calculated to a point deemed consistent with data given.

EXERCISE

1. Find angle x in Fig. 4·1.

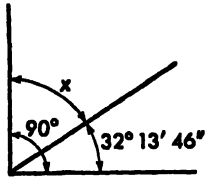


Fig. 4·1

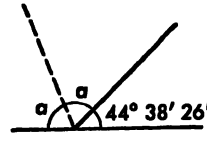


Fig. 4·2

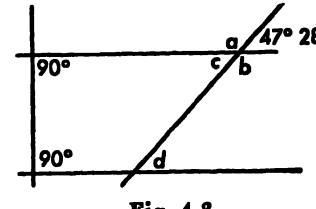


Fig. 4·3

2. Find angle a in Fig. 4·2.
3. Find angles a, b, c, d in Fig. 4·3.

[1] See, for example, Wells and Hart, "Modern Plane Geometry," D. C. Heath & Co., 1926, R. P. Keniston and J. Tully, "Plane Geometry," rev. ed., Ginn & Co., 1953; V. S. Mallory, "New Plane Geometry," rev. ed., Benjamin H. Sanborn & Co., 1943; C. I. Palmer and S. F. Bibb, "Geometry with Applications," 4th ed., McGraw-Hill Book Company, Inc., 1941.

4. Find angles a and b and side x in Fig. 4·4.
5. Find x and y in Fig. 4·5.
6. Find angle a in Fig. 4·6.

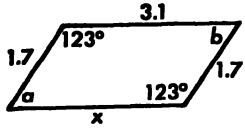


Fig. 4·4

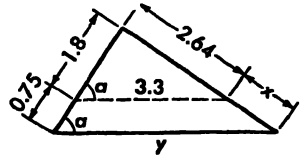


Fig. 4·5

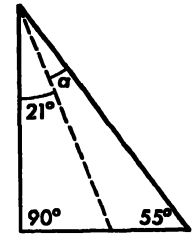


Fig. 4·6

7. An exterior angle of a triangle is 73°. One opposite interior angle is 39°. Find the other opposite interior angle.
8. Find angle a in Fig. 4·7.
9. Find angle x in Fig. 4·8.
10. Find angles a and b in Fig. 4·9.

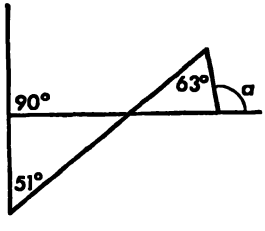


Fig. 4·7

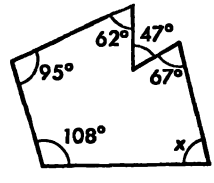


Fig. 4·8

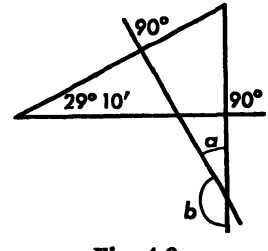


Fig. 4·9

11. Two angles of a triangle are 41°30′ and 97°. The included side is 4.75. Find the shorter missing side.
12. Find the angle between the bisectors of the two angles of a trapezoid which are formed by the bases and one nonparallel side.
13. Find the angle between the bisectors of the acute angles of a right triangle.
14. Two consecutive angles of a quadrilateral are 83 and 105°. Find the angle between the bisectors of the other two angles.
15. An isosceles triangle has a base $4\frac{5}{8}$ in. long and two equal sides each $3\frac{1}{4}$ in. long. Find, to the nearest $\frac{1}{32}$ in., the altitude to the base.
16. The base angles of a trapezoid are 60 and 45°. The upper base is 15, and the leg adjacent to the 60° angle is 10. Find the area of the trapezoid.

17. The altitude to the hypotenuse of a right triangle divides the hypotenuse into segments 3.70 and 8.43 in. long. Find the altitude.

18. Find to the nearest 0.1 sq in. the area of a triangle whose sides are 19, 28, and 39 in.

19. The area of a square is $\frac{2}{3}$ sq in. Find its side to the nearest $\frac{1}{64}$ in.

20. If the angles of a triangle are in the ratio of 1:2:3, find the ratio of the longest side to the shortest side.

21. Find the area of a square whose diagonal is 30 in.

22. Find the angles a and b in Fig. 4·10.

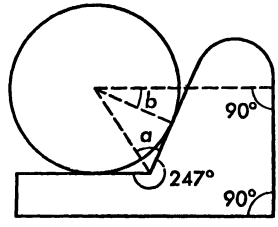


Fig. 4·10

23. Find the angle a in Fig. 4·11.

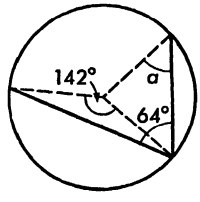


Fig. 4·11

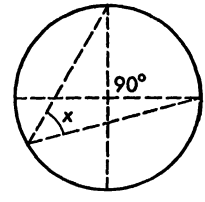


Fig. 4·12

24. Find the angle x in Fig. 4·12.

25. The sides of a triangle are 9, 11, and 14 in. long. A new triangle is formed by joining the mid-points of the sides of the original triangle. Find the perimeter of the new triangle.

26. The diagonals of a quadrilateral are 16 and 22 in. and are perpendicular to each other. Find the area of the quadrilateral.

27. Two sides of a parallelogram are 10 and 12 in. long, and the included angle is 135°. Find the length of the longer diagonal.

28. In an isosceles trapezoid the parallel sides are 11 and 19 in. long, respectively, and the diagonals are each 17 in. long. Find its area.

29. The lengths of the sides of a triangle are 3, 4, and 5. What is the ratio of the areas of the two smaller triangles into which it is divided by a perpendicular drawn to the longest side from the opposite vertex?

30. Two sides of a triangle are 13 and 15 in., and the altitude upon the third side is 12 in. Find the third side and the area.
31. Find the altitude to one of the equal sides of an isosceles triangle whose sides are 5, 5, and 6.
32. The areas of two similar polygons are to each other as 16 is to 25. The perimeter of the first polygon is 10. Find the perimeter of the second.
33. An equilateral triangle and a regular hexagon are inscribed in the same circle. Find the ratio between the areas of hexagon and triangle.
34. Find the height of a segment whose width is 24 in. if the diameter of the circle is 40 in.
35. Find the length of an arc of 28°34′ in a circle of diameter 1,500 yd.
36. Find the area of the room the plan of which is shown in Fig. 4·13.

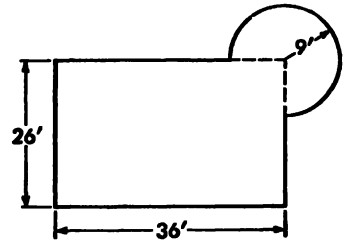


Fig. 4-13

37. A counterweight on a 60-in. driving wheel is in the form of a segment subtending a central angle of 60°. Find the weight if it is 4 in. thick and made of cast iron (density 0.26 lb per cu in.).
38.[1] Find the area of a segment 3½ in. high in a circle 18 in. in diameter [see Eq. (87), Appendix B].
39.[1] Find the area of a segment having a height of 4 in. and a width of 25 in. [see Eq. (86), Appendix B].
40. Find the area of a 51°24′ sector of a 15-in. circle.
41. A circle of what diameter has numerically the same diameter in inches as it has area in square inches?
42. Two circles of radii 3 and 9 in. are externally tangent to each other. Find the length of the common external tangent (*i.e.*, of a line whose extremities are externally tangent to these circles at points other than the point of their mutual tangency).
43. Find, in terms of π, the area of a circle inscribed in a rhombus whose side is 8 in. and whose obtuse angle is 120°.

[1] In both Probs. 38 and 39, the smaller h is in relation to the other quantities, the closer is the approximation in the formulas used.

44. A regular hexagonal pyramid has a base edge 10 in. long. If the lateral faces are inclined 60° to the base, find the altitude of the pyramid.

45. An empty rectangular swimming pool is 2 ft deep at one end, and the bottom slopes uniformly to a depth of 10 ft at the other end. After cleaning, the pool is being refilled at a constant rate. If after $2\frac{1}{2}$ hr the water is 2 ft deep at the deep end, how much longer will it take to fill the pool?

46. A pyramid is 5 in. square on the base and 7 in. high. Find the volume, total area, and lateral edge.

47. Find the volume and total surface of the frustum of a square pyramid whose height is 12 in. and whose bases are 14 and 24 in. square, respectively.

48. Find the volume and lateral surface of the frustum of a hexagonal pyramid if the height is 28 in. and the upper and lower bases are 12 and 22 in. across flats, respectively.

49. The eight corners of a rectangular solid 3 by 4 by 12 in. all lie in the surface of a sphere. Find the surface of the sphere in terms of π.

50. A $\frac{3}{8}$-in. brass rod $76\frac{1}{2}$ in. long weighs 3 lb 12 oz. How long a piece should be cut off to make a $\frac{1}{2}$-oz weight? What tolerance in the length is allowable if the weight is to be in error by not more than 0.01 oz?

51. Find the total surface and volume of a cone whose height and base diameter are 17 and 20 in., respectively.

52. Find the volume and lateral surface of the frustum of a cone having base diameters of 13 and 22 in. and height 20 in.

53. Find angle β in Fig. 4·14.

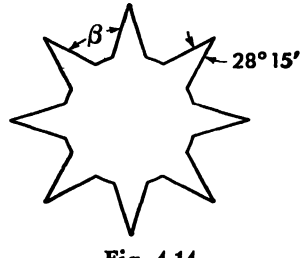


Fig. 4·14

54. A wooden vat 4 ft 0 in. by 3 ft 2 in. by 2 ft 8 in. deep (open at top) is to be lined with $\frac{1}{8}$-in. sheet lead. Find the cost of the lead if a cubic foot weighs 700 lb and a pound of lead costs \$0.12.

55. The frustum of a cone is $5\frac{1}{4}$ in. high. The diameters of its bases are $8\frac{3}{4}$ and $12\frac{1}{2}$ in. Find the height of the original cone.

56. In Fig. 4·15 two cuts are made approximately 11 in. from each corner of a 30-in. square to make a regular duodecagon (12 sides). What distance should be measured when working with a 14-in. square? How many degrees are there in angle a?

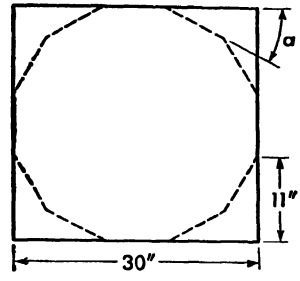


Fig. 4·15

57. If lead weighs 700 lb per cu ft, how long a piece must be cut from a long sheet 20 in. wide and $\frac{1}{16}$ in. thick to weigh 175 lb?

58. The scale ratio of a certain aerial photograph is 1:25,000. What is the area in acres of a town if the area in the photograph is 34.23 sq in.?

59. What is the area of an orchard which appears in the photograph of Prob. 58 as a square 0.1 in. on a side?

60. A farmer traced the map of his farm on a sheet of No. 4 zinc and cut out the outline of his map. Then, to the same scale (1 in. = 50 ft), he cut out a 4-in. square. The 4-in. square weighed $\frac{1}{2}$ oz, while the zinc map weighed 3 lb $10\frac{1}{2}$ oz. How many acres were there in the farm?

61. The U.S. Coast and Geodetic Survey maps have a scale of about 1 in. to the mile. It is desired to measure areas on one of these maps by laying a grid made of celluloid over the area to be measured. The lines on the grid form squares. What should be the spacing of the rulings if each square represents 100 acres?

62. Find the roof area the top view of which is given in Fig. 4·16

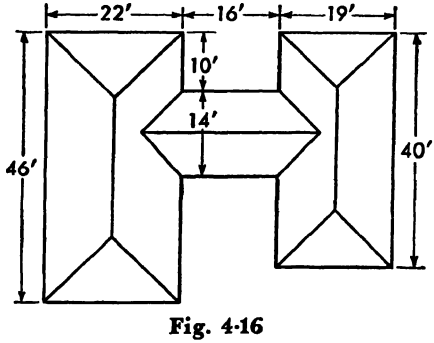


Fig. 4·16

if all parts of the roof have a $\frac{3}{4}$ pitch (3 in. rise in every 4 in. taken horizontally).

63. A lean-to roof is 22 ft long. The rafters are 13 ft long, with a $\frac{2}{3}$ pitch (2 in. rise in every 3 in. horizontally). If all the rain which falls on the roof is collected, how many gallons will have been collected after a rainfall of 0.82 in.?

64. In Fig. 4·17 find the length of a cross brace *ab* and the distance *aw* of the rivet from one end.

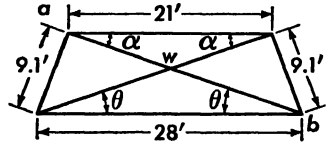


Fig. 4·17

65. A regular hexagon is to be made of four sections which are to be cut from one piece, as shown in Fig. 4·18. Find the length and width of this piece (no allowance for saw kerfs).

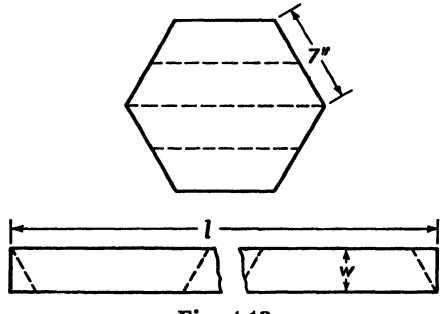


Fig. 4·18

66. Find the length of one side and the area of a rhombus whose diagonals are 15 and 22 in.

67. A pumping station at *O* on a river is to deliver water to *A* and to *B*. When angle *KOB* equals angle *AOM*, the total length of piping is a minimum. Find this length in Fig. 4·19.

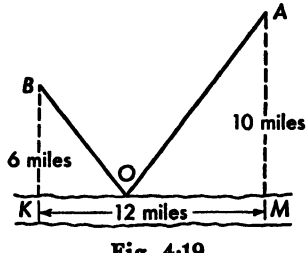


Fig. 4·19

68. A regular octagon may be cut from a square by the method shown in Fig. 4·20. Find the length of a side of the octagon.

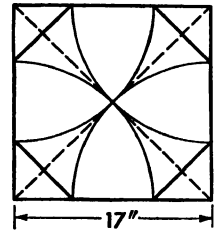


Fig. 4·20

69. A wrench is to have a total play of 30° in a $\frac{5}{8}$-in.-square socket. Find the size of the wrench (Fig. 4·21).

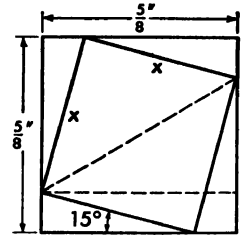


Fig. 4·21

70. Find the length of a common rafter c in the gable roof shown in Fig. 4·22.

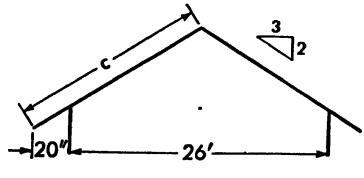


Fig. 4·22

71. A plot of land has the shape of a quadrilateral whose sides in order are 3,470, 1,980, 2,540, and 3,720 ft. If the last two sides form a right angle, find the area of the plot in acres.

72. Find the altitude to the longest side of a triangle whose sides are 8, 12, and 15 in.

73. Find the tensile strength of a hexagonal brass bar $\frac{5}{8}$ in. across flats. Assume 1 sq in. will support 30,000 lb.

74. Eight centers are evenly spaced on a $3\frac{1}{4}$-in. circle. Find to the nearest $\frac{1}{64}$ in. the center-to-center distance.

75. Find the center-to-center distance from A to B in Fig. 4·23.

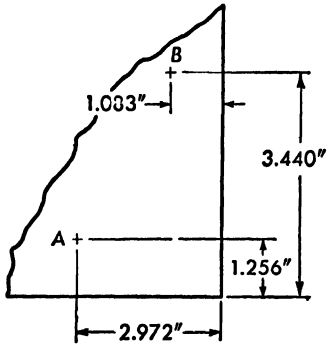


Fig. 4·23

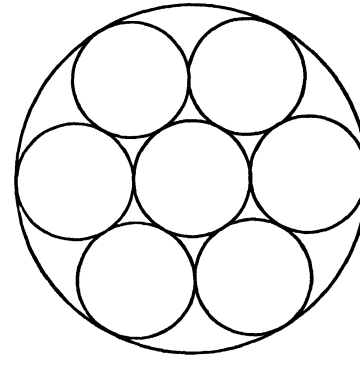


Fig. 4·24

76. Find the fraction of the large circle occupied by the seven small circles in Fig. 4·24.

77. A circle is inscribed in an isosceles triangle whose equal sides are each 1.875 in. long and whose vertex angle is 48°14′. Find the angle formed at the center of the circle by lines drawn to the extremities of the base.

78. Find the angle θ in Fig. 4·25.

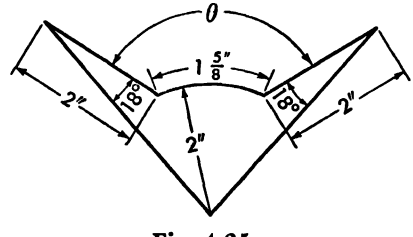


Fig. 4·25

79. Two 10-in. pulleys are connected by a belt which crosses itself at an angle of 60°. Find the length of the belt.

80. Find R in Fig. 4·26.

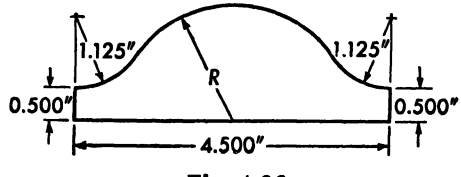


Fig. 4·26

81. Find x in Fig. 4·27.

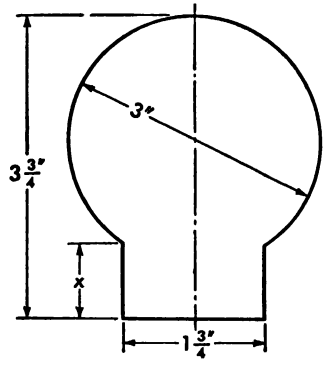


Fig. 4·27

82. Using 3,960 miles as the radius of the earth, find the length in miles of 1′ longitude (**a**) at 45°N; (**b**) at 60°N.

83. In Fig. 4·28 find angle α if the total length $ABCD$ = 6.426 in.

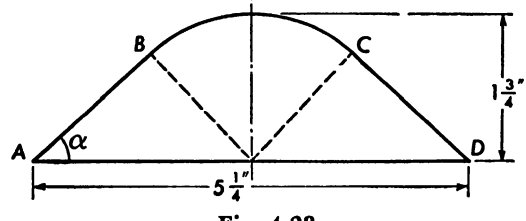


Fig. 4·28

84. Find the angle ϕ in Fig. 4·29.

85. Find the diameters of the circles in Fig. 4·30.

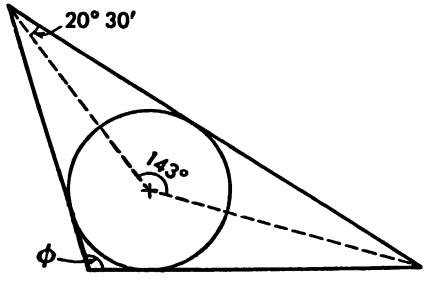


Fig. 4·29

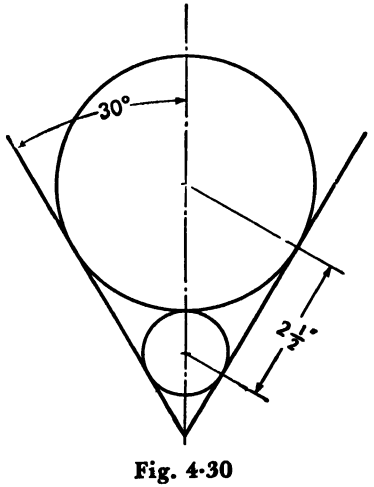


Fig. 4·30

86. Find the distance x in Fig. 4·31.

87. Find the angle β in Fig. 4·32. (There are 21 teeth.)

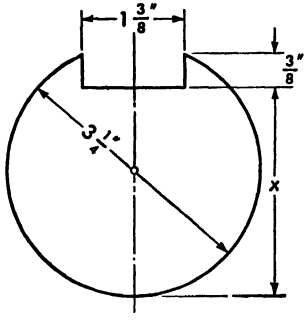


Fig. 4·31

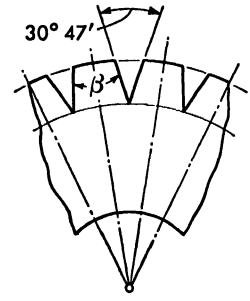


Fig. 4·32

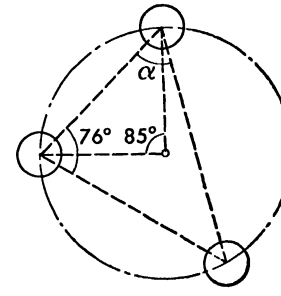


Fig. 4·33

88. Find the angle α in Fig. 4·33.

89. Find the diameter of the circle in Fig. 4·34. (Occurs in caliper measurement of inside diameters. See also Prob. 62, page 204.)

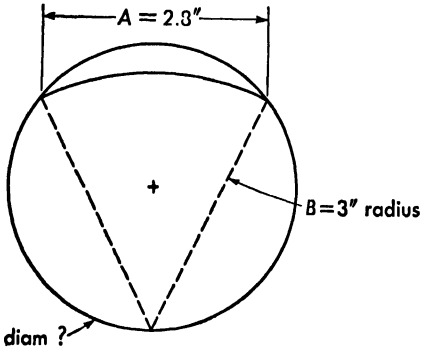


Fig. 4·34

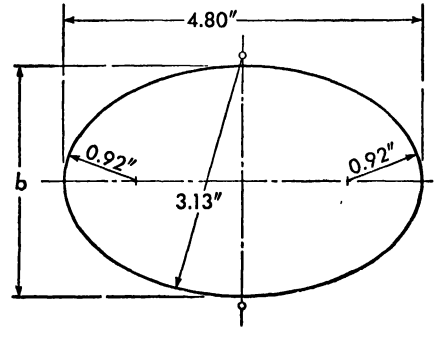


Fig. 4·35

90. Find the short diameter b in the approximate ellipse in Fig. 4·35.

91. A wash boiler having semicircular ends is 20 in. deep, 18 in. wide, and 25 in. long. Find its capacity in gallons.

92. Find the allowable revolutions per minute for a 7½-in.-diameter grindstone if the rim speed must not exceed 72 fps.

93. What diameter of hole should be drilled clear through a ⅜-in. plate of cast iron to remove 2 oz? Density of cast iron = 0.26 lb per cu in.

94. An approximate formula for determining how many (N) small circles of diameter d can be contained by a large circle of diameter D is $N = 0.907\ (D/d - 0.94)^2 + 3.7$. Use this formula to calculate the number of ⅛-in. wires that may be enclosed in a 1 5/16-in. conduit.

95. A Norman window has the shape of a rectangle of base 4 ft 4 in. capped by a semicircle of diameter equal to the base. Find the window area if the over-all height is 8 ft 10 in.

96. Three circular ducts whose diameters are 4, 4½, and 5½ in., respectively, are supplied by a single larger duct. Find its diameter to the nearest 1/16 in.

97. A circular pool 65 ft in diameter is surrounded by a walk having an area of 600 sq ft. Find the width of the walk to the nearest ½ in.

98. A horizontal cylindrical tank 30 in. in diameter contains 400 gal when full. How many gallons remain when the depth of liquid is 8 in.? (See Prob. 38, page 35.)

99. Find the weight of an 18-ft section of concrete pipe having 44 in. ID and a 3½-in. wall. One cubic foot of concrete weighs 154 lb.

100. A ¼-in. wire is wound around a 1½-in. cylinder with a lead of ⅝ in. How much wire is required for 12 turns? (Compute the length of the center line of the wire.)

101. Find the area of the three-cornered section enclosed by three mutually tangent 15-in. circles.

102. Find the length of stock needed for the piece shown in Fig. 4·36. (Find the length of the center line.)

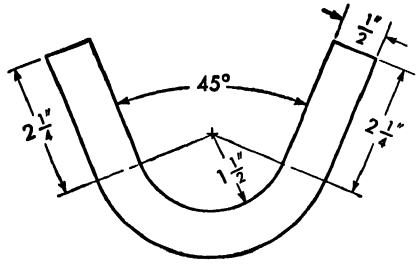


Fig. 4·36

103. Two 15-in. pulleys connected by an open belt are 10 ft apart (center-to-center). If the belt splice passes a given point 125 times a minute, find the revolutions per minute of one of the pulleys.

104. A 12,000-gal cylindrical tank is to be installed in a factory. Because of headroom the height is restricted to 15 ft. What should be the diameter of the tank to the nearest inch?

105. A roll of paper 4 ft 6 in. in diameter is wound on a 6-in. hub. Find the length of the paper in the roll if the sheet is 0.0080 in. thick. What assumption is made here?

106. **a.** If a number of plants are to be set out 8 in. apart each way in a 9½-ft-diameter circular flower bed, how many plants are needed?

b. If the plants were set out on 8-in. equilateral-triangular centers, how many would be required?

107. Using the approximate formula $l = \pi\left(\frac{D+d}{2}\right) + 2a$, where l is the length of an open belt connecting two pulleys whose diameters D and d differ only slightly and whose center-to-center distance is a, find the length of belting required for a 12-in. and a 16-in. pulley 5 ft apart (a, l, D, and d are expressed in like units).

108. When an open belt connects two pulleys of radically different sizes, the formula in Prob. 107 will give a better approximation if modified to the following:

$$l = 2\sqrt{\left(\frac{D-d}{2}\right)^2 + a^2} + \pi\left(\frac{D+d}{2}\right)$$

Find l when D is 18, d is 5, and a is 30 in.

109. Repeat Prob. 108, using a crossed belt in which the approximate formula applying is as follows:

$$l = 2\sqrt{\left(\frac{D+d}{2}\right)^2 + a^2} + \pi\left(\frac{D+d}{2}\right)$$

110. The following data were obtained in finding the diameter of the bore in a capillary glass tube: Weight of empty tube = 14.603 g; weight of tube containing a thread of mercury 11¼ in. long = 16.385 g. Find the diameter of the bore to the nearest 0.001 in. if 1 g = 0.0353 oz and 1 cu in. mercury weigh 7.84 oz.

111. In Fig. 4·37 the speed of the upper pulley is 450 rpm. Find the speed of the lower pulley when the idler belt is in the position

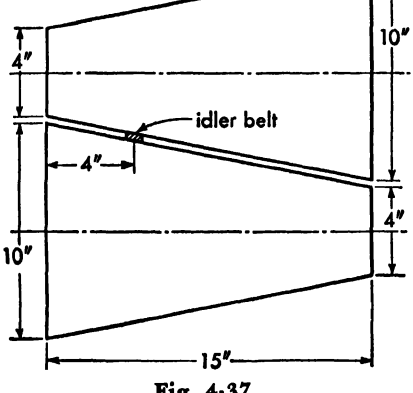


Fig. 4·37

shown. What is the extreme range of speeds possible in the lower pulley?

112. In the drawing of the wire gauge in Fig. 4·38, $d_1 = 0.525$ in. when $w_1 = 3.12$ in. Find d_2 when $w_2 = 2.87$ in.

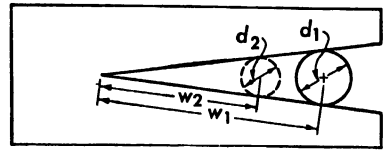
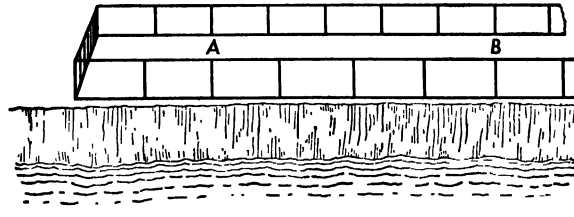


Fig. 4·38

113. On one side of a river is a rectangular playground 100 ft wide measured back from the riverbank. An observer on the opposite bank notices that the fence posts of the sides parallel to the riverbank line up at A and B, as illustrated in Fig. 4·39. Assuming the fence posts to be uniformly spaced, find the width of the river.



observer

Fig. 4·39

114. If water is flowing through an 8-in. pipe with an average linear rate of 0.35 fps, find the rate of flow in gallons per minute.

115. What is the linear rate of flow in Prob. 114 when water is passing through a ¾-in.-diameter metering orifice?

116. Find the diameter of the flywheel a fragment of which is shown in Fig. 4·40.

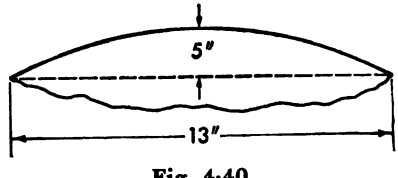


Fig. 4·40

117. A cone is to have an altitude of 10 in. and a base diameter of 8 in. Find the radius and central angle of its development sector.

118. Find the cross section of the storm sewer illustrated in Fig. 4·41.

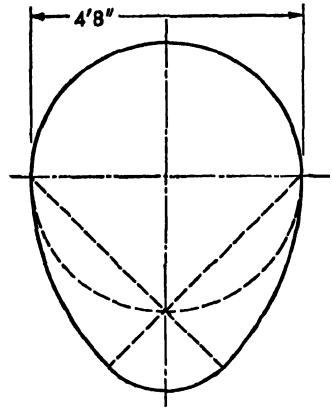


Fig. 4·41

119. A semicircular piece of copper having a 6-in. radius is rolled into a cone by bringing the two radii together. Find the capacity of the cone in cubic inches.

120. Find m in Fig. 4·42.

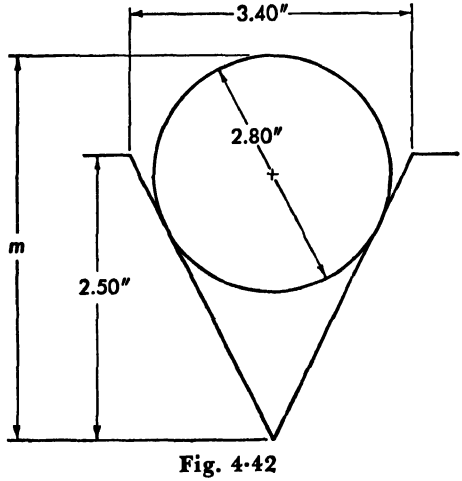


Fig. 4·42

121. A slab of granite is being moved on rollers 6 in. in diameter. How far will the slab advance when a roller revolves once?

122. How large a cylindrical tank can be placed in a triangular plot 42 by 50 by 55 ft?

Show that the radius r of the inscribed circle $= A/s$ [see Eq. (5), Appendix B].

123. Find, in terms of s, the area of the regular duodecagon shown in Fig. 4·43.

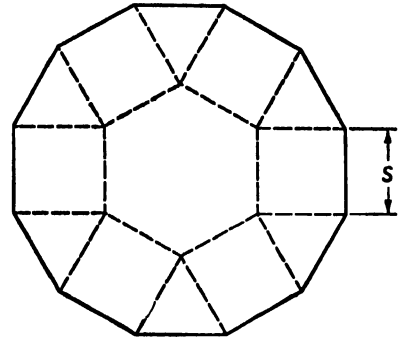


Fig. 4·43

124. Find the area of the arch in Fig. 4·44.

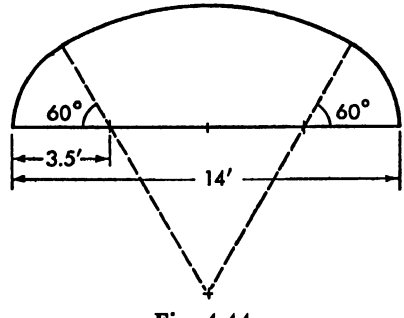


Fig. 4·44

125. Find the percentage of error in taking $4(PM)$ as equal to the circumference of the circle in Fig. 4·45. Assume the radius = 1.

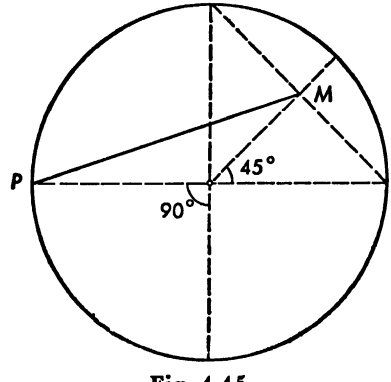


Fig. 4·45

Would the final answer be affected by assuming a different value for the radius?

126. Find the percentage of error in taking the area of the square as equal to that of the circle in Fig. 4·46 (see Prob. 125).

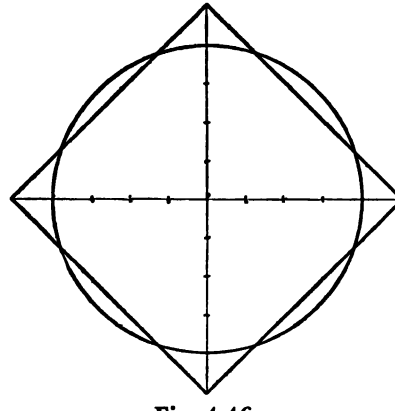


Fig. 4·46

127. Two 12-in. circles have their centers 6 in. apart. What is the area of the portion common to both?

128. A 1½-in. lead sphere is to be flattened into a circular sheet 14 in. in diameter. Find the thickness of the sheet.

129. A copper rod 24 in. long and ½ in. in diameter is to be drawn into wire $\frac{3}{64}$ in. in diameter. Find the length of the wire.

130. Water is flowing out of a 14-in.-diameter pipe into a circular reservoir 350 ft in diameter. If the average linear flow through the pipe is 9 mph, how much will the surface of the reservoir rise in 24 hr?

131. A 3-in.-diameter piston ring should have an end clearance of 0.010 to 0.015 in. If the end clearance was 0.010 in when installed, how much could the diameter increase with wear before the clearance exceeded 0.015 in.?

132. Find the weight supported by a hollow copper sphere completely submerged in water. The average diameter of the sphere is 4 in., and the metal is 0.04 in. thick. The densities of copper and water are 0.317 and 0.0361 lb per cu in., respectively.

133. A circular pan is 3 in. deep and 8 in. in diameter at the base. The sides flare out at an angle of 45°. If 1 in. of water collected in the pan during a rainstorm, how many inches of rain actually fell?

134. A cylindrical tank having a hemispherical base is 12 ft high over all and 8 ft in diameter. Find its capacity in gallons.

135. A horizontal 300-gal cylindrical tank has a diameter of 40 in. A gauge is to be made by locating depth marks corresponding to 50, 100, 150, 200, and 250 gal. How far from the bottom of the gauge should each of these marks be located? Use Eq. (87), Appendix B, to determine volumes corresponding to various depths. From these values plot a graph of volume against depth. Read off the depths corresponding to the given volumes.

136. A hollow bronze sphere is 11 in. in diameter and weighs 52 lb. The wall thickness is assumed to be uniform and is to be determined without mutilating the sphere. Assume the density of bronze = 0.30 lb per cu in. How might we test for uniformity of wall thickness?

137. Find the volume of a square pyramid each of whose edges is 7 in. long.

138. A pile of coal is 14 ft wide and 40 ft long over all, with semicircular ends. Find the number of tons in the pile if the angle of repose is 30° and a cubic foot of broken coal weighs 38 lb.

139. Find the capacity in gallons of a pail 11 in. in diameter at the top, 9 in. in diameter at the bottom, and 10 in. deep.

140. The profile AB of the nose of an experimental rocket in Fig. 4·47 is an arc of a circle whose center C is on the line BC. If AD is 24 ft and BD is $2\frac{1}{2}$ ft, find the radius r of the cross section at distances x from the nose tip, making x successively equal 6 ft, 12 ft, 18 ft.

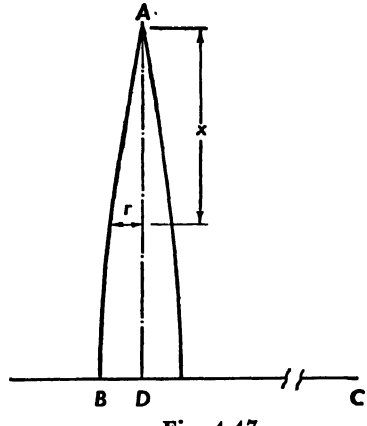


Fig. 4·47

141. In Fig. 4·48 a ship AB lies anchored at right angles to the shore line CE. Find the distance CD corresponding to the maximum subtended angle ADB (target in best position for a direct hit).

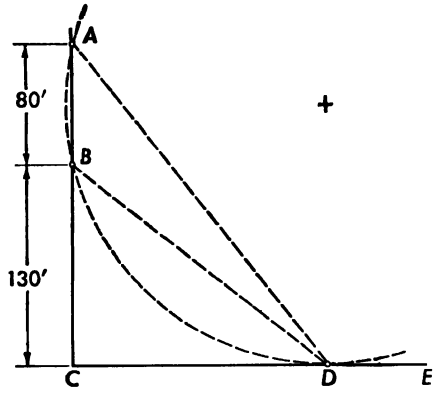


Fig. 4·48

142. As an auto traveled at constant speed, a point P on the tread of the tire traveled through space with a varying speed, as shown by values taken at intervals during one revolution of the wheel.

sec	0	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16
fps	0	38	69.5	91.5	99	91.5	69.5	38	0

Find the speed of the car and the diameter of the wheel.

143. A strip of metal 12 in. wide is bent along the center line to form a V trough. What angle between the sides would correspond to the maximum water capacity of the trough?

UNIT TWO

BASIC ALGEBRA

Algebra which the student should know already, but which he may have forgotten

Chapter 5 INTRODUCTION TO ALGEBRA

In this chapter we shall attempt to set forth some of the first principles of algebra. Their mastery is essential to the successful handling of the more involved applications in later chapters.

Mathematics differs from some other studies in that it is built layer upon layer. Once a particular mathematical operation is learned, it will be used repeatedly thereafter in subsequent operations. In fact, a study of a later operation would be futile without the support of the thoroughly mastered rudiments. By way of analogy it might be pointed out that the construction of the foundation of a house must precede that of the first floor. The foundation supports the superstructure for all time —unlike a scaffolding, which is removed after it has served its purpose.

5·1 Explicit and Literal Numbers. In arithmetic we represent quantities by *explicit* numbers such as 3, 8, 0, whereas in algebra we also make use of various letters of the alphabet to designate a general quantity. Such number symbols as a, r, x, and y will be called *literal* numbers. For example we use the letter r to represent the number of units of length in the radius of a circle. Its value changes for different circles. It might be 5 yd, 8 in., or any number of length units of any denomination, but in a general way it represents the radius of a circle. When we see the general formula $C = 2\pi r$ for the circumference of a circle, we do not think of a circle of any particular size but rather of circles in general.

Numbers used in elementary algebra are called *real* numbers, in contrast with *imaginary* numbers, which we shall consider at a later stage. Real algebraic numbers may be positive, negative, or zero, whereas arithmetic numbers do not include negative values. A number written without a sign is understood to be positive; thus the number 5 is understood to be $+5$.

Real numbers may also be classified as *rational* or *irrational*. The former can be expressed as the ratio of two integers; *e.g.*, $\frac{2}{3}$, 5, $\frac{7}{4}$. The latter cannot be so expressed; *e.g.*, $\sqrt{2}$, $\sqrt[3]{6}$, π, e.

5.2 Absolute Numerical Values. Positive and negative signs are used to convey the idea of measuring in opposite directions from a zero value or reference point. For example, $+12°$ and $-7°$ refer to 12° *above* and 7° *below* 0°, respectively. Likewise, if +140 lb designates a *lifting* force of 140 lb, then −38 lb refers to a dead *weight* of 38 lb. Further, if +$4 denotes a *credit* of $4, then −$6 represents a *debit* of $6. The velocity of a guided missile might be +3,400 fps on the upward trajectory, and a little later a similar velocity on the descent would be referred to as −3,400 fps. However, if we simply wish to designate the rate of motion without specifying sense or direction (that is the *speed*), we write |3,400| fps, which is called the *absolute numerical value* of either +3,400 or −3,400 fps.

5·3 Addition. The operation of addition of signed numbers may well be illustrated by the balancing of credits and debits.

For example,

$7 + 8 = 15$ (Adding credits of $7 and $8 results in a total credit of $15)
$-3 - 1 = -4$ (Adding debits of $3 and $1 gives us a total debit of $4)
$-4 + 7 = +3$ (Adding a debit of $4 and a credit of $7 gives us a net credit of $3)
$-9 + 2 = -7$ (Adding a debit of $9 and a credit of $2 results in a net debit of $7)

From the above it will be seen that the following rules are valid:

To add two quantities of the same sign, add their absolute values and prefix the sum by their common sign.

To add two quantities of opposite sign, find the difference of their absolute values and prefix the result by the sign of the quantity having the greater absolute value.

To find the sum, or net value, of several quantities of both positive and negative sign, add all the positive quantities and all the negative quantities and find the sum of these two results.

EXAMPLE 1. Add 5, −3, −6, +2, −1, +7, −8. We find that we are adding 5, +2, +7 and −3, −6, −1, −8, or +14 and −18, which = −4.

5·4 Subtraction

Subtraction of two quantities is performed by reversing the sign of the subtrahend and proceeding as with the addition of signed numbers.

Examples of this rule follow (in which the lower number is subtracted from the upper number):

$+8$	$+4$	-7	-2	-5	$+7$
$+5$	$+9$	-3	-9	$+6$	-3
$+3$	-5	-4	$+7$	-11	$+10$

Note that in the second example it is possible to subtract a larger number from a smaller. Consider the embarrassing situation of the bank depositor who, although he has only a \$4 balance, draws a check of \$9 and is therefore \$5 overdrawn!

In the last example a man may figure that he is \$7 to the good after taking account of all his debits and credits. If, after taking stock, an outstanding debt of \$3 is canceled (subtract $-\$3$), or \$3 in cash is received (add $+\$3$), the result is the same in either case his net worth is \$10.

5·5 Multiplication. The multiplication of signed quantities is governed by the following rules:

1. *The product of two quantities of like sign is positive.*
2. *The product of two quantities of opposite sign is negative.*
3. *The continued product of three or more quantities is positive if there is an even number of negative quantities, and negative if there is an odd number of negative quantities.*

To this rule we may add a corollary:

Even powers of both positive and negative quantities are positive. Odd powers of positive quantities are positive, and odd powers of negative quantities are negative.

EXAMPLE 2

a. The product of $+2$ and $+3$ is $+6$.
b. The product of $+2$ and -3 is -6.
c. The product of -2 and $+3$ is -6.
d. The product of -2 and -3 is $+6$.

These examples may be illustrated by the following commonplace situations:

a. The temperature is rising at the rate of 2° an hour. At this rate it will be 6° warmer 3 hr hence.

b. At the same rate it was 6° colder 3 hr ago.

c. The temperature is falling at the rate of 2° an hour. At this rate it will be 6° colder 3 hr hence.

d. At the rate in *c*, it was 6° warmer 3 hr ago.

4. *Changing the sign of a quantity is equivalent to multiplying that quantity by* -1.

5. *The product of zero by any number is equal to zero.*

6. *If the product of two or more quantities is zero, at least one of the quantities is zero.*

5·6 Division. The rules of sign governing division resemble those for multiplication, since dividing by a quantity x is equivalent to multiplying by the fraction $1/x$ (the reciprocal of x).

In addition to the foregoing, the following principles apply to division:

1. *A quantity cannot be divided by zero.*

For example, assume $5 \div 0 = n$. Then $5 = n \times 0$. But this is impossible since $n \times 0 = 0$.

2. *The value of a fraction is unchanged if both numerator and denominator are multiplied, or divided, by the same number.*

3. *Changing the sign of either the numerator or denominator of a fraction reverses the sign of the fraction. Reversing the sign of both numerator and denominator does not change the sign of the fraction.*

4. *Division of zero by zero is indeterminate.*

For example, assume $0 \div 0 = n$. Then $0 = 0 \times n$. But $0 \times n = 0$ whatever the value of n; hence n is indeterminate.

5·7 Signs of Operation. The signs $+$, $-$, $\times$, $\div$, and $\sqrt{\ }$ have the same meaning as in arithmetic. However, the sign $\times$ is not commonly used in algebra. For example, multiplication of a by b is customarily written $a \cdot b$, or simply ab. Thus abc has the same meaning as $a \times b \times c$. If we wish to multiply the expression $a + b$ by the expression $x + y$, we write $(a + b)(x + y)$, the absence of any connecting sign indicating multiplication.

EXERCISE 1

1. Find the sum of each of the following pairs of numbers:

a. $+8, +5$	**b.** $+7, -2$	**c.** $+4, +10$	**d.** $+6, -9$
e. $-3, -11$	**f.** $-12, -8$	**g.** $-9, +13$	**h.** $-11, +4$

2. Subtract the second number from the first in each pair of numbers in Prob. 1.
3. Find the algebraic sum of the following:

a. $8 + 2 - 3 - 4 + 6 - 1 - 9 + 5$
b. $-3 + 7 + 2 - 6 - 5 + 1 - 8$
c. $4 + 10 - 3 + 7 - 6 - 9 + 2$

4. Find the product of each of the following groups of numbers:

a. $(8)(5)$ **b.** $(-6)(7)$ **c.** $(-12)(-4)$
d. $(9)(-6)$ **e.** $(2)(-3)(4)$ **f.** $(-5)(3)(-4)$
g. $(-3)(-6)(2)(-1)$

5. Perform the indicated operations:

a. $20 \div 5$ **b.** $24 \div (-8)$ **c.** $(-30) \div 6$
d. $(-35) \div (-7)$ **e.** $\dfrac{-28}{(-7)(3)}$ **f.** $\dfrac{(36)(15)}{-81}$
g. $\dfrac{(42)(-35)}{(-49)(-18)}$

5·8 Algebraic Expression. An *algebraic expression* is one that represents an operation according to the signs and symbols of algebra.

In an algebraic expression, a *term* is any part that is not broken by a plus or minus sign. Thus, in the expression $ab - ac + bcxy$, the terms are ab, $-ac$, and $bcxy$.

An algebraic expression containing only one term is called a *monomial*, *e.g.*, bcm or $-ax$. An expression containing two terms is called a *binomial*, *e.g.*, $mn + xyz$; one containing three terms is called a *trinomial*, *e.g.*, $k - hn + w$. In general, expressions containing more than one term are called *polynomials*, or *multinomials*.

A *coefficient* is the multiplier of a term. In the term $5xyz$, 5, x, y, and z are called the factors of the expression. Usually the numerical part of the term is regarded as the coefficient, although any one or more of the factors could be taken as the coefficient of the remaining factors. Thus, in the term $5xyz$, $5y$ could be considered the coefficient of xz. When the expression contains no numerical coefficient, a coefficient of 1 is understood. Thus bcm is the same as $1\ bcm$.

An *exponent* is a figure or letter placed to the right of and somewhat above a quantity called the *base* to indicate how many times the base is to be taken as a factor. Thus a^4 means $a \cdot a \cdot a \cdot a$ and is read "a fourth" or "a to the fourth *power*."

It is important to maintain a clear distinction between coefficient and exponent: $3x$ means $x + x + x$, whereas x^3 means $x \cdot x \cdot x$.

When a quantity is written without an exponent, the exponent 1 is understood. Thus y means y^1.

Similar monomials, having a common literal part, can be combined by combining the coefficients and affixing the common literal part. If the monomials are unlike, the operations can only be indicated.

EXAMPLE 3

$$\begin{array}{r} 5cd^2 \\ cd^2 \\ -2cd^2 \\ \hline 4cd^2 \end{array}$$

EXAMPLE 4

$$\begin{array}{r} 7xy \\ 5x^2 \\ -2y^2 \\ \hline 7xy + 5x^2 - 2y^2 \end{array}$$

EXERCISE 2

Write the expressions in Probs. 1 to 8 in the most concise form possible by using coefficients and exponents.

1. $a + a + a - a + a$
2. $x - x - x + x - x - x$
3. $y \cdot y \cdot y \cdot y$
4. $a \cdot a \cdot b \cdot b \cdot b$
5. $4x \cdot y \cdot y \cdot z \cdot z \cdot z$
6. $(a + c)(a + c)(a + c)$
7. $w(xw)w(xw)$
8. $a(ab)(abc)(bc)c$

Find the numerical values in Probs. 9 to 30 if $a = 3, b = 4, c = -5$, and $x = 1$.

9. $a + b$
10. $2c - a$
11. $7(2a - c)$
12. $3a^2bc$
13. $4a^2 - b^2$
14. $(a + c)^2$
15. $b^2 - x^2$
16. $(3a - b)^3$
17. $(a + b)(a + b)$
18. $a + b(a + b)$
19. $(a + b)a + b$
20. $a + ba + b$
21. $(2a - b + c)^2$
22. $(c + b)^3x$
23. $\sqrt{a^2 + b^2}$
24. $\sqrt{(3a)^2 - b^2 - x^2}$
25. $\dfrac{4a + b}{c + 3x}$
26. $\dfrac{4a}{c} + \dfrac{b}{3x}$
27. $(4b + 3c - x)^4$
28. $\left(\dfrac{4a + 2b}{3x - c}\right)^2$
29. $3abc^2$
30. $(3abc)^2$

Perform the indicated additions in Probs. 31 to 42.

31.
$$\begin{array}{r} 7x \\ 5x \\ -x \\ \hline \end{array}$$

32.
$$\begin{array}{r} -11ab \\ 4ab \\ ab \\ -2ab \\ \hline 8ab \end{array}$$

33.
$$\begin{array}{r} 9mn \\ -2mn \\ -mn \\ \hline 7mn \end{array}$$

34.
$$\begin{array}{r} 6cd \\ -4cd \\ 7cd \\ \hline -8cd \end{array}$$

35.
$$\begin{array}{r} 5x + 4y \\ x + 5y \\ \hline \end{array}$$

36.
$$\begin{array}{r} -3a + 2c \\ 4a - 3c \\ \hline \end{array}$$

37. $6m + 10p$
$\underline{6m - 2p}$

38. $4x^2 - 2xy + y^2$
$\underline{5x^2 + 3xy - 2y^2}$

39. $3k^2 - 2km + m^2$
$\underline{k^2 \qquad + 4m^2}$

40. $2p^2 \qquad - 3q^2$
$\underline{5p^2 + 4pq - 2q^2}$

41. $4x - 2y$
$\underline{5x \qquad - 3z}$

42. $10c^2 + cm - 3m^2$
$\underline{3c^2 + cm - 4m^2}$

43–50. In Probs. 35 to 42 subtract the second expression from the first.

51. From the sum of $a^2 - 2a + 3$ and $2a^2 - a + 5$ subtract $4a^2 - 4a + 7$.

52. From $8c^2 - 2c + 11$ subtract the sum of $c^2 - 4c + 6$ and $5c^2 + 3c + 5$.

53. Subtract the sum of $9x^2 - xy$ and $3x^2 + xy - 2y^2$ from $6x^2 - 3xy - y^2$.

5·9 Some Fundamental Rules of Exponents

1. *In multiplying two powers having the same base, add the exponents.*

EXAMPLE 5. $c^2 \cdot c^3 = c^{2+3} = c^5$
because
$$c^2 \cdot c^3 = (c \cdot c)(c \cdot c \cdot c) = c \cdot c \cdot c \cdot c \cdot c = c^5$$

EXAMPLE 6. $a \cdot a^2 \cdot a^5 = a^1 \cdot a^2 \cdot a^5 = a^{1+2+5} = a^8$

2. *To find the power of a power of a quantity, multiply the exponents.*

EXAMPLE 7. $(y^2)^3 = y^{2\times 3} = y^6$
because
$$(y^2)^3 = y^2 \cdot y^2 \cdot y^2 = (y \cdot y)(y \cdot y)(y \cdot y)$$
$$= y \cdot y \cdot y \cdot y \cdot y \cdot y = y^6$$

3. *To find the quotient of two powers of the same base, take the difference between the exponents.*

EXAMPLE 8. $m^5 \div m^3 = m^{5-3} = m^2$ for $m^5 \div m^3 = \dfrac{m \cdot m \cdot m \cdot m \cdot m}{m \cdot m \cdot m}$
$$= m \cdot m = m^2$$

A further definition will be given [Eq. (3), page 155] to cover cases in which the power of the numerator is less than that of the denominator.

EXERCISE 3

Perform the indicated operations:

1. $x^2 \cdot x^4$	**2.** $c^3 \cdot c^5$	**3.** $y^6 \cdot y$
4. $k^3 \cdot k^4 \cdot k^2$	**5.** $a \cdot a^2 \cdot a^3 \cdot a^4$	**6.** $(b^2)^4$
7. $(m^5)^3$	**8.** $(3a)^2$	**9.** $(5x^3)^2$
10. $(-4c^2)^3$	**11.** $(-2b^4)^6$	**12.** $-(-5w^4)^3$
13. $x^7 \div x^4$	**14.** $h^{12} \div h^3$	**15.** $y^6 \div y$
16. $(-a)^5 \div a^2$	**17.** $(-b)^6 \div (-b)^5$	**18.** $(-c)^9 \div (-c)^3$

5·10 Symbols of Grouping. Parentheses ( ), brackets [ ], or braces { } indicate that the expression which they enclose is to be treated as a unit. The less frequently used vinculum, $\overline{\quad}$, written over an expression has the same meaning.

EXAMPLE 9. $5 + (2 + 7) = 5 + 9 = 14$
EXAMPLE 10. $8 - (3 + 10) = 8 - 13 = -5$
EXAMPLE 11. $6 - \overline{7 - 2} = 6 - 5 = 1$
EXAMPLE 12. $4(5 + 2) = 4(7) = 28$
Note that we might write

$$\begin{aligned} 4(5 + 2) &= 4(5) + 4(2) \\ &= 20 + 8 = 28 \end{aligned}$$

The following remarks concerning parentheses will be understood to apply equally well to brackets and braces.

Parentheses are useful in enclosing expressions to be multiplied:

EXAMPLE 13. $5(-4)$ means 5 times -4, or -20.

EXAMPLE 14. To multiply the expression $a + 3$ by the expression $a - 5$, we write $(a + 3)(a - 5)$ (read "the quantity $a + 3$ times the quantity $a - 5$"). If $a = 9$, then $(9 + 3)(9 - 5) = (12)(4) = 48$.

Parentheses can be used to avoid ambiguity regarding the order of application of the fundamental operations.

EXAMPLE 15. Does the expression $14 - 6 \div 2$ mean that 8 is to be divided by 2 or that 3 is to be subtracted from 14? If written $(14 - 6) \div 2 = 8 \div 2 = 4$ in one case or $14 - (6 \div 2) = 14 - 3 = 11$ in the other, the meaning is no longer questionable.

To insert or remove parentheses preceded by a plus sign, rewrite the affected quantities with their respective signs unchanged.

EXAMPLE 16. $8 + (3 - 4) = 8 + 3 - 4 = 7$

To insert or remove parentheses preceded by a minus sign, rewrite the affected quantities with their respective signs reversed.

EXAMPLE 17. $11 - (7 - 2) = 11 - 7 + 2 = 6$

The last example may be illustrated by the following common situation. A man with \$11 in his possession buys a \$7 article and returns for credit a \$2 article.

If the purchase and refund were made at the same counter, the \$2 refund would probably be applied at the time of purchase, so that $11 - (7 - 2) = 11 - 5 = 6$; *i.e.*, only \$5 would change hands. If, on the other hand, the refund were made in another department after the \$7 purchase, we would write $11 - 7 + 2 = 6$. (In either case the customer still has \$6 in cash.)

Note that if we started with the expression $11 - 7 + 2$ and wished to enclose the last two quantities with a pair of parentheses preceded by a minus sign, we should have to write $11 - 7 + 2 = 11 - (7 - 2)$, since we have already seen that $11 - (7 - 2) = 11 - 7 + 2$ and the operation must be reversible.

When one set of grouping symbols is contained within another, the innermost set is removed first. Usually, parentheses are enclosed within brackets, and brackets within braces.

5·11 Similar Terms. *Similar terms* are terms containing the same letters with the same respective exponents, but not necessarily with the same coefficients. *Dissimilar terms* are terms containing different letters or the same letters with different exponents. Thus $7a^2b$, $-3a^2b$, and a^2b are similar terms, but $5ab^2$, $2a^2b^2$, and $5abx$ are dissimilar terms.

To collect a group of similar terms, add their coefficients and multiply this sum by the common factor.

EXAMPLE 18. $3cd + 5cd - cd = 7cd$

EXAMPLE 19. $axy + bxy = xy(a + b)$ (considering a and b the coefficients of xy)

EXAMPLE 20. Perform the indicated operations and collect (similar) terms in the following expression: $3x + 2 - [2y + 5 - (6y - 2x + 3)]$

Beginning with the inner group and performing the indicated multiplications with due attention to signs, we obtain

$$3x + 2 - [2y + 5 - 6y + 2x - 3]$$
$$= 3x + 2 - 2y - 5 + 6y - 2x + 3 = x + 4y$$

EXERCISE 4

In Probs. 1 to 10, simplify by removing the symbols of grouping and collecting and combining similar terms.

1. $3a + 4b + (2a - b)$
2. $5a - 3c - (2a - 4c)$
3. $4x - 7y - (3x - 4y) + x + 3y$
4. $6u + 5 - (3v + 4w - 2) - 2u + 4v$
5. $8m - [4n - (-3m + 3n) + m]$
6. $a - \{2x + [3a - 5x - (4a + x)] - 2a\}$
7. $-7b + \{6c - [2b - c + (8c - 4b) + b]\}$
8. $\{8c + 3k - [5c + 2k]\} - \{3c + k - [7c - (c - 4k)]\}$
9. $11ab - 3a^2b - [5ab^2 - (6ab + 2a^2b) + ab] + 3ab^2$
10. $\{4x^2 - xy + [y^2 - 2xy] - x^2\} - \{3x^2 - [y^2 - (2xy + x^2) + xy]\}$

In Probs. 11 to 16, enclose the last three terms in parentheses preceded by (**a**) a minus sign and (**b**) a plus sign.

11. $5a + 2b - 4c + m - x$
12. $2a^2 + 3ab - 7b^2 + 4b$
13. $7mn + 3m^2 - 4n^2 - 8m + 5n + 2mn$
14. $6c^2 + 5cd + d^2 - 2m^2$
15. $9w^2 + 5y^3 - 2w^2y - 3wy^2 + w^3$
16. $-7x^2 - 5xy - 2y^2 + 4yz - z^2$

5·12 Multiplying by a Monomial

To multiply a monomial by a monomial, multiply the product of the numerical coefficients by the continued product of the literal factors, making use of the law of exponents where it applies. Determine and apply the proper sign.

EXAMPLE 21. Find the indicated product of

$$(-3axy^2)(5a^3x^2) \qquad \text{ANS.: } -15a^4x^3y^2$$

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial and write the sum of the products.

EXAMPLE 22. $4x^2y(5a^2x - 7b^2y) = 20a^2x^3y - 28b^2x^2y^2$

EXERCISE 5

Perform the following indicated multiplications:

1. $(8mn)(4mx)$
2. $(-3h^2x)(6hx^2)$
3. $(9abc)(-4bcd)$
4. $(-2ck^2x)(-7c^2x^5)$
5. $(a^2m)(m^2x)(ax^2)$
6. $(3ac^2d)(-4cd^3)(-2a^4cd^2)$

7. $(-9a^2b^3c^4)(-4a^3b^4m^5)(-2b^4c^5m^6)$

8. $5(4h - 6k)$ **9.** $-3y(6m - 5t)$

10. $4ax(5ay + 9mx)$ **11.** $6a^2b(5a^2 - 7ab - 9b^2)$

12. $-2xy^2(1 - 2x + 3x^2 - 4x^3)$ **13.** $(-3hk)(-2hk^3)(4a^3h^2 - 11bk)$

14. $\frac{2}{3}ab^2c^3(12ax - 21by)$

5·13 Multiplying One Polynomial by Another Polynomial

In order to multiply one polynomial by another, arrange the terms in descending powers of one of the letters involved. Multiply the first polynomial by the successive terms of the second, arranging like terms of the product in columns, and add.

EXAMPLE 23. Multiply $10x - 2x^3 + x^4 - 1$ by $x^2 + 2 + 6x^3$.

Arranging in descending powers of x,

$$
\begin{array}{lll}
x^4 - 2x^3 + 10x - 1 & & \\
6x^3 + x^2 + 2 & & \\
\hline
6x^7 - 12x^6 + 60x^4 - 6x^3 & & \text{(multiplying by } 6x^3\text{)} \\
\quad + x^6 - 2x^5 + 10x^3 - x^2 & & \text{(multiplying by } x^2\text{)} \\
\quad + 2x^4 - 4x^3 + 20x - 2 & & \text{(multiplying by 2)} \\
\hline
6x^7 - 11x^6 - 2x^5 + 62x^4 - x^2 + 20x - 2 & &
\end{array}
$$

The answer may be checked with reasonable certainty by substituting some simple number for x. Substitution by the number 1 should be avoided since any power of 1 is still 1.

Let us check by letting $x = 2$; then

$$x^4 - 2x^3 + 10x - 1 = (2)^4 - 2(2)^3 + 10(2) - 1 = 16 - 16 + 20 - 1 = 19$$

and

$$6x^3 + x^2 + 2 = 6(2)^3 + (2)^2 + 2 = 48 + 4 + 2 = 54$$

$$(19)(54) = 1{,}026$$

$$6x^7 - 11x^6 - 2x^5 + 62x^4 - x^2 + 20x - 2$$
$$= 6(2)^7 - 11(2)^6 - 2(2)^5 + 62(2)^4 - (2)^2 + 20(2) - 2$$
$$= 768 - 704 - 64 + 992 - 4 + 40 - 2 = 1{,}026$$

EXERCISE 6

Perform the following multiplications:

1. $(a + b)(c + d)$ **2.** $(m + x)(m - y)$

3. $(h + k)(h - k)$ **4.** $(y^4 - y^2)(y^5 + y^3)$

5. $(2m + 5w)(3m - 7w)$ **6.** $(4x - 6y)(6x + 9y)$

7. $(4x - 2y - 13)(6x + 3y)$ **8.** $(3p^2 + 4pq + 5q^2)(2p - 3q)$

9. $(b - x - y)(b + x + y)$ **10.** $(c + d - 5)(c - d + 4)$

11. $(n^2 + 2n + 4)(n - 2)$ **12.** $(a^2 - 5ab + b^2)(a^2 + 5ab - b^2)$
13. $(x^4 - x^3 + 4x^2 - x + 1)(x + 1)$
14. $(c^3 + 2c^2 - 9c - 18)(c - 2)$

5·14 Removing Parentheses Used to Indicate Multiplication. In removing parentheses used to indicate multiplication, the usual rules of multiplication apply.

EXAMPLE 24. Simplify the expression

$$3 + 2\{5 - 4x[1^* - 6(x + 2)]\}$$

Performing the indicated multiplications beginning with the innermost group, we obtain

$$\begin{aligned} 3 + 2[5 - 4x(1 - 6x - 12)] &= 3 + 2(5 - 4x + 24x^2 + 48x) \\ &= 3 + 10 - 8x + 48x^2 + 96x \\ &= 13 + 88x + 48x^2 \end{aligned}$$

EXERCISE 7

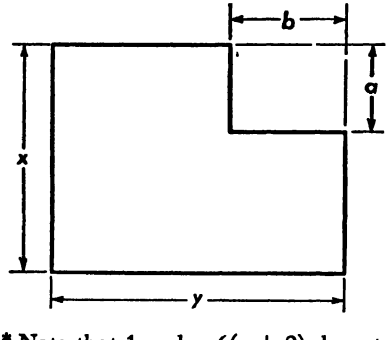
Simplify the following expressions:

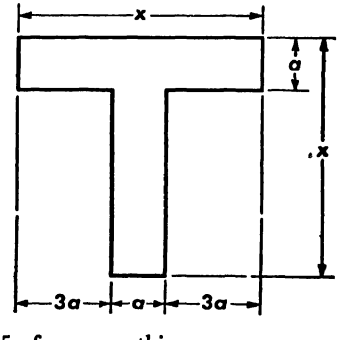
1. $1 - 2\{1 + 3[1 - 4(1 - 5x)]\}$
2. $a - \{[b - c] - [a + b - c - 2(a - b + c)]\}$
3. $5\{4[3(2 + x)]\} - 5\{-4[-3(2 - x)]\}$
4. $y^2 - y\{y + z[x(y - z) + y(z - x) + z(x - y)]\}$

EXERCISE 8

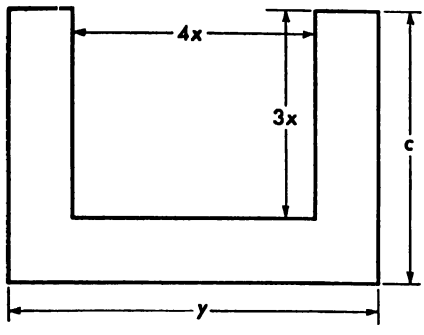
In the following problems, in cases where the result can be expressed in more than one way, the use of the letter representing the smaller dimension is to be preferred.

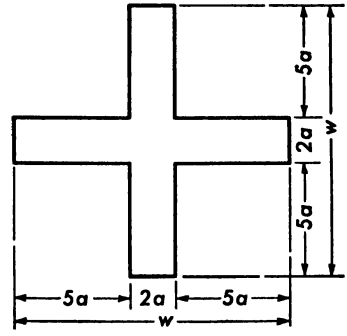
1–4. Express in the simplest possible form the perimeter of each of the following figures.





* Note that 1 and $-6(x + 2)$ do not make -5 of any one thing.





5–8. Express in the simplest possible form the area of each of the above.

9. Find the area of the form in Fig. 5·1.

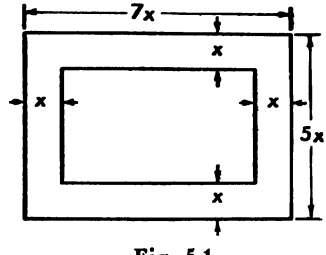


Fig. 5·1

10. Find the area of overlap in Fig. 5·2.

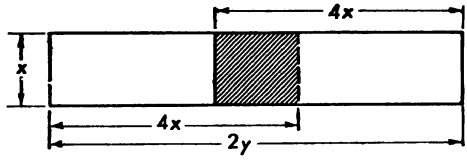


Fig. 5·2

11. Find the outer surface and volume of the open-top box shown in Fig. 5·3.

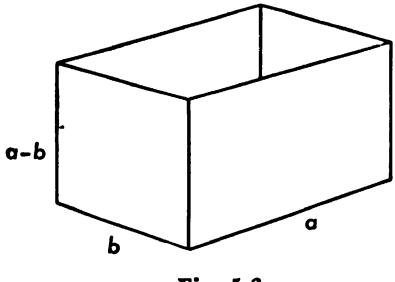


Fig. 5·3

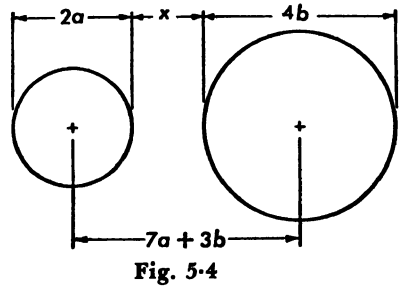


Fig. 5·4

12. Find in terms of a and b the distance x in Fig. 5·4.

13. Find in terms of D and d the wall thickness t in Fig. 5·5.

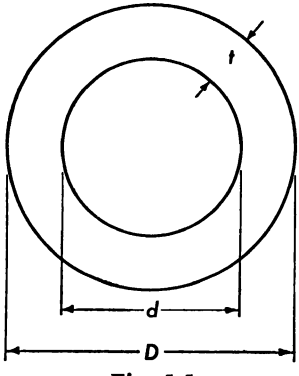


Fig. 5·5

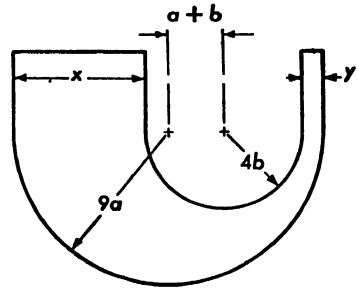


Fig. 5·6

14. Find in Fig. 5·6 the values of x and y in terms of the other letters.

5·15 Dividing One Monomial by Another Monomial

To divide one monomial by another monomial, divide the coefficient of the dividend by the coefficient of the divisor, apply the rule of exponents to the literal factors, and prefix the proper sign.

EXAMPLE 25. Divide $-20x^3yz^2$ by $-5xyz$.

Regarding the problem as a cancellation, we write

$$\frac{-\overset{4}{\cancel{20}}\cancel{x}\cdot x\cdot x\cdot\cancel{y}\cdot z\cdot\cancel{z}}{-\cancel{5}\cancel{x}\qquad\cancel{y}\qquad\cancel{z}} = 4x^2z$$

EXAMPLE 26. Divide $-12a^2bc^3$ by $16a^3cx^2$.

Cancellation gives us

$$\frac{-\overset{3}{\cancel{12}}\cancel{a}\cdot\cancel{a}\cdot b\cdot\cancel{c}\cdot c\cdot c}{\underset{4}{\cancel{16}}\cancel{a}\cdot\cancel{a}\cdot a\cdot\cancel{c}\cdot x\cdot x} = -\frac{3bc^2}{4ax^2}$$

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial and write the sum of the quotients.

EXAMPLE 27. $(24c^4x - 21c^2x^2 + 18cx^2) \div (-6c^2x) = -4c^2 + \frac{7x}{2} - \frac{3x}{c}$

EXERCISE 9

Obtain the following quotients:

1. $15c^3 \div 5c^2$
2. $-24h^3 \div 8h$
3. $35c^3d^2 \div (-7cd^2)$
4. $-42ab^2c^3 \div (-3ac^2)$

5. $12x^{12} \div 3x^3$
6. $-12ab^2m^3 \div 18a^3bm^2$
7. $40x^2y^3z \div 5wxz^3$
8. $-28a^4h^2k^3 \div (-20a^3h^3n^3)$
9. $(6ab - 8ac) \div 2a$
10. $(12c^2y + 20cw) \div 4c$
11. $(16km^2 - 24k^2m) \div (-8km)$
12. $(6xy - 30x^2y - 6xy^2) \div 6xy$
13. $(\pi r^2h + 2\pi rh) \div \pi rh$
14. $(35xy^4 - 20x^2y^3) \div 10xy^4$
15. $(20hk^3r^4 + 30h^2k^2r^2 - 21h^3k^2r^2) \div 12h^2k^2r^3$
16. $(10a^3bc - 12ab^2c^2 + 16ab^4) \div 8a^2b^2c^2$

5·16 Division of a Polynomial by a Polynomial. Division of a polynomial by a polynomial is governed by essentially the same rules as long division in arithmetic.

EXAMPLE 28. Divide $3x^2 + 8x + 9$ by $x + 2$.

$$
\begin{array}{r|l}
x + 2 & 3x + 2 \\
\hline
3x^2 + 8x + 9 & \\
3x^2 + 6x & \\
\hline
2x + 9 & \\
2x + 4 & \\
\hline
5 &
\end{array}
$$

ANS.: $3x + 2 + \dfrac{5}{x + 2}$

If we let $x = 10$, we obtain the very similar long division in arithmetic.

$$
\begin{array}{r|l}
12 & 32 \\
\hline
389 & \\
36 & \\
\hline
29 & \\
24 & \\
\hline
5 &
\end{array}
$$

ANS.: $32\frac{5}{12}$

In each case we note that

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Therefore we proceed to divide one polynomial by another as follows:

1. *Arrange each polynomial in either descending or ascending powers of some letter common to each expression.*

2. *Divide the first term of the dividend by the first term of the divisor to find the first term of the quotient.*

3. *Multiply the entire divisor by the first term of the quotient and subtract the result from the dividend.*

4. *Consider the remainder obtained in step* 3 *as a new dividend and repeat steps* 2 *and* 3 *continuing the process until there is either no remainder or a remainder whose first term cannot be evenly divided by the first term of the divisor.*

EXAMPLE 29. Divide $8x - 13x^2 + 6x^3 - 12$ by $2x - 3$.
Rearranging, we obtain

$$
\begin{array}{r|l}
2x - 3 & 3x^2 - 2x + 1 \\
\hline
 & 6x^3 - 13x^2 + 8x - 12 \\
 & \underline{6x^3 - 9x^2} \\
 & \quad -4x^2 + 8x \\
 & \quad \underline{-4x^2 + 6x} \\
 & \qquad\qquad 2x - 12 \\
 & \qquad\qquad \underline{2x - 3} \\
 & \qquad\qquad\quad -9
\end{array}
$$

ANS.: $3x^2 - 2x + 1 - \dfrac{9}{2x - 3}$

Note that while remainders in arithmetical division are always positive, they may be either positive or negative in algebraic division.

To check, (quotient)(divisor) + remainder = dividend. In this illustration,

$$(3x^2 - 2x + 1)(2x - 3) - 9 = 6x^3 - 13x^2 + 8x - 12$$

A check may also be obtained which is usually valid as in multiplication by numerical substitution (avoiding use of 0 and 1, or any value of x which makes the divisor equal zero).

Occasionally the dividend will be lacking one or more intermediate powers of a letter. In this case space should be left where such gaps occur.

EXAMPLE 30. Divide $36x^4 - 25x^2 + 4$ by $3x - 2$. It will be noted that there are no x^3 and x terms in the dividend. Hence we arrange as follows:

$$
\begin{array}{r|l}
3x - 2 & 12x^3 + 8x^2 - 3x - 2 \\
\hline
 & 36x^4 \quad - 25x^2 \quad + 4 \\
 & \underline{36x^4 - 24x^3} \\
 & \quad + 24x^3 - 25x^2 \\
 & \quad \underline{24x^3 - 16x^2} \\
 & \qquad -9x^2 \\
 & \qquad \underline{-9x^2 + 6x} \\
 & \qquad\qquad -6x + 4 \\
 & \qquad\qquad \underline{-6x + 4}
\end{array}
$$

EXERCISE 10

Divide

1. $a^2 + 11a + 24$ by $a + 3$
2. $c^3 - 2c^2 - 2c + 1$ by $1 + c$
3. $8m^2 - 22mw + 15w^2$ by $2m - 3w$
4. $16z^4 - 1$ by $2z - 1$
5. $6k^6 - 48$ by $3k^2 - 6$
6. $8q^2 - q - 19q^3 + 15q^4 - 1$ by $5q^2 - 3q - 1$
7. $h^6 - 2h^3 + 1$ by $h^2 + h + 1$
8. $y^5 - 2y^4 - 13y^2 + 17y + 5$ by $3y + 1 + y^3 - 4y^2$
9. $64x^6 - 1$ by $2x - 1$
10. 1 by $1 + x$ (to 4 terms)
11. 1 by $x + 1$ (to 4 terms)

5·17 Special Products or Expansions. The following special products should be verified by the student and memorized both as a literal formula and as an equivalent descriptive sentence.

$$a(x + y + z) = ax + ay + az \tag{1}$$

Example: $3ab^2(5ax - 8b) = 15a^2b^2x - 24ab^3$

$$(x + y)(x - y) = x^2 - y^2 \tag{2}$$

Example: $(3cm^2 + 4x)(3cm^2 - 4x) = 9c^2m^4 - 16x^2$

$$(x + y)^2 = x^2 + 2xy + y^2 \tag{3}$$

Example: $(5ab + 4w)^2 = 25a^2b^2 + 40abw + 16w^2$

$$(x - y)^2 = x^2 - 2xy + y^2 \tag{4}$$

Example: $(2ax - 7x^2y)^2 = 4a^2x^2 - 28ax^3y + 49x^4y^2$

$$(x + a)(x + b) = x^2 + (a + b)x + ab \tag{5}$$

Example: $(x + 5)(x - 2) = x^2 + (5 - 2)x - 10$
$= x^2 + 3x - 10$

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd \tag{6}$$

Example: $(3x + 8)(2x - 7) = 6x^2 + (-21 + 16)x - 56$
$= 6x^2 - 5x - 56$

There are additional special products which have not been listed above, as they arise too infrequently to be memorized. Some of these are as follows:

$$(a + b)(c + d) = ac + ad + bc + bd \tag{7}$$

$$(x - y)(x^2 + xy + y^2) = x^3 - y^3 \tag{8}$$

$$(x + y)(x^2 - xy + y^2) = x^3 + y^3 \tag{9}$$

$$(x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) = x^n - y^n \tag{10}$$

$$(x + y)(x^{n-1} - x^{n-2}y + \cdots + xy^{n-2} - y^{n-1}) = x^n - y^n \tag{11}$$

(for even values of n only)

$$(x + y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1}) = x^n + y^n \tag{12}$$

(for odd values of n only)

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \tag{13}$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3 \tag{14}$$

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz \tag{15}$$

$$(x^2 + xy + y^2)(x^2 - xy + y^2) = x^4 + x^2y^2 + y^4 \tag{16}$$

EXERCISE 11

Obtain the following special products as far as possible as an oral exercise:

1. $4xy(5a - 7b)$
2. $-5m^2x(3bm - 8cx)$
3. $2ab^2c(3ax - 4by + 7cz)$
4. $-8hk(-2ah^2 - 9b^2k)$
5. $(x - 3)(x + 3)$
6. $(7 - c)(7 + c)$
7. $(8k - 3m)(8k + 3m)$
8. $(5c^2 - 4n)(5c^2 + 4n)$
9. $(9pq - 7r)(9pq + 7r)$
10. $(c + 7)^2$
11. $(8 + n)^2$
12. $(3a + 1)^2$
13. $(4b + 3)^2$
14. $(6c + 5k)^2$
15. $(12am + 7y)^2$
16. $(m - 6)^2$
17. $(6 - m)^2$
18. $(h - 8)^2$
19. $(5x - 1)^2$
20. $(1 - 4c)^2$
21. $(7t - 9)^2$
22. $(11w - 4)^2$
23. $(5h - 12n)^2$
24. $(d + 3)(d + 4)$
25. $(m + 7)(m - 3)$
26. $(y + 2)(y - 9)$
27. $(a - 4)(a - 11)$
28. $(x + 1)(x + 6)$
29. $(z - 4)(z - 12)$
30. $(2b + 3)(3b + 1)$
31. $(4c - 1)(2c + 5)$
32. $(5m + 3)(4m - 7)$
33. $(6x - 1)(5x - 2)$

5·18 Factoring. Factoring is the inverse of the problems in Exercise 11. We shall now start with a given product and determine what quantities or factors were multiplied to give this product. We are interested in determining any monomial factors, also any prime polynomial factors, *i.e.*, polynomial factors which are divisible by no rational integral expression except themselves or 1.

By reversing each of the expansion rules in Sec. 5·17, we obtain

rules for factoring. Each example, upon interchanging right and left members of the identity, becomes an example of factoring.

In factoring out a common monomial factor of a polynomial each literal factor of the monomial is taken the least number of times that it occurs in any one of the terms of the original polynomial.

EXAMPLE 31. Factor $18ab^4x^3 - 30a^2b^2c$.

We obtain

$$6ab^2(3b^2x^3 - 5ac)$$

since 6 is the highest common factor of 18 and 30 and the lowest power of a in any term is 1 and of b is 2, with c and x missing entirely from one term each.

Hint on Factoring: If an expression appears to be of the type illustrated in Eq. (6), page 67, after removing any monomial factor test out the various factor combinations of the coefficients. We might go further and state that the factoring of any polynomial is simplified by first removing any monomial factors.

EXAMPLE 32. Factor $12x^2 + x - 6$.

Writing the factors in a general form, we have

$$(ax + b)(cx - d)$$

We note that ac must equal 12, $(b)(-d)$ must equal -6, and $bc - ad$ must equal 1. Two possible factors *might* be $(6x - 2)$ and $(2x + 3)$. Instead of actually multiplying, all we need to do is to write $\begin{matrix} 6 & -2 \\ 2 & +3 \end{matrix}$, representing the coefficients and constants arranged for multiplication. A glance at the cross products (6)(3) and (2)(−2) whose sum is $18 - 4$ or 14 shows that this is not the answer. A more careful inspection would have eliminated this choice as a possibility, since $6x - 2$ contains a factor of 2, whereas $12x^2 + x - 6$ does not. It will be seen from the arrangement $\begin{matrix} 3 & -2 \\ 4 & +3 \end{matrix}$ that $(3x - 2)(4x + 3)$ are the required factors (see Note, page 193).

EXERCISE 12

Factor out the common monomial factor from each of the following:

1. $15x - 18y$ **2.** $16x + 40y$ **3.** $30ab - 42km$

4. $ax + ay$ **5.** $a^2 - ac$ **6.** $12xy - 20x^2$

7. $abc + bcm + cmx$ **8.** $a^2b^2 + a^2b + ab^2$

9. $42ax^2y - 24bx^3y^2 + 18cx^4y$ **10.** $15ab^2c^2 - 12a^2bc^2 + 8a^2b^2c$

11. $10a^2x^2 - 15abxy + 20b^2y^2$ **12.** $30ax^2y + 48bxy^2 - 36cxyz$

EXERCISE 13

Factor completely:

1. $x^2 - 4$ **2.** $c^2 - 25$ **3.** $36 - y^2$
4. $16z^2 - 1$ **5.** $49m^2 - 1$ **6.** $4a^2 - 9b^2$
7. $25d^2 - 64m^2$ **8.** $36 - 121a^2b^2$ **9.** $100y^2z^2 - 49c^2d^2$
10. $121k^4 - 1$ **11.** $16n^8 - 1$ **12.** $16 - 81x^{12}$
13. $(25)^2 - (24)^2$ **14.** $(10\frac{1}{2})^2 - (8\frac{1}{2})^2$

EXERCISE 14

Find the missing terms which are needed to make the following expressions take the form of the trinomial squares $x^2 + 2xy + y^2$ or $x^2 - 2xy + y^2$. Insert only positive quantities within the parentheses.

1. $x^2 + (\ ) + 25$ **2.** $y^2 + (\ ) + 81$
3. $z^2 - (\ ) + 49$ **4.** $m^2 - (\ ) + 144$
5. $4a^2 + (\ ) + 1$ **6.** $36c^2 - (\ ) + 1$
7. $9n^2 + (\ ) + 25$ **8.** $16 - (\ ) + 121k^2$
9. $64b^2 - (\ ) + 25h^2$ **10.** $d^2 + 4d + (\ )$
11. $p^2 - 10p + (\ )$ **12.** $a^2 + 6ab + (\ )$
13. $c^2 - 14cq + (\ )$ **14.** $4w^2 - 28w + (\ )$
15. $9m^2 + 30mx + (\ )$ **16.** $25t^2 - 50tr + (\ )$

EXERCISE 15

Factor each of the following trinomial squares:

1. $x^2 - 4x + 4$ **2.** $b^2 + 10b + 25$
3. $36 - 12m + m^2$ **4.** $4y^2 - 20y + 25$
5. $9 + 42z + 49z^2$ **6.** $16c^2 + 24cd + 9d^2$
7. $64a^2 - 80an + 25n^2$ **8.** $49a^2b^2 - 112abc + 64c^2$
9. $81m^4 - 72m^2p + 16p^2$ **10.** $121x^2 + 154xyz^2 + 49y^2z^4$

EXERCISE 16

Factor each of the following expressions:

1. $y^2 + 7y + 12$ **2.** $a^2 + 9a + 20$
3. $m^2 - 12m + 35$ **4.** $c^2 - 15c + 36$
5. $x^2 + 19x + 48$ **6.** $b^2 - 2b - 24$
7. $h^2 + h - 30$ **8.** $w^2 - 11w - 42$
9. $k^4 + k^2 - 72$ **10.** $z^6 - 3z^3 - 108$
11. $x^2 + 21xy + 54y^2$ **12.** $y^2 - 25yz + 66z^2$

13. $2a^2 + 3a + 1$
14. $3c^2 + 10c + 3$
15. $4m^2 - 11m + 6$
16. $7k^2 - 16k + 4$
17. $8b^2 + 18b + 9$
18. $17n^2 + 6n - 11$
19. $54h^2 - 15h - 50$
20. $48x^2 + 14x - 45$

EXERCISE 17

In the following miscellaneous expressions the student is first to determine whether any factors exist. If an expression is factorable, it is suggested that any monomial factor be removed first. Factor completely into prime factors.

EXAMPLE 33. Factor:

$$\begin{aligned} 3a^2x^4 - 15a^2x^2 + 12a^2 &= 3a^2(x^4 - 5x^2 + 4) \\ &= 3a^2(x^2 - 4)(x^2 - 1) \\ &= 3a^2(x + 2)(x - 2)(x + 1)(x - 1) \end{aligned}$$

1. $9ac + 15bc$
2. $2c^3 - 32c^2 + 128c$
3. $4a^2 - 100$
4. $4m^2 - 20m + 100$
5. $6b^2 - 18b + 12$
6. $4y^3 - 8y^2 - 32y$
7. $16k^2 + 36$
8. $9am^2 - 30am + 9a$
9. $9h^2 - 34h + 25$
10. $100z^2 - 60z + 36$
11. $81x^4 - 16$
12. $35a^2b^2 - 80a^2b + 20a^2$
13. $4y^2 - 30y + 36$
14. $24h^2 + 12h - 36$
15. $36t^2 + 120t + 100$
16. $16rx^2 - 30rx + 9r$
17. $9m^2 - 12m + 16$
18. $6c^2 - c + 4$

5·19 Properties of Fractions. The basic properties of algebraic fractions resemble those of arithmetical fractions.

1. *If both numerator and denominator of a fraction are multiplied or divided by the same quantity (other than zero), the value of the fraction remains unchanged.*

2. *Multiplying the numerator or dividing the denominator of a fraction by a given quantity multiplies the fraction by that quantity.*

3. *Dividing the numerator or multiplying the denominator of a fraction by a given quantity divides the fraction by that quantity.*

4. *If an even number of factors in a fraction are multiplied by* (-1), *the sign of the fraction remains unchanged. If an odd number of factors are multiplied by* (-1), *the sign before the fraction must be reversed* (see Sec. 5·5).

EXAMPLE 34. The rule just expressed states in effect that we must introduce an even number of (-1)'s as a multiplier. Given the expression

$$\frac{(a-3)(a+2)}{a-8}$$

the following modifications are all equivalent:

$$*-\frac{(3-a)(a+2)}{a-8} \qquad -\frac{(a-3)(-a-2)}{a-8} \qquad -\frac{(a-3)(a+2)}{8-a}$$

$$\frac{(3-a)(-a-2)}{a-8} \qquad \frac{(3-a)(a+2)}{8-a} \qquad \frac{(a-3)(-a-2)}{8-a}$$

$$-\frac{(3-a)(-a-2)}{8-a}$$

The original expression with its minimum of minus signs is usually preferred because of its simplicity.

To reduce a fraction to its lowest terms, express the numerator and denominator as products of their prime factors and cancel all factors common to both.

EXAMPLE 35. $\dfrac{3a+6b}{a^2+3ab+2b^2} = \dfrac{3(\cancel{a+2b})}{(a+b)(\cancel{a+2b})} = \dfrac{3}{a+b}$.

EXAMPLE 36. $\dfrac{a^2+a-12}{9-a^2} = \dfrac{(a-3)(a+4)}{(3-a)(3+a)}$.

Multiplying by -1 in two places, we obtain

$$-\frac{(\cancel{a-3})(a+4)}{(\cancel{a-3})(3+a)} = -\frac{a+4}{a+3}$$

Note that any quantity canceled must be a factor of the entire numerator and denominator.

EXAMPLE 37. $\dfrac{3bx}{a+b}$ does *not* equal $\dfrac{3x}{a}$, for b is not a factor of $a+b$; here we have *subtracted* b instead of dividing.

EXAMPLE 38. $\dfrac{3bx}{ab+bc} = \dfrac{3\cancel{b}x}{\cancel{b}(a+c)} = \dfrac{3x}{a+c}$.

EXERCISE 18

Express each of the following negative fractions as an equivalent fraction containing a minimum of negative signs:

1. $-\dfrac{y}{5}$ 2. $-\dfrac{3}{-a}$ 3. $-\dfrac{-x}{-2}$ 4. $-\dfrac{1}{1-x}$

5. $-\dfrac{c-3}{4}$ 6. $-\dfrac{m+8}{7}$ 7. $-\dfrac{a-1}{a+1}$ 8. $-\dfrac{1+k}{k-1}$

* In this illustration the binomial $a-3$ in the original expression was multiplied by (-1) to give us $3-a$, which is called the *negative* of $a-3$ (and vice versa).

9. $-\dfrac{-b-c}{a-c}$ **10.** $-\dfrac{1}{(a-b)(a-c)}$ **11.** $-\dfrac{x-y}{(x-z)(y+z)}$

12. $-\dfrac{(k-m)^2}{h-k}$ **13.** $-\dfrac{(c-d)^3}{p+r}$ **14.** $-\dfrac{(a+b)(b-c)}{(b+c)(c-d)}$

EXERCISE 19

Reduce the following fractions to lowest terms where possible, using a minimum of negative signs in your answer:

1. $\dfrac{xy}{wx+xz}$ **2.** $\dfrac{a}{a^2+a}$ **3.** $\dfrac{am+an}{ab-ac}$

4. $\dfrac{8a-12b}{16a+24b}$ **5.** $\dfrac{a+b}{a^2+2ab+b^2}$ **6.** $\dfrac{k-m}{k^2-m^2}$

7. $\dfrac{2a+4b}{a^2-4b^2}$ **8.** $\dfrac{x^2-4x+4}{x^2-4}$ **9.** $\dfrac{6a+9b}{4a^2-9b^2}$

10. $\dfrac{2m^2-4m}{m^2+4m-12}$ **11.** $\dfrac{9-a^2}{3a^2-9a}$ **12.** $\dfrac{a^2+3a+2}{a^2+2a+1}$

13. $\dfrac{x^2-8x+16}{x^2-4}$ **14.** $\dfrac{5cm^3-20c^3m}{10cm^2-10c^2m-20c^3}$

15. $\dfrac{8a^2-16ab+8b^2}{4b^2-4a^2}$ **16.** $\dfrac{6x^2-12xy+18xz}{9mx+27mz-18my}$

17. $\dfrac{18x^2-24x+32}{9x^3-16x}$ **18.** $\dfrac{4c^2-17c+4}{12-3c}$

19. $\dfrac{(b-2c)^3}{2c^2+3bc-2b^2}$ **20.** $\dfrac{3x^2-10xy+3y^2}{6y^2-18xy}$

5·20 Least Common Multiple. The *least common multiple* (LCM) of several algebraic expressions is the product of all the factors of the various expressions, each factor being taken the greatest number of times that it occurs in any one of the given expressions. In other words the least common multiple is the smallest quantity which is divisible by all the several algebraic expressions.

EXAMPLE 39. Find the LCM of $6ax-9x^2$, $8a^4-18a^2x^2$, and $16a^3+48a^2x+36ax^2$. The factors of each are $3x(2a-3x)$, $2a^2(2a-3x)(2a+3x)$, and $4a(2a+3x)^2$. The LCM of 3, 2, and 4 is 12; hence the required LCM is $12a^2x(2a-3x)(2a+3x)^2$, each factor appearing to the highest power to which it occurs in any one expression.

5·21 Addition and Subtraction of Fractions. These operations in algebra are entirely comparable to the corresponding arithmetical procedure.

To add or subtract fractions:

1. *Reduce to lowest terms any fractions which may be so reduced.*

2. *Find the least common denominator, which is the LCM of all the denominators.*

3. *For each fraction, divide the least common denominator (LCD) by its own denominator and multiply both numerator and denominator by this quotient.*

4. *Write the new numerators obtained in step* 3 *over a single denominator (LCD). Each polynomial numerator is placed in parentheses preceded by the sign of the fraction from which it originated.*

5. *Remove the grouping signs, collect terms, and reduce the resulting fraction to lowest terms.*

EXAMPLE 40. Simplify $\frac{2x^2 - 1}{6x^2} - \frac{2x - 5}{10x} - \frac{2}{15}$.

The LCM of the denominators is $30x^2$. Hence by step 3 we obtain

$$\frac{5(2x^2 - 1)}{30x^2} - \frac{3x(2x - 5)}{30x^2} - \frac{4x^2}{30x^2}$$

and by step 4

$$\frac{5(2x^2 - 1) - 3x(2x - 5) - 4x^2}{30x^2} = \frac{10x^2 - 5 - 6x^2 + 15x - 4x^2}{30x^2}$$

$$= \frac{15x - 5}{30x^2}$$

$$= \frac{3x - 1}{6x^2}$$

EXAMPLE 41. Add $\frac{5x}{2x + 4} + \frac{3x - 1}{x^2 - 4}$. The factors of the denominators are $2(x + 2)$ and $(x - 2)(x + 2)$. Hence the LCM (and LCD) is $2(x - 2)(x + 2)$. By step 3 we obtain

$$\frac{5x(x - 2)}{2(x - 2)(x + 2)} + \frac{2(3x - 1)}{2(x - 2)(x + 2)}$$

and by step 4

$$\frac{5x(x - 2) + 2(3x - 1)}{2(x - 2)(x + 2)} = \frac{5x^2 - 10x + 6x - 2}{2(x - 2)(x + 2)} = \frac{5x^2 - 4x - 2}{2(x - 2)(x + 2)}$$

EXAMPLE 42. Combine $\frac{3x^2 - 4}{x^2 - 10x + 25} + \frac{7x - 9}{20 - 4x} + 2$.

Factoring denominators,

$$\frac{3x^2 - 4}{(x-5)^2} + \frac{7x - 9}{4(5-x)} + \frac{2}{1}$$

(Note that we may write integral expressions like 2 as $\frac{2}{1}$.) If we write the fraction $+\frac{7x-9}{4(5-x)}$ as $-\frac{7x-9}{4(x-5)}$, we obtain a common factor $x - 5$. Therefore, the LCD is $4(x-5)^2$, and we obtain

$$\begin{aligned}
&\frac{4(3x^2-4)}{4(x-5)^2} - \frac{(x-5)(7x-9)}{4(x-5)^2} + \frac{8(x-5)^2}{4(x-5)^2} \\
&\quad = \frac{4(3x^2-4) - (x-5)(7x-9) + 8(x-5)^2}{4(x-5)^2} \\
&\quad = \frac{12x^2 - 16 - (7x^2 - 44x + 45) + 8(x^2 - 10x + 25)}{4(x-5)^2} \\
&\quad = \frac{12x^2 - 16 - 7x^2 + 44x - 45 + 8x^2 - 80x + 200}{4(x-5)^2} \\
&\quad = \frac{13x^2 - 36x + 139}{4(x-5)^2}
\end{aligned}$$

EXAMPLE 43. Occasionally, two fractions that have the same denominator or one denominator the negative of the other may first be combined to advantage.

Simplify:

$$\begin{aligned}
1 + \frac{2a}{2a-1} + \frac{8a^2-4}{1-4a^2} + \frac{2}{4a^2-1} &= 1 + \frac{2a}{2a-1} - \frac{8a^2-4}{4a^2-1} + \frac{2}{4a^2-1} \\
&= 1 + \frac{2a}{2a-1} - \frac{8a^2-2}{4a^2-1} \\
&= 1 + \frac{2a}{2a-1} - 2 \\
&= \frac{2a}{2a-1} - \frac{2a-1}{2a-1} = \frac{1}{2a-1}
\end{aligned}$$

EXERCISE 20

Simplify each of the following:

1. $\frac{7x}{10} - \frac{x}{6}$
2. $\frac{5w}{8y} - \frac{7w}{12y}$
3. $\frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy}$
4. $\frac{3x-2y}{6} - \frac{5x+y}{10}$
5. $\frac{7a+b}{8} + \frac{5a-4b}{12} - \frac{-2a+3b}{9}$

6. $\dfrac{7x - 3a^2y}{6a^2b} - \dfrac{3x - 4by}{8b^2}$

7. $\dfrac{2a + 3b - c}{4ab} - \dfrac{a - 2b + 3c}{6bc} + \dfrac{-3a - b + 2c}{8ac}$

8. $\dfrac{1}{a + b} - \dfrac{1}{a + c}$

9. $\dfrac{m + 5}{5} - \dfrac{m + 6}{6}$

10. $\dfrac{x + y}{x - y} - \dfrac{x - y}{x + y}$

11. $a - \dfrac{c^2}{a}$

12. $\dfrac{4}{5a} - 3$

13. $b + 2 + \dfrac{1}{b}$

14. $\dfrac{6}{k + 3} + k - 2$

15. $\dfrac{a - 2b}{a^2 - b^2} + \dfrac{2}{a - b}$

16. $\dfrac{k + 2m}{k^2 - 9m^2} + \dfrac{4}{3m - k}$

17. $\dfrac{x - 5}{x - 6} + \dfrac{2x - 8}{x^2 - 10x + 24}$

18. $y^2 + y + 1 + \dfrac{1}{y - 1}$

19. $\dfrac{2 - c}{c^2 + c - 6} - \dfrac{5}{9 - c^2} - \dfrac{4 - c}{c^2 - 7c + 12}$

20. $\dfrac{a + b}{(b - c)(c - a)} - \dfrac{b + c}{(a - c)(a - b)} + \dfrac{a + c}{(a - b)(b - c)}$

21. $\dfrac{3}{c - d} + \dfrac{4d}{(c - d)^2} - \dfrac{5d^2}{(c - d)^3}$

22. $\dfrac{1}{a - b} - \dfrac{2b}{a^2 - ab} + \dfrac{b^2}{a^3 - a^2b}$

5·22 Multiplying Two or More Fractions

To multiply two or more fractions involving polynomials, write the indicated product of all the factors of the numerators over the indicated product of all the factors of the denominators. Cancel out all common factors of the result.

EXAMPLE 44.
$$\frac{24a^2 - 54b^2}{35x^2y} \cdot \frac{28xy^2}{30a^2 + 15ab - 90b^2}$$
$$= \frac{6(\cancel{2a - 3b})(2a + 3b)}{35x^2y} \cdot \frac{28xy^2}{15(\cancel{2a - 3b})(a + 2b)}$$
$$= \frac{8y(2a + 3b)}{25x(a + 2b)}$$

5·23 Dividing one Fraction by Another

To divide one fraction by another, multiply the first fraction by the second inverted and proceed as in Sec. 5·22.

EXAMPLE 45. $\dfrac{k^2+9km+18m^2}{k^2-9km+20m^2} \div \dfrac{k^2+6km+9m^2}{km^2-4m^3}$

$$= \frac{k^2+9km+18m^2}{k^2-9km+20m^2} \cdot \frac{km^2-4m^3}{k^2+6km+9m^2}$$

$$= \frac{(k+6m)(k+3m)}{(k-4m)(k-5m)} \cdot \frac{m^2(k-4m)}{(k+3m)(k+3m)}$$

$$= \frac{m^2(k+6m)}{(k-5m)(k+3m)}$$

EXERCISE 21

Carry out the indicated operations:

1. $\dfrac{a^2-b^2}{16} \cdot \dfrac{12}{a-b}$

2. $\dfrac{x^2+2x+1}{18cw^3} \cdot \dfrac{12c^3w}{x^2-1}$

3. $\dfrac{x^2-x+12}{abc} \cdot \dfrac{bcd}{x^2+x-6}$

4. $\dfrac{6x+12y}{10a+5} \cdot \dfrac{100a^2-25}{9x^2-81y^2}$

5. $\dfrac{a^2+2ab+b^2}{a^2-b^2} \cdot \dfrac{a^2-2ab+b^2}{a^2-b^2}$

6. $\dfrac{x+3y}{2a+1} \cdot \dfrac{1-4a^2}{x^2-9y^2}$

7. $\dfrac{m^2+2mn+n^2}{4m^2+4mn} \cdot \dfrac{6m^2}{n^2-m^2}$

8. $\dfrac{x^2+x-2}{7b^2x^2-14b^2x+7b^2} \cdot \dfrac{14abx-28ab}{1-2x+x^2}$

9. $\dfrac{a+2}{15} \div \dfrac{a^2-4}{10a}$

10. $\dfrac{a-b}{x+y} \div \dfrac{x-y}{a+b}$

11. $\dfrac{4a-6x}{9ay} \div \dfrac{9x-6a}{12y^2}$

12. $\dfrac{15ax^2}{x^2-9} \div \dfrac{25a^2bx}{x^2+x-12}$

13. $\dfrac{14xy^2z}{4x^2-9} \div \dfrac{10x^2y}{6x+9}$

14. $\dfrac{3c^2-12cm+12m^2}{2c^2+2c-12} \div \dfrac{6c^2-24m^2}{8ac+24a}$

15. $\dfrac{a^2-25}{4a-8} \cdot \dfrac{8-2a^2}{5a^2+5a-10} \div \dfrac{5+a}{15a^2-15}$

16. $\dfrac{4x^2-20x+9}{4x^2-15x+9} \cdot \dfrac{5x-15}{4x-18} \div (12x^2-3)$

5·24 Complex Fractions

A complex fraction is a fraction whose numerator or denominator or both contain fractions.

EXAMPLE 46. $\dfrac{\dfrac{a^2}{b^2} - 1}{\dfrac{a}{b} + 1}$ is a complex fraction. For convenience we shall refer to $a^2/b^2 - 1$ as the primary numerator and $a/b + 1$ as the primary denominator, while b^2 and b will be known as *secondary denominators.*

To simplify a complex fraction reduce the primary numerator and the primary denominator each to a single fraction and divide the primary numerator by the primary denominator.

EXAMPLE 47. Simplify $\dfrac{4 - \dfrac{4}{a} + \dfrac{1}{a^2}}{1 - \dfrac{1}{4a^2}}$.

Simplifying the primary numerator, we obtain

$$4 - \frac{4}{a} + \frac{1}{a^2} = \frac{4a^2}{a^2} - \frac{4a}{a^2} + \frac{1}{a^2} = \frac{4a^2 - 4a + 1}{a^2} = \frac{(2a - 1)^2}{a^2}$$

Simplifying the primary denominator, we have

$$1 - \frac{1}{4a^2} = \frac{4a^2}{4a^2} - \frac{1}{4a^2} = \frac{4a^2 - 1}{4a^2} = \frac{(2a + 1)(2a - 1)}{4a^2}$$

Then

$$\frac{(2a - 1)^2}{a^2} \div \frac{(2a + 1)(2a - 1)}{4a^2} = \frac{\cancel{(2a - 1)}(2a - 1)}{\cancel{a^2}} \cdot \frac{4\cancel{a^2}}{(2a + 1)\cancel{(2a - 1)}}$$

$$= \frac{4(2a - 1)}{2a + 1}$$

This operation can often be performed more easily and quickly by finding the LCM of the *secondary* denominators. Multiply the *primary* numerator and denominator by the LCM just found and reduce to lowest terms.

Using the same complex fraction $\dfrac{4 - \dfrac{4}{a} + \dfrac{1}{a^2}}{1 - \dfrac{1}{4a^2}}$ as in the preceding illustration, we note the LCM of the secondary denominators is $4a^2$. We then write

$$\frac{4a^2\left(4 - \dfrac{4}{a} + \dfrac{1}{a^2}\right)}{4a^2\left(1 - \dfrac{1}{4a^2}\right)} = \frac{16a^2 - 16a + 4}{4a^2 - 1} = \frac{4(4a^2 - 4a + 1)}{(2a + 1)(2a - 1)}$$

$$= \frac{4(2a - 1)\cancel{(2a - 1)}}{(2a + 1)\cancel{(2a - 1)}} = \frac{4(2a - 1)}{2a + 1}$$

The short method often makes it possible to simplify less involved complex fractions at sight; *e.g.*,

$$\frac{x-\frac{1}{x}}{1+\frac{1}{x}}=\frac{x\left(x-\frac{1}{x}\right)}{x\left(1+\frac{1}{x}\right)}=\frac{x^2-1}{x+1}=x-1$$

A special type of complex fraction is the continued fraction, which is typified by the expression

$$\cfrac{1}{1+\cfrac{2}{1+\cfrac{1}{1+\cfrac{3}{x}}}}$$

Simplification is best effected by starting with the end fraction.

$$\cfrac{1}{1+\cfrac{2}{1+\cfrac{1}{1+\frac{3}{x}}}}=\cfrac{1}{1+\cfrac{2}{1+\frac{x}{x+3}}}=\cfrac{1}{1+\cfrac{2}{\frac{2x+3}{x+3}}}=\cfrac{1}{1+\frac{2x+6}{2x+3}}$$

$$=\frac{1}{\frac{4x+9}{2x+3}}=\frac{2x+3}{4x+9}$$

EXERCISE 22

Simplify the following complex fractions:

1. $\dfrac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x}+\frac{1}{y}}$ **2.** $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{z}}$ **3.** $\dfrac{\frac{c}{d}-\frac{d}{c}}{c-d}$ **4.** $\dfrac{m+w}{\frac{1}{m}+\frac{1}{w}}$

5. $\dfrac{\frac{1}{h}-\frac{1}{k}}{\frac{h-k}{hk}}$ **6.** $\dfrac{\frac{b}{c}+\frac{b}{d}}{\frac{b}{cd}}$ **7.** $\dfrac{q-\frac{1}{q}}{1+\frac{1}{q}}$ **8.** $\dfrac{\frac{a^2-x^2}{a}}{\frac{a+x}{a^2}}$

9. $\dfrac{1-\frac{1}{r}}{r-2+\frac{1}{r}}$ **10.** $\dfrac{b+\frac{b}{c-1}}{\frac{c}{c-1}}$

11. $\dfrac{\dfrac{m}{1+m} - \dfrac{1-m}{m}}{\dfrac{m}{1+m} + \dfrac{1-m}{m}}$

12. $\dfrac{a - \dfrac{ab}{b-a}}{\dfrac{a^2}{a^2-b^2} - 1}$

13. $1 - \dfrac{1}{1 - \dfrac{1}{1 - \dfrac{1}{x}}}$

14. $1 + \dfrac{1}{2 + \dfrac{1}{3 + \dfrac{1}{4 + \dfrac{5}{x}}}}$

Chapter 6 LINEAR EQUATIONS IN ONE UNKNOWN

In this chapter we shall continue with the study of the "language" of algebra. We shall see how it is developed from our ordinary "language" and thus makes possible the solution of "word" problems.

6·1 Equations. An *equation* is a statement that two expressions are equal. The two expressions are called the *members* or *sides* of the equation. However, an equation containing one or more unknowns is not only a statement of fact; it also proposes a problem.

The equation $x + 5 = 13$ states the fact that 13 is 5 more than some number x, as yet unknown. The equation also proposes the problem of finding what number x is. If 13 is 5 more than x, it follows that x is 5 less than 13, or $x = 13 - 5 = 8$, and the problem is solved.

To *solve* an equation, then, we find a numerical value for the letter which makes the members obviously equal.

To put the same idea somewhat more formally, we might say that the equation

$$x + 5 = 13 \tag{1}$$

has been converted to the equation

$$x = 13 - 5$$

or

$$x = 8 \tag{2}$$

by subtracting 5 from both members.

If we had subtracted, say, 2 from each member of Eq. (1), we would have obtained the equation

$$x + 5 - 2 = 13 - 2$$

or

$$x + 3 = 11 \tag{3}$$

However, in solving Eq. (1) we chose to subtract 5, instead of 2, from each number because 5 and only 5, when so used, will make x

appear alone on one side. The process of subtracting the same quantity from both members of an equation is also known as *transposition.*

Consider now the equation

$$x - 2 = 6 \tag{4}$$

The statement of fact is that 6 is 2 less than some as yet unknown number x. The proposed problem is to find what number x is. If 6 is 2 less than x, it follows that x is 2 more than 6, or

$$x = 6 + 2 = 8$$

More formally, if we add 2 to both sides of the equation, we obtain

$$\begin{aligned} x - 2 + 2 &= 6 + 2 \\ x &= 8 \end{aligned} \tag{5}$$

The equation

$$x/4 = 2 \tag{6}$$

states that 2 is one-fourth of x and implies the question, "What is x?" If one-fourth of a single x is 2, then one whole x must be 4 times 2, or 8. In other words, we can multiply both members of Eq. (6) by 4 and obtain

$$\begin{aligned} 4(x/4) &= (4)(2) \\ x &= 8 \end{aligned}$$

Finally, let us consider the equation

$$3x = 24 \tag{7}$$

This equation states that 24 is equal to 3 times a certain number x, whose value is at present unknown. It also implies the question, "What is x?" If 3 x's are equal to 24, then a single x must be equal to one-third of 24, or 8. We can obtain the same result by dividing both sides of the equation by 3.

$$\begin{aligned} 3x \div 3 &= 24 \div 3 \\ x &= 8 \end{aligned}$$

Equations may be classified into conditional, equivalent, or identical. A *conditional* equation is one whose members are equal only for a certain value, or certain values, of the unknown letter. Equation (1) illustrates this type. The members are equal only when $x = 8$. Most of the equations we shall deal with will be conditional equations. *Equivalent* equations are those whose respective members are equal when the same value of the letter is used in each case. Equations (1), (3), (4), (6), and (7) are all equivalent equations. Each may be obtained from any of the others by one or more of the processes of sub-

traction, addition, multiplication, and division illustrated above. An *identical* equation, or identity, is one in which the members are equal no matter what value is assigned to the unknown letter. An example of an identical equation is

$$x^2 - 9 = (x + 3)(x - 3)$$

An equation is said to be satisfied by certain values of the letter or unknown if, on substituting these values for the unknown, the members of the equation become obviously equal. A *solution* of an equation consists of the values of the unknowns (called *roots*) which satisfy the equation.

6·2 Operations on Equations. Summarizing Sec. 6·1, we may state that both sides of an equation may be increased, decreased, multiplied, or divided by the same quantity without destroying the equality. Division by zero or by an expression containing the unknown is excluded. Later we shall see that both members may be raised to the same power.

6·3 Degree of an Equation. The *degree of a term* in an equation is obtained by adding the exponents of the unknown literal quantities in that term. The *degree of an equation* corresponds to the term of highest degree occurring in the equation. For example, if x and y are unknown, all other letters being considered as known quantities, the following equations are of the first degree: $3x - 4 = b + x$, $x + y = 7$. Second-degree equations may be exemplified by

$$x^2 - 3x + xy - y = 10, \quad y^2 + 2y = x - 3$$

A typical third-degree equation is $x^2 - x^2y = y + 6$.

A first-degree equation is usually called a *linear equation.*

We may transpose a term from one side of the equation to the other if we change its sign. This is equivalent to a process of subtraction (see page 81). Thus in the equation $6x - 5 = 13$ we may transpose the -5 from left to right, obtaining $6x = 13 + 5$, or $6x = 18$. We have actually subtracted -5 from each member of the equation.

6·4 To Solve a Linear Equation

1. *If fractions occur in the equation, first make sure that they are in their lowest terms. Clear the equation of fractions by multiplying both members by the LCD.*

2. *Remove any parentheses.*

3. *Transpose all terms containing the unknown to one side of the equation and all other terms to the other side.*

4. *Collect like terms, expressing in a factored form the collection of terms containing the unknown.*

5. *Divide each member of the equation by the coefficient of the unknown.*

6. *Check the solution by substituting it for the unknown in the original equation.*

EXAMPLE 1. Solve the equation

$$\frac{3x}{4} + \frac{1}{6} = 2x - \frac{7}{3}$$

Multiplying by the LCD, 12, we obtain

$$9x + 2 = 24x - 28$$

Transposing terms containing x to the right and other terms to the left (the reverse procedure might have been followed but would have resulted in a greater number of negative values), we have

$$2 + 28 = 24x - 9x$$

Combining like terms,

$$30 = 15x$$

Dividing by 15,

$$\tfrac{30}{15} = 2 = x$$

Substituting 2 for x in the original equation produces the identity $\frac{5}{3} = \frac{5}{3}$.

EXAMPLE 2. Solve the equation

$$\frac{8}{x-4} - \frac{6}{x-3} = \frac{2}{x-6}$$

Multiplying by the LCD $(x - 4)(x - 3)(x - 6)$,

$$8(x-3)(x-6) - 6(x-4)(x-6) = 2(x-4)(x-3)$$

Removing parentheses,

$$8x^2 - 72x + 144 - 6x^2 + 60x - 144 = 2x^2 - 14x + 24$$

Transposing,

$$8x^2 - 6x^2 - 2x^2 - 72x + 60x + 14x = 24 - 144 + 144$$

Collecting like terms,

$$2x = 24$$

Dividing by 2,

$$x = 12$$

EXAMPLE 3. Solve for x the equation

$$3(3x - a) + 2a = a(ax - 3) + 6^*$$

Removing parentheses,

$$9x - 3a + 2a = a^2x - 3a + 6$$

Transposing,

$$-3a + 2a + 3a - 6 = a^2x - 9x$$

Collecting like terms,

$$2a - 6 = x(a^2 - 9)$$

Dividing by $a^2 - 9$,

$$\frac{2a - 6}{a^2 - 9} = \frac{2}{a + 3} = x$$

We may check by substituting $\frac{2}{a + 3}$ for x in the original equation, but such a check in a literal equation is apt to entail more work than the original solution. A reasonably certain check may be obtained by assigning to a some arbitrary simple numerical value. We should avoid using 1 or 0 or any number which will make any denominator equal to zero.

In this illustration, if we let $a = -2$, then $x = 2$. Substituting in the original equation,

$$3[(3)(2) - (-2)] + 2(-2) = (-2)[(-2)(2) - 3] + 6$$

or

$$3(6 + 2) - 4 = (-2)(-4 - 3) + 6$$

or

$$20 = 20$$

EXERCISE 1

Solve the following linear equations. When more than one letter appears, consider the last letter (in alphabetical order) as the unknown quantity and all other letters as known quantities.

1. $13x - 8 = 8x + 2$ **2.** $9x - 1 = 2x + 6$
3. $7x + 4 = x - 8$ **4.** $5 - 2x = x + 20$
5. $2 - 3x + 7 = 8x + 3 - x$
6. $11x + 3 - 4x = 16 - 2x + 2$
7. $5y - (3y - 2) = 10$ **8.** $7 - (8x + 1) = 18$
9. $6(w + 5) - 12 = 3(3w - 1) + 4w$

* This is called a *literal equation*, since in addition to the unknown x it contains a letter a, considered here as a known quantity.

10. $26 - 5(3 - 2z) = z - 4(z + 9)$

11. $(r + 1)^2 = r^2 + 9$

12. $(2x - 3)^2 = 4x^2 - 15$

13. $(x - 2)^3 = x^2(x - 6)$

14. $(y + 1)(y - 2) = y^2 + 5$

15. $(z + 1)(z + 5) = (z + 2)(z + 3)$

16. $(2w + 1)(3w + 1) = (6w - 1)(w + 2)$

17. $\frac{x}{3} + \frac{x}{4} = \frac{7}{2}$

18. $\frac{5}{8}x - 1 = 1 + \frac{7}{10}x$

19. $\frac{x}{10} + \frac{x}{12} + \frac{x}{15} = x - 6$

20. $\frac{1}{6}(x - 2) = \frac{1}{8}(x + 1)$

21. $\frac{1}{8}(1 - y) - \frac{1}{10}(2 - y) - \frac{1}{12}(3 + y) = 0$

22. $0.2x = 46 - 0.03x$

23. $0.103 - 0.1x = 0.02x - 0.13x + 0.11$

24. $\frac{2}{x} + \frac{3}{x} = 10$

25. $\frac{4}{w} + 3 = 4 - \frac{3}{w}$

26. $\frac{1}{4y} - \frac{1}{6y} = \frac{1}{8}$

27. $\frac{3}{5x} - \frac{1}{2x} = \frac{1}{40}$

28. $\frac{8}{x + 4} = \frac{6}{x - 4}$

29. $\frac{5}{3y + 2} = \frac{7}{5y - 2}$

30. $\frac{4}{7z + 3} = \frac{3}{6z + 2}$

31. $\frac{4x - 3}{2x + 6} = \frac{6x - 2}{3x + 11}$

32. $\frac{6y - 3}{3y + 2} = \frac{2y + 1}{y + 2}$

33. $\frac{4}{5x + 5} - \frac{7}{10x + 10} = \frac{1}{20}$

34. $\frac{8}{x - 2} - \frac{5}{x - 11} = \frac{3}{x - 5}$

35. $\frac{7}{x + 1} - \frac{4}{x - 1} = \frac{3}{x + 5}$

36. $\frac{2x}{x^2 - 4} - \frac{4}{x^2 - 4} = \frac{2}{2x - 3}$

37. $\frac{2}{1 - 2w} + \frac{2}{7 - 2w} = 1 - \frac{4w^2 - 1}{4w^2 - 16w + 7}$

38. $ax + b = c$

39. $t - sx = r$

40. $6abx = 9a^3b^2c$

41. $\frac{x}{a} = \frac{a}{4}$

42. $\frac{b}{y} = \frac{c}{d}$

43. $\frac{m}{x} = k$

44. $\frac{c}{x} = a + 1$

45. $ax + bx = 3a + 3b$

46. $mx - h = hx - m$

47. $2mx + n^2 = 2nx + m^2$

48. $3ay = 5by + 2$

49. $ax + 2 = bx + 8$

50. $b(x + 1) = c$

51. $\frac{x}{c} - a = b$

52. $\frac{x}{a} + \frac{x}{b} = a^2 - b^2$ **53.** $\frac{a}{x} = b + c$

54. $\frac{m+n}{x} = m^2 + mn$ **55.** $(a-b)x - a^2 = (a+b)x$

56. $c(1+w) + d(1+w) = w(c+d+1)$

57. $b(b-2y) + c(c-2y) + 2bc = 0$

58. $mn(z^2-1) = (m+nz)(n+mz)$

59. $(p-y)(y+q) - r(y+r) = (r-y)(y+r) + pq$

60. $\frac{ax}{b} + \frac{bx}{a} = 1$ **61.** $\frac{c+x}{m} = \frac{x}{c+m}$

62. $a^2 = \frac{a+c}{x} + c^2$ **63.** $\frac{w-c}{w-d} = \frac{c^2}{d^2}$

64. $\frac{by^2}{c-ay} + b + \frac{by}{a} = 0$

65. $\frac{m}{n-p} + \frac{n-p}{z} = \frac{m}{n+p} + \frac{n+p}{z}$

6·5 Formulas. Perhaps the most common occurrence of the literal equation is the formula. In Appendix B are listed a number of formulas for determining length, area, volume, etc. Frequently it will be desirable to transform a formula or solve for one of the factors previously regarded as a known quantity. For example, the volume of a cone is expressed by the formula $V = \pi r^2 h/3$, in which it is assumed that the radius r and the height h are known and the volume V is to be found. Should there be repeated occasions where the volume and radius are known and the height is to be found, we should solve the formula for h. Following the usual procedure for solving literal linear equations, we obtain $h = 3V/\pi r^2$.

In Exercise 2 the student will be expected to leave his transformed version of the formula in the most convenient form for computation. Few specific directions can be offered, but it is suggested that a number of constants related to each other by multiplication or division may well be combined into a single constant. Furthermore, a multiplying constant is usually more easily handled than a dividing constant. The student's most reliable guide is his imagination. If he will regard himself as the user of the formula he has produced, he will so arrange it that it requires a minimum of effort for actual computation.

If we consider the recently discussed formula $h = 3V/\pi r^2$, we shall see that possible variations are

$$h = \frac{3}{\pi} \cdot \frac{V}{r^2} = \frac{0.9549V}{r^2} \qquad (a)$$

and

$$h = \frac{V}{\frac{\pi}{3}r^2} = \frac{V}{1.047r^2} \qquad (b)$$

For a student having a table of areas of circles the original form would probably be best; otherwise Eq. (*a*) would be preferable. On the other hand, it is doubtful that Eq. (*b*) would be a good choice under any circumstances.

EXERCISE 2

Solve the following formulas for the letter indicated in parentheses. Leave your answer in the most convenient form for computation, and be prepared to justify your choice. The formulas are identified in the right-hand column.

1. $Q = \frac{WL}{T}$	(T)	Latent heat of vaporization.
2. $X = \frac{1}{2\pi fC}$	(C)	Reactance of a capacitor.
3. $I = \frac{E - e}{R}$	(e)	Current flowing through armature of generator.
4. $V = \frac{V_t + V_0}{2}$	(V_0)	Average speed of uniformly accelerating body.
5. $\frac{E}{e} = \frac{R + r}{r}$	(r)	Voltage drop.
6. $T = \frac{1}{a} + t$*	(a)	Temperature conversion formula.
7. $C = \frac{Kab}{b - a}$	(a)	
8. $S = \frac{rl - a}{r - 1}$	(a,r)	Geometric progression.
9. $\rho = \frac{m}{d - L} - \frac{m}{d + L}$	(m)	
10. $\frac{e}{x} = C(e - b) + \frac{b}{x}$	(x)	
11. $Q = 0.000477EIT$	(T)	Electrical equivalent of heat.

* First solve for $\frac{1}{a}$.

12. $d = \frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2$	(t)	Distance covered by falling body.
13. $C = \frac{5}{9}(F - 32)$	(F)	Centigrade-Fahrenheit temperature conversion.
14. $A = \frac{m}{t}(p + t)$	(t)	Thickness of pipe.
15. $H = \frac{0.4\pi NI}{L}$	(I)	Magnetic intensity.
16. $M = 10.5C + 35.2\left(W - \frac{C}{8}\right)$	(C)	Theoretical amount of air required to burn solid fuel.
17. $\frac{1}{x} + \frac{1}{nx} = \frac{1}{f}$	(x)	Photographic enlargement.
18. $S = \left(\frac{\pi d^2}{2} + \frac{\pi dl}{r}\right) \div \frac{\pi d^2 l}{4rc}$	(r)	Exposed surface of cylinders.
19. $S = T - \frac{1.299}{N}$	(N)	Tap-size drill for U.S. standard thread.
20. $W = \frac{2PR}{R - r}$	(R)	Differential pulley.
21. $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$	(R)	Parallel resistances.
22. $I = \frac{E}{r + \frac{R}{n}}$	(n)	Current produced by cells in parallel.
23. $T = T_1\left(1 - \left[\frac{n-1}{n}\right]\left[\frac{h}{h_0}\right]\right)$	(n)	
24. $x - y = xy$	(y)	
25. $V = \frac{h}{6}(B + 4M + b)$	(M)	Prismoidal formula.
26. $wf = \left(\frac{w}{k} - 1\right)\left(\frac{1}{k}\right)$	(w)	
27. $V_1 = V_0(1 + 0.00365t)$	(t)	Expansion of gases.

6·6 Mathematical Operations with Dimensional Units. In setting up equations derived from physical problems, it must be remembered that each side of the equation must reduce to like units or dimensions; *i.e.*, the equation must be dimensionally correct. Operations with dimensional symbols are subject to the usual laws of algebra. Some common examples are listed below:

Length × length = area	$(\text{ft})(\text{ft}) = \text{ft}^2$
Length × length × length = volume	$(\text{ft})(\text{ft})(\text{ft}) = \text{ft}^3$

Area $\times$ length = volume	$\text{ft}^2 \times \text{ft} = \text{ft}^3$
$\frac{\text{Weight}}{\text{Volume}}$ = density	$\text{lb} \div \text{ft}^3 = \frac{\text{lb}}{\text{ft}^3}$
$\frac{\text{Distance}}{\text{Time}}$ = average velocity	$\text{ft} \div \text{sec} = \frac{\text{ft}}{\text{sec}}$
$\frac{\text{Change in velocity}}{\text{Time}}$ = average acceleration	$\frac{\text{ft}}{\text{sec}} \div \text{sec} = \frac{\text{ft}}{\text{sec}^2}$
$\frac{\text{Force}}{\text{Area}}$ = pressure	$\text{lb} \div \text{ft}^2 = \frac{\text{lb}}{\text{ft}^2}$
Density $\times$ height = pressure	$\frac{\text{lb}}{\text{ft}^3} \times \text{ft} = \frac{\text{lb}}{\text{ft}^2}$
Mass* $\times$ acceleration = force	$\text{lb} \times \frac{\text{ft}}{\text{sec}^2} = \frac{(\text{ft})(\text{lb})}{\text{sec}^2}$
Pressure $\times$ volume = work	$\frac{\text{lb}}{\text{ft}^2} \times \text{ft}^3 = (\text{ft})(\text{lb})$
Distance $\times$ force = work	$\text{ft} \times \text{lb} = (\text{ft})(\text{lb})$
Power $\times$ time = work	$\frac{(\text{ft})(\text{lb})}{\text{sec}} \times \text{sec} = (\text{ft})(\text{lb})$
$\frac{\text{Work}}{\text{Time}}$ = power	$(\text{ft})(\text{lb}) \div \text{sec} = \frac{(\text{ft})(\text{lb})}{\text{sec}}$
Force* $\times$ time = momentum	$\frac{(\text{ft})(\text{lb})}{\text{sec}^2} \times \text{sec} = \frac{(\text{ft})(\text{lb})}{\text{sec}}$
Height $\times$ weight = potential energy	$\text{ft} \times \text{lb} = (\text{ft})(\text{lb})$
Weight $\times$ (velocity)^2 $\div$ gravitational constant = kinetic energy	$\text{lb} \times \left(\frac{\text{ft}}{\text{sec}}\right)^2 \div \frac{\text{ft}}{\text{sec}^2} = \text{lb} \times \frac{\text{ft}^2}{\text{sec}^2} \times \frac{\text{sec}^2}{\text{ft}} = (\text{ft})(\text{lb})$

6·7 Suggestions for the Solution of Applied Problems Involving Linear Equations. There is no royal road to the solution of "word" problems. The student should not feel discouraged if he finds this section of the work difficult, for this reaction is the rule rather than the exception. This situation is due largely to the fact that the

* Absolute units.

translation of a verbal or descriptive problem (commonly called a "word" problem) involves a minimum of dependence on mechanical rules. Such translation requires some exercise of ingenuity and of the skill that comes only with experience. If this seems a bleak outlook, it should be remembered that these remarks apply equally well to carpentry, machine design, music, painting, etc. The feeling of accomplishment in solving the more interesting applied problems is not experienced by those content merely to follow mechanical procedures.

While a rigid, ironclad procedure cannot be offered, a few general suggestions can be outlined which have proved helpful.

1. *Read the problem through once to get the general idea.*

2. *Carefully reread the problem, noting what is given and what is wanted.*

3. *A rough graphical representation will often suggest relationships leading to a solution* (see Sec. 8·9).

4. *Represent the unknown quantity by some appropriate symbol, such as* v *for velocity,* t *for time,* w *for weight.*

The symbol should be quantitative: it should refer to the number of units in the unknown; *e.g.*, "Let t represent the time for the trip in hours"; "Let w represent the weight of copper in pounds." Be explicit.

All related quantities should be expressed in consistent units. For example, if it has been decided to express a velocity in miles per hour, all distances must be expressed in miles and all times must be expressed in hours.

1. *Try to discover two expressions which are equal. Translate this into an equation.*

2. *Solve the equation, and check the solution against the original worded statement,* not *the equation you derived from it.*

Numerous laws of mathematics, natural sciences, etc., may serve as the basis of equality, such as the following:

The sum of the parts equals the whole.

Distance equals rate times time.

Weight equals volume times density.

Number of units times the unit cost equals the total cost.

Principal times the rate of interest times the time equals the interest.

Amount of material times the fraction of a particular ingredient equals the amount of ingredient.

Square of the hypotenuse of a right triangle equals the sum of the squares of the legs.

EXAMPLE 4. A tourist, having $4\frac{1}{2}$ hr at his disposal, rides out into the country on a bus averaging 19 mph. He plans to walk back at $3\frac{1}{2}$ mph. How far can he ride without arriving late at his starting point?

The one-way distance is required. The total time and the outgoing and return rates are given. Let d = number of miles one way. We know that distance = rate × time, or $\frac{\text{distance}}{\text{rate}}$ = time. We can set up an equation:

$$\text{Outbound time} + \text{return time} = \text{total time}$$

$$\frac{\text{Distance}}{\text{Rate}}_{\text{(outbound)}} + \frac{\text{distance}}{\text{rate}}_{\text{(return)}} = \text{total time}$$

$$\frac{d}{19} + \frac{d}{3\frac{1}{2}} = 4\frac{1}{2}$$

Then

$$3\tfrac{1}{2}d + 19d = (19)(3\tfrac{1}{2})(4\tfrac{1}{2})$$

$$22\tfrac{1}{2}d = 299\tfrac{1}{4}$$

$$d = 13.3 \text{ miles}$$

Checking against the original statement,

$$\frac{13.3}{19} = 0.7 \text{ hr outbound}$$

$$\frac{13.3}{3.5} = \frac{3.8 \text{ hr return}}{4.5 \text{ hr total}}$$

EXAMPLE 5. How much 80 per cent antifreeze must be added to 5 qt of 30 per cent antifreeze (alcohol) to raise the strength to 65 per cent?

We wish to know the amount of 80 per cent antifreeze needed. We are given concentrations of the initial solutions and the final mixture, also the amount of the weaker solution. Let n = the number of quarts of 80 per cent solution required. Then $n + 5$ = the number of quarts of mixture finally obtained. We can set up an equation saying that quarts of pure alcohol in the 5 qt of the 30 per cent solution plus the quarts of pure alcohol in the n qt of 80 per cent solution taken equals the quarts of pure alcohol in the $(n + 5)$ qt of 65 per cent final mixture.

Hence $(0.30)(5) + 0.80n = 0.65(n + 5)$

$$1.5 + 0.8n = 0.65n + 3.25$$

$$0.15n = 1.75$$

$$n = \frac{1.75}{0.15} = 11\tfrac{2}{3} \text{ qt of 80 per cent antifreeze required}$$

CHECK: $0.80 \times 11\frac{2}{3} = 9\frac{1}{3}$ qt pure alcohol in the 80 per cent solution

$0.30 \times 5 = 1\frac{1}{2}$ qt pure alcohol in the 30 per cent solution

$10\frac{5}{6}$ qt pure alcohol in mixture

$0.65 \times 16\frac{2}{3} = 10\frac{5}{6}$ qt pure alcohol in the mixture (check)

EXERCISE 3

Set up linear equations in terms of one unknown for the following problems. Do not solve.

1. The difference between two numbers is 7. Their sum is 53. Find the numbers.
2. One number is $1\frac{1}{2}$ times as large as another. The sum of the numbers is 35. Find the numbers.
3. The sum of two numbers is 45. The quotient obtained by dividing the larger by the smaller is 2, with a remainder of 3. Find the numbers.
4. Separate 22 into two parts such that one part is 3 times the other.
5. Find the angles of a triangle if the first angle is $2\frac{1}{2}$ times the second and the third angle is $\frac{1}{4}$ of the second.
6. Find three consecutive numbers whose sum is 72.
7. Find four consecutive even numbers whose sum is 68.
8. Find five consecutive odd numbers whose sum is 85.
9. One side of a right triangle is 8 in., and the hypotenuse is 2 in. longer than the other side. Find the sides of the triangle.
10. The difference between the squares of two consecutive numbers is 37. Find the numbers.
11. The denominator of a fraction exceeds the numerator by 8. The reduced value is $\frac{7}{9}$. What is the fraction?
12. The reduced value of a fraction is $\frac{2}{3}$. If 16 be added to both numerator and denominator, the reduced value of the new fraction is $\frac{10}{13}$. What was the original fraction?
13. The reduced value of a fraction is $\frac{4}{5}$. If 21 be deducted from both numerator and denominator, the reduced value of the new fraction is $\frac{1}{3}$. Find the original fraction.
14. The length of a rectangle exceeds the width by 14 ft. The perimeter is 160 ft. Find the dimensions.
15. The circumference of a circle exceeds its diameter by 13. Find the diameter.
16. The hypotenuse of a 45° right triangle exceeds one of the equal sides by 9. Find one of the equal sides. [See Eq. (6), Appendix B.]
17. The perimeter of an isosceles triangle is 50 in. The base is 11 in. longer than one of the equal sides. Find the sides of the triangle.
18. The side of an equilateral triangle exceeds the altitude by 5. Find a side of the triangle. [See Eq. (31), Appendix B.]
19. Ten boys agreed to buy a canoe dividing the expense equally. Two more boys joined the group, reducing the share of each by \$1.50. How much did the canoe cost?

20. A boy worked 16 days for a certain amount of money. Had he received \$0.50 a day more he could have earned his money in 1 day less time. Find his daily wages.

21. A man bought 30 acres of land for \$1,280. Part of the land cost \$40 an acre; the rest cost \$50 an acre. How much land was sold at \$40 an acre?

22. A sum of money consists of dimes and quarters. There are 9 more dimes than quarters. If the total value is \$10, find the number of dimes.

23. A man invested \$7,000, part at 2¼ per cent and the rest at 3 per cent. If the interest for one year was \$174, find the amount invested at 3 per cent.

24. How much solder containing 50 per cent tin and how much type metal containing 15 per cent tin must be mixed to make 80 lb of solder containing 40 per cent tin?

25. A 12-qt cooling system is filled with 25 per cent antifreeze. How many quarts must be drawn off and replaced with pure antifreeze to raise the strength to 45 per cent?

26. Two trains leave opposite terminals of a 200-mile line at the same time. If the rates are 40 and 35 mph, respectively, when will they meet?

27. A group of bicyclists maintains an average rate of 12 mph. One hour and 45 min after they leave, a motorist sets out to overtake them. If his rate is 40 mph, how long will he take?

EXERCISE 4

Solve the following problems and check against the original statement.

1. A grocer estimated that his supply of sugar would last 30 days. Because he sold 20 lb a day more than he expected, it lasted only 24 days. How many pounds did he have?

2. The reduced value of a certain fraction is ⅔, and its denominator exceeds its numerator by 4. Find the fraction.

3. How many gallons of water must be mixed with 500 gal of 96 per cent sulfuric acid to reduce the strength to 80 per cent?

4. A uniform 20-in. bar weighing 1 lb is in equilibrium when a 3-lb weight is suspended from one end and a 2-lb weight from the other. Find the position of the point of support.

5. A bar of metal contains 20 per cent silver and a second bar 12 per cent. How many ounces of each must be taken to make a 40 oz bar containing 14½ per cent silver?

6. Find the length of one of the equal sides and the area of an isosceles right triangle having a perimeter of 20 ft.
7. The area of a 4-ft walk surrounding a circular pond is 1,496 sq ft. Using $\pi = {}^{22}/_{7}$, find the diameter of the pond.
8. A length of rope hangs from the top of a flagpole to the ground. Six feet of rope lies coiled up on the ground. It is found that the free end of the rope can be brought out 30 ft from the base of the pole and yet be touched to the ground. How high is the flagpole?
9. Eight thousand dollars is invested, some at 6½ and the rest at 4½ per cent. How much must be invested at 6½ per cent to ensure a total annual income of $420?
10. How much high-speed tool steel containing 18 per cent tungsten and how much steel containing 12 per cent tungsten should be mixed to make 3,000 lb containing 14.6 per cent tungsten?
11. What will be the final temperature when 42 lb of water at 135°F are mixed with 70 lb of water at 60°F?[1]
12. A bookrack is to be 44 in. high over all. The stock is ⅞ in. thick. There are to be five shelf spaces, each one having ¾ in. more vertical space than the one above it. Find the height of each shelf space. There are six thicknesses of stock.
13. At a high school game the price of admission was $0.25 for each adult and $0.10 for each child. If the turnstile showed 397 persons at the game and the gate receipts were $56.80, how many adults attended?
14. If the equation $C = {}^{5}/_{9}\ (F - 32)$ represents the relationship between the Centigrade and Fahrenheit readings for any temperature, find the temperature at which the two will be equal.
15. A machine having an initial value of $1,450 depreciates in value each year $50 less than during the preceding year. At the end of the sixth year it has a value of $190. What is its value at the end of the first year?
16. A 12-lb steel gear is to be tempered in an oil bath containing 150 lb of oil. If the temperature of the gear is 800 and that of the oil 20°C, to what temperature will the gear be cooled? Neglect the heat capacity of the oil container. The specific heats of steel and oil are 0.13 and 0.50, respectively (see Note, Prob. 11).
17. In an approximate method of determining furnace temperature, a piece of nickel weighing 1¼ lb is taken from the furnace and quickly plunged into water at 65°F. The combined heat capacity

[1] In problems of heat exchange involving no change of state, weight × specific heat × temperature change for the warm body equals weight × specific heat × temperature change for the cooler body.

of the water and container is equivalent to that of 14 lb of water. If the water temperature rises to 82°F, what is the temperature of the furnace? The specific heats of nickel and water are 0.13 and 1.0, respectively (see Note, Prob. 11.)

18. A 39¾-in.-diameter iron tire is to be shrunk onto a 40-in. wheel. If the linear coefficient of expansion of iron is 0.000006 per degree Fahrenheit, to what temperature must the tire be heated from 70°F to fit on the wheel?

19. A poorly compensated watch, when carried vertically in the pocket, gains 11 sec in 9 hr and, when laid down horizontally, loses 28 sec in 13 hr. How many hours out of 24 in the horizontal position would result in no net gain or loss in a 24-hr period?

20. In order to measure the flow of chlorine which would corrode an ordinary mechanical meter, pure oxygen is "bled" into the stream of chlorine gas at the rate of 5.3 cfpm. If the chlorine, before dilution, contained 2.8 per cent oxygen and afterward contained 12.1 per cent oxygen, find the original flow of chlorine gas in cubic feet per minute.

21. In the gambrel roof in Fig. 6·1 the upper set of rafters has a 1/2 pitch and the lower set a pitch of 2. Find the lengths of A and B.

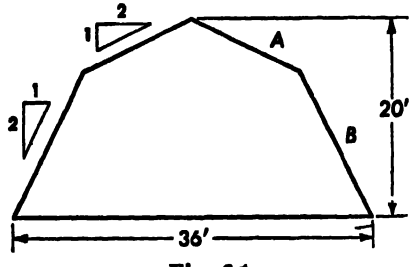


Fig. 6·1

22. An isosceles triangle has a perimeter of 32 in. and the altitude to the base is 8 in. Find the length of one of the equal sides.

23. A group of men pay \$0.75 each toward the expenses of a luncheon. An excess of \$1.25 results. If each had paid out \$0.50, there would have been \$2.50 lacking. How many men were there?

24. An oval running track has semicircular ends and straight sides. The over-all length is 3 times the width. If one lap is 600 ft, find the over-all length and width of the oval.

25. In the formula $N = 0.907(D/d - 0.94)^2 + 3.7$, N is the number of wires of diameter d that can be contained in a conduit of diameter D. What must be the ratio of conduit diameter to wire diameter in order that the conduit may hold 100 wires?

26. A guy rope runs from the top of a derrick to a stake 7 ft from the base. If the rope is lengthened 15 ft, it will reach a stake 32 ft from the base. Find the height of the derrick.

27. A manufacturer wishes to make a competitive line of copper-coated iron wire which he can sell for \$0.20 a pound and make a gross profit of 60 per cent above the cost of materials. If copper costs \$0.24 a pound and iron costs \$0.09 a pound, find the percentage of copper in the wire.

28. A bonus of \$30,000 is to be distributed among 500 employees of a factory. There are 50 men with 20 years' service, 100 men with 10 years' service, and 350 men with 5 years' service. Each 20-year man is to receive twice as much as a 10-year man, and each 10-year man is to receive twice as much as a 5-year man. How much should each of the 5-year, 10-year, and 20-year men receive?

29. Fifty pounds of high-grade solder spatters containing 50 per cent lead and 50 per cent tin are to be melted with some type metal containing 90 per cent lead and 10 per cent tin. The resulting alloy is to be a low-grade solder containing 75 per cent lead and 25 per cent tin. How many pounds of type metal will be needed?

30. A 4-in. circle is inscribed in a right triangle ABC. The circle is tangent to the hypotenuse AB at point D. If $AD = 8$-in., find BC.

31. A utility company petitioned to change its rate on electrical energy from a straight $5\frac{1}{2}$ cents per kilowatthour to $4\frac{1}{4}$ cents per kilowatthour plus 75 cents per month service charge. What monthly cost would be unaffected by the change in rate?

32. A man pays an income tax of \$118.06 on an income which is taxed as follows: 1 per cent on all income over \$1,500, plus 3 per cent on all income over \$2,500, plus 5 per cent on all income over \$4,000. What is his income?

33. A stick of wood is to be cut into four equal sections for braces, as shown in Fig. 6·2. Find x and y, making no allowance for saw kerfs.

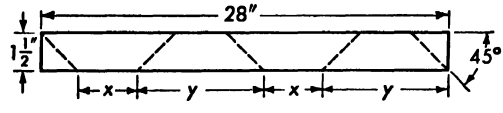


Fig. 6·2

34. Find the radius R used in the construction of the approximate ellipse shown in Fig. 6·3.

35. How many gallons of 35 per cent antifreeze solution must be added to 3 gal of 80 per cent antifreeze solution to reduce the strength to 60 per cent?

36. A dairyman has 1,000 qt of milk containing 4.8 per cent butterfat, but the city in which he sells his milk requires only 4 per cent butterfat. How many quarts of cream testing 20 per cent butterfat may be separated from the milk and still satisfy the legal minimum requirement?

37. Given a square $ABCD$, in which $AB = 3\frac{1}{2}$ in., a circle is drawn passing through vertex C and tangent to AB and AD. Find the diameter of the circle.

38. Find the diameter of the circle shown in Fig. 6·4.

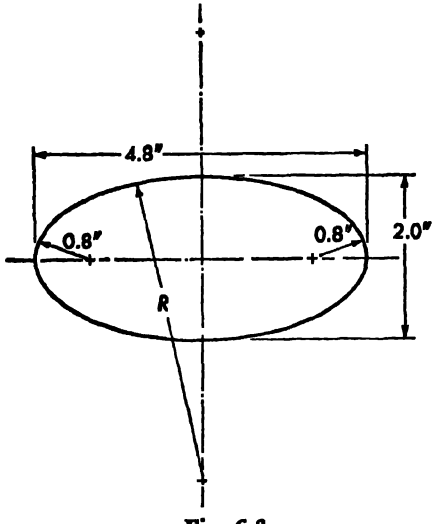


Fig. 6·3

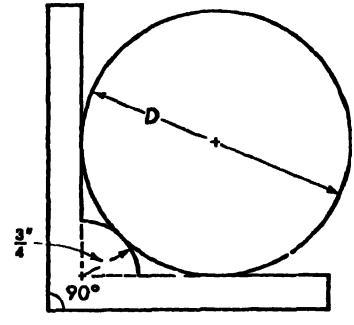


Fig. 6·4

39. Three equal circles are externally tangent, each to each. A belt passed around these three circles is 60 in. long. Find, in terms of π, the radius of a circle.

40. The outbound trip of a bus was made at 24 mph, while the return trip was made at 30 mph. Find the one-way distance if the total running time was $4\frac{1}{2}$ hr.

41. A farmer bought 100 acres of land for \$3,880. Part cost \$50 an acre, and the balance \$18 an acre. Find the number of acres bought at each price.

42. Position B is 5 miles due east of position A. An enemy battery is firing due north of A. The sound of the firing reaches A 2.5 sec before it reaches B. Find the distance of the battery from A if sound travels at 1,100 fps.

43. In order to weigh a uniform bar of iron 8 ft long, a 5-lb weight was hung 8 in. from one end, whereupon the bar was found to balance $45\frac{1}{2}$ in. from that end. Find the weight of the bar.

44. The airspeed of a Piper Cub plane is 120 mph. If a 40-mph west wind is blowing, how far west can the plane fly in order to return to the airport 2¼ hr after taking off?

45. A planer has a ratio of cutting speed to return speed of 1:2.75. Find the cutting speed in feet per second for a 5-ft stroke when making 15 cycles every 2 min.

46. A valve operates off a camshaft turning at a speed of 1,200 rpm. What is the travel of the valve if it opens at the rate of 2 fps and closes at the rate of 10 fps (no rest time)?

47. A man has 4½ hr at his disposal. He can ride out on a bus at 16 mph and walk back at 3½ mph. How far can he ride and be able to return to his starting point within the allotted time?

48. Two cylindrical containers, the larger 10 in. in diameter and the smaller 8 in. in diameter, are connected by a tube at their bases. If the water is initially 9 in. deep in the first cylinder and 5 in. deep in the second, find the common level when equilibrium is reached.

49. At two stations A and B on a railroad line, the prices of soft coal are \$3.50 and \$4 per ton, respectively. If the distance between A and B is 180 miles and coal can be shipped for ½ cent per ton per mile, find the location on the line between A and B at which it will be immaterial to a consumer whether he buys from A or from B.

50. A man can clear a wood lot in 6 days, while his son could do it alone in 12 days. How long would the job take if they worked together?

51. A tank can be filled by two pipes in 6 min, while the first pipe alone would require 10 min. How long would it take the second pipe alone to fill the tank?

52. How much time would be required to fill the tank in Prob. 51 if the first pipe operated as an inlet and the second pipe as an outlet? The tank is being used for leaching purposes.

53. A job can be completed by A in 1½ hr, by B in 2⅓ hr, and by C in 3 hr. Find the time required when all are working together.

54. A and B have a pile of 825 pieces to stamp. A works alone for 1½ hr, during which time he stamps 105 pieces. Then he is joined by B, and together they finish the job in 4 hr more. At the rate of \$0.016 a piece, how much should each man receive?

55. A contractor must finish a job in 35 days. Only one steam shovel can be used on the job at a time. The contractor owns a shovel that can do the job in 40 days and can rent, for \$50 a day, a second shovel that can do the work in 20 days. What is the smallest amount the contractor will have to pay for the rented shovel?

56. A range burner uses up a 54-gal drum of fuel oil in 31 days. The drum is filled the first of the month; and after 10 days a water heater is also started, and the oil lasts 8 days longer. How many gallons did the heater use per day?

57. A contractor agrees to put in a concrete foundation within a certain time limit. He is to receive \$12 per day worked, plus \$40 for each day by which completion of the job precedes the time limit. He finished the job in $\frac{2}{3}$ of the time allowed and received \$384. What was the time limit?

58. An excavation job could be done by a steam shovel alone in 12 days or by a gang of laborers alone in 28 days. After being used a certain time, the shovel was disabled, and the work was then completed by the men who worked 2 days less than the time of steam-shovel operation. How long was the shovel used?

59. A circle of diameter D is inscribed in a 45° right triangle of leg S. Derive a formula for D in terms of S.

60. An irregular aluminum-bronze casting weighed 8.20 lb in air and 5.35 lb in water. If the densities of copper, aluminum, and water are 0.32, 0.097, and 0.036 lb per cu in., respectively, find the percentage by weight of copper in the casting.[1]

61. It is to be determined whether a medal is solid gold or gold-plated brass. If it is plated, the percentage by weight of gold is to be found. In order to avoid mutilating the medal, the following data were taken: Weight of medal in air = 42.68 g; weight of medal in water = 39.25 g. The specific gravity of brass = 8.2, specific gravity of gold = 19.3.[1]

62. Using the formula for the length of a pulley belt in Prob. 107, page 44, find the missing diameters of the step-cone pulleys in Fig. 6·5 so that the same belt may be used in each of the three positions.

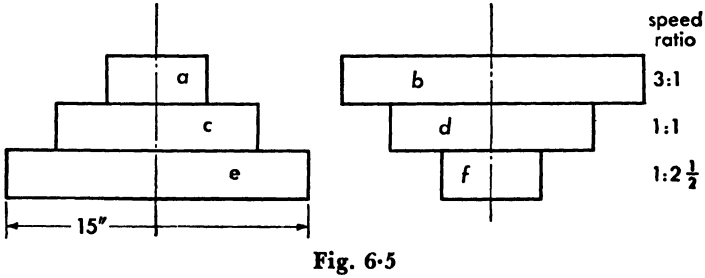


Fig. 6·5

63. A closed manometer tube is shown in Fig. 6·6. A length of air column AB is sealed off at atmospheric pressure with mercury. More mercury is then poured into the open arm at D, with new

[1] Apparent loss of weight in water equals the weight of water displaced.

levels at C and D. Find, to the nearest 0.1 in. of mercury, the original atmospheric pressure.

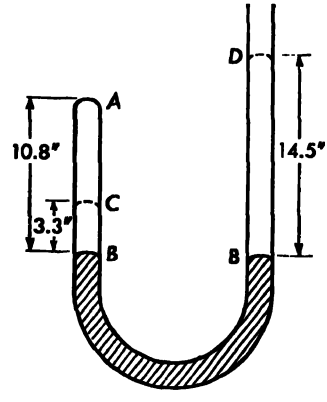


Fig. 6·6

64. A man pays 25 cents a gallon for gasoline. He gets 16 miles per gallon from his car. He can have his engine overhauled for \$110, after which he can expect to get 22 miles per gallon of gasoline and save 30 cents per hundred miles on oil. After how many miles will he recover the cost of overhauling?

65. In Prob. 141, page 49, show that $CD = \sqrt{(AC)(BC)}$.

66. A plane has an airspeed of 300 mph. It flies directly into a head wind for 48 min and returns along the same route in 42 min. Find the wind speed.

67. A family has 1 qt of milk delivered daily. The bill for the month of November was \$6.96. It is known that the price advanced 2 cents a quart at some time before the middle of the month. What was the price per quart before the increase? When did the increase become effective? Prices are in integral cents.

68. The explosion of a floating mine was heard 8 sec sooner through the water than through the air. If the speed of sound was 4,800 ft per sec through the water and 1,125 ft per sec through the air, how far away was the explosion?

69. A 6-ft offset in a pipe line consists of two equal reverse arcs. If the offset is accomplished in a distance of 10 ft (measured parallel to the straight pipe), find the radius of curvature of an arc.

70. A crossover in a double-track railroad line (Fig. 6·7) consists of two equal reverse arcs. Find the radius of curvature R of the inner rail.

71. Using the three-wire system for determining the root diameter r of a 60° sharp-V thread, develop a formula for r in terms of c and w (Fig. 6·8).

In Probs. 72 to 76 express the answer in the most convenient form for computation. Justify your choice.

72. A washer (annulus) has a face area A and an inside diameter d and is made of stock of width w. Derive a formula for d in terms of A and w.

73. If the prismoidal formula [Eq. (75), Appendix B] is applied to the wedge in Fig. 6·9, develop a formula for V in terms of w, h, l, and L.

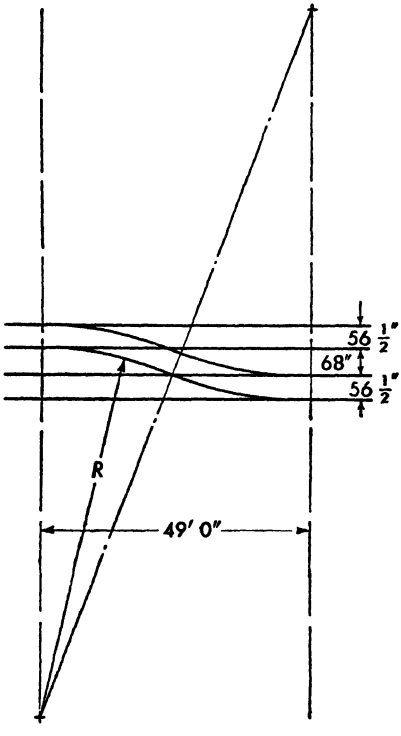


Fig. 6·7

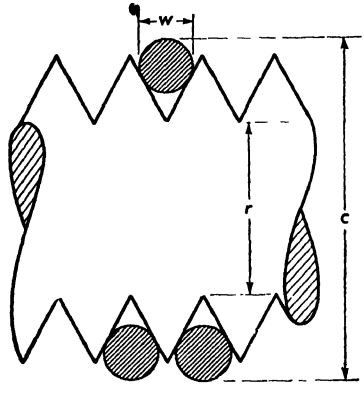


Fig. 6·8

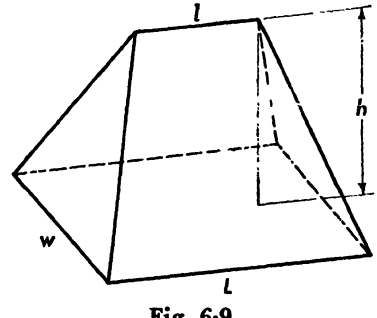


Fig. 6·9

74. A cone of base diameter d and slant height S is developed from a sector of central angle θ. Derive a formula for θ (in degrees) in terms of d and S.

75. Using the prismoidal formula, develop a formula for the volume of a barrel in terms of the height h, end diameter d, and bung diameter D.

76. Two equal pulleys of diameter d in. are l in. from center to center. If the splice on an open belt passes a given point r times a minute, derive a formula for the revolutions per minute of a pulley in terms of d, l, and r (see Prob. 107, page 44).

UNIT THREE

ANALYTIC GEOMETRY, MORE ADVANCED ALGEBRA, AND LOGARITHMS

Sections 14·13 to 14·27 deal with the use of logarithms in computation. While it is not recommended, these articles may be assigned without regard to the rest of Unit Three

Chapter 7 FUNCTIONS AND GRAPHS

A graph or curve is a forceful way of representing data and portraying the relation between them. In this chapter, suggestions will be given for effective graphical presentation of data.

In the second part of this chapter, which deals with the properties of the straight line, we shall see how the more direct methods of algebra may be applied to the solution of geometrical problems. This offspring of algebra and geometry is called *analytic geometry*.

7·1 Constants and Variables. A *constant* is a quantity which does not change in value during the course of a problem. Some quantities are inherently constant, such as $\sqrt{2}$, -7, or π. Other constants, usually represented by the first few letters of the alphabet, a, b, c, etc., retain their values without change throughout a particular problem but may vary from one problem to another.

A *variable* is a quantity that is free to change during the course of a problem. The change may take place according to some simple law, or it may be the result of a combination of influences too complex to be represented by a formula.

When we notice, for example, that the freezing point of antifreeze depends on its strength, the length of a clock pendulum changes with the temperature, or the time required to cover a certain distance depends on the rate, we are then regarding freezing point or pendulum length or time as a variable.

Variables are usually represented by one of the last several letters of the alphabet, such as x, y, or z. It is often desirable to use letters suggestive of the unit represented, such as v for a volume unit, t for a time unit, and w for a weight unit.

7·2 Functions. If two variables, say x and y, are so related that whenever a value is assigned to x, a corresponding value (or values) of y can be determined, we say that y is a *function* of x. Since the value of the function y is considered to depend upon the value of the variable

x, we refer to y as the *dependent variable* and to x as the *independent variable.*

In each of the following illustrations it will be noted that the dependent variable is mentioned before the independent variable.

Rate of growth of vegetation depends upon the temperature.

Time of high tide depends upon the position of the moon.

Price of an article depends upon supply and demand.

Flow of electricity through a fixed resistance depends upon the voltage.

Postage to a given destination depends upon the weight of the package.

Sometimes the roles of dependent and independent variables are reversible. To cite a simple example: In using the formula $C = \pi D$ to determine the circumference of a circle when the diameter is given, D would be considered the independent variable and C the dependent variable. If, however, the circumference were given and the diameter were to be found, we might use the formula $D = C/\pi$ in which to substitute the value of C as the independent variable to find D, the dependent variable.

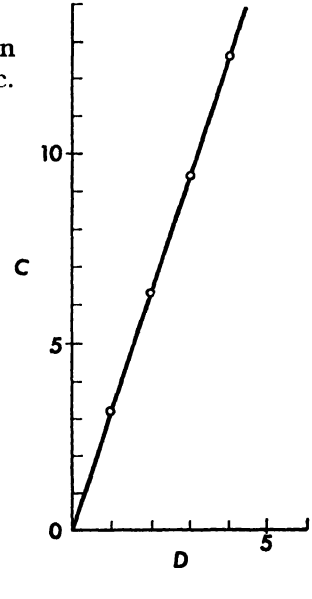
Functional relationships between two variables may be expressed in these ways:

1. By a worded statement: The circumference of a circle is π times the diameter.
2. By a formula or equation: $C = \pi D$.
3. By a table of values:

If $D =$	0	1	2	3	4	5
Then $C =$	0	3.14	6.28	9.42	12.6	15.7

4. By a graph:

This is discussed in greater detail in Sec. 7·5.



7·3 Functional Notation. When, for any reason, the exact relation between variables is not to be expressed, a general form is used. Thus, instead of writing $A = \pi r^2$, we might use the more general form $A = f(r)$, which states that A is a function of r without giving the exact relation. It is read, "A equals the f function of r," or, more simply, "A equals f of r."

Suppose that y is some function of x. We may then write $y = f(x)$. Then $f(6)$ is the value of y when $x = 6$; likewise $f(-1)$ is the value y assumes when $x = -1$; and so on. In general, $f(a)$ is the value of y which results when a is substituted for x in the expression for $f(x)$.

It should be remembered that $f(x)$ does not mean f times x; in fact, f does not represent a number. In the expression $y = f(x)$, f simply indicates that some kind of relationship exists between y and x.

EXAMPLE 1. Suppose that $y = x^2 - 3x - 4$. We may then write

$$y = f(x) = x^2 - 3x - 4$$

Then, when $x = 5$,

$$y = f(5) = (5)^2 - 3(5) - 4 = 6$$

Also, when $x = -6$,

$$y = f(-6) = (-6)^2 - 3(-6) - 4 = 50$$

EXERCISE 1

If $y = 3x^2 - 2x - 5$, find

1. $f(0)$ **2.** $f(-1)$ **3.** $f(\frac{2}{3})$ **4.** $f(a + 1)$ **5.** $f(4) - f(-2)$

If $y = \dfrac{12x}{16 - x^2}$, find

6. $f(-2)$ **7.** $f(5)$ **8.** $f(\frac{1}{2})$ **9.** $f(3a - 4)$ **10.** $f(2a) - f(a)$

7·4 Rectangular Coordinates. To form a *rectangular system of coordinates* we draw a horizontal line $X'X$ and a vertical line $Y'Y$. Their point of intersection O is called the *origin.* $X'X$ is called the X axis, and $Y'Y$ is called the Y axis.

The location of any point in the plane of the coordinate axes may be expressed by stating its distance and direction from each of the two axes. Distances above $X'X$ and to the right of $Y'Y$ are positive, whereas distances below $X'X$ and to the left of $Y'Y$ are negative.

The distance of a point measured to the right or left of the Y axis is called the *abscissa*, or X coordinate, of the point. The distance of a point from the X axis (upward or downward) is called the *ordinate*, or Y coordinate, of the point. In Fig. 7·1, point P is said to have the

coordinates (x,y) and may be referred to as the point (x,y). In the same figure, points (3,4), (4,−2), (0,−3) (−5,−1), and (−2,3) are also designated. It should be noted that the coordinates are enclosed in parentheses and separated by a comma and that the abscissa is always written first. It will also be seen that the axes divide the plane into four quadrants, numbered as shown. The signs of the coordinates of points in each quadrant are also indicated.

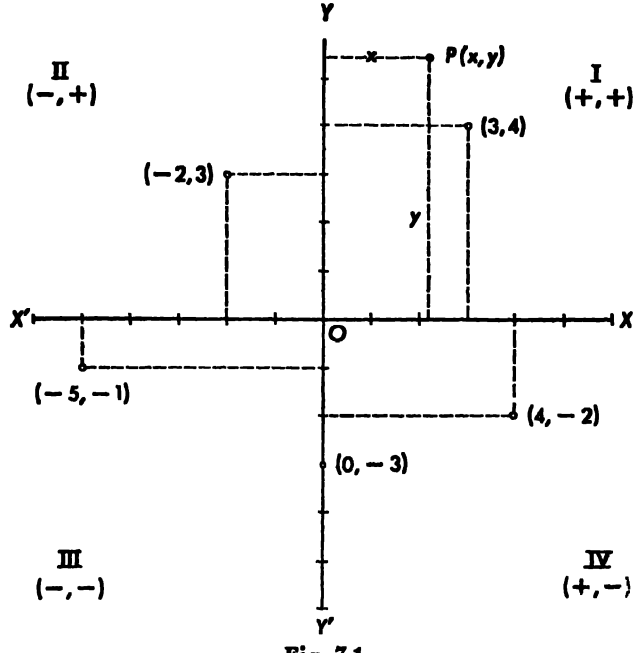


Fig. 7·1

The position of any point can be determined if its coordinates are known, and conversely the coordinates of a point appearing on the plane can be determined by measurement.

EXERCISE 2

Locate the points having the coordinates given below.

1. (2,5)	**2.** $(4,\sqrt{3})$	**3.** (−2,6)	**4.** $(-\sqrt{2},3)$
5. (8,−3)	**6.** (7,−4)	**7.** (−3,−5)	**8.** (−6,−2)
9. (0,5)	**10.** $(-5\frac{1}{2},0)$	**11.** (0,0)	**12.** $(1\frac{1}{2},-\frac{2}{3})$

EXERCISE 3

Identify the closed figures formed by plotting the given points and joining in order with straight lines.

1. (5,2), (5,−4), (−1,−4), (−1,2)
2. (0,4), (7,4), (7,−1), (0,−1)
3. (3,2), (8,0), (0,−9)
4. (2,4), (5,−3), (−1,−3)
5. (−1,6), (5,2), (−1,2)
6. (5,5), (7,−3), (−4,−3), (−6,5)

7·5 Graph of a Function. When we sketch the graph of a function, we say that we "graph the function."

Suppose that we are given some function $y = f(x)$. The first step in graphing the function is to make a table listing a number of different values of x and opposite them writing the corresponding values of y. We take these pairs of corresponding values of x and y as the coordinates of points which we plot. These points all lie on the graph of $y = f(x)$. Accordingly, we may define the *graph*, or *locus*, of a function as consisting of a system of points whose coordinates satisfy the relation $y = f(x)$.

A *linear function* of x is a first-degree polynomial in x having the form $ax + b$, where a and b are constants. The graph of such a function is always a straight line; hence the name linear function.

EXAMPLE 2. Graph the function $y = f(x) = \frac{2}{3}x - 4$. Form a table of arbitrarily chosen values of x together with the corresponding values of $f(x)$, *i.e.*, of y.

x	−6	−3	0	$1\frac{1}{2}$	3	5	8	12
y	−8	−6	−4	−3	−2	$-\frac{2}{3}$	$1\frac{1}{3}$	4

In Fig. 7·2 the points (−6,−8), (−3,−6), etc., are plotted, and a smooth line is drawn through these points.

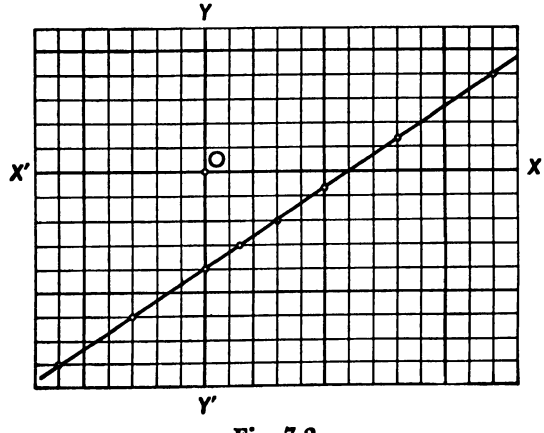


Fig. 7·2

It is evident that any number of points might be plotted whose coordinates satisfy the equation $y = \frac{2}{3}x - 4$. We infer, then, that we have a straight line which may be extended indefinitely in either direction. Theoretically, two points are sufficient to determine a straight line. However, in graphing any linear function, we should compute a minimum of three values of the function. For the sake of accuracy, these points should be located some distance apart. Any points not lying on the same straight line should be checked for errors.

For a given section of coordinate paper the position of the axes and the number of units assigned to a division on the paper should be chosen so as to make the desired portion of the graph as large as possible.

In general, if a function of x is defined by a formula, its graph is a smooth curve or, in some cases, two or more disconnected smooth curves. The term *smooth curve* is used in a broad sense to denote a line which may be straight or curved but which does not show any abrupt change in direction. It is often desirable to use different scales on the X and Y axes.

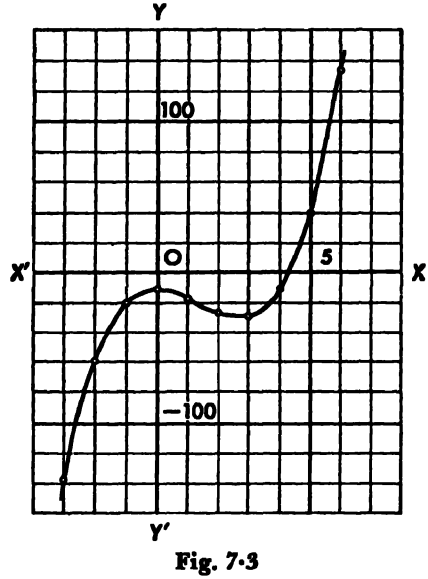


Fig. 7·3

EXAMPLE 3. Graph the function $y = f(x) = 2x^3 - 8x^2 - 11$. Form a table listing arbitrary values of x and corresponding values of y:

x	6	5	4	3	2	1	0	−1	−2	−3
y	133	39	−11	−29	−27	−17	−11	−21	−59	−137

From the plotting in Fig. 7·3, it will be noticed that, in order to open up the curve, the same distance that represents 20 vertical units is used to represent

1 horizontal unit. It will be noted that, although the curve continues indefinitely above and below the portion shown, we have included only the interesting part of the graph.

That part of a curve which is higher than points immediately adjacent to it on either side is called a *maximum* point. In our illustration it occurs at $x = 0$, for which the corresponding maximum value of the function is -11. On the other hand, that part of the curve which is lower than points immediately adjacent to it on either side is called a *minimum* point. In the illustration it occurs at $x = \frac{8}{3}$, for which the corresponding minimum value of the function is about -30.

The student will do well to see that each division on the axes corresponds to 1, 2, or 5 times an integral power of 10, as is the case with the scales of a slide rule.

EXERCISE 4

In drawing the following curves, graph that portion of the function which includes any maximum or minimum points. Locate axes and choose scales to display the curve to the best advantage. Indicate the coordinates of those points corresponding to zero, maximum, or minimum values of $f(x)$.

Plot the graphs of the following equations where $y = f(x)$:

1. $y = x$ **2.** $y = 3x$ **3.** $y = \frac{1}{2}x$
4. $y = -x$ **5.** $y = -2x$ **6.** $y = -\frac{1}{3}x$
7. $y = 2$ **8.** $y = -3$ **9.** $x = 4$
10. $x = -2\frac{1}{2}$ **11.** $y = x + 3$ **12.** $y = 2x - 1$
13. $y = \frac{1}{2}x + 2$ **14.** $y = x^2$ **15.** $y = \frac{1}{4}x^2$
16. $y = \frac{1}{2}x^2 - 8$ **17.** $y = x^2 + 2x$ **18.** $y = -x^2$
19. $y = -x^2 + 3x$ **20.** $y = x^2 + 5$ **21.** $y = -x^2 - 4$
22. $y = x^2 - 2x - 15$ **23.** $y = 6x^2 - x - 12$
24. $y = x^2 - 6x + 9$ **25.** $y = 4x^2 - 7x - 36$
26. $y = x^3 - 4x$ **27.** $y = x^3 - 4x^2 - 3x - 10$
28. $y = \dfrac{6}{x}$ **29.** $y = \dfrac{1}{x^2 + \frac{1}{4}}$
30. $y = x + \dfrac{1}{x}$ **31.** $y = \dfrac{12x}{x^2 + 4}$

7·6 Functions Not Defined by Equations. In laboratory and statistical work, observations of a number of variables often indicate that although they are related, there is no simple, obvious formula which expresses that relationship. These data are usually arranged in a table of values in which the independent variable is paired with corresponding values of the function.

To previous comments made on graphing functions the following suggestions are added:

1. *Plot values of the independent variable as abscissas.* (*Time units are perhaps the most common example.*)

2. *Data obtained from some field of natural science* (e.g., *physics, chemistry, biology, or engineering*) *should be connected by a smooth curve.*

3. *Data representing relationships established by man* (e.g., *stock-market quotations or utility rates*) *should be represented by a broken-line graph,* i.e., *by points connected by straight lines. Sometimes a series of disconnected lines is used* (e.g., *the cost of sending a package by parcel post as a function of the weight*).

4. *Data taken at intervals should be represented by a broken-line graph when intermediate values of the time element have no significance. An example is the year-by-year plotting of the volume of Christmas retail sales.*

5. *If all the values of the dependent variable* (*ordinates*) *are far above or below zero, a break should be indicated as in Fig.* 7·4. *In this way desirable*

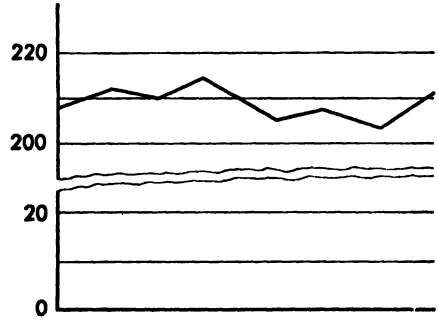


Fig. 7·4

magnification of fluctuations may be portrayed, without giving the erroneous impression made by a base line located at, say, y = 200.

EXERCISE 5

Graph the data of the following tables in the clearest and most effective manner.

1. The characteristics of a radio tube are given in the following table:

Grid volts	−60	−50	−40	−30	−20	−10	0
Plate current, amp	0.010	0.021	0.034	0.050	0.068	0.092	0.124

Grid volts	10	20	30	40	50	60	70	80
Plate current, amp	0.164	0.200	0.234	0.262	0.282	0.294	0.300	0.303

2. Voltage-current characteristics of a 60-watt, 120-volt tungsten lamp:

Volts	2	10	30	50	70	80
Amperes	0.049	0.117	0.227	0.311	0.383	0.414

Volts	90	100	110	120	130	150	170
Amperes	0.443	0.473	0 501	0.526	0.553	0.597	0.639

3. The following weight-height tables of boys in the United States is based on government statistics:

Age, years	1	2	3	4	5	6	7	8
Weight, lb	21.9	27.1	32.2	35.9	41.1	45.2	49.1	53.9
Height, in.	29.4	33.8	37.1	39.5	41.6	43.8	45.7	47.8

Age, years	9	10	11	12	13	14	15	16
Weight, lb	59.2	65.3	70.2	76.9	84.4	94.9	107.1	121.0
Height, in.	49.7	51.7	53.3	55.1	57.2	59.9	62.3	65.0

4. The time required for a sum of money to double itself at various rates of interest compounded semiannually:

Interest rate, %	*Time, years*
1½	46.4
2	34.8
2½	27.9
3	23.3
3½	20.0
4	17.5
4½	15.6
5	14.0
5½	12.8
6	11.7

5. If interest on a $100 loan is 2 per cent monthly, the total amount required to repay loan and interest varies with the monthly payment, as shown in the table.

Monthly payment	*Total amount paid*
$26.26	$105.05
12.25	110.25
6.67	120.00
4.83	130.50
4.00	140.00
3.39	152.55
2.90	171.10
2.60	192.40
2.36	224.20

6. The American experience mortality table indicates that of 100,000 average people living at age 10, the following number will die at various ages:

Age, years	10	20	25	30	40	50	60	65	70	75	80	85	90	95
No. of deaths	749	723	718	720	765	962	1,546	1,980	2,391	2,476	2,091	1,292	385	3

Explain the shape of this curve.

7. The number of square inches of material required to make a gallon cylindrical container varies with the diameter.

Diameter, in.	*Amount of material, sq in.*
4	255
4.5	237
5	223
5.5	215.5
6	210.5
6.5	208.7
7	209.6
7.5	212
8	215.5
8.5	221

8. The melting point of lead-tin alloys for various percentages of tin:

Tin, %	0	10	20	30	40	50	60	70	80	90	100
Melting point, °C	326	295	276	262	240	220	190	185	200	216	232

9. The freezing point of glycerin antifreeze solution for various specific gravities:

Specific gravity	1.000	1.024	1.049	1.075	1.101	1.128	1.156	1.184	1.211	1.238	1.264
Freezing point, °F	32.0	29.1	23.4	14.9	4.3	−9.4	−30.5	−38.0	−5.5	29.1	62.6

10. Manufacturer's literature indicates the length of straight No. 22 chromel wire to be used for constructing heating units of various wattages when used on a 110-volt circuit.

Length, in.	13.9	14.7	15.9	17.1	18.6	20.3	22.2	24.7	27.7
Watts	800	750	700	650	600	550	500	450	400

11. The percentage of light of various wavelengths transmitted by a blue-glass filter:

Wavelength of light, μ	0.28	0.30	0.32	0.34	0 36	0.42	0.46	0.54	0.60
Transmission, %	0	8	65	87	93	98	98	69	62

Wavelength of light, μ	0.64	0.70	1.00	1.50	2.00	2.50	3.00	4.00	5.00
Transmission, %	60	93	99	89	92	87	55	31	12

12. The temperature and rainfall for San Francisco (a "maritime" climate) and Indianapolis (a "continental" climate):

	Indianapolis		*San Francisco*	
	Temp, F°	Rainfall, in.	Temp, F°	Rainfall, in.
Jan.	29	2.9	50	4.7
Feb.	31	2.6	53	3.8
Mar.	41	3.9	54	3.1
Apr.	52	3.6	56	1.5
May	63	3.9	57	0.7
June	72	4.1	59	0.2
July	76	3.8	59	0.0
Aug.	74	3.2	59	0.0
Sept.	67	3.2	62	0.3
Oct.	56	2.7	61	1.0
Nov.	42	3.2	57	2.5
Dec.	32	2.8	52	4.4

13. The birth rate per 1,000 in the United States:

Year	1936	'37	'38	'39	'40	'41	'42	'43
Birth rate	16.7	17.1	17.6	17.3	17.9	18.9	20.9	21.5

Year	'44	'45	'46	'47	'48	'49	'50	'51
Birth rate	20.2	19.5	23.3	25.8	24.4	24.0	23.5	24.5

14. Insurance fees for third- and fourth-class mail:

Amount of insurance	\$0.01–5.00	\$5.01–10.00	\$10.01–25.00	\$25.01–50.00	\$50.01–100	\$100.01–200
Insurance fee	\$0.05	\$0.10	\$0.15	\$0.20	\$0.25	\$0.30

15. Price-index number for cotton (period of 1910–1914 = 100):

Year	1915	'18	'20	'25	'30	'32	'33	'34	'36	'38
Index	73	238	252	179	100	47	66	97	95	67

Year	'40	'41	'42	'44	'46	'47	'48	'49	'50	'51
Index	77	107	149	164	228	261	259	235	269	322

16. According to the 1954 World Almanac, the following running times were the record performance up to November 1, 1953:

Distance	*Time*
60 yd	6.1 sec
100 yd	9.3 sec
220 yd	20.2 sec
300 yd	29.8 sec
440 yd	46.0 sec
600 yd	1 min 9.2 sec
880 yd	1 min 49.2 sec
1,000 yd	2 min 9.3 sec
¾ mile	2 min 58.2 sec
1 mile	4 min 1.4 sec
2 miles	8 min 40.4 sec
4 miles	19 min 17.3 sec
8 miles	40 min 48.8 sec
15 miles	1 hr 17 min 28 6 sec

(*From The World Almanac.*)

Calculate the average speed for each run and plot both time and speed over a common distance base.

17. If we cut squares out of the corners of a 13-by-20-in. sheet of zinc (Fig. 7.5) and turn up the resulting flaps to form an open-top

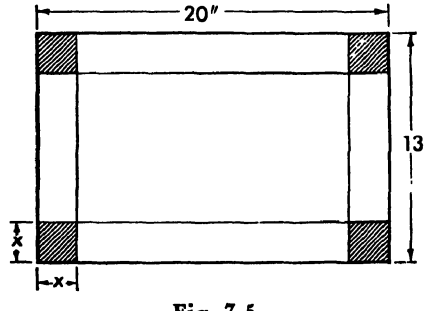


Fig. 7·5

box, the volume of the box will be given by the equation $V = 4x^3 - 66x^2 + 260x$. Plot V against x, and determine the dimensions of the box for maximum capacity.

7·7 Slope of a Straight Line. Referring to Fig. 7·6, draw any straight line not parallel to the axes. Let P and Q be any two points on the line, denoting the coordinates of P by (x_1, y_1) and the coordinates of Q by (x_2, y_2).

The steepness of slope of the line is then expressed by the equation

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \star (1)$$

Hence the *slope* of the straight line passing through two given points is equal to the difference of the ordinates of the points divided by the difference of their abscissas taken in the same order. (For the student's convenience, key equations will be marked by a star.)

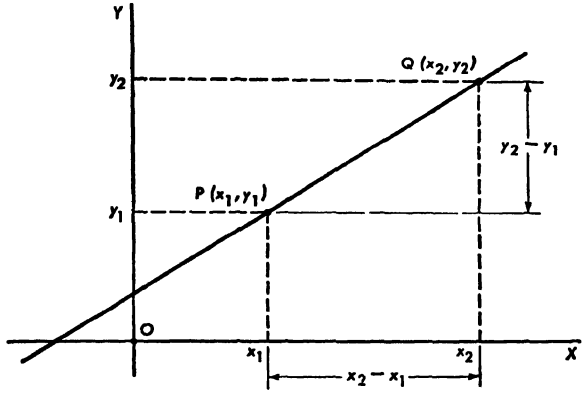


Fig. 7·6

It is apparent from the rule just stated that the slope of a line could also be expressed by the fraction $(y_1 - y_2)/(x_1 - x_2)$. This is, of course, in agreement with the law of signs as applied to fractions.

EXAMPLE 4. Find the slope of the line passing through (−3,5) and (9,11) (Fig. 7·7).

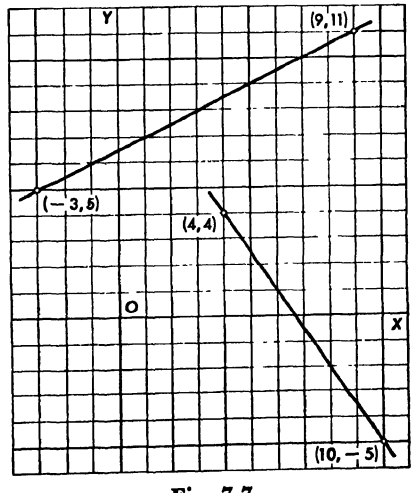


Fig. 7·7

Referring to Eq. (1), we may consider $x_1 = -3$, $x_2 = 9$, $y_1 = 5$, $y_2 = 11$. The slope is therefore

$$\frac{11 - 5}{9 - (-3)} = \frac{6}{12} = \frac{1}{2}$$

EXAMPLE 5. Find the slope of the line passing through (4,4) and (10,−5) (Fig. 7·7).

Taking $x_1 = 4$, $x_2 = 10$, $y_1 = 4$, and $y_2 = -5$, we have

$$\text{slope} = \frac{-5-4}{10-4} = \frac{-9}{6} = -\frac{3}{2}$$

The following characteristics of slope may be cited:

1. *A line sloping upward to the right has a positive slope, a line sloping downward to the right has a negative slope.*

2. *The slope of a horizontal line is zero.*

3. *The slope of a vertical line is undefined* (since $x_2 - x_1 = 0$ and we cannot divide by zero).

4. *If we move a point to the right along a straight line, the slope corresponds to the progress made in a vertical direction when the point has advanced one unit horizontally.* Thus if the point rises $1\frac{1}{2}$ units while at the same time moving one unit to the right the slope of the line is $1\frac{1}{2}$; while if the point falls $\frac{4}{5}$ of a unit while advancing one unit to the right the slope is $-\frac{4}{5}$.

EXERCISE 6

Find the slope of the straight line passing through the following points:

1. (−2,1) and (4,3)
2. (1,−3) and (4,3)
3. (0,4) and (8,−2)
4. (−2,5) and (3,0)
5. Through origin and (4,10)
6. (−3,−1) and (5, -1)
7. What is the slope of (*a*) a 30° line; (*b*) a 45° line; (*c*) a 60° line?

If three points $P(x_1, y_1)$, $Q(x_2, y_2)$, and $R(x_3, y_3)$ lie in a straight line (Fig. 7·8) the slope of PQ must be equal to that of QR. Hence, by (1) for three points lying in a straight line

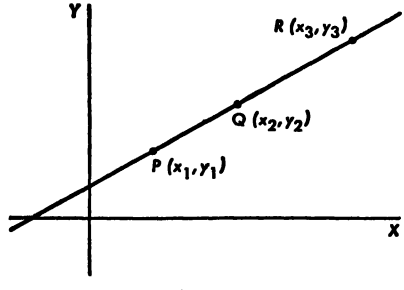


Fig. 7·8

$$\frac{y_3 - y_2}{x_3 - x_2} = \frac{y_2 - y_1}{x_2 - x_1} \qquad *(2)$$

The student will recognize that this is the type of straight-line equation which forms the basis of interpolation.

7·8 Determination of a Straight Line. A straight line may be determined (1) if the slope of the line and a point on the line are known or (2) if two points on the line are known.

Slope-point form. According to Eq. 1 the slope of the straight line in Fig. 7·9 is

$$m = \frac{y - y_1}{x - x_1} \qquad (3)$$

where the coordinates (x_1, y_1) of the fixed point P are given and $Q(x,y)$ is any variable point on the line.

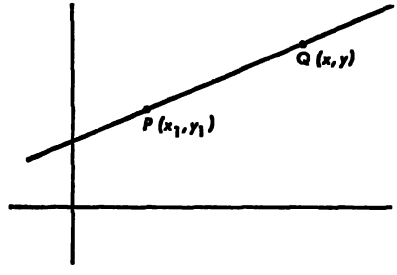


Fig. 7·9

Rearranging (3) we obtain the slope-point form:

$$y - y_1 = m(x - x_1) \qquad *(4)$$

EXAMPLE 6. Find the equation of the straight line having a slope of $\frac{2}{3}$ and passing through point (4,5).

Substituting $y_1 = 5$, $m = \frac{2}{3}$, and $x_1 = 4$ in (4) we obtain

$$y - 5 = \frac{2}{3}(x - 4)$$

whence

$$3y - 15 = 2x - 8 \text{ or } 3y - 2x = 7$$

A familiar example of this relationship is found in the equation $r_1 - r_2 = k(t_1 - t_2)$ which states that the change in electrical resistance of a conductor is proportional to the temperature change.

Slope-intercept form. If we substitute in Eq. (4), above, the coordinates of point P (Fig. 7·10), we obtain

$$y - b = m(x - 0)$$

or, rearranging, we have the slope-intercept form

$$y = mx + b \qquad \star (5)$$

NOTE: *The ordinate of the point of intersection of a straight line with the* Y *axis is called the* y-*intercept. The abscissa of its point of intersection with the* X *axis is called the* x-*intercept.*

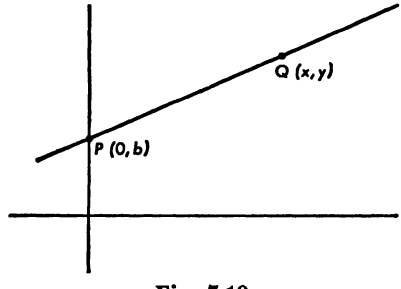


Fig. 7·10

EXAMPLE 7. Find the slope and y-intercept of the line whose equation is $3x + 2y = 8$.

Solving the equation for y, we obtain

$$y = -3/2x + 4$$

Hence the line has a slope of $-3/2$ and a y-intercept of 4.

A common example of the slope-intercept form occurs in the equation $l_1 = kW + l$, which states that in a spring balance the length of the relaxed spring is extended by k length units per W units of weight applied, producing a length l_1 under tension.

If the student has any doubt that the graph of $y = mx + b$ is a straight line he may be reassured by noting the similarity to a flight of stairs. If we tabulate a series of values as in plotting a graph we obtain the following table:

Δx	1	1	1	1	1	
x	0	1	2	3	4	5 $\cdots$
y	b	$m + b$	$2m + b$	$3m + b$	$4m + b$	$5m + b \cdots$
Δy	m	m	m	m	m	

The change in x from point to point has been represented by Δx (read "delta x").

The change in y from point to point has been represented by Δy.

It will be noted that when x increases uniformly in steps of 1, y also increases uniformly, but in steps of m. The ratio of vertical progress to horizontal progress is, therefore, uniform and the slope is every-

where equal to m. Referring to Fig. 7·11, we may draw an analogy to a flight of steps, each tread being equal to 1 and each riser equal to m.

The two-point form. If we have given the coordinates (x_2,y_2) and (x_1,y_1) of the fixed points P_2 and P_1 and if $P(x,y)$ is any variable point

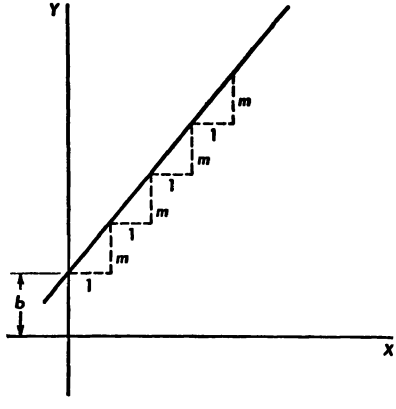


Fig. 7·11

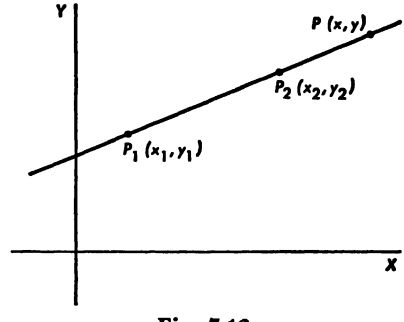


Fig. 7·12

on the line (Fig. 7·12), we may eliminate m from formulas (1) and (4) by substituting $\dfrac{y_2 - y_1}{x_2 - x_1}$ for m in the formula

$$y - y_1 = m(x - x_1)$$

obtaining

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \star (6)$$

This is called the *two-point form* of a straight-line equation. [Note the similarity to Eq. (2).]

EXAMPLE 8. Find the equation of the straight line passing through the points $(-2,3)$ and $(6,9)$.

Here we have $x_1 = -2$, $y_1 = 3$, $x_2 = 6$, and $y_2 = 9$. Substituting in Eq. (6),

$$y - 3 = \frac{9 - 3}{6 - (-2)}[x - (-2)]$$

or

$$y - 3 = \frac{6}{8}(x + 2)$$

Multiplying by 4,

$$4y - 12 = 3(x + 2)$$

or

$$4y - 3x = 18$$

which is the desired equation.

An everyday example of the operation of this formula is found with a slight rearrangement by writing

$$\frac{l - l_1}{t - t_1} = \frac{l_2 - l_1}{t_2 - t_1}$$

which states that the linear expansion of a metal rod obeys the law that changes in length are proportional to changes in temperature.

The intercept form. One of the most convenient ways of plotting the graph of a linear equation is to find the intercepts on the X and Y axes and draw a straight line through them.

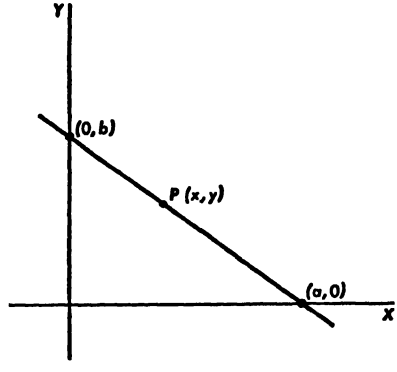


Fig. 7·13

To derive the intercept form we shall rearrange Eq. (6) in the form

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Referring to Fig. 7·13, $x_1 = 0$, $y_1 = b$, $x_2 = a$, and $y_2 = 0$. Hence we obtain (on substituting)

$$\frac{y - b}{x - 0} = \frac{0 - b}{a - 0}$$

or $$\frac{y - b}{x} = \frac{-b}{a}$$

or $$a(y - b) = -bx$$

or $$bx + ay = ab$$

Dividing through by ab we have

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \star (7)$$

This is known as the *intercept form* of the linear equation.

EXAMPLE 9. Find the equation of the line having an x-intercept of -4 and a y-intercept of 10.

Substituting $a = -4$ and $b = 10$ in Eq. (7), we obtain

$$\frac{x}{-4} + \frac{y}{10} = 1$$

or multiplying by -20, we have

$$5x - 2y = -20$$

which is the required equation.

EXAMPLE 10. Find the intercepts of the line $3x - 4y = 12$.

Dividing through by 12, we have

$$\frac{x}{4} - \frac{y}{3} = 1 \text{ or } \frac{x}{4} + \frac{y}{-3} = 1$$

Hence $a = 4 =$ the x-intercept and $b = -3 =$ the y-intercept.

EXERCISE 7

Write the equations for lines passing through the given point P and having the slope indicated (see Fig. 7·14, where the slope is given in degrees):

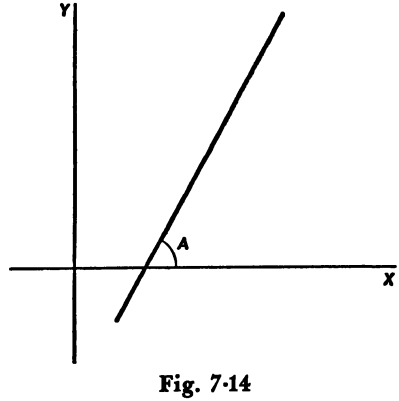


Fig. 7·14

1. $P(4,6)$ $m = \frac{1}{2}$
2. $P(-2,3)$ $m = 1$
3. $P(5,-1)$ $m = 2$
4. $P(-3,-2)$ $m = -\frac{2}{3}$
5. $P(3,7)$ $\angle A = 45°$
6. $P(-3,6)$ $\angle A = 30°$
7. $P(2,-5)$ $\angle A = 60°$
8. $P(0,0)$ $\angle A = 135°$
9. $P(2,7)$ horizontal
10. $P(4,-3)$ vertical

Convert each of the following equations to the slope-intercept form $y = mx + b$, finding the slope and y-intercept in each case:

11. $x + y = 8$ **12.** $2x - y = 6$ **13.** $x - 3y = 12$
14. $2x + 3y = 6$ **15.** $4y - x = 10$ **16.** $3x - 4y - 6 = 0$
17. $7x - 5y + 14 = 0$ **18.** $\frac{1}{2}x - \frac{1}{3}y = 1$ **19.** $4y = 6$

Find the equation of the line passing through each of the following given pairs of points, leaving the answer in the form $Ax + By = C$, where A and B are integers:

20. (3,1), (5,3) **21.** (3,2), (6,3) **22.** (1,−1), (3,3)
23. (2,−4), (−6,2) **24.** (0,0), (−4,−6) **25.** (3,0), (−5,2)
26. (6,3), (−4,−2) **27.** (−2,5), (4,5)

Find the equations of the lines having the following given intercepts:

28. x-intercept 3, y-intercept −6.
29. x-intercept −4, y-intercept 2.
30. x-intercept 10, y-intercept 4.
31. x-intercept −9, y-intercept −6.
32. x-intercept 6, y-intercept −10.
33. x-intercept −8, y-intercept 6.
34. x-intercept 8, y-intercept 14.
35. x-intercept −15, y-intercept −10.

7·9 Lines Parallel to the Axes. If a line is parallel to the X axis its slope m is equal to zero. Substituting $m = 0$ in $y = mx + b$, we obtain $y = b$. This is reasonable, for it indicates that everywhere on the line the ordinate is b. Hence the line is parallel to the X axis and b units above it.

Since a line parallel to the Y axis is vertical it has no finite slope and the form $y = mx + b$ does not apply. However, it is obvious that a vertical line a units to the right of the Y axis must have the equation $x = a$.

7·10 Lines Passing through the Origin. It is evident that, since the origin (0,0) is one of the points on the line, if we substitute $x = 0$ in the equation, y must also equal zero, and there will be no constant term in the simplified equation.

EXAMPLE 11. Show that the line passing through the points (12,8) and (9,6) also passes through the origin.

Substituting $x_1 = 9$, $y_1 = 6$, $x_2 = 12$, and $y_2 = 8$ in Eq. (6), we obtain

$$y - 6 = \frac{8 - 6}{12 - 9}(x - 9)$$

$$y - 6 = \frac{2}{3}(x - 9)$$

$$3(y - 6) = 2(x - 9)$$

$$3y - 18 = 2x - 18$$

$$3y - 2x = 0$$

Since the values $x = 0$, $y = 0$ satisfy the equation, the line must pass through the origin.

7·11 Parallel Lines. *If two lines are parallel their slopes are equal.* The comparison is most conveniently made by reducing both equations to the slope-intercept form, $y = mx + b$ [Eq. (5)].

7·12 Perpendicular Lines. In Fig. 7·15 let AC and CD be two perpendicular lines, where $C(b,c)$ is their common point and $A(a,0)$

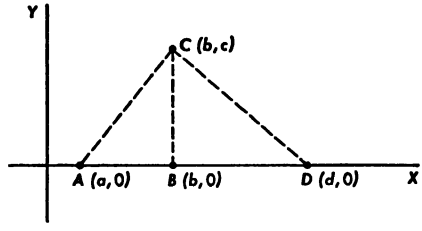


Fig. 7·15

and $D(d,0)$ are their x-intercepts. Let $B(b,0)$ be the foot of the perpendicular CB drawn from C to AD.

Since the altitude to the hypotenuse of a right triangle is a mean proportional between the segments of the hypotenuse, we may write

$$\overline{CB}^2 = \overline{AB} \cdot \overline{BD}$$

Expressed in terms of the coordinates, this relationship becomes

$$c^2 = (b - a)(d - b) \tag{8}$$

The slope of $AC = (c - 0)/(b - a)$ or

$$m_1 = \frac{c}{b - a}.$$

The slope of $CD = (c - 0)/(b - d)$ or

$$m_2 = \frac{c}{b - d}$$

Multiplying the values of the two slopes by each other, we obtain

$$m_1 m_2 = \frac{c^2}{(b-a)(b-d)} = -1 \qquad \text{from (8)}$$

or

$$m_1 = -\frac{1}{m_2} \qquad \star (9)$$

Hence the slopes of perpendicular lines are negative reciprocals.

7·13 Distance between Two Points. Given the points $P_1(x_1y_1)$ and $P_2(x_2y_2)$ in Fig. 7·16, locate the point $Q(x_1y_2)$ and draw lines

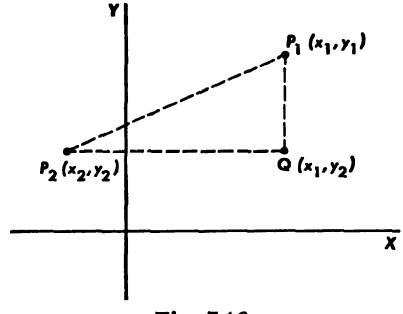


Fig. 7·16

P_1P_2, P_1Q and P_2Q. Then in right triangle P_1P_2Q

$$\overline{P_1P_2}^2 = \overline{P_1Q}^2 + \overline{P_2Q}^2$$

In terms of the coordinates this becomes

$$\overline{P_1P_2}^2 = (y_1 - y_2)^2 + (x_1 - x_2)^2$$

or the distance between P_1 and P_2 is

$$\overline{P_1P_2} = \sqrt{(y_1 - y_2)^2 + (x_1 - x_2)^2} \qquad \star (10)$$

This rule is valid regardless of the quadrants in which P_1 and P_2 are located, provided only that due regard is paid to signs when evaluating $y_1 - y_2$ and $x_1 - x_2$.

EXERCISE 8

1. Given the equation $3x - 4y = 24$, corresponding to the general equation $Ax + By = C$, modify only one quantity at a time (A, B, or C) to make the line (*a*) pass through the origin; (*b*) vertical; (*c*) horizontal; (*d*) parallel to $6x - y = 11$; and (*e*) perpendicular to $8x - 12y = 13$.

2. Given the equation $5x + 8y = 20$, proceed as in Prob. 1 to make the line (*a*) pass through the origin; (*b*) vertical; (*c*) horizontal; (*d*) parallel to $15x - 10y = 23$; and (*e*) perpendicular to $4x - 15y = 17$.

Find the distance between P_1 and P_2 in the following problems:

3. $P_1(2,5)$, $P_2(2,11)$
4. $P_1(3,-4)$, $P_2(3,3)$
5. $P_1(-4,7)$, $P_2(5,7)$
6. $P_1(6,1)$, $P_2(0,9)$
7. $P_1(1,5)$, $P_2(4,9)$
8. $P_1(-2,6)$, $P_2(3,9)$
9. $P_1(8,-4)$, $P_2(-1,-6)$
10. $P_1(7,-8)$, $P_2(3,-5)$

11. Referring to Eq. (2) and Fig. 7·8, if point Q were located midway between points P and R, prove that the mid-point of line segment PR is located at $x_2 = (x_1 + x_3)/2$, $y_2 = (y_1 + y_3)/2$.

7·14 Area of a Triangle. The area of the triangle PQR (Fig. 7·17) is evidently equal to the area of triangle PQM plus the area of the trapezoid $MQRN$ minus the area of the triangle PRN.

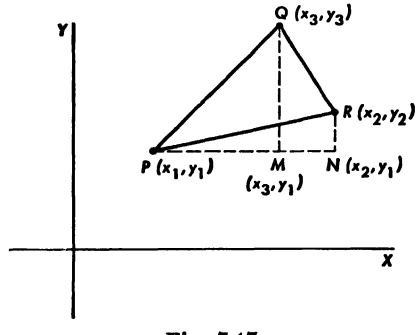


Fig. 7·17

In terms of the coordinates we have:

$$\text{Area of } PQM = \tfrac{1}{2}(x_3 - x_1)(y_3 - y_1)$$
$$\text{Area of } MQRN = \tfrac{1}{2}(x_2 - x_3)[(y_2 - y_1) + (y_3 - y_1)]$$
$$\text{Area of } PRN = \tfrac{1}{2}(x_2 - x_1)(y_2 - y_1)$$

(The points must be taken in counterclockwise order.) Therefore

$$\begin{aligned}\text{Area of } PRQ &= PQM + MQRN - PRN \\ &= \tfrac{1}{2}(x_3 - x_1)(y_3 - y_1) + \tfrac{1}{2}(x_2 - x_3)[(y_2 - y_1) \\ &\qquad + (y_3 - y_1)] - \tfrac{1}{2}(x_2 - x_1)(y_2 - y_1) \\ &= \tfrac{1}{2}(x_3y_3 - x_1y_3 - x_3y_1 + x_1y_1 + x_2y_2 - x_3y_2 - 2x_2y_1 \\ &\qquad + 2x_3y_1 + x_2y_3 - x_3y_3 - x_2y_2 + x_1y_2 + x_2y_1 - x_1y_1) \\ &= \tfrac{1}{2}(-x_1y_3 - x_3y_2 - x_2y_1 + x_3y_1 + x_2y_3 + x_1y_2) \\ &= \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \qquad \star (11)\end{aligned}$$

If any of the vertices lie outside of the first quadrant, proper attention must be paid to signs. (For an application of this principle to surveying see Sec. 20·7.)

EXAMPLE 12. Find the area of the triangle in Fig. 7·18. From the figure we write

$$x_1 = -3 \qquad y_1 = -5 \qquad x_2 = 17 \qquad y_2 = 2 \qquad x_3 = 3 \qquad y_3 = 8$$

Substituting in Eq. (11),

$$\begin{aligned} \text{Area of } \Delta PQR &= \tfrac{1}{2}\{(-3)(2-8) + 17[8-(-5)] + 3(-5-2)\} \\ &= \tfrac{1}{2}(18 + 221 - 21) \\ &= 109 \end{aligned}$$

The area of a polygon of n sides may be found in like manner by first dividing by diagonals into $n - 2$ triangles.

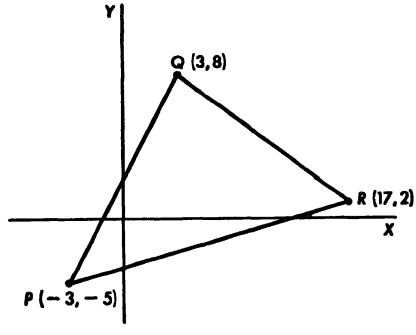


Fig. 7·18

EXERCISE 9

Find the area of the triangles represented by the following vertices:

1. (0,0), (0,8), (12,0)
2. (−2,−5), (−4,4), (7,1)
3. (−5−3), (2,3), (9,−7)
4. (−5,6), (3,8), (4,−4)
5. (−5,0), (0,7), (8,−4)
6. (1,4), (10,5), (3,−4)
7. Prove that the triangle in Prob. 5 is isosceles.

Find the areas of the polygons formed by joining the following points in the order given:

8. (−2,2), (2,6), (8,3), (5,−3)
9. (0,−4), (3,4), (12,0), (10,−3)
10. (−3,−1), (0,8), (12,4), (9,−5)
11. (−2,−4), (−3,3), (4,6), (7,2), (4,−3)
12. Prove that the figure in Prob. 10 is a rectangle.

Chapter 8 SYSTEMS OF LINEAR EQUATIONS

Technical students constantly deal with formulas and equations. In engineering, technology, business statistics, etc., systems of linear equations are frequently encountered. In this chapter we shall discuss graphical and algebraic methods (including determinants) for solving such equations.

8·1 Graphical Solution of a System of Two Equations. A solution of two *simultaneous equations* in two unknowns, x and y, is a pair of corresponding values of x and y which simultaneously satisfies both equations.

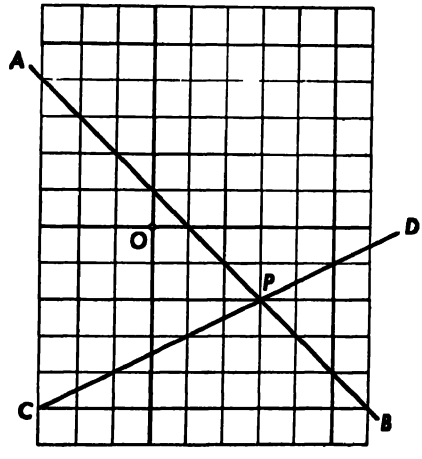


Fig. 8·1

EXAMPLE 1. Solve graphically:

$$x + y = 1 \tag{1}$$
$$x - 2y = 7 \tag{2}$$

In Fig. 8·1, AB is the graph of Eq. (1), and CD is the graph of Eq. (2). All points on AB have coordinates satisfying (1), and all points on CD have

coordinates satisfying (2); therefore intersection point P has coordinates satisfying the two equations *simultaneously*. Since two straight lines can intersect in but one point, P is the only point having the property of a common solution. In this example the coordinates of P are $(3,-2)$; therefore $(x = 3, y = -2)$ is the only solution of this system of equations.

If the graphs of the equations are parallel lines, there is no solution and the equations are termed *inconsistent.*

EXAMPLE 2. Given the system of equations

$$3x - 5y = 6 \tag{3}$$
$$6x - 10y = 10 \tag{4}$$

The graphs of these two equations are parallel lines, each having a slope of 3/5; and since the lines cannot intersect, the equations have no solution. This conclusion might have been reached without plotting, since on dividing Eq. (4) by 2 we have $3x - 5y = 5$. Since the expression $3x - 5y$ cannot, at the same time, be equal to 6 and to 5, there are no values of x and y satisfying both equations.

If the graphs of the two equations are the same line, any solution of one equation is also a solution of the other and the system has an unlimited number of solutions. Such equations are known as *dependent* equations.

EXAMPLE 3. Given the system

$$2x - 5y = 4 \tag{5}$$
$$6x - 15y = 12 \tag{6}$$

Since Eq. (6) is reducible to Eq. (5) by dividing by 3, the equations are not independent but are equivalent.

It will be appreciated that graphical solutions are only approximate; when we have to estimate fractional scale divisions, the approximation is rougher.

EXERCISE 1

In the following problems determine by inspection whether each pair of equations is inconsistent or dependent or possesses one common solution. In the last case solve graphically, estimating fractional answers to the nearest 0.1.

1. $x + y = 8$
$x - y = 2$

2. $x + y = -4$
$x - y = 8$

3. $2x - y = 3$
$2x - 3y = 11$

4. $3x - y = 4$
$2y - 6x = -8$

5. $2x + 5y = 8$
$3x - 2y = -7$

6. $4x - 6y = 8$
$6x - 9y = 14$

7. $2x = 7y$
$6x - 5y = 16$

8. $3x + 2y = 12$
$3y - 2x = 4$

9. $4x - 3y = 10$
$6y - 8x = -20$

10. $5x - 8y = 0$
$7y - 4x = 3$

11. $2x - y = 5$
$x + y = 2$

12. $8x - 12y = 10$
$18y - 12x = 15$

13. $12x + 5y = 0$
$8x - 10y = 8$

14. $2x - 7y + 9 = 0$
$5x + 3y - 6 = 0$

8·2 Algebraic Methods of Solution. Algebraic methods of solving systems of simultaneous equations depend on eliminating one of the unknowns. There are three methods of solution: addition or subtraction; substitution; and comparison.

8·3 Elimination by Addition or Subtraction

EXAMPLE 4. Solve for x and y:

$$4x - 7y = 29 \tag{7}$$
$$6x + 5y = -3 \tag{8}$$

Multiplying Eq. (7) by 3,

$$12x - 21y = 87 \tag{9}$$

Multiplying Eq. (8) by 2,

$$12x + 10y = -6 \tag{10}$$

Subtracting Eq. (10) from Eq. (9),

$$-31y = 93 \tag{11}$$

Dividing Eq. (11) by -31,

$$y = -3 \tag{12}$$

Substituting -3 for y in Eq. (7),

$$4x + 21 = 29$$

or

$$x = 2 \tag{13}$$

Check by substituting $x = 2$ and $y = -3$ in Eq. (8).

$$12 - 15 = -3 \qquad \text{(check)}$$

The above procedure may be summarized as follows:

To solve a system of linear equations by elimination by addition or subtraction:

1. *Multiply both members of both equations, if necessary, by numbers which will cause the coefficients of one of the unknowns to have the same absolute value in both equations.*

2. *To eliminate this unknown, the two equations obtained in step* 1 *are added if the matched coefficients have opposite signs. They are subtracted if they have like signs.*

3. *Solve the equation formed in step* 2 *for the unknown contained in it. Substitute the result in one of the original equations to obtain the other unknown.*

4. *Check by substituting the values of the unknowns in the original equation not used in step* 3.

NOTE: *If both members of an equation contain a common factor, it may be advisable to divide through by that factor.*

Clear of any fractions before eliminating an unknown (see exception, Sec. 8·7).

It will usually be preferable to eliminate the unknown whose coefficients have the smaller LCM.

8·4 Elimination by Substitution

EXAMPLE 5. Solve for x and y:

$$7x - 3y = 10 \tag{14}$$
$$5x - 2y = 8 \tag{15}$$

Solve Eq. (15) for y:

$$5x - 8 = 2y \tag{16}$$

or

$$y = \frac{5x - 8}{2} \tag{16}$$

Substitute $\frac{5x - 8}{2}$ for y in Eq. (14):

$$7x - 3\left(\frac{5x - 8}{2}\right) = 10 \tag{17}$$

Solve Eq. (17) for x, first eliminating fractions:

$$14x - 3(5x - 8) = 20$$
$$14x - 15x + 24 = 20$$
$$x = 4$$

Substitute $x = 4$ in Eq. (16):

$$y = \frac{(5)(4) - 8}{2}$$

or

$$y = 6$$

Hence the solution of the system is $x = 4, y = 6$.

Check by substituting $x = 4$ and $y = 6$ in Eq. (14):

$$(7)(4) - (3)(6) = 10 \qquad 28 - 18 = 10$$

To solve a system of two linear equations by elimination by substitution:

1. *Solve one equation for one unknown (preferably the unknown having the simplest coefficient) in terms of the other.*

2. *The expression obtained in step* 1 *is to be substituted for the equivalent unknown in the other equation.*

3. *Solve the equation obtained in step* 2 *for the second unknown.*

4. *Substitute the value of the second unknown in the expression obtained in step* 1, *and solve for the first unknown.*

5. *Check by substituting values of both unknowns in the original equation not used in step* 1.

8·5 Solution by Elimination by Comparison

EXAMPLE 6. Solve the following for x and y:

$$4x + 7y = 3 \tag{18}$$
$$6x - 5y = 20 \tag{19}$$

Solving Eq. (18) for x,

$$x = \frac{3 - 7y}{4} \tag{20}$$

Solving Eq. (19) for x,

$$x = \frac{5y + 20}{6} \tag{21}$$

Equating Eqs. (20) and (21),

$$\frac{3 - 7y}{4} = \frac{5y + 20}{6}$$
$$3(3 - 7y) = 2(5y + 20)$$
$$9 - 21y = 10y + 40$$
$$-31y = 31$$
$$y = -1$$

Complete the solution as in elimination by addition or subtraction, obtaining $x = 2\frac{1}{2}$.

NOTE: *The method of addition or subtraction is the most generally useful, especially when the coefficients of one unknown have the same absolute value in both equations.*

The method of substitution is most convenient to apply when the unknown which is to be expressed in terms of the other has a coefficient of one.

There is seldom a case of solution by comparison which could not have been done at least as easily by either of the first two methods.

EXERCISE 2

Solve the following systems of equations by the most convenient method:

1. $3x - 2y = 4$
$x = y$

2. $9x - 5y = 42$
$x + y = 0$

3. $8y - 5x = 18$
$x - y = 0$

4. $7x - 5y = 161$
$x = 4y$

5. $9x + 8y = 77$
$x - y = 1$

6. $3x - 2y = 10$
$x - y = 1$

7. $8x - 5y = 58$
$x + y = 4$

8. $6x - 10y = 4$
$x + y = 14$

9. $2x + y = 5$
$x + 2y = 19$

10. $4x + 5y = 30$
$6x + y = 19$

11. $x + y = 13$
$x - y = 5$

12. $5x + 2z = 61$
$3x - 2z = 43$

13. $11y - 4z = 41$
$7y + 2z = 42$

14. $5w + 6y = 17$
$3w - 2y = 27$

15. $9x - 13w = -3$
$6x - 7w = 3$

16. $12x - 11z = -5$
$9x + 4z = 33$

17. $7x - 3y = 41$
$4x + 5y = 10$

18. $9w + 8z = 15$
$5w + 7z = 16$

19. $11x - 3y + 34 = 0$
$8x + 10y - 24 = 0$

20. $13y + 5z - 33 = 0$
$7y - 9z - 47 = 0$

21. $3(x + y) = 33$
$5(x - y) = 25$

22. $4(x + z) = 22$
$6(x - z) = 15$

23. $0.5x + 0.4w = 0.26$
$0.7x + 0.3w = 0.26$

24. $0.3x + 0.4y = 14.5$
$0.4x + 0.3y = 13.5$

25. $\frac{x}{4} + 4y = 28$

$7x - \frac{y}{5} = 36$

26. $\frac{x}{3} - 4y = -58$

$5x - \frac{y}{6} = 27\frac{1}{2}$

27. $\frac{w + 3}{2} + 5z = 9$

$\frac{z + 9}{10} - \frac{w - 2}{3} = 0$

28. $\frac{x - 2}{10} + 3y = 8$

$\frac{x + 2}{6} - \frac{y - 7}{9} = 2$

8·6 Literal Linear Equations. A system involving letters other than the unknowns is usually best solved by finding each unknown independently by elimination by addition or subtraction.

EXAMPLE 7. Solve for x and y:

$$mx + ny = 2(m^2 - n^2) \tag{22}$$
$$x - y = m + n \tag{23}$$

Multiply Eq. (23) by n:

$$nx - ny = mn + n^2 \tag{24}$$

Add Eqs. (22) and (24):

$$\begin{array}{rl} mx + ny = & 2m^2 \quad - 2n^2 \\ nx - ny = & \quad mn + n^2 \\ \hline mx + nx = & 2m^2 + mn - n^2 \end{array} \tag{25}$$

Factor Eq. (25):

$$x(m + n) = (m + n)(2m - n) \tag{26}$$

Divide Eq. (26) through by $m + n$:

$$x = 2m - n$$

Multiply Eq. (23) by m:

$$mx - my = m^2 + mn \tag{27}$$

Subtract Eq. (27) from Eq. (22):

$$\begin{array}{rl} mx + ny = & 2m^2 \quad - 2n^2 \\ mx - my = & m^2 + mn \\ \hline ny + my = & m^2 - mn - 2n^2 \end{array} \tag{28}$$

Factor Eq. (28):

$$y(n + m) = (m + n)(m - 2n) \tag{29}$$

Divide Eq. (29) through by $m + n$:

$$y = m - 2n$$

If answers are at all involved, checking may best be done by numerical substitution, as in Example 3, page 85.

8·7 Equations Linear in the Reciprocals of the Unknowns. Equations of the type $a/x + b/y = c$ are usually best solved without removing the unknowns from the denominators.

EXAMPLE 8. Solve the system

$$\frac{9}{x} - \frac{15}{y} = 1 \tag{30}$$
$$\frac{14}{x} + \frac{20}{y} = 3 \tag{31}$$

Multiply Eq. (30) by 4:

$$\frac{36}{x} - \frac{60}{y} = 4 \tag{32}$$

Multiply Eq. (31) by 3:

$$\frac{42}{x} + \frac{60}{y} = 9 \tag{33}$$

Add Eqs. (32) and (33):

$$\frac{78}{x} = 13$$
$$78 = 13x$$

or

$$x = 6$$

Substitute $x = 6$ in Eq. (30):

$$\frac{9}{6} - \frac{15}{y} = 1$$
$$-\frac{15}{y} = -\frac{1}{2}$$
$$y = 30$$

The solution of the system is $x = 6, y = 30$.

EXAMPLE 9. Solve the system

$$\frac{3}{8x} - \frac{1}{2y} = 2 \tag{34}$$
$$\frac{5}{6x} - \frac{5}{3y} = -8\frac{1}{3} \tag{35}$$

Multiply Eq. (34) by 10:

$$\frac{15}{4x} - \frac{5}{y} = 20 \tag{36}$$

Multiply Eq. (35) by 3:

$$\frac{5}{2x} - \frac{5}{y} = -25 \tag{37}$$

Subtract Eq. (37) from Eq. (36):

$$\frac{15}{4x} - \frac{5}{2x} = 45 \tag{38}$$

Multiply Eq. (38) by $4x$:

$$15 - 10 = 180x$$
$$x = \tfrac{1}{36}$$

Substitute $x = \frac{1}{36}$ in Eq. (34):

$$\frac{3}{8(\frac{1}{36})} - \frac{1}{2y} = 2$$
$$\frac{27}{2} - \frac{1}{2y} = 2$$
$$-\frac{1}{2y} = -\frac{23}{2}$$
$$y = \frac{1}{23}$$

Hence the solution of the system is $x = \frac{1}{36}, y = \frac{1}{23}$.

EXERCISE 3

Solve the following systems for x, y, z, or w, as the case may be.

1. $3x + y = 4c$
$x - y = 4d$

2. $x + y = 5a$
$2x - 3y = 5b$

3. $2x - 3y = a + 4b$
$x + 2y = 4a - 5b$

4. $5x - 4y = 6c - 15d$
$3x + y = 7c - 9d$

5. $ax + 2by = 4$
$2ax - 6by = 3$

6. $12ax - 5by = -16$
$6ax + 7by = 30$

7. $2ax + by = 10c$
$ax - 3by = -9c$

8. $3ax - 4by = 18ab$
$2ax + by = ab$

9. $bx + cy = 2bc$
$cx + by = b^2 + c^2$

10. $ax - by = 2a^2 - 6ab + 2b^2$
$bx + ay = 3a^2 - 3b^2$

11. $2nx + 2my = m^2 + n^2$
$4nx - mx + my = mn + 2n^2$

12. $ax + 2cy = 5ac$
$\frac{2x}{c} - \frac{y}{a} = \frac{-5}{ac}$

13. $\frac{1}{x} + \frac{1}{y} = 6$
$\frac{1}{x} - \frac{1}{y} = 4$

14. $\frac{3}{x} - \frac{2}{y} = 10$
$\frac{4}{x} + \frac{1}{y} = 28$

15. $\frac{10}{x} + \frac{6}{y} = 7$
$\frac{14}{x} - \frac{9}{y} = 4$

16. $\frac{15}{x} + \frac{14}{w} = 5$
$\frac{25}{x} - \frac{6}{w} = 1$

17. $\frac{1}{2x} + \frac{1}{3y} = 12$
$\frac{1}{4x} + \frac{1}{9y} = 5$

18. $\frac{5}{6x} - \frac{3}{4z} = 6$
$\frac{7}{9x} + \frac{1}{6z} = 7\frac{1}{3}$

8·8 Linear Equations in More than Two Unknowns. Like a system in two unknowns, a system of linear equations in more than two unknowns may have a solution or may be dependent or inconsistent. Only the first case will be considered here. For a solution to be possible, there must be as many independent equations as unknowns.

EXAMPLE 10. Solve for x, y, and z:

$$6x - 5y - 2z = 2 \tag{39}$$
$$4x + y + 3z = 10 \tag{40}$$
$$5x + 3y + 7z = 13 \tag{41}$$

Multiply Eq. (40) by 5:

$$20x + 5y + 15z = 50$$

Add Eq. (39):

$$26x + 13z = 52 \tag{42}$$

Multiply Eq. (40) by 3:

$$12x + 3y + 9z = 30$$

Subtract Eq. (41):

$$\begin{array}{r} 5x + 3y + 7z = 13 \\ \underline{7x \qquad + 2z = 17} \end{array} \qquad (43)$$

Note that Eq. (42) may be divided throughout by 13. Therefore we will obtain smaller coefficients in Eq. (42) by multiplying by $\frac{2}{13}$ rather than by 2.

$$4x + 2z = 8 \qquad (44)$$

Subtract Eq. (43):

$$\begin{array}{r} 7x + 2z = 17 \\ \underline{-3x \qquad = -9} \\ x = 3 \end{array}$$

Substitute $x = 3$ in Eq. (43):

$$\begin{aligned} (7)(3) + 2z &= 17 \\ z &= -2 \end{aligned}$$

Substitute $x = 3$ and $z = -2$ in Eq. (40):

$$\begin{aligned} (4)(3) + y + 3(-2) &= 10 \\ y &= 4 \end{aligned}$$

Check the solution by substituting the values of x, y, and z in Eq. (39) or (41).

The procedure for solving a system of three linear equations in three unknowns may be summarized as follows:

1. *Select the unknown most easily eliminated and eliminate this unknown from one pair of equations.*

2. *Eliminate the same unknown from one of the equations just used and the third equation.*

3. *Solve the two equations so obtained as in Sec.* 8·2.

4. *Substitute the two values found in step* 3 *in the simplest of the original equations and solve for the third unknown.*

5. *Check by substituting the values of the unknowns in either of the two original equations not used in step* 4.

Simultaneous linear equations in more than three unknowns may be solved by an extension of the above method but a more effective method will be explained in Secs. 8·12 to 8·14.

EXERCISE 4

Solve the following systems of equations:

1. $x + y = 7$
$x + z = 8$
$y + z = 9$

2. $x + y = 4$
$x + z = 7$
$y + z = 1$

3. $x + y + z = 8$
$x - y + z = 2$
$x + y - z = 12$

4. $x - y - z = 1$
$x + y - z = 6$
$x - y + z = -1$

5.* $2x + y + z = 7$
$x + 2y + z = 15$
$x + y + 2z = 18$

6.* $x + y + 3z = 9$
$x + 3y + z = 7$
$3x + y + z = 7$

7. $2x + 3y - 4z = 11$
$3x - 5y + z = -3$
$6x + 2y - 7z = 18$

8. $7x + 4y - 3z = -15$
$5x - 6y + 2z = 41$
$x - 10y - 9z = -8$

9. $6x - 8y + 3z = -5$
$9x + 20y - 4z = 25$
$15x + 12y + 7z = 12$

10. $10x + 9y + 7z = 32$
$11x - 6y - 8z = 20$
$-9x + 12y + 10z = -22$

*SUGGESTION: Add the equations and divide the sum through by the common factor.

8·9 Graphical Solutions of Simultaneous Linear Equations. Two-component mixture problems may be solved approximately and quickly by means of the graphical procedure outlined in Appendix A. .

Distance, time, and rate linear problems are especially adaptable to graphical solution. Time, the independent variable, is usually the abscissa; and distance, the dependent variable, the ordinate.

EXAMPLE 11. A westbound New York Central express leaves Albany at 1 P.M. and maintains an average speed of 50 mph; an eastbound train leaves Buffalo at 1:30 P.M. and maintains an average speed of 40 mph. If the distance from Albany to Buffalo is 280 miles, when and where will they pass?

Reading answers from Fig. 8·2, we find that the trains pass at 4:20 P.M. at a point $166\frac{2}{3}$ miles west of Albany and $113\frac{1}{3}$ miles east of Buffalo. A rough sketch of the graphical solution may be used to indicate the relationships upon which to base an algebraic solution. By dimensioning the sketch (Fig. 8·3), which need not be drawn to scale, we obtain an equation $280 = 40t + 50(t + \frac{1}{2})$ (the whole equals the sum of the parts).

EXAMPLE 12. A Pennsylvania local leaves New York City for Philadelphia at 9:30 A.M. and maintains an average speed of 30 mph. An express leaves

New York for Philadelphia at 10:15 A.M. and maintains an average speed of 40 mph for the first 10 miles, after which the average speed is 60 mph. When and where will the express overtake the local?

Figure 8·4 indicates that the express overtakes the local at 11:10 A.M. at a point 50 miles out of New York City.

EXAMPLE 13. A plane has enough fuel for 5 hr of flying at a cruising speed of 240 mph. If a 30-mph west wind is blowing, how far east can the plane

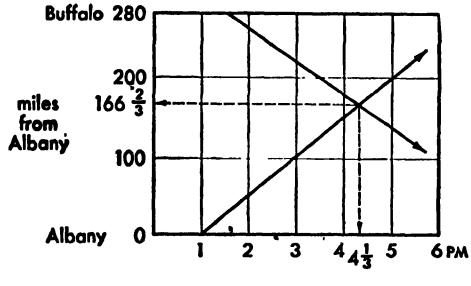


Fig. 8·2

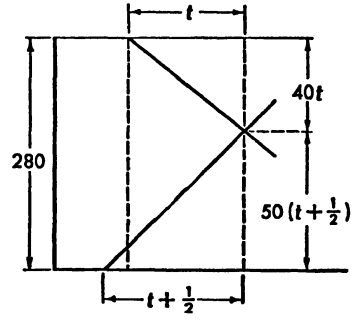


Fig. 8·3

fly and return to the airport before the fuel is exhausted? (That is, find the "point of no return.") When must the pilot reverse his course?

Figure 8·5 shows that the point of no return is reached in about 2 hr 11 min at about 591 miles from the airport.

EXAMPLE 14. A time-honored problem is that of the motorcycle courier who starts from the rear of a troop column, rides to the head of the column,

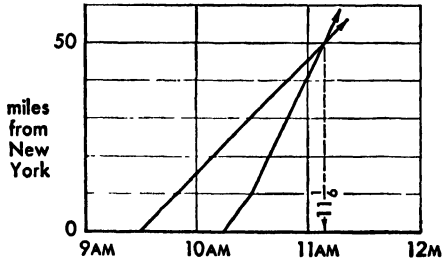


Fig. 8·4

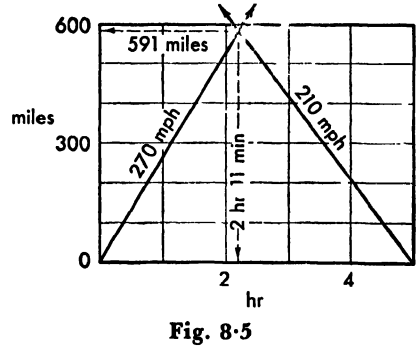


Fig. 8·5

and immediately returns to the rear. How far did the courier ride if the column is 6 miles long and marching at 4 mph? The speed of the motorcycle is 30 mph. How long was the courier riding?

According to Fig. 8·6, the courier traveled about 12.2 miles (6.9 miles to the head of the column and 5.3 miles to the rear). The total time was about 0.4 hr, or 24 min.

This problem is a good example of the opportunity offered by maintaining the proper relative values while changing other conditions.

In this illustration let us consider rates of the cyclist relative to that of the troop column. It is evident the cyclist is riding at 30 − 4, or 26, mph relative to the troop column while riding to the head of the column and 30 + 4, or 34, mph relative to the column while riding to the rear. If the 6-mile column were stationary, then our problem would be to find the total time for the

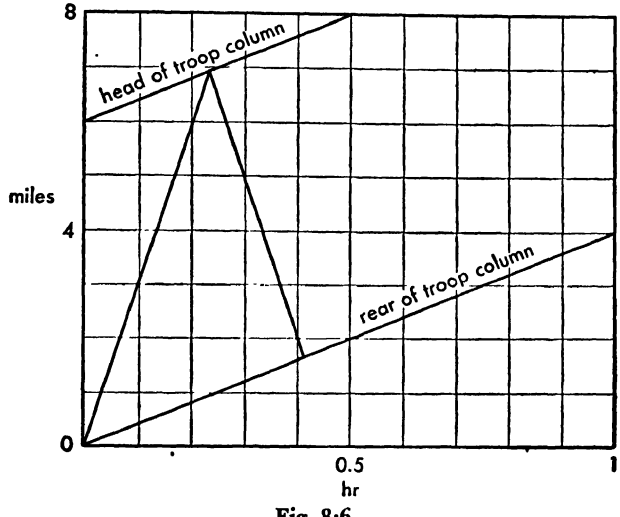


Fig. 8·6

cyclist to advance 6 miles at 26 mph and return at 34 mph. The solution is obvious:

$$\frac{6}{34} + \frac{6}{26} = 0.1765 \text{ hr} + 0.231 \text{ hr, or } 0.408 \text{ hr}$$

and confirms the answer already obtained by graphical means.

EXERCISE 5

1. A plumber and his helper together receive \$19.20, the plumber working 5 hr and the helper 6 hr. At another time the plumber works 8 hr and the helper 7½ hr, and they receive together \$28.20. What are the hourly wages of each?
2. Three men and 6 boys can do in 2 days what 1 man and 8 boys can do in 3 days. Find the time required for 1 man alone and for 1 boy alone to do the work.
3. Six men and 5 boys take 2 days to do what 9 men and 15 boys can do in 1 day. If the labor cost for the job is \$216, find the daily wage for a man and for a boy.

4. Three disks A, B, and C are externally tangent each to each. If the center-to-center distances AB, AC, and BC are $6\frac{3}{4}$, $6\frac{1}{4}$, and $8\frac{1}{2}$ in., respectively, find the diameters.
5. A marksman fires at a target 480 yd away. He hears the bullet strike 2.4 sec after he fires. An observer standing 420 yd from the target and 240 yd from the marksman hears the bullet strike 1.6 sec after he hears the report of the rifle. Find the velocities of sound and of the bullet, assuming each to be uniform.
6. A gas company charges $\$a$ service charge plus $\$b$ a thousand cubic feet. Find a and b if 12,000 cu ft of gas costs \$1.46 and 20,000 cu ft costs \$2.10.
7. Find the diameters of the disks in Fig. 8·7.

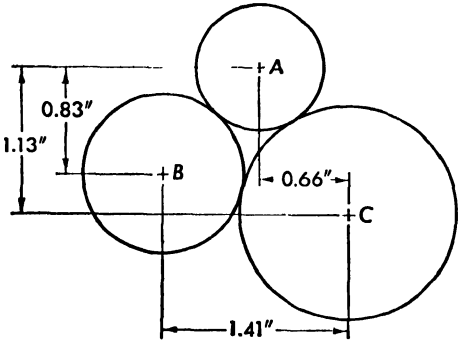


Fig. 8·7

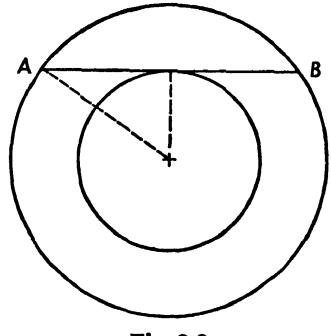


Fig. 8·8

8. Express the area of the ring in Fig. 8·8, in terms of AB. Given a regular 6-in. pentagon, find the difference between the areas of the circumscribed and inscribed circles. How does your answer compare with that obtained by taking a 6-in. heptagon?
9. In sending a telegram, it costs a cents for a minimum number of words and b cents for each word above the minimum. Seven words cost \$0.35, 28 words cost \$0.80, and 46 words cost \$1.25. Find the minimum charge and corresponding number of words and the cost of each word above the minimum.
10. The difference between two quantities equals their product. Their sum less their product equals the smaller divided by the larger. What are the numbers?
11. The product of two quantities equals their difference. Their product plus their sum equals the larger divided by 3 times the smaller. Find the numbers.
12. There are three bars of metal A, B, and C having the following weight composition: A contains 6 parts gold, 2 parts silver, and 1 part copper; B, 3 parts gold, 4 parts silver, and 2 parts copper;

C, 1 part gold, 3 parts silver, and 5 parts copper. How many ounces of each must be taken to make 28 oz of alloy containing equal parts of gold, silver, and copper?

13. The following table gives the weight composition of three alloys *A*, *B*, and *C*.

	Copper, %	*Tin*, %	*Zinc*, %
A	80	10	10
B	20	40	40
C	50	..	50

How many pounds each of *A*, *B*, and *C* must be mixed to produce 600 lb of an alloy containing 60 per cent copper, 10 per cent tin, and 30 per cent zinc?

14. A waste mixed acid left over from nitrating is composed of 61.8 per cent sulfuric acid, 20.1 per cent nitric acid, and 18.1 per cent water. It is required to make a mixture of 1,000 lb containing 60 per cent sulfuric acid, 23 per cent nitric acid, and 17 per cent water. Solutions of sulfuric acid (98 per cent) and nitric acid (90 per cent) are available. How many pounds each of waste acid, sulfuric acid, and nitric acid must be taken if no additional water is used?

15. A New York Central passenger train running between New York City and Albany can make up 20 min time lost in starting by averaging 6 mph faster than usual. On another occasion, by giving another train the right of way, it reduces its average speed by 8 mph and will finish its run 36 min late. Find the distance and usual running time from New York to Albany.

16. A traveler has 400 lb of baggage, for which he must pay \$4.50 transportation because of excess over the weight carried without charge. If a friend takes charge of part of the baggage, the traveler will pay \$1.80 and the friend \$1.20 for excess above the weight carried free. How much baggage is allowed to go free on one ticket?

17. A girder AE weighing p lb per ft is supported at five equally spaced points A, B, C, D, and E. $AB = d$ ft. To determine how much of the load each support bears, solve the following equations for W_2, W_3, and W_4 in terms of p and d. ($W_1 = W_5 = 0$.)

$$W_1 + 4W_2 + W_3 = -\frac{pd^2}{2}$$

$$W_2 + 4W_3 + W_4 = -\frac{pd^2}{2}$$

$$W_3 + 4W_4 + W_5 = -\frac{pd^2}{2}$$

18. A beam AF weighing 400 lb per ft is supported at points A, B, C, D, E, and F. $AB = 5$ ft, $BC = 6$ ft, $CD = 7$ ft, $DE = 5$ ft, and $EF = 4$ ft. To determine the load carried by each support, the following equations must be solved for W_2, W_3, W_4, and W_5.

$$\begin{aligned} 11W_2 + 3W_3 &= -34{,}100 \\ 6W_2 + 26W_3 + 7W_4 &= -118{,}000 \\ 7W_3 + 24W_4 + 5W_5 &= -93{,}600 \\ 5W_4 + 16W_5 &= -37{,}800 \end{aligned}$$

19. Two workmen A and B were employed at different wages. Over a given pay period, A received \$24 more than B. If A had been idle two days while B worked the entire period, they would have received equal amounts. If B had been idle two days with A working the entire period, A would have received \$45 more than B. How long was a pay period and what was the daily wage of each man?

20. In a race of 100 yd, A beats B by $\frac{1}{5}$ of a second. In the second trial, A gives B a start of 4 yd and B wins by $\frac{1}{5}$ of a second. Find the time required for A and B each to run 100 yd.

21. A man ordered a number of pamphlets from a job printer. The printer made a fixed charge for each order plus a certain amount for each pamphlet. The total cost amounted to \$42. Several months later the customer placed a somewhat larger order for additional pamphlets. This order cost \$54. Had the customer been able to anticipate his needs and ordered all the pamphlets at one time, they would have cost only \$84. How much was the fixed charge? If the second order contained 80 more pamphlets than the first, find the number of pamphlets in each order and the unit cost.

22. Fifty people wish to reach a place 27½ miles away. The only available transportation is a bus having a capacity of 30 people and a speed of 35 mph. The party is divided into two roughly equal groups, which start at the same time. The first group starts on foot, walking at an average rate of 4 mph. The second group rides on the bus a certain distance and then walks the rest of the way at an average rate of 3 mph. The bus returns to meet the first group and to carry it the rest of the way. How far should each group walk in order that all may arrive at their destination at the same time? How many hours are required for the trip?

23. A man wishes to apply to his garden 50 lb of a fertilizer containing 5 per cent phosphorus and 12 per cent nitrogen. He has available brand A containing 4 per cent phosphorus and 10 per cent nitrogen, brand B containing 7 per cent phosphorus and 11 per cent nitrogen, and brand C containing negligible phosphorus

and 18 per cent nitrogen. How much of each brand should be taken?

24. Two stations A and B on the same railroad line are on opposite sides of a mountain pass. A is 10 miles east of the divide, and B is 15 miles west of the divide. The grade is the same on both sides. If the running time from A to B is 44 min and the running time from B to A is 50 min, find the upgrade and downgrade speeds.

25. Two cars are traveling in the same direction around a 1-mile circular race track. The faster car overtakes the slower car every 3 min. When the cars are traveling in opposite directions, they pass each other every 18 sec. Find their speeds.

26. The resistance R_t of a nickel wire at any Centigrade temperature t is equal to $R_0(1 + at)$, where R_0 is the resistance at 0°C and a is the temperature coefficient of resistance. If the resistance at 20°C is 26.82 ohms and at 32°C is 28.10 ohms, find a and R_0.

27. In balancing the equation for the combustion of methyl alcohol, the coefficients a, b, and c must be determined in the equation

$$a\text{CH}_3\text{OH} + b\text{O}_2 \rightarrow c\text{H}_2\text{O} + a\text{CO}_2$$

According to a hydrogen balance,

$$4a = 2c$$

An oxygen balance requires that

$$a + 2b = c + 2a$$

Find the smallest integral values of a, b, and c satisfying the above equations.

28. The solution of copper in nitric acid proceeds according to the equation

$$a\text{Cu} + 2b\text{HNO}_3 \rightarrow a\text{Cu(NO}_3)_2 + c\text{NO} + b\text{H}_2\text{O}$$

The nitrogen balance requires that

$$2b = 2a + c$$

The oxygen balance requires that

$$6b = 6a + c + b$$

Find the smallest integral values of a, b, and c satisfying the above equations.

8·10 Determinants. Simultaneous linear equations containing three or more unknowns may entail considerable computation in their solution. A common example may be found in the application

of Kirchhoff's laws to the determination of the current and voltage in various parts of an electrical network. Expressions called *determinants* are particularly useful in facilitating such solutions.

8·11 Determinants of the Second Order. If a_1, b_1, a_2, and b_2 represent any numbers, the arrangement

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

is called a *determinant of the second order*, in which a_1, b_1, a_2, and b_2 are the *elements*.

The value of the determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

is, by definition, equal to

$$a_1b_2 - a_2b_1 \tag{45}$$

EXAMPLE 15: Find the value of $\begin{vmatrix} 4 & 5 \\ 3 & 7 \end{vmatrix}$

By Eq. (45)

$$\begin{vmatrix} 4 & 5 \\ 3 & 7 \end{vmatrix} = (4)(7) - (3)(5) = 28 - 15 = 13$$

EXAMPLE 16: Evaluate $\begin{vmatrix} 6 & 3 \\ -2 & 5 \end{vmatrix}$

$$\begin{vmatrix} 6 & 3 \\ -2 & 5 \end{vmatrix} = (6)(5) - (-2)(3) = 30 + 6 = 36$$

EXAMPLE 17: Evaluate $\begin{vmatrix} 7 & x \\ 5 & -y \end{vmatrix}$

$$\begin{vmatrix} 7 & x \\ 5 & -y \end{vmatrix} = (7)(-y) - (5)(x) = -7y - 5x$$

EXERCISE 6

Evaluate the following determinants:

1. $\begin{vmatrix} 3 & 5 \\ 5 & 8 \end{vmatrix}$
2. $\begin{vmatrix} 3 & a \\ 4 & b \end{vmatrix}$
3. $\begin{vmatrix} 4 & -6 \\ 2 & 5 \end{vmatrix}$
4. $\begin{vmatrix} 7 & -x \\ 6 & y \end{vmatrix}$
5. $\begin{vmatrix} 9a & 5b \\ 4b & a \end{vmatrix}$
6. $\begin{vmatrix} 6x & -7x \\ 5y & -6y \end{vmatrix}$

8·12 Solution of Simultaneous Linear Equations by Determinants. Given the system of linear equations in x and y

$$\begin{aligned} a_1x + b_1y &= k_1 \\ a_2x + b_2y &= k_2 \end{aligned} \tag{46}$$

the solution can be shown to be

$$x = \frac{k_1b_2 - k_2b_1}{a_1b_2 - a_2b_1}, \; y = \frac{a_1k_2 - a_2k_1}{a_1b_2 - a_2b_1} \tag{47}$$

provided $a_1b_2 - a_2b_1$ is not equal to zero.

In determinant notation we can write

$$k_1b_2 - k_2b_1 = \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}, \; a_1k_2 - a_2k_1 = \begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}$$

and

$$a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

We can, therefore, write the solution of Eq. (47) in the form

$$x = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \; y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ if } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0 \qquad \star (48)$$

The solution [Eq. (48)] is, therefore, a general formula applicable to any system [Eq. (46)] of two simultaneous linear equations in two unknowns when arranged in standard form. (This equation has been starred to indicate its importance.) A system of linear equations is said to be in *standard form* when, as in Eq. (46), the terms containing the unknowns are in the left-hand members and in the same order in both equations, and the right-hand members contain only constants.

It will be noted that both denominators of Eq. (48) are the same. This determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

is called the *determinant of the system.* Its elements are the coefficients of the unknown quantities arranged in the same relative position as in the system of equations when the equations are arranged in standard form.

The numerator of the fraction which expresses the value of x differs from the denominator in that the constants k_1 and k_2 replace a_1 and a_2, the coefficients of x; and the numerator of the fraction which expresses the value of y differs from the denominator in that the constants k_1 and k_2 replace b_1 and b_2, the coefficients of y. This procedure may be summarized in *Cramer's rule:*

Arrange the given equations in standard form. In the solution, the value of an unknown is given by a fraction whose denominator is the determinant of the system and whose numerator is the same determinant except that the coefficients of the unknown have been replaced by the constants k_1 *and* k_2.

EXAMPLE 18: Solve by determinants

$$\begin{aligned} 4x + 7y &= -19 \\ 5x - 3y &= 35 \end{aligned}$$

We have

$$\begin{aligned} a_1 &= 4,\ b_1 = 7,\quad k_1 = -19 \\ a_2 &= 5,\ b_2 = -3,\ k_2 = 35 \end{aligned}$$

Hence

$$x = \frac{\begin{vmatrix} -19 & 7 \\ 35 & -3 \end{vmatrix}}{\begin{vmatrix} 4 & 7 \\ 5 & -3 \end{vmatrix}} = \frac{57 - 245}{-12 - 35} = \frac{-188}{-47} = 4$$

$$y = \frac{\begin{vmatrix} 4 & -19 \\ 5 & 35 \end{vmatrix}}{\begin{vmatrix} 4 & 7 \\ 5 & -3 \end{vmatrix}} = \frac{140 + 95}{-12 - 35} = \frac{235}{-47} = -5$$

EXAMPLE 19: Solve by determinants

$$\begin{aligned} x - y &= m - n \\ mx - ny &= 2m^2 - 2n^2 \end{aligned}$$

Here

$$\begin{aligned} a_1 &= 1,\ b_1 = -1,\ k_1 = m - n \\ a_2 &= m,\ b_2 = -n,\ k_2 = 2m^2 - 2n^2 \end{aligned}$$

Hence

$$x = \frac{\begin{vmatrix} m - n & -1 \\ 2m^2 - 2n^2 & -n \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ m & -n \end{vmatrix}} = \frac{-mn + n^2 + 2m^2 - 2n^2}{-n + m}$$

$$= \frac{2m^2 - mn - n^2}{m - n} = 2m + n$$

$$y = \frac{\begin{vmatrix} 1 & m - n \\ m & 2m^2 - 2n^2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ m & -n \end{vmatrix}} = \frac{2m^2 - 2n^2 - m^2 + mn}{-n + m}$$

$$= \frac{m^2 + mn - 2n^2}{m - n} = m + 2n$$

If the equations $a_1x + b_1y = k_1$ and $a_2x + b_2y = k_2$ have not just one simultaneous solution—they are either inconsistent or dependent.

Graphically, this means that the lines are either parallel or coincident. In either case the slopes are equal, and $-\frac{a_1}{b_1} = -\frac{a_2}{b_2}$ or $a_1b_2 - a_2b_1 = 0$.

If the two lines coincide their y intercepts are equal or $\frac{k_1}{b_1} = \frac{k_2}{b_2}$; that is $k_1b_2 - k_2b_1 = 0$

These facts may be summarized as follows:

If $a_1b_2 - a_2b_1 \neq 0$	*The equations have just one unique simultaneous solution.*
If $a_1b_2 - a_2b_1 = 0$ and $k_1b_2 - k_2b_1 \neq 0$	*The equations are inconsistent and have no solution.*
If $a_1b_2 - a_2b_1 = 0$ and $k_1b_2 - k_2b_1 = 0$	*The equations are dependent and have an infinite number of solutions.*

EXERCISE 7

In the following systems, use determinants to ascertain whether the equations are inconsistent, dependent, or simultaneous. If simultaneous, determine the solutions.

1. $2x - 3y = 5$
   $4x - 6y = 8$

2. $3x - 7y = 1$
   $6x + 5y = 40$

3. $6x + 9y = 15$
   $8x + 12y = 20$

4. $3x - 4y - 15 = 0$
   $7x + 2y - 52 = 0$

5. $4x + 15y = 7a$
   $10x - 9y = 2a$

6. $\frac{2x - y}{3} = 5$
   $\frac{11x + 2y}{5} = 6$

7. $ax + by = (a - b)^2$
   $ax - by = a^2 - b^2$

8. $bx + ay = 2ab$
   $ax + by = a^2 + b^2$

9. $17x + 11y = 13$
   $7x - 9y = 73$

8·13 Determinants of the Third and Higher Order. If a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, and c_3 represent any numbers, the arrangement

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is called a *determinant of the third order*. By definition its value may be written

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \qquad (49)$$

The first of these second-order determinants is formed by removing from the third-order determinant the row and column containing the element a_1; the second by removing the row and column containing the element a_2; and the third by removing the row and column containing the element a_3.

The determinant that remains after removing the row and column containing a particular element is called the *minor* of that element. Thus in Eq. (49)

$$\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$$

is the minor of a_2.

In Eq. (49) the third-order determinant has been expanded by minors of the first column. Other definitive equations which are algebraically equivalent could be written in place of Eq. (49). For example, by methods discussed in Sec. 8·11, Eq. (49) is equivalent to

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

or

$$a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 \qquad (50)$$

Eq. (50) can be regrouped in a variety of ways, for example

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

which is equivalent to

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \qquad (51)$$

Therefore Eq. (51), in which we have expanded the determinant by minors of the first row, is equivalent to the third-order determinant in Eq. (49).

Eq. (50) could also be regrouped to give

$$a_3(b_1c_2 - b_2c_1) - b_3(a_1c_2 - a_2c_1) + c_3(a_1b_2 - a_2b_1)$$

which is equivalent to

$$a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad (52)$$

Therefore Eq. (52), in which we have expanded by minors of the third row, is also equivalent to the third-order determinant in Eq. (49). Other equivalent defining equations can be developed by the student from Eq. (50).

When expanding a determinant by minors, the signs can be determined as follows:

Add the number of the column and the number of the row in which an element lies. If the sum is even, that element is preceded by a plus sign, if the sum is odd, by a minus sign. Thus in Eq. (49) the coefficient of the second group is preceded by a minus sign since a_2 is in the first column and the second row.

The general procedure of expansion by minors is applicable to determinants of fourth or higher order. Thus a fourth-order determinant will have as minors of its elements four determinants of the third order. Each of these third-order determinants will expand into three second-order determinants. In this way a fourth-order determinant expands into 12 second-order determinants; a fifth-order determinant into 60 of the second order and so on.

If one of the elements of a determinant is zero the work may be simplified by using minors of that element (see Example 21).

EXAMPLE 20: Evaluate the determinant

$$\begin{vmatrix} 8 & -2 & 3 \\ 5 & -4 & 1 \\ -6 & 7 & 2 \end{vmatrix}$$

Expanding by minors of the third column (we might have chosen any other column, or any row) we obtain

$$\begin{vmatrix} 8 & -2 & 3 \\ 5 & -4 & 1 \\ -6 & 7 & 2 \end{vmatrix} = 3\begin{vmatrix} 5 & -4 \\ -6 & 7 \end{vmatrix} - 1\begin{vmatrix} 8 & -2 \\ -6 & 7 \end{vmatrix} + 2\begin{vmatrix} 8 & -2 \\ 5 & -4 \end{vmatrix}$$

$$= 3[(5)(7) - (-6)(-4)] - 1[(8)(7) - (-6)(-2)] + 2[(8)(-4) - (5)(-2)]$$

$$= 3(35 - 24) - 1(56 - 12) + 2(-32 + 10) = 33 - 44 - 44 = -55$$

EXAMPLE 21: Evaluate the determinant

$$\begin{vmatrix} 3 & 0 & -2 & 1 \\ 4 & -5 & 2 & -3 \\ -1 & 6 & 0 & -4 \\ 7 & 1 & -6 & 5 \end{vmatrix}$$

Expanding by minors of the first row since it contains a zero:

$$3\begin{vmatrix} -5 & 2 & -3 \\ 6 & 0 & -4 \\ 1 & -6 & 5 \end{vmatrix} - 0\begin{vmatrix} 4 & 2 & -3 \\ -1 & 0 & -4 \\ 7 & -6 & 5 \end{vmatrix} + (-2)\begin{vmatrix} 4 & -5 & -3 \\ -1 & 6 & -4 \\ 7 & 1 & 5 \end{vmatrix} - 1\begin{vmatrix} 4 & -5 & 2 \\ -1 & 6 & 0 \\ 7 & 1 & -6 \end{vmatrix}$$

Taking minors of the second row since it contains a zero, the first group equals

$$3\left\{-6\begin{vmatrix} 2 & -3 \\ -6 & 5 \end{vmatrix} + 0\begin{vmatrix} -5 & -3 \\ 1 & 5 \end{vmatrix} - (-4)\begin{vmatrix} -5 & 2 \\ 1 & -6 \end{vmatrix}\right\}$$
$$= 3\{-6(10 - 18) + 0 + 4(30 - 2)\} = 3\{48 + 112\} = 480$$

The value of the second group is zero.

Taking minors of the first row, the third group equals

$$-2\left\{4\begin{vmatrix} 6 & -4 \\ 1 & 5 \end{vmatrix} - (-5)\begin{vmatrix} -1 & -4 \\ 7 & 5 \end{vmatrix} + (-3)\begin{vmatrix} -1 & 6 \\ 7 & 1 \end{vmatrix}\right\}$$
$$= -2\{4(30 + 4) + 5(-5 + 28) - 3(-1 - 42)\}$$
$$= -2\{136 + 115 + 129\} = -760$$

Taking minors of the second row, the fourth group equals

$$-1\left\{-(-1)\begin{vmatrix} -5 & 2 \\ 1 & -6 \end{vmatrix} + 6\begin{vmatrix} 4 & 2 \\ 7 & -6 \end{vmatrix} - 0\begin{vmatrix} 4 & -5 \\ 7 & 1 \end{vmatrix}\right\}$$
$$= -1\{30 - 2 + 6(-24 - 14) - 0\} = -1\{28 - 228\} = 200$$

Hence the value of the fourth-order determinant is

$$480 - 0 - 760 + 200 = -80$$

EXERCISE 8

Evaluate each of the following determinants. Select rows or columns having the most zeros.

1. $\begin{vmatrix} 2 & -1 & 3 \\ 4 & 5 & -2 \\ 1 & -3 & 1 \end{vmatrix}$

2. $\begin{vmatrix} 5 & 4 & 3 \\ 0 & 6 & 2 \\ 1 & 1 & 7 \end{vmatrix}$

3. $\begin{vmatrix} 4 & 0 & -2 \\ 7 & 8 & 3 \\ -5 & 0 & 1 \end{vmatrix}$

4. $\begin{vmatrix} 8 & 7 & -6 \\ 2 & 1 & 4 \\ 3 & -1 & 0 \end{vmatrix}$

5. $\begin{vmatrix} 6 & 2 & 1 \\ -3 & 4 & -5 \\ 3 & -1 & 2 \end{vmatrix}$

6. $\begin{vmatrix} 7 & 0 & -2 \\ -3 & 1 & 4 \\ 6 & 3 & 5 \end{vmatrix}$

7. $\begin{vmatrix} 4 & -4 & a \\ 3 & 5 & a \\ 2 & -1 & a \end{vmatrix}$

8. $\begin{vmatrix} 11 & 2 & -3 \\ -10 & 1 & 4 \\ 9 & 5 & 3 \end{vmatrix}$

9. $\begin{vmatrix} 2 & 3 & a+b \\ 1 & -4 & b+c \\ 5 & -1 & a+c \end{vmatrix}$

10. $\begin{vmatrix} m & -2 & 3 \\ 4 & 2 & 5 \\ 1 & -1 & m \end{vmatrix}$

11. $\begin{vmatrix} 1 & 2 & 0 & 5 \\ 3 & 0 & 7 & 4 \\ 0 & -6 & 1 & -1 \\ 4 & 3 & 0 & -2 \end{vmatrix}$ **12.** $\begin{vmatrix} 7 & 2 & 5 & -1 \\ 4 & 3 & 6 & 1 \\ 8 & 0 & 9 & -2 \\ -3 & 4 & 2 & 1 \end{vmatrix}$

13. $\begin{vmatrix} 5 & 2 & -3 & 8 \\ 4 & -1 & 0 & -2 \\ 0 & 3 & -4 & 1 \\ 6 & 0 & 3 & 5 \end{vmatrix}$ **14.** $\begin{vmatrix} 6 & 4 & 0 & -1 \\ 1 & 3 & -5 & m \\ 2 & 0 & -4 & 0 \\ 7 & -3 & 1 & -2m \end{vmatrix}$

8·14 Determinant Formulas for the Solution of Linear Equations in Three Unknowns. The application of Cramer's rule to the solution of a system of three linear equations in three unknowns is entirely comparable to that used in the solution of two simultaneous equations in two unknowns.

In solving the system

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}$$

we obtain by a method entirely analogous to that used in Sec. 8·12

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad \star (53)$$

As in second-order determinants the common denominator is called the *determinant of the system.*

As in the case of two equations in two unknowns, if the determinant of the system equals zero the equations are either inconsistent, with no solution, or dependent, with infinitely many solutions.

EXAMPLE 22: Use determinants to solve the system

$$\begin{aligned} 10x + 3y - 6z &= -9 \\ 7x + 5y + 4z &= 12 \\ 8x - 2y - 9z &= -2 \end{aligned}$$

We note the following values

$$\begin{aligned} a_1 &= 10, & b_1 &= 3, & c_1 &= -6, & k_1 &= -9 \\ a_2 &= 7, & b_2 &= 5, & c_2 &= 4, & k_2 &= 12 \\ a_3 &= 8, & b_3 &= -2, & c_3 &= -9, & k_3 &= -2 \end{aligned}$$

Hence

$$x = \frac{\begin{vmatrix} -9 & 3 & -6 \\ 12 & 5 & 4 \\ -2 & -2 & -9 \end{vmatrix}}{\begin{vmatrix} 10 & 3 & -6 \\ 7 & 5 & 4 \\ 8 & -2 & -9 \end{vmatrix}}$$

Expanding each determinant by minors of the first row

$$x = \frac{(-9)\begin{vmatrix} 5 & 4 \\ -2 & -9 \end{vmatrix} - (3)\begin{vmatrix} 12 & 4 \\ -2 & -9 \end{vmatrix} + (-6)\begin{vmatrix} 12 & 5 \\ -2 & -2 \end{vmatrix}}{(10)\begin{vmatrix} 5 & 4 \\ -2 & -9 \end{vmatrix} - (3)\begin{vmatrix} 7 & 4 \\ 8 & -9 \end{vmatrix} + (-6)\begin{vmatrix} 7 & 5 \\ 8 & -2 \end{vmatrix}}$$

$$x = \frac{\begin{array}{l} -9[(5)(-9) - (-2)(4)] - 3[(12)(-9) - (-2)(4)] \\ \qquad - 6[(12)(-2) - (-2)(5)] \end{array}}{10[(5)(-9) - (-2)(4)] - 3[(7)(-9) - (8)(4)] - 6[(7)(-2) - (8)(5)]}$$

$$x = \frac{-9[-45 + 8] - 3[-108 + 8] - 6[-24 + 10]}{10[-45 + 8] - 3[-63 - 32] - 6[-14 - 40]}$$

$$= \frac{333 + 300 + 84}{-370 + 285 + 324} = \frac{717}{239} = 3$$

By a similar method y and z can be calculated, giving

$$y = \frac{\begin{vmatrix} 10 & -9 & -6 \\ 7 & 12 & 4 \\ 8 & -2 & -9 \end{vmatrix}}{\begin{vmatrix} 10 & 3 & -6 \\ 7 & 5 & 4 \\ 8 & -2 & -9 \end{vmatrix}} = -5, \text{ and } z = \frac{\begin{vmatrix} 10 & 3 & -9 \\ 7 & 5 & 12 \\ 8 & -2 & -2 \end{vmatrix}}{\begin{vmatrix} 10 & 3 & -6 \\ 7 & 5 & 4 \\ 8 & -2 & -9 \end{vmatrix}} = 4$$

EXERCISE 9

Solve the following systems of simultaneous linear equations by determinants.

1. $x + 2y + z = 20$
$3x - y + 2z = 22$
$2x + y - 4z = 7$

2. $4x + 7y + 2z = 21$
$5x + 8y - 3z = 16$
$-3x - 5y + 9z = 5$

3. $5x + 2y = -17$
$3x + 7z = 23$
$4y + 6z = 36$

4. $3x + 5y - 4z = 9a$
$2x - 3y + 4z = 7b$
$-6x + y + 3z = b - a$

5.
$$\begin{aligned} -7x + y + 4z &= -16a \\ x + y - 4z &= -8b \\ -x + 7y - 4z &= 16c \end{aligned}$$

6.
$$\begin{aligned} \frac{1}{x} - \frac{1}{y} + \frac{2}{z} &= 7 \\ \frac{2}{x} + \frac{2}{y} - \frac{3}{z} &= -2 \\ -\frac{3}{x} + \frac{1}{y} + \frac{1}{z} &= 1 \end{aligned}$$

Suggestion: Let $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$ and solve for a, b, and c.

7.
$$\begin{aligned} x - 2y - 3z + 4w &= -4 \\ 3x - z + 2w &= 16 \\ 5y + 2z - 3w &= 20 \\ 4x + y - 7w &= 10 \end{aligned}$$

8.
$$\begin{aligned} x + 2y + 3z &= 11 \\ -x + 2z + 3w &= 9 \\ 2x - 3y - w &= -23 \\ y - z + 2w &= -1 \end{aligned}$$

Problems in Exercise 5 may be worked by determinants as directed by the instructor.

8·15 Application of Simultaneous Linear Equations to Kirchhoff's Law. Figure 8·9 is an idealized diagram of considerable importance if batteries are operated in parallel. Ordinarily, in such problems the given circuit parameters are the resistances R_1, R_2, R_3 and the battery voltages E_1 and E_2. The problem is to solve for the currents I_1 and I_2. The arrows next to the resistors indicate the direction in which the current is assumed to flow. Should either I_1 or I_2 come out negative it simply means that the assumed direction of current flow was wrong but if the solution is otherwise correct the absolute value of the current will be correct.

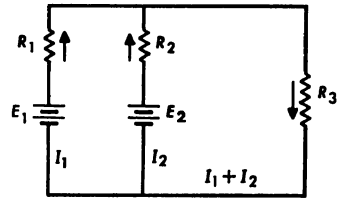


Fig. 8·9

The basic equations for the solution of this and similar problems are

$$E_1 = R_1I_1 + R_3(I_1 + I_2) \quad (54)$$
$$E_2 = R_2I_2 + R_3(I_1 + I_2) \quad (55)$$

EXERCISE 10

1. Find I_1 and I_2 when the circuit parameters are as follows:

	R_1	R_2	R_3	E_1	E_2
(*a*)	0.05	0.10	0.75	4.5	5.2
(*b*)	1.50	2.30	0.50	32.0	35.5
(*c*)	0.10	0.01	5.00	6.0	7.5

2. (*a*) Solve Eq. (54) for I_1 and find under what conditions I_1 must be negative.

(*b*) Solve Eq. (55) for I_2 and find under what conditions I_2 must be negative.

3. If $E_1 = 5$, $E_2 = 6$, and $R_3 = 10$ find the value of R_2 so that $I_1 = 0$.

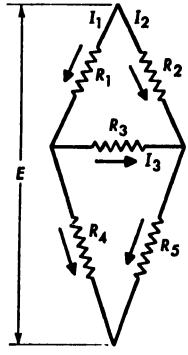


Fig. 8·10

4. Show that if

$$\frac{R_2}{R_3} + 1 = \frac{E_2}{E_1}$$

then $I_1 = 0$.

Figure 8·10 illustrates a "bridge" circuit which is widely used for electrical measurements as well as other applications.

We shall assume that the known quantities are E, R_1, R_2, R_3, R_4, and R_5. Having been given these quantities we shall solve for I_1, I_2, and I_3. As in the previous group of problems the arrows next to the resistors indicate the assumed direction of current flow. Should either I_1, I_2, or I_3 come out to be negative it means that the actual direction of current flow is exactly opposite to the assumed direction. This will not affect the absolute value of the current.

The basic equations involved are the following:

$$E = R_1I_1 + R_4(I_1 - I_3)$$
$$E = R_2I_2 + R_5(I_2 + I_3)$$
$$E = R_1I_1 + R_3I_3 + R_5(I_3 + I_2)$$

EXERCISE 11

1. Find I_1, I_2, and I_3 from the following data:

	E	R_1	R_2	R_3	R_4	R_5
(*a*)	10	2	3	4	5	6
(*b*)	5	1	5	3	8	4
(*c*)	1	2	6	9	5	3
(*d*)	2	5	9	5	3	9
(*e*)	25	6	9	2	5	8

2. If $I_3 = 0$ show that $R_1/R_4 = R_2/R_5$.

Chapter 9 EXPONENTS AND RADICALS

In Chap. 5 we discussed integral exponents (positive, negative, and zero). In this chapter we shall extend our study to cover fractional exponents and the equivalent radical notation.

Fractional exponents and radicals find frequent application in engineering and mathematics.

9·1 Laws of Positive Integral Exponents. We recall from Sec. 5·8 that an exponent of a number indicates how many times that number is to be taken as a factor. Thus

$$a^1 = a,\ a^2 = a \cdot a,\ a^3 = a \cdot a \cdot a$$

and in general

$$a^n = a \cdot a \cdot a \cdot \cdots a \ (n \text{ factors})$$

a^n is read "the nth power of a," or "a to the nth power." The quantity a is called the base and n the exponent (of the power).

The following relationships are either set forth in Sec. 5·9 or can easily be demonstrated, where m and n are assumed to be positive integers and a and b are any real numbers other than zero. The term *real number* is used here to distinguish between the kind of number which we have been using in ordinary practice from a new kind of number which we shall introduce in Secs. 9·14 and 9·15.

(1) ★ $\qquad a^m \cdot a^n = a^{m+n}$

Examples: $x^7 \cdot x^5 = x^{12} \qquad 2^3 \cdot 2^4 = 2^7 = 128$

(2) ★ $\qquad (a^m)^n = a^{mn}$

Examples: $(x^2)^3 = x^6 \qquad (3^2)^3 = 3^6 = 729$

(3) ★ $\qquad \dfrac{a^m}{a^n} = a^{m-n}$ if $m > n$ or $\dfrac{1}{a^{n-m}}$ if $n > m$

Examples: $\dfrac{x^8}{x^5} = x^3 \qquad \dfrac{x^5}{x^8} = \dfrac{1}{x^3} \qquad \dfrac{5^7}{5^4} = 5^3 = 125 \qquad \dfrac{5^6}{5^8} = \dfrac{1}{5^2} = \dfrac{1}{25}$

(4) ★ $\qquad (ab)^n = a^n b^n$

Examples: $(2x^2)^3 = (2)^3(x^2)^3 = 8x^6 \qquad (3xy^3)^4 = 3^4x^4(y^3)^4 = 81x^4y^{12}$

(5) ★ $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Examples: $\left(\frac{3}{5}\right)^4 = \frac{3^4}{5^4} = \frac{81}{625} \qquad \left(\frac{4x^2}{3y^3}\right)^3 = \frac{4^3(x^2)^3}{3^3(y^3)^3} = \frac{64x^6}{27y^9}$

(6) An even power of any number is positive; an odd power of any number has the same sign as the number.

Examples: $(-2)^4 = 16 \qquad (2)^4 = 16$
$(-2)^5 = -32 \qquad (2)^5 = 32$

EXERCISE 1

Carry out the indicated operations according to the laws of exponents.

1. $x^4 \cdot x^2$	**2.** $y \cdot y^6$	**3.** $b^x \cdot b^3$	**4.** $c^4 \cdot c^n$
5. $3^2 \cdot 3^3$	**6.** $a^x \cdot a^{1-x}$	**7.** $a^{2y} \cdot a^{3y}$	**8.** $(-2)^4$
9. $(-1)(-4)^2$	**10.** $-(-5)^2$	**11.** $(-2x)^3$	**12.** $-(2x)^3$
13. $-(-2x)^3$	**14.** $(a^2)^5$	**15.** $(y^3)^4$	**16.** $(-x^3)^2$
17. $(a^n)^2$	**18.** $(a^2)^n$	**19.** $(a^n)^n$	**20.** $(-5x^4)^2$
21. $(\frac{5}{3})^2$	**22.** $(-\frac{3}{4})^3$	**23.** $(\frac{2}{3}x)^5$	**24.** $-(\frac{3}{2}x)^3$
25. $(0.2x)^3$	**26.** $(-0.3x^2)^4$	**27.** $(a^nb^2)^3$	**28.** $(a^2b^3)^n$
29. $(4x^2y^n)^n$	**30.** $(-2x^3)(-2x)^3$	**31.** $xy^4(xy)^4$	**32.** $\frac{x^7}{x^4}$
33. $\frac{a^5}{a^3}$	**34.** $\frac{b^8}{b}$	**35.** $\frac{y^2}{y^6}$	**36.** $\frac{x^3}{x^3}$
37. $\frac{a}{a^5}$	**38.** $\frac{a^2b^3}{b^5}$	**39.** $\frac{x^3y^3}{x^2y^4}$	**40.** $\frac{x^5}{x^7y^3}$
41. $\frac{(6x^3)^2}{(4x^2)^3}$	**42.** $\frac{(8a^2b^3)^2}{(6ab^2)^3}$	**43.** $\frac{(xy^2)^n}{(x^ny)^2}$	**44.** $\frac{a^nb^{n+3}}{a^{n+1}b^{n+2}}$
45. $\frac{10ab^3}{15(ab)^3}$	**46.** $\frac{0.6ab^2}{(0.2ab)^2}$	**47.** $\frac{(ab^2c^3)^n}{(a^nb^nc^n)^2}$	**48.** $\frac{(3)^{4m}}{(9)^{2m}}$

9·2 Roots and Radicals. If $a^n = b$ where n is a positive integer, not only is b equal to the nth power of a, but by definition a is said to be the nth root of b. If $n = 2$, a is a square root of b; if $n = 3$, a is a cube root of b; and so on.

EXAMPLE 1. Since $2^3 = 8$, 2 is a cube root of 8.

EXAMPLE 2. Since $(-2)^3 = -8$, -2 is a cube root of -8.

EXAMPLE 3. Since $(-3)^2 = 9$, and $3^2 = 9$, both -3 and 3 are square roots of 9.

EXAMPLE 4. Since $2^5 = 32$, 2 is a fifth root of 32.

From the above examples it is apparent that a number may have more than one nth root. On the other hand we have, at this point, no method at all for finding an *even* root of a *negative* number. That is, we have no method for finding a square root, fourth, sixth, eighth, etc., root of any negative number. For example, we can find the square root of $+25$, and we know that there are two such roots. They are $+5$ and -5. However, we have, at the moment, no method for finding two *equal* factors which will multiply and give -25. The fact that $(-5)(+5) = -25$ has no bearing on the problem, because -5 and $+5$ are not equal factors.

We shall deal with the problem of finding even roots of negative numbers in Sec. 9·14, and we shall present some extremely important applications of these methods in Chap. 24.

For the present we shall exclude from the discussion any consideration of even roots of negative numbers.

This brings us to the definition of *principal root.*

9·3 Principal Roots. The nth root of the quantity b is written $\sqrt[n]{b}$, except that when $n = 2$, we simply write $\sqrt{b}$. However, without further definition, this symbol may be somewhat ambiguous. For example, does $\sqrt{9}$ mean $+3$ or -3? The answer is found in the concept of *principal root.*

Excluding even roots of negative numbers, which we are not ready to handle yet, the principal root of a number has the same sign as the number itself. Thus: $\sqrt{+16} = +4$, and $+4$ is the principal square root of $+16$; $\sqrt[3]{-8} = -2$, and the principal cube root of -8 is -2; $\sqrt[3]{+8} = +2$, and the principal cube root of $+8$ is $+2$.

EXAMPLE 5. $\sqrt{9} = 3$. Note that $\sqrt{9}$ is not equal to -3 even though -3 is one of the square roots of 9. When we wish to designate the negative square root of 9, we write $-\sqrt{9}$, so that $-\sqrt{9} = -3$ and $\sqrt{9} = 3$.

EXAMPLE 6. $\sqrt{36} = 6$ since $6^2 = 36$

EXAMPLE 7. $\sqrt[3]{-125} = -5$ since $(-5)^3 = -125$

EXAMPLE 8. $\sqrt[4]{16} = 2$ since $2^4 = 16$

EXAMPLE 9. $\sqrt[5]{243} = 3$ since $3^5 = 243$

EXAMPLE 10. $\sqrt{\frac{9}{25}} = \frac{3}{5}$ since $(\frac{3}{5})^2 = \frac{9}{25}$

EXAMPLE 11. $\sqrt[3]{8x^6y^9} = 2x^2y^3$ since $(2x^2y^3)^3 = 8x^6y^9$

These examples illustrate the following rules:

(7) If b is positive and n even or odd, $\sqrt[n]{b}$ is positive.

(8) If b is negative and n is odd, $\sqrt[n]{b}$ is negative.
(9) If b is negative and n is even, no real root exists.

In the notation $\sqrt[n]{b}$, the symbol $\sqrt{\ }$ is called the *radical* sign, n is the *index* of the root, and b is the *radicand*. Where n does not appear, as in $\sqrt{b}$, the square root of b is understood.

EXERCISE 2

In Probs. 1 to 8, write in radical form the principal roots of the numbers indicated and find their values.

1. The cube root of -8
2. The cube root of 8
3. The fourth root of 81
4. The fifth root of -32
5. The square root of $\frac{1}{25}$
6. The sixth root of $\frac{1}{64}$
7. The square root of 49
8. The cube root of $2\frac{7}{125}$

Find the principal value of each of the following:

9. $\sqrt{25}$
10. $\sqrt{121}$
11. $\sqrt{64}$
12. $\sqrt{225}$
13. $\sqrt[3]{27}$
14. $\sqrt[3]{64}$
15. $\sqrt{144}$
16. $\sqrt[3]{-27}$
17. $\sqrt[4]{16}$
18. $\sqrt[3]{1}$
19. $\sqrt[3]{-1}$
20. $\sqrt[5]{32}$
21. $\sqrt{0.04}$
22. $\sqrt[3]{0.125}$
23. $\sqrt{1600}$
24. $\sqrt{\frac{9}{49}}$
25. $\sqrt[3]{-2\frac{7}{125}}$
26. $\sqrt[3]{\frac{1}{8}}$
27. $\sqrt{4a^2b^4c^6}$
28. $\sqrt[3]{8m^6n^9}$
29. $\sqrt[4]{\dfrac{a^4x^8}{c^{12}}}$
30. $\sqrt[5]{\dfrac{-x^{10}y^{20}}{32}}$
31. $\sqrt{0.0169x^{16}}$
32. $\sqrt[3]{0.008a^{27}}$

9·4 Rational and Irrational Numbers. In Sec. 5·1 the distinction between rational and irrational numbers was brought out. A real *rational* number can be expressed as the *ratio* of one integer to another. *Irrational* numbers are numbers that cannot be so expressed.

An example of an irrational number is $\sqrt{2}$. There exists a real number, approximately 1.414, which when squared equals 2. But this number ($\sqrt{2}$), though real, cannot be expressed by an integer or the ratio of two integers. Other examples of real but irrational numbers are $\sqrt[3]{5}$, $\sqrt{6}$, $\sqrt[5]{-7}$, π, and e.

9·5 Fractional Exponents. Our discussion of n thus far has been limited to the consideration of values which are positive integers. We shall require that the rules for positive integral exponents shall apply to negative, fractional, and zero exponents and thus see what significance we must, as a consequence, attribute to these kinds of exponents.

At present the expression a^2 indicates that a is taken twice as a factor, but it is meaningless to say that $a^{\frac{1}{2}}$ indicates that a is taken one-half a time. However, if we require that Sec. 5·9 hold, we may write

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$$

Evidently, then, $a^{\frac{1}{2}}$ is one of two equal factors of a, and hence

$$a^{\frac{1}{2}} = \sqrt{a}$$

Similarly, $a^{\frac{1}{3}}$ assumes a definite meaning if we write

$$a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a$$

The expression $a^{\frac{1}{3}}$ is one of three equal factors of a, and hence

$$a^{\frac{1}{3}} = \sqrt[3]{a}$$

In general, if n is a positive integer,

$$a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot \cdots \cdot a^{\frac{1}{n}} \quad (n \text{ factors}) = a^{\frac{n}{n}} = a^1 = a$$

Therefore,

(10) ★ $$a^{\frac{1}{n}} = \sqrt[n]{a}$$

In a corresponding way we can define an expression like $a^{\frac{3}{4}}$ by writing:

$$a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} = a^{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}} = a^3$$

or

$$a^{\frac{3}{4}} = \sqrt[4]{a^3}$$

In general, if m and n are positive integers and a is a real number,

$$a^{\frac{m}{n}} \cdot a^{\frac{m}{n}} \cdot a^{\frac{m}{n}} \cdot \cdots \cdot a^{\frac{m}{n}} \quad (n \text{ factors}) = a^{\frac{mn}{n}} = a^m$$

Hence

(11) ★ $$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

We conclude from the foregoing that a fractional exponent is equivalent to a radical. The numerator m indicates a power, and the denominator n a root. It is understood in this discussion that $a^{\frac{m}{n}}$ will be defined as the principal nth root of a^m. Unless it is specified, we shall exclude the case where a is negative and n is an even integer.

9·6 Zero Exponent. If a is any number except zero, we can define the symbol a^0 by writing

(12) ★ $$a^0 = 1 \qquad (a \neq 0)$$

This interpretation is entirely consistent, since by Sec. 5·9

$$\frac{a^m}{a^m} = 1$$

But by Eq. (3), page 155

$$\frac{a^m}{a^m} = a^{m-m} = a^0$$

Hence the two results must be equivalent.

9·7 Negative Exponents. We may define the symbol a^{-n} as follows:

(13) ★ $$a^{-n} = \frac{1}{a^n}$$

and

$$a^n = \frac{1}{a^{-n}}$$

This conclusion follows from Eq. (3), page 155, by letting $m = 0$.

$$\frac{a^m}{a^n} = a^{m-n}$$

and if $m = 0$,

$$\frac{1}{a^n} = \frac{a^0}{a^n} = a^{0-n} = a^{-n}$$

Note that from Eq. (13) any *factor* (not any term) of the numerator or denominator of a fraction may be transferred to the denominator or numerator, as the case may be, by reversing the sign of its exponent.

EXAMPLE 12. $16^{\frac{3}{4}} = \sqrt[4]{16^3}$, or, more conveniently, $(\sqrt[4]{16})^3 = 2^3 = 8$

EXAMPLE 13. $27^{\frac{2}{3}} = \sqrt[3]{27^2}$, or, more conveniently, $(\sqrt[3]{27})^2 = 3^2 = 9$

EXAMPLE 14. $12^0 = 1$

EXAMPLE 15. $x^0 = 1$

EXAMPLE 16. $(4x - 7)^0 = 1$

EXAMPLE 17. $3x^{-2} = \frac{3}{x^2}$ (note that we do not obtain $\frac{1}{9x^2}$)

EXAMPLE 18. $(3x)^{-2} = \frac{1}{(3x)^2} = \frac{1}{9x^2}$

EXAMPLE 19. $8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{2^2} = \frac{1}{4}$

EXAMPLE 20. $\frac{5x^3y^2}{z^4} = 5x^3y^2z^{-4}$

EXAMPLE 21. $\dfrac{4a^3c}{by^4} = 4a^3b^{-1}cy^{-4}$

EXAMPLE 22. $\dfrac{1}{x^{-1} - y^{-1}} = \dfrac{1}{\frac{1}{x} - \frac{1}{y}} = \dfrac{1}{\frac{y - x}{xy}} = \dfrac{xy}{y - x}$

(A simple solution at sight depends upon multiplying numerator and denominator by xy. See page 78.)

EXERCISE 3

Find the value of each expression.

1. $16^{\frac{1}{4}}$	2. $36^{\frac{1}{2}}$	3. 5^{-2}	4. 7^{-1}
5. $(-3)^{-3}$	6. $(49)^{-\frac{1}{2}}$	7. $(\frac{2}{3})^{-3}$	8. $(\frac{1}{32})^{\frac{1}{5}}$
9. $(-\frac{1}{32})^{-\frac{1}{5}}$	10. $(0.09)^{\frac{1}{2}}$	11. $(0.16)^{-\frac{1}{2}}$	12. $\dfrac{3a^0 - b^0}{a^0 + (3b)^0}$
13. $(-3)^4$	14. $-(-3)^4$	15. $(-3)^{-4}$	16. $-(3)^{-4}$
17. 10^{-3}	18. $(-\frac{1}{6})^{-1}$	19. $4^{-1} + 2^{-2}$	20. $(0.001)^{-\frac{1}{3}}$
21. $(-8)^{\frac{2}{3}}$	22. $(-27)^{\frac{1}{3}}$	23. $(8^{-1} - 4^{-1})^{\frac{1}{2}}$	24. $(-\frac{27}{64})^{-\frac{2}{3}}$

Write each expression without negative or zero exponents and simplify.

25. $x^{\frac{3}{4}} \cdot x^{\frac{1}{2}}$	26. $y^5 \cdot y^{-5}$	27. $(16x^{16})^{\frac{1}{2}}$
28. $(3x^{\frac{4}{3}})^3$	29. $(-5^{\frac{1}{2}}x^{\frac{1}{4}}y)^4$	30. $(2^{\frac{1}{6}}x^{\frac{1}{4}}y^{\frac{1}{16}})^{12}$
31. $-(-3a^{0.4}b^{0.6})^5$	32. $4^{-2}ab^{-3}c$	33. $\dfrac{3x^{-3}y^2}{6^{-1}z^{-2}}$
34. $\dfrac{3^{-1}x^6}{8y^{-3}}$	35. $\dfrac{x^0y^{-1}}{x^2w^{-4}}$	36. $\left(\dfrac{10^{-3}a^6}{90}\right)^{\frac{1}{2}}$
37. $(x^{\frac{1}{2}} - x^{-\frac{1}{2}})^2$	38. $(m^{\frac{1}{2}} + n^{\frac{1}{2}})(m^{\frac{1}{2}} - n^{\frac{1}{2}})$	
39. $(a^{\frac{1}{2}} - 3a^{-\frac{1}{2}})(a^{\frac{1}{2}} + 3a^{-\frac{1}{2}})$	40. $(x^{\frac{3}{2}} + x^{\frac{1}{2}})^4$	

Write the following expressions without denominators, using negative exponents where necessary.

41. $\dfrac{a}{b^2}$	42. $\dfrac{1}{xy}$	43. $\dfrac{z^2}{4^2}$
44. $\dfrac{3k}{ab^2}$	45. $\dfrac{x^{\frac{1}{2}}y^2}{x^{\frac{1}{4}}y^{\frac{3}{2}}}$	46. $\left(\dfrac{a^{\frac{1}{2}}b^{\frac{1}{3}}}{x^2}\right)^{-6}$
47. $\left(\dfrac{a^3x^{-1}}{2by^{-2}}\right)^{-3}$	48. $\left(\dfrac{64x^{\frac{1}{2}}}{y^{\frac{3}{4}}z^{\frac{1}{4}}}\right)^{\frac{1}{6}}$	

Write the following expressions in radical notation.

49. $x^{\frac{1}{2}}$ **50.** $y^{\frac{1}{3}}$ **51.** $a^{-\frac{1}{2}}$ **52.** $8x^{\frac{2}{3}}$
53. $x^{\frac{3}{4}}y^{\frac{1}{4}}$ **54.** $(4xy^2)^{\frac{1}{3}}$ **55.** $(5x^2)^{-\frac{1}{3}}$ **56.** $(a^3b^2)^{\frac{1}{6}}$

Using fractional exponents, write the following expressions without radicals.

57. $\sqrt{a^3}$ **58.** $\sqrt[3]{x^2y}$ **59.** $\sqrt[4]{x+y}$ **60.** $\sqrt[6]{a^3b^2}$
61. $\sqrt[4]{a^2b^4c^6}$ **62.** $\sqrt{9ab^2c^3}$ **63.** $\sqrt[3]{\dfrac{a^2}{8x^6}}$ **64.** $\sqrt{16x^2y^3z^4}$

9·8 Laws of Radicals. The following laws of radicals follow directly from the laws of exponents.

(14) ★ $$(\sqrt[n]{a})^n = \sqrt[n]{a^n} = a$$

(15) ★ $$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

(16) ★ $$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

(17) ★ $$(\sqrt[n]{a})^m = \sqrt[n]{a^m} = a^{\frac{m}{n}}$$

These laws are valid if we assume (1) that m and n are positive integers and (2) that the radicand is a real number. We are now in a position to write radicals in a number of equivalent ways. Application of Eq. (14) is illustrated in the following example.

EXAMPLE 23. $(\sqrt[3]{x^2})^3 = x^2$

To remove factors from the radicand, apply Eq. (15) and remove from the radicand any factor which is a perfect nth power.

EXAMPLE 24

$$\sqrt{45} = \sqrt{(3)(3)(5)} = \sqrt{(3)^2(5)} = \sqrt{3^2} \cdot \sqrt{5} = 3\sqrt{5}$$

EXAMPLE 25

$$\sqrt{4a^2x^2 - 36a^2y^2} = \sqrt{4a^2(x^2 - 9y^2)} = \sqrt{4a^2} \cdot \sqrt{x^2 - 9y^2}$$
$$= 2a\sqrt{x^2 - 9y^2}$$

EXAMPLE 26

$$\sqrt[3]{48x^4y^6z^8} = \sqrt[3]{(2)(2)(2)(2)(3)x^4y^6z^8}$$
$$= \sqrt[3]{(2)^3x^3y^6z^6(2)(3)xz^2}$$
$$= \sqrt[3]{(2)^3x^3y^6z^6} \cdot \sqrt[3]{(2)(3)xz^2} = 2xy^2z^2\sqrt[3]{6xz^2}$$

Note that this example is worked very conveniently by shifting temporarily to exponential notation.

$$\sqrt[3]{48x^4y^6z^8} = [(2)^4(3)x^4y^6z^8]^{\frac{1}{3}} = (2)^{\frac{4}{3}}(3)^{\frac{1}{3}}x^{\frac{4}{3}}y^{\frac{6}{3}}z^{\frac{8}{3}}$$

In any case where an exponent is an improper fraction, divide the expression involved into two factors, one raised to an integral power, the other raised to a power expressed by an exponent in the form of a proper fraction. Thus, from the above example, we obtain

$$2^1 \cdot 2^{\frac{1}{3}} \cdot 3^{\frac{1}{3}}x^1 \cdot x^{\frac{1}{3}} \cdot y^2 \cdot z^2 \cdot z^{\frac{2}{3}}$$

Rearranging, we obtain

$$(2xy^2z^2)(2^{\frac{1}{3}} \cdot 3^{\frac{1}{3}}x^{\frac{1}{3}}z^{\frac{2}{3}})$$
$$2xy^2z^2(2 \cdot 3xz^2)^{\frac{1}{3}}$$
$$2xy^2z^2\sqrt[3]{6xz^2}$$

EXERCISE 4

Simplify the following expressions by removing factors from the radicands.

1. $\sqrt{8}$	**2.** $\sqrt{12}$	**3.** $\sqrt{40}$	**4.** $\sqrt{54}$
5. $5\sqrt{18}$	**6.** $2\sqrt{27}$	**7.** $3\sqrt{32}$	**8.** $4\sqrt{63}$
9. $\sqrt[3]{24}$	**10.** $\sqrt[3]{-32}$	**11.** $5\sqrt[3]{54}$	**12.** $7\sqrt[3]{192}$
13. $\sqrt[3]{-250}$	**14.** $4\sqrt[3]{135}$	**15.** $5\sqrt[4]{96}$	**16.** $3\sqrt[5]{96}$
17. $\sqrt{x^7}$	**18.** $a^2\sqrt{c^9}$	**19.** $\sqrt{x^3y^3}$	**20.** $a\sqrt{a^2b}$
21. $\sqrt{12x^3y^5}$	**22.** $\sqrt{50a^3b^6c^9}$	**23.** $\sqrt{ax^2 + bx^2}$	
24. $\sqrt{a^2b^2 + a^2c^2}$	**25.** $\sqrt{4m^2 - 16a^2}$	**26.** $\sqrt{18x^3 - 27x^2y}$	
27. $\sqrt[n]{a^{n+1}}$	**28.** $\sqrt[n-1]{a^n}$	**29.** $\sqrt[n+1]{a^{n+3}}$	
30. $\sqrt[n-1]{a^{2n}}$	**31.** $\sqrt[n+1]{a^{2n+3}}$	**32.** $\sqrt[n-2]{a^{2n-3}}$	
33. $\sqrt[3]{x^6y^{6+a}}$	**34.** $\sqrt[4]{x^9y^{10+a}}$	**35.** $\sqrt[3n]{x^{8n}y^{6n}}$	
36. $\sqrt[3k-1]{x^{6k}}$			

In Probs. 37 to 42, simplify the radicals as before; then compute their values by using a table of square roots and cube roots.

37. $\sqrt{5300}$	**38.** $\sqrt{0.000105}$	**39.** $\sqrt{1375}$
40. $\sqrt{1648}$	**41.** $\sqrt[3]{6032}$	**42.** $\sqrt[3]{1875}$

Any quantity multiplying a radical may be introduced under the radical sign by raising to a power corresponding to the index of the radical.

EXAMPLE 27. $5\sqrt{2} = \sqrt{5^2}\cdot\sqrt{2} = \sqrt{(25)(2)} = \sqrt{50}$

EXAMPLE 28. $3ab^2\sqrt[3]{2a^2b} = \sqrt[3]{(3ab^2)^3}\cdot\sqrt[3]{2a^2b} = \sqrt[3]{(27a^3b^6)(2a^2b)}$
$= \sqrt[3]{54a^5b^7}$

9·9 Rationalizing the Denominator in a Radical. This term refers to the elimination of fractions within a radical. The procedure is desirable in that it simplifies numerical computations.

To rationalize the denominator of a radical having an index n, multiply numerator and denominator of the fraction by the smallest quantity which will make the denominator a perfect nth power. Extract the nth root of the denominator [see Eq. (16)].

EXAMPLE 29. $\sqrt{\frac{3}{7}} = \sqrt{\frac{3}{7}\cdot\frac{7}{7}} = \sqrt{\frac{21}{49}} = \frac{\sqrt{21}}{\sqrt{49}} = \frac{\sqrt{21}}{\sqrt{7^2}} = \frac{\sqrt{21}}{7}$

EXAMPLE 30. $\sqrt{\frac{c^3}{x^3}} = \sqrt{\frac{c^3\cdot x}{x^3\cdot x}} = \frac{\sqrt{c^3x}}{\sqrt{x^4}} = \frac{c\sqrt{cx}}{x^2}$

EXAMPLE 31. $\sqrt[3]{\frac{5m^4}{16a^2x^4}} = \sqrt[3]{\frac{5m^4\cdot 4ax^2}{16a^2x^4\cdot 4ax^2}} = \frac{\sqrt[3]{20am^4x^2}}{\sqrt[3]{64a^3x^6}} = \frac{m\sqrt[3]{20amx^2}}{4ax^2}$

EXAMPLE 32. $\sqrt{\frac{xy}{x+y}} = \sqrt{\frac{xy}{x+y}\cdot\frac{x+y}{x+y}} = \frac{\sqrt{xy(x+y)}}{\sqrt{(x+y)^2}} = \frac{\sqrt{xy(x+y)}}{x+y}$

EXAMPLE 33. $\sqrt{\frac{a}{b}-\frac{b}{a}} = \sqrt{\frac{a^2-b^2}{ab}} = \sqrt{\frac{ab(a^2-b^2)}{a^2b^2}} = \frac{\sqrt{ab(a^2-b^2)}}{\sqrt{a^2b^2}}$
$= \frac{\sqrt{ab(a^2-b^2)}}{ab}$

Note that, without rationalizing the denominator, we should have to evaluate an expression such as $\sqrt{\frac{3}{7}}$ by writing

$$\sqrt{\tfrac{3}{7}} = \sqrt{0.42857} = 0.6547$$

or

$$\frac{\sqrt{3}}{\sqrt{7}} = \frac{1.732}{2.646} = 0.6547$$

It can readily be appreciated how much easier and faster it is to rationalize and evaluate $\frac{\sqrt{21}}{7} = \frac{4.5826}{7} = 0.6547$. Tables are assumed to be used in all cases.

EXERCISE 5

In the following expressions, rationalize the denominators and remove any perfect powers from the radicands. Where possible, use tables to compute the decimal values of Probs. 1 to 17.

1. $\sqrt{1/2}$	2. $\sqrt{2/3}$	3. $6\sqrt{1/8}$	4. $9\sqrt{1/12}$
5. $6\sqrt{1/15}$	6. $15\sqrt{5/18}$	7. $\sqrt[3]{1/2}$	8. $\sqrt[3]{1/4}$
9. $8\sqrt[3]{1/16}$	10. $10\sqrt[3]{1/25}$	11. $6\sqrt[4]{1/27}$	12. $6\sqrt[4]{1/8}$
13. $\frac{1}{2}\sqrt{8/3}$	14. $\sqrt{25/32}$	15. $4\sqrt{27/80}$	16. $\sqrt[3]{16/9}$
17. $\frac{1}{3}\sqrt[3]{36/25}$	18. $\sqrt{\frac{1}{a}}$	19. $\sqrt{\frac{1}{x^3}}$	20. $x\sqrt{\frac{1}{x^5}}$
21. $\sqrt[3]{\frac{1}{a^2}}$	22. $ab\sqrt[3]{\frac{1}{a^4}}$	23. $\sqrt[3]{\frac{m}{mx^2}}$	24. $\sqrt{\frac{3x}{2y}}$
25. $\sqrt{\frac{8a^3b}{27c^3}}$	26. $\frac{a}{b}\sqrt[3]{\frac{1}{ab}}$	27. $\sqrt{\frac{a-b}{a+b}}$	
28. $\sqrt{x-2+\frac{1}{x}}$	29. $\sqrt[n]{\frac{1}{x}}$	30. $\sqrt[n+1]{\frac{1}{c^n}}$	
31. $\sqrt{\frac{1}{a^{2n+1}}}$	32. $\sqrt[3]{\frac{1}{x^{3n-1}}}$		

9·10 Reduction of the Order of a Radical. The order (index) of a radical can be reduced if the radicand can be expressed as a perfect power whose exponent contains a factor which is also contained in the index of the radical.

To reduce the order of a radical:

1. *Express the radicand as a perfect power of some rational number.*
2. *Transform the radical to equivalent fractional-exponent notation.*
3. *Reduce the fractional exponent to lowest terms.*
4. *Convert back to radical form.*

EXAMPLE 34. Reduce the order of $\sqrt[6]{8a^3b^3}$.

$$\sqrt[6]{8a^3b^3} = \sqrt[6]{(2ab)^3} = (2ab)^{3/6} = (2ab)^{1/2} = \sqrt{2ab}$$

In general, a radical may be said to be simplified if the procedures of the preceding sections have been carried out where applicable. That is, it contains no negative or fractional exponents, the denomi-

nator of the radicand has been rationalized, all possible factors have been removed from the radicand, and the order is as small as possible.

EXERCISE 6

Simplify each of the following:

1. $\sqrt[4]{25}$
2. $\sqrt[4]{49}$
3. $\sqrt[6]{16}$
4. $\sqrt[6]{36}$
5. $\sqrt[6]{x^3y^3}$
6. $\sqrt[4]{49x^2y^4}$
7. $\sqrt[6]{25x^4y^6z^8}$
8. $\sqrt[10]{x^6y^8}$
9. $\sqrt[2n]{x^n}$
10. $\sqrt[3n]{b^3}$
11. $\sqrt[6]{1/4}$
12. $\sqrt[4]{1/25}$
13. $\sqrt[6]{\frac{1}{c^2}}$
14. $\sqrt[6]{\frac{1}{x^3}}$

9·11 Addition and Subtraction of Radicals. Two radicals are said to be the *same* if they have the same radicand and the same index. Multiples of the same radical can be added or subtracted by combining their coefficients. In order to add or subtract expressions containing radicals, simplify each radical and collect and combine all multiples of the same radical. Addition of radicals which remain unlike after simplification can only be indicated. The decimal value of numerical problems can be computed approximately by the use of tables.

EXAMPLE 35. Add $\sqrt{96} - 3\sqrt{2/3} + 1/2\sqrt{150} - 2\sqrt{3/8}$.
Simplifying each radical,

$$4\sqrt{6} - \sqrt{6} + 5/2\sqrt{6} - 1/2\sqrt{6}$$

Adding coefficients,

$$(4 - 1 + 5/2 - 1/2)\sqrt{6} = 5\sqrt{6}$$

EXAMPLE 36. Add $\sqrt{5x} - \sqrt{5/x} + 2\sqrt{x/5} - \sqrt[6]{25x^2}$.
Simplifying each radical,

$$\sqrt{5x} - \frac{1}{x}\sqrt{5x} + \frac{2}{5}\sqrt{5x} - \sqrt[3]{5x}$$

Adding coefficients of like radicals,

$$\left(1 - \frac{1}{x} + \frac{2}{5}\right)\sqrt{5x} - \sqrt[3]{5x} = \left(\frac{7x - 5}{5x}\right)\sqrt{5x} - \sqrt[3]{5x}$$

EXERCISE 7

Simplify and combine where possible.

1. $\sqrt{2} + 6\sqrt{2}$
2. $3\sqrt{2} - \sqrt{18}$
3. $3\sqrt{75} - 2\sqrt{27} + \sqrt{48}$
4. $\sqrt{40} - 2\sqrt{90} + \sqrt{\frac{2}{5}}$
5. $\sqrt{98} + \frac{1}{3}\sqrt{72} - 2\sqrt{\frac{1}{8}}$
6. $3\sqrt{20} - \sqrt[4]{25} - \frac{2}{3}\sqrt{180}$
7. $3\sqrt{28} + 4\sqrt{\frac{1}{7}} + \sqrt{112}$
8. $\sqrt{60} - \sqrt{\frac{3}{5}} - \sqrt{\frac{5}{27}}$
9. $\frac{1}{2}\sqrt{24} + 3\sqrt{\frac{2}{3}} - 2\sqrt{\frac{3}{2}}$
10. $3\sqrt[3]{6} + 2\sqrt[3]{48} - 4\sqrt[6]{36}$
11. $2\sqrt{28} - 4\sqrt{63} + 14\sqrt{\frac{1}{7}}$
12. $5\sqrt[3]{108} - 2\sqrt[3]{32} - 4\sqrt[3]{\frac{1}{2}}$
13. $\sqrt{x} + \sqrt{x^3} + \sqrt{x^5}$
14. $\sqrt{12x} - \sqrt{75x^3} + \sqrt{\frac{3}{x}}$
15. $\sqrt{\frac{x+y}{x-y}} - \sqrt{\frac{x-y}{x+y}}$
16. $\sqrt{\frac{x+2}{x-1}} - \sqrt{\frac{x-1}{x+2}}$
17. $\sqrt{(a+b)^3} - \sqrt{a+b}$
18. $\sqrt{x^5} - 4\sqrt{x^3} + 4\sqrt{x}$
19. $\sqrt{18} - \sqrt{\frac{1}{3}} + \sqrt{12} - \sqrt{\frac{1}{8}}$
20. $\sqrt{12x^4 + 48x^3 + 48x^2} - \sqrt{12x^4 + 12x^3 + 3x^2}$
21. $\sqrt{\frac{a}{b} - 3 + 2\sqrt{1 - \frac{3b}{a}}} - \frac{3}{a}\sqrt{a^3b^2 - 3a^2b^3}$

9·12 Multiplication and Division of Radicals. A radical may be multiplied or divided by another radical of the same order by using the formulas

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

and

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

When radicals of different orders are multiplied or divided, they must first be reduced to radicals of the same order. (The common index is usually the LCM of the original indices.)

EXAMPLE 37. Multiply $\sqrt[3]{4x^2y^2}$ by $\sqrt[3]{12a^2x^2}$.

$$\sqrt[3]{4x^2y^2} \cdot \sqrt[3]{12a^2x^2} = \sqrt[3]{48a^2x^4y^2} = \sqrt[3]{(8x^3)(6a^2xy^2)} = 2x\sqrt[3]{6a^2xy^2}$$

EXAMPLE 38. Divide $\sqrt[4]{8a^3c^2}$ by $\sqrt[4]{54ac^3}$.

$$\frac{\sqrt[4]{8a^3c^2}}{\sqrt[4]{54ac^3}} = \sqrt[4]{\frac{8a^3c^2}{54ac^3}} = \sqrt[4]{\frac{4a^2}{27c}} = \sqrt[4]{\frac{4a^2}{27c} \cdot \frac{3c^3}{3c^3}} = \frac{1}{3c}\sqrt[4]{12a^2c^3}$$

EXAMPLE 39. Multiply $2\sqrt{6} - \sqrt{3}$ by $3\sqrt{3} - \sqrt{2}$.

$$\begin{array}{r} 2\sqrt{6} - \sqrt{3} \\ 3\sqrt{3} - \sqrt{2} \\ \hline 6\sqrt{18} - 3(3) - 2\sqrt{12} + \sqrt{6} = 18\sqrt{2} - 9 - 4\sqrt{3} + \sqrt{6} \end{array}$$

EXAMPLE 40. Multiply $\sqrt{2}$ by $\sqrt[3]{4}$.

$$\sqrt{2} \cdot \sqrt[3]{4} = \sqrt[6]{8} \cdot \sqrt[6]{16} = \sqrt[6]{128} = 2\sqrt[6]{2}$$

EXAMPLE 41. Divide $\sqrt[3]{9}$ by $\sqrt{3}$.

$$\sqrt[3]{9} \div \sqrt{3} = \sqrt[6]{81} \div \sqrt[6]{27} = \sqrt[6]{81/27} = \sqrt[6]{3}$$

ALTERNATIVE SOLUTION:

$$\begin{aligned} \sqrt[3]{9} \div \sqrt{3} &= \sqrt[3]{3^2} \div \sqrt{3} \\ &= 3^{\frac{2}{3}} \div 3^{\frac{1}{2}} \\ &= 3^{\frac{2}{3}-\frac{1}{2}} \\ &= 3^{\frac{1}{6}} = \sqrt[6]{3} \end{aligned}$$

9·13 Division by an Irrational Polynomial. In one important case of division of radicals, the divisor is a polynomial, most commonly a binomial, in which one or more terms contain a second-degree radical. The division is effected by rationalizing the binomial divisor. This is done by multiplying the numerator and denominator of the fraction by the *conjugate* of the denominator, *i.e.*, the denominator with the sign between its terms reversed. Simplify the new numerator and denominator. Similar methods can be applied to other special forms. This method will be extremely important in complex numbers and vector division (see Secs. 9·16 and 24·6).

EXAMPLE 42. Rationalize the denominator of $\dfrac{12}{5 - \sqrt{7}}$.

Multiplying numerator and denominator by the conjugate of $5 - \sqrt{7}$, we obtain

$$\frac{12}{5 - \sqrt{7}} = \frac{12(5 + \sqrt{7})}{(5 - \sqrt{7})(5 + \sqrt{7})}$$

$$\frac{12(5 + \sqrt{7})}{5^2 - (\sqrt{7})^2} = \frac{12(5 + \sqrt{7})}{25 - 7} = \frac{12(5 + \sqrt{7})}{18} = \frac{2(5 + \sqrt{7})}{3}$$

EXAMPLE 43. Rationalize the denominator of $\dfrac{\sqrt{6} + 3\sqrt{2}}{3\sqrt{6} + 2\sqrt{3}}$.

Multiplying numerator and denominator by $3\sqrt{6} - 2\sqrt{3}$,

$$\begin{aligned}\frac{\sqrt{6} + 3\sqrt{2}}{3\sqrt{6} + 2\sqrt{3}} &= \frac{(\sqrt{6} + 3\sqrt{2})(3\sqrt{6} - 2\sqrt{3})}{(3\sqrt{6} + 2\sqrt{3})(3\sqrt{6} - 2\sqrt{3})} \\ &= \frac{(3)(6) - 2\sqrt{18} + 9\sqrt{12} - 6\sqrt{6}}{(3\sqrt{6})^2 - (2\sqrt{3})^2} \\ &= \frac{18 - 6\sqrt{2} + 18\sqrt{3} - 6\sqrt{6}}{54 - 12} \\ &= \frac{6(3 - \sqrt{2} + 3\sqrt{3} - \sqrt{6})}{42} \\ &= \frac{3 - \sqrt{2} + 3\sqrt{3} - \sqrt{6}}{7}\end{aligned}$$

EXAMPLE 44. Rationalize the denominator of $\dfrac{\sqrt{a+1} - \sqrt{a-1}}{\sqrt{a+1} + \sqrt{a-1}}$.

Multiplying numerator and denominator by $\sqrt{a+1} - \sqrt{a-1}$,

$$\begin{aligned}&\frac{(\sqrt{a+1} - \sqrt{a-1})(\sqrt{a+1} - \sqrt{a-1})}{(\sqrt{a+1} + \sqrt{a-1})(\sqrt{a+1} - \sqrt{a-1})} \\ &\qquad = \frac{a + 1 - 2\sqrt{a^2 - 1} + a - 1}{(a+1) - (a-1)} \\ &\qquad = \frac{2a - 2\sqrt{a^2 - 1}}{2} = a - \sqrt{a^2 - 1}\end{aligned}$$

EXERCISE 8

Perform the indicated multiplications and divisions and simplify the results. Where possible, use tables to compute the decimal values of answers containing no literal expressions.

1. $\sqrt{2} \cdot \sqrt{3}$
2. $\sqrt{3} \cdot \sqrt{6}$
3. $\sqrt{7} \cdot \sqrt{14}$
4. $3\sqrt{2} \cdot \sqrt{10}$
5. $2\sqrt{11} \cdot 3\sqrt{11}$
6. $4\sqrt{15} \cdot \sqrt{21}$
7. $\sqrt{6a} \cdot \sqrt{2ab}$
8. $2\sqrt{xy} \cdot \sqrt{yz}$
9. $\sqrt{abc} \cdot \sqrt{bcd}$
10. $\sqrt[3]{4} \cdot \sqrt[3]{6}$
11. $\sqrt[3]{18} \cdot \sqrt[3]{15}$
12. $\sqrt[3]{xy^2z} \cdot \sqrt[3]{x^2yz}$
13. $\sqrt[3]{4} \cdot \sqrt{6}$
14. $\sqrt[3]{9} \cdot \sqrt{3}$
15. $\sqrt[4]{8} \cdot \sqrt{2}$
16. $\sqrt[4]{27} \cdot \sqrt{3}$
17. $(\sqrt{10} + \sqrt{3})(\sqrt{10} - \sqrt{3})$
18. $(7 + \sqrt{5})(7 - \sqrt{5})$
19. $(5 - 3\sqrt{2})(5 + 3\sqrt{2})$
20. $(5\sqrt{6} - 2\sqrt{10})(5\sqrt{6} + 2\sqrt{10})$
21. $(\sqrt{6} - \sqrt{3})^2$

22. $(7 - 2\sqrt{5})^2$

23. $(\sqrt{3} + \sqrt{2})(3\sqrt{3} - 2\sqrt{2})$

24. $(2\sqrt{6} - \sqrt{3})(\sqrt{6} + 3\sqrt{2})$

25. $(\sqrt{6} + 2\sqrt{10})(2\sqrt{15} - \sqrt{6})$

26. $\dfrac{\sqrt{6}}{\sqrt{2}}$ **27.** $\dfrac{\sqrt{35}}{\sqrt{15}}$ **28.** $\dfrac{5\sqrt{22}}{\sqrt{11}}$

29. $\dfrac{6\sqrt{3}}{\sqrt{15}}$ **30.** $\dfrac{\sqrt{ab}}{\sqrt{ac}}$ **31.** $\dfrac{\sqrt{21xy}}{\sqrt{14yz}}$

32. $\dfrac{\sqrt[3]{4}}{\sqrt{2}}$ **33.** $\dfrac{\sqrt{6}}{\sqrt[3]{3}}$ **34.** $\dfrac{\sqrt[4]{a^3}}{\sqrt[4]{a}}$

35. $\dfrac{\sqrt[4]{27}}{\sqrt{3}}$ **36.** $\dfrac{\sqrt{2}}{\sqrt[4]{8}}$ **37.** $\dfrac{6}{\sqrt{7} - 2}$

38. $\dfrac{10}{\sqrt{13} - 3}$ **39.** $\dfrac{24}{10 - \sqrt{28}}$ **40.** $\dfrac{15}{\sqrt{17} - \sqrt{7}}$

41. $\dfrac{21}{2\sqrt{5} - \sqrt{6}}$ **42.** $\dfrac{\sqrt{6}}{2\sqrt{3} - \sqrt{2}}$ **43.** $\dfrac{\sqrt{10}}{3\sqrt{5} + 2\sqrt{2}}$

44. $\dfrac{3\sqrt{2} - \sqrt{3}}{2\sqrt{3} + \sqrt{2}}$ **45.** $\dfrac{4\sqrt{5} + 3\sqrt{3}}{3\sqrt{5} - 2\sqrt{3}}$ **46.** $\dfrac{\sqrt{a - b}}{\sqrt{a} - \sqrt{b}}$

47. $\dfrac{\sqrt{xy} + \sqrt{yz}}{\sqrt{xyz}}$ **48.** $\dfrac{a - x}{\sqrt{a} + \sqrt{x}}$

9·14 Imaginary Numbers. Classical mathematicians (*i.e.*, ancient Greek mathematicians) knew only the kind of numbers which we use in enumeration and in the ordinary elementary mathematical calculations. The letters which we have used thus far in algebra represent this kind of number.

The nomenclature and symbolism of our Western mathematics are dominated by the conventions of classical mathematics. Yet the Western mind has created a technology which makes demands on its mathematical symbolism not anticipated by the originators of this symbolism.

Thus, nomenclature which, from a classical point of view, is characteristically descriptive may, in the light of present usage, be ambiguous or even definitely misleading.

The words *imaginary* and *real*, as used to distinguish between two kinds of numbers, are an illustration of the ambiguity resulting from our unfortunate attempt to make classical mathematical symbols

convey a meaning which they were never intended to convey. All numbers are imaginary to the extent that they are all creations of the human mind and have no physical existence. All numbers are real to the extent that they represent ideas and may be used as aids in consolidating familiar ideas and developing new ones.

Within the limits prescribed by classical mathematics, an even root of a negative number is an impossibility. That is, $\sqrt{-25}$ does not exist, for, again within the limits of classical mathematics, there are no two equal numbers which will multiply and yield -25 as a product. However, a symbolism has been developed to cover this situation. We have come to let the symbol j (sometimes i) be defined in such a way that

$$j^2 = -1 \tag{18}$$

Then

$$j = \sqrt{-1} \tag{19}\ \star$$

Now, following Eq. (15), where $a = -1$, $b = 25$, and $n = 2$,

$$\sqrt{-25} = \sqrt{-1} \times 25 = \sqrt{-1} \times \sqrt{25} = \sqrt{-1} \times 5 \tag{20}$$

or substituting Eq. (19) in Eq. (20),

$$\sqrt{-25} = j \times 5 \tag{21}$$

At this point the conservatives are willing to retreat slightly and concede that we can after all find the square root of a negative number but that the result is "imaginary."

Actually, $\sqrt{-1}$ is an extremely useful number. True, it has no use in measuring some physical quantities like bushels of apples nor in banking nor in many other fields where mathematics is used. Yet, as we shall discover in Chap. 24, the number $\sqrt{-1}$ is a natural device for measuring vectors so important in alternating-current-circuit problems.

We shall use the traditional terms *real* and *imaginary* to designate each of two different kind of numbers, but we shall bear in mind that these terms, as used in this connection, are not descriptive of everyday usage.

By definition, an *imaginary number* is a symbol in the form

$$\sqrt[n]{-b} \tag{22}$$

where n is an even integer and b is a positive real number.

By definition, a *real number* is the kind of number useful in enumeration, arithmetic, and the simple calculations.

The *imaginary unit* is, by definition, $\sqrt{-1}$. This we shall denote by the letter j, since that is the more common designation in works on engineering. Strictly mathematical texts usually use the letter i.

9·15 Operations with Imaginary Numbers. As a general policy, we shall require that imaginary numbers obey the same basic laws of operation that real numbers obey. Then we shall determine what meaning we may in consequence assign to these operations and hence what practical use they may be made to serve.

The Imaginary Unit

As was illustrated in Eq. (21), an imaginary number in the form $\sqrt{-b}$ can be written

$$\sqrt{-b} = \sqrt{-1} \times \sqrt{b} = j\sqrt{b} \tag{23}$$

Thus the square root of any negative number can be indicated as the product of the imaginary unit j and the principal root of the absolute value of the number under the radical.

Powers of j

By definition,

$$j^2 = -1$$

Then it follows from the basic rules of algebra that

$$\begin{aligned} j^0 &= 1 \\ j^1 &= j \\ j^2 &= -1 \\ j^3 &= j^2 \times j = -j \\ j^4 &= j^2 \times j^2 = 1 \\ j^5 &= j^4 \times j = j \\ j^6 &= j^4 \times j^2 = -1 \end{aligned}$$

In general, where n is any integer,

$$j^{4n} = +1 \tag{24}$$
$$j^{4n+1} = +j \tag{25}$$
$$j^{4n+2} = -1 \tag{26}$$
$$j^{4n+3} = -j \tag{27}$$

By using Eqs. (24) through (27), any power of j may be determined at a glance. Observe that any real, integral power of j must be either unity or j or their negatives.

EXERCISE 9

Find the value of

1. j^{10} **2.** j^{6} **3.** j^{15} **4.** j^{985} **5.** j^{-25}

Addition and Subtraction of Imaginary Numbers

Imaginary numbers may be added and subtracted according to the rules of algebra already discussed.

EXAMPLE 45. Add $\sqrt{-9}$ and $\sqrt{-49}$.

$$\sqrt{-9} = j\sqrt{9} = j3$$
$$\sqrt{-49} = j\sqrt{49} = j7$$
$$\sqrt{-9} + \sqrt{-49} = j3 + j7 = j10$$

NOTE: An alternative expression is $3j + 7j = 10j$.

EXAMPLE 46. Subtract $\sqrt{-25}$ from $\sqrt{-121}$.

$$\sqrt{-25} = j\sqrt{25} = j5$$
$$\sqrt{-121} = j\sqrt{121} = j11$$
$$\sqrt{-121} - \sqrt{-25} = j11 - j5 = j6$$

Multiplication of Imaginary Numbers

Multiplication of imaginary numbers can be accomplished as is illustrated below.

EXAMPLE 47. Multiply $\sqrt{-5}$ by $\sqrt{-7}$. Both $\sqrt{-5}$ and $\sqrt{-7}$ are imaginary numbers and consequently must be changed to the form $j\sqrt{b}$ before multiplying.

$$\sqrt{-5} = j\sqrt{5}$$
$$\sqrt{-7} = j\sqrt{7}$$
$$\sqrt{-5} \times \sqrt{-7} = j\sqrt{5} \times j\sqrt{7} = j^2\sqrt{35} = -\sqrt{35}$$

If we had ignored the fact that $\sqrt{-5}$ and $\sqrt{-7}$ are imaginary numbers, we would have been inclined to write, according to Eq. (15),

$$\sqrt{-5} \times \sqrt{-7} = \sqrt{(-5)(-7)} = \sqrt{35}$$

which is incorrect.

EXAMPLE 48. Multiply $\sqrt{-5}$, $\sqrt{-7}$, and $\sqrt{-15}$.

$$\sqrt{-5} = j\sqrt{5}$$
$$\sqrt{-7} = j\sqrt{7}$$
$$\sqrt{-15} = j\sqrt{15}$$
$$\sqrt{-5} \times \sqrt{-7} \times \sqrt{-15} = j\sqrt{5} \times j\sqrt{7} \times j\sqrt{15}$$
$$= j^3\sqrt{5} \times \sqrt{7} \times \sqrt{15} = j^3\sqrt{525} = j^3 5\sqrt{21} = -j5\sqrt{21}$$

EXAMPLE 49. Multiply $3\sqrt{-6} + 2\sqrt{-5}$ by $3\sqrt{-3} - 7\sqrt{-11}$.

$$3\sqrt{-6} = j3\sqrt{6}$$
$$2\sqrt{-5} = j2\sqrt{5}$$
$$3\sqrt{-3} = j3\sqrt{3}$$
$$7\sqrt{-11} = j7\sqrt{11}$$
$$j3\sqrt{6} + j2\sqrt{5}$$
$$j3\sqrt{3} - j7\sqrt{11}$$
$$j^2 9\sqrt{18} + j^2 6\sqrt{15} - j^2 21\sqrt{66} - j^2 14\sqrt{55}$$

Substituting for j^2 its numerical value, we obtain

$$-9\sqrt{18} - 6\sqrt{15} + 21\sqrt{66} + 14\sqrt{55}$$

Evaluating the radicals, we obtain

$$(-9 \times 4.243) - (6 \times 3.873) + (21 \times 8.124) + (14 \times 7.416)$$
$$= -38.187 - 23.238 + 170.604 + 103.824 = 213.003$$

Division of Imaginary Numbers

EXAMPLE 50. Divide $\sqrt{-30}$ by $\sqrt{-15}$.

$$\frac{\sqrt{-30}}{\sqrt{-15}} = \frac{j\sqrt{30}}{j\sqrt{15}} = \frac{\sqrt{30}}{\sqrt{15}} = \sqrt{2}$$

EXERCISE 10

Simplify:

1. $\sqrt{-27} + \sqrt{-147} - \sqrt{12}$
2. $\sqrt{-150} - \sqrt{-54} + \sqrt{216}$
3. $\sqrt{-18} + \sqrt{-72} - \sqrt{128} + \sqrt{32}$
4. $\sqrt{-63} - \sqrt{-567} + \sqrt{112}$
5. $\sqrt{-80} + \sqrt{-20} + \sqrt{-320} - \sqrt{-245}$
6. $\sqrt{-128} + \sqrt{-72} - \sqrt{-800} + \sqrt{-392}$
7. $\sqrt{-196} - \sqrt{-64} + \sqrt{-80} + \sqrt{80}$

8. $\sqrt{-192} + \sqrt{-12} - \sqrt{-48} + \sqrt{-256}$

9. $\sqrt{-32} - \sqrt{-392} + \sqrt{-128}$

10. $\sqrt{-3} \times \sqrt{-3}$ **11.** $\sqrt{-25y^4} \times \sqrt{-9x^2}$

12. $(-4\sqrt{-4})(-3\sqrt{-9})$ **13.** $-\sqrt{-27a^2} \times \sqrt{-75b^2}$

14. $j\sqrt{3} \times j\sqrt{5}$ **15.** $-j\sqrt{7} \times j\sqrt{15}$

16. $-j\sqrt{2} \times (-j\sqrt{5})$ **17.** $j5 \times j8$

18. $j5 \times (-j8)$ **19.** $(-j2) \times (-j3)$

20. $(2\sqrt{-5} - 3\sqrt{-2})(2\sqrt{-5} + 3\sqrt{-2})$

21. $(\sqrt{-2} + \sqrt{-3})^2$ **22.** $\sqrt{-5} \times \sqrt{-7} \times \sqrt{-2}$

23. $\sqrt{-3} \times \sqrt{-10} \times \sqrt{-5}$ **24.** $\sqrt{-15} \times \sqrt{-5} \times \sqrt{-3}$

25. $(\sqrt{-5} + \sqrt{-3})(\sqrt{-6} - \sqrt{-2})$

26. $(\sqrt{-2} + \sqrt{-5})(\sqrt{-7} + \sqrt{-6})(\sqrt{-4} - \sqrt{-9})$

27. $(\sqrt{-12} + \sqrt{-11})(\sqrt{-3} + \sqrt{-15})(\sqrt{-10} - \sqrt{-19})$

28. $\sqrt{-144} \div \sqrt{-100}$ **29.** $\sqrt{-294} \div \sqrt{-54}$

30. $j3 \div j^25$ **31.** $\sqrt{-160} \div \sqrt{8}$

32. $\sqrt{-320} \div \sqrt{-128}$ **33.** $\sqrt{-63} \div \sqrt{-18}$

34. $\sqrt{96} \div \sqrt{-32}$ **35.** $\sqrt{48} \div \sqrt{96}$

36. $[\sqrt{-392} \div \sqrt{-49}] \times \sqrt{196}$

37. $[\sqrt{-128} \div \sqrt{-80}][\sqrt{32} \div \sqrt{-16}]$

9·16 Complex Numbers. A *complex number* is an expression in the form $a + jb$, where a and b are real numbers and $j = \sqrt{-1}$.

Complex numbers are used extensively in the solution of alternating-current-circuit problems and will be discussed further in Chap. 24. For the present, we shall concentrate on the purely manipulative techniques involving complex numbers, leaving the applications until we have had a chance to cover more advanced algebra and trigonometry.

Notice that in the general form of a complex number $a + jb$, if $a = 0$, we have a pure imaginary number. If $b = 0$, we have a real number.

Addition of Complex Numbers

EXAMPLE 51. Add the complex numbers $2 + j6$, $3 - j5$, and $4 + j2$.

$$\begin{array}{r} 2 + j6 \\ 3 - j5 \\ 4 + j2 \\ \hline 9 + j3 \end{array}$$

Subtraction of Complex Numbers

EXAMPLE 52. Subtract $5 + j9$ from $8 + j4$.

$$\begin{array}{r} 8 + j4 \\ 5 + j9 \\ \hline 3 - j5 \end{array}$$

EXAMPLE 53. Subtract $8 - j5$ from $3 + j2$.

$$\begin{array}{r} 3 + j2 \\ 8 - j5 \\ \hline -5 + j7 \end{array}$$

Multiplication of Complex Numbers. Complex numbers are multiplied according to procedure already described in Sec. 5·13. After multiplication any powers of j except the first are replaced by their equivalent (see page 172).

EXAMPLE 54. Multiply $2 + j7$ by $3 - j6$.

$$\begin{array}{r} 2 + j7 \\ \underline{3 - j6} \\ 6 + j21 - j12 - j^2 42 \end{array}$$

Since $j^2 = -1$, the product can be written

$$6 + j21 - j12 + 42$$

or

$$48 + j9$$

EXAMPLE 55. Find the product of the general complex numbers $(a + jb)$ and $(c + jd)$ (also see Sec. 24·5). The multiplication is performed exactly as is the multiplication of any two binomials.

$$\begin{array}{r} a + jb \\ \underline{c + jd} \\ ac + jbc + jad + j^2 bd \end{array} \tag{28}$$

By definition, $j^2 = -1$; therefore expression (28) becomes

$$ac + jbc + jad - bd \tag{29}$$

Grouping real and imaginary numbers, expression (29) becomes

$$(ac - bd) + j(bc + ad)$$

Division of Complex Numbers

In division of complex numbers we use the principle of rationalization described in Sec. 9·13.

EXAMPLE 56. Divide $5 + j8$ by $2 - j7$.

The problem can be arranged as

$$\frac{5 + j8}{2 - j7}$$

If we multiply the denominator by the conjugate of the denominator, *i.e.*, by $(2 + j7)$ (see Sec. 9·13), we obtain

$$(2 - j7) \times (2 + j7) = 4 + 49 = 53$$

Thus it is a basic principle of complex numbers that the *product of a complex number and its conjugate is a real number.*

In the present problem we can multiply both numerator and denominator by the conjugate of the denominator and obtain a fraction with a complex numerator and a real denominator.

$$\frac{(5 + j8) \times (2 + j7)}{(2 - j7) \times (2 + j7)} = \frac{(5 + j8) \times (2 + j7)}{53}$$

The numerator can now be multiplied in the usual way.

$$\begin{array}{r} 5 + j8 \\ \underline{2 + j7} \\ 10 + j16 + j35 + j^2 56 \\ 10 + j16 + j35 - 56 \\ -46 + j51 \end{array}$$

The original fraction is now

$$\frac{-46 + j51}{53}$$

and can be written

$$-\frac{46}{53} + j\frac{51}{53}$$

or

$$-0.868 + j0.962$$

EXAMPLE 57. Divide the complex number $(a + jb)$ by the complex number $(c + jd)$.

Rewriting the problem in the form of a fraction,

$$\frac{a + jb}{c + jd} \tag{30}$$

Multiplying numerator and denominator of expression (30) by $c - jd$, which is called the *complex conjugate* of $c + jd$ (see Sec. 9·13),

$$\frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} \tag{31}$$

Performing the indicated multiplication to obtain the numerator of Eq. (31),

$$\begin{array}{r} a + jb \\ \underline{c - jd} \\ ac + jbc - jad - j^2bd \end{array}$$

Since by definition $j^2 = -1$, the above product becomes

$$ac + jbc - jad + bd$$

or after simplification the numerator of (31) becomes

$$(ac + bd) + j(bc - ad)$$

Performing the indicated multiplication to obtain the denominator of (31),

$$\begin{array}{r} c + jd \\ \underline{c - jd} \\ c^2 + jcd - jcd - j^2d^2 \end{array} \tag{32}$$

Simplifying expression (32), we obtain

$$c^2 + d^2$$

Equation (31) can now be written

$$\frac{a + jb}{c + jd} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$$

or

$$\frac{a + jb}{c + jd} = \frac{ac + bd}{c^2 + d^2} + j\,\frac{bc - ad}{c^2 + d^2}$$

EXERCISE 11

Multiply the following:

1. $(2 + j3)(7 + j5)$
2. $(-3 - j6)(2 + j8)$
3. $(2 + j7)(4 - j5)(3 + j10)$
4. $(3 + j5)(3 - j5)$
5. $-(2 + j7)(2 - j7)$
6. $(5 + \sqrt{-6})(3 - \sqrt{-8})(2 + \sqrt{-5})$
7. $(2 + j5)^5$
8. $(1.4 + j3.6)(0.7 + j9.1)$
9. $(-0.42 - j2.7)(0.63 + j4.2)$
10. $(-8.6 + j3.6)(4.7 + j5.1)$
11. $\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)^3$

Divide the following:

12. $\dfrac{1}{1+j}$

13. $\dfrac{3+j6}{4-j8}$

14. $\dfrac{2-j9}{3+j5}$

15. $\dfrac{1+j}{2j^5}$

16. $\sqrt{4^{-2}} \div \sqrt{0.625}$

17. $(3-\sqrt{-5}) \div (3+\sqrt{-5})$

18. $\dfrac{1}{4+2\sqrt{-3}-\sqrt{-7}}$

19. $\dfrac{1}{\sqrt{-2}-1}$

20. $\dfrac{2\sqrt{-5}+5\sqrt{-2}}{2\sqrt{-5}-5\sqrt{-2}}$

21. $\dfrac{\sqrt{5}-\sqrt{-5}}{\sqrt{2}+\sqrt{-2}}$

22. $\dfrac{3+j2}{j}$

23. $\dfrac{j}{5-3j}$

24. $\left[\dfrac{2+6j}{2-7j}\right]^2$

Simplify:

25. $\dfrac{(2+6j)(3-7j)}{(5-2j)+(6-4j)}$

26. $\dfrac{(1+7j)(2-3j)}{(4-6j)-(3-5j)}$

27. $\dfrac{(1.8-3.2j)(0.8+0.6j)}{(1.8-3.2j)+(0.8+0.6j)}$

28. $\dfrac{(2+3j)(6-4j)}{(2+3j)+(6-4j)}$

29. $\dfrac{(7.6-4.2j)(3.8+5.6j)}{(7.6-4.2j)-(3.8+5.6j)}$

30. $\dfrac{(10.5-31.6j)(24+15j)}{(10.5-31.6j)+(24+15j)}$

Chapter 10 QUADRATIC EQUATIONS IN ONE UNKNOWN

Quadratic equations are so called because the highest power of the unknown is the *square;* in general, they are second-degree equations. These are of considerable interest in that they usually contain maximum or minimum values (maximum return, minimum cost, etc.). Secondary answers often suggest a broader scope of a problem than is indicated by the original question.

10·1 Quadratic Equations. An equation which can be reduced to the form

$$ax^2 + bx + c = 0$$

is called a *quadratic equation* in x and is the type form for the general quadratic. If either b or $c = 0$ (note that a cannot be zero), the equations

$$ax^2 + c = 0$$

and

$$ax^2 + bx = 0$$

are known as *incomplete quadratic equations.* (The first type is also known as a *pure quadratic.*) An equation containing all the terms, such as

$$ax^2 + bx + c = 0 \qquad (1)$$

is called a *complete* or *affected quadratic equation.*

10·2 Solution of Incomplete Quadratics. Both types of incomplete quadratic equations are easily solved.

EXAMPLE 1. Solve the equation $4x^2 - 9 = 0$.

Transposing and dividing by 4,

$$x^2 = \tfrac{9}{4}$$

Extracting square root,

$$x = \pm\tfrac{3}{2}$$

That is,

$$x = \tfrac{3}{2} \text{ or } -\tfrac{3}{2}$$

Note that both answers, or roots, are possible, since either $\frac{3}{2}$ or $-\frac{3}{2}$ upon squaring produces $\frac{9}{4}$. Note also that while one might logically write $\pm x = \pm\frac{3}{2}$, no additional answers are obtained.

The same example might have been solved by the factoring method as follows:

$$4x^2 - 9 = 0$$
$$(2x + 3)(2x - 3) = 0$$

whereby

$$2x + 3 = 0$$

or

$$x = -\frac{3}{2}$$

$$2x - 3 = 0$$

or

$$x = \frac{3}{2} \text{ as above}$$

This procedure is justified by recalling from past experience that if the product of two quantities is zero, one or both of those quantities must have been equal to zero. Therefore we set each factor equal to zero in turn.

EXAMPLE 2. Solve the equation $2x^2 + 6x = 0$.

Factoring,

$$2x(x + 3) = 0$$

Setting each factor equal to zero,

$$2x = 0 \quad \text{or} \quad x = 0$$
$$x + 3 = 0 \quad \text{or} \quad x = -3$$

EXERCISE 1

Solve the following incomplete quadratic equations, regarding x, y, z, and w as unknown quantities.

1. $y^2 - 9 = 0$	**2.** $z^2 - 16z = 0$	**3.** $x^2 = 49$
4. $144 - w^2 = 0$	**5.** $y^2 = 121$	**6.** $x^2 - 81 = 0$
7. $z^2 = 8$	**8.** $12 - w^2 = 0$	**9.** $x^2 - 30 = 0$
10. $z^2 = c^2$	**11.** $y^2 = 9a^2$	**12.** $x^2 - 16b^2 = 0$
13. $w^2 = aw$	**14.** $y^2 = a^2 + b^2$	**15.** $z^2 = \dfrac{1}{c^2}$
16. $\dfrac{a^2}{b^2} - y^2 = 0$	**17.** $16x^2 = 1$	**18.** $9y^2 - 25 = 0$
19. $12w^2 = 1$	**20.** $9z^2 = 8$	**21.** $w^2 + 121 = 5w^2$
22. $3y^2 - 17 = 8 - y^2$	**23.** $\dfrac{x^2}{3} = 48$	

24. $\frac{z^2}{b} = 4b$ **25.** $\frac{4}{y^2} = 49$ **26.** $\frac{1}{y^2} = 36$

27. $\frac{m^2}{x^2} = 25$ **28.** $\frac{1}{20y^2} = 5$ **29.** $\frac{1}{z^2} = \frac{1}{16}$

30. $\frac{a^2}{c^2} = \frac{1}{w^2}$ **31.** $\frac{4}{x} = \frac{x}{9}$ **32.** $\frac{1}{2x} = \frac{8x}{9}$

33. $\frac{3w}{4} - \frac{9}{2w} = \frac{5w}{8}$ **34.** $\frac{x^2 - 1}{6} - \frac{x^2 + 2}{9} = 1$

10·3 Solution of Complete or Affected Quadratics. The solution of the complete equation $ax^2 + bx + c = 0$ may be accomplished by the following means:

1. Plotting (approximate values—limited by accuracy of reading).
2. Factoring.
3. Completing the square.
4. Quadratic formula.
5. Short graphical method (approximate values—limited by accuracy of reading).

10·4 Solution by Plotting. While exact answers can be obtained by plotting if they are integers, fractional values are apt to be approximated, and irrational answers will always be approximations. Imaginary roots cannot normally be found by plotting.[1]

EXAMPLE 3. Solve the equation $x^2 - 2x - 11 = 0$ by plotting.

Set $x^2 - 2x - 11 = y$. Substitute various values of x, and determine the corresponding values of y. For instance, if $x = 7$, then $y = (7)^2 - 2(7) - 11 = 49 - 14 - 11 = 24$. In like manner make out a table, continuing to substitute various values of x until it becomes evident that y is receding from zero at an increasing rate.

x	6	5	4	3	2	1	0	−1	−2	−3	−4	−5
y	13	4	−3	−8	−11	−12	−11	−8	−3	4	13	24

Plot these values on coordinate paper. Answers correspond to the intersection of the curve and the X axis. This is reasonable because, in effect, we are solving for the simultaneous equations $y = x^2 - 2x - 11$ and $y = 0$.

Figure 10·1 shows the result obtained by plotting these data. The indicated answers are roughly $x = 4.5$ and $x = -2.5$.

Tabulation may be facilitated if we take uniformly spaced values of x and list increments in y (represented by Δy) and also increments in Δy [that is, $\Delta(\Delta y)$], which we abbreviate to $\Delta^2 y$.

[1] However by a suitable shift in the relative positions of the curve and the X axis such roots may be approximated.

Δx		-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
x	6	5	4	3	2	1	0	-1	-2	-3	-4
y	13	4	-3	-8	-11	-12	-11	-8	-3	4	13
Δy		-9	-7	-5	-3	-1	1	3	5	7	9
$\Delta^2 y$		2	2	2	2	2	2	2	2	2	

Taking advantage of the fact that $\Delta^2 y$ constantly equals 2, the table may be extended quite rapidly by this means.

Note also that the curve is symmetrical about the line $x = 1$. If a series of points has been plotted on one side of the axis of symmetry, corresponding

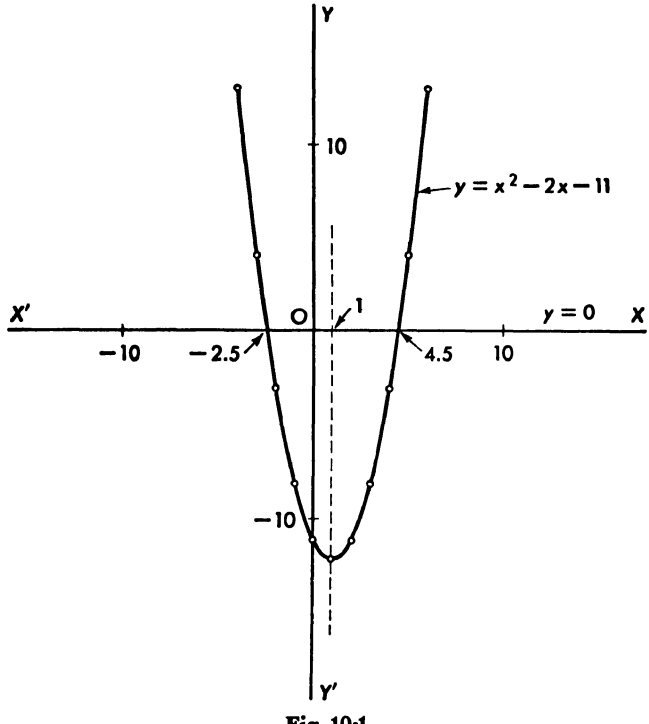


Fig. 10·1

points may be located on the opposite side, as in a mirror image. The edge of the mirror rests on the axis of symmetry.

EXERCISE 2

Solve the following complete quadratic equations by plotting.

1. $x^2 - 2x - 3 = 0$
2. $x^2 + x - 6 = 0$
3. $x^2 + x = 56$
4. $x^2 - 21 = 4x$
5. $x(x + 5) = 24$
6. $2x^2 - 3x + 1 = 0$
7. $3x^2 - x - 2 = 0$
8. $5x^2 - 11x + 2 = 0$
9. $(x - 2)(x - 5) = x - 5$
10. $(x - 5)(x + 8) = x - 16$

10·5 Solution by Factoring. This method is useful when factors of an expression are fairly obvious. However, the student should not spend too much time looking for factors that may not exist.

EXAMPLE 4. Solve the equation $x(2x - 1) = 15$ by factoring.

Carrying out the indicated multiplication,

$$2x^2 - x = 15$$

Transposing,

$$2x^2 - x - 15 = 0$$

Inspection will show that the factored form is

$$(2x + 5)(x - 3) = 0$$

Therefore,

$$2x + 5 = 0 \quad \text{and} \quad x = -\tfrac{5}{2}$$
$$x - 3 = 0 \quad \text{and} \quad x = 3$$

Note that, if the equation had been written $12x^2 - 6x - 90 = 0$, our first step would be the division by a common monomial factor—in this case, by 6.

EXERCISE 3

Solve the following quadratic equation by the method of factoring:

1. $x^2 - 4x + 4 = 0$ **2.** $x^2 - x - 30 = 0$ **3.** $x^2 + x = 72$
4. $x^2 - 2x = 15$ **5.** $x^2 + 7x = -12$ **6.** $x^2 - 33 = 8x$
7. $x^2 - 48 = 2x$ **8.** $x(x - 5) = 36$ **9.** $x(x + 3) = 40$
10. $x^2 + 3(x - 18) = 0$ **11.** $2x^2 + 3x + 1 = 0$
12. $5x^2 - 2x - 3 = 0$ **13.** $8x^2 + 10x - 12 = 0$
14. $21x^2 - 12x - 9 = 0$ **15.** $(x - 4)(x - 7) = x - 4$
16. $(x + 6)(x - 6) = 5x$

10·6 Solution by Completing the Square. Under certain special conditions this method offers a ready solution. It will be used in Sec. 10·7 to derive the quadratic formula.

EXAMPLE 5. Solve the equation $x^2 + 8x + 52 = 0$ by completing the square.

Rearranging so that all terms containing the unknown are on one side of the equation and all constant terms on the other, we have

$$x^2 + 8x = -52$$

Making the left side a perfect square by adding 16 to both sides,

$$x^2 + 8x + 16 = -36$$

Extracting square root,

$$x + 4 = \pm 6\sqrt{-1} = \pm j6$$
$$x = -4 + j6 \quad \text{or} \quad x = -4 - j6$$

The selection of the quantity to be added to both sides before taking square root is governed by considering the identity $(x + k)^2 = x^2 + 2kx + k^2$. Here it is evident that if we have the expression $x^2 + 2kx$, the missing quantity which will complete the square is found by evaluating $(2k/2)^2$ or k^2.

Thus in our example we square half the coefficient of x, or $(8/2)^2 = 16$.

EXAMPLE 6. Solve, by completing the square, the equation $3x(x - 3) = 2(1 - 2x)$.

Carrying out indicated multiplications,

$$3x^2 - 9x = 2 - 4x \tag{2}$$

Transposing, leaving the constant term on the right,

$$3x^2 - 5x = 2 \tag{3}$$

Dividing by the coefficient of x^2,

$$x^2 - \frac{5x}{3} = \frac{2}{3} \tag{4}$$

The term to be added to both sides will be the square of half the coefficient of $x = \left(\frac{5/3}{2}\right)^2 = 25/36$.

Adding $25/36$ to both sides of Eq. (4),

$$x^2 - \frac{5x}{3} + \frac{25}{36} = \frac{2}{3} + \frac{25}{36} = \frac{49}{36} \tag{5}$$

Extracting the square root,

$$x - 5/6 = \pm 7/6 \tag{6}$$
$$x = 5/6 + 7/6 \quad \text{or} \quad x = 2$$
$$x = 5/6 - 7/6 \quad \text{or} \quad x = -1/3$$

EXAMPLE 7. Solve by the method of completing the square the equation

$$x^2 - 6cx = 4a^2 - 12ac \tag{7}$$

In this example the terms not containing x have already been separated from those containing x. The missing quantity which will make the left side a perfect square is $(6c/2)^2 = 9c^2$. Adding $9c^2$ to both sides of Eq. (7), we obtain

$$x^2 - 6cx + 9c^2 = 4a^2 - 12ac + 9c^2 \tag{8}$$

Extracting the square root of both sides,

$$x - 3c = \pm(2a - 3c) \tag{9}$$

Hence
$$x = 3c + (2a - 3c) = 2a$$
Also
$$x = 3c - (2a - 3c) = 6c - 2a$$

Each of these answers will be found to check the original equation.

It is evident that the method of completing the square can be used to the greatest advantage when the coefficient of the second-degree term is unity and that of the first-degree term contains 2 as a factor.

To summarize, the solution of a quadratic by the method of completing the square consists of the following steps:

1. *Transpose, if necessary, so that the constant term is on one side of the equation, and terms containing the unknown are on the other side.*
2. *Make the coefficient of* x^2 *equal to unity by dividing by the coefficient of* x^2 *unless it is already equal to one. The equation now has the form* $x^2 + mx = k$.
3. *Add to both sides of the equation the square of half the coefficient of* x. *This transforms the left side of the equation to* $x^2 + mx + m^2/4$, *a perfect square.*
4. *Extract the square roots of both sides, remembering to place a plus or minus sign before the square root of the right-hand number.*
5. *Solve the two equations so formed for the unknown.*
6. *Check the results in the original equation.*

EXERCISE 4

Solve the following equations by means of completing the square, leaving any irrational answers in radical form.

1. $x^2 + 2x - 3 = 0$
2. $x^2 - 4x - 21 = 0$
3. $x^2 + 6x - 7 = 0$
4. $x^2 - 8x - 20 = 0$
5. $x^2 + 10x - 24 = 0$
6. $y^2 - 12y + 27 = 0$
7. $x(x - 10) = 39$
8. $z(z + 8) = 48$
9. $x^2 + 2bx = a^2 - b^2$
10. $x^2 - 2dx = c^2 + 2cd$
11. $x^2 + 2ax = -6ab + 9b^2$
12. $x^2 - 4cx = 9b^2 + 12bc$
13. $x^2 + \frac{2x}{3} = \frac{15}{9}$
14. $y^2 - \frac{4y}{5} = \frac{9}{5}$
15. $y^2 - 20y + 105 = 0$
16. $x^2 - \frac{6x}{5} = \frac{8}{5}$
17. $w^2 - 4w = 8$
18. $x^2 + 10x = 55$
19. $y^2 - 14y = 11$
20. $x^2 - 8x + 27 = 0$
21. $x^2 + 8x = 32$
22. $w^2 - 12w = -16$
23. $x^2 - 14x = -59$
24. $z^2 - 12z = -43$

10·7 Solution by the Formula. The most generally applicable method of solving a quadratic equation is that employing the *quadratic formula*. The derivation of this formula begins with the general quadratic

$$ax^2 + bx + c = 0$$

Transposing,

$$ax^2 + bx = -c \tag{10}$$

Dividing by the coefficient of x^2,

$$x^2 + \frac{bx}{a} = -\frac{c}{a} \tag{11}$$

Adding the square of half the coefficient of x to both sides,

$$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \tag{12}$$

Combining fractions on the right,

$$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2} \tag{13}$$

Taking square roots of both sides,

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \tag{14}$$

Transposing,

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \tag{15}$$

Combining fractions,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{16}$$ ★

This formula has the advantage of being applicable to any quadratic equation. Under special conditions, the methods of factoring or completing the square may be more convenient. With practice the student will learn to select the shortest method for the equation at hand.

It should be remembered that a is the coefficient of the second-degree term, b is the coefficient of the first-degree term, and c includes all terms not containing the unknown.

EXAMPLE 8. Use the quadratic formula to solve the quadratic equation $2x(4x - 1) = 15$.

Performing the indicated multiplication,

$$8x^2 - 2x = 15$$

Transposing, leaving zero on the right,

$$8x^2 - 2x - 15 = 0$$

According to the formula, $a = 8$, $b = -2$, $c = -15$.

Then

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(8)(-15)}}{(2)(8)}$$

$$x = \frac{2 \pm \sqrt{4 + 480}}{16} = \frac{2 \pm 22}{16}$$

$$x = \frac{2 + 22}{16} = \frac{3}{2}$$

$$x = \frac{2 - 22}{16} = -\frac{5}{4}$$

EXERCISE 5

Solve the following equations by the formula, leaving any irrational answers in the radical form.

1. $x^2 - 17x + 60 = 0$
2. $x^2 + x - 156 = 0$
3. $x^2 - 4x = 165$
4. $3x^2 + 7x = 6$
5. $5y - 6y^2 + 1 = 0$
6. $12z^2 + 24z = -9$
7. $3w^2 - 6w + 5 = 0$
8. $5x^2 - 55x - 5 = 0$
9. $3x^2 - 1 = \frac{11x}{12}$
10. $30y^2 + 76y + 48 = 0$
11. $w^2 + 5w + 7 = 0$
12. $7y^2 + 12y + 4 = 0$
13. $5p^2 - 3p + 1 = 0$
14. $\frac{1}{y - 1} + \frac{2}{y + 2} = \frac{1}{y + 1}$

10·8 Short Graphical Method. This method is designed for rapid, routine approximation. It depends upon separating the second-degree term from the rest of the expression and dividing by the coefficient of the second-degree term so that the equation is of the type $x^2 = mx + k$. From this equation we form the two equations $y = x^2$ and $y = mx + k$. If we were to plot the graphs of these simultaneous equations on a common coordinate system, the roots would be indicated by the abscissas of the intersections (see Fig. 10·2).

It will be observed that the plot of $y = x^2$ is a parabola, while that of $y = mx + k$ is a straight line. The advantage of this method lies

in the fact that the graph of $y = x^2$ may be drawn once and for all. A straight line scratched on the underside of a transparent strip will, upon proper positioning, represent the graph of $y = mx + k$. The abscissas of the intersections will represent the roots of the equation $x^2 = mx + k$.

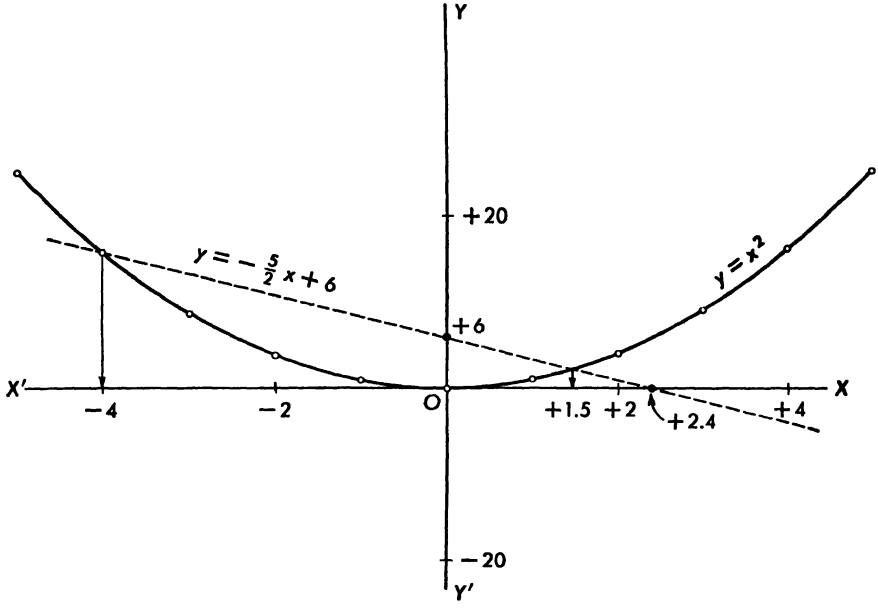


Fig. 10·2

EXAMPLE 9. Solve the equation

$$2x^2 + 5x - 12 = 0$$

Transposing,

$$2x^2 = -5x + 12$$

Dividing by the coefficient of x^2,

$$x^2 = \frac{-5x}{2} + 6$$

It is assumed that the graph of $y = x^2$, having been used in other problems, has already been drawn. The plastic strip is then laid on the graph paper so that the straight-line scratch passes through the points (0, 6) and (2.4, 0) (the y and x intercepts of the graph of $y = -5x/2 + 6$). The intersections of the two curves have abscissas of $\frac{3}{2}$ and -4, which are the roots.

Note that, for convenience in reading, the horizontal scale is 10 times as open as the vertical scale.

10·9 Literal Quadratic Equations. In a *literal* quadratic equation some or all of the constants are letters. These constants are usually represented by the first several letters of the alphabet or may involve π. In some cases it is preferable not to substitute the numerical value of π until a later stage of the solution.

The formula usually offers the easiest means of solution in the long run unless one of the other methods clearly offers a more convenient solution.

EXAMPLE 10. Solve the equation $2(x^2 + c^2) + d(x + c - d) = 5cx$.

Carrying out the indicated multiplication,

$$2x^2 + 2c^2 + dx + dc - d^2 = 5cx$$

Transposing and arranging in descending powers of x,

$$2x^2 + dx - 5cx + 2c^2 + cd - d^2 = 0$$

Grouping and factoring x^2 and x terms,

$$2x^2 + (d - 5c)x + 2c^2 + cd - d^2 = 0$$

Tabulate the values **a**, **b**, and **c** of the quadratic formula, remembering that all terms not containing the unknown, no matter how many there may be, are grouped under **c**. Thus $\mathbf{a} = 2$, $\mathbf{b} = d - 5c$, $\mathbf{c} = 2c^2 + cd - d^2$.

Then

$$x = \frac{-(d-5c) \pm \sqrt{(d - 5c)^2 - 4(2)(2c^2 + cd - d^2)}}{2(2)}$$

$$= \frac{-d + 5c \pm \sqrt{d^2 - 10cd + 25c^2 - 16c^2 - 8cd + 8d^2}}{4}$$

$$= \frac{-d + 5c \pm \sqrt{9d^2 - 18cd + 9c^2}}{4}$$

$$= \frac{-d + 5c \pm (3d - 3c)}{4}$$

$$x = \frac{-d + 5c + (3d - 3c)}{4} = \frac{2d + 2c}{4} = \frac{d + c}{2}$$

$$x = \frac{-d + 5c - (3d - 3c)}{4} = \frac{-4d + 8c}{4} = 2c - d$$

In cases where the equation or the roots are more involved, checking by direct substitution may entail as much time and effort as the original solution. The process may be facilitated by assigning numerical values to the literal constants. Avoid assigning numerical values which will make a denominator equal to zero.

EXAMPLE 11. Check the equation $(n - k)x^2 + (k - m)x + m - n = 0$ whose roots are $x = \dfrac{m - n}{n - k}$ and $x = 1$. Assigning arbitrary simple numerical

values (other than 0 or 1) to m, n, and k such that x will also be a simple integer, we might select $m = 10$, $n = 4$, $k = 2$, and therefore $x = \frac{10 - 4}{4 - 2}$ $= 3$. Substituting in the original equation,

$$(4 - 2)(3)^2 + (2 - 10)(3) + 10 - 4 = 0$$

or

$$(2)(9) + (-8)(3) + 6 = 0$$

or

$$18 - 24 + 6 = 0$$
$$0 = 0$$

EXERCISE 6

Solve the following literal quadratic equations for w, x, y, or z as the case may be:

1. $15x^2 - mx - 2m^2 = 0$
2. $4x^2 + (x + 3a)^2 = 17a^2$
3. $w^2 + aw + bw - cw = c(a + b)$
4. $abcx^2 + b^2cx + ac^2x + bc^2 = 0$
5. $(b - c)y^2 + (c - a)y + a - b = 0$
6. $w^2 - \left(\frac{a}{b} + \frac{b}{a}\right) w + 1 = 0$
7. $3z^2 - hz - 5 = 0$
8. $\frac{c^2}{x^2} = \frac{c + 1}{x + 1}$
9. $(x - k)^2 + (x - h)^2 = k^2 + h^2$
10. $a = \pi x(x + 2h)$

10·10 Precautions Regarding Roots. At this point the student is urged to exercise vigilance against two common errors resulting from apparently legitimate operations. These pitfalls are vanishing roots and extraneous roots. *Extraneous roots* are values which are derived in the process of solving of an equation but which do not check the original equation.

10·11 Vanishing Roots. The student is cautioned against dividing an equation through by an expression containing the unknown. By such division a root may be lost.

EXAMPLE 12. Solve the equation $x^2 - 5x = 0$.

Dividing through by x,

$$x - 5 = 0$$

or

$$x = 5$$

It is evident that another root, $x = 0$, has been lost in this process.

EXAMPLE 13. Solve the equation $2x^2 + 3x - 2 = x^2 - 4$.

Dividing through by $x + 2$,

$$2x - 1 = x - 2$$

Transposing,

$$x = -1$$

In this case, by dividing by $x + 2$ we have lost the root $x = -2$.

10·12 Extraneous Roots. These fictitious values usually arise from multiplying by an expression containing the unknown.

EXAMPLE 14. Solve the equation $3 + \dfrac{4}{x+2} = \dfrac{x^2}{x+2} - 1$.

Multiplying by $x + 2$,

$$3(x + 2) + 4 = x^2 - (x + 2)$$

Clearing of parentheses,

$$3x + 6 + 4 = x^2 - x - 2$$

Transposing and collecting terms,

$$0 = x^2 - 4x - 12$$

Factoring,

$$0 = (x - 6)(x + 2)$$

$$x - 6 = 0 \quad \text{or} \quad x = 6$$

$$x + 2 = 0 \quad \text{or} \quad x = -2$$

The root $x = 6$ satisfies the equation, but the root $x = -2$ does not.

Note that this situation could have been avoided by transposing and combining fractions. Thus

$$3 + 1 = \frac{x^2}{x+2} - \frac{4}{x+2}$$

Combining fractions and reducing,

$$4 = \frac{x^2 - 4}{x+2} = x - 2$$

or

$$x = 6$$

10·13 Checking. Checking is usually accomplished by direct substitution of the answers, or roots, in the original equation. However, if the original equation is in the type form for the general quad-

ratic, a shorter method is available, especially if one or both roots are fractions. It depends upon the fact that if $(x - m)(x - n) = 0$, the roots are $x = m$ and $x = n$. Conversely, if the roots are m and n, we may write $(x - m)(x - n) = 0$.* Referring back to Eq. 3, we may check by writing $(x - 2)(x - [-\frac{1}{3}]) = 0$. Multiplying by 3,

$$(x - 2)(3x + 1) = 0$$
$$3x^2 - 5x - 2 = 0$$

which checks all but the preliminary multiplication and transposition.

NOTE: *The alert student will appreciate that we have here an aid to factoring expressions of the type* $acx^2 + (ad + bc)x + bd$ (expansion 6, Chap. 5, Sec. 5·17). *If necessary, first factor the expression to free it of monomial factors. Set the resulting expression, such as* $3x^2 - 5x - 2$, *equal to zero, determine the roots by Eq.* (16), *the quadratic formula, and derive the factored form of the expression* $(x - 2)(3x + 1)$ *as above.*

10·14 Summary of Procedure for Solving Quadratic Equations

1. *Simplify any reducible fractions and combine any fractions having a common denominator.*
2. *Clear of fractions.*
3. *If there is a factor, not containing the unknown, common to all terms, divide through by that factor.*
4. *The choice of the method of solution might well be based upon consideration of the following suggestions in the order given:*
   a. *Use the factoring method if the factors are readily discernible.*
   b. *Use the method of completing the square if the coefficient of the second-degree term is unity and that of the first-degree term is divisible by 2.*
   c. *If neither* (a) *nor* (b) *is applicable, use the quadratic formula. This is somewhat longer but has general application to all cases.*
5. *Check all answers, discarding any extraneous roots.*

EXERCISE 7

Solve the following equations by the most convenient method, rejecting any extraneous roots and leaving any irrational answers in the radical form.

1. $\dfrac{y}{y+1} = \dfrac{y+2}{3y}$

2. $(w + 2)^3 - w^3 = 56$

3. $\dfrac{1}{z-3} + \dfrac{1}{z+4} = \dfrac{1}{12}$

4. $x + \dfrac{1}{5} = 5 + \dfrac{1}{x}$

* The *factor theorem:* If r is a root of the equation $f(x) = 0$, then $x - r$ is a factor of the polynomial $f(x)$. See also Appendix C.

5. $y + \frac{mn}{y} = m + n$

6. $\frac{1}{r} + r = 3 + \frac{3}{r}$

7. $11d^2 + 7d + 1 = 0$

8. $\frac{10}{x} - \frac{9}{x+1} - \frac{8}{x+2} = 0$

9. $\frac{h}{5} - \frac{5}{6} = \frac{6}{5} - \frac{5}{h}$

10. $\frac{1}{x} - \frac{1}{d} = \frac{1}{x+d}$

11. $\frac{1}{c} + \frac{1}{8-c} = \frac{1}{8}$

12. $x(2x - c) + x(x - c) = bx$

13. $2y(7y - a) = (a + y)(a - y)$

14. $\frac{1}{x+3} + \frac{1}{x+2} - \frac{1}{x+1} = 0$

15. $\pi x^2 + 2\pi nx - A = 0$

16. $8w = -3(1 + 4w^2)$

17. $\frac{2}{m+5} - \frac{m+3}{(m+4)(m+5)} = \frac{1}{4(m-8)}$

18. $\frac{a}{x-b} + \frac{b}{x-a} = 2$

19. $c(x^2 - 1) = (ax + b)(x - 1)$

20. $\frac{1}{w-2} + 1 = \frac{6-w}{w^2-4} + \frac{1}{w+2}$

21. $\frac{a}{x} + \frac{x}{a} = \frac{33a^2 - x^2}{ax}$

22. $\frac{1}{c+d+x} = \frac{1}{c} + \frac{1}{d} + \frac{1}{x}$

23. $9x^2 - hx + h^2 = 0$

10·15 Discriminant. In the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

the expression $b^2 - 4ac$ is called the *discriminant*. The value of the discriminant may usually be determined by inspection and indicates the

Table 10·1

Discriminant	Positive and a perfect square	Positive but not a perfect square	Zero	Negative
Character of the roots	Real, rational, and unequal	Real, irrational, and unequal	Real, rational, and equal	Complex and unequal if $b \neq 0$. Imaginary if $b = 0$
Graph of $ax^2 + bx + c = y$	Cuts X axis in two points		Tangent to X axis	Does not intersect X axis

nature of the roots of the equation $ax^2 + bx + c = 0$. The relationships are summarized in Table 10·1.

EXAMPLE 15. Determine the nature of the roots of the equation $2x^2 - 7x - 5 = 0$. Here $b^2 - 4ac = (-7)^2 - 4(2)(-5) = 49 + 40 = 89$. Since 89 is positive but not a perfect square, the roots are real, irrational, and unequal.

EXAMPLE 16. Determine the nature of the roots of the equation $25x^2 - 30x + 9 = 0$. In this example, $b^2 - 4ac = (-30)^2 - 4(25)(9) = 0$. Hence the roots are real, rational, and equal.

EXERCISE 8

Calculate the discriminant and describe the character of the roots without solving:

1. $x^2 + 4x - 12 = 0$
2. $x^2 - x - 20 = 0$
3. $x^2 - 6x + 9 = 0$
4. $x^2 - 3x - 5 = 0$
5. $2x^2 + 5x - 3 = 0$
6. $4x^2 + 20x + 25 = 0$
7. $6x^2 - x - 2 = 0$
8. $x^2 + 7x + 18 = 0$
9. $3x^2 - 18x + 27 = 0$
10. $2x^2 + 7x + 9 = 0$
11. $5x^2 - x - 10 = 0$
12. $28x^2 + 84x + 63 = 0$
13. $20x^2 - 7x - 3 = 0$
14. $14x^2 + 13x - 12 = 0$

10·16 Axis of Symmetry—Extreme Value. It will be recalled from Sec. 10·4 that the equation $ax^2 + bx + c = y$ has a vertical axis

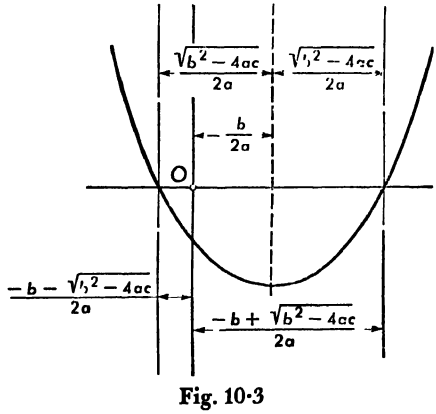


Fig. 10·3

of symmetry and that the extreme value of y or the turning point of the curve lies on this axis.

Taking advantage of the properties of symmetry, the abscissa of the axis of symmetry, and therefore of the extreme point, must

be the average of the x intercepts or roots $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$ (Fig. 10·3). The average of these values is clearly $x = -\frac{b}{2a}$. Substitution of $-\frac{b}{2a}$ for x in the equation $ax^2 + bx + c = y$ gives us $y = c - \frac{b^2}{4a}$, which may be confirmed by the student.

Hence the extreme value or turning point of the curve will correspond to the point

$$x = -\frac{b}{2a},\ y = c - \frac{b^2}{4a} \qquad \star (17)$$

A brief check will show that if a is positive the curve opens upward, ⌣, and the extreme is a minimum. On the other hand, if a is negative, the curve opens downward, ⌢, and the extreme is a maximum.

EXERCISE 9

Determine the nature and coordinates of the extreme point of the graphs of the following equations.

1. $x^2 - 2x - 15 = y$
2. $x^2 + 4x - 12 = y$
3. $x^2 + 7x + 6 = y$
4. $2x^2 + 5x - 3 = y$
5. $-x^2 + 6x - 8 = y$
6. $5x^2 - 6x - 8 = y$
7. $-6x^2 + 7x - 2 = y$
8. $3x^2 - x - 10 = y$
9. $-4x^2 - 4x + 15 = y$
10. $-2x^2 + x + 28 = y$

10·17 Equations of the Quadratic Type. Equations of the quadratic type may be represented by the general equation $ax^{2n} + bx^n + c = 0$. Such equations may be solved by methods comparable to those used in the solution of ordinary quadratics.

EXAMPLE 17. Solve the equation $x^4 - 29x^2 + 100 = 0$.
Factoring,

$$(x^2 - 25)(x^2 - 4) = 0$$
$$(x + 5)(x - 5)(x + 2)(x - 2) = 0$$
$$x = \pm 5, \pm 2$$

EXAMPLE 18. Solve the equation $6x + 7x^{\frac{1}{2}} - 20 = 0$.
Let $y^2 = x$. Then

$$6y^2 + 7y - 20 = 0$$

By the formula

$$y = \frac{-7 \pm \sqrt{49 - 4(6)(-20)}}{(2)(6)}$$

$$= \frac{-7 \pm \sqrt{529}}{12}$$

$$y = \frac{4}{3}, \frac{-5}{2}$$

$$y^2 = x = \tfrac{16}{9}, \tfrac{25}{4}$$

EXAMPLE 19. Solve the equation $x^6 + 7x^3 - 8 = 0$.

Factoring,

$$(x^3 - 1)(x^3 + 8) = 0$$

or

$$(x - 1)(x^2 + x + 1)(x + 2)(x^2 - 2x + 4) = 0$$

Hence the roots are

$$x = 1, \frac{-1 \pm \sqrt{-3}}{2}, -2, 1 \pm \sqrt{-3}$$

EXERCISE 10

Solve the following equations of the quadratic type.

1. $x^4 - 10x^2 + 9 = 0$
2. $z^4 - 13z^2 + 36 = 0$
3. $x^6 - 28x^3 + 27 = 0$
4. $y + 2y^{\frac{1}{2}} - 15 = 0$
5. $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 3 = 0$
6. $2w^{\frac{1}{2}} - 5w^{\frac{1}{4}} + 2 = 0$
7. $x^{-2} + x^{-1} - 6 = 0$
8. $z^{-1} + z^{-\frac{1}{2}} - 6 = 0$

10·18 Applications Leading to Quadratic Equations Containing One Unknown. In the following problems the student should decide whether one answer or both are consistent with the physical conditions of the problem. It is frequently interesting and instructive to determine the significance of the secondary answer.

Remember that a "check," unless made on the original stated problem, is no check at all.

EXERCISE 11

1. Find two numbers whose sum is 23 and whose product is 126.
2. Separate 19 into two parts whose product is 84.
3. The difference of two numbers is 7 and their product is 78. Find the numbers.
4. Separate 132 into two parts such that one part is the square of the other.
5. The sum of a number and its square is 72. Find the number.

6. Find two consecutive positive integers whose product is 156.
7. Find two consecutive positive even integers whose product is 224.
8. Find two consecutive positive odd integers whose product is 195.
9. Find a number whose square exceeds 36 by as much as 36 exceeds the number.
10. What number added to its reciprocal equals 2.9?
11. How high is a tree if it takes 3 sec for a stone thrown over it to return to the ground? Use the formula $s = 16t^2$, where s is the distance in feet covered by a freely falling body in t seconds (starting from rest).
12. A bomber traveling at 375 mph releases a bomb at 8,200 ft. How long after releasing the bomb will it land? How far ahead of the target is the bomb released? Neglect air resistance.
13. If the cross-sectional area of the angle beam in Fig. 10·4 is $6\frac{1}{4}$ sq in., find the thickness x.

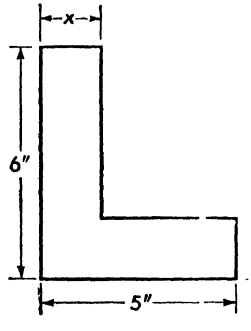


Fig. 10·4

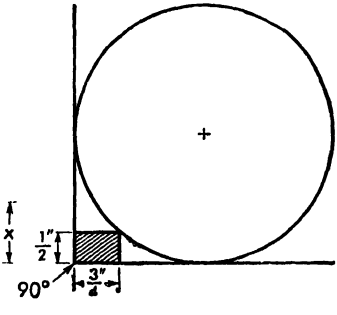


Fig. 10·5

14. A group of boys bought a canoe for \$70, planning to divide the expense equally. However, two boys dropped out, increasing the share of each boy by \$1.75. How many boys were there in the original group?
15. A ball is thrown upward from a building 380 ft high with a speed of 112 fps. Under these conditions the height h of the ball in feet at any time t seconds after throwing the ball is given by the equation $h = -16t^2 + 112t + 380$. When will the ball be 476 ft above the ground?
16. Repeat Prob. 15 for the time at which the ball is 252 ft above ground. Can you explain the secondary (negative) answer?
17. Find the diameter of the circle in Fig. 10·5.
18. Two resistances in parallel have a joint resistance of 4.2 ohms. The same two resistances in series have a resistance of 20 ohms. Find the value of each resistance.

19. A park is 480 yd long by 320 yd wide. It is decided to double its area, retaining the rectangular shape, by adding strips of equal width to one end and one side. Find the width of the strip.

20. A man bought two farms for \$3,600 each. The larger, which contained 15 acres more than the smaller, cost \$8 an acre less than the smaller. How many acres did each contain?

21. Telephone poles are placed at equal intervals along a road. If 2 fewer poles per mile were used, the intervals between poles would have to be increased by 24 ft. Find the original number of poles per mile.

22. The perimeter of a rectangular field is 274 yd and the diagonal is 97 yd. Find its dimensions.

23. A triangle has a 12-in. altitude and a 20-in. base. How wide a strip should be cut off by a line parallel to the base to leave 55 sq in. at the top?

24. A 50-cp lamp and a 120-cp lamp are 30 ft apart. Find the point on a line between them which is equally illuminated by both lights. When an object is equally illuminated by two light sources, the light intensities vary directly as the squares of their respective distances from the object. What significance attaches to the secondary solution of this problem? What can you say about the locus of all such equally illuminated points (*a*) in the same plane, (*b*) in three-dimensional space?[1]

25. The sum of the areas of the two inner circles in Fig. 10·6 is $\frac{3}{4}$ the area of the outer circle. Find the diameter of the smallest circle.

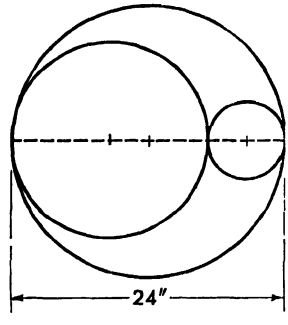


Fig. 10·6

26. Prove Eq. (24), Appendix B, for the length of a chord.

27. Referring to the triangle illustrating Eqs. (29–33), Appendix B,

[1] The relationship in this problem might have been restated to read that "the intensity of illumination varies inversely as the square of the distance from the light source." A similar relationship holds for other forms of energy or force, *e.g.*, heat, electricity, magnetism, and gravity.

denote the half base of the triangle by m. Express h and m in terms of s. Prove (*a*) Eq. (29), (*b*) Eq. (30), (*c*) Eq. (31), (*d*) Eq. (32), and (*e*) Eq. (33).

28. Referring to the hexagon illustrating Eqs. (33–44), Appendix B, express c and a in terms of s. Prove (*a*) Eq. (34), (*b*) Eq. (35), (*c*) Eq. (36), (*d*) Eq. (37), (*e*) Eq. (38), and (*f*) Eq. (39).

29. Referring to the octagon illustrating Eqs. (45–55), Appendix B, and to Fig. 4·20, page 39, confirm (*a*) Eq. (45), (*b*) Eq. (46), (*c*) Eq. (47), (*d*) Eq. (48), (*e*) Eq. (49), (*f*) Eq. (51), and (*g*) Eq. (55).

30. Derive the formula for the frustum of a cone, Appendix B, from Eq. (73), Appendix B.

31. A strip of metal 8 in. wide is to be bent into a trough of rectangular cross section (open top) whose cross-sectional area is to be $7\frac{1}{2}$ sq in. Find the depth and width of the trough. How many answers?

32. The outer portion of a garden 22 by 30 ft is to be occupied by a walk of uniform width. If the garden is to be reduced to $\frac{3}{4}$ of its original area, find the width of the walk.

33. How large a cube can be covered with 387 cu in. of insulation if the insulation is $1\frac{1}{2}$ in. thick? Edges and corners are not to be rounded.

34. An object is weighed on a platform balance and found to balance 236.0 g. When the positions of object and weights are reversed, the indicated weight is 249.0 g because of unequal balance arms. Find the true weight correct to 0.1 g.

35. When a flexible rope or wire t ft long is strung between two poles l ft apart (points of attachment at same elevation) the formula $t = \sqrt{l^2 + 5.3s^2}$ applies where s is the sag in feet. A copper telegraph wire is being installed at 60°F, the poles being 140 ft part. Find the necessary sag (in inches) in order to provide against a temperature of −20°F. The linear coefficient of expansion of copper is 0.00001 per Fahrenheit degree.

36. An express train makes the run between two cities 250 miles apart in $1\frac{1}{4}$ hr less time than a local whose speed is 10 mph less. Find the average speed of the express.

37. A motorboat takes 2 hr 8 min longer to make a trip of 48 miles up a stream than it takes on the return trip downstream. If the average rate of the current is 4 mph, find the rate of the boat in still water.

38. A bomb was released from a plane in a glide at an altitude of 20,000 ft. If the vertical component of the speed of the plane was 150 mph, find the time required for the bomb to strike the target.

$s = vt + \frac{1}{2}gt^2$, where s is the vertical distance in feet, v is the initial vertical component of the speed in feet per second, t is the time in seconds, and g is the gravitational constant (= 32 ft per sec per sec). Air resistance is neglected.

39. Find the radius of curvature in Fig. 10·7.

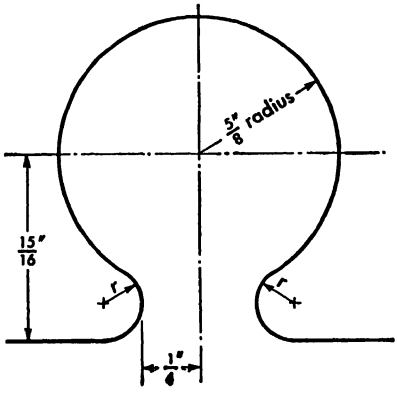


Fig. 10·7

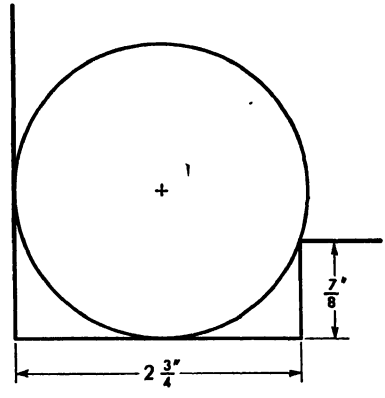


Fig. 10·8

40. Find the diameter of the circle in Fig. 10·8.

41. Find the radius R of the arc in Fig. 10·9.

42. A cube of cast aluminum has a volume shrinkage of 2 cu in. and a linear shrinkage of $\frac{1}{8}$ in. Find the original dimensions before cooling.

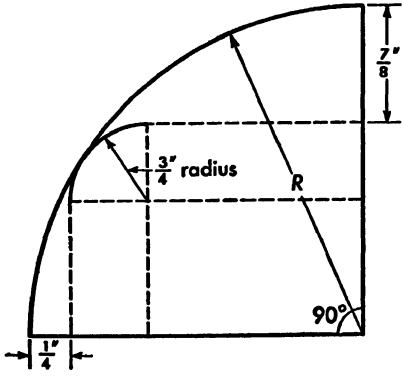


Fig. 10·9

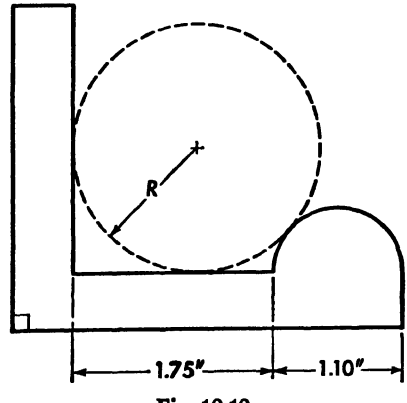


Fig. 10·10

43. Find the radius R of the cylindrical gauge in Fig. 10·10.

44. Two steamers ply between two ports 450 miles apart. One steamer runs $2\frac{1}{2}$ mph faster and takes $2\frac{1}{2}$ hr less for the trip than the other. Find their rates in miles per hour.

45. A man travels 30 miles by bus and returns by a train which runs 15 mph faster. If the total running time is 1 hr 57 min, find the rates of the bus and the train.

46. Find the length of the bar in Fig. 10·11 to balance a steam pressure of 18 lb per sq in. An 8-lb weight is attached at the free end of the bar. The bar weighs 1.2 lb per lin ft.

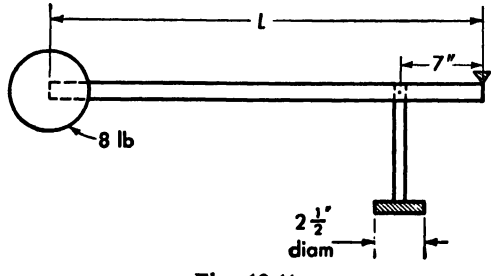


Fig. 10·11

47. A 2-ft walk surrounds a circular flower bed. If the area of the walk is $\frac{1}{10}$ the area of the flower bed, find the diameter of the bed.

48. A right triangle containing 210 sq ft is roped off by a line 70 ft long. Find the three sides of the triangle.

49. Six seconds after a stone is dropped into a mine shaft the sound of the impact at the bottom reaches the top. If the velocity of sound is 1,120 ft per sec and the usual formula for a freely falling body applies to the falling stone, find the depth of the shaft.

50. A flat disk $1\frac{3}{8}$ in. in diameter is to be pressed into an open-top cylinder $\frac{5}{8}$ in. deep. If the total surface of the metal is unchanged during the operation, find the diameter of the cylinder.

51. The hypotenuse of a right triangle is 10 ft longer than the shorter side and 5 ft longer than the longer side. Find the sides of the triangle.

52. A diesel-electric coach has an acceleration of 0.8 ft per sec per sec and a braking deceleration of 1.2 ft per sec per sec. If two stations are $1\frac{1}{4}$ miles apart, what is the maximum speed reached when the power is shut off and the brakes applied? What time is required between the two stops?

53. Ice on a power-transmission line may be melted off by increasing the voltage. If $\dfrac{E^2}{a(1 + bt)} = K(t - t_0)$, find the temperature t(°F) of the wire if the voltage E is 580; a is the resistance of the line at 0°F and is equal to 52 ohms; b is the temperature coefficient of resistance of the wire, or 0.003; K is the coefficient of heat

transfer by convection and conduction, or 350; and t_0 is the air temperature, or 24°F.

54. The specific heat of a gas at any Centigrade temperature t is $s = 0.8 + 0.0005t$. To what temperature will 1,000 cal heat 4 g of gas originally at 20°C? (First find an expression for the average specific heat of the gas over the temperature range involved.)

55. Find the diameter of the circle in Fig. 10·12.

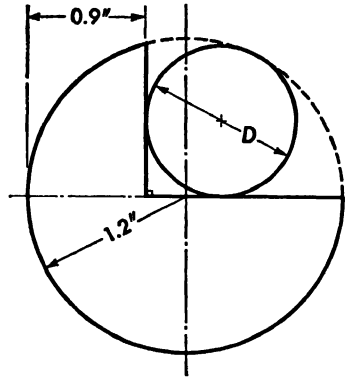


Fig. 10·12

56. Two boats leave simultaneously the opposite shores of a bay which is 2¼ miles wide and pass each other in 6 min. The faster boat completes the trip 4½ min before the other boat docks. Find the rates of the boats in miles per hour.

57. The specific heat of iron is given as $0.0001135t + 0.1055$, where t is the temperature in Centigrade degrees. A block of iron weighing 60 g is transferred from a heat-treating furnace to 500 g of water. The water temperature rises from 18.0 to 26.4°C. Find the temperature of the furnace. (See Prob. 54.)

58. An open-top box is to be made from a 15-in.-square sheet of brass by cutting out squares of a certain size from each corner and turning up the resulting tabs. It was found that the capacity of the box would remain unchanged whether the squares to be cut out were of a given size or ½ in. larger. What was the smaller size of square cut out? What was the capacity of the box?

59. In the trapezoid shown in Fig. 10·13 find x such that a line drawn parallel to the base shall divide the area in half.

60. The length L of a parabolic bridge cable of span a and sag h is given approximately by the formula $L = a[1 + \frac{8}{3}(h/a)^2 - \frac{32}{5}(h/a)^4]$. Determine the sag if a 200-ft cable is suspended from two points 198 ft apart and in the same horizontal plane ($a =$

198). Solve first for h/a, then for h, (**a**), by using the entire formula, (**b**) by neglecting the term containing $(h/a)^4$.

61. If the radius of the earth is 3,960 miles, h is the elevation of the observer (feet above sea level), and d is the distance in miles of the horizon at sea, derive an equation for d in terms of h (see Fig. 10·14). In this approximation a term is dropped, since its importance is considered to be less than that of light refraction. Show that this assumption is probably justified for such values of h as 100 ft, 1,000 ft, or even 5 miles.

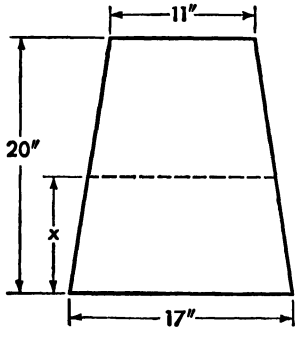


Fig. 10·13

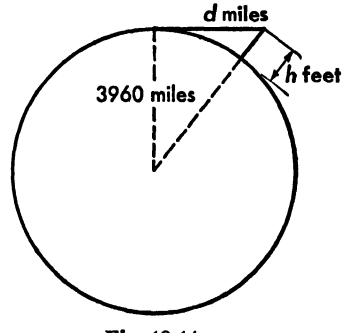


Fig. 10·14

62. The following is an approximate rule[1] for obtaining the variation in the size of a tube corresponding to a given amount of side play in the calipers (see Prob. 89, page 42). The rule has the merit of extreme simplicity and can be applied equally well to all diameters, except the very smallest. In most cases, the calculation is so simple that it can be done without recourse to pencil and paper.

A = side play of calipers or end-measuring rod, in sixteenths of an inch

B = dimensions to which calipers are set, or length of measuring rod, in inches

C = difference between diameter of tube and length of B, in thousandths of an inch

Then

$C = A^2/2B$, within a very small limit

Show that the above approximation is valid when A is relatively small, say not over one-eighth the diameter of the tube.

Specifically, find the error in the tube diameter, as calculated by this approximation, when $A = \frac{3}{8}$ in. and $B = 3.5000$ in.

[1] F. H. Colvin and F. A. Stanley, "American Machinists' Handbook," 8th ed., McGraw-Hill Book Company, Inc., 1945, p. 721.

10·19 Equations involving Radicals. In the simplest cases the equation may be solved by squaring both sides to remove the radical sign. In some other cases the solution can be effected by making substitutions according to the method of Sec. 10·17, Example 18. Regardless of the method chosen, each root should be tested by substituting in the original equation. Any root not satisfying the original equation is extraneous and should be rejected.

EXAMPLE 20. Solve the equation $2\sqrt{x+4} = x + 1$.

Squaring both sides,

$$4(x + 4) = (x + 1)^2$$

or

$$x^2 - 2x - 15 = 0$$

whence

$$x = 5 \quad \text{and} \quad x = -3$$

Only the root $x = 5$ satisfies the original equation, since $2\sqrt{9} = 5 + 1$, but $2\sqrt{1} \neq -3 + 1$.

If we solve the foregoing example by the substitution method, we let $u = \sqrt{x+4}$ and $u^2 = x + 4$. Then the equation

$$2\sqrt{x+4} = x + 1$$

becomes

$$2u = u^2 - 3$$

or

$$u = 3 \text{ or } -1$$

whence $x + 4 = 9$ or 1 and $x = 5$ or -3 as before.

If the expression contains more than one radical, transpose so that the most complicated radical expression is alone on one side. Square both sides of the equation, combine terms, and repeat the foregoing process until the resulting equation contains no radicals involving the unknown. Solve and check as before.

EXAMPLE 21. Solve the equation $3\sqrt{x} - \sqrt{5x-4} = 2$.

Transposing, we get

$$3\sqrt{x} - 2 = \sqrt{5x-4}$$

Squaring both sides,

$$9x - 12\sqrt{x} + 4 = 5x - 4$$

Transposing, we have

$$4x + 8 = 12\sqrt{x}$$

Dividing by 4,

$$x + 2 = 3\sqrt{x}$$

Squaring both sides,

$$x^2 + 4x + 4 = 9x$$

or

$$x^2 - 5x + 4 = 0$$

whence

$$x = 4 \quad \text{and} \quad x = 1$$

Both roots satisfy the original equation.

EXERCISE 12

In Probs. 1 to 14 find the real roots.

1. $\sqrt{3x + 7} = x + 1$
2. $\sqrt{x^2 - 28} - 6 = 0$
3. $\sqrt{x + 7} + 2 = \sqrt{3x - 5}$
4. $\sqrt{5x - 6} = \sqrt{3x + 7} + 1$
5. $\sqrt{3x + 4} + \sqrt{4x + 1} = 1$
6. $\sqrt{x^2 + 24} - \sqrt{x^2 - 21} = 5$
7. $\sqrt{x + 10} - \sqrt{x + 1} - \sqrt{x - 14} = 0$
8. $\sqrt{2x + 5} - \sqrt{4x + 3} = 1$
9. $\sqrt{x - 3} = \sqrt{x + 42} - \sqrt{3x + 4}$
10. $\sqrt{x + 2} + \sqrt{x} = \dfrac{3}{\sqrt{x + 2}}$
11. $\sqrt{2x - 2} + \sqrt{3x - 2} = \dfrac{4x + 9}{\sqrt{3x - 2}}$
12. $(x^2 + 4x - 7) - 4\sqrt{x^2 + 4x - 7} = 5$
13. $3x^2 + 5x - 4\sqrt{3x^2 + 5x + 3} = 2$
14. $2x^2 - 6x - 7\sqrt{x^2 - 3x - 2} = 8$

Solve the following problems, rejecting roots incompatible with the dimensions given.

15. Determine the distance c in Fig. 10·15.
16. Determine the radius r in Fig. 10·16.
17. Determine the distance m in Fig. 10·17.

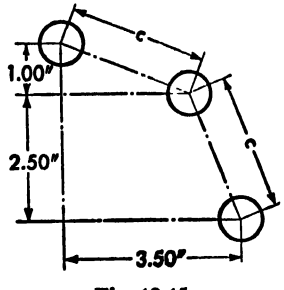


Fig. 10·15

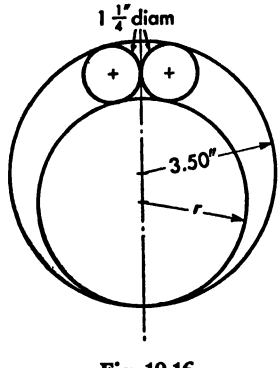


Fig. 10·16

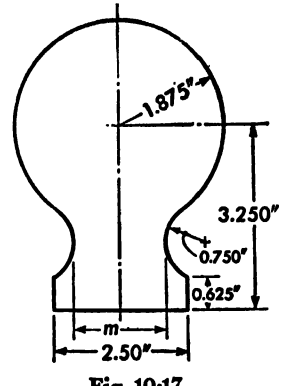


Fig. 10·17

18. Determine the radius r in Fig. 10·18.

19. Determine a side a in the equilateral triangle in Fig. 10·19.

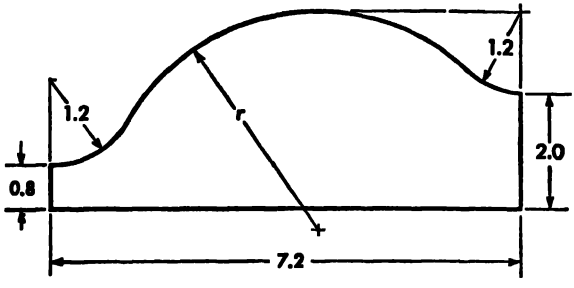


Fig. 10·18

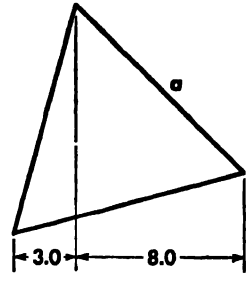


Fig. 10·19

20. Find the diameter of the small circle in Fig. 10·20.

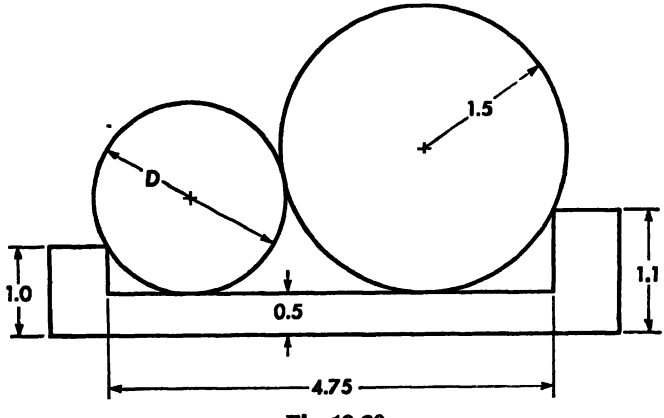


Fig. 10·20

10·20 Applied Problems in Maxima and Minima. A variety of interesting applications involving maxima and minima can be solved by making use of the principles set forth in Sec. 10·16.

EXAMPLE 22. The average attendance at a moving-picture theater at which the admission fee is 60 cents is 400. The manager has good reason to believe that with each reduction of 5 cents in the admission fee the attendance would increase by 50. What admission fee and what corresponding attendance would produce the maximum return? What is the maximum return for a performance?

Let n = the number of admissions and t = the cost of one admission (in cents). Then the total receipts $R = nt$. But

$$n = 400 + 50\frac{(60 - t)}{5} = 400 + 10(60 - t)$$

Therefore

$$R = t[400 + 10(60 - t)] = -10t^2 + 1{,}000t$$

Referring back to Eq. (17) (page 196), we read that if $y = ax^2 + bx + c$,

$$y_{(\text{max or min})} = c - \frac{b^2}{4a}$$

when $x = -\frac{b}{2a}$.

In our illustration, $y = R$, $x = t$, $a = -10$, $b = 1{,}000$, $c = 0$. Therefore

$$R_{(\text{max or min})} = 0 - \frac{(1{,}000)^2}{4(-10)} = 25{,}000 \text{ cents, or } \$250$$

$$t = \frac{-1{,}000}{2(-10)} = 50 \text{ cents optimum admission charge}$$

The attendance is evidently $\frac{250}{0.50}$, or 500 people.

Note that this problem is not concerned at all with incidental factors such as increased operating costs or ability to take care of increased patronage.

EXERCISE 13

1. Divide 17 into two parts such that their product will be a maximum.
2. Find the dimensions of the largest possible rectangular area that may be enclosed by 140 ft of fencing.
3. The work done by exploding a mixture of 1 cu ft of water gas with v cu ft of air is $w = 84v - 3.15v^2$. What value of v will lead to the maximum value of w? What is the maximum value of w?
4. Show that if P is the perimeter of a rectangle, the maximum possible area is represented by a square whose side equals $P/4$ and whose area equals $P^2/16$.
5. A ball is thrown upward with a speed of 112 ft per sec from a building 340 ft high. If the height of the ball in feet h at any time t seconds after throwing is given by the equation $h = 340 + 112t - 16t^2$, find the maximum height reached.
6. The power delivered to an external circuit by a 32-volt generator whose internal resistance is 2 ohms is $32a - 2a^2$ watts, where a is the current in amperes. At what current will this generator deliver the maximum power?
7. What would be the diameter of the inner circles in Prob. 25 in Exercise 11 to make their combined area a minimum?
8. A long sheet of copper 22 in. wide is to be made into a gutter by turning strips up vertically along the two sides. How many inches should be turned up at each side to obtain the greatest carrying capacity?
9. The manager of a theater found that with an admission charge of 60 cents per person the average daily attendance was 2,000, while with every decrease of 5 cents the attendance increased by 250.

What admission price will provide the maximum daily receipts? What will be the maximum daily receipts?

10. A telephone company can get 1,200 subscribers at a monthly rate of \$5 each. It can get 60 more subscribers for each decrease of \$0.10 in the rate. What rate will yield the maximum gross income? What will the income be?

11. An impulse turbine with a peripheral speed of t ft per sec is driven by a jet of water flowing at the rate of w ft per sec. Under these conditions the power P generated is proportional to $t(w - t)$. If 12 kw are generated where $t = 120$ ft per sec and $w = 300$ ft per sec, find the turbine speed required to generate the maximum power. What is the maximum power? (w remains constant at 300 ft per sec.)

12. An automobile manufacturer estimates that he can sell 5,000 cars a month at \$1,800 each and that he can sell 800 more cars a month for each \$100 decrease in price. If each car costs the manufacturer \$500 to make, at what price should he sell to obtain the maximum profit?

13. A printer contracts to print 3,000 circulars or less at the rate of \$1 per hundred. If the number of circulars exceeds 3,000, he will deduct 1 cent per hundred on the whole contract for each hundred printed in excess of 3,000. For what number of circulars would the printer realize maximum gross receipts, and what would the receipts be?

14. A Norman window has vertical sides and a horizontal base, but the top is a semicircle. If the perimeter is 14 ft, what dimensions will lead to the largest possible area?

15. Road tests show that the gas mileage of a certain automobile may be expressed by the formula $M = \dfrac{20}{1 + 0.0043A}$, where M is the miles per gallon and A is the percentage by volume of alcohol in an alcohol-gasoline blend. The cost (c cents) of a gallon of such a blend may be expressed by the formula $c = 0.24(100 - A) + 0.14A$. Derive, in terms of A, a formula for the fuel cost F in cents per mile.

16. **a.** In Prob. 15 find the percentage by volume of alcohol resulting in the least economical performance in cents of fuel cost per mile. **b.** What percentage of alcohol must be contained in the alcohol-gasoline blend before any saving in fuel costs over that of straight gasoline is realized?

17. A 20-in. length of wire is to be cut into two parts, one of which is formed into a circle, the other into a square. Find the diameter of the circle and the side of the square if their combined area is to be a minimum.

Chapter 11 SIMULTANEOUS QUADRATIC EQUATIONS

There is no simple general algebraic solution for two simultaneous quadratic equations of the type

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = F$$

An approximate graphical solution may be obtained by plotting. This is also useful at any time as a rough check on the algebraic solution. The algebraic solutions will be limited to the simpler cases.

11·1 Characteristic Types of Quadratic Curves. In more advanced mathematics it is stated that the graph of any quadratic equation in two variables is always a parabola, ellipse, circle, hyperbola, two straight lines, or a single point.

The following statements are confirmed in analytical geometry:

Parabola. An equation in x and y that is linear in one variable and quadratic in the other and contains no xy term represents a parabola whose axis of symmetry is parallel to either the X axis or the Y axis (Fig. 11·1). Plotting of the parabola is discussed in Sec. 10·4.

Ellipse. An equation of the type

$$Ax^2 + Cy^2 = F$$

where A, C, and F all have the same sign and $A \neq C$, represents an ellipse symmetrical about both axes.

EXAMPLE 1. Graph the equation

$$4x^2 + 9y^2 = 100 \text{ (Fig. 11·2)}$$

In order to outline the curve roughly, find the intercepts as follows:

Substitute $y = 0$, obtaining

$$4x^2 = 100$$

whereby

$$x = \pm 5$$

Substituting $x = 0$ in the original equation, we have

$$9y^2 = 100 \text{ or } y = \pm 10/3$$

If a more accurate curve is required, other substitutions may be made. For example if we let $x = \pm 3$, then $y = \pm 8/3$ giving us the four points A, B, C, and D.

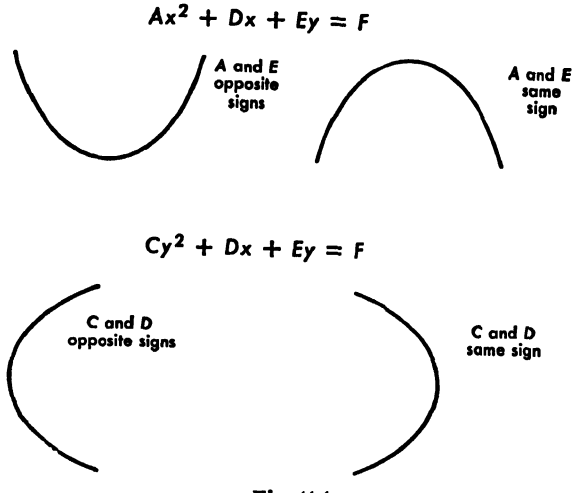


Fig. 11·1

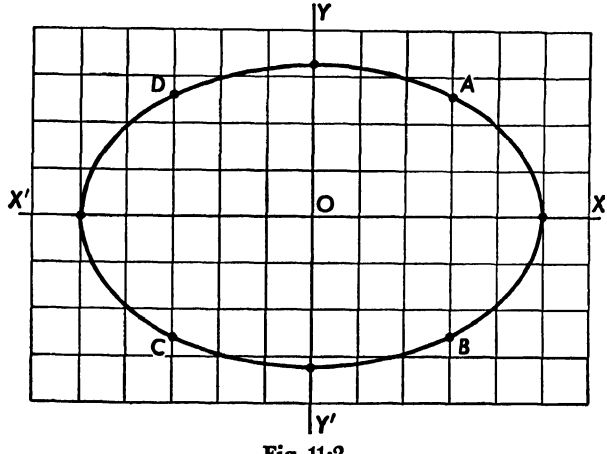


Fig. 11·2

Circle. An equation of the type $Ax^2 + Cy^2 = F$, where A, C, and F all have the same sign and $A = C$, represents a circle of radius $\sqrt{F/A}$ with its center at the origin. (NOTE: the circle is a special case of the ellipse.)

EXAMPLE 2. Graph the equation

$$4x^2 + 4y^2 = 81 \text{ (Fig. 11·3)}$$

Dividing through by 4 we obtain

$$x^2 + y^2 = 81/4$$

or

$$\sqrt{x^2 + y^2} = 9/2$$

Since the distance from the origin to any point (x, y) is $\sqrt{x^2 + y^2}$ (Sec. 7·13), the graph is the locus of all points whose distances from the origin

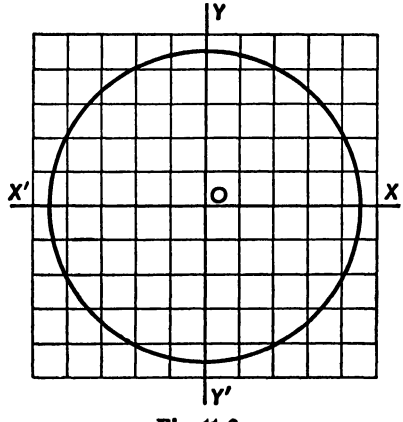


Fig. 11·3

are $9/2$. Hence we have a circle of radius $9/2$ with its center at the origin. Obviously this curve is best drawn with a compass.

Two Intersecting Straight Lines. An equation of the type

$$Ax^2 + Cy^2 = 0$$

where A and C have opposite signs, represents a pair of straight lines intersecting at the origin.

EXAMPLE 3. Plot the locus of

$$9x^2 - 25y^2 = 0 \tag{1}$$

Factoring (1) we obtain

$$(3x + 5y)(3x - 5y) = 0 \tag{2}$$

This equation may be transformed into two equations (Example 1, page 181):

$$3x + 5y = 0$$
$$3x - 5y = 0$$

These straight lines constitute the locus of Eq. (1). They are drawn as dotted lines in Fig. 11·4.

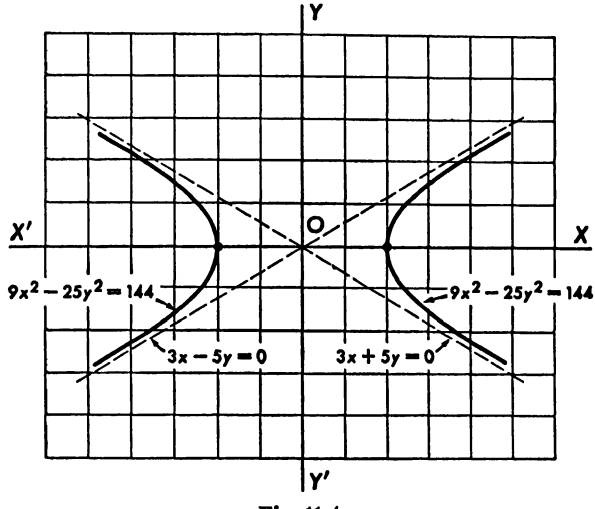


Fig. 11·4

Hyperbola. An equation of the type

$$Ax^2 + Cy^2 = F$$

where A and C have opposite signs and $F \neq 0$, represents a hyperbola symmetrical about the coordinate axes.

EXAMPLE 4. Plot the locus of

$$9x^2 - 25y^2 = 144 \text{ (Fig. 11·4)} \tag{3}$$

Substituting $y = 0$ we find $x = \pm 4$

If we attempt to find the y = intercepts by substituting $x = 0$ in Eq. (3) we find $y = \pm 1\frac{2}{5}\sqrt{-1}$, which, being an imaginary quantity, cannot be plotted.

If we solve Eq. (3) for y we obtain

$$y = \pm \tfrac{3}{5}\sqrt{x^2 - 16}$$

It is apparent that the curve does not occur in the band between the lines $x = 4$ and $x = -4$ and that there is no y intercept.

Replacing the value of the constant F in Eq. (3) by 0 we obtain Eq. (1) discussed above and plotted as dotted lines in Fig. 11·4.

These lines are the *asymptotes* of the graph of Eq. (3). (An *asymptote* of a curve is a line which the curve approaches but never reaches as the absolute numerical value of one of the variables increases without limit.)

A hyperbola may be sketched roughly by locating x or y intercepts and using the asymptotes as guiding lines. As in our previous discussion, additional points may be obtained by substitution. For this purpose it is suggested that an equation such as (3) be converted to the form

$$x = \pm\tfrac{1}{3}\sqrt{25y^2 + 144}$$

The advantage of this transformation lies in the fact that all values of y will correspond to real points on the curve.

Simple forms of hyperbola are illustrated in Fig. 11·5.

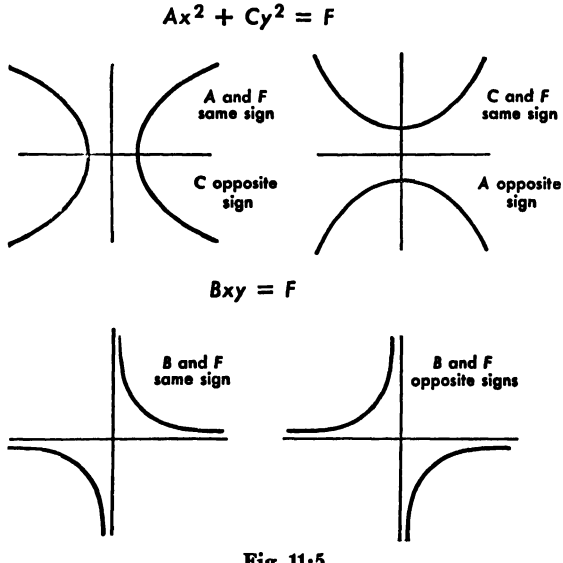


Fig. 11·5

11·2 Addition of Ordinates. Plotting of a curve can often be facilitated by the addition of ordinates. This process may be best illustrated by taking as an example the equation

$$y = \frac{x}{2} + \frac{6}{x}$$

Here we separately plot (Fig. 11·6)

$$y_1 = \frac{x}{2} \tag{4a}$$

$$y_2 = \frac{6}{x} \tag{4b}$$

Algebraic addition of ordinates gives us the equation

$$y_1 + y_2 = y = \frac{x}{2} + \frac{6}{x} \tag{4c}$$

while by graphical addition we obtain the graph of Eq. (4c).

It will be seen that large absolute values of x make $x/2$ the dominant term, while small absolute values of x make the term $6/x$ dominant.

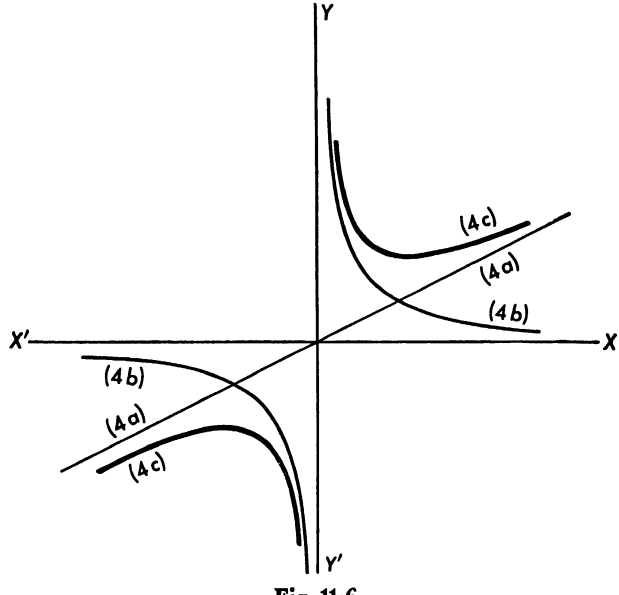


Fig. 11·6

The graph emphasizes this relationship in a slightly different way. The curves $y = x/2$ and $y = 6/x$ appear as the asymptotes.

The choice of scales will be dictated by the range of the variables to be covered. They may not be equal. In any event one scale division should equal 1, 2, or 5 times some integral power of 10.

EXERCISE 1

Plot the graphs of the following equations. Show the asymptotes, if any, and name the intercepts.

1. $y = -x^2 + 9$ **2.** $x^2 - y^2 = 25$ **3.** $y = 2x^2 + 6$
4. $x^2 + 4y^2 = 36$ **5.** $x^2 + y^2 = 25$ **6.** $y = 3x^2 - 12$
7. $y^2 - x^2 = 16$ **8.** $y = x^2 + 2x$ **9.** $9x^2 + y^2 = 16$
10. $4x^2 - y^2 = 9$ **11.** $3x^2 + 3y^2 = 33\frac{1}{3}$
12. $x^2 - 9y^2 = 0$ **13.** $y = x^2 - 3x - 4$
14. $x^2 - 9y^2 = 36$ **15.** $y = -x^2 + x + 6$

16. $\frac{x^2}{25} + \frac{y^2}{64} = 1$

17. $x^2 = (7 + y)(7 - y)$

18. $x = 4y^2 - 9$

19. $y^2 - 16x^2 = 25$

20. $x = -y^2 + 16$

21. $y^2 - 4x^2 = 0$

22. $9y^2 - 4x^2 = 36$

23. $x = y^2 - 4y$

24. $xy = 12$

25. $\frac{x}{8 + y} = \frac{8 - y}{x}$

26. $3x^2 - 48y^2 = 0$

27. $y = -\frac{10}{x}$

28. $\frac{x^2}{36} + \frac{y^2}{25} = 1$

11·3 Graphical Solution. An approximate graphical solution of a system of two quadratic equations (or one quadratic and one linear equation) may be obtained by plotting their loci. Estimate the coordi-

Four real distinct roots.

Four real roots—two identical.

Two real roots—two imaginary.

Four imaginary roots.

a. Two imaginary roots.
b. Two real identical roots.
c. Two real distinct roots.

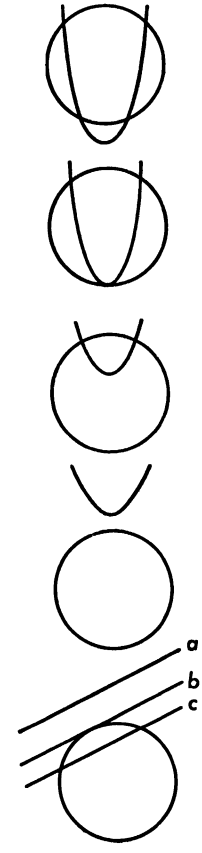


Fig. 11·7

nates of the points of intersection or tangency of the two curves. The coordinates of each such point, satisfying both equations, constitute a solution.

It is evident that two quadratic curves intersect in at most four points, while a system consisting of one linear and one quadratic curve has at most two real solutions (Fig. 11·7).

More accurate information may be obtained by plotting more points in the vicinity of the intersection, and often it will be advisable to plot the critical portion to a larger scale.

EXAMPLE 5. Solve graphically the system

$$x^2 - 4x + y^2 - 4y = 9 \tag{5}$$
$$y = x^2 - 3 \tag{6}$$

Rearranging Eq. (5) and solving for y, we have

$$\begin{aligned} y^2 - 4y &= -x^2 + 4x + 9 \\ y^2 - 4y + 4 &= -x^2 + 4x + 13 \\ y - 2 &= \pm\sqrt{-x^2 + 4x + 13} \\ y &= 2 \pm \sqrt{-x^2 + 4x + 13} \end{aligned} \tag{7}$$

By the usual process of substituting arbitrary values of x, solving for y, and plotting the pairs of values as coordinate points, we obtain a circle with center at 2, 2 (Fig. 11·8). Equation (6) is suitable for substitution of x values as it is.

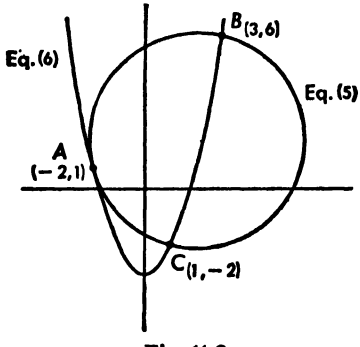


Fig. 11·8

The point of tangency A corresponds to two real identical roots. Hence the solution is

$$\begin{aligned} A \quad & x = -2, y = 1 \\ & x = -2, y = 1 \\ B \quad & x = 3, y = 6 \\ C \quad & x = 1, y = -2 \end{aligned}$$

EXERCISE 2

Solve the following systems graphically, estimating any fractional or irrational answers to two significant figures.

1. $xy = 6$
   $3x - y = 7$

2. $x^2 + y^2 = 20$
   $x - y = 6$

3. $y = x^2 - 4x - 1$
   $2x + y = -2$

4. $x^2 + 4y^2 = 36$
   $2x - 3y = 18$

5. $x^2 - y^2 = 9$
   $5x + 4y = -9$

6. $3x^2 + 3y^2 = 40$
   $3x + 2y = -18$

7. $x^2 + y^2 + 6x - 8y = 0$
   $16x^2 + 9y^2 = 144$

8. $xy = -12$
   $x^2 + y^2 = 25$

9. $y = x^2 - 4$
   $x = y^2 - 4$

10. $4x^2 + 9y^2 = 36$
    $9y - x^2 = 0$

11. $4y^2 - x^2 = 25$
    $x^2 + y^2 = 4$

12. $9x^2 + 25y^2 = 144$
    $xy = 6$

13. $x^2 + xy + y^2 = 7$
    $x + 4y = -1$

14. $x + y + 2xy = 25\frac{1}{6}$
    $5x - 2y = \frac{1}{3}$

11·4 Algebraic Solutions of Quadratic Systems. Since the graphical solution of a quadratic system leads to only approximate real solutions and yields no imaginary solutions at all, we shall require algebraic solutions for more accurate results. Our study of algebraic solutions will be confined to the simple types of systems.

11·5 Solution of Systems Consisting of One Linear and One Quadratic Equation. In order to solve a system consisting of one linear and one quadratic equation, solve the linear equation for one unknown in terms of the other and substitute the expression in the quadratic equation. Solve the resulting quadratic equation in one unknown and substitute each value in turn in the linear equation to find values of the unknown.

EXAMPLE 6. Solve the system

$$x^2 - 3xy - 2y^2 = 4 \tag{8}$$
$$4x + y = 5 \tag{9}$$

SOLUTION: Solve Eq. (9) for y:

$$y = 5 - 4x \tag{10}$$

(Note that we chose y in preference to x because of the simpler resulting expression.) Substituting Eq. (10) in Eq. (8), we obtain

$$x^2 - 3x(5 - 4x) - 2(5 - 4x)^2 = 4$$

or, simplifying,

$$19x^2 - 65x + 54 = 0 \tag{11}$$

Solving Eq. (11), we find

$$x = 2 \quad \text{and} \quad x = {}^{27}\!/_{19}$$

Substituting these values succcssively in Eq. (10), we obtain

$$y = -3 \quad \text{and} \quad y = -{}^{13}\!/_{19}$$

The solutions are therefore

$$x = 2, y = -3 \quad \text{and} \quad x = {}^{27}\!/_{19}, y = -{}^{13}\!/_{19}$$

EXERCISE 3

In each of the Probs. 1 to 10 solve for x and y.

1. $xy = 48$
$3x - y = 0$

2. $xy = -36$
$4y + x = 0$

3. $xy = 80$
$4x - 5y = 0$

4. $xy = -96$
$3x + 2y = 0$

5. $x^2 + y^2 = 25$
$x + y = -1$

6. $x^2 + y^2 = 100$
$x - y = 2$

7. $x^2 - y^2 = 40$
$x + 2y = 13$

8. $x^2 - 2y^2 = -34$
$5x - y = 15$

9. $x^2 + xy + y^2 = 12$
$5x + 2y = 2$

10. $5x^2 - xy = 15$
$3x + 2y = 26$

11·6 Equations of the Form $ax^2 + by^2 = c$. When both equations have the form $ax^2 + by^2 = c$, the system is linear in x^2 and y^2 and can be solved for x^2 and y^2 by the usual procedure applicable to systems of linear equations. From the values of x^2 and y^2, usually four solutions will be obtained which, if real, will correspond to four points symmetrically grouped about the origin.

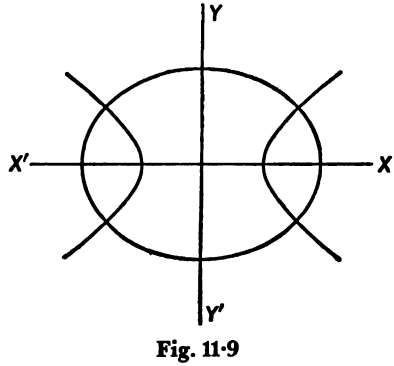


Fig. 11·9

EXAMPLE 7. Solve the system (Fig. 11·9)

$$5x^2 + 7y^2 = 47 \tag{12}$$
$$6x^2 - 10y^2 = 15 \tag{13}$$

SOLUTION: Multiply Eq. (12) by 6:

$$30x^2 + 42y^2 = 282 \tag{14}$$

Multiply Eq. (13) by 5:

$$30x^2 - 50y^2 = 75 \tag{15}$$

Subtract Eq. (15) from Eq. (14):

$$92y^2 = 207$$

Solving for y,

$$y = \pm\tfrac{3}{2}$$

Substitute $y^2 = \frac{9}{4}$ in Eq. (13):

$$6x^2 - 10(\tfrac{9}{4}) = 15$$
$$x = \pm\tfrac{5}{2}$$

There are four solutions:

$$x = \tfrac{5}{2}, y = \tfrac{3}{2} \qquad x = \tfrac{5}{2}, y = -\tfrac{3}{2}$$
$$x = -\tfrac{5}{2}, y = \tfrac{3}{2} \qquad x = -\tfrac{5}{2}, y = -\tfrac{3}{2}$$

EXERCISE 4

Solve for all values of x and y.

1. $3x^2 + y^2 = 12$, $x^2 + 2y^2 = 19$

2. $2x^2 - 3y^2 = 5$, $4x^2 + y^2 = 17$

3. $5x^2 - 2y^2 = 20$, $x^2 + 3y^2 = 4$

4. $2y^2 - 3x^2 = 18$, $x^2 + y^2 = 9$

5. $4x^2 + 4y^2 = 15$, $6x^2 - 5y^2 = 20$

6. $6x^2 + 6y^2 = 25$, $9y^2 - 5x^2 = 30$

7. $4x^2 - y^2 = 20$, $2y^2 - x^2 = 8$

8. $3x^2 + 8y^2 = 20$, $5x^2 + 5y^2 = 28$

11·7 Elimination of Constants. A quadratic system in two unknowns in which all terms containing the variables are of the second degree may often be solved by first eliminating the constant terms from the original system. The method is as follows:

1. *Using the method of addition or subtraction, combine the two equations, eliminating the constant terms and obtaining a single equation of the type*

$$Ax^2 + Bxy + Cy^2 = 0$$

2. *Factor this quadratic equation and express one of the unknowns as a multiple of the other.*

3. *Substitute the relations in step* 2 *successively in one of the original equations, and solve the resulting equations in one unknown.*

4. *Substitute in step* 2 *the values obtained in step* 3 *to determine the other unknown.*

EXAMPLE 8. Solve the system

$$2x^2 - 3xy + 2y^2 = 28 \tag{16}$$
$$4x^2 + 3xy - y^2 = 21 \tag{17}$$

Multiplying Eq. (17) by 4 and Eq. (16) by 3, we obtain;

$$16x^2 + 12xy - 4y^2 = 84 \tag{18}$$
$$6x^2 - 9xy + 6y^2 = 84 \tag{19}$$

Subtracting,

$$10x^2 + 21xy - 10y^2 = 0 \tag{20}$$

Factoring Eq. (20), we obtain

$$(5x - 2y)(2x + 5y) = 0 \tag{21}$$

From Eq. (21) we get two solutions:

$$x = \tfrac{2}{5}y \tag{22}$$
$$x = -\tfrac{5}{2}y \tag{23}$$

Substituting Eq. (22) in Eq. (16), we obtain

$$2(\tfrac{2}{5}y)^2 - 3y(2/5y) + 2y^2 = 28$$
$$8y^2 - 30y^2 + 50y^2 = 700$$

from which

$$y^2 = 25$$
$$y = \pm 5$$

Since $x = \tfrac{2}{5}y$, the corresponding values of x are ± 2. Substituting Eq. (23) in Eq. (16), we have

$$2(-\tfrac{5}{2}y)^2 - 3y(-\tfrac{5}{2}y) + 2y^2 = 28$$
$$25y^2 + 15y^2 + 4y^2 = 56$$

from which

$$y^2 = \tfrac{14}{11}$$
$$y = \pm\tfrac{1}{11}\sqrt{154}$$

Since $x = -\tfrac{5}{2}y$, the corresponding values of x are $\mp\tfrac{5}{22}\sqrt{154}$. Pairing the values of x and y, we obtain

From $x = \tfrac{2}{5}y$			From $x = -\tfrac{5}{2}y$	
x	2	-2	$\tfrac{5}{22}\sqrt{154}$	$-\tfrac{5}{22}\sqrt{154}$
y	5	-5	$-\tfrac{1}{11}\sqrt{154}$	$\tfrac{1}{11}\sqrt{154}$

EXERCISE 5

Solve by eliminating the constant terms.

1. $x^2 + xy = 3$
   $xy + y^2 = 10$

2. $4x^2 + 3xy + y^2 = 22$
   $x^2 + xy = 4$

3. $x^2 - 8xy + 17y^2 = 5$
$x^2 - 5xy + 9y^2 = 15$

4. $2x^2 + 3xy + 2y^2 = 64$
$x^2 - 6xy + 6y^2 = -8$

5. $(2x - 9y)(x - 2y) = -12$
$(2x + y)(x - 3y) = 16$

6. $x + y = \dfrac{20}{x}$
$x - y = \dfrac{3}{y}$

7. $3x^2 + 3xy + y^2 = 7$
$x^2 + 11xy + 5y^2 = 19$

8. $4x + 5y = \dfrac{84}{x}$
$3x + 16y = \dfrac{28}{y}$

11·8 Reduction to Simpler Systems. Two equations may often be combined in such a way as to result in one or more linear equations after which the method of Example 6, page 218, may be applied. In general, our objective is to obtain by addition or division an expression which is factorable or of lower degree. The method of elimination of constants is one example of this principle. In any case, the solutions should be tested by substitution in the original equations. Some typical solutions are illustrated in the following examples.

EXAMPLE 9. Solve the system

$$x^2 + xy = 10 \tag{24}$$
$$xy + y^2 = -6 \tag{25}$$

Factoring Eqs. (24) and (25), we obtain

$$x(x + y) = 10 \tag{26}$$
$$y(x + y) = -6 \tag{27}$$

Dividing Eq. (26) by Eq. (27),

$$\frac{x}{y} = \frac{10}{-6}$$
$$y = -\frac{3x}{5} \tag{28}$$

Substituting Eq. (28) in Eq. (24),

$$x^2 + x\left(-\frac{3x}{5}\right) = 10$$

whence

$$x = \pm 5$$

and from Eq. (28)

$$y = \mp 3$$

Hence the solution is

$$x = 5, y = -3 \qquad x = -5, y = 3$$

which checks the original equations.

EXAMPLE 10. Solve the system

$$x^2 - 3xy + y^2 = 59 \tag{29}$$
$$xy = -10 \tag{30}$$

Adding Eqs. (29) and (30), we have

$$x^2 - 2xy + y^2 = 49 \tag{31}$$

Extracting the square root of each side,

$$x - y = 7 \tag{32}$$
$$x - y = -7 \tag{33}$$

Multiplying Eq. (30) by 5 and adding to Eq. (29),

$$x^2 + 2xy + y^2 = 9$$

Extracting the square root of each side,

$$x + y = 3 \tag{34}$$
$$x + y = -3 \tag{35}$$

Combining Eqs. (30) and (32) by the method of substitution (Example 6, page 218),

$$x(x - 7) = -10$$

or

$$x = 5 \text{ or } 2 \qquad \text{and} \qquad y = -2 \text{ or } -5$$

Similarly combining Eqs. (30) and (33),

$$x(x + 7) = -10$$

or

$$x = -5 \text{ or } -2 \qquad \text{and} \qquad y = 2 \text{ or } 5$$

Combining Eqs. (30) and (34),

$$x(3 - x) = -10$$

or

$$x = 5 \text{ or } -2 \qquad \text{and} \qquad y = -2 \text{ or } 5$$

Combining Eqs. (30) and (35),

$$x(-3 - x) = -10$$

or

$$x = -5 \text{ or } 2 \qquad \text{and} \qquad y = 2 \text{ or } -5$$

We note that the last two combinations have given us no information not revealed by the first two. Hence the solution is

$$x = 5, y = -2$$
$$x = -5, y = 2$$
$$x = 2, y = -5$$
$$x = -2, y = 5$$

EXAMPLE 11. Solve the system

$$x^3 - y^3 = 91 \tag{36}$$
$$x^2 + xy + y^2 = 13 \tag{37}$$

Factoring Eq. (36) and dividing by Eq. (37),

$$\frac{(x - y)(x^2 + xy + y^2)}{x^2 + xy + y^2} = \frac{91}{13}$$

or

$$x - y = 7 \tag{38}$$

The solution may be completed by combining Eqs. (37) and (38) by substitution. The solution is

$$x = 3, y = -4 \qquad x = 4, y = -3$$

EXAMPLE 12. Solve the system

$$\frac{1}{x} + \frac{1}{y} = 1 \tag{39}$$
$$\frac{1}{x^2} + \frac{1}{y^2} = 41 \tag{40}$$

It is not advisable to clear of fractions. Instead we may temporarily substitute, say, a for $\frac{1}{x}$ and b for $\frac{1}{y}$ and write

$$a + b = 1 \tag{41}$$
$$a^2 + b^2 = 41 \tag{42}$$

Equations (41) and (42) may be solved by substitution, from which we obtain

$$a = 5 \text{ and } -4 \qquad \text{and} \qquad b = -4 \text{ and } 5$$

whence the solution is

$$x = \tfrac{1}{5}, y = -\tfrac{1}{4} \qquad x = -\tfrac{1}{4}, y = \tfrac{1}{5}$$

EXERCISE 6

Solve the following systems of equations by any convenient method.

1. $y(x + y) = 7$
$x - y = 5$

2. $y = x(x - 1)$
$y(y - 3) + x = 0$

3. $m^2 + 2mn + n^2 = 64$
$m^2 - 2mn + n^2 = 36$

4. $x = \dfrac{12}{y}$
$\dfrac{y}{x} = \dfrac{1}{3}$

5. $\dfrac{1}{y} = 1 - x$
$\dfrac{1}{x} = 4 - y$

6. $y = \dfrac{1}{4 - 9x}$
$15x = \dfrac{3y + 2}{y}$

7. $\dfrac{x}{4y} = \dfrac{y}{25x}$
$1 + \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{17}{xy}$

8. $\dfrac{1}{u} = \dfrac{3z + 1}{z}$
$\dfrac{1}{1 - u} + \dfrac{1}{1 - z} = \dfrac{18}{5}$

9. $x(x - y) = y^2$
$y(x + 1) = x^2$

10. $r^2 - t^2 = 24$
$rt = 35$

11. $m(m + n) = 20$
$n(n + m) = 16$

12. $\dfrac{u + w}{u - w} + \dfrac{u - w}{u + w} = \dfrac{10}{3}$
$u^2 + w^2 = 45$

13. $xy(3xy - 1) = 14$
$x + y + 1 = 0$

14. $w^3 + z^3 = 126$
$wz(w + z) = 30$

15. $n(m + n) + m(n + m) = 4mn$
$n(m + n) + m + n = 24$

16. $r^3 + t^3 = 152$
$rt = 15$

17. $\dfrac{u + 1}{z + 1} = \dfrac{2}{3}$
$\dfrac{z^2 + z + 1}{u^2 + u + 1} = \dfrac{31}{13}$

18.* $\dfrac{p - q}{\sqrt{p} - \sqrt{q}} = 8$
$\sqrt{pq} = 12$

19. $x^2 = 5x + 3y$
$y^2 = 3x + 5y$

20. $x(x - 5y) = -3.71$
$10x + y = 8.2$

*SUGGESTION: Let $a = \sqrt{p}$ and $b = \sqrt{q}$.

EXERCISE 7

1. The product of two numbers is 48, and the sum of their squares is 10 times the sum of the numbers. Find the numbers.
2. A fence enclosing a rectangular lot 60 yd wide by 80 yd long is torn down and, with 16 extra yards of fencing, is used to enclose a second lot having the same area as the first. Find the dimensions of the second lot.
3. The difference of two quantities equals their product. Their sum plus their product equals 1. Find the numbers.

4. By increasing its speed by 6 mph a train can cover its run of 168 miles in ½ hr less time. Find its original rate.
5. The diagram shown in Fig. 11·10 is the writing diagram for a

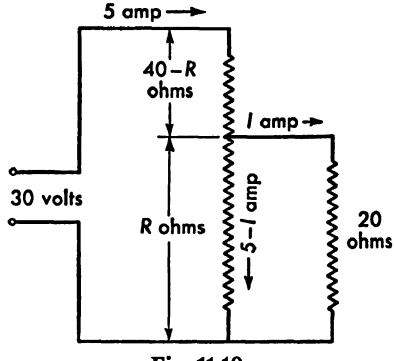


Fig. 11·10

potentiometer circuit. According to Kirchhoffs' law, the equations will be

$$30 - 5(40 - R) - (5 - I)R = 0$$
$$30 - 5(40 - R) - 20I = 0$$

Eliminate R from these equations and solve the resulting quadratic equation for I.

6. A group of men hiring a motorboat shared the cost equally. If there had been 4 fewer men in the group, they could have used a smaller boat whose rental was only three-fourths that of the first boat, and the individual charge would have been \$0.25 less. Had there been 9 more men in the group than originally, a larger boat renting for 10 per cent more than the first boat would have been required, but the individual expense would have been \$0.50 less than originally. How many men were there in the original group?
7. A motorboat whose speed in still water is 12 mph goes 6 miles down a stream and then heads up a tributary for 4½ miles. If the trip takes 52½ min and returning takes 1 hr 3 min, calculate the rates of the currents in the main stream and in the tributary.
8. A motorcycle messenger left the rear of a motorized troop column 10 miles long and rode to the head of the column, returning at once to the rear. The column advanced 24 miles during this time. Assuming uniform speeds for the column and messenger, how far did the messenger ride?

*

Chapter 12 RATIO, PROPORTION, VARIATION

The comparison of quantities, often indicated by their ratio, is a very common occurrence. In business it often takes the form of percentage (*e.g.*, profit, loss, discount, interest rate). In engineering, physics, and other sciences, we find it in such characteristics as specific gravity, specific heat, and atomic weight.

12·1 Quantities of the Same Kind. Quantities of the same kind may be compared (*a*) by stating the difference in magnitude of the two quantities or (*b*) by finding the quotient of the two quantities when expressed in the same units.

Thus in a group consisting of 100 men and 20 women we may say (*a*) that there are 80 more men than women or (*b*) that there are 5 times as many men as women.

The second type of comparison is called the *ratio* of the quantities. It is expressed as a fraction, in this illustration $\frac{100}{20}$, which is then reduced to lowest terms, $\frac{5}{1}$. We may say that the number of men is to the number of women as 5 is to 1.

If the numbers are prime to each other, the division indicated by the fraction is often carried out; *e.g.*, the rate of 56 to 39 is 1.436 to 1 (to four significant figures).

Note that the comparison must be made between like dimensions, *e.g.*, between a length and a length—not between, say, a length and a weight. The dimensions must be expressed in the same units; *e.g.*, the ratio between 2 lb and 12 oz is $\frac{32}{12}$ or $\frac{8}{3}$.

It will be seen that percentage simply expresses the ratio between a given number of units and 100 units of the same kind.

It should be noted also that the ratio of two like quantities has no units of its own—it is said to be dimensionless. A good illustration of

this is π, the ratio between the circumference and diameter of any particular circle.

EXERCISE 1

Express each of the following ratios in the simplest form:

1. 4 hr to 40 min
2. 5 lb to 10 oz
3. 8 in. to 2 yd
4. 330 ft to 1 mile
5. 480 sq in. to 1 sq ft
6. 6 cu ft to 1 cu yd
7. 13,200 sq ft to 1 acre
8. 1°7′30″ to 360° (angular measurement)
9. Circumference of a 9-in. circle to the circumference of a 2-ft circle
10. Area of an 8-in. square to the area of a 12-in. square
11. $6ax^2$ to $8a^2x$
12. $6xy - 4y^2$ to $9x^2 - 6xy$
13. $\frac{5}{8}$ to $\frac{15}{32}$
14. $3\frac{1}{3}$ to $6\frac{1}{4}$

12·2 Proportions. A statement of equality between two ratios is called a *proportion*. Thus $\frac{\$3}{\$2} = \frac{84 \text{ miles}}{56 \text{ miles}}$ is a proportion and is read "\$3 is to \$2 as 84 miles is to 56 miles." The four quantities are called the *terms* of the proportion. The first and fourth terms are called the *extremes*, and the second and third terms are called the *means*, of a proportion.

Among the properties of proportions, perhaps the most important may be listed in terms of the proportion

$$\frac{a}{b} = \frac{c}{d}$$

1. *The product of the extremes is equal to the product of the means. Thus* ad = bc.
2. *The terms are in proportion by inversion. Thus* b/a = d/c.
3. *The terms are in proportion by alternation. Thus* a/c = b/d.

Each of these statements may be confirmed by the ordinary operations of equations.

If the means of a proportion are equal, as in the proportion $\frac{m}{x} = \frac{x}{n}$, it is called a mean proportion and x is said to be the *mean proportional* between m and n.

A *continued proportion* between six or more quantities is illustrated by the expression $a:b:c = x:y:z$. This is a compact notation for expressing the simultaneous proportions

$$\frac{a}{b}=\frac{x}{y} \quad \frac{a}{c}=\frac{x}{z} \quad \frac{b}{c}=\frac{y}{z} \quad \text{or} \quad \frac{a}{x}=\frac{b}{y}=\frac{c}{z}$$

EXAMPLE 1. \$4,628.20, representing the profits of a venture, is to be divided among three partners in the ratio of 2:3:7. How much does each receive?

We may represent the amounts by $2x$, $3x$, and $7x$ since $2x:3x:7x = 2:3:7$. Therefore,

$$\begin{aligned} 2x + 3x + 7x &= \$4{,}628.20 \\ 12x &= 4{,}628.20 \\ x &= 385.683 \\ 2x &= 771.37 \\ 3x &= 1{,}157.05 \\ 7x &= 2{,}699.78 \end{aligned}$$

EXERCISE 2

In Probs. 1 to 16 solve the proportions for x:

1. $\frac{x}{6}=\frac{4}{3}$ **2.** $\frac{x}{8}=\frac{9}{16}$ **3.** $\frac{10}{x}=\frac{15}{4}$ **4.** $\frac{12}{x}=\frac{20}{9}$

5. $\frac{21}{10}=\frac{x}{25}$ **6.** $\frac{28}{15}=\frac{x}{9}$ **7.** $\frac{3}{4}=\frac{x}{14-x}$ **8.** $\frac{5}{3}=\frac{20-x}{x}$

9. $\frac{x+3}{x-3}=\frac{4}{3}$ **10.** $\frac{x-5}{x+5}=\frac{2}{7}$ **11.** $\frac{a}{b}=\frac{x}{c}$ **12.** $\frac{m}{n}=\frac{k}{x}$

13. $\frac{a}{b}=\frac{c-x}{x}$ **14.** $\frac{h}{k}=\frac{x}{k-x}$

15. $\frac{x+a}{x+b}=\frac{x-c}{x-d}$ **16.** $\frac{x-a}{x+b}=\frac{x+a}{x-k}$

In Probs. 17 to 26 find the mean proportional between

17. 6 and 24 **18.** 8 and 50 **19.** $2\frac{1}{2}$ and $\frac{5}{8}$

20. $8\frac{1}{3}$ and 12 **21.** $2a^2x$ and $8b^2x$ **22.** $\frac{x}{y}$ and $\frac{y}{x}$

23. x^2-y^2 and $\frac{4(x+y)}{x-y}$

24. $x^2-2xy+y^2$ and $x^2+2xy+y^2$

25. $\frac{18ab^3}{c}$ and $\frac{2ac}{b}$

26. $x+a$ and $x-a$ (Find the approximate value if a is very small relative to x.)

12·3 Variation. If two variables are so related that their ratio is constant, such as the relation $y/x = k$, we say that y *varies directly* as x, or simply y *varies* as x, or y is *proportional* to x. The constant k is called the *constant of proportionality*, or the constant of variation.

An equivalent notation is $y \propto x$, in which the symbol $\propto$ is read "varies as" or "is proportional to." This is *not* an equation but can be converted to the notation of an equation $y = kx$ by introducing the proportionality constant k.

For example, we say that the circumference of a circle varies as the diameter, or $C \propto D$, or $C = kD$. In this instance the proportionality constant $k = \pi$.

We note that the relation between y and x is a mutual one. That is, if y varies as x, then $y = kx$; but x is also proportional to y, or $x = (1/k)y$. Therefore, if y varies as x with the constant of proportionality equal to k, then x varies as y with the constant of proportionality equal to $1/k$.

Inverse Variation

If one variable varies directly as the reciprocal of a second, then the first is said to *vary inversely* as the second, or the first is said to be *inversely proportional* to the second. Thus $y = k \cdot 1/x$ or $y = k/x$. Since it is evident that $xy = k$, it follows that if the product of two variables is constant, either variable varies inversely as the other.

A common example of inverse variation is found in the inverse relationship between rate and time when a fixed distance is being covered. Here $rt = d$ or $t = d/r$. The constant of proportionality here is d.

Joint Variation

If a variable x varies directly as the product of y and z, *i.e.*, if $x = kyz$, we say that x *varies jointly* as y and z.

If $x = ky \cdot 1/z = ky/z$, it may be said that x varies directly as y and inversely as z.

Types of variation can be combined. For example, the electrical resistance of a wire varies directly as the length and inversely as the square of the diameter. Translated into algebraic notation, this may be written $R = kl/d^2$.

Solution of Variation Problems

Usually we know the type of variation involved and a set of corresponding values of the variables. From these data we can determine the proportionality constant k and also any one missing value in a second set of values of the variables.

EXAMPLE 2. We are given the relation that x is directly proportional to the square root of y and inversely proportional to z. Further, it is known that $x = 4$ when $y = 9$ and $z = 15$. Find the formula for x and find x when $y = 25$ and $z = 40$.

Expressing the relationship in algebraic notation, we have the equation of variation

$$x = \frac{k\sqrt{y}}{z} \tag{1}$$

Substituting in Eq. (1) $x = 4$, $y = 9$, and $z = 15$, we have

$$4 = \frac{k\sqrt{9}}{15} = \frac{3k}{15}$$

or

$$k = 20 \tag{2}$$

From Eqs. (1) and (2) we write the formula

$$x = \frac{20\sqrt{y}}{z} \tag{3}$$

To solve the last part of our problem, we substitute $y = 25$ and $z = 40$ in Eq. (3), obtaining

$$x = \frac{20\sqrt{25}}{40} = \frac{(20)(5)}{40} = 2.5 \tag{4}$$

The illustration points up the following rules:

1. *Translate the statement of variation into an equation involving an unknown constant of proportionality.*

2. *Solve for the proportionality constant by substituting given data.*

3. *Substitute the value of the constant of proportionality in the equation of variation.*

4. *Use the formula or equation in step* 3 *to obtain the missing value of a variable when a second set of values of the other variables is given.*

There will be occasions when there are insufficient data to evaluate k or when we are not interested in determining k. To continue with the illustration just given, we represent the first set of values by x_1, y_1, and z_1, and the second set by x_2, y_2, and z_2. Hence we obtain

$$x_1 = \frac{k\sqrt{y_1}}{z_1} \tag{5}$$

and

$$x_2 = \frac{k\sqrt{y_2}}{z_2} \tag{6}$$

Dividing Eq. (5) by Eq. (6),

$$\frac{x_1}{x_2} = \frac{z_2\sqrt{y_1}}{z_1\sqrt{y_2}} \tag{7}$$

Substituting values in Eq. (7),

$$\frac{4}{x_2} = \frac{40\sqrt{9}}{15\sqrt{25}}$$

$$\frac{4}{x_2} = \frac{120}{75}$$

$$x_2 = 2.5$$

It should be noted the computation in Eq. (7) above will often be facilitated if we write it in the form

$$\frac{x_1}{x_2} = \frac{z_2}{z_1}\sqrt{\frac{y_1}{y_2}}$$

If, for example, $y_1 = 63$ and $y_2 = 28$, the preference is obvious when we write

$$\frac{\sqrt{63}}{\sqrt{28}} = \sqrt{\frac{63}{28}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$$

Similarly, if the equation includes the expression $\frac{{w_1}^2}{{w_2}^2}$, the alternative form $\left(\frac{w_1}{w_2}\right)^2$ is often preferable. For example, suppose $w_1 = 15/32$ and $w_2 = 9/16$. Then

$$\frac{(15/32)^2}{(9/16)^2} = \left(\frac{15/32}{9/16}\right)^2 = (15/32 \times 16/9)^2 = (5/6)^2 = 25/36$$

The process of eliminating the proportionality constant may be used to demonstrate that variation and proportion are equivalent ideas.

Given

$$y_1 = kx_1 \tag{8}$$

and

$$y_2 = kx_2 \qquad (y \text{ varies directly as } x) \tag{9}$$

Dividing Eq. (8) by Eq. (9),

$$\frac{y_1}{y_2} = \frac{x_1}{x_2} \qquad (y \text{ is directly proportional to } x) \tag{10}$$

Given

$$y_1 = \frac{k}{x_1} \tag{11}$$

and

$$y_2 = \frac{k}{x_2} \qquad (y \text{ varies inversely as } x) \tag{12}$$

Dividing Eq. (11) by Eq. (12),

$$\frac{y_1}{y_2} = \frac{x_2}{x_1} \qquad (y \text{ is inversely proportional to } x) \tag{13}$$

12·4 Dimensionality of the Proportionality Constant. It is frequently instructive to determine the dimensionality of k or the units in which k is expressed (see page 90).

EXAMPLE 3. It is known that the weight of a rectangular block of wood varies jointly with the thickness, width, and length. If the weight is expressed in pounds and the dimensions in inches, determine in what units k is expressed.

The equation of variation is

$$W = ktwl$$

Solving for k,

$$\frac{W}{twl} = k$$

Substituting units,

$$\frac{\text{lb}}{(\text{in.})(\text{in.})(\text{in.})} = \frac{\text{lb}}{\text{in.}^3}$$

Hence k is expressed in pounds per cubic inch—a density figure.

EXAMPLE 4. The volume of a given quantity of a gas varies inversely as the pressure (temperature remaining constant). Determine the units of k if volume is expressed in cubic inches and pressure in pounds per square inch.

The equation of variation is

$$V = \frac{k}{P} \qquad \text{or} \qquad PV = k$$

Substituting units,

$$\left(\frac{\text{lb}}{\text{in.}^2}\right)(\text{in.}^3) = (\text{in.})(\text{lb})$$

Hence k is expressed in inch-pounds—a work unit.

EXERCISE 3

Express each of the relations in Probs. 1 to 5 by an equation containing an unknown constant of proportionality.

1. W varies jointly as x and y.
2. Q varies directly as x and inversely as y.

3. V varies directly as the cube of x and inversely as d.
4. M varies directly as b and inversely as the square root of c.
5. R varies directly as w and the square root of x and inversely as the cube of h.

In Probs. 6 to 10 write the formula for the first variable in terms of the other variables and the computed value of k.

6. H varies directly as x. $H = 8$ when $x = 20$.
7. N varies inversely as y. $N = 20$ when $y = 0.35$.
8. Q varies jointly as a, b, and c. $Q = 300$ when $a = 3$, $b = 7.5$, $c = 8$.
9. V varies directly as m and inversely as the square of t. $V = 2$ when $m = 15$ and $t = 6$.
10. R varies directly as the fourth power of T and inversely as the square root of x. $R = \frac{1}{3}$ when $T = 2$ and $x = 36$.

EXERCISE 4

In Probs. 1 to 7 determine the numerical value and the units of k that refer to the units in which the first set of values of the variables is expressed. Solve for the unknown value of the variable in the second set of values.

1. P varies inversely as V. If $V = 30$ cu in. when $P = 84$ psi, find V when $P = 63$ psi.
2. R varies directly as l. If $R = 6.8$ ohms when $l = 23.5$ ft, find R when $l = 31.8$ ft.
3. v varies directly as t; if $v = 45$ fps when $t = 25$ sec, find v when $t = 1$ min.
4. W varies directly as d^2. If $W = 12$ oz when $d = 8$ in., find W when $d = 1$ ft.
5. N varies inversely as d^2. If $N = 10{,}890$ plants per acre when set $d(= 2$ ft) apart, find N when $d = 5\frac{1}{2}$ ft.
6. m varies inversely as d. If $m = 12$ men when $d = 10$ days, find m when $d = 8$ days.
7. v varies jointly as the square root of g and the square root of h. If $v = 3.8$ fps when $g = 32$ ft per sec^2 and $h = 0.17$ ft, find v when $g = 30$ ft per sec^2 and $h = 8$ in.

In Probs. 8 to 12 determine the unknown quantity without solving for k.

8. C varies directly as d^2. If $C = 80$ when $d = 12$, find C when $d = 15$.
9. v varies directly as $\sqrt{h}$. If $v = 28$ when $h = 3$, find v when $h = 12$.

10. R varies directly as l and inversely as d^2. If $R = 35$ when $l = 110$ and $d = 0.006$, find R when $l = 75$ and $d = 0.004$.

11. V varies directly as r^4 and p, and inversely as l. If $V = 120$ when $r = 0.012$, $p = 20$, and $l = 30$, find V when $r = 0.016$, $p = 36$, and $l = 25$.

12. a varies directly as v^2 and inversely as r. If $a = 540$ when $v = 84$ and $r = 5$, find a when $v = 119$ and $r = 4$.

12·5 Applications. A few of the more common principles of mathematics and the physical sciences that may be expressed as variations are as follows:

Areas of similar figures vary as the squares of corresponding dimensions.

Volumes of similar solids vary as the cubes of corresponding dimensions.

Volumes of gases vary inversely as the absolute pressure and directly as the absolute temperature.

In any given chemical reaction between substances A and B the reacting amount of A varies directly as the reacting amount of B.

The time required to finish a given job varies inversely as the number of men working on the job.

The rate of energy reception (heat, light, magnetism, etc.) varies inversely as the square of the distance from the source of energy.

The revolutions per minute of two pulleys belted together vary inversely as their diameters.

The revolutions per minute of two gears in mesh vary inversely as the number of teeth.

Rate of heat conduction through a flat plate varies jointly as the area of one face of the plate and the difference between the temperatures of the opposite faces and inversely as the thickness of the plate.

EXERCISE 5

1. Hydrogen used for inflation of balloons may be made by passing steam over red-hot scrap iron. If 8.5 lb of iron will make 78 cu ft of hydrogen, how much iron would be needed to make 500 cu ft of hydrogen?

2. If 4,250 ft of $\frac{5}{32}$-in. wire has a resistance of 13.6 ohms, how long a wire $\frac{3}{16}$ in. in diameter will have a resistance of 4.2 ohms? In what units would the proportionality constant be expressed? The resistance of a wire of a given material varies directly as the length and inversely as the square of the diameter.

3. Seven men take eighteen 8-hr days to finish a job. How many men will be needed to finish a like job in twelve $7\frac{1}{2}$-hr days?

4. A train usually makes its run in 2 hr and 25 min at an average speed of 42 mph. How long would it take if the speed were reduced to 34 mph?
5. The weight of 195 machine screws is $8\frac{1}{2}$ oz. Find the number in 2 lb 7 oz.
6. The resistance of a spool of enameled magnet wire was 955 ohms. A piece 1 ft $9\frac{1}{2}$ in. long was cut off and found to have a resistance of 5.37 ohms. Find the length of wire originally on the spool.
7. From a 1:2:3 mixture of cement, sand, and gravel, 84 cu yd of concrete are to be made. How many cubic yards of each will be needed, allowing for a 20 per cent shrinkage on mixing?
8. Given $D = kWL^3/th^3$, where D is the deflection (in inches) of a beam loaded at the center and supported at the ends. k is a constant, L is the distance between the supports in feet, t is the width of the beam in inches, h is the depth in inches, and W is the load in pounds. If D is 4 when W is 250, L is 12, h is 3, and t is $2\frac{1}{2}$, find D when W is 400, L is 10, h is 4, and t is 2.
9. The air-line distance between two points is 235 miles. They are $5\frac{1}{8}$ in. apart on the map. What is the air-line distance between two points $3\frac{13}{16}$ in. apart on the same map?
10. The areas of two similar polygons are 162 and 242 sq in., respectively. If one side of the first polygon is 27 in. long, find the corresponding side of the second polygon.
11. A 250-ml Erlenmeyer flask has a height of 5.0 in. How high must a 500-ml Erlenmeyer flask be if it has the same shape as the smaller one?
12. A 16-in. disk cut from a piece of sheet steel weighs 5.65 lb. What will be the diameter of a disk weighing 2.08 lb cut from the same piece of stock?
13. A $2\frac{1}{2}$-in. cast-iron sphere weighs 1.87 lb. How much will a $3\frac{1}{2}$-in. cast-iron sphere weigh?
14. An inverted cone is 32 in. high and 18 in. in diameter at the top. If water flowing into the cone at a uniform rate reaches a depth of 12 in. in 5 min, how much longer would it take to reach a depth of 15 in.?
15. A piece of land is represented on a map by a rectangle $8\frac{5}{32}$ by $10\frac{5}{8}$ in. Find the area of the land in square feet if the scale is $\frac{3}{16}$ in. to the foot.
16. The outline of an estate is cut out from a map and found to weigh 42.78 g. A rectangular section 5 by 8 in. was cut from the same sheet and found to weigh 5.31 g. If the scale of the map is 1 in. to 150 ft, find the number of acres in the estate.
17. The gravity rate of flow of water from the bottom of a tank varies

directly as the square root of the depth h of the water in the tank. The rate of flow v was 20 gpm when the water was 9 ft deep.

a. Derive a formula for v in terms of h.

b. Find v when $h = 15$ ft.

c. Find h when $v = 12$ gpm.

18. Large oranges were selling at 50 cents a dozen at a fruit market. A sale on smaller oranges offered two dozen for 65 cents. What was the percentage of saving when buying the latter if the outside diameters were 3 and $2\frac{1}{2}$ in., respectively, assuming the rind to be $\frac{1}{4}$ in. thick in both cases?

19. A man 5 ft 4 in. tall weighs 140 lb. Another man, of about the same build, is just 6 ft tall. How much would you expect the taller man to weigh?

20. An 8-in. model of a 5-ft statue weighs $1\frac{1}{2}$ lb. If both are made of the same material, estimate the weight of the finished statue.

21. The distance of the horizon at sea varies directly as the square root of the elevation of the observer above sea level. If the horizon is $4\frac{1}{2}$ miles distant at $13\frac{1}{2}$ ft elevation, find the distance at 380 ft elevation.

22. In Prob. 21 how high above sea level must a lighthouse be to be visible 15 miles out to sea?

23. If extra-large eggs averaging 2 in. in length sell for 85 cents a dozen, find a comparable price for pullet eggs averaging $1\frac{1}{2}$ in. long.

24. Two sections of a regular hexagonal pyramid are made by parallel planes 8 in. apart. The areas of the sections are 175 and 63 sq in., respectively. How far from the vertex is the smaller section?

25. A cone of slant height 12 in. holds 3 qt of water. How far from the vertex should marks be placed on the slant height to indicate 1 qt? 2 qt?

26. A city of 50,000 people is supplied by a 25-in. water main. If a future population of 120,000 is anticipated within 30 years, what size main (to the nearest inch) will provide for this population?

27. The particle size of a powder averages 0.0006 in. How is the total surface affected by grinding to an average particle size of 0.0003 in.?

28. The intensity of illumination at a given point is directly proportional to the intensity of the light source and inversely proportional to the square of the distance from the light source. If a reader obtains adequate illumination with a 60-watt bulb 3 ft from the page, what size bulb will be needed to provide proper lighting at a distance of $4\frac{1}{2}$ ft?

29. The horsepower required to drive a motorboat varies as the cube

of the speed through the water. If 5 hp drives a boat at 10 mph, what size motor is needed to maintain a speed of 14 mph?

30. The profits of a partnership amounted to $1,283.67. This is to be divided among four partners in proportion to their investment. Johnson, Miller, Spencer, and Weston invested $842, $1,363, $1,759, and $1,876, respectively. Find each man's share of the profits to the nearest cent.

31. A pilot flew a glider plane a distance of 87 miles in 50 min. During this time he descended from 12,800 to 8,000 ft. How much longer could he have remained aloft, and how far would he have glided?

32. The air pressure in an automobile tire was tested after a summer day's drive and found to be 30-lb gauge. Assuming its temperature to be 120°F, what would the pressure be the next morning when the temperature is 60°F? (Absolute pressure varies directly as the absolute temperature, volume remaining constant.)

33. The analysis of a paint shows 46 per cent vehicle and 54 per cent pigment. The analysis of the pigment shows 15 per cent zinc oxide, 60 per cent titanium dioxide, and 25 per cent lithopone. What is the percentage of each pigment in the ready-mixed paint?

34. A lot of soda ash containing 52% by weight of water of crystallization is bought at 17½ cents a pound. When the material is sold at retail, the moisture content is found to have dropped in storage to 37 per cent. What should be the retail price per pound to realize a 40 per cent profit?

35. A formula calls for 22½ lb of soda ash as originally bought in Prob. 34. Find the proper amount of the material to use after partially drying in storage.

36. It required 18 sec to make a photographic enlargement with the light source 33 in. from the paper. How long would it take to make a larger print with the light source placed 41 in. from the paper? Length of exposure is inversely proportional to the intensity of illumination, which, in turn, is inversely proportional to the square of the distance from the light source.

37. The gravitational acceleration at the earth's surface may be taken as 32.2 ft per sec^2. What will be the value of the gravitational constant for a guided missile 150 miles above the earth's surface? Gravitational force is inversely proportional to the square of the distance from the center of the earth. Assume earth's radius = 3,960 miles.

38. In the equation $P_1V_1/T_1 = P_2V_2/T_2$, where P is the absolute pressure in pounds per square inch, V is the volume in cubic feet, and T is the absolute temperature (F° + 460), find the

volume of oxygen at 60°F and atmospheric pressure which may be stored in a steel tank at 100°F if the volume of the tank is 6.5 cu ft and it can safely withstand 820 psi gauge.

39. A 25-ft wire $\frac{1}{16}$ in. in diameter stretches 0.3 in. under a force of 15 lb. Find the tension required to stretch, by 0.5 in., a 15-ft wire $\frac{1}{32}$ in. in diameter. (Stretch varies directly as the force and original length and inversely as the square of the diameter.)

40. The United States makes 77 per cent of the world's automobiles yet contains only 7 per cent of the world's population. How many times as likely is a man to be employed in the motor industry in the United States as elsewhere in the world?

41. Given the formula $x = \dfrac{m\sqrt{f}}{W\left(a + \dfrac{b^2}{k}\right)}$. Indicate, quantitatively where possible, the effect on x of doubling each of the other letters one at a time.

42. The velocity of sound in air is independent of the density and pressure of the air and varies directly as the square root of the absolute temperature. An experimental unit carried on a rocket indicated a sound velocity of 977 fps at an altitude of 20 miles. What was the Fahrenheit temperature at that altitude if at 70°F the speed of sound in the air is 1,130 fps?

43. A car was stopped 60 ft from the point where the driver first sighted an obstruction. If the driver's mental reaction time was 0.4 sec, find the initial speed of the car. The braking distance is directly proportional to the square of the speed of the car and was 22 ft at 20 mph.

Chapter 13 THE BINOMIAL THEOREM—PROGRESSIONS

The binomial theorem and the progressions find application in the ready calculation of any power or root of a number and also serve as the basis of derivation of many formulas and tables used in financial transactions. These expressions introduce us to the series and the sequence. A *sequence* is a succession of quantities arranged in a definite order or pattern. A *series* is the indicated sum of the quantities comprising a sequence.

13·1 The Binomial Theorem. The binomial theorem enables us to expand any power of a binomial into a series of terms. Powers of binomials occur frequently, and the application of the theorem can effect a substantial saving in time and labor. An important application of the binomial expansion will be made in Chap. 24, in which we shall use it to develop equations occurring in alternating-current theory. The binomial expansion is also the basis of several useful approximations.

13·2 Expansion of a Positive Integral Power of a Binomial. By actual multiplication the student may verify that

$$(a + b)^0 = 1$$
$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

We see from the foregoing that the expansion of the general expression $(a + b)^n$ has the following properties:

1. *The expansion contains* n + 1 *terms.*

2. *The first term is* a^n, *and in each successive term the exponent of* a *decreases by* 1.

3. *The second term is* $na^{n-1}b$, *and in each successive term the exponent of* b *increases by* 1, *making the last term* b^n.

4. *In any term the sum of the exponents of* a *and* b *is* n.

5. *The coefficients of terms equidistant from the ends are equal.*

6. *If the coefficient of any term is multiplied by the exponent of* a *in that term and the product divided by the number of that term, the result is the coefficient of the next term.*

Application of the above rules leads to the following formula:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 \cdots b^n \quad \star (1)$$

The notation $r!$ is read "factorial r" and denotes the product of the first r integers. Thus $3! = 1 \cdot 2 \cdot 3$; $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$; etc.

In this text we shall assume without proof that the theorem holds not only for all positive integers but also for all rational numbers negative as well as fractional. When n is negative or fractional, we obtain an unending, or infinite, series, of which we compute as many terms as desired.

If we arrange in rows the coefficients of the expansion of a positive integral power of a binomial, we obtain the following arrangement, known as *Pascal's triangle.*

$n = 0$				1				
$n = 1$			1		1			
$n = 2$		1		2		1		
$n = 3$	1		3		3		1	
$n = 4$	1		4		6		4	1
$n = 5$	1	5		10		10	5	1
$n = 6$	1	6	15		20	15	6	1

It will be seen that the first and last coefficient in each row is 1 and that the second and next-to-the-last coefficients equal n. Any other coefficients may be obtained by adding the two nearest coefficients in the row above.

EXAMPLE 1. Expand $(2x + 3y)^4$.

Substituting in Eq. (1) $2x$ for a, $3y$ for b, and 4 for n, we obtain

$$(2x+3y)^4 = (2x)^4 + 4(2x)^3(3y) + \frac{4 \cdot 3}{2!}(2x)^2(3y)^2 + \frac{4 \cdot 3 \cdot 2}{3!}(2x)(3y)^3 + \frac{4 \cdot 3 \cdot 2 \cdot 1}{4!}(3y)^4 \quad (2)$$

$$= (2x)^4 + 4(2x)^3(3y) + 6(2x)^2(3y)^2 + 4(2x)(3y)^3 + (3y)^4 \quad (3)$$

$$= 16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$$

Note that if we had referred to Pascal's triangle, we could have written expression (3) directly. However, the utility of Pascal's triangle is restricted by inapplicability to expansions of negative and fractional powers.

EXAMPLE 2. Expand $\left(3x - \frac{1}{2x^2}\right)^6$.

Substituting, in Eq. (1), $3x$ for a, $\left(-\frac{1}{2x^2}\right)$ for b, and 6 for n, we obtain

$$\left(3x - \frac{1}{2x^2}\right)^6 = (3x)^6 + 6(3x)^5\left(-\frac{1}{2x^2}\right) + \frac{6\cdot 5}{2!}(3x)^4\left(-\frac{1}{2x^2}\right)^2$$
$$+ \frac{6\cdot 5\cdot 4}{3!}(3x)^3\left(-\frac{1}{2x^2}\right)^3 + \frac{6\cdot 5\cdot 4\cdot 3}{4!}(3x)^2\left(-\frac{1}{2x^2}\right)^4 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{5!}$$
$$(3x)\left(-\frac{1}{2x^2}\right)^5 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{6!}\left(-\frac{1}{2x^2}\right)^6$$
$$= 729x^6 - 729x^3 + \frac{1{,}215}{4} - \frac{135}{2x^3} + \frac{135}{16x^6} - \frac{9}{16x^9} + \frac{1}{64x^{12}}$$

Two significant points are brought out by Example 2. (*a*) If the terms of the binomial are at all complicated, each term should be enclosed in parentheses, with the proper coefficients and exponents as required by the binomial rule, after which the expansion may be computed. (*b*) If the terms of the binomial are opposite in sign and n is a positive integer, the terms of the expansion will alternate in sign, since odd powers of a negative quantity are negative and even powers are positive.

EXERCISE 1

Expand each expression by the binomial theorem.

1. $(x + 3y)^4$
2. $(2m + k)^5$
3. $(w^2 + \frac{1}{2}x)^6$
4. $(\frac{2}{3}x - 6)^4$
5. $\left(4x + \frac{1}{2x}\right)^5$
6. $(\frac{2}{3}x - \frac{3}{4}y)^5$
7. $\left(\frac{x}{y} + \frac{y}{z}\right)^6$
8. $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^6$
9. $(100 - 2)^3$

13·3 The General Term of the Binomial Expansion. In the binomial theorem [Eq. (1)] the term containing b^r, *i.e.*, the $(r + 1)$st term, is

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}a^{n-r}b^r \qquad \star (4)$$

There are several types of computation, notably in probabilities, where it is necessary to calculate one particular term. With the foregoing formula, any term of an expansion can be found without deriving all the preceding terms.

EXAMPLE 3. Find the term involving y^5 in the expansion of $(x + y)^8$.

Substituting $n = 8, r = 5, a = x$, and $b = y$, we have, from expression (4),

$$\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} x^3y^5 = 56x^3y^5$$

EXAMPLE 4. Find the term not containing x in the expansion of $(x^2 - 1/2x)^6$.

According to the conditions, $(x^2)^{6-r}(-\frac{1}{2}x^{-1})^r$ must produce a term containing no x, that is x^0. Therefore we may write

$$2(6 - r) + (-1)r = 0$$

or

$$r = 4$$

Hence we have

$$\frac{6 \cdot 5 \cdot 4 \cdot 3}{4 \cdot 3 \cdot 2 \cdot 1} (x^2)^{6-4} \left(-\frac{1}{2x}\right)^4 = 15x^4 \frac{1}{16x^4} = \frac{15}{16}$$

EXERCISE 2

Write the specified term in each of the following without calculating the preceding terms:

1. $(x + y)^9$; term in y^6
2. $(5 - x)^7$; term in x^4
3. $(w + 3z)^8$; term in w^5
4. $(2x - 3)^6$; term in x^4
5. $(x/2 - 2y/3)^{10}$; fourth term
6. $(3x/4 + y/3)^8$; fifth term
7. $(3x - 4y)^6$; middle term
8. $(5m + 2w)^4$; middle term
9. $(2/x + 3x)^8$; term containing no x
10. $(x - 3/x^2)^9$; term containing no x

13·4 Computation of Powers and Roots. Many powers and roots may be found by calculating as many terms in the series as needed to give the required accuracy. The smaller the second term in the binomial relative to the first, the fewer terms will be required. In computing a root, *i.e.*, a fractional power, the first term of the binomial should be a perfect power of the order corresponding to the index of the root.

EXAMPLE 5. Find $\sqrt[6]{1.04}$ to six significant figures.

Arranging in a form suitable for expansion, we have $(1 + 0.04)^{1/6}$. Substituting in the formula $a = 1$, $b = 0.04$, and $n = 1/6$, we obtain

$$(1)^{\frac{1}{6}} + \tfrac{1}{6}(1)^{-\frac{5}{6}}(0.04) + \frac{(\frac{1}{6})(-\frac{5}{6})(1)^{-1\frac{1}{6}}(0.04)^2}{2!}$$
$$+ \frac{(\frac{1}{6})(-\frac{5}{6})(-1\frac{1}{6})(1)^{-1\frac{7}{6}}(0.04)^3}{3!} + \cdots = 1 + 0.006667$$
$$-0.000111 + 0.000003 \cdots \text{(negligible terms)} = 1.00656$$

We note that the terms in the foregoing series become rapidly smaller. The series is said to converge rapidly and relatively few terms are needed.

EXAMPLE 6. Find $\dfrac{1}{(0.985)^3}$ to six significant figures.

Our problem is evidently to expand $(1 - 0.015)^{-3}$. Substituting in the formula $a = 1$, $b = -0.015$, and $n = -3$, we write

$$(1)^{-3} + (-3)(1)^{-4}(-0.015) + \frac{(-3)(-4)(1)^{-5}(-0.015)^2}{2!}$$
$$+ \frac{(-3)(-4)(-5)(1)^{-6}(-0.015)^3}{3!} + \cdots = 1 + 0.045$$
$$+ 0.00135 + 0.00003375 \cdots \text{(negligible terms)} = 1.04639$$

EXAMPLE 7. Find $\sqrt[3]{200}$ correct to six significant figures.

Since the perfect cube nearest to 200 is 216, our problem is to evaluate $(216 - 16)^{\frac{1}{3}}$. The expansion will be

$$(216)^{\frac{1}{3}} + \tfrac{1}{3}(216)^{-\frac{2}{3}}(-16) + \frac{(\frac{1}{3})(-\frac{2}{3})(216)^{-\frac{5}{3}}(-16)^2}{2!}$$
$$+ \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})(216)^{-\frac{8}{3}}(-16)^3}{3!} + \cdots = 6 - 0.14815$$
$$- 0.00366 - 0.00015 - \cdots = 5.84804$$

EXERCISE 3

In Probs. 1 to 6 compute the first four terms in each expansion.

1. $(1 + y)^{-1}$	**2.** $(1 + z)^{\frac{1}{2}}$	**3.** $(1 - x)^{-\frac{1}{4}}$
4. $(x^2 + 2)^{-\frac{1}{2}}$	**5.** $(8 - 3x)^{\frac{2}{3}}$	**6.** $(2w + 5)^{-2}$

In Probs. 7 to 15 compute each value to five significant figures, using no tables.

7. $\sqrt{50}$	**8.** $\sqrt{35}$	**9.** $\sqrt{1.5}$
10. $\sqrt[3]{9}$	**11.** $\sqrt[3]{60}$	**12.** $\sqrt[3]{120}$
13. $\sqrt[5]{35}$	**14.** $1/\sqrt{5}$	**15.** $(25)^{\frac{3}{5}}$

13·5 Arithmetic Progression. An *arithmetic progression* (abbreviated AP) is a succession of terms so arranged that each term differs from the term immediately preceding by a fixed amount called the *common difference*. The following are arithmetic progressions:

1. 7, 10, 13, 16, 19
2. 11, 7, 3, -1, -5
3. -4, $-2\frac{5}{6}$, $-1\frac{2}{3}$, $-\frac{1}{2}$, $\frac{2}{3}$, $1\frac{5}{6}$
4. $x - y$, x, $x + y$, $x + 2y$

The common difference is obtained by subtracting any term from the term immediately following it. Thus the common differences in the examples just given are

$$(1)\ 3,\ (2)\ -4,\ (3)\ 1\tfrac{1}{6},\ (4)\ y$$

The five *elements* of an AP and the letters usually denoting them are:

1. The first term a
2. The common difference d
3. The number of terms n
4. The last (or nth) term l
5. The sum of n terms S

Any three of the above elements are usually sufficient to determine the other two.

13·6 The nth Term of an AP. If a is the first term and d is the common difference, the second term is $a + d$. The third is $a + 2d$, and in general the nth is $a + (n - 1)d$. Hence, if l is the last or nth term, we have

$$l = a + (n - 1)d \qquad \star(5)$$

EXAMPLE 8. Find the 43d term of the AP 11, $12\frac{1}{2}$, 14, ...
Substituting 11 for a, 43 for n, and $1\frac{1}{2}$ for d, we obtain by Eq. (5)

$$l = 11 + (43 - 1)(1\tfrac{1}{2}) = 74$$

EXAMPLE 9. Find the 39th term of the AP 193, 187, 181, ...
Substituting 193 for a, 39 for n, and -6 for d, we have

$$l = 193 + (39 - 1)(-6) = -35$$

13·7 The Sum of the First n Terms of an AP. Let us consider an arithmetic progression of n terms, whose first term is a, last term is l, and common difference d. If we write the sum S first in the usual order, then in reverse order, we have

$$S = a + (a + d) + (a + 2d) + \cdots (l - 2d) + (l - d) + l \quad (6)$$
$$S = l + (l - d) + (l - 2d) + \cdots (a + 2d) + (a + d) + a \quad (7)$$

If we add Eqs. (6) and (7), all terms containing d cancel and we obtain

$$2S = (a + l) + (a + l) + (a + l) \cdots (a + l) + (a + l) + (a + l) = n(a + l)$$

Hence

$$S = \frac{n}{2}(a + l) \qquad \star (8)$$

EXAMPLE 10. Find the sum of an AP of 15 terms which begins with 19 and ends with 61.

Substituting in (8), $n = 15$, $a = 19$, and $l = 61$, we have

$$S = {}^{15}\!/_{2}(19 + 61) = 600$$

If we substitute in Eq. (8) the expression for l in Eq. (5), we obtain

$$S = \frac{n}{2}[2a + (n - 1)d] \qquad \star (9)$$

EXAMPLE 11. Find the number of terms in an AP whose sum is 126, common difference 11, and first term -30.

Substituting, in Eq. (9), $S = 126$, $a = -30$, and $d = 11$, we have

$$126 = \frac{n}{2}[-60 + (n - 1)11]$$
$$252 = n(11n - 71)$$
$$0 = 11n^2 - 71n - 252$$

The solution of this quadratic is $n = 9$, $n = -{}^{28}\!/_{11}$. Since we must have a whole number of terms, $-{}^{28}\!/_{11}$ is inadmissible and there are 9 terms.

EXAMPLE 12. Find the common difference in an AP of 17 terms whose first term is 22 and sum is 272.

Substituting, in Eq. (9), 17 for n, 22 for a, and 272 for S, we have

$$272 = {}^{17}\!/_{2}[2(22) + 16d]$$

or

$$d = -{}^{3}\!/_{4}$$

13·8 Arithmetic Means. The first term a and the last term l of an AP are called the *extremes*. The intervening terms are called *arithmetic means* between a and l. To insert k arithmetic means between a and l is to determine an AP of $k + 2$ terms having a as the first term and l as the last term. If we substitute $k + 2$ for n in Eq. (5), we have

$$l = a + (k + 1)d$$

or

$$d = \frac{l - a}{k + 1} \qquad \star (10)$$

EXAMPLE 13. Insert 5 arithmetic means between -19 and 23. Substituting, in Eq. (10), 23 for l, -19 for a, and 5 for k, we have

$$d = \frac{23 - (-19)}{5 + 1}$$

or

$$d = 7$$

and the AP is $-19, -12, -5, 2, 9, 16, 23$.

If a single arithmetic mean is to be inserted between a and l, it is called *the* arithmetic mean of a and l, or the average of a and l, and is equal to $\frac{a+l}{2}$.

If we divide both sides of Eq. (8) by n, we obtain

$$\frac{S}{n} = \frac{a+l}{2} \qquad \star (11)$$

or the average of all the terms of an AP, however many, is the average of the first and last terms. If m denotes the middle term of an AP, then $m - a = l - m$, or $m = \frac{a+l}{2}$. When m is substituted for $\frac{a+l}{2}$ in Eq. (8) we have

$$S = nm \qquad \star (12)$$

That is, the sum of an AP is equal to the number of terms multiplied by the middle term.

EXAMPLE 14. The sum of 81, 89, 97, 105, 113 is $(5)(97) = 485$.

EXAMPLE 15. Find the sum of 73, 67, 61, 55, 49, 43, 37, 31. The mid-point or hypothetical middle term is 52. Hence $S = (8)(52) = 416$.

EXERCISE 4

In Probs. 1 to 7 find the required terms:

1. 9th term of 2, 5, 8, . . .
2. 12th term of 3, 7, 11, . . .
3. 10th term of $-9, -7, -5, \ldots$
4. 16th term of 93, 86, 79, . . .
5. 12th term of $-\frac{3}{2}, -\frac{2}{3}, \frac{1}{6}, \ldots$
6. 8th term of $a - 3b, a - b, a + b, \ldots$
7. 7th term of $3x - y, x, y - x, \ldots$

In Probs. 8 to 13 find the sum of the indicated AP's.

8. 5, 8, 11 . . . to 13 terms
9. 23, 19, 15 . . . to 16 terms
10. $-11, -6, -1$ . . . to 21 terms
11. $a + b, a, a - b$ . . . to 11 terms
12. $\frac{1}{6}, \frac{4}{3}, \frac{5}{2}, \ldots$ to 14 terms
13. $5a - 3b, 4a - 2b, 3a - b$ . . . to 10 terms.

In Probs. 14 to 23 determine the missing quantities:

	S	n	a	l	d
14.			9	25	2
15.		12		64	6
16.		15	71		-4
17.		10	61	16	
18.	231			11	-2
19.	-132		11		-4
20.	84		5	19	
21.	80	8			4
22.	77	7		20	
23.	63	7	3		

In Probs. 24 to 33 find the arithmetic mean between

24. 3 and 13 **25.** 5 and 31 **26.** -2 and 15
27. -8 and 3 **28.** -17 and -9 **29.** $\frac{5}{6}$ and $\frac{3}{10}$
30. $-\frac{3}{8}$ and $\frac{1}{2}$ **31.** $2x$ and $2y$ **32.** $m + n$ and $m - n$
33. y and $1/y$

In Probs. 34 to 38 insert the required arithmetic means:

34. 3 arithmetic means between 5 and 29
35. 4 arithmetic means between 17 and 32
36. 6 arithmetic means between -9 and 26
37. 5 arithmetic means between -10 and 17
38. 4 arithmetic means between 13 and -5.

13·9 Applications of Arithmetic Progressions. In a problem involving an arithmetic progression, set down the first few terms of the progression. Evaluate as many as possible of the elements of the AP in terms of the data given. At times it may be advisable to divide the computation into two portions, one including the progression, the other representing a value easily found without using our AP formulas. Occasionally it will not be possible to meet the stated requirements exactly, in which case we shall have to seek the best possible compromise.

EXAMPLE 16. A home freezer costing \$560 is bought under an agreement to pay \$40 per month plus interest on the unpaid balance at the rate of 18 per cent. If the first payment is to be made one month after purchase of the unit, find: (*a*) the first payment, (*b*) the last payment, and (*c*) the total amount paid.

Since $560 \frac{9}{40} = 14$, there will be 14 payments made. For the time being we shall set aside the \$40 payments made on the principal and turn our attention to interest payments. At the end of the first month the interest payment will be

$$(0.18)(\tfrac{1}{12})(560) = (0.015)(560) = \$8.40$$

(*a*). Therefore the first payment will be \$40 + \$8.40, or \$48.40.

The unpaid balance during the second month will be \$520, and the interest payment at the end of the second month will be

$$(0.18)(\tfrac{1}{12})(520) = (0.015)(520) = \$7.80$$

Similarly, the interest payment at the end of the third month will be (0.015)(480); at the end of the fourth month, (0.015)(440); and so on.

(*b*). It is evident that the last payment will be \$40 on the principal and (0.015)(40) = \$0.60 interest, or a total of \$40.60.

(*c*). Clearly the interest payments will form the AP

$$(0.015)(560),\ (0.015)(520),\ (0.015)(480),\ \ldots\ (0.015)(40)\ (14\ \text{terms})$$

From Eq. (8), page 246,

$$S = \frac{n}{2}(a + l) = 14\tfrac{1}{2}[(0.015)(560) + (0.015)(40)] = \$63$$

Therefore the total amount paid will be 560 + 63, or \$623.

EXAMPLE 17. Raffle tickets are to be sold at all prices ranging from 1 cent to not more than 99 cents. It is desired to realize \$300 from the sale of tickets. How can this be done if the same number of tickets is sold at each price?

Let us first determine how much the sale of one ticket at each price from 1 cent to 99 cents would realize. According to Eq. (8), $S = \frac{n}{2}(a + l)$; in this example, $n = 99$, $a = 1$, and $l = 99$. Making these substitutions, we find $S = 4{,}950$, or \$49.50.

Since 300/49.5 = 6.1, we shall need to sell 7 tickets at each price from 1 cent to some figure under \$1. Therefore we shall have 7 identical AP's whose combined sum is 30,000 (\$300). To evaluate one of the AP's, we shall use Eq. (9) and substitute $S = \frac{30{,}000}{7}$, $a = 1$, and $d = 1$. Thus we obtain

$$\frac{30{,}000}{7} = \frac{n}{2}[2(1) + (n - 1)(1)]$$

or

$$n^2 + n = 8{,}571$$

Solving for n, we find n = approximately 92.1, or -93.1.

Only the positive root 92.1 has any application to this problem, and the next-higher integer, 93, indicates that each identical AP will contain 93 terms, with 93 cents as the highest-priced ticket. Using Eq. (8), we see that the combined sum of the 7 identical series will be

$$S = (7)(93\tfrac{1}{2})(1 + 93) = 30{,}597, \text{ or } \$305.97$$

EXERCISE 5

1. A machine having an initial value of \$1,450 depreciates in value each year \$50 less than during the preceding year. At the end of the sixth year it has a scrap value of \$190. What is its value at the end of the first year?
2. A ball rolling up an incline covers 24 ft during the first second, 21 ft during the next second, 18 ft during the next, etc. When will it be 99 ft (measured up the incline) from the starting point? (Two answers—explain.)
3. Find the sum of all the multiples of 6, beginning with zero and ending with 240.
4. If 250 raffle tickets are sold at all prices from 1 cent to \$2.50, what are the total receipts?
5. Find the sum of all the integers between 1 and 500 which end in 3.
6. A ball, starting from rest, rolls down a uniform slope. It passes over distances of 4, 12, 20 in., etc., in successive seconds. How long will it take to roll a distance of 22 ft?
7. A man repays a \$1,200 debt by paying \$150 at the end of each year, plus 6 per cent interest on the balance remaining unpaid during that year. Find the total of his payments.
8. In a potato race, 20 potatoes are placed in a line at intervals of 4 ft, with the first potato 24 ft from the starting point. A contestant is required to bring the potatoes back to the starting place one at a time. How far would he run in bringing back all the potatoes?
9. A man accepts a position with an initial salary of \$2,600 per year. It is understood that he will receive an automatic increase of \$160 at the end of the first year and each year thereafter through the end of the tenth year.
   a. Find his salary for the tenth year.
   b. What are his total earnings during the first eleven years?
10. A television set costing \$450 is bought under an agreement to repay the principal at the rate of \$30 at the end of each month plus 6 per cent interest on the balance remaining unpaid during that month.
    a. Find the first payment (made a month after purchase of the set).
    b. Find the last payment.
    c. Find the total amount paid.
11. The product of three numbers in arithmetic progression is 224, and the largest number is 7 times the smallest. Find the numbers.

12. Show that the sum of the first n odd numbers is equal to n^2.

13. Find the sum, last term, and number of terms in the arithmetic progression 33, 40, 47, . . . , whose sum best approximates 4,000.

14. A vacation club can be used to build up a fund of \$200 in 40 weekly installments which form a decreasing arithmetical progression. With the payment of the twentieth installment three-fifths of the fund will have been accumulated. What will be the first and second installments?

13·10 Geometric Progression. A *geometric progression* (abbreviated GP) is a succession of terms each of which may be obtained by multiplying the preceding term by a fixed quantity called the *common ratio*. Some examples of geometric progressions are

1. 4, 12, 36, 108, . . . (common ratio = 3)
2. $-24, 12, -6, 3, -\frac{3}{2}$, . . . (common ratio = $-\frac{1}{2}$)
3. $5x^3, 10x^2, 20x, 40, \ldots$ (common ratio = $2/x$)

The five elements of a GP and their customary symbols are

1. The first term a
2. The common ratio r
3. The number of terms n
4. The last (or nth) term l
5. The sum of n terms S

13·11 The *n*th Term of a GP. If a is the first term and r is the common ratio, the second term is ar, the third is ar^2, and in general the nth term is ar^{n-1}. Therefore l, the last or nth term, is

$$l = ar^{n-1} \qquad \star (13)$$

EXAMPLE 18. Find the 8th term in the GP $27, -18, 12, -8, \ldots$.

In this progression, $r = -\frac{2}{3}$. Hence, substituting, in Eq. (13), $a = 27$, $r = -\frac{2}{3}$, and $n = 8$, we obtain

$$l = 27\left(-\frac{2}{3}\right)^7 = 27\left(-\frac{128}{2{,}187}\right) = -\frac{128}{81}$$

EXAMPLE 19. Find the 10th term in the GP $k^{11}m^2, k^9m^3, k^7m^4, \ldots$.

In this progression $r = \frac{m}{k^2}$. Hence, substituting, in Eq. (13), $a = k^{11}m^2$, $r = \frac{m}{k^2}$, and $n = 10$, we have

$$l = k^{11}m^2\left(\frac{m}{k^2}\right)^9 = k^{11}m^2 \cdot \frac{m^9}{k^{18}} = \frac{m^{11}}{k^7}$$

13·12 The Sum of the First n Terms of a GP. If we write the sum S of the first n terms of a GP whose first term is a and common ratio r and then multiply both sides of this equation by r, we have

$$S = a + ar + ar^2 + ar^3 \cdots ar^{n-2} + ar^{n-1} \tag{14}$$
$$rS = ar + ar^2 + ar^3 + ar^4 \cdots ar^{n-1} + ar^n \tag{15}$$

On subtracting Eq. (15) from Eq. (14), all the intermediate terms cancel out and we obtain

$$S - rS = a - ar^n \tag{16}$$

Solving Eq. (16) for S, we obtain

$$S = \frac{a - ar^n}{1 - r} \tag{17}$$

or

$$S = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1} \quad \star (18)$$

Since from Eq. (13) $l = ar^{n-1}$

$$rl = ar^n \tag{19}$$

and, substituting Eq. (19) in Eq. (17), we have

$$S = \frac{a - rl}{1 - r} = \frac{rl - a}{r - 1} \quad \star (20)$$

EXAMPLE 20. Find the sum of the first seven terms of the GP 5, 15, 45, ... Substituting, in Eq. (18), $a = 5$, $r = 3$, and $n = 7$, we have

$$S = \frac{5(3^7 - 1)}{3 - 1} = \frac{5(2{,}187 - 1)}{2} = 5{,}465$$

EXAMPLE 21. Find the sum of the GP 45, -30, 20, . . . , $-640\frac{640}{243}$. Substituting, in Eq. (20), $a = 45$, $r = -\frac{2}{3}$, and $l = -640\frac{640}{243}$, we write

$$S = \frac{45 - (-\frac{2}{3})\left(-\frac{640}{243}\right)}{1 - (-\frac{2}{3})} = \frac{45 - \frac{1{,}280}{729}}{\frac{5}{3}} = \frac{6{,}305}{243}$$

EXAMPLE 22. If $S = 468$, $r = 5$, and $a = 3$, find n and l. Substituting in Eq. (20), we have

$$468 = \frac{5l - 3}{5 - 1}$$

or

$$l = 375$$

Substitution in Eq. (13) gives

$$375 = 3(5)^{n-1}$$

or

$$5^{n-1} = \frac{375}{3} = 125 = 5^3$$

Hence

$$5^{n-1} = 5^3$$

or

$$n = 4$$

13·13 Geometric Means. In a GP the first term a and the last term l are called the *extremes* of the progression. The intermediate terms are called *geometric means* between a and l. To insert k geometric means between two given quantities a and l, we must find k numbers which, inserted between a and l, form a GP of $k + 2$ terms, with a the first term and l the last. Substituting $k + 2$ for n in Eq. (13), we have

$$l = ar^{k+1}$$

or

$$r = \sqrt[k+1]{\frac{l}{a}} \qquad \star (21)$$

If $k + 1$ is even and l/a is positive, r may be either positive or negative and two sets of means can be determined. It is understood that only real values of r will be considered.

EXAMPLE 23. Insert three geometric means between 80 and 0.648.

Substituting $k = 3$, $a = 80$, and $l = 0.648$ in Eq. (21), we find

$$r = \sqrt[4]{\frac{0.648}{80}} = \sqrt[4]{0.0081} = 0.3 \text{ or } -0.3$$

Hence the required GP is either

$$80,\ 24,\ 7.2,\ 2.16,\ 0.648$$

or

$$80,\ -24,\ 7.2,\ -2.16,\ 0.648$$

If a and l have the same sign and we insert between them one geometric mean, m of the same sign as a and l, m is called the *geometric mean* of a and l. Since $l/m = m/a$, it follows that $m = \pm\sqrt{al}$. Hence

$$m = \pm\sqrt{al} \qquad \star (22)$$

where the sign agrees with that of a and l.

EXAMPLE 24. Find the geometric mean of $25/6$ and $3/8$. The geometric mean is $m = \sqrt{25/6 \cdot 3/8} = 5/4$.

The geometric mean or geometric average of n quantities is the nth root of their product.

EXAMPLE 25. Find the geometric average of 9, 15, and 200.

$$\sqrt[3]{(9)(15)(200)} = \sqrt[3]{27{,}000} = 30$$

EXERCISE 6

In Probs. 1 to 8 find the specified terms.

1. 3, 6, 12, ... Find the 7th term.
2. $-2, -6, -18, \ldots$. Find the 8th term.
3. 8, 4, 2, ... Find the 10th term.
4. $1/8, -1/4, 1/2, \ldots$. Find the 11th term.
5. $1/16, 3/8, 9/4, \ldots$. Find the 9th term.
6. $ax^3, ax^2, ax, \ldots$. Find the 8th term.
7. $\sqrt{2}, 2\sqrt{3}, 6\sqrt{2}, \ldots$. Find the 5th term.
8. $-\sqrt[3]{2}, \sqrt[3]{4}, -2, \ldots$. Find the 7th term.

In Probs. 9 to 14 determine the numerical value of the letter if a GP is to be formed. Determine the common ratio.

9. 8, 12, x
10. 45, x, 20
11. y, 24, -30
12. 32, k, $12\frac{1}{2}$
13. $2k - 2, 2k + 2, 2k + 8$
14. $4k + 4, 6k - 2, 9k - 13$

Determine the common ratio in Probs. 15 to 17.

15. $a, a - 2b, b$ **16.** $9, c - 3d, d^2$ **17.** $a + 3b, 4\sqrt{ab}, a + 5b$

In Probs. 18 to 24 determine the sum of the GP.

18. 3, 15, 75, . . . , to 6 terms
19. 24, 12, 6, . . . , to 7 terms
20. 5, -10, 20, . . . , to 7 terms
21. $27/32, 9/16, 3/8, \ldots$, to 6 terms
22. $32/45, -16/15, 8/5, \ldots$, to 5 terms
23. 1536, 768, 384, . . . , to 9 terms
24. 1215, -405, 135, . . . , to 7 terms

In Probs. 25 to 32 determine the missing quantities.

	S	n	a	l	r
25.			6	750	5
26.		8		384	2
27.		5	2		-3
28.		6	192	6	
29.	20			$\frac{1}{2}$	$\frac{1}{3}$
30.	508		4		2
31.	$59\frac{1}{9}$		108	$-14\frac{2}{9}$	
32.	$277\frac{1}{12}$	6			$\frac{3}{2}$

Find the geometric mean of each pair of quantities in Probs. 33 to 41.

33. 2 and 18 **34.** -3 and -27 **35.** 50 and 8
36. 6 and 54 **37.** $-\frac{1}{3}$ and -48 **38.** $\frac{2}{3}$ and $\frac{1}{6}$
39. $2\frac{7}{8}$ and $\frac{2}{3}$ **40.** a^3b and abc^2 **41.** x^{n-1} and x^{n+1}

In Probs. 42 to 47 insert geometric means as directed.

42. Two means between 5 and 40
43. Two means between 4 and -108
44. Three means between 48 and 3
45. Three means between -405 and -5
46. Four means between $3h^8$ and $96h^3$
47. Five means between $5x^4$ and $320x^{16}$.

13.14 Infinite Geometric Progressions. If we add the first n terms of the GP $a, ar, ar^2, \ldots, ar^{n-1}$, the sum ($S_n$) by Eq. (18) is

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

As an illustration of this formula consider the unending GP

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots, \frac{1}{2^{n-1}} \qquad (n \text{ terms}) \tag{23}$$

In progression (23), $a = 1$ and $r = \frac{1}{2}$, which, substituted in Eq. (18), gives us

$$S_n = \frac{1[1 - (\frac{1}{2})^n]}{1 - \frac{1}{2}} = 2[1 - (\tfrac{1}{2})^n] \tag{24}$$

As n increases and we continue to write more terms in Eq. (23), the value of the expression $(\frac{1}{2})^n$ in Eq. (24) becomes continually smaller.

By taking n large enough, we can make $(\frac{1}{2})^n$ as near zero as we please. Thus, if we find the sum of the first 24 terms, $(\frac{1}{2})^n = \frac{1}{16{,}777{,}216}$, so that $2[1 - (\frac{1}{2})^{24}]$ is very nearly equal to 2. If we find the sum of the first 48 terms, the value of $(\frac{1}{2})^{48} = \frac{1}{281{,}474{,}976{,}710{,}656}$, which is practically zero, and the value of $2[1 - (\frac{1}{2})^{48}]$ is practically 2. To summarize, we say that as n becomes infinitely large, S_n approaches the limit 2, and 2 then represents the sum of the progression 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, . . . , to infinitely many terms. This relation may be compactly expressed "lim $S_n = 2$."

$$\lim_{n \to \infty} S_n = 2$$

It will be apparent that if r is any quantity between -1 and 1, the value of r^n may be made as close to zero as we please by making n sufficiently large. Hence from Eq. (18) we see that as n increases without limit, the value of S_n approaches $\frac{a(1 - 0)}{1 - r}$, or $\frac{a}{1 - r}$. That is,

$$\lim_{n \to \infty} S_n = \frac{a}{1 - r} \tag{25}$$

The limit of the sum of n terms, as n becomes infinitely large, is called the sum of the GP with infinitely many terms. If we represent the sum by S, then

$$S = \frac{a}{1 - r} \qquad (-1 < r < 1) \qquad \star (26)$$

The indicated sum of a succession of numbers is usually called a *series*. If infinitely many terms are taken, it is known as an *infinite series*. Accordingly, the expression $a + ar + ar^2 \cdots$ is called an *infinite geometric series*. If the terms continually decrease and S approaches a limiting value as n becomes infinitely large, it is said to be a *convergent infinite series*.

Note that if $r < -1$ or $r > 1$, as in the GP 1, 2, 4, 8, . . . , where $r = 2$, S increases without limit as $n \to \infty$.

EXAMPLE 26. Find a rational number equivalent to the endless repeating decimal 0.53333 . . . . We may express 0.53333 . . . as 0.5 plus the series $0.03 + 0.003 + 0.0003 + \cdots$ to infinitely many terms. Here we evidently have an infinite geometric series in which $a = 0.03$ and $r = 0.1$. According

to Eq. (26), the sum of the series is

$$\frac{0.03}{1 - 0.1} = \frac{0.03}{0.9} = \frac{1}{30}$$

Hence $0.53333 \cdots = 0.5 + \frac{1}{30} = \frac{1}{2} + \frac{1}{30} = \frac{8}{15}$.

It should be noted that any endlessly repeating decimal is a rational number.

EXAMPLE 27. Find the fractional equivalent of the repeating decimal 0.61363636 . . . . Rearranging, we have

$$0.61 + 0.0036 + 0.000036 + 0.00000036 \cdots$$

After the first term, we have an infinite geometric series in which $a = 0.0036$ and $r = 0.01$. The sum of this portion of the series is

$$\frac{0.0036}{1 - 0.01} = \frac{0.0036}{0.99} = \frac{1}{275}$$

Hence $0.61363636 \cdots = \frac{61}{100} + \frac{1}{275} = \frac{671}{1100} + \frac{4}{1100} = \frac{675}{1100} = \frac{27}{44}$

EXERCISE 7

In Probs. 1 to 8 find the sum of an infinitely large number of terms.

1. 12, 6, 3, . . .
2. 15, 5, $\frac{5}{3}$, . . .
3. 75, -15, 3, . . .
4. 54, 36, 24, . . .
5. $\frac{3}{4}$, $-\frac{1}{2}$, $\frac{1}{3}$, . . .
6. 5, $-\frac{1}{2}$, $\frac{1}{20}$, . . .
7. $1 + x + x^2 + x^3, \ldots x < |1|$
8. $1 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} \cdots x > |1|$

In Probs. 9 to 16 find the fractional equivalent, in lowest terms, of each of the following repeating decimals.

9. 0.6666 . . .
10. 0.8888 . . .
11. 0.272727 . . .
12. 0.8333 . . .
13. 0.135135 . . .
14. 0.185185 . . .
15. 0.4090909 . . .
16. 0.41666 . . .

13·15 Applications of Geometric Progressions. The comments made regarding AP's in Sec. 13·9 apply in general to GP's. To these remarks it might be added that many problems involving the exponential type of equation $y = ba^x$ (see Sec. 14·29) may be solved by application of a GP. Typical of this type is a problem in which a pump removes $\frac{1}{10}$ of the air from a tank at each stroke. After the nth stroke the remaining fraction of the original air will evidently be $(0.9)^n$.

EXAMPLE 28. If 50 ml of a 15 per cent aqueous solution of a uranium salt is shaken with 10 ml of ether, which is then drawn off, the concentration of uranium salt in the aqueous layer is reduced to 12 per cent; After the second extraction with 10 ml of ether, the concentration is reduced to 9.6 per cent; and so on. Find the concentration of uranium salt in the aqueous layer after 12 extractions.

We shall need to find the 13th term of the GP 0.15, 0.12, 0.096, ... Referring to Eq. (13), we shall substitute $a = 0.15$, $r = 0.8$, and $n = 13$ and obtain

$$l = (0.15)(0.8)^{12} = 0.0103, \text{ or } 1.03 \text{ per cent.}$$

EXAMPLE 29. A block of metal at 200°C is placed in a room at 20°C. After 10 min it has cooled to 140°C; in another 10 min the temperature is 100°C. What will be its temperature after 2 hr?

We note that the original temperature differential is $200 - 20 = 180$. After 10 min it is $140 - 20 = 120$, and after 20 min it is reduced to $100 - 20 = 80$. It is evident that in any 10-min interval the temperature differential is reduced to two-thirds of its value at the beginning of that interval. Thus we see that the geometrical progression is a property of Newton's law of cooling. This law also states that the rate of cooling at any instant is proportional to the temperature differential (between object and surrounding medium) at that instant. (See Prob. 8, page 330.)

In our problem we are dealing with twelve 10-min intervals. Hence we wish to find the 13th term of the GP 180, 120, 80, ...

Applying Eq. (13) and substituting $a = 180$, $r = \frac{2}{3}$, and $n = 13$, we obtain

$$l = (180)(\tfrac{2}{3})^{12} = 1.39$$

or

$$20 + 1.4 = 21.4°\text{C temperature after 2 hrs}$$

EXERCISE 8

1. Find two numbers whose difference is 14 and whose geometric mean is 24.
2. What number must be subtracted from each of the numbers 15, 5, and 20 so that the resulting numbers, taken in the same order, will form a geometric progression?
3. At the end of each year the value of a certain machine has depreciated by 20 per cent of its value at the beginning of that year. If its initial value was \$1,250, find its value at the end of 5 years.
4. If \$1,000 is deposited in a bank at 3 per cent interest compounded annually, find the amount in the account at the end of 4 years.
5. A tank full of a 12 per cent salt solution is emptied of one-third of its contents, after which it is filled up with fresh water and con-

tents well mixed. If this is done 6 times, what is the final concentration of the salt solution?

6. The air in a tank is at atmospheric pressure (29.92 in. of mercury). If the volume of the tank is 800 cu in. and the pump removes 50 cu in. of air at each stroke, find the air pressure in the tank after 200 strokes, given that $(15/16)^{50} = 0.03967$.
7. A flask of salt solution, originally at 95°C, cooled to 71°C after standing for 10 min in a laboratory at 23°C. Find its temperature after another hour.
8. How many ancestors have you had in the ten preceding generations if no ancestor appears in more than one line of descent?
9. Find the sum of the first 19 positive integral powers of 1.02. SUGGESTION: Evaluate $(1.02)^{20}$ by the binomial theorem.
10. The radiator system of a car contains 4 gal of water. At the approach of the winter season the driver draws off a gallon and replaces it with alcohol. Subsequently a gallon of the mixture is drawn off and replaced with alcohol. By midwinter this process has been performed 5 times. Assuming no leakage or evaporation, what was the final percentage of alcohol in the cooling system?
11. Repeat Prob. 10, using 90 per cent alcohol as the replacement solution.
12. A ball is dropped from a height of 96 ft. If it always rebounds to half the height from which it falls, how far will it have traveled when it reaches the top of the fourth bounce? What distance will it have covered before coming to rest?
13. A saturated (4 per cent) aqueous solution of bromine is extracted with 20 ml of chloroform, whereupon the concentration of bromine in the aqueous layer is reduced to 1.2 per cent. Find the concentration of bromine in the aqueous layer after the sixth extraction with 20 ml of chloroform.
14. In the 60° right triangle in Fig. 13·1 find the sum of all the perpendiculars to the base a, a_1, a_2, . . . if the series is continued indefinitely.

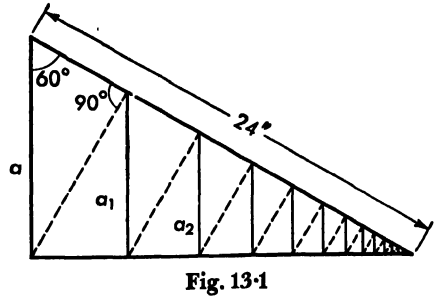


Fig. 13·1

15. The intensity of sunlight upon penetrating a layer of water 1 ft thick is diminished by 2 per cent of its value at the top of that layer. What percentage of the original light intensity remains at a depth of 50 ft? SUGGESTION: Calculate by the binomial theorem.

16. When the air pressure at sea level is 750 mm of mercury, it is 600 mm at 850 m elevation. If a like percentage reduction in pressure occurs with each increase of 850 m in altitude, find the air pressure at 8500 m.

17. The period of half-life for radium F (polonium) is 136 days. That is, in 136 days the disintegration of radium F will be one-half accomplished, in another 136 days the disintegration will be three-quarters complete, in another 136 days seven-eighths, etc. What percentage will have disintegrated in 68 days?

Chapter 14 LOGARITHMS

One of the most effective devices for saving time and effort in mathematical computations is the logarithm. By means of logarithms, processes of multiplication and division are replaced by addition and subtraction, and those of raising to a power and extracting a root are replaced by multiplication and division. In fact, some mathematical operations, previously impossible, admit of ready solution by logarithms.

14·1 Definition of a Logarithm. Given the equation $x = b^y$, the following assertions may be made:

1. If b is any positive number other than zero or unity, then for every real value of y there exists one and only one positive number x, defined by the equation $x = b^y$.

2. Conversely, for every positive value of x there exists a unique number y such that $x = b^y$.

If $x = b^y$, then y is called the logarithm of x to the base b and is written

$$y = \log_b x \tag{1}$$

The logarithm of a number is the exponent indicating the power to which it is necessary to raise the base to equal the given number.

The above assertions imply that

1. For any given logarithm y there exists a unique number x, corresponding to that logarithm;

2. Every positive number x has a unique logarithm for any given positive base other than zero or unity.

14·2 Graphical Representation. The graph shown in Fig. 14·1 is at once the graph for $x = b^y$ and $y = \log_b x$, since the two equations are equivalent. Here $b > 1$.

The shape of this curve remains substantially unchanged for all values of $b > 1$. The Y axis is an asymptote, with y approaching $-\infty$ as x approaches zero. The only intercept is $x = 1, y = 0$. y continually increases with increasing values of x. No portion of the curve exists

for which x is negative, for there is no negative quantity representing a power of a positive number.

If we consider the equations $x = b^y$ or $y = \log_b x$, in which the base $b = 3$, Table 14·1 may be used for the purpose of plotting the graph in Fig. 14·1.

Table 14·1

x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81
y	−3	−2	−1	0	1	2	3	4

It will be noted that the series of y values forms an arithmetic progression, whereas the corresponding values of x form a geometric progression.

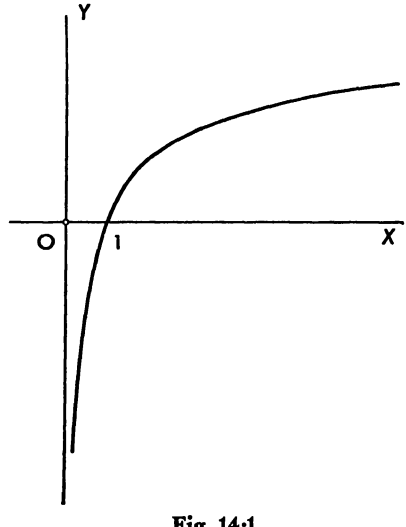


Fig. 14·1

By determining the means between the existing terms, we obtain

y	4	3.5	3	2.5	2	1.5	1	0.5	0	etc.
x	81	46.8	27	15.6	9	5.20	3	1.73	1	etc.

Repeating the process,

y	4	3.75	3.5	3.25	3	2.75	2.5	2.25	2	etc.
x	81	61.7	46.8	35.6	27	20.6	15.6	11.8	9	etc.

A continuation of this process would enable us to plot the graph in Fig. 14·1 as accurately as we wish.

14·3 Formation of a Table of Logarithms. From this graph it is possible to read and tabulate approximate values of y corresponding

to assigned values of x. A portion of such a table is reproduced as Table 14·2.

Table 14·2

x	1	2	3	4	5	6	7	8	9
y	0	0.63	1.00	1.26	1.46	1.63	1.77	1.89	2.00
x	10	11	12	.....	24	.....	27	.....	60
y	2.09	2.18	2.26	.....	2.89	.....	3.00	.....	3.72

This table, when sufficiently extended, becomes a table of logarithms to the base 3.

We may make use of this table to multiply or divide, remembering to observe the usual laws of exponents.

EXAMPLES. Multiply 2×5.

$$2 = 3^{0.63} \quad \text{and} \quad 5 = 3^{1.46}$$

Therefore

$$2 \times 5 = 3^{0.63} \times 3^{1.46} = 3^{2.09} = 10$$

Multiply 3×8. Since $8 = 3^{1.89}$, we obtain

$$3 \times 8 = 3^{1} \times 3^{1.89} = 3^{2.89} = 24$$

Divide 60 by 12.

$$60 = 3^{3.72} \quad \text{and} \quad 12 = 3^{2.26}$$
$$60 \div 12 = 3^{3.72} \div 3^{2.26} = 3^{1.46} = 5$$

Divide 24 by 6.

$$24 = 3^{2.89} \quad \text{and} \quad 6 = 3^{1.63}$$
$$24 \div 6 = 3^{2.89} \div 3^{1.63} = 3^{1.26} = 4$$

14·4 Use of Exponents in Determining Logarithms. From the preceding discussion it can be seen that an exponential relationship may be expressed as a logarithm, and vice versa.

EXAMPLES

$3^2 = 9$ is equivalent to $\log_3 9 = 2$
$5^3 = 125$ is equivalent to $\log_5 125 = 3$
$4^{-1} = \frac{1}{4}$ is equivalent to $\log_4 \frac{1}{4} = -1$
$5^{-2} = \frac{1}{25}$ is equivalent to $\log_5 \frac{1}{25} = -2$
$49^{\frac{1}{2}} = 7$ is equivalent to $\log_{49} 7 = \frac{1}{2}$
$15^0 = 1$ is equivalent to $\log_{15} 1 = 0$*

EXERCISE 1

In Probs. 1 to 9 write each exponential expression in logarithmic notation.

* Since any number, other than zero, raised to the zero power, equals 1, it follows that the logarithm of 1 to any base equals zero.

1. a. $2^3 = 8$	**b.** $2^{-6} = \frac{1}{64}$	**c.** $7^{-2} = \frac{1}{49}$
2. a. $3^5 = 243$	**b.** $10^{-2} = 0.01$	**c.** $(\frac{1}{6})^2 = \frac{1}{36}$
3. a. $5^4 = 625$	**b.** $3^{-3} = \frac{1}{27}$	**c.** $(\frac{2}{3})^3 = \frac{8}{27}$
4. a. $2^5 = 32$	**b.** $7^0 = 1$	**c.** $b^0 = 1$
5. a. $3^{-4} = \frac{1}{81}$	**b.** $6^2 = 36$	**c.** $(\frac{1}{5})^{-2} = 25$
6. a. $4^{\frac{1}{2}} = 2$	**b.** $8^{\frac{1}{3}} = 2$	**c.** $9^{\frac{3}{2}} = 27$
7. a. $16^{\frac{3}{4}} = 8$	**b.** $25^{\frac{1}{2}} = 5$	**c.** $27^{\frac{2}{3}} = 9$
8. a. $125^{\frac{2}{3}} = 25$	**b.** $32^{\frac{3}{5}} = 8$	**c.** $64^{\frac{5}{6}} = 32$
9. a. $27^{-\frac{1}{3}} = \frac{1}{3}$	**b.** $16^{-\frac{3}{4}} = \frac{1}{32}$	**c.** $36^{-\frac{3}{2}} = \frac{1}{216}$

Let it be required to find the value of $\log_7 49$. Setting $\log_7 49 = y$, we paraphrase the question by asking ourselves what power of 7 equals 49. Since $7^2 = 49$, it is apparent that $\log_7 49 = 2$.

EXERCISE 2

Find the values of the following logarithms:

1. a. $\log_5 25$	**b.** $\log_2 8$	**c.** $\log_3 81$
2. a. $\log_4 8$	**b.** $\log_8 16$	**c.** $\log_{16} 8$
3. a. $\log_9 27$	**b.** $\log_{27} 81$	**c.** $\log_{125} 25$
4. a. $\log_7 \sqrt{7}$	**b.** $\log_{12} 1$	**c.** $\log_3 \frac{1}{3}$
5. a. $\log_2 \frac{1}{8}$	**b.** $\log_{10} 0.001$	**c.** $\log_4 \frac{1}{32}$
6. a. $\log_{125} 5$	**b.** $\log_{16} 4$	**c.** $\log_{16} \sqrt{4}$

Another version of the logarithmic relationship is exemplified by the equation $\log_5 x = 3$. Here we wish to know what number is equal to the third power of 5. Since $5^3 = 125$, we have $\log_5 125 = 3$.

It may be required to find the base of a logarithmic equation. For example, $\log_b 16 = 4$. This is read "log 16 to what base equals 4?" In other words, what number b, raised to the fourth power, equals 16? The answer is evidently 2. Hence $\log_2 16 = 4$.

EXERCISE 3

Find the value of x in Probs. 1 to 4.

1. a. $\log_2 x = 3$	**b.** $\log_5 x = 2$	**c.** $\log_3 x = 4$
2. a. $\log_5 x = 0$	**b.** $\log_6 x = -1$	**c.** $\log_8 x = \frac{4}{3}$
3. a. $\log_{10} x = 2$	**b.** $\log_3 x = -2$	**c.** $\log_{27} x = \frac{2}{3}$
4. a. $\log_{16} x = \frac{3}{4}$	**b.** $\log_8 x = -\frac{2}{3}$	**c.** $\log_{25} x = -\frac{1}{2}$

In Probs. 5 to 10 find the value of the base b.

5.	**a.** $\log_b 9 = 2$	**b.** $\log_b 8 = 3$	**c.** $\log_b 4 = \frac{2}{3}$
6.	**a.** $\log_b 1{,}000 = 3$	**b.** $\log_b 25 = 2$	**c.** $\log_b 3 = \frac{1}{2}$
7.	**a.** $\log_b 15 = 1$	**b.** $\log_b 1 = 0$	**c.** $\log_b \frac{1}{4} = -2$
8.	**a.** $\log_b \frac{1}{27} = -\frac{3}{2}$	**b.** $\log_b \frac{1}{25} = -\frac{2}{3}$	**c.** $\log_b 7 = -\frac{1}{2}$
9.	**a.** $\log_b 9 = -\frac{2}{3}$	**b.** $\log_b 0.01 = 2$	**c.** $\log_b \frac{1}{16} = -\frac{4}{3}$
10.	**a.** $\log_b 1{,}000 = \frac{3}{2}$	**b.** $\log_b 6 = -\frac{1}{2}$	**c.** $\log_b \frac{1}{64} = -6$

14·5 Properties of Logarithms. Since logarithms are exponents, the rules relating to the use of logarithms resemble the laws of exponents.

The most common relationships are the following:

$$\log_b MN = \log_b M + \log_b N \qquad \star (2)$$

$$\log_b \left(\frac{M}{N}\right) = \log_b M - \log_b N \qquad \star (3)$$

$$\log_b M^n = n \log_b M \qquad \star (4)$$

$$\log_b \sqrt[n]{M} = \frac{1}{n} \log_b M \qquad \star (5)$$

$$\log_b 1 = 0 \qquad \star (6)$$

$$\log_b \left(\frac{1}{M}\right) = -\log_b M \qquad \star (7)$$

These rules may be readily derived.

14·6 Multiplication. Let

$$M = b^c \quad \text{or} \quad \log_b M = c$$

and

$$N = b^d \quad \text{or} \quad \log_b N = d$$

Then

$$MN = b^c \cdot b^d = b^{c+d}$$

In logarithmic notation, this becomes

$$\log_b MN = c + d$$

but

$$c + d = \log_b M + \log_b N$$

Therefore

$$\log_b MN = \log_b M + \log_b N$$

or

The logarithm of a product is equal to the sum of the logarithms of the separate factors.

14·7 Division. Let

$$M = b^c \quad \text{or} \quad \log_b M = c$$

and

$$N = b^d \quad \text{or} \quad \log_b N = d$$

Then

$$\frac{M}{N} = \frac{b^c}{b^d} = b^{c-d}$$

In logarithmic notation, this becomes

$$\log_b \frac{M}{N} = c - d$$

But

$$c - d = \log_b M - \log_b N$$

Therefore

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

or

The logarithm of a quotient is equal to the logarithm of the numerator minus the logarithm of the denominator.

14·8 Powers. If

$$M = b^c$$

then

$$\log_b M = c$$

Also

$$M^n = b^{cn}$$

which in logarithmic notation is

$$\log_b M^n = cn$$

But

$$c = \log_b M$$

Therefore

$$\log_b M^n = n \log_b M$$

or

The logarithm of a power of a number is equal to the exponent times the logarithm of the number.

14·9 Roots. This is a form of the power rule, for if we write

$$\log_b \sqrt[n]{M} \text{ as } \log_b M^{\frac{1}{n}}$$

we obtain

$$\log_b M^{\frac{1}{n}} = \frac{1}{n} \log_b M$$

Therefore

$$\log_b \sqrt[n]{M} = \frac{1}{n} \log_b M$$

The logarithm of a root of a number is the logarithm of that number divided by the index of the root.

14·10 Logarithm of Unity. This is also a special case of the power rule in which $n = 0$.

$$\log_b 1 = \log_b M^0 = 0 \log_b M = 0$$

Therefore

The logarithm of unity to any base is zero.

14·11 Reciprocal. This is another variation of the power law. In this case $n = -1$.

$$\log_b \frac{1}{M} = \log_b M^{-1} = (-1) \log_b M = - \log_b M$$

The logarithm of the reciprocal of a number is the negative logarithm of that number.

EXERCISE 4

Transform the equations in Probs. 1 to 4 into exponential form.

1. $\log_b (a - x) = c$ **2.** $\log_b (x^2 + 3) - \log_b (2x + 1) = k$

3. $2 \log_b (3x - 2) = a + c$ **4.** $\log_b y = \sqrt{2x + 5}$

5. If $\log \sqrt{x} = 0.35$, find $\log x^2$

6. If $\log_b bx = 5$, find $\log_b x$.

7. If $\log \frac{1}{x} = \frac{1}{4}$, find $\log x$.

8. If $\log x^2 = 16$, find $\log x$.

9. If $\log (x^2 - 9) - \log (x + 3) = 0$, find x.

Rewrite the expressions in Probs. 10 to 12 without exponents or radicals.

10. **a.** $\log x^5$ **b.** $\log 3x^2$ **c.** $\log \frac{5}{x^2}$

11. a. $\log \sqrt[3]{x}$ **b.** $\log \frac{1}{x^3}$ **c.** $\log \frac{1}{2\sqrt{x}}$

12. a. $\log \frac{x^5}{8}$ **b.** $\log_b bx^2$ **c.** $\log_b \frac{2x^3}{b^2}$

Convert the expressions in Probs. 13 and 14 to single positive logarithms whose coefficient is 1.

13. a. $6 \log x$ **b.** $2 \log x - \log 3$ **c.** $\frac{\log_b x}{2} + 1$

14. a. $3 - \log_b x$ **b.** $\frac{1}{2}(\log x + \log 2)$ **c.** $-10 \log x$

For additional short problems illustrating the properties of logarithms, see Exercise 17.

14·12 Change of Base. At times it is necessary to change from one base to another, especially when dealing with sound levels, mechanical shock, and transient electric currents. The formula is

$$\log_a M = \frac{1}{\log_b a} \cdot (\log_b M) \qquad \star (8)$$

Since a and b are both constants, $\log_b a$ is also a constant. Hence, if only a table to the base b is available, *the logarithm of a number to the base* b *can be converted to the logarithm of the same number to the base* a *by dividing the former by the constant* $\log_b$ a.

The proof of the statement follows:

Let

$$M = a^x$$

so that

$$\log_a M = x \qquad (9)$$

Also if

$$M = a^x$$

then

$$\log_b M = \log_b a^x = x \log_b a$$

or

$$x = \frac{1}{\log_b a} (\log_b M) \qquad (10)$$

Then, from Eqs. (9) and (10),

$$\log_a M = \frac{1}{\log_b a} (\log_b M)$$

14·13 Systems of Logarithms. From the problems in Exercises 1 through 3 it is evident that any positive number, except 1, can be

used as a base for a system of logarithms. The number 1 cannot be used as a base because 1, when raised to any power whatever, is still equal to 1.

There are two systems of logarithms in general use: the system of common logarithms to the base 10, and the system of natural logarithms to the base e, or 2.71828. . . . Natural logarithms will be discussed further in Chap. 16.

The system of common logarithms is most convenient for general computation because it is best suited to the decimal system. Hereafter, unless otherwise indicated, the word *logarithm* will be understood to mean common logarithm. When the base is not expressed, the base 10 is understood. Thus log 3 = 0.47712 means $\log_{10} 3 = 0.47712$.

From the laws of exponents we know that $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1{,}000$, $10^4 = 10{,}000$, $10^5 = 100{,}000$, etc.

In logarithmic notation the above equations become $\log_{10} 1 = 0$, $\log_{10} 10 = 1$, $\log_{10} 100 = 2$, $\log_{10} 1000 = 3$, $\log_{10} 10{,}000 = 4$, $\log_{10} 100{,}000 = 5$, etc.

14·14 Graphical Derivation of Table. If we use 10 for the base instead of the general value b, we obtain the graph shown in

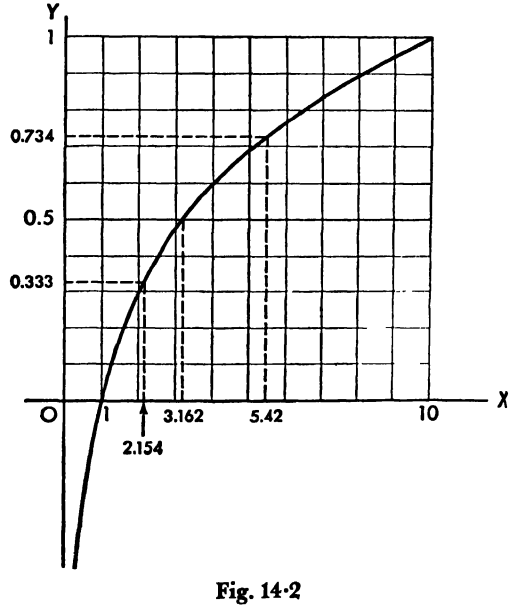


Fig. 14·2

Fig. 14·2. This, then, is the graph of $x = 10^y$ or $y = \log_{10} x$. It is apparent that when $y = 0$, $x = 1$. When $y = \frac{1}{3}$, $x = 10^{\frac{1}{3}}$ or $\sqrt[3]{10}$ or 2.154. When $y = \frac{1}{2}$, $x = 10^{\frac{1}{2}}$ or $\sqrt{10}$ or 3.162. When $y = 1$, $x = 10$, etc.

For any given value of y, there is only one value of x; and conversely for any given positive value of x, there is only one value of y (Sec. 14·1). Thus any positive number x may be expressed as a power of 10, just as we have considered 2.154 as the 0.333 power of 10 and 3.162 as the 0.500 power of 10.

If this curve were drawn to a large enough scale, we should be able to read that when $x = 5.42$, $y = 0.734$ (approximately). Evidently, then, $10^{0.734} = 5.42$, or $\log_{10} 5.42 = 0.734$.

14·15 Characteristic and Mantissa. If we know that

$$10^{0.734} = 5.42$$

or that $\log_{10} 5.42 = 0.734$, we can then write the logarithms of 5.42 times a power of 10. Thus

$$\log 54.2 = \log (10 \times 5.42) = \log 10 + \log 5.42 = 1 + 0.734$$
$$\log 542 = \log (100 \times 5.42) = \log 100 + \log 5.42 = 2 + 0.734$$
$$\log 5420 = \log (1{,}000 \times 5.42) = \log 1{,}000 + \log 5.42 = 3 + 0.734$$
$$\log 0.542 = \log (0.1 \times 542) = \log 0.1 + \log 5.42 = -1 + 0.734$$
$$\log 0.0542 = \log (0.01 \times 5.42) = \log 0.01 + \log 5.42 = -2 + 0.734$$
$$\log 0.00542 = \log (0.001 \times 5.42) = \log 0.001 + \log 5.42 = -3 + 0.734$$

Each logarithm will be seen to consist of a whole number and a decimal value. *The whole number is called the characteristic, and the decimal value is called the mantissa.* The mantissa is usually an unending decimal which, by more advanced mathematics, may be computed to as many places as desired.

It will be noted that the mantissa depends only on a given sequence of digits, in this case 542. All numbers having a given sequence of digits will have the same mantissa. On the other hand the characteristic is seen to be related to the position of the decimal point.

Table 14·3

		Characteristic		*Mantissa*	*Common method of writing logarithm*
log 5,420	=	3	+	0.734	3.734
log 542	=	2	+	0.734	2.734
log 54.2	=	1	+	0.734	1.734
log 5.42	=	0	+	0.734	0.734
log 0.542	=	−1	+	0.734	0.734 − 1
log 0.0542	=	−2	+	0.734	0.734 − 2
log 0.00542	=	−3	+	0.734	0.734 − 3

The relationships set forth in the beginning of this section are summarized in Table 14·3.

From this table it will be evident that *for numbers larger than* 1 *the characteristic of the logarithm is positive and one less than the number of digits at the left of the decimal point in the number. If the number is smaller than* 1, *the characteristic is negative and corresponds to the position of the first significant figure.* For example, in the number 0.00542 the first significant digit is in the third place to the right of the decimal point; therefore the characteristic of the logarithm is -3.

By common agreement the mantissa in a common logarithm table (and elsewhere except as noted below) is always considered to be positive.

Alternative conventions of writing logarithms containing negative characteristics are illustrated below. For the most part we shall confine ourselves in this text to the first form.

$0.734 - 1$	$9.734 - 10$	$\bar{1}.734$	-0.266*
$0.734 - 2$	$8.734 - 10$	$\bar{2}.734$	-1.266
$0.734 - 3$	$7.734 - 10$	$\bar{3}.734$	-2.266

If the number is expressed in scientific notation (Sec. 1·3), the characteristic corresponds to the power of 10 (see Table 14·4).

Table 14·4

Number	*Scientific notation*	*Logarithm*
5,420	5.42×10^3	3.734
542	5.42×10^2	2.734
54.2	5.42×10^1	1.734
5.42	5.42×10^0	0.734
0.542	5.42×10^{-1}	$0.734 - 1$
0.0542	5.42×10^{-2}	$0.734 - 2$
0.00542	5.42×10^{-3}	$0.734 - 3$

A table of mantissas, or logarithms of numbers between 1 and 10, will be seen to be sufficient to determine the logarithm of any positive number, the proper positive or negative characteristic being supplied by inspection when writing the logarithms of numbers greater than 10 or less than 1.

14·16 The Antilogarithm of an Expression. The antilogarithm of an expression is the number whose logarithm is the given expression. For example, to find the antilogarithm of 1.734 is to find the number whose logarithm is 1.734, *i.e.*, 54.2.

* Used only under special conditions (Sec. 14·29).

EXERCISE 5

1. Complete the following table:

	Number	*Scientific Notation*	*Logarithm*
a.	8.61	$8.61\ (\times 10^0)$	0.935
b.	861	8.61×10^2	2.935
c.	0.0861	8.61×10^{-2}	$0.935 - 2$
d.	8,610		
e.		8.61×10^5	
f.			4.935
g.	0.00861		
h.		8.61×10^{-1}	
i.			$0.935 - 4$
j.	8,610,000		
k.		8.61×10^8	
l.			$0.935 - 3$

Given log 3.54 = 0.549 and log 7.98 = 0.902 in Probs. 2 to 4, find the logarithms of the given numbers. In Probs. 5 to 7 find the antilogarithms of the given logarithms.

2.	**a.** 35.4	**b.** 798	**c.** 79,800
3.	**a.** 0.0354	**b.** 0.354	**c.** 7,980
4.	**a.** 0.00798	**b.** 3,540,000	**c.** 0.000354
5.	**a.** 2.549	**b.** 1.902	**c.** $8.902 - 10$
6.	**a.** $0.902 - 3$	**b.** 4.549	**c.** $0.549 - 1$
7.	**a.** $6.549 - 10$	**b.** 3.902	**c.** $0.549 - 3$

14·17 Role of Logarithms in Rough Approximation. The order of magnitude of a quantity may be readily approximated by logarithms.

EXAMPLE 1. Given only that log 2 = 0.301 and log 3 = 0.477, find the position of the first significant digit of N if $N = 6^7$.

Applying Eq. (2), Sec. 14·5,

$$\log 6 = \log 2 + \log 3 = 0.778$$

By Eq. (4), Sec. 14·5,

$$\log 6^7 = 7 \log 6 = 5.446$$

Since the characteristic equals 5, there must be 5 + 1 or 6 digits to the left of the decimal point (Sec. 14·15); hence the first significant digit is in the sixth place to the left of the decimal point.

EXAMPLE 2. Repeat Example 1 for $N = (0.45)^8$.

$$4.5 = \frac{(3)(3)}{2}$$

$$\log \frac{(3)(3)}{2} = 2 \log 3 - \log 2 = 0.653 \qquad \text{[Eqs. (1) and (3), Sec. 14·5]}$$

Therefore

$$\log 0.45 = 0.653 - 1 \qquad \text{(Sec. 14·15)}$$
$$\log (0.45)^8 = 8 \log 0.45 \qquad \text{[Eq. (4), Sec. 14·5]}$$
$$8(0.653 - 1) = 5.224 - 8 = 0.224 - 3$$

By Sec. 14·15 a characteristic of -3 indicates that the first significant digit of $(0.45)^8$ is in the third place to the right of the decimal point.

EXERCISE 6

Given only that log 2 = 0.301 and log 3 = 0.477, determine the order of magnitude (position of the first significant figure) of the following.

1. 2^{14}	**2.** 6^{10}	**3.** $(0.8)^{20}$	**4.** 30^9
5. 5^{13}	**6.** $(0.15)^8$	**7.** 3^{11}	**8.** $(200)^6$
9. $(0.02)^8$	**10.** $(0.5)^{12}$	**11.** 9^9	**12.** $(1.5)^{15}$

14·18 Using a Table of Logarithms. A portion of a five-place table of logarithms is reproduced in Table 14·5.

Table 14·5

N	0	1	2	3	4	5	6	7	8	9
310	49136	150	164	178	192	206	220	234	248	262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	831	845	859	872	886	900	914	927	941	955
316	969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50106	120	133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637

From this table we find that the mantissa of the logarithm of 3,124 is 0.49471. We have obtained this value by reading in from the number representing the first three digits, 312, and under the fourth digit, 4. Two space-saving conventions will be noted: The decimal point is not shown; neither are the first two digits, 49, of the mantissa shown.

Another space-saving device is revealed if it be required to look up the mantissa of the logarithm of 3,168. Reading opposite 316 and under 8, we find *079. The asterisk tells us that we have moved from mantissas 0.49 . . . to mantissas 0.50 . . . Therefore this particular mantissa is 0.50079.

Since $3{,}168 = 3.168 \times 10^3$, the characteristic of the logarithm is seen to be 3, and the complete logarithm of 3,168 is 3.50079.

EXERCISE 7

Referring to Table 14·5, determine the complete logarithms (characteristic and mantissa) of the following numbers.

1. 3,103	**2.** 313.6	**3.** 310	**4.** 3.190
5. 31,160	**6.** 318,200	**7.** 3.152	**8.** 3,167
9. 3,100,000	**10.** 31.63	**11.** 31	**12.** 3.1

In looking up the logarithm of a number smaller than unity, we may first express the number in scientific notation and then determine the mantissa and characteristic from the respective first and second parts of that number.

EXAMPLE 3. Use Table 14·5 to read the logarithm of 0.003148. In scientific notation this becomes 3.148×10^{-3}. Therefore by inspection the characteristic is -3, and, from the table, the mantissa is 0.49803. Hence the logarithm of 0.003148 is $0.49803 - 3$.

EXERCISE 8

Referring to Table 14·5, determine the logarithms of the following numbers in Probs. 1 to 12:

1. 0.003122	**2.** 0.03196	**3.** 0.3144	**4.** 0.00031
5. 0.03167	**6.** 0.0000315	**7.** 0.3171	**8.** 0.003185
9. 0.00000319	**10.** 0.3111	**11.** 0.03105	**12.** 0.003162

Referring to Table 14·5, determine the numbers corresponding to the logarithms (*i.e.*, find the antilogarithms) in Probs. 13 to 24.

13. 1.49178	**14.** 2.49374	**15.** 3.50065
16. $0.50092 - 2$	**17.** $0.50515 - 3$	**18.** $0.50284 - 1$
19. 4.50569	**20.** 0.50243	**21.** 5.49290
22. $0.50325 - 4$	**23.** 1.50010	**24.** $0.49996 - 1$

EXERCISE 9

Referring to a five-place log table, look up the logarithms of the numbers in Probs. 1 to 3.

1. a. 1,234	**b.** 100.3	**c.** 0.01234	**d.** 37.05	**e.** 398.4
2. a. 79,020	**b.** 0.3076	**c.** 3,000	**d.** 0.005006	**e.** 1,003,000
3. a. 179.7	**b.** 0.1042	**c.** 0.01090	**d.** 709.3	**e.** 0.0007065

Find the numbers corresponding to the logarithms in Probs. 4 to 6.

4. a. 3.75051	**b.** 0.56038 − 2	**c.** 1.55096		
d. 0.00043	**e.** 0.93997 − 1			
5. a. 7.93475 − 10	**b.** 0.06333	**c.** 5.60206		
d. 2.96440	**e.** 4.76005			
6. a. 1.21537	**b.** 3.61794	**c.** 0.27161 − 4		
d. 5.99502	**e.** 2.47012			

14·19 Interpolation. It is often necessary to find the logarithm of a number which does not appear in the table. If we are satisfied to use the logarithm of the nearest number, this choice would reduce the accuracy of a five-place table to that of a four-place table. It is not necessary to sacrifice this extra degree of accuracy if we resort to interpolation. In a five-place table interpolation will determine for us the fifth digit in the mantissa, but not a sixth. Perhaps this process is best illustrated by an example.

EXAMPLE 4. Find by interpolation log 25.813.

First find the mantissas of 25,810 and 25,820. These are 41179 and 41196, respectively. (Decimal points are disregarded for the time being.) The *tabular difference* between the mantissas is 17, whereas the difference between 25,810 and 25,820 is 10. It is evident that the given number 25,813 is 0.3 of the way from 25,810 to 25,820. Therefore the desired mantissa must be about 0.3 of the way from 41179 to 41196. 0.3 of 17 is 5.1, which is rounded off to 5. 41179 + 5 is 41184. Therefore the mantissa is 0.41184, and by inspection the characteristic is 1. Hence log 25.813 = 1.41184.

The data used in interpolation might well be laid out as follows:

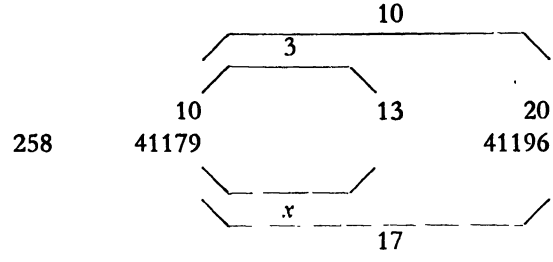
Write down the bracketing numbers and their mantissas in the same relative order as they occur in the table, leaving space for the intermediate number and its mantissa, which is sought.

	1	2
258	41179	41196

Annex ciphers to the numbers at the top of the columns, and fill in the fourth and fifth digits of the given number.

	10	13	20
258	41179		41196

Indicate the differences, or proportional parts.



By proportion, $3/10 = x/17$. $x = 5.1$, by mental calculation or slide rule. Observe that since the mantissas are unending decimals and are rounded off to five digits, there would be no point in adding 41179. and 5.1 and calling the result 41184.1. Hence we calculate x only to the nearest whole number.

In most tables there are included tables of proportional parts in which x in the above proportion may be looked up. This is simply a table of tenths of the tabular difference, which is 17 in this example. Usually, mental computation is the quickest way to determine the proportional part x.

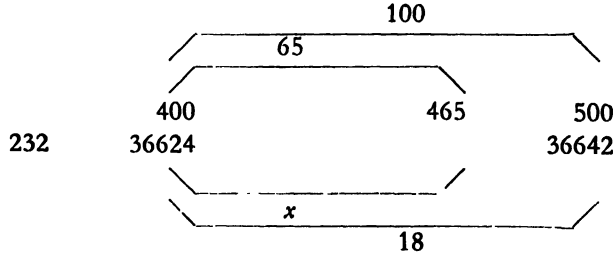
	17
1	1.7
2	3.4
3	5.1

etc.

If the given number contains more than five significant figures and and the tabular difference is 15 or more, round off to six significant figures and proceed as in the following example.

EXAMPLE 5. Find log 232.464973.

Rounding off to six digits, we get 232.465.

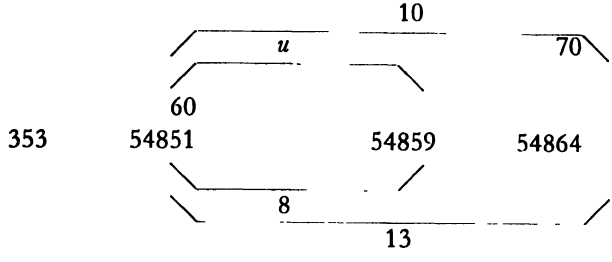


By proportion, $65/100 = x/18$. $x = 11.7$, which we call 12. Therefore, the mantissa is 36624 + 12, or 0.36636. The complete logarithm is 2.36636.

The process of interpolation as applied to antilogarithms is simply the reverse of that used in looking up logarithms.

EXAMPLE 6. Find antilog 3.54859.

We do not find 54859 in the table of mantissas, but we do find the bracketing mantissas 54851 and 54864. These correspond to the numbers 3,536 and 3,537, respectively. The operation is illustrated below:



By proportion, $u/10 = 8/13$. $u = 6.2$, which we call 6. Therefore the digits of the number are 35,360 + 6, or 35,366. The characteristic 3 determines the number to be 3,536.6.

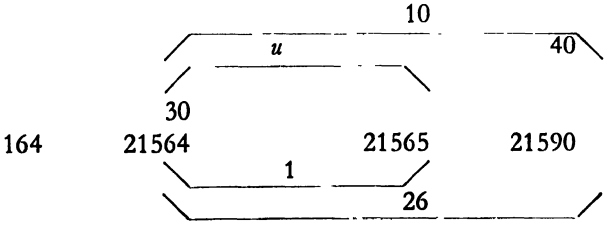
If we choose to use the proportional-parts table, we scan the values listed under 13 until we come to the value nearest 8. This is 7.8, which corresponds to 6 (0.6 of 13).

	13
—	—
5	6.5
6	7.8
7	9.1
—	—

Note that if the difference x is less than 1 part out of 20, interpolation is omitted, since if $x < 1$, $(x/20)(10)$ is < 0.5 or less than 5 in the sixth place.

In this case, simply the nearest mantissa is taken.

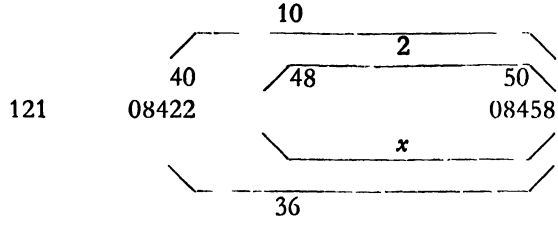
EXAMPLE 7. Find the antilogarithm of 1.21565.



$u/10 = 1/26$. $u = 0.4$, which is less than 0.5. 21564. + 0.4 = 21564. Therefore we ignore interpolation and take the nearest mantissa, 21564. Thus we get antilog 1.21565 = 16.430.

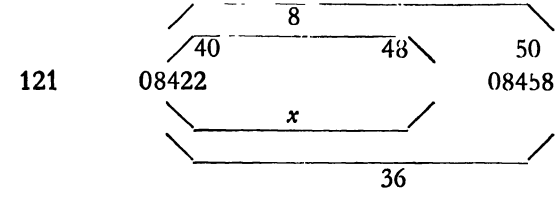
Interpolation may sometimes be facilitated by approaching our answer from the upper value instead of the lower.

EXAMPLE 8. Find log 0.012148.



Here we write $x/36 = 2/10$. $x = 7.2$ or 7. 08458 − 7 = 08451.

This is easier than the equivalent operation:



$x/36 = 8/10$. $x = 28.8$ or 29. 08422 + 29 = 08451.

In either case the complete logarithm is 0.08451 − 2.

EXERCISE 10

Using a five-place log table, look up the logarithms of the numbers in Probs. 1 to 3, obtaining the last digit by interpolation.

1. a. 268.18 **b.** 0.0043356 **c.** 21.355 **d.** 9.2354 **e.** 4.1497

2. a. 68.173 **b.** 0.030689 **c.** 55,071 **d.** 0.13309 **e.** 8,585.4

3. a. 11,958 **b.** 37.596 **c.** 0.00079364 **d.** 2,090.2 **e.** 118,210,000

Find to five significant figures the numbers whose logarithms are given in Probs. 4 to 6. Obtain the fifth significant figure by interpolation.

4. a. 3.20756 **b.** 0.29577 − 1 **c.** 1.79652
d. 4.38896 **e.** 2.59460
5. a. 0.00309 − 3 **b.** 0.47562 **c.** 0.89964 − 4
d. 0.69250 − 2 **e.** 5.14521
6. a. 0.01997 − 4 **b.** 1.27461 **c.** 5.93376
d. 2.00215 **e.** 0.37965 − 1

14·20 Computation Using Logarithms. In carrying out computations involving logarithms, we shall make use of the equations set forth in Sec. 14·5.

14·21 Multiplication by the Use of Logarithms. According to Eq. (2), the logarithm of a product equals the sum of the logarithms of the separate factors.

EXAMPLE 9. Find the product $(238.24)(0.072495)(9.5668) = N$.
From the tables,

$$\begin{aligned} \log 238.24 &= 2.37701 \\ \log 0.072495 &= 0.86031 - 2 \\ \log 9.5668 &= 0.98077 \\ \log N &= 4.21809 - 2 \\ &= 2.21809, \text{ or } N = 165.23 \end{aligned}$$

If a computation involves negative quantities, they should be treated as though they were positive, since a negative number does not have a logarithm. The sign of the answer can be determined by inspection after completing the computation.

EXERCISE 11

Using logarithms, perform the following computations to five significant figures:

1. $47.69 \times 32.41 \times 76.48 \times 1.932$
2. 3.7596×159.06
3. 10.365×0.42659
4. $2{,}586 \times 169.2 \times 1.423 \times 0.96$
5. $73.65 \times 9.9 \times 0.9999 \times 1.001 \times 0.01309$
6. $70.603 \times 0.55146 \times 0.000213 \times 1.402 \times 40.326 \times 743.01$
7. $25.726 \times 5.0789 \times 2{,}188.5 \times 5.2525 \times 4{,}660.1$
8. $2{,}208.6 \times 0.04368 \times 2{,}781{,}200{,}000 \times 0.000016976$
9. $0.0018746 \times 15.856 \times 2{,}650 \times 2.046 \times 50{,}620{,}000$
10. $3.6492 \times 1{,}006.4 \times 0.00037964 \times 0.001796 \times 36.592$

14·22 Division by the Use of Logarithms. From Eq. (3) we find that the logarithm of a quotient equals the logarithm of the numerator minus the logarithm of the denominator.

EXAMPLE 10. Evaluate $\dfrac{845.67}{0.68332} = N$.

$$\begin{aligned} \log 845.67 &= 2.92720 \\ \log 0.68332 &= 0.83462 - 1 \\ \hline \log N &= 2.09258 + 1 \\ &= 3.09258, \text{ or } N = 1237.6 \end{aligned}$$

If the denominator is larger than the numerator, we must devise some way to subtract a larger logarithm from a smaller one. This can be accomplished by adding enough to the characteristic of the logarithm of the numerator to make it larger than the logarithm of the denominator. To compensate for the increase in the characteristic, a like amount is written in as a negative characteristic of the logarithm of the numerator.

EXAMPLE 11. Evaluate $\dfrac{37.844}{510.36} = N$.

$$\begin{aligned} \log 37.844 &= 1.57800 \\ \log 510.36 &= 2.70788 \end{aligned}$$

If we increase the characteristic in 1.57800 by 2, it will be larger than 2.70788. A compensating characteristic -2 is appended.

$$\begin{aligned} \log 37.844 &= 3.57800 - 2 \\ \log 510.36 &= 2.70788 \\ \hline \log N &= 0.87012 - 2, \text{ or } N = 0.074152 \end{aligned}$$

EXERCISE 12

Use logarithms to compute the following quotients to five significant figures:

1. 3,796 ÷ 472	**2.** 7,943.2 ÷ 25.793
3. 7.5437 ÷ 97.604	**4.** 0.04792 ÷ 6.3951
5. 4,605.2 ÷ 56,517	**6.** 99.976 ÷ 10.532
7. 1,000.1 ÷ 3,576.5	**8.** 0.3796 ÷ 0.02071
9. 12.147 ÷ 0.0004963	**10.** 5,076 ÷ 0.043297
11. 1 ÷ 53.827	**12.** 1 ÷ 0.082941

14·23 Combined Multiplication and Division. This is simply a combination of Eqs. (2) and (3) for multiplication and division.

EXAMPLE 12. Evaluate $\dfrac{75.282\pi}{(6.754)(0.38949)(0.01217)} = N.$

$$\begin{array}{lll} \log 75.282 & = 1.87669 & \\ \log 3.1416 & = 0.49715 & \\ \hline \log \text{num.} & = 2.37384 & \rightarrow 2.37384 \end{array}$$

$$\begin{array}{lll} \log 6.754 & = 0.82956 & \\ \log 0.38949 & = 0.59050 - 1 & \\ \log 0.01217 & = 0.08529 - 2 & \\ \hline \log \text{den.} & = 1.50535 - 3 & \rightarrow \underline{1.50535 - 3} \\ & & \overline{0.86849 + 3} \end{array}$$

$$\log N = 3.86849, \text{ or } N = 7387.3$$

14·24 Cologarithms. The *cologarithm* of a number is the logarithm of the reciprocal of the number. It is therefore equal to the negative of the logarithm of the number.

$$\operatorname{colog} N = \log\left(\frac{1}{N}\right) = -\log N \qquad \star (11)$$

It is evident that division by a given number may be accomplished by adding the cologarithm instead of subtracting the logarithm. A combined multiplication and division problem may therefore be evaluated by adding up only one column of logarithms.

A separate table of cologarithms is not usually available, but by subtracting the logarithm from zero the student can write down the cologarithm at sight without writing down the logarithm. This is easily accomplished by subtracting each figure of the mantissa from 9 except the last, which is subtracted from 10.

EXAMPLE 13. Find colog 66.32.

$$\begin{array}{ll} \log 1 & = 2.00000 - 2 \\ \log 66.32 & = 1.82164 \\ \hline \operatorname{colog} 66.32 & = 0.17836 - 2 \end{array}$$

EXAMPLE 14. Find colog 0.7042.

$$\begin{array}{ll} \log 1 & = 1.00000 - 1 \\ \log 0.7042 & = 0.84770 - 1 \\ \hline \operatorname{colog} 0.7042 & = 0.15230 \end{array}$$

Note that the zero value of log 1 in each case is written in such a way as to avoid a split characteristic in the colog.

EXAMPLE 15. Evaluate the expression in Example 12, using cologs.

$$\begin{aligned}
\log\ 75.282 &= 1.87669\\
\log\ \ 3.1416 &= 0.49715\\
\operatorname{colog}\ \ 6.754 &= 0.17044 - 1\\
\operatorname{colog}\ 0.38949 &= 0.40950\\
\operatorname{colog}\ 0.01217 &= 1.91471\\
&\overline{4.86849 - 1}\\
\log N &= 3.86849, \text{ or } N = 7387.3
\end{aligned}$$

EXERCISE 13

Evaluate the following to five significant figures.

1. $\dfrac{576.43 \times 976.52 \times 1.4962}{3.7425 \times 0.009652 \times 0.01736}$ 2. $\dfrac{57.04 \times 25.936 \times 0.48352}{764.32 \times 97.63 \times 0.007986}$

3. $\dfrac{601.47 \times 93.276 \times 2.5037 \times 79.631}{927.43 \times 26.485 \times 0.001793 \times 62}$

4. $\dfrac{10.243 \times 1.5006 \times 0.96016 \times 9.1207}{59.329 \times 3.8421 \times 0.027532 \times 0.50008}$

14·25 Raising to a Power by the Use of Logarithms. According to Eq. (4), the logarithm of the power of a number is found by multiplying the logarithm of the number by the exponent.

EXAMPLE 16. Evaluate $(1.8653)^8 = N$.

$$\begin{aligned}
\log 1.8653 &= 0.27075\\
&\qquad\quad 8\\
\log N &= \overline{2.16600}, \text{ or } N = 146.56
\end{aligned}$$

EXAMPLE 17. Evaluate $(5.167)^{-2.8} = \mathrm{N}$.

$$\begin{aligned}
(5.167)^{-2.8} &= \frac{1}{(5.167)^{2.8}} \text{ by law of exponents}\\
\log 5.167 &= \ \ 0.71324\\
&\qquad\quad 2.8\\
&\qquad\overline{570592}\\
&\qquad\ 142648\\
\log (5.167)^{2.8} &= \overline{1.997072}\\
\log 1 &= 2.00000 - 2\\
\log (5.167)^{2.8} &= 1.99707\\
\log N &= 0.00293 - 2, \text{ or } N = 0.010068
\end{aligned}$$

ALTERNATIVE SOLUTION: $-2.8 \log 5.167 = (-2.8)(0.71324) = -1.99707$.
Adding,

$$\log 1 = 0 = \begin{array}{r} 2.00000 - 2 \\ -1.99707^{*} \\ \hline 0.00293 - 2 \end{array} \text{ or } N = 0.010068$$

EXAMPLE 18. Evaluate $(0.17528)^{1.75} = N$.

$$\log 0.17528 = 0.24373 - 1$$

$$\begin{array}{r} 1.75 \\ \hline 121865 \\ 170611 \\ 24373 \\ \hline 0.4265275 - 1.75 \end{array}$$

Note that the entire logarithm, both mantissa and characteristic, is multiplied by 1.75.

Add zero in a form to make the characteristic a whole number.

$$\begin{array}{rrr} 0.42653 & - 1.75 & \\ + 0.25 & - 0.25 & (= 0) \\ \hline 0.67653 & - 2, & \end{array}$$

$$\log N = 0.67653 - 2, \text{ or } N = 0.047482$$

EXAMPLE 19. Evaluate $(0.4627)^{-3.2}$.

$$(0.4627)^{-3.2} = \frac{1}{(0.4627)^{3.2}} = \left(\frac{1}{0.4627}\right)^{3.2}$$

$$\begin{array}{rl} \log 1 & = 1.00000 - 1 \\ \log 0.4627 & = 0.66530 - 1 \\ & \hline 0.33470 \\ & \quad 3.2 \\ & \hline 66940 \\ & 100410 \end{array}$$

$$\log N = 1.07104, \text{ or } N = 11.777$$

ALTERNATIVE SOLUTION: $\log 0.4627 = 0.66530 - 1$

Multiplying entire logarithm by

$$\begin{array}{r} 0.66530 - 1 \\ - 3.2 \\ \hline 133060 \\ 199590 \\ \hline -2.12896 \quad + 3.2 \end{array}$$

Combine

$$\begin{array}{r} +3.20000 \\ -2.12896 \\ \hline 1.07104 \end{array}$$

$$\log N = 1.07104, \text{ or } N = 11.777$$

* -1.99707 means that we have not only -1, but also -0.99707. Since mantissas in the table are positive, we must convert to a positive mantissa by a device such as that illustrated.

EXERCISE 14

Use logarithms to evaluate the following to five significant figures:

1. a. $(17.56)^2$	**b.** $(8.493)^3$	**c.** $(6.072)^4$
2. a. $(4.875)^5$	**b.** $(-11.83)^3$	**c.** $(0.6432)^4$
3. a. $(0.09458)^3$	**b.** $(0.14732)^2$	**c.** $(-0.25713)^4$
4. a. $(23.805)^{-1}$	**b.** $(7.432)^{-2}$	**c.** $(16.031)^{-3}$
5. a. $(0.34771)^{-3}$	**b.** $(0.084263)^{-2}$	**c.** $(82.517843)^2$
6. a. $(17.584)^{\frac{1}{2}}$	**b.** $(405.76)^{\frac{2}{5}}$	**c.** $(1807.4)^{\frac{2}{5}}$
7. a. $(0.6432)^{2.2}$	**b.** $(0.09473)^{-1.8}$	**c.** $(2.75)^{2.75}$
8. a. $(0.16223)^{\frac{3}{4}}$	**b.** $(0.074962)^{\frac{4}{5}}$	**c.** $(0.25814)^{\frac{5}{2}}$
9. a. $(70.58)^{0.2}$	**b.** $(13.625)^{1.3}$	**c.** $(48.461)^{-0.3}$
10. a. $(14.83)^{-1.62}$	**b.** $(0.3447)^{-2.22}$	**c.** $(0.0962)^{3.15}$
11. a.* $(1.030)^6$	**b.*** $(1.0075)^{14}$	**c.*** $(1.025)^{11}$
12. a.* $(1.00625)^{17}$	**b.*** $(1.0275)^{21}$	**c.*** $(1.0005)^{33}$

14·26 Extracting Roots by the Use of Logarithms. Referring to Eq. (5), Sec. 14·5, we read that the logarithm of the root of a number equals the logarithm of that number divided by the index of the root.

EXAMPLE 20. Evaluate $\sqrt[4]{737.12} = N$.

$$\log 737.12 = 2.86754$$
$$\log N = \frac{2.86754}{4}$$
$$= 0.71688, \text{ or } N = 5.2106$$

EXAMPLE 21. Evaluate $\sqrt[3]{0.00028864} = N$.

$$\log 0.00028864 = 0.46036 - 4$$

If we were to divide the entire logarithm by 3 as it stands, the characteristic -4 would give us the awkward -1.33333. To avoid this, we add at the left, and subtract at the right, a quantity which will make the characteristic divisible by 3. The simplest such number is 2.

$$\begin{aligned} \log 0.00028864 &= \quad 0.46036 - 4 \\ &\quad +2 \qquad\quad -2 \\ &\quad 3/2.46036 - 6 \\ \log N &= \quad 0.82012 - 2, \text{ or } N = 0.066087 \end{aligned}$$

* Evaluate to six significant figures where the table allows.

EXERCISE 15

Evaluate the following expressions:

1. a. $\sqrt[3]{972}$ b. $\sqrt[3]{97.2}$ c. $\sqrt[3]{9.72}$
2. a. $\sqrt[3]{0.972}$ b. $\sqrt[3]{0.0972}$ c. $\sqrt[3]{0.00972}$
3. a. $\sqrt{73.464}$ b. $\sqrt[3]{168.15}$ c. $\sqrt[4]{3051.8}$
4. a. $\sqrt{0.082167}$ b. $\sqrt[3]{0.22469}$ c. $\sqrt[4]{0.000058274}$
5. $(3.142)^2 \times (1.344)^3$
6. $(8.527)^3 \times (0.7161)^2$
7. $(32.00)^4 \times (0.03125)^5$
8. $\sqrt{58.43} \times (6.710)^2$
9. $(23.876 \times 1.0554)^2$
10. $(185.2 \times 0.071828)^3$
11. $\sqrt{97.422 \times 4.7881}$
12. $\sqrt[3]{496.7} \times (0.6645)^2$
13. $\sqrt{\dfrac{219.6}{0.005733}}$
14. $\sqrt[3]{\dfrac{52.482}{371.05 \times 0.18061}}$
15. $\sqrt[3]{\dfrac{3 \times 0.28617}{4\pi}}$
16. $[\pi(0.07112)(8.859)]^2$
17. $\sqrt[3]{\dfrac{(427.5)^2}{(13.482)^5}}$
18. $\sqrt[6]{\left(\dfrac{27.42}{0.03966}\right)^5}$
19. $\sqrt[5]{-381.72}$
20. $(-3.0655)^4$
21. $\sqrt[3]{\dfrac{-17.477}{(-0.84991)^2}}$
22. $\left(\dfrac{64.66}{-1009}\right)^{2/3}$
23. $(-2.8857)^{-3}$
24. $\sqrt[4]{\dfrac{1}{8.3567}}$

14·27 Logarithmic Computation of Expressions Involving Addition and Subtraction. Since numbers cannot be added or subtracted by logarithms, we must convert to antilogarithms before performing such an operation.

EXAMPLE 22. Evaluate $\sqrt{\dfrac{a^5+1}{a^5-1}} = N$, where $a = 1.0037$.

$$\log 1.0037 = 0.0016039$$

$$\log (1.0037)^5 = \overline{5}.0080195 \text{ or } (1.0037)^5 = 1.01864 \text{ (seven-place table)}$$

$$\sqrt{\frac{1.01864+1}{1.01864-1}} = \sqrt{\frac{2.01864}{0.01864}}$$

$$\log 2.0186 = 0.30505$$

Subtracting $\log 0.01864 = \overline{0.27045-2}$

$$\overline{2}/2.03460$$

$$1.01730 \text{ or } N = 10.41$$

Note that although we have been able to find $(1.0037)^5$ to six figures using a seven-place table, after we subtract 1 we get a denominator 0.01864 having only four significant figures. Therefore, we can retain only four significant figures in our answer. This situation points up the need of extra accuracy in reading logarithms of numbers near unity. The mantissas of such numbers may have only two or three significant figures when taken from a five-place table.

EXERCISE 16

Use logarithms to compute the value of the following to five significant figures:

1. $\sqrt{(8.315)^2 - (5.024)^2}$
2. $\sqrt{(11.964)^2 + (4.4857)^2}$
3. $\sqrt[3]{(4.115)^3 + (6.382)^3}$
4. $\sqrt[5]{(1.486)^7 - 20}$
5. $\sqrt[3]{\dfrac{(1.389)^3 + 1}{(1.389)^3 - 1}}$

EXERCISE 17

The short problems in this exercise may be used to test the student's understanding of the properties of logarithms without spending the time usually required in consulting tables.

In Probs. 1 to 10, given only that $\log 2 = 0.301$ and $\log 3 = 0.477$, find without using tables:

1.	**a.** $\log 2{,}000$	**b.** $\log 0.03$	**c.** $\log \sqrt{30}$
2.	**a.** $\log \frac{1}{16}$	**b.** $\log (3)^7$	**c.** $\log 6$
3.	**a.** $\log 1$	**b.** $\log 20$	**c.** $\log 5$
4.	**a.** $\log 1.5$	**b.** $\log 8$	**c.** $\log 0.0003$
5.	**a.** $\log \sqrt{15}$	**b.** $\log \sqrt[3]{60}$	**c.** $\log 3\frac{1}{3}$
6.	**a.** $\log 39 - \log 13$	**b.** $\log (\frac{1}{3})^4$	**c.** $\log \sqrt[5]{\frac{1}{2}}$
7.	**a.** $\log \sqrt[4]{\frac{1}{3}}$	**b.** $\log \sqrt[3]{0.0002}$	**c.** $\log (24)^{1.65}$
8.	**a.** $\log 46 - \log 23$	**b.** antilog 3.477	**c.** antilog $0.301 - 3$
9.	**a.** antilog 2.301	**b.** antilog $0.176 - 2$	**c.** antilog 3.01
10.	**a.** antilog 1.778	**b.** antilog $7.778 - 10$	**c.** antilog 0.0477

14·28 Accuracy in Logarithmic Computations. The student should bear in mind that in a multiplication or division the accuracy of the result is determined by the least accurate individual factor. If several six-digit numbers are to be multiplied by a number of four significant figures, all the individual factors should be rounded off to four significant figures, since the answer will be limited to this degree of accuracy. Consequently, a four-place table of logarithms will suffice. By the same token, if the utmost accuracy is to be secured

from a combination of numbers of seven significant figures, a table to at least seven places must be used.

14·29 Exponential Equations. An equation in which the unknown appears in the exponent is called an exponential equation. An example of the simplest of this type is $2^x = 8$. Here x is obviously equal to 3, since $2^3 = 8$.

In equations where the result cannot be determined by inspection, we must take logarithms of both sides of the equation and obtain the final solution by ordinary algebraic methods.

EXAMPLE 23. Solve the following equation for x:

$$32^x = 512$$

Taking the logarithm of each member, we have

$$\log 32^x = \log 512$$

Then

$$x \log 32 = \log 512$$
$$1.50515x = 2.70927$$

In order to avoid confusion, it is suggested that the student consider this equation as an ordinary linear equation, which it is. Had this equation appeared in the section of elementary algebra devoted to linear equations in one unknown, there would be no doubt as to the method of solving for x. The possible origin of the equation did not affect the procedure there; neither does it in this case. Accordingly,

$$x = \frac{2.70927}{1.50515} = 1.8000$$

This result might have been anticipated, since $32^{\frac{9}{5}} = 512$.

If only two- or three-place accuracy is desired, this division may be performed on the slide rule.

EXAMPLE 24. Solve the equation $9^{2x-7} = 27^{x-3}$ for x.

Taking logarithms of each member, we obtain

$$\log 9^{2x-7} = \log 27^{x-3}$$
$$(2x - 7) \log 9 = (x - 3) \log 27$$
$$(2x - 7)(0.95424) = (x - 3)(1.43136)$$

Here again we have an ordinary linear equation and proceed accordingly. Carrying out the indicated multiplication,

$$1.90848x - 6.67968 = 1.43136x - 4.29408$$
$$0.47712x = 2.38560$$
$$x = \frac{2.38560}{0.47712} = 5.0000$$

ALTERNATIVE SOLUTION: Since in this example both 9 and 27 are powers of the same number, 3, we might proceed as follows without tables:

$$9^{2x-7} = 27^{x-3}$$

but

$$9 = 3^2 \quad \text{and} \quad 27 = 3^3$$

Therefore

$$(3^2)^{2x-7} = (3^3)^{x-3}$$
$$3^{4x-14} = 3^{3x-9}$$

It is evident that $4x - 14$ must equal $3x - 9$, or $x = 5$.

In some cases a few preliminary steps are necessary to obtain one term on each side of the equation before taking logarithms.

EXAMPLE 25. Solve the equation $\dfrac{(1.05)^x - 1}{0.05} = 18$ for x.

$$(1.05)^x - 1 = (0.05)(18) = 0.9$$
$$(1.05)^x = 1.9$$

And proceed as before.

Some equations in which the value of the exponent ultimately proves to be negative depend for their solution upon the deliberate conversion of the mantissa of a logarithm to the negative form (Sec. 14·15).

EXAMPLE 26. Solve the equation $6^x = 0.0208$ for x.

Taking logarithms of both sides,

$$x \log 6 = \log 0.0208$$

Then

$$x(0.77815) = 0.31806 - 2$$

Treating this as an ordinary linear equation, we obtain

$$0.77815x = 0.31806 - 2.00000$$
$$0.77815x = -1.68194$$
$$x = \frac{-1.68194}{0.77815}$$
$$x = -2.1615$$

EXERCISE 18

Use logarithms to solve for x in the following exponential equations:

1. $2^x = 75$ **2.** $(1.07)^x = 3$ **3.** $125^x = 48(5)^x$

4. $5^x = 1{,}000$ **5.** $(1.05)^{-x} = 0.36$ **6.** $3^{x+1} = 5^{x-1}$

7. $4^{x+2} = 8^{2x-1}$ **8.** $6^{-x} = 0.02$ **9.** $(1.04)^{-x} = 0.75$

10. $\dfrac{(1.06)^x - 1}{0.06} = 12$ **11.** $8^x = \dfrac{8}{2^{x-3}}$ **12.** $5^x = \sqrt[x]{20}$

14·30 Aids to Logarithmic Computation. Computation involving logarithms may be expedited by observing a few details of technique, of which the following are representative:

1. Follow some orderly plan of development, such as that set forth in Example 12, page 281.
2. Cancel or combine simple numbers where possible to reduce the number of logarithms to be handled.
3. Improve the arrangement for logarithmic computation, *e.g.*,
   (*a*) $2\pi r^2 + 2\pi rh = 2\pi r(r + h)$. Once r and h are added, there will be no further obstacle to logarithmic computation.
   (*b*) $\sqrt{c^2 - a^2} = \sqrt{(c + a)(c - a)}$. Here we enter a "straight-away" as soon as $c + a$ and $c - a$ are evaluated.
4. Negative mantissas or characteristics may often be avoided by multiplying by 10^n; *e.g.*, $\dfrac{0.03152}{0.00567} = \dfrac{31.52}{5.67}$.
5. Another device to circumvent negative logarithms is multiplication by a^n; *e.g.*, the equation $47.59 = (1.52)^{-3}\,(N)$ may be multiplied throughout by $(1.52)^3$; thus we get $(47.59)(1.52)^3 = N$.

EXERCISE 19

The following problems are drawn from various fields of technology. In each case compute the answer to the degree of accuracy you think is warranted by the data.

1. The population of a small city in 1940 was 27,000; in 1950 it was 38,500. Find the annual rate of increase, assuming it to be uniform. What was the probable population in 1947?
2. A solid cast-iron sphere $4\frac{3}{4}$ in. in diameter weighs 14.6 lb. Find to the nearest 0.01 in. the diameter of a cast-iron sphere weighing 32 lb.
3. The speed of a chemical reaction doubles for each 18°F rise in temperature. What temperature rise will be needed to produce a 20-fold increase in the speed of the reaction?
4. By the use of logarithms find the radius r of a circle inscribed in a triangle whose sides are a, b, and c if

$$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}$$

where s is half of the perimeter and the sides are 53.6, 41.9, and 38.4.

5. Find the diameter of a circle inscribed in a triangle whose sides are 287.6, 303.1, and 365.9.
6. In the formula $P = 29.92e^{-h/5}$, P is the barometer reading in inches of mercury, h is the altitude in miles, and $e = 2.718$ (base natural logarithms).
   a. What will the pressure be at 28,000 ft?
   b. What altitude in feet corresponds to a pressure of 19.50 in.?
7. In Fig. 14·3 a weight W is suspended by a rope over a round beam. The rope is prevented from slipping by a force F at the other end.

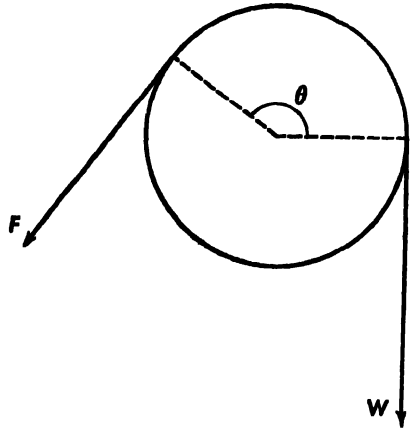


Fig. 14·3

   a. Using the formula $F = We^{-a\theta}$ where $e = 2.718$, a is a constant, and θ is the angle of contact of the rope on the beam (expressed in turns), find a to 0.01 if 170 lb supports 200 lb when $\theta = 150°$.
   b. What force is needed to support 200 lb, using two complete turns of rope?
   c. How many turns would be required for 50 lb to support 200 lb?
8. The electrical resistance of a wire doubles (approximately) as the B & S gauge number increases 3 units. If the resistance of No. 10 copper wire is 1 ohm per 1,000 ft, find the resistances of No. 13 and No. 4 wire.
9. Find the resistance of No. 5 copper wire.
10. What is the resistance ratio between two successive B & S wire sizes?
11. The diameters of successive wire gauges are in geometric progression. If the diameter of No. 10 wire is 0.102 in. and that of No. 4 is 0.204 in., find the diameter of No. 6 wire.

12. What is the diameter ratio between two successive B and S wire sizes?

13. What is the resistance ratio between two wires five sizes apart?

14. What is the diameter ratio between two wires five sizes apart?

15. Find the diameter of a hemispherical cup holding exactly one-half a pint.

16. In public-address systems the decibel gain $D = 10 \log P_A/P_B$ where P_A and P_B are the power levels of the two sounds A and B, and D is the difference in loudness (in decibels) between the sounds A and B. Find the decibel gain in an electrical circuit if the input is 2 watts and the output is 5 watts.

17. What is the decibel loss in a circuit if the output is 0.07 watts and the input is 0.10 watts?

18. If the power ratio between input and output of an amplifier is $\frac{1}{3}$, what is the decibel gain?

19. A loudspeaker requires 1.2 watts to produce the proper volume of sound. If we assign to this sound level an arbitrary value of 1, calculate the volume range within which the speaker operates if the power varies between 1.0 and 1.5 watts.

20. If the sound output of an amplifier varies from 4 db above to 3 db below the normal value, between what power levels does the amplifier operate?

21. The ratio of sound volume in speech may range as high as 250 to 1. What is the range in decibels?

22. The normal power rating of an amplifier is 1.6 watts. To what should this be increased to produce a perceptible (*i.e.*, one-decibel) increase in sound level?

For radio and public-address work, etc., 0.006 watt is a common reference level from which to figure sound-energy levels. If the output of an amplifier is 0.24 watt, the decibel difference between this level and 0.006 watt is

$$L_{db} = 10 \log \frac{0.24}{0.006} = 10 \log 40 = 10 \times 1.60206 = 16 \text{ db}$$

where L_{db} is the so-called power level at the amplifier. For convenience, the power level of this amplifier is said to be 16 db. We understand, however, that we really should say that the power level of the amplifier output is 16 db above 0.006 watt.

23. What is the decibel level equivalent to 0.0005 watt?

24. What is the decibel level equivalent to 0.46 watt?

25. What is the decibel gain in amplifier with 0.005 watt input and 0.46 watt output?

26. What is the difference between the decibel levels in Probs. 23 and 24?

27. What power level is equivalent to 59 watts?

Problems having to do with air conditioning, refrigeration, internal-combustion engines, air compressors, etc., often use formulas based on the following fundamental types of equations:

$$V = \frac{KT}{P} \tag{12}$$

(By Charles' and Boyles' laws, the volume V of a given weight of gas varies directly as the absolute temperature T and inversely as the absolute pressure P.)

$$\frac{PV}{T} = K \tag{13}$$

According to Prob. 38, page 238, under two separate sets of conditions of pressure, volume, and temperature for the *same weight* of gas we may write

$$\frac{P_1V_1}{T_1} = K \text{ for the first set of conditions} \tag{14}$$

and

$$\frac{P_2V_2}{T_2} = K \text{ for the second set of conditions} \tag{15}$$

Any consistent set of units may be used as long as P and T are expressed on an absolute scale. Usually P is expressed in pounds per square inch absolute (equals pounds per square inch gauge $+14.7$, abbreviated psia), and T is expressed in Rankine or Fahrenheit absolute (equals Fahrenheit $+460$).

From Eqs. (14) and (15) we obtain

$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2} \tag{16}$$

If we wish to express the pressure-volume relationship of a given weight of gas undergoing compression or expansion without being concerned about any accompanying temperature change, we could write from thermodynamic considerations

$$PV^n = C \tag{17}$$

where n and C are constants.

Then for two distinct sets of conditions for the same weight of gas we may write

$$P_1V_1^n = C \tag{18}$$

and

$$P_2V_2^n = C \tag{19}$$

From Eqs. (18) and (19) we obtain

$$P_1V_1^n = P_2V_2^n \tag{20}$$

28.[1] From Eqs. (16) and (20) show that

$$\frac{T_1}{T_2} = \left(\frac{V_2}{V_1}\right)^{n-1}$$

29.[1] From Prob. 28 show that

$$\frac{V_2}{V_1} = \left(\frac{T_1}{T_2}\right)^{\frac{1}{n-1}}$$

30.[2] From Eq. 20 and Prob. 29 show that

$$\frac{T_1}{T_2} = \left(\frac{P_1}{P_2}\right)^{\frac{n-1}{n}}$$

31.[2] From Prob. 30 show that

$$\frac{P_1}{P_2} = \left(\frac{T_1}{T_2}\right)^{\frac{n}{n-1}}$$

32. Find the final gauge pressure of a quantity of gas if the volume is compressed from 220 to 150 cu ft. P_1 is atmospheric pressure and $n = 1.39$.

33. Show that if $n = 0$, the pressure is constant, that if $n = 1$, the temperature is constant, and that if $n = \infty$, the volume is constant (see formulas in Probs. 28 to 31).

34. If $n = 1.64$, find the final volume of 68 cu ft of gas when, under compression, the temperature rises from 135 to 200°F.

35. A gas is compressed from 4.32 to 2.09 cu ft. Find the final temperature if the original temperature was 30°F and $n = 1.4$.

36. When 10 cu ft of a vapor expanded to 32 cu ft, the temperature fell from 191 to 0°F. Find n to 0.01.

[1] In Probs. 28 and 29 we develop formulas to be used when we are not interested in pressure changes involved or when the pressure is unknown.

[2] Formulas in Probs. 30 and 31 are used when we are not concerned with changes in volume or when data are unavailable.

37. The pressure on 220 cu ft of a gas is increased from one to two atmospheres, while the volume decreases to 127 cu ft. Find n.

38. Following is a formula for the velocity of air being discharged through an orifice to the atmosphere.

$$V = 108.9\sqrt{T\left[1 - \left(\frac{14.7}{P}\right)^n\right]}$$

Find n if the air is being discharged at the rate of 323 cfm through a $1\frac{5}{8}$-in.-diameter orifice. The pressure in the tank is 1.2 (psi gauge), and temperature in the tank is 65°F.

V = linear air flow, feet per second
T = air temperature in tank, degrees Rankine (Fahrenheit absolute)
P = absolute air pressure in tank, pounds per square inch

39. A 10-oz spherical weight is to be cast of lead. Find its diameter to 0.001 in. if a cubic foot of lead weighs 700 lb.

40. 1,000 ml of a salt solution contains a suspended sludge. The sludge is to be washed by allowing it to settle, part of the clear, overlying salt solution is poured off, replaced by an equal volume of pure water, stirred, and allowed to settle. This cycle is to be repeated until the necessary amount of salt is washed out.

The required equation is

$$C_n = \left(\frac{1{,}000 - w}{1{,}000}\right)^n C_0$$

where C_0 = original concentration of salt = 15.8 g per liter
C_n = final concentration of salt
w = milliliters of solution removed and replaced by water each cycle = 400 ml
n = number of washings

a. If $n = 8$, find C_n.
b. Find n to reduce C_n to less than 0.085 g per liter.

41. In the measurement of heat flow through pipe insulation the log mean diameter is computed. This is

$$D_{av} = \frac{D_2 - D_1}{2.3 \log D_2/D_1}$$

where D_1 = outer diameter of the pipe
D_2 = outer diameter of the insulation

Find D_{av} when the outer pipe diameter is $2\frac{5}{8}$ in. and the insulation is $2\frac{5}{16}$ in. thick. Compare with the arithmetic mean.

42. The equation $h = \dfrac{0.22C_pT^{\frac{3}{4}}v^{0.8}}{D^{0.2}}$ is used to determine the coefficient of heat transfer h for turbulent flow of gas inside a pipe.

C_p is the average specific heat at constant pressure = 0.53
T is the temperature of the gas film = 695°R
v is lb gas per second per square foot of free area = 0.38
D is the equivalent diameter of the pipe = 5½ in.

Find h to 0.01.

43. Find to the nearest ¼ in. the pipe diameter required to deliver natural gas. Use the formula

$$D = 0.56\left(\frac{Q^2GL}{H}\right)^{0.225}$$

where D = pipe diameter, inches
Q = rate of gas flow = 280 cu ft per hr
G = specific gravity of the gas relative to air = 0.433
L = pipe length in yards = 5½ miles
H = pressure drop = 8¼ in. water

44. Find the size (in circular mils) of a bare copper wire required to transmit safely 8½ amp. Use the formula

$$I = 0.00004584(\text{cir mils})^{1.5}$$

where I = safe current carrying capacity, amperes

45. The equation $\dfrac{T_1}{T_2} = \left(\dfrac{P_1}{P_2}\right)^{\frac{n-1}{n}}$ relates to the temperature changes occurring during adiabatic[1] compression or expansion of a gas where T_1 and T_2 are the absolute temperatures at absolute pressures P_1 and P_2, respectively. n is a constant for any particular gas; for dry air, $n = 1.40$. (See Prob. 30.)

The equation $H = 60{,}470 \log \dfrac{B_2}{B_1}$ is used in the determination of altitudes by the use of barometer readings, where H is the difference in elevation (in feet) between two points at which the respective barometer readings are B_1 and B_2.

Derive an expression for H in terms of T_1 and T_2 only, eliminating P_1, P_2, B_1, and B_2.

[1] Adiabatic: *with no gain or loss of heat from the system.*

46. A westerly wind, locally known as the *chinook*, or *snow eater*, descending the east slope of the Rocky Mountains, is subjected to adiabatic compression and becomes relatively warm and dry. Use the formula derived in Prob. 45 to compute the final temperature at 3,000 ft altitude of an air stream originally at 0°F at 12,000 ft.

47. The average molecular weight M of a petroleum fraction may be approximately determined from its atmospheric boiling point t (degrees Centigrade) by the equation

$$\log M = 2.51 \log (t + 393) - 4.7523$$

a. Calculate M, to the nearest unit, when $t = 160°C$.

b. Solve the above equation for t (exponential form).

48. The magnitude, or brightness, of a star is given by the equation

$$M = 2.5 \log \left(\frac{I_1}{I}\right)$$

where M = the magnitude of a star whose light intensity is I
I_1 = the light intensity of a star of the first magnitude

Determine the ratio of the light intensities $\frac{I}{I_1}$ for a star of the fourth magnitude ($M = 4$). (Note that the brighter the star, the lower the magnitude: in fact, for the brighter planets M is negative.)

49. The following equations apply to a gas-filled Mazda lamp:

$$\frac{L_1}{L_0} = \left(\frac{E_1}{E_0}\right)^{-13.1} \tag{21}$$

$$\frac{Q_1}{Q_0} = \left(\frac{E_1}{E_0}\right)^{3.38} \tag{22}$$

$$\frac{W_1}{W_0} = \left(\frac{E_1}{E_0}\right)^{1.54} \tag{23}$$

where E_0 = rated voltage
E_1 = applied voltage
L_0 = average life at rated voltage
L_1 = average life at applied voltage
Q_0 = light output (lumens) at rated voltage
Q_1 = light output at applied voltage
W_0 = power consumption at rated voltage
W_1 = power consumption at applied voltage

K_0 = lumens[1] per watt at rated voltage
K_1 = lumens per watt at applied voltage

Show that $\frac{K_1}{K_0} = \left(\frac{E_1}{E_0}\right)^{1.84}$

50. If a Christmas-tree bulb may be expected to give 150 hr of operation when series-connected, eight bulbs to a string, on a 110-volt line, find the probable life when two extra bulbs are added to a string.

51. If the life of a lamp is expected to be about 1,000 hr at a rated voltage of 115 volts, find the hours of service to be expected at 120 volts (see Prob. 49).

52. A bulb rated at 115 volts is used on a 105-volt circuit. Find the percentage loss in light intensity.

53. What is the relative light efficiency K, taking $K_0 = 1$, when a 115-volt bulb is used on a 110-volt line?

54. The rate of heat transfer by radiation is given by the equation

$$Q = K\left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right]$$

where K = a constant depending upon the physical properties of the objects involved
T_1 = absolute temperature in degrees Fahrenheit of the hot body
T_2 = absolute temperature in degrees Fahrenheit of the cooler body

If the rate of heat transfer is 7500 Btu per hr when the hot and cold objects are at 1050 and 70°F, respectively, find the necessary hot-body temperature (in degrees Fahrenheit) to double the rate of radiant heat transfer, keeping the cooler body at 70°F.

55. Under certain conditions the drying of sheets of wallboard may be expressed by the equation

$$\log\left(\frac{T_0 - E}{T - E}\right) = K\theta$$

where T_0 = pounds of moisture per pound of bone-dry stock before drying
T = amount of moisture remaining (same basis) after θ hours
E = pounds of moisture per pound of bone-dry stock remaining in the stock after coming to equilibrium with the air

[1] Qualitative definition of a lumen: *a unit of light intensity emitted by a standard light source.*

used for drying, *i.e.*, E represents the limit to the drying possible under the given conditions

K = a constant

If the moisture content drops from 1.2 to 0.8 in 3 hr, with a further drop to 0.64 in another 6 hr, find the amount of moisture remaining in the stock after a drying period of indefinitely great length.

56. The resistance of a tungsten lamp filament is given by

$$\frac{R_1}{R_2} = \left(\frac{T_1}{T_2}\right)^{1.2}$$

where R_1 = resistance at room temperature (T_1)

R_2 = resistance at operating temperature (T_2)

Temperatures are both Kelvin (Centigrade absolute) or both Rankine (Fahrenheit absolute). If the resistance at 20°C is 16 ohms and the operating resistance is 232 ohms, find the operating temperature in degrees Centigrade.

57. A tungsten lamp is rated at 60 watts at 115 volts. If the resistance measured at 75°F is 15.2 ohms, find the operating temperature in degrees Fahrenheit. Ohms $= \dfrac{(\text{volts})^2}{\text{watts}}$ (see Prob. 56).

58. The horsepower necessary to compress a gas in a single-stage compressor is given by the formula

$$\text{hp} = \frac{144nP_1V_1}{33{,}000(n-1)}\left[\left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} - 1\right]$$

Find the horsepower necessary to compress in one minute 240 cu ft of helium from atmospheric pressure (14.7 psia) to 250 psig. For helium, $n = 1.66$. Other symbols have same meaning as in Probs. 28 to 31, page 293.

59. The weight P in pounds that will crush a solid cylindrical cast-iron column is given by the formula

$$P = 6.759 \times 10^6 \frac{d^{3.55}}{l^{1.7}}$$

where d and l are the diameter and length of the cylinder in inches.

What weight will crush a cast-iron column 72 in. high and 4.3 in. in diameter?

60. The weight in pounds W of a cubic foot of saturated steam depends upon the boiler pressure P (pounds per square inch abso-

lute) according to the formula

$$W = \frac{P^{0.941}}{330.36}$$

Find W when P is 265 psig. Assume atmospheric pressure is 15 psi.

14·31 Relation of Logarithms to the Slide Rule. An arrangement of uniform scales similar to those illustrated in Fig. 14·4 could be used for mechanically adding or subtracting numbers.

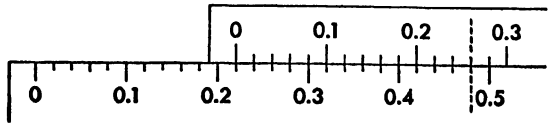


Fig. 14·4

This figure illustrates the addition $0.22 + 0.26 = 0.48$. It also illustrates the subtraction $0.48 - 0.26 = 0.22$.

As an example of multiplication, multiply 1.66 by 1.82. Looking up three-place logarithms, we find 0.220 and 0.260 to be the respective mantissas. We may use the scales to add 0.220 and 0.260, obtaining

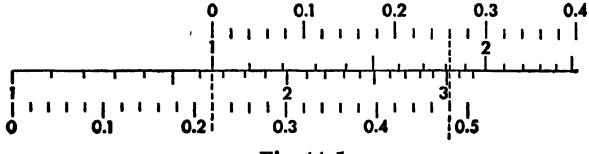


Fig. 14·5

0.480, as in Fig. 14·4. Looking up the antilogarithm of 0.480, we obtain 3.02.

Now, such a slide rule would not be of much help to us. However, if we replace the uniform scales which we have been using to represent mantissas by scales bearing the numbers to which these mantissas

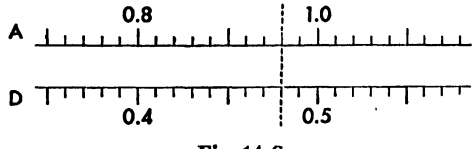


Fig. 14·6

correspond, we will have eliminated the necessity of looking up logarithms and antilogarithms (Fig. 14·5). Thus the scheme for mechanically adding 0.22 and 0.26 becomes a means of multiplying the antilogarithms 1.66 and 1.82 to get 3.02. Evidently, by reversing the procedure, $0.48 - 0.26 = 0.22$ corresponds to $3.02 \div 1.82 = 1.66$.

In Fig. 14·6 it will be observed that opposite any given number n on the D scale we read $2n$ on the A scale.

If we wished to square 3.02, we could look up the (three-place) mantissa = 0.480, double by the arrangement in Fig. 14·6, and look up the antilogarithm of 0.96, obtaining 9.12. Obviously we stand to gain here also by placing the antilogarithms in the place of the corresponding mantissas. Thus the process of mechanically multiplying 0.48

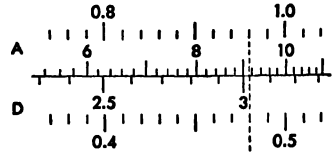


Fig. 14·7

by 2 to get 0.96 becomes the operation of squaring 3.02 to obtain 9.12. Likewise the operation $0.96/2 = 0.48$ is replaced by $\sqrt{9.12} = 3.02$ (Fig. 14·7).

These illustrations will serve to indicate that the slide rule is basically a device to add or subtract quantities (mantissas) mechanically. A knowledge of the laws of logarithms (Sec. 14·5) will be naturally helpful in developing facility in the use of the slide rule.

14·32 Logarithmic Computations on the Slide Rule. It will be noticed that the uniform scale we have been discussing in Sec. 14·31

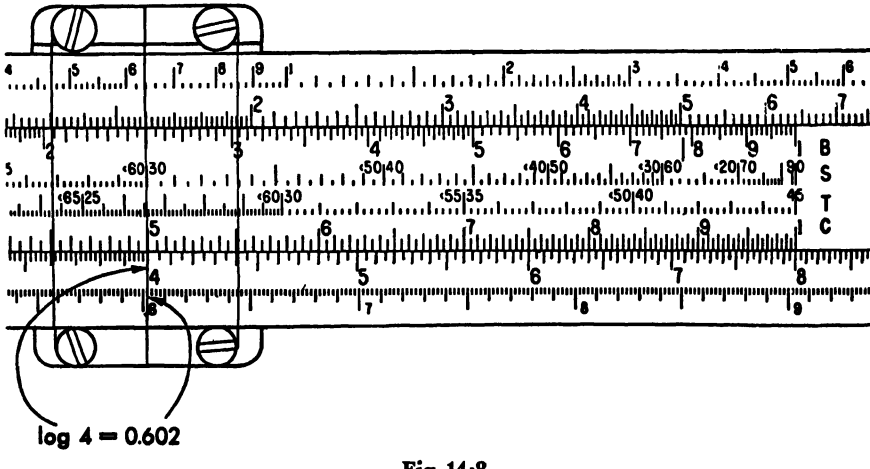


Fig. 14·8

is engraved on the slide rule as the L scale. If we set the hairline over a given number on the D scale, we shall at the same time find under the hairline on the L scale the mantissa of the logarithm of that

number. Thus in Fig. 14·8 we find illustrated log 4 = 0.602; in Fig. 14·9, log 8 = 0.903; in Fig. 14·10, the mantissa of log 16 = 0.204, or log 16 = 1.204.

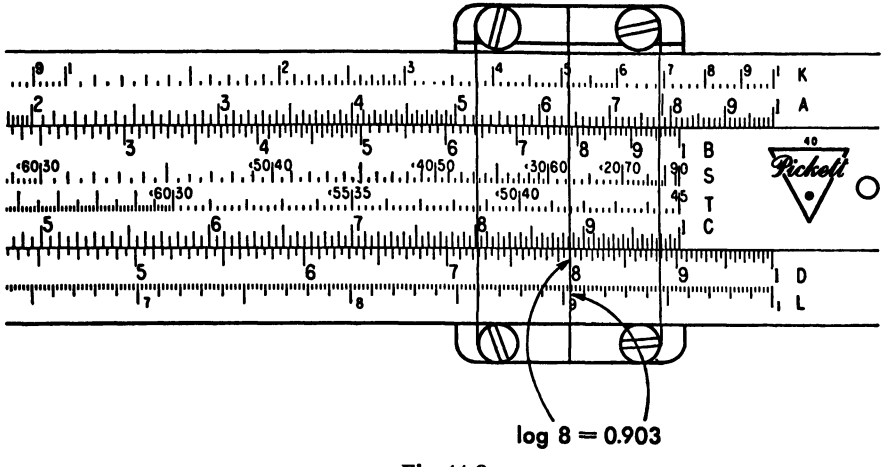


Fig. 14·9

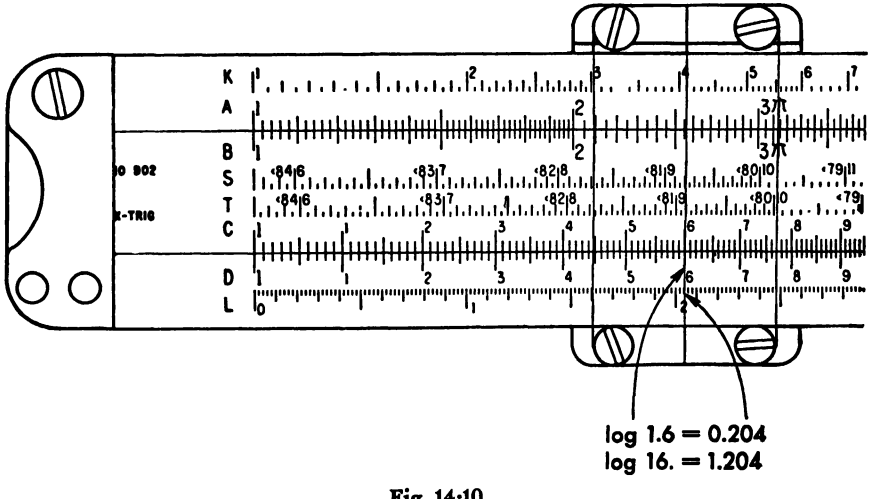


Fig. 14·10

Let us illustrate the use of the L scale in certain logarithmic computations.

EXAMPLE 27. Solve the equation $4^{3/2} = x$. Taking logarithms, we have 1.5 log 4 = log x (Sec. 14·25). In Fig. 14·8 we have shown that log 4 = 0.602. Figure 14·11 shows that 1.5 × 0.602 = 0.903. If log x = 0.903, then antilog 0.903 = x = 8. This step is illustrated in Fig. 14·9.

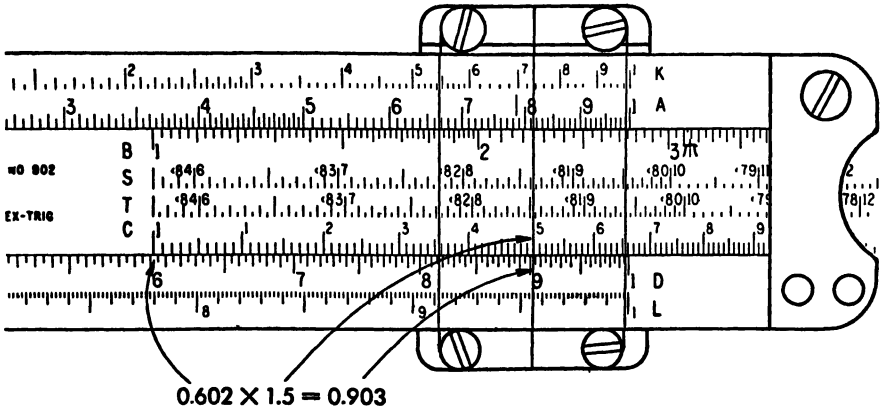


Fig. 14·11

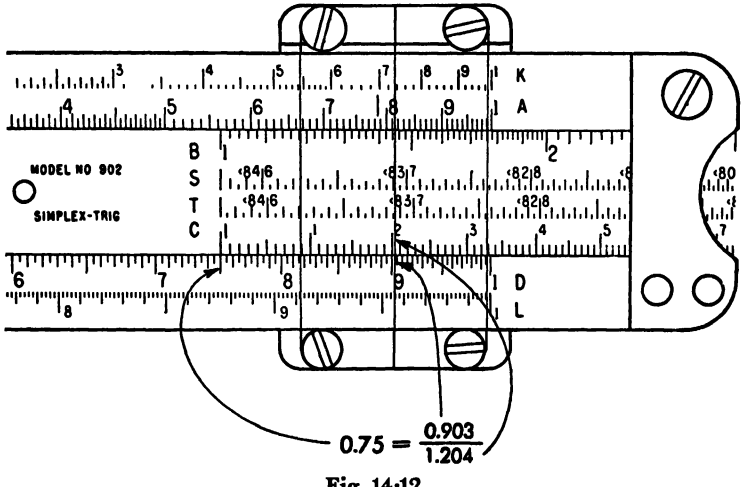


Fig. 14·12

EXAMPLE 28. Solve the equation $16^x = 8$. Taking logs of both sides, we have $x \log 16 = \log 8$. Since the settings in Fig. 14·10 and 14·9 show that $\log 16 = 1.204$ and $\log 8 = 0.903$, we may write $1.204x = 0.903$. Finally Fig. 14·12 illustrates the division $0.903 \div 1.204 = 0.75 = x$.

Chapter 15 MATHEMATICS OF INVESTMENT

It is a rare individual who has not at some time made his acquaintance with the problem of loaning or borrowing money. Such transactions include the accumulation of funds in a building and loan association, the payment of premiums on an insurance policy, purchase of a car on a time-payment basis, and the reduction of a mortgage on (and building up an equity in) a house.

Many such problems require for their solution a knowledge of logarithms, binomial theorem, and the progressions. While tables for the solution of such problems are in common use, they do not cover all cases. Ability to select and use a formula, then, becomes essential.

15·1 Principal and Interest. When money is loaned, a fee is charged for its use. The money loaned is called the *principal*, and the fee for its use is called the *interest*. The *rate* of interest is the ratio of the interest earned in a unit time to the principal. This is usually expressed as a percentage paid for the use of the money for 1 year. Interest charged may be either simple or compound.

15·2 Simple Interest. Interest which is paid only on the original principal is called *simple interest*. On a principal P borrowed at rate of interest i for n years, the simple interest I is

$$I = Pin \qquad \star (1)$$

This formula indicates that the interest is jointly proportional to the principal, rate, and time.

The rate of interest is substituted as a decimal. The number of years may be, and frequently is, a fractional value. Simple interest is not so commonly used for terms longer than a year.

When the interest is added to the original principal, the total is called the *amount* A. Thus

$$A = P + I = P(1 + in) \qquad \star (2)$$

Ordinary simple interest is based on a year of 360 days divided into 12 equal months of 30 days each.

EXAMPLE 1. Find the ordinary simple interest and amount on \$500 borrowed from April 1 to September 15 of the same year. The rate of interest is 6 per cent.

Here $P = 500$, $i = 0.06$, and $n = {}^{165}\!/_{360}$. Then

$$I = Pin = 500(0.06)({}^{165}\!/_{360}) = \$13.75 \text{ interest}$$
$$A = P + I = 500 + 13.75 = \$513.75 \text{ amount}$$

15·3 Compound Interest. In this chapter we are more interested in the applications of compound interest. When interest is added to the principal at the end of an interest period and interest during the next period is calculated on this newly augmented principal, the interest is said to be *converted* or *compounded*. Interest so computed is called *compound interest*. The amount accumulated in a given time is the total of compound interest and original principal and is called *compound amount*.

Compound interest is quite commonly converted or compounded more often than once a year. In our discussion of compound interest we shall employ the following nomenclature:

P = principal
S = compound amount
j = rate of interest
n = length of term of loan of money in years
m = number of interest conversions per year

The time between successive compoundings or conversions of interest to principal is called the *conversion period* and is $1/m$ years.

Perhaps an illustration of some particular case will serve to clarify the computation of compound interest.

Let us consider a principal P invested at 6 per cent compounded annually. At the end of 1 year the amount will be $P(1.06)$. According to our definition of compound interest, this value now becomes the principal on which to compute interest for the second year. Therefore, we multiply $P(1.06)$ by 1.06 and obtain $P(1.06)^2$ as the compound amount at the end of the second year. Continuing the process, we get $P(1.06)^3$ at the end of 3 years and so on to $P(1.06)^n$ at the end of n years.

Now, if interest is compounded semiannually, the principal will earn 3 per cent in a half year, but in n years there will be $2n$ interest conversions. Accordingly, our compound amount, which under annual compounding is $P(1.06)^n$, becomes $S = P(1.03)^{2n}$.

If interest is compounded quarterly, the principal will earn $1\frac{1}{2}$

per cent in a period of a quarter year, but in n years there will be $4n$ conversion periods. Hence we get

$$S = P(1.015)^{4n}$$

15·4 General Formula. From consideration of this specific case it appears that the compound amount S, derived from the investment of principal P, at rate j, converted m times per year for n years, is

$$S = P\left(1 + \frac{j}{m}\right)^{mn} \qquad \star (3)$$

EXAMPLE 2. Find the compound amount of \$450 invested for 13 years at 5 per cent interest compounded semiannually.

$$S = P\left(1 + \frac{j}{m}\right)^{mn}$$

$$= 450\left(1 + \frac{0.05}{2}\right)^{2\times 13}$$

$$= 450(1.025)^{26}$$

$$\log 1.025 = 0.0107239 \qquad \text{(seven-place table)}$$

$$\log (1.025)^{26} = 26 \log 1.025 = 0.27882$$

$$\log 450 = 2.65321$$

$$\log S = \overline{2.93203}$$

or

$$\$855.12 = S$$

Note that if log 1.025 had been read from a five-place table, we would have read log 1.025 = 0.01072, a value of only four significant figures. When this is multiplied by 26, the discrepancy between the five-place and the seven-place logarithms is also multiplied by 26. (Note remark in Appendix A, Sec. A·19 on errors of powers.) Because of the compounding of this error introduced by the five-place table, we obtain \$854.92 as the answer to the above example.

Numbers near unity, whose mantissas are very small, should therefore be looked up in a more accurate table than is required by the rest of the computation. Accordingly, if the principal computation is based on a five-place table, log $(1 + j/m)$ should be read from a table of at least seven places.

The value of $(1 + 0.05/2)^{2\times 13}$ can be read directly from a table of amounts at compound interest. Referring to the values $j/m = 0.025$ and $mn = 26$, the compound amount of \$1 is found to be 1.90029270. Since (450)(1.90029270) will be around \$900, our answer must be correct to five significant figures if it is to be correct to the nearest cent. Accordingly, we retain six significant figures and obtain S = (450)(1.90029) = \$855.13.

EXAMPLE 3. A municipal \$1,000 bond is sold for \$800 and matures in 12 years. If interest is compounded semiannually, find the rate.

$$1{,}000 = 800\left(1 + \frac{j}{2}\right)^{2\times 12}$$

$$1.25 = \left(1 + \frac{j}{2}\right)^{24}$$

$$\log 1.25 = 0.09691 = 24 \log\left(1 + \frac{j}{2}\right)$$

$$\frac{0.09691}{24} = 0.004038 = \log\left(1 + \frac{j}{2}\right)$$

$$1 + \frac{j}{2} = 1.00934$$

$$\frac{j}{2} = 0.00934$$

$$j = 0.0187, \text{ or } 1.87 \text{ per cent}$$

EXAMPLE 4. How long will it take \$1,500 to amount to \$2,014.20 at 2¼ per cent interest compounded semiannually?

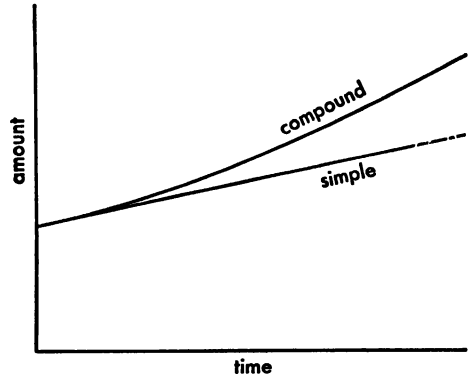


Fig. 15·1

Substituting $S = 2{,}014.20$ and $P = 1{,}500$ in Eq. (3), we obtain

$$2{,}014.20 = 1{,}500\left(1 + \frac{0.0225}{2}\right)^{2n}$$

or

$$(1.01125)^{2n} = 1.3428$$

Referring to $j/m = 1\frac{1}{8}$ per cent in a table of amounts at compound interest, we find that 1.3428 corresponds to a value of mn between 26 and 27.

Therefore, $2n = 26+$, or n is between 13 and 13½ years.

It will be observed that the logarithmic solution illustrated in Example 26, page 288, might be employed here to obtain greater

accuracy. However, the solution based on the use of the table is quicker and easier and is sufficiently accurate for the purpose.

The contrast between simple interest, in which the earning power is confined to the original principal, and compound interest is shown in Table 15·1 and in the accompanying graph (Fig. 15·1). The interest rate in this illustration is 6 per cent compounded annually. The principal is taken as \$10.

Table 15·1

n	*Simple amount* $= 10(1 + 0.06n)$ *by Eq.* (2)	*Compound amount* $= 10(1.06)^n$ *by Eq.* (3)
1	\$10.60	\$10.60
2	11.20	11.24
3	11.80	11.91
4	12.40	12 62
5	13.00	13 38
6	13.60	14.19
7	14.20	15.04
8	14.80	15.94
9	15.40	16 89
10	16.00	17 91
11	16.60	18 98
12	17.20	20.12
13	17.80	21.33
14	18.40	22.61
15	19.00	23.97

It will readily be seen that the higher the interest rate and the longer the term of the loan, the greater will be the difference between the simple and compound amounts.

15·5 Present Value. P may also be defined as the *present value* of a sum S due in n years at compound interest j. In other words, if a man signs a note calling for payment of S dollars in n years, the obligation could be discharged by payment of P dollars now.

Solving Eq. (3) for P, we obtain for present value

$$P = \frac{S}{(1 + j/m)^{mn}} \qquad \star (4)$$

EXAMPLE 5. What sum must be invested now at 3 per cent compounded quarterly to amount to \$2,000 in 15 years? (Find the present value.)

Substituting $S = 2{,}000$, $j = 0.03$, $m = 4$, and $n = 15$ in Eq. (4),

$$P = \frac{S}{(1 + j/m)^{mn}}$$

we have

$$P = \frac{2{,}000}{(1 + 0.03/4)^{4\times 15}}$$

or

$$P = \frac{2{,}000}{(1.0075)^{60}}$$

$$\log 1.0075 = 0.0032451$$

$$\log (1.0075)^{60} = 60 \log 1.0075 = 0.19471$$

$$\log 2{,}000 = 3.30103$$

$$\log (1.0075)^{60} = 0.19471$$

$$\log P = \bar{3}.10632$$

or

$$\$1{,}277.40 = P$$

The value of $\frac{1}{(1.0075)^{60}}$ may be read from a table of present values by referring to a rate of ¾ per cent and a time of 60. It is found to be 0.63869970. Since our answer is to be correct to six significant figures, we take P as equal to (2,000)(0.6386997) and obtain \$1,277.40.

EXERCISE 1

In the following problems any one of the quantities S, P, j, or n may be unknown. Of these the determination of time is perhaps least often required.

S or P may be obtained either from logarithmic computation or from tables. n may be approximated well enough from tables as a rule, but unless an extensive table of values of j/m is available, j had best be calculated by logarithms.

1. Compare the compound amounts of \$1,000 invested for 5 years at 1, 2, 3, 4, 5, and 6 per cent compounded annually. How do these values compare with the corresponding amounts at simple interest?
2. Compare the compound amounts of \$1,000 invested at 5 per cent compounded annually for 1, 2, 3, 4, 5, and 6 years. How do these values compare with the corresponding amounts at simple interest?
3. Find the rate of interest when \$590 in 9½ years amounts to \$1,072 with interest compounded quarterly.
4. How long will it take \$1,750 to amount to \$2,240 at 5 per cent compounded semiannually?
5. Find the compound amount when \$2,500 is invested for 17 years at 6 per cent compounded quarterly.
6. A savings E bond costing \$18.75 will be redeemed in 10 years for \$25. Find the rate of interest if compounded annually.

7. Find the compound amount if \$500 is invested for 7 years at 4 per cent interest compounded, (**a**) annually, (**b**) semiannually, (**c**) quarterly.
8. How long will it take \$730.10 to amount to \$1,200 at 5 per cent interest compounded quarterly?
9. A man wishes to purchase an endowment of \$7,500 to be paid to his son, who is now 8 years old, on the boy's twenty-first birthday. How much must he pay for it if interest is at 4 per cent compounded semiannually?
10. If a note calls for a payment of \$4,000 at the end of 10 years, what is the present value if interest is at 6 per cent compounded quarterly?
11. A man, wishing to provide for the education of his daughter, deposits a sum of money on her third birthday. How much should be deposited at 5 per cent compounded annually to amount to \$6,000 when the daughter reaches 18 years of age?
12. A man wishes to present each of his three grandchildren aged 2, 7, and 9 years with \$1,000 when each reaches the age of 18. If he can invest his money at 4½ per cent compounded semiannually, how much must he set aside for that purpose now?
13. Calculate the compound amount of \$1,000 for 5 years at 6 per cent compounded annually, semiannually, quarterly, and monthly. Compare the results.
14. Which yields the better annual return, and by how much, on \$400: investment (**a**) in a savings bank paying 3 per cent convertible quarterly or (**b**) in bonds yielding 3¼ per cent annually?
15. Calculate the final compound amount on \$1,400 which was invested for 4 years at 3½ per cent compounded semiannually and then reinvested at 5 per cent compounded quarterly for the next 3 years.
16. At what rate of interest convertible quarterly will a sum of money triple itself in 22 years?
17. A sum of money at compound interest is found to double itself every 14½ years. How long would it take for the sum to triple itself under the same conditions?
18. A man leaves a \$12,000 investment to a college with the provision that it remain undisturbed until the investment has amounted to \$19,000. If the investment yields 5 per cent compounded quarterly, when will the bequest become available?
19. Find the rate of interest to 0.01 per cent if \$850 amounts to \$1,225 in 12 years, interest being compounded quarterly.
20. The compound interest on \$625 for 5 years at a certain rate compounded semiannually is \$187.50. Find the rate.

15·6 Nominal vs. Effective Rate. When interest at a stated rate is converted to principal more often than once a year, the interest actually earned by \$1 in a year is called the *effective rate*. The stated or quoted rate is known as the *nominal rate*. When interest is compounded annually, the nominal and effective rates are equal.

Let us consider the specific case of \$1 invested for one year at 5 per cent compounded quarterly.

$$S = P\left(1 + \frac{j}{m}\right)^{mn} \qquad (3)$$

$$= 1\left(1 + \frac{0.05}{4}\right)^{4\times 1}$$

$$= (1.0125)^4 = 1.0509$$

Therefore, \$1 earned \$0.0509 interest, and the effective rate is 5.09 per cent, although the nominal or stated rate j is 5 per cent.

The value 1.0509 may be read directly from a table of amounts at compound interest by referring to a rate of 1¼ per cent and a time of 4.

If i represents the effective or equivalent rate, we have

$$1 + i = \left(1 + \frac{j}{m}\right)^m$$

or

$$i = \left(1 + \frac{j}{m}\right)^m - 1 \qquad \star (5)$$

EXERCISE 2

Find to 0.01 per cent the nominal or effective rate as indicated.

1. Find the effective rate when interest is compounded quarterly at the nominal rate of 8 per cent.
2. What effective rate is equivalent to a nominal rate of 6 per cent compounded monthly?
3. What effective rate corresponds to 5½ per cent compounded quarterly?
4. Calculate the effective rate equivalent to a nominal rate of 4½ per cent compounded (**a**) annually, (**b**) semiannually, (**c**) quarterly, (**d**) monthly, (**e**) weekly, (**f**) daily.
5. What nominal rate compounded quarterly is equivalent to an effective rate of 6¼ per cent?
6. An effective rate of 7½ per cent corresponds to what nominal rate compounded (**a**) annually, (**b**) semiannually, (**c**) quarterly, (**d**) monthly?

15·7 Use of Binomial Theorem. When neither compound interest tables nor adequate logarithm tables are available, the binomial theorem may be used to calculate the expression $(1 + j/m)^{mn}$, called the *accumulation factor.*

The need of improved accuracy is particularly evident in the evaluation of the expression $(1 + j/m)^{mn} - 1$. If we wish five significant figures in this expression, even a seven-place table of logarithms may be inadequate.

EXERCISE 3

In the following problems use the binomial theorem to calculate the value of $(1 + j/m)^{mn}$ to the extent necessary to yield five significant figures in the expression $(1 + j/m)^{mn} - 1$ or $1 - (1 + j/m)^{-mn}$, as the case may be.

1. $(1.02)^{14}$ **2.** $(1.015)^{16}$ **3.** $(1.005)^{20}$

4. $\left(1 + \frac{0.04}{12}\right)^{60}$ **5.** $\left(1 + \frac{0.02}{12}\right)^{90}$ **6.** $(1.0075)^{80}$

7. $(1.025)^{-8}$ **8.** $(1.04)^{-5}$ **9.** $\left(1 + \frac{0.035}{12}\right)^{-180}$

15·8 Annuities. A series of payments (usually equal) made at regular time intervals is called an *annuity*. Annuities are used in making time payments, payments on pensions, mortgages, etc. They are also involved in the accumulation of funds, such as sinking funds and shares in building and loan associations.

Payments are made at the *end* of each payment period. This is called an *ordinary* annuity as distinguished from an *annuity due*, in which payments are made at the *beginning* of each period. In this discussion the term *annuity* will be assumed to mean ordinary annuity.

Payment periods are usually a year, a half year, a quarter year, or a month. The *term* of an annuity is the time between the beginning of the first payment period and the end of the last. The amount to which the payments and their earned interest accumulate is called the *amount of the annuity*. In other words, the amount of annuity is the sum of the compound amounts of all the payments computed at the end of the term.

15·9 Amount of Annuity. In order to develop a formula for the amount of an annuity, we must find the sum of the terms of a geometric progression.

Consider the annuity whose amount is S; one of the equal payments

is p, the dollars interest accumulated by \$1 over one interest period is r, and the time expressed in interest periods is t.

It will be noted in our illustration that payments are made and interest is converted at equal intervals simultaneously. This limitation will apply throughout our treatment of annuities.

Remembering that payments are made at the end of a payment period, the payments and their compound amounts may be indicated as in Table 15·2.

Table 15·2

End of period	*Payments* 1*st*	2*d*	3*d*	4*th*	*Last* (t)
1	p				
2	$p(1+r)$	p			
3	$p(1+r)^2$	$p(1+r)$	p		
4	$p(1+r)^3$	$p(1+r)^2$	$p(1+r)$	p	
.	.	.	.	.	
.	.	.	.	.	
.	.	.	.	.	
t	$p(1+r)^{t-1}$	$p(1+r)^{t-2}$	$p(1+r)^{t-3}$	$p(1+r)^{t-4}$	p

Note that the last payment is made at the end of the term and therefore accumulates no interest.

From the last line of Table 15·2 we may write

$$S = p(1+r)^{t-1} + p(1+r)^{t-2} + p(1+r)^{t-3} \cdots p \qquad (6)$$

Evidently we have a geometric series in which the ratio of the progression is $(1+r)^{-1}$, the first term is $p(1+r)^{t-1}$, and the last is p.

Substituting the values in Eq. (6) in the formula

$$S = \frac{a - rl}{1 - r} \qquad \text{[Eq. (20), page 252]}$$

we get

$$S = \frac{p(1+r)^{t-1} - (1+r)^{-1}p}{1 - (1+r)^{-1}} \qquad (7)$$

Multiplying numerator and denominator by $1 + r$, we obtain

$$S = \frac{p(1+r)^t - p}{1 + r - 1}$$

or

$$S = \frac{p[(1+r)^t - 1]}{r} \qquad (8)$$

But according to our definitions of r and t, $r = j/m$ and $t = mn$. Therefore the amount of annuity is given by

$$S = \frac{p[(1 + j/m)^{mn} - 1]}{j/m} \qquad \star (9)$$

Figure 15·2 shows the stepwise manner in which the amount of the annuity is accumulated. Note the increasing rate of growth in dollars per year as S is being built up.

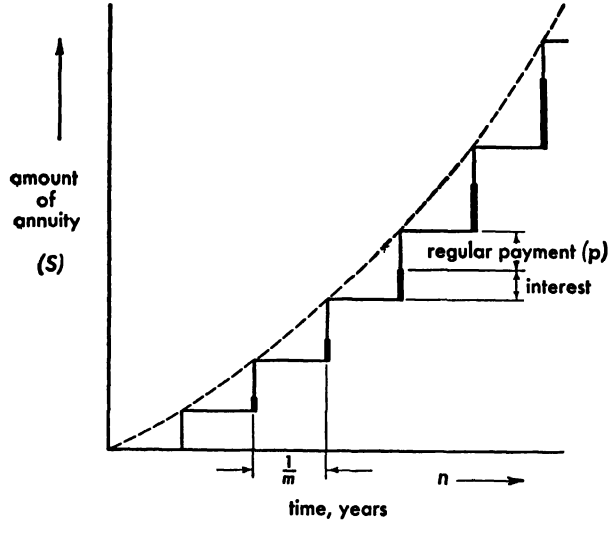


Fig. 15·2

EXAMPLE 6. A man deposits \$30 at the end of each month with a building and loan association. Interest is 4½ per cent compounded monthly. Find the amount of the annuity after 11 years.

$$\begin{aligned} S &= \frac{p[(1 + j/m)^{mn} - 1]}{j/m} \qquad (9) \\ &= \frac{30[(1 + 0.045/12)^{12\times 11} - 1]}{0.045/12} \\ &= \frac{30[(1.00375)^{132} - 1]}{0.00375} \\ &= 8{,}000[(1.00375)^{132} - 1] \end{aligned}$$

Completing the solution by logarithms, we obtain $S = \$5{,}112$.

Our shareholder deposited $(12)(11)(30) = \$3{,}960$ of his own money. The earned interest came to \$1,152. If amount-of-annuity tables are available, we refer to a rate of ⅜ per cent and a time of 132 and read the value of

$\frac{(1.00375)^{132} - 1}{0.00375}$ as equal to 170.3947. Therefore, for \$30 deposits instead of \$1 deposits, we obtain

$$S = (30)(170.3947) = \$5{,}111.84$$

EXAMPLE 7. A man deposits \$100 every 3 months with a building and loan association. In 8 years the amount of the annuity is \$4,000. Find the nominal rate of interest if converted quarterly.

There is no direct solution for j. We shall find j by a trial in which we substitute various values of j and solve for S, regarding it as unknown. Interpolation may be used in the final step to determine the value of j to produce the desired value of S.

As a first approximation, we can assume the average amount accumulated over the entire term is \$2,000 (the average of \$0 at the beginning and \$4,000 at the end of the term). The interest earned is $4{,}000 - 32(100) = 800$. Using the formula for simple interest, $I = Pin$, we obtain $800 = 2{,}000\ (i)(8)$ or $i = 0.05$ or 5 per cent. Hence we shall calculate S from Eq. (9), using $j = 0.05$ as a first approximation.

This calculation indicates $S = 3{,}905$, which is too small; therefore, we shall try a larger value of j. This and other trials are tabulated in Table 15·3.

Table 15·3

j	S
0.05	3,905
0.06	4,069
0.055	3,986
0.056	4,003

By interpolation it would appear that a choice of $j = 0.0558$ would make $S = 4{,}000$. Therefore, we consider that the nominal rate is 5.58 per cent, approximately.

EXERCISE 4

Find the amount of the annuity, the term, the periodic payment, or the rate as required.

1. \$60 is invested quarterly at 5 per cent interest compounded quarterly. Find the amount of the annuity after 6½ years.
2. A man pays \$25 quarterly into a building and loan association which pays 4½ per cent compounded quarterly. How much will his stock be worth in 7½ years?
3. Every six months \$25 is deposited with interest at 5 per cent compounded semiannually. How long will it take the compound amount to equal \$1,150?

4. A machine costing \$3,000 has a probable life of 20 years. At the end of this time its scrap value is expected to be \$500. If interest is paid at 3 per cent compounded annually, what equal annual deposits must be made, beginning 1 year from the time the machine is bought, to provide for its replacement?
5. A man pays \$15 per month for 12 years into a building and loan association which pays 5½ per cent compounded monthly. What will be the value of his stock at the end of that time?
6. A member of a building and loan association has paid \$10 at the end of each month for the last 6 years and now has \$917 to his credit. Find the nominal rate of interest if compounded monthly. To what effective rate of interest does this correspond?
7. In 15 years \$2,000 is to be accumulated by making regular quarterly deposits, interest being 4½ per cent compounded quarterly. What regular deposit is made?
8. \$10 deposited each month would take how long to amount to \$3,000 with interest at 4.8 per cent compounded monthly?

15·10 Present Value of an Annuity. The *present value* of an annuity is the sum, at the beginning of the term, of the present values of all the future payments. In other words, if we elected to pay cash for a car now in one lump sum instead of on the installment plan, the cash payment would be the present value of all the installments.

Consider the annuity of which the present value is P, one of the equal payments is p, the dollars interest accumulated by \$1 over one interest period is r, and the time expressed in interest periods is t. Then

$$P = p(1+r)^{-1} + p(1+r)^{-2} \cdots p(1+r)^{-t+1} + p(1+r)^{-t}$$

Here we have a geometric progression in which the first term is $p(1+r)^{-1}$, the last term is $p(1+r)^{-t}$, and the ratio is $(1+r)^{-1}$. Accordingly, the sum P equals

$$\frac{p(1+r)^{-1} - (1+r)^{-1}[p(1+r)^{-t}]}{1-(1+r)^{-1}} \text{ [Eq. (20), page 252]}$$

$$= \frac{p - p(1+r)^{-t}}{1+r-1} = \frac{p[1-(1+r)^{-t}]}{r}$$

Substituting j/m for r and mn for t, we obtain for present value of annuity

$$P = \frac{p[1-(1+j/m)^{-mn}]}{j/m} \qquad \star (10)$$

15·11 Time Payments. It is commonly necessary to find the regular payment, or installment p. The formula then becomes

$$p = \frac{P(j/m)}{1 - (1 + j/m)^{-mn}} \qquad \star (11)$$

Figure 15·3 indicates the stepwise fashion in which an indebtedness is reduced by regular time payments. Note that the reduction of P

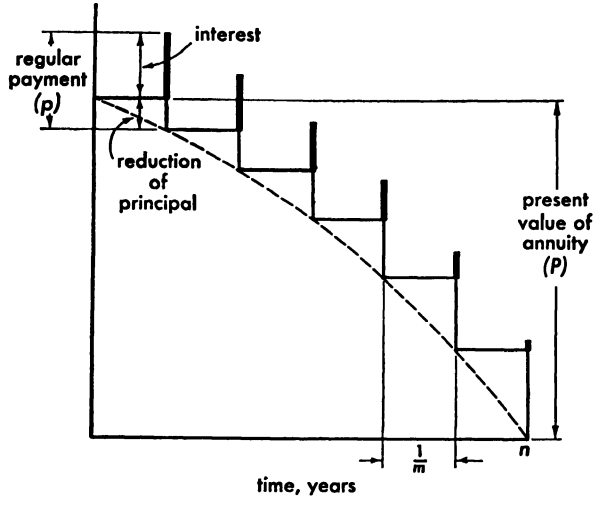


Fig. 15·3

accelerates as a smaller and smaller proportion of the payments is required for the payment of interest and a correspondingly larger portion is applied to reduction of the principal.

EXAMPLE 8. A man buys a house and lot for \$1,000 cash and pays the balance in 32 equal quarterly payments of \$200 each, the first payment being due 3 months after purchase of the property. If interest is 7 per cent compounded quarterly, determine a fair equivalent cash price.

$$P = \frac{p[1 - (1 + j/m)^{-mn}]}{j/m} \qquad (10)$$

$$= \frac{200[1 - (1 + 0.07/4)^{-4\times 8}]}{0.07/4}$$

$$= \frac{200[1 - (1.0175)^{-32}]}{0.0175}$$

$$\log 1.0175 = 0.0075344$$

$$-32(0.0075344) = -0.24110 = 0.75890 - 1$$

$$\text{antilog } 0.75890 - 1 = 0.57399$$

$$P = \frac{200(1 - 0.57399)}{0.0175}$$

$$= \frac{(200)(0.42601)}{0.0175}$$

Solving by logarithms,

$$P = 4{,}868.70$$

$$4{,}868.70 + 1{,}000 \text{ (down payment)} = \$5{,}868.70, \text{ cash price}$$

Note that the privilege of making time payments cost our property owner (32)(200) − 4,868.70, or \$1,531.30.

From a table of "present value of annuity" (referring to rate = $1\frac{3}{4}$ per cent and time = 32) we observe that $\frac{1-(1.0175)^{-32}}{0.0175} = 24.34385897$, or the present value of an annuity of 32 quarterly payments of \$1 each. For \$200 payments we obtain

$$P = 200(24.34386) = \$4{,}868.77$$

EXAMPLE 9. A man has \$22,000 in an investment trust which pays $5\frac{1}{2}$ per cent interest compounded semiannually. He wishes to withdraw equal semiannual amounts for 15 years, at the end of which time principal and interest on the diminishing principal will have been exhausted. Find the amount of each payment.

$$p = \frac{P(j/m)}{1-(1+j/m)^{-mn}} \tag{11}$$

$$= \frac{22{,}000(0.055/2)}{1-(1+0.055/2)^{-2\times 15}}$$

$$= \frac{605}{1-(1.0275)^{-30}}$$

$$\log 1.0275 = 0.0117818$$

$$-30(0.0117818) = -0.35345 = 0.64655 - 1$$

$$\text{antilog } 0.64655 - 1 = 0.44315$$

$$p = \frac{605}{1-0.44315} = \frac{605}{0.55685} = \$1{,}086.50$$

Referring to a rate of $2\frac{3}{4}$ per cent and a time of 30 in table of "annuities whose present value is 1," we read that one of the equal payments equals $\frac{0.0275}{1-(1.0275)^{-30}} = 0.04938442$. Therefore, when the present value is \$22,000, we have

$$p = (22{,}000)(0.04938442) = \$1{,}086.46$$

EXERCISE 5

Find the present value, the term, the periodic payment, or the rate as required. (When money is said to be worth r per cent the effective rate is indicated.) Assume annual compounding except where otherwise noted.

1. A man assumes a \$5,000 mortgage on a house. Ten equal annual payments, beginning a year from the time of purchase of the

house, are made, reducing the principal and paying interest on the unpaid principal. Find the amount of each payment if interest is compounded annually at 6 per cent.

2. A property owner pays an assessment of $68.75 for street improvements at the end of each year for 8 years. If money is worth 4½ per cent, what is the cash equivalent of this assessment?
3. A man receives a legacy in the form of an annuity of $5,500 payable each year for 12 years, but he wishes to realize its actual cash value now. What is the present value, or cash value received, if interest is at 3½ per cent compounded annually?
4. If money is worth 5 per cent, find the cash value of a machine that can be bought for $200 down and $150 at the end of each year for 3 years.
5. A contractor can buy a steam shovel for $4,000 cash and $1,800 at the end of each year for 3 years. What would be a fair cash settlement on an 8 per cent basis?
6. A tractor costs $1,500 down and $1,000 at the end of 2 years and $1,000 at the end of 4 years. What should be the cash price if money is worth 7½ per cent?
7. The sum of $5,600 is borrowed at 6 per cent compounded quarterly, to be repaid in 4½ years in quarterly payments. Find the quarterly payment.
8. The sum of $3,250 is borrowed at 7 per cent compounded semiannually, to be repaid in 6 years in semiannual payments. Find the amount of each payment.
9. A life-insurance policy matures with a value of $12,000, and the insurance company offers the option between a lump cash payment and 12 equal annual installments, the first installment payable 1 year after maturity of the policy. What will be the annual installment if money is worth 4¼ per cent?
10. Which is the better offer for a piece of property: (**a**) $2,500 cash and $1,000 at the end of each year for 5 years, or (**b**) $1,300 cash and $1,300 at the end of each year for 5 years?—both offers being computed on a 5 per cent basis.
11. A man is to receive an income of $1,800 at the end of each year for 10 years and wishes to reduce the term to 7 years. What is the annual income in the second plan if money is worth 4¾ per cent?
12. An endowment insurance policy for $1,000 having matured, the policyholder is offered the option of a cash settlement or of an annuity of $175 a year for 7 years beginning a year hence. What interest rate (compounded annually) is used by the company?
13. A finance company requires that a loan of $1,000 be repaid in

quarterly payments of \$95 for 3 years. If interest is compounded quarterly, find the rate.

15·12 Amortization and Sinking Funds. Two common ways of repaying a debt over a period of time are by means of (*a*) amortization and (*b*) a sinking fund. The most common problem is to find the periodic payments, other factors being given.

Amortization of a debt usually requires repayment in a series of regular, equal payments which include payment of interest as well as reduction of the principal. Since this is merely a variation of the problem of present value, installment buying, etc. (Sec. 15·10), it will not be discussed further, except for the case in which it is desired to close the transaction before the maturity of the debt.

If, during the term of amortization, it is desired to liquidate the debt in one final lump payment of outstanding principal, such a payment is the present value of an annuity of the remaining payments.

EXAMPLE 10. A debt of \$11,250 bearing interest at 3½ per cent compounded annually is to be amortized in 12 equal annual installments. Immediately after the eighth payment has been made, it is desired to pay the balance due in a single final payment. What final payment is required?

$$p = \frac{(11{,}250)(0.035/1)}{1 - (1 + 0.035/1)^{-1\times 12}} = \$1{,}164.20 \qquad (11)$$

The present value of an annuity of four annual payments of \$1,164.20 each is

$$P = \frac{1{,}164.20[1 - (1 + 0.035/1)^{-1\times 4}]}{0.035/1} \qquad (10)$$

$$= \$4{,}276.20 \text{ liquidating payment}$$

(Compare 4(1,164.20) or \$4,656.80 total of last four payments.)

When a *sinking fund* is used to repay a debt, the entire principal remains outstanding until maturity of the debt. The interest on the original principal is paid when due. Equal periodic payments into the sinking fund accumulate with interest to an amount (of an annuity) which will completely repay the debt at maturity. Interest payments on the debt form no part of the sinking-fund payments and may be subject to a different rate from that applied to the sinking-fund payments.

EXAMPLE 11. A debt of \$7,000 is to be repaid at the end of 4 years by means of a sinking fund. Payments into the sinking fund can be invested at 4 per cent compounded annually, while the interest on the debt is 5 per cent due annually. Find the combined annual payment.

$$\text{Payment into the sinking fund } p = \frac{S(j/m)}{(1+j/m)^{mn}-1} \text{ from Eq. (9)}$$

$$= \frac{7{,}000(0.04)}{(1.04)^4-1}$$

$$= \$1{,}648.43$$

$$\text{Interest on \$7,000 for a year at 5 per cent} = \quad 350.00$$

$$\$1{,}998.43 \text{ combined annual payment}$$

Suppose it had been decided to amortize the debt instead of using a sinking fund. Then

$$p = \frac{P(j/m)}{1-(1+j/m)^{-mn}} \tag{11}$$

$$= \frac{7{,}000(0.05)}{1-(1.05)^{-4}}$$

$$= \$1{,}928.43 \text{ annual amortization payment}$$

It can be seen that unless the sinking fund earns a higher rate of interest than accrues on the debt, amortization is a cheaper form of repayment than the sinking fund.

EXERCISE 6

1. A man buys a farm for \$22,500, giving a mortgage for 8 years at 5½ per cent payable annually. If funds are invested at 4 per cent compounded annually, what sum must be set aside annually to provide for payment of interest as it comes due and the repayment of the debt at the end of the term?
2. A firm buys equipment for a machine shop costing \$35,000, agreeing to pay off the debt in 6 years with interest at 6½ per cent payable semiannually. What sum must be set aside semiannually to provide for payment of interest as it comes due and the repayment of the debt at the end of the 6-year period, if funds are invested at 5 per cent compounded semiannually?
3. A man buys a machine valued at \$10,500, agreeing to repay the debt in 5 years, with interest at 7 per cent payable quarterly. What quarterly payment must be made to provide for the interest as it comes due and the repayment of the debt at the end of the 5-year term if funds can be invested at 5½ per cent compounded quarterly?
4. Referring to Prob. 1, what periodic payment would be required if the term were extended to 10 years?

15·13 Perpetuities. A perpetuity is an annuity whose payments are to be continued forever. A depositor who is paid the interest on a sum of money without disturbing the principal itself is receiving the payments of a perpetuity. Compare Fig. 15·4 with Figs. 15·1 and 15·3.

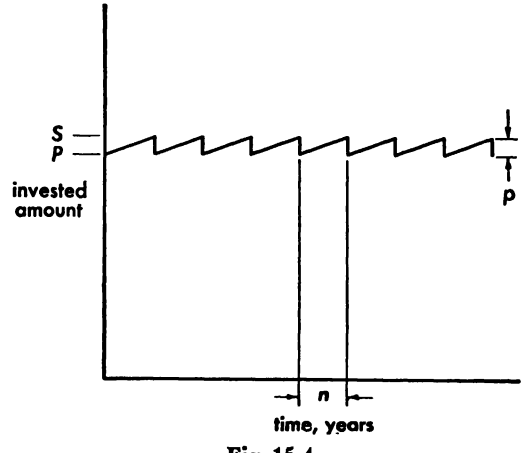


Fig. 15·4

According to Eq. (3), compound amount is

$$S = P(1 + j/m)^{mn}$$

After n years a payment p of the perpetuity is made, reducing the invested amount to the original value P (present value). Hence

$$P\left(1 + \frac{j}{m}\right)^{mn} - p = P$$

$$P\left[\left(1 + \frac{j}{m}\right)^{mn} - 1\right] = p$$

$$P = \frac{p}{\left(1 + \frac{j}{m}\right)^{mn} - 1} \qquad \star\ (12)$$

EXAMPLE 12. A pier must be rebuilt every 40 years at a cost of \$45,000. When the pier is first built, a sum is set aside to provide for an indefinite number of renewals. Assuming that the rate of interest is constant at 4½ per cent compounded annually and no change in the cost of renewals is contemplated, what sum is necessary?

$$P = \frac{45{,}000}{(1.045)^{40} - 1} = \$9{,}343.14$$

15·14 Capitalized Cost. The *capitalized cost* of an asset is equal to the first cost plus the present value of all indefinite renewals. Since the first cost is equal to the cost of a renewal (one of the perpetual payments), it follows that

$$C = p + P \qquad \star (13)$$

where C is the capitalized cost of the asset.

EXAMPLE 13. On a 5 per cent basis find the capitalized cost of a machine costing \$3,500 and lasting for 8 years.

$$P = \frac{3{,}500}{(1.05)^8 - 1} = \$7{,}330.53$$

Capitalized cost $C = p + P = 3{,}500 + 7{,}330.53 = \$10{,}830.53$.

The selection of the most attractive of a number of possible choices is often based on the capitalized cost.

EXAMPLE 14. A machine can be bought for \$800. It will last 3 years, at the end of which time it will have a negligible scrap value. A second machine which will perform the same work can be bought for \$1,300 and will last 5 years. It will also have a negligible scrap value. Which is the better buy if money is worth 6 per cent?

$$C_1 = 800 + \frac{800}{(1.06)^3 - 1} = \$4{,}988.14$$

$$C_2 = 1{,}300 + \frac{1{,}300}{(1.06)^5 - 1} = \$5{,}143.60$$

Hence, on this basis, the first machine is the better choice. However, it should be remembered that if transportation and installation costs, lost time, etc., are included, the more frequent renewals of the first machine could easily make it the more expensive purchase.

EXAMPLE 15. A machine which may be bought for \$3,000 has a probable life of 7 years and a scrap value of \$400. Use the principle of equal capitalized costs to determine a fair price for a machine which is expected to last 10 years and also has a scrap value of \$400. Money is worth 5 per cent.

The capitalized cost of the first machine is

$$3{,}000 + \frac{3{,}000 - 400}{(1.05)^7 - 1} \qquad (13)$$

The capitalized cost of the second machine is

$$x + \frac{x - 400}{(1.05)^{10} - 1}$$

where x = the first cost.

Equating the two expressions, we obtain

$$3{,}000 + \frac{2{,}600}{(1.05)^7 - 1} = x + \frac{x - 400}{(1.05)^{10} - 1}$$

$$3{,}000 + \frac{2{,}600}{0.407100} = x + \frac{x - 400}{0.628895}$$

$$x = \$3{,}869.62$$

EXERCISE 7

1. To maintain a fellowship in a university costs \$1,800 a year. Find the present value of the perpetuity (endowment fund), assuming that it is invested at 5 per cent compounded annually.
2. Find the payments of a perpetuity having a present value of \$60,000 whose payments are made every 3 years. Interest is at 4 per cent compounded semiannually.
3. Find the capitalized cost of a machine costing \$8,500 and lasting for 11 years on a 6 per cent basis.
4. Referring to Prob. 3, what is a fair purchase price of a machine expected to last 15 years if the capitalized cost is to be the same?
5. A house requires painting every 5 years at a cost of \$400. What would be a comparable cost of an asbestos-shingle siding job lasting 20 years if money is worth 5½ per cent? (Equate capitalization costs.)
6. A wooden bridge costs \$65,000 to build and must be renewed every 20 years. Find the comparable cost of a steel and concrete bridge having a probable life of 35 years. Money is worth 5 per cent.

15·15 Depreciation. The *total depreciation* of an asset is the difference between the original value and the estimated salvage value. Among the various methods of calculating depreciation perhaps the most common are (*a*) the straight-line method and (*b*) constant-percentage-of-book-value method.

In the *straight-line* method of computing depreciation, it is assumed that the depreciation or loss of value is the same for each year of the life of the asset. A table of data showing the year-by-year value of an asset would thus form an arithmetic progression. Thus, if a machine originally costs \$4,000, has an estimated life or 8 years, and has an estimated scrap value of \$400, the total depreciation will be \$3,600 and the annual depreciation will be \$450.

In the *constant-percentage* method the value of an asset at the end of a given year is computed at a fixed percentage of its value at the beginning of that year. Thus at the end of the first year

$$V_1 = F(1 - D)$$

at the end of the second year

$$V_2 = V_1(1 - D) = F(1 - D)^2$$

at the end of the third year

$$V_3 = V_2(1 - D) = F(1 - D)^3$$

and, therefore, after n years we may write

$$V = F(1 - D)^n$$

where F is the first cost, D is the rate of depreciation (as a decimal), n is the number of years, and V is the scrap value. Rearranging this equation gives us

$$D = 1 - \sqrt[n]{\frac{V}{F}} \qquad \star (14)$$

It will be seen that a table of figures listing the year-by-year value of an asset would constitute a geometric progression for which the usual rules apply.

EXAMPLE 16. Compute the rate of depreciation on our previous illustration by the fixed percentage method. From Eq. (14) we obtain

$$D = 1 - \sqrt[8]{\frac{400}{4{,}000}}$$
$$= 1 - 0.7499 = 0.2501$$

or 25.01 per cent per year.

Thus the loss in value during the first year will be (0.2501)(4,000), or \$1,000.40. The depreciation during the last year will be $\frac{400}{0.7499} - 400 =$ 533.40 − 400, or \$133.40.

It can be seen that this method results in a relatively high depreciation during the first year with progressively diminishing depreciation during subsequent years. Therefore, it is more conservative than the straight-line method in the event retirement occurs before the completion of the estimated term of service.

EXERCISE 8

1. A lathe costs \$1,425 and is estimated to last 14 years, at which time its scrap value will be \$175. If depreciation is figured on a constant-percentage basis, calculate the value of the lathe at the end of the fourth year.[1]

[1] How could semilog paper be used to estimate the value of the lathe at any time during the 14 years? (See Appendix A, Sec. A·23.)

2. A machine originally cost \$2,200 and had an estimated salvage value of \$400 at the end of 20 years. Compute the book value at the end of 5 years (**a**) by the straight-line method; (**b**) by the fixed-percentage-of-diminishing-balance method.
3. Compute by both methods the book values of the machine in Prob. 2 at the end of 15 years.
4. When will the difference between the straight-line and constant-percentage book values be a maximum in Prob. 1? Solve graphically.
5. A machine cost \$35,000 when new and is expected to last 16 years, after which time the salvage value will be \$1,000. Repairs on the machine are estimated to be negligible for the first 3 years, \$300 for the fourth year, \$600 for the fifth, \$900 for the sixth, etc. If depreciation is to be based on the fixed-percentage method and the machine is to be sold when the combined annual expense due to depreciation and repairs is a minimum, determine the optimum time to replace the machine. Solve graphically.
6. Compute the optimum time of replacement as in Prob. 5, assuming repairs to be \$200 during the first year, \$400 during the second year, \$600 during the third year, etc. Other conditions are to remain unchanged. Solve graphically.

Chapter 16 EXPONENTIAL FUNCTIONS—RATE OF GROWTH

Many phenomena of the physical sciences are governed by laws of growth and decay which involve e, the base of natural logarithms. It develops in calculus that this is indeed a "natural" logarithm.

The natural logarithm is the logical outgrowth of the situation in which the rate of growth or depreciation of a variable at any time is proportional to the amount of the variable itself at that time. To cite some fields in which e commonly occurs, we might mention heat transfer, radioactive materials, rate of chemical reaction, thermodynamics, transient electric currents, and characteristic impedance of transmission lines.

16·1 Power Function vs. Exponential Function. An expression such as x^3 in which a variable is raised to a fixed power is called a *power function.* An expression in which the variable occurs in the exponent is called an *exponential function.* Table 16·1 contrasts these functions.

Table 16·1

x	0	1	2	3	4	5	6	7
x^3	0	1	8	27	64	125	216	343
3^x	1	3	9	27	81	243	729	2,187

16·2 Compound-interest Law. Many quantities in nature grow in much the same way as a sum of money at compound interest—*i.e.*, the rate of growth is a fixed percentage of the amount on hand at the beginning of the interest period in question. However, there is one important difference. The growth of money occurs stepwise at the end of each interest-conversion period, whereas growth (and decay) in nature is usually a smooth, continuous operation; *i.e.*, the interest is converted exceedingly often (or, in effect, continuously), and the rate of growth at any time is a fixed percentage of magnitude of the quantity at that time.

16·3 Significance of e. It will be recalled from Sec. 15·4 that when a sum of money is allowed to grow undisturbed at compound interest, the compound amount S is given by the formula $S = P(1 + j/m)^{mn}$. In this connection j is the nominal rate of interest and not the imaginary unit. This nomenclature will apply throughout this chapter.

If we consider the growth of \$1 at 100 per cent compounded m times a year for one year, P, j, and n each become equal to 1, and we obtain the equation $S = (1 + 1/m)^m$.

It is apparent that for semiannual compounding $m = 2$ and $S = 2.25$. For quarterly compounding $m = 4$ and $S = 2.441$. If it were possible to compound 100 times a year, S would equal 2.705, and when $m = 10{,}000$, $S = 2.718$. Evidently S is approaching a limit of some kind as m becomes extremely large.

To arrive at a more accurate value of this limit, we shall expand $\left(1 + \frac{1}{m}\right)^m$ by the binomial theorem and allow m to increase indefinitely in the result. Thus we obtain

$$\left(1 + \frac{1}{m}\right)^m = 1 + m\left(\frac{1}{m}\right) + \frac{m(m-1)}{2!} \cdot \frac{1}{m^2} + \frac{m(m-1)(m-2)}{3!} \cdot \frac{1}{m^3} + \frac{m(m-1)(m-2)(m-3)}{4!} \cdot \frac{1}{m^4} + \cdots = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{m}\right) + \frac{1}{3!}\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right) + \frac{1}{4!}\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\left(1 - \frac{3}{m}\right) + \cdots$$

As m increases indefinitely $1/m$, $2/m$, $3/m$, etc., approach zero and we have

$$e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = 2.7183 \text{ approx} = e \quad \star (1)$$

which is the base of natural logarithms and occurs frequently in problems concerned with natural growth or decay, *e.g.*, speed of chemical reactions, decrease of atmospheric pressure with increasing altitude, radioactive decay, and rate of cooling of a warm object.

A compact arrangement for calculating e to six figures is shown below:

$$
\begin{array}{r}
1.000000 \\
2/1.000000 \\
3/0.500000 \\
4/0.166667 \\
5/0.041667 \\
6/0.008333 \\
7/0.001389 \\
8/0.000198 \\
9/0.000025 \\
\hline
0.000003 \\
\hline
2.718282
\end{array}
$$

Or $e = 2.71828$ to six significant figures.

A value for any power of e may be found by evaluating

$$\left[\left(1 + \frac{1}{m}\right)^m\right]^x = \left(1 + \frac{1}{m}\right)^{mx} \text{ as above.}$$

Thus we obtain

$$\left(1 + \frac{1}{m}\right)^{mx} = 1 + mx\left(\frac{1}{m}\right) + \frac{mx(mx-1)}{2!} \cdot \frac{1}{m^2} + \frac{mx(mx-1)(mx-2)}{3!} \cdot \frac{1}{m^3} \cdots$$

$$= 1 + x + \frac{x^2}{2!}\left(1 - \frac{1}{mx}\right) + \frac{x^3}{3!}\left(1 - \frac{1}{mx}\right)\left(1 - \frac{2}{mx}\right) \cdots$$

As m increases without limit, $\frac{1}{mx}$, $\frac{2}{mx}$, etc., approach zero, and we obtain the series for e^x:

$$e^x = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{mx} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots \quad \star (2)$$

In using Eq. (1) or (2) for computation, enough terms should be used so that the first term omitted is less than half the allowable error.

16·4 General Formula. In order to determine the effect of continuous compounding at any rate for any length of time, we shall again refer to the formula $S = P\left(1 + \frac{j}{m}\right)^{mn}$, expand by the binomial theorem, and allow m to increase without limit in the result. Thus

we obtain the expansion

$$\left(1+\frac{j}{m}\right)^{mn} = 1 + mn\left(\frac{j}{m}\right) + \frac{mn(mn-1)}{2!}\cdot\frac{j^2}{m^2} + \frac{mn(mn-1)(mn-2)}{3!}\cdot\frac{j^3}{m^3} + \cdots$$

$$= 1 + nj + \frac{n^2j^2}{2!}\left(1-\frac{1}{mn}\right) + \frac{n^3j^3}{3!}\left(1-\frac{1}{mn}\right)\left(1-\frac{2}{mn}\right) + \cdots$$

and

$$\lim_{m\to\infty}\left(1+\frac{j}{m}\right)^{mn} = 1 + nj + \frac{n^2j^2}{2!} + \frac{n^3j^3}{3!} + \cdots$$

However it can be shown by analogy to Eq. (2) that

$$e^{nj} = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mnj}$$

also equals

$$1 + nj + \frac{n^2j^2}{2!} + \frac{n^3j^3}{3!} + \cdots$$

Hence we may write

$$\lim_{m\to\infty}\left(1+\frac{j}{m}\right)^{mn} = e^{nj}$$

or

$$S = \lim_{m\to\infty} P\left(1+\frac{j}{m}\right)^{mn} = Pe^{nj}(\text{growth}) \qquad \star\ (3)$$

Observe that in computing continuous depreciation or decay instead of growth j will be negative and

$$S = \lim_{m\to\infty} P\left(1-\frac{j}{m}\right)^{mn} = Pe^{-nj}(\text{depreciation}) \qquad \star\ (4)$$

The proof of Eq. (4) is left as an exercise for the student.

EXAMPLE 1. The number of bacteria in a colony increased at an instantaneous rate (per hour) continuously equal to 20 per cent of the number at that time. Starting from a colony of 1,000 bacteria, how many would there be in 6 hr?

Substituting in Eq. (3) $P = 1{,}000$, $n = 6$, and $j = 0.2$, we have

$$S = 1{,}000e^{1.2}$$

From tables of e^x,

$$e^{1.2} = 3.320$$

Hence

$$S = 1{,}000(3.320) = 3{,}320$$

EXAMPLE 2. Radium decomposes at an instantaneous rate (per century) continuously equal to 4.1 per cent of the amount remaining at that time. If 25 mg are present at the start, how much will be left after 2,000 years?

Substituting in Eq. (4) $P = 25$, $n = 20$, and $j = 0.041$, we have

$$S = 25e^{-0.82}$$

From tables of e^{-x},

$$e^{-0.82} = 0.44043$$

Hence

$$S = 25(0.44043) = 11.011$$

EXERCISE 1

1. Using the series, calculate the value of $e^{1.6}$ to five significant figures. Check your answer by the table.
2. Repeat for $e^{0.8}$.
3. Repeat for $e^{-1.3}$.
4. Repeat for $e^{-0.35}$.
5. The speed of a given chemical reaction depends upon the temperature t according to the formula $V = 0.8e^{0.15t}$. Find V (**a**) when $t = 20°$; (**b**) when $t = 60°$.
6. The speed V of a rotating wheel after the power was cut off decreased at a rate (per minute) which at every instant was 35 per cent of V itself. If the original value of V was 1,500 rpm, find its value after (**a**) 5 min; (**b**) 10 min; (**c**) 15 min.
7. If 80 gm of cane sugar is inverted by acid so that the inversion rate is constantly 13.5 per cent (per hour) of the remaining cane sugar, find the amount of cane sugar remaining after 12 hr.
8. The temperature of a warm object initially 150° warmer than the surroundings is falling at an instantaneous rate of 60 per cent (per hour) of the temperature difference at the time. Find the temperature difference remaining after (**a**) 40 min; (**b**) 2 hr.
9. If a plant is growing so that its rate of growth is constantly 40 per cent (per week) of its weight, find the actual percentage increase in weight in a week.
10. According to Halley's law the atmospheric pressure P in inches is related to the height above sea level by the following equation: $P = 29.92e^{-h/5}$, where h is the altitude in miles. What is the barometer reading at (**a**) sea level; (**b**) 2,640 ft; (**c**) 3 miles?

11. The curve which a chain or rope assumes when hanging under its own weight is called a catenary. The equation for the catenary is $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$. If we take $a = 1$, plot the graph from $x = 2$ to $x = -2$.

16·5 Natural Logarithms. Common logarithms are based upon the fact that every positive number can be expressed as a power of 10. Similarly, it is true that every such number may be expressed as a power of e. For example, $3.8 = e^{1.335}$, and we say that the logarithm of 3.8 to the base e is 1.335. This is written $\log_e 3.8 = 1.335$ or $\ln 3.8 = 1.335$.

Natural logarithms (ln), or logarithms to the base e, follow the same rules of combination that apply to common logarithms as set forth in Sec. 14·5.

16·6 Use of the Table. A few lines of a table of natural, or Napierian, logarithms are reproduced as Table 16·2 below:

Table 16·2

N	0	1	2	3	4	5	6	7	8	9
7.0	1.9459	473	488	502	516	530	544	559	573	587
7.1	601	615	629	643	657	671	685	699	713	727
10.0	2.3026	036	046	056	066	076	086	096	106	115

From this table we read that

$$\ln 7.16 = 1.9685 \quad \text{or} \quad 7.16 = e^{1.9685}$$
$$\ln 10 = 2.3026 \quad \text{or} \quad 10 = e^{2.3026}$$

In order to find the natural logarithm of a number which lies outside the limits of the table, we cannot simply change the characteristic as in common logs. The common logarithm of 716 is $\log 7.16 + \log 100$, or $0.8549 + 2$; but since e^2 is not 100, the natural logarithm of 100 is not 2. However, we can make use of scientific notation to carry out much the same idea.

Hence we write

$$716 = 7.16 \times 10^2$$

and

$$\ln 716 = \ln 7.16 + 2 \ln 10 = 1.9685 + 2(2.3026) = 6.5737$$

Conversely, to find anti-ln 6.5737 (or the value of $e^{6.5737}$), we would subtract from 6.5737 as many 2.3026's as possible, which would be two. Therefore we would write $6.5737 - 2(2.3026) = 1.9685$ or $\ln N - 2 \ln 10 = \ln N/100$. Hence the anti-ln $1.9685 = 7.16 = N/100$, and $N = 716$.

If we wish to find ln 0.703, we write $0.703 = 7.03 \times 10^{-1}$ and ln 0.703 = ln 7.03 − ln 10 = 1.9502 − 2.3026 = −0.3524. Note that with natural logarithms we make no attempt to make the mantissa positive, since the mantissas for, say, ln 70.3, ln 7.03, or ln 0.703 would bear no resemblance to one another.

Let it be required to find anti-ln −4.9505. We shall add only enough 2.3026's to −4.9505 to make the total positive. We shall need three. Hence we write −4.9505 + 3(2.3026) = 1.9573, or ln N + 3 ln 10 = ln (1,000 N) = 1.9573. Anti-ln 1.9573 = 7.08 = 1,000 N. Hence N = 0.00708.

EXERCISE 2

1. Look up the natural logarithms of

a. 6.14 **b.** 9.08 **c.** 1.97 **d.** 33.8
e. 417 **f.** 0.216 **g.** 0.0117

2. Use interpolation to obtain the natural logarithms of

a. 49.73 **b.** 0.06492 **c.** 3,584 **d.** 0.1856

3. Look up the antilogarithms of

a. 3.6533 **b.** 9.0688 **c.** −3.0038 **d.** 5.9429 − 10
e. 11.2836

4. Use interpolation where necessary to find the antilogarithms of

a. 4.0877 **b.** 10.0623 **c.** −6.7211 **d.** 9.8754 − 10
e. 7.5873

5. Using natural logarithms, calculate

a. $e^{3.8}$ **b.** $e^{2.6}$ **c.** $e^{-1.4}$ **d.** $e^{-0.9}$
e. $56e^{-2.2}$ **f.** $452e^{1.85}$ **g.** $0.087e^{4.732}$ **h.** $5300e^{-6.185}$

6. Solve for n

a.* $346 = 195e^{10n}$ **b.** $14.7 = 68.3e^{-0.6n}$
c. $0.0183 = 0.087e^{-0.02n}$ **d.** $0.0964 = 0.0529e^{1.28n}$
e. $450 = 2{,}000e^{20n}$ **f.** $3{,}000 = 11.8e^{-50n}$

7. In Prob. 5 in Exercise 1, determine the temperature at which V = 100.

8. In the same problem determine what temperature rise is required to double V.

* Note that our task of looking up natural logarithms will be facilitated by first dividing through by 100, giving us $3.46 = 1.95e^{10n}$.

9. In Prob. 6 in Exercise 1, how long will it take for the speed to drop to one-tenth of its original value?

10. In Example 2, page 330, what is the period of half-life; *i.e.*, when will the radium be 50 per cent disintegrated?

11. In Prob. 10 in Exercise 1, what is the elevation in feet if the barometer reads 23.62 in.?

16·7 Relation between Common and Natural Logarithms. The general formula for converting the logarithm of a number from one base to another is

$$\log_a M = \frac{1}{\log_b a} \cdot (\log_b M) \quad \text{[Eq. (10), p. 268]}$$

Rearranging, we obtain

$$(\log_b a)(\log_a M) = \log_b M \tag{5}$$

Hence it follows that if $b = e$ and $a = 10$, substitution in Eq. (5) produces the formula for converting common logarithms to natural logarithms:

$$(\log_e 10)(\log_{10} M) = \log_e M$$

or

$$2.3026 \log M = \ln M \qquad \star (6)$$

Hence also

$$\frac{1}{2.3026} \ln M = \log M$$

or

$$0.43429 \ln M = \log M \qquad \star (7)$$

EXERCISE 3

1. Convert the following to common logarithms:

a. $\ln x = 3.0425$ **b.** $\ln x = 0.00488$
c. $\ln x = 7.9286 - 10$ **d.** $\ln x = 0.6493 - 2$
e. $\ln x = -1.7106$ **f.** $\ln x = -3.3822$

2. Convert the following to natural logarithms:

a. $\log x = 2.1734$ **b.** $\log x = 0.01007$
c. $\log x = 8.7466 - 10$ **d.** $\log x = 0.7843 - 2$
e. $\log x = -1.2755$ **f.** $\log x = 9.0843 - 10$

3. In Prob. 1 find x, using the values of common logarithms obtained.

4. Using only a table of common logarithms, find

a. ln 5.828 **b.** ln 73.46 **c.** ln 10,000
d. ln 0.08469 **e.** ln 0.005497 **f.** ln $\frac{1}{15}$

16·8 Equivalent Expressions. Because of an occasional lack of a table of natural logarithms or for other reasons, it is desirable to be able to express an equation in the exponential form to any required base or in terms of either system of logarithms. The following typical problem will illustrate the steps involved.

EXAMPLE 3. Given the equation $y = 8.3e^{1.2x}$.

a. Convert to the form $y = kb^x$.

b. Express as a natural logarithmic equation.

c. Express as a common logarithmic equation.

a. Since we may write $y = 8.3(e^{1.2})^x$, we can find from tables that $e^{1.2} = 3.3201$. Hence $y = 8.3(3.3201)^x$.

b. Taking natural logarithms of both sides of the equation,

$$\ln y = \ln [8.3e^{1.2x}] = \ln 8.3 + \ln e^{1.2x}$$

or

$$\ln y = 2.1163 + 1.2x$$

c. Taking common logarithms of both sides, we obtain

$$\begin{aligned} \log y &= \log [8.3e^{1.2x}] \\ &= \log 8.3 + 1.2x \log e \\ &= \log 8.3 + 1.2x(0.43429) \\ &= 0.91908 + 0.52115x \end{aligned}$$

Perhaps the most frequently used of these procedures is *c*, in which the equation is immediately expressed in terms of common logs.

EXAMPLE 4. Solve the equation in Example 2, page 330, by the use of common logs.

$$\begin{aligned} S &= 25e^{-0.82} \\ \log S &= \log 25 + \log e^{-0.82} \\ &= \log 25 - 0.82 \log e \\ &= 1.39794 - 0.82(0.43429) \\ &= 1.39794 - 0.35612 \\ &= 1.04182 \\ S &= 11.011 \end{aligned}$$

EXERCISE 4

1. Prove, without using tables, that $\dfrac{1}{\ln 10} = \log e$.

2. Express the following in exponential form:

a. $\ln y = x$ **b.** $\ln y + \ln a = x$ **c.** $a \ln y = x$

3. Express the following in natural logarithmic form:

a. $z = e^w$ **b.** $y = ae^{cx}$ **c.** $y = \frac{3}{\sqrt{e}}$

4. Find the value of the following, first using only a table of natural logarithms, then using only a table of common logarithms:

a. $e^{2.5}$ **b.** $e^{-3.5}$ **c.** $0.08552e^{3.52}$

5. Convert the following to the form $y = be^{kx}$:

a. $y = 5(1.6)^x$ **b.** $y = 3.2(0.275)^x$

6. Convert the following to the form $y = kb^x$:

a. $y = 0.63e^{2.7x}$ **b.** $y = 1.62e^{-1.4x}$

In the following problems, given only that $e = 2.718$, $\ln 10 = 2.303$, $\ln 7 = 1.946$, $\ln 17 = 2.833$, $\log 8 = 0.903$, and $\log 9 = 0.954$, calculate without tables or slide rule the values of

7. $\log e$ **8.** $\ln 8$ **9.** $\ln 9$ **10.** $\ln 3$
11. $\ln 0.008$ **12.** $\ln 90$ **13.** $\ln \frac{1}{8}$ **14.** $\ln 4$
15. $\ln 27.18$ **16.** $\log 0.2718$ **17.** $\log 170$ **18.** anti-ln 2.946
19. anti-ln 1.833 **20.** anti-ln $8.054 - 10$
21. t if $8 = 1/e^{3t}$ **22.** y if $y = 7e^{0.887}$

EXERCISE 5

1. In a circuit containing a capacitor C and a resistance R in series, the instantaneous value of the current i at any time t after the circuit has been closed is given by the equation

$$\ln i = -\frac{t}{RC} + \ln E - \ln R$$

Convert this equation to the exponential form (i.e. solve for i rather than $\ln i$).

2. Use the answer obtained in Prob. 1 to find the current 0.00002 sec after the circuit has been closed, if $E = 24$ volts, $R = 9.5$ ohms, $C = 0.000029$ farad.

3. Find i in Prob. 1 when $E = 100$, $R = 250$, $C = 25 \times 10^{-6}$, and $t = 3 \times 10^{-5}$.

4. Find i in Prob. 1 when $t = 0$.

5. In Prob. 1 what is the approximate value of i when t is very large?

6. In Prob. 1 what percentage is i of E/R when $t = RC$?

7. a. The emission current for a heated filament is given by the equation

$$i = AT^2e^{-B/T}$$

where i is expressed in amperes, A is a constant and is equal to 60, B is a constant and is equal to 52.4×10^3 for a tungsten filament, T is the temperature in degrees Kelvin. Find i when $T = 2400°\text{K}$.

b. How would you proceed to solve an equation of the type in **a** in which T is the unknown?

8. In a simple steam engine the equation for the mean effective pressure is

$$P_m = P\left(\frac{1 + \ln r}{r}\right) - p$$

Determine P_m if $P = 100$ psi, $p = 14.7$, and $r = 3.54$.

9. When light passes through a transparent medium, its intensity is reduced according to the equation

$$I = I_oe^{-kd}$$

where I_o is the initial light intensity, I is the intensity after passing through a medium of thickness d, and k is a constant depending upon the nature of the light and of the medium. If the intensity of sunlight is reduced to half of its original value after penetrating water to a depth of 4 ft, at what depth will the light intensity be 10 per cent of that at the surface?

10. The temperature of a body surrounded by a medium at a different temperature is given by the equation

$$T = T_oe^{-kt}$$

where T_o is the initial difference in the temperatures of body and surroundings, T is the temperature difference at any time t, k is a heat-transfer constant dependent upon various physical conditions and the units involved. If, during a power failure, a home freezer warms up from -10 to $+2°\text{F}$ after standing for 3 hr in a room at 50°F, how much longer will it take to warm up to 32°F?

11. A radioactive material disintegrates according to the relation

$$A = A_oe^{-kt}$$

where A_o is the original amount of active material, A is the amount of active material remaining at time t, and k is a constant characteristic of the given material. It is frequently desirable to deter-

mine the period of half-life, *i.e.*, that time at which half of the original material still remains.

a. Show that the period of half-life is equal to $0.6932/k$.

b. If 3 per cent of a certain radioactive material disintegrates in 1 hr, find the period of half-life.

12. A problem in electrical engineering required the solution of the following system of simultaneous equations:

$$E_m(1 - e^{-3n}) = 13$$
$$E_m(1 - e^{-n}) = 8$$

Solve for E_m and n to four significant figures.

13. If

$$i = \frac{E}{R}(1 - e^{-\frac{Rt}{L}})$$

find i when $E = 10$, $L = 0.9$, $R = 12$, and $t = 0.005$.

14. Find i in Prob. 13 if $t = 0$.

15. Approximately what is the value of E/R in Prob. 13 when t is very large?

16. If you were given i, E, L, and t, how would you go about solving for R?

17. When $t = L/R$, what percentage is i of E/R? Is this percentage independent of the actual values of the variables?

18. If $x = y$, refer to the principles relating e and natural logarithms to show that $x = e^{\ln y}$. (This problem is referred to in Chap. 24.)

UNIT FOUR

INTRODUCTION TO ANALYTICAL TRIGONOMETRY

This unit will be of particular interest to electrical and mechanical students

Chapter 17 VECTORS AND TRIGONOMETRY

Vectors are mathematical tools necessary for the student who intends to study the principles of structures, mechanics, electricity, etc. A knowledge of vectors is equally important for the engineer who uses these principles.

17·1 Vector and Scalar Quantities. For our purpose we shall define a vector as follows: A vector is a line segment or simply a line whose distinctive properties are *length*, *direction*, and *sense*. The *length* of a vector is the linear distance between the extremities. *Direction* is measured in terms of the angular orientation of the vector with respect to some reference line. If we consider the vector to have been generated by a moving point, the *sense* of the vector is determined by the order in which the point coincides with the extremities.

A *vector quantity* is a quantity whose pertinent characteristics can be represented by a vector.

Quantities which have no pertinent relation to direction are called *scalar quantities*.

Examples of vector quantities are force, velocity, acceleration, voltage, current, and (more loosely) impedance.

Examples of scalar quantities are such quantities as arise in the process of simple enumeration. Other examples are volume, area, and prices.

EXERCISE 1

Which of the following quantities would you expect to be vector quantities, and which would you expect to be scalar quantities?

1. The path of an airplane in flight.
2. The weight of an automobile.
3. The stress in a member of a bridge truss.
4. The area of a plot of land.

5. A magnetic field.
6. The salinity of sea water.
7. The field strength of an antenna.
8. The number of teeth on a gear.

17·2 Vector Representation. The line $\overrightarrow{AB}$ in Fig. 17·1 is a vector since it is distinguished by a *length* (7 units), a *direction* (30° referred to *xy*), and a *sense* as indicated by the arrow.

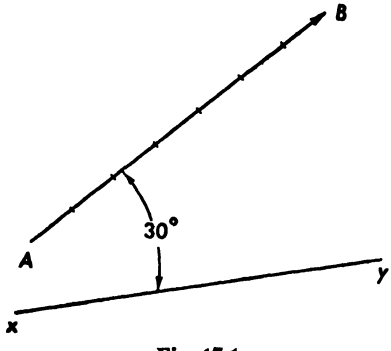


Fig. 17·1

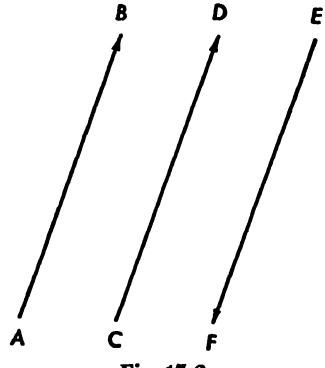


Fig. 17·2

The vector would be referred to in written symbols as $\overrightarrow{AB}$. The arrow above the letters indicates that it is a vector quantity which is under consideration and not a scalar quantity.

It is convenient to think of the vector $\overrightarrow{AB}$ as the path generated by a point which moved from position A to position B. Consequently the sense is from A toward B and is so indicated by the sequence of the letters AB.

If we happen to be interested only in the length, or magnitude, of $\overrightarrow{AB}$, we use the symbol $|AB|$.

Two vectors are said to be equal if their lengths are equal and if their direction and sense are identical (see Fig. 17·2). Here $\overrightarrow{AB} = \overrightarrow{CD}$ because of three facts: (1) their lengths are equal; (2) their directions are the same; (3) their sense is identical.

It is true that

$$|AB| = |CD| = |EF| \tag{1}$$

It is also true that the direction of all three vectors is the same, since they are parallel. However, the sense of $\overrightarrow{EF}$ is not the same as the sense of $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Therefore

$$\overrightarrow{EF} \neq \overrightarrow{AB} \tag{2}$$

$$\overrightarrow{EF} \neq \overrightarrow{CD} \tag{3}$$

The sense of $\overrightarrow{EF}$ is exactly opposite to the sense of $\overrightarrow{AB}$ and $\overrightarrow{CD}$. By definition, one vector is said to be the negative of another if their senses are exactly opposite while their magnitudes and directions are equal. Therefore

$$\overrightarrow{EF} = -\overrightarrow{AB} \tag{4}$$

$$\overrightarrow{EF} = -\overrightarrow{CD} \tag{5}$$

EXERCISE 2

1. Which of the vectors in Fig. 17·3 are equal?

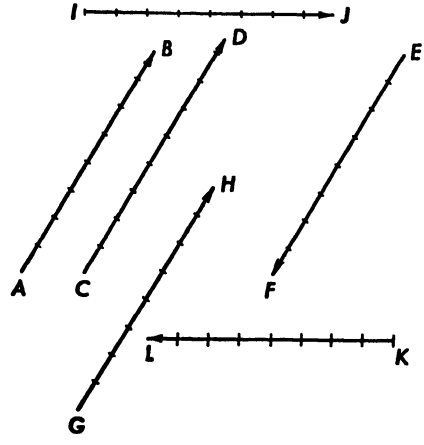


Fig. 17·3

2. Which of these vectors are the negative of $\overrightarrow{AB}$?
3. Which of these vectors have equal magnitudes?

17·3 Vectors and Rectangular Coordinates. Practical experience in engineering has demonstrated that a close association between vectors and rectangular coordinates leads to a powerful technique for the solution of many engineering problems (see Sec. 7·4).

The vector $\overrightarrow{OP}$ in Fig. 17·4 is on the positive side of the X axis. Now we shall rotate this vector about the point O in a counterclockwise direction to the position shown in Fig. 17·5. During this rotation it has generated the angle $+\phi$. The angle ϕ then defines the direction of $\overrightarrow{OP}$ with reference to the positive side of the X axis.

An angle is said to be drawn in standard position if, as in Fig. 17·5, it has its vertex at the origin of a system of rectangular coordinates and its initial side coincident with the positive side of the X axis. The terminal side of the angle may occupy any position in the coordinate plane, depending on the size of the angle.

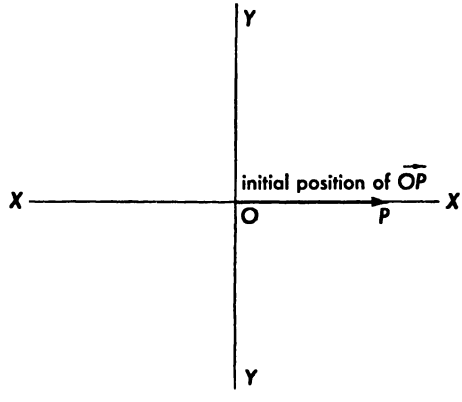


Fig. 17·4

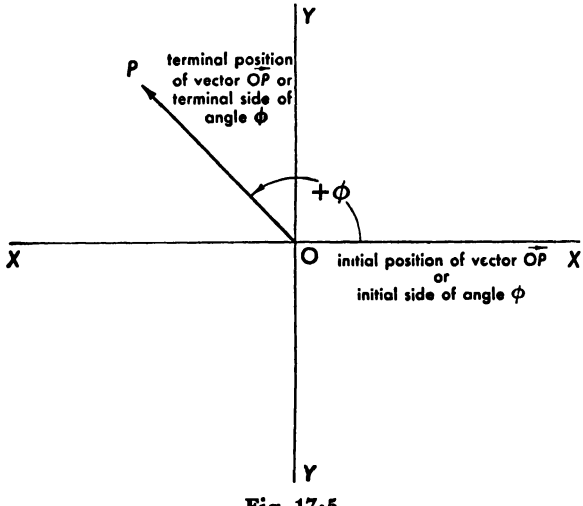


Fig. 17·5

17·4 Positive and Negative Angles. The relative direction of the terminal position of the vector $\overrightarrow{OP}$ compared with its initial position is important, but so also is the direction of rotation which brought this particular relative position into being.

A *positive angle* is defined as an angle generated by the counterclockwise rotation of a vector.

A *negative angle* is defined as an angle generated by the clockwise rotation of a vector.

17·5 Units of Angular Measure. Angles are commonly measured in either of two systems of units, the degree system or the radian system.

The Degree System

The vector $\overrightarrow{OP}$ in Fig. 17·4 is shown in its initial position. If now it is rotated counterclockwise one complete revolution until it again coincides with its initial position, then, by definition, an angle of $+360$ degrees has been generated. The angle 360 degrees is usually written 360°. One degree is by definition $\frac{1}{360}$ revolution.

The Radian System

As the vector $\overrightarrow{OP}$ rotates, the point P moves in a circular path (see Fig. 17·6). If the arc length traversed by P is $\overset{\frown}{XP}$ (where $\overset{\frown}{XP}$ is measured in the same linear units as $\overrightarrow{OP}$, then ϕ measured in radians is

$$\phi_r = \frac{\overset{\frown}{XP}}{|OP|} \qquad \star (6)$$

and

$$\overset{\frown}{XP} = |OP| \times \phi_r \qquad (7)$$

where the subscript r indicates that the angle is measured in radians rather than in degrees.

In other words, the quotient of the arc length divided by the radius is the measure of the central angle in radians.

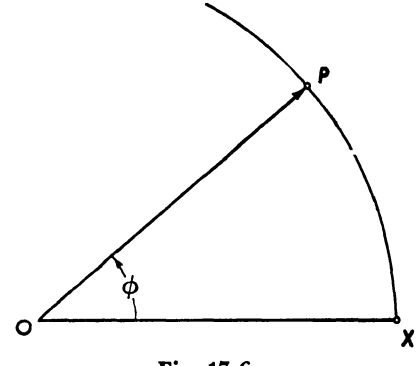


Fig. 17·6

If the vector $\overrightarrow{OP}$ makes one complete revolution, then the arc length becomes the circumference of the circle and

$$\phi_r = \frac{2\pi|OP|}{|OP|} = 2\pi \qquad (8)$$

There are 2π radians in one revolution, and one radian is $1/2\pi$ revolution.

Since by definition one revolution of the vector is equivalent to an angle of 360° and since by definition one revolution is also equivalent to 2π radians, therefore

$$2\pi_{(r)} = 360^\circ$$

$$\pi_{(r)} = 180^\circ$$

$$1 \text{ radian} = \frac{180^\circ}{\pi} = 57.2957795 \text{ degrees} \qquad \star (9)$$

$$1 \text{ degree} = \frac{\pi}{180} = 0.01745329 \text{ radians} \qquad \star (10)$$

Ordinarily, fractions of a degree are not written in decimals but rather in sub-units called minutes and seconds. These will be discussed in the next chapter. There are no sub-units attached to radian measure. For purposes of numerical calculation in solving triangles, etc., the degree system is ordinarily used. In more analytical work the radian system is common.

For simplicity, we usually choose to write π radians rather than 3.14159 radians, $\pi/2$ radians rather than 1.57080 radians, $3\pi/2$ radians rather than 4.71239 radians, etc. In such cases it is easy to change from radian measure to degree measure by substituting for π radians the equivalent, 180°.

$$\pi_{(r)} = 180^\circ \qquad (11)$$

$$\frac{\pi}{2}(r) = \frac{180^\circ}{2} = 90^\circ \qquad (12)$$

$$\frac{3\pi}{2}(r) = \frac{3 \times 180^\circ}{2} = 270^\circ \qquad (13)$$

EXAMPLE 1. Change 56.73° to radians.

$$1^\circ = \frac{1}{360} \text{ revolutions}$$

$$1 \text{ revolution} = 2\pi \text{ radians}$$

$$1^\circ = \frac{1}{360} \times 2\pi \text{ radians}$$

$$56.73^\circ = 56.73 \times 2\pi \times \frac{1}{360} = 0.99013 \text{ radians}$$

EXAMPLE 2. Change 0.792 radians to degrees. Leave the answer a decimal.

$$1 \text{ radian} = \frac{1}{2\pi} \text{ revolutions}$$

$$1 \text{ revolution} = 360^\circ$$

$$1 \text{ radian} = \frac{1}{2\pi} \times 360^\circ$$

$$0.792 \text{ radians} = 0.792 \times \frac{360}{2\pi} = \frac{0.792 \times 360}{2\pi} = 45.3783^\circ$$

EXAMPLE 3. How many degrees are there in $7\pi/15$ radians?

$$\pi \text{ radians} = 180°$$

$$\frac{7}{15}\pi \text{ radians} = \frac{7}{15} \times 180° = \frac{1{,}260}{15} = 84°$$

EXERCISE 3

Change the following examples of angular measure from degrees to radians:

1. 27° 2. 137° 3. 251° 4. 322°

Change the following examples of angular measure from radians to degrees:

5. 1.36_r 6. 0.37_r 7. 3.67_r 8. 0.592_r

17·6 Angles of any Magnitude. In the process of making one complete revolution, the rotating vector assumes all directions in the

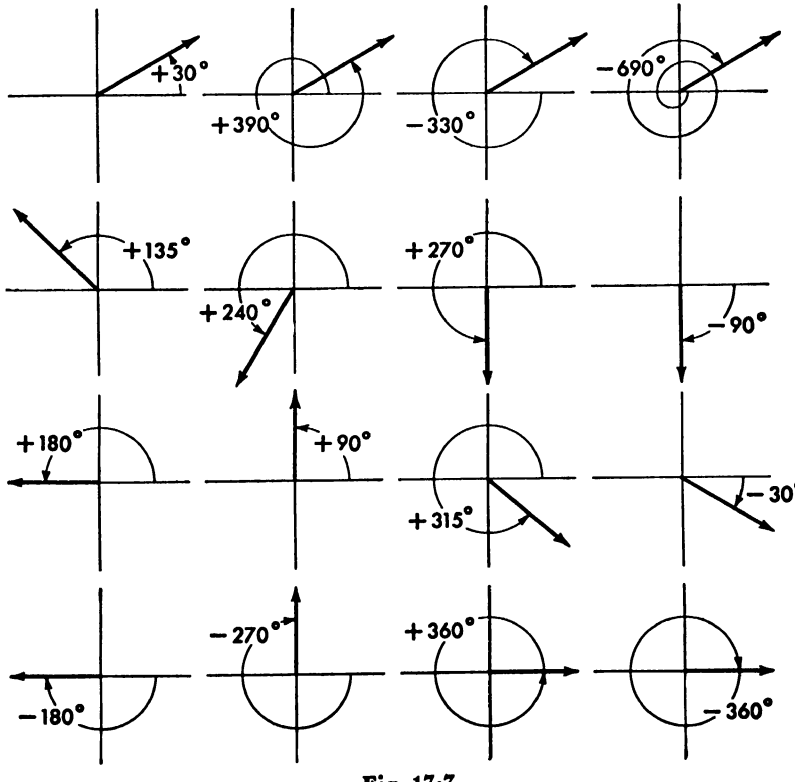


Fig. 17·7

coordinate plane. However, the vector can be made to continue its rotation through the 360° position and go around again generating angles between 360 and 720°. During another revolution it would generate angles between 720 and 1,080° and so on indefinitely. The vector would then repeat each direction in the coordinate plane periodically, once, and only once, for each revolution.

Figure 17·7 illustrates angles of several magnitudes. Note particularly that the angles +30, +390, −330, and −690° all define the same direction.

EXERCISE 4

Using a protractor, draw the following angles in standard position on a system of rectangular coordinates. Dimension the angles by a curved arrow, as in Fig. 17·7. Let the arrow indicate the direction of rotation. All angles are in radians unless degree measure is specifically indicated.

1.	35°	110°	185°	275°	472°	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$
2.	70°	138°	200°	300°	1,596°	$\frac{\pi}{5}$	$\frac{5\pi}{2}$	$\frac{2\pi}{3}$
3.	10°	175°	260°	315°	720°	$\frac{\pi}{6}$	$\frac{3\pi}{4}$	π
4.	28°	135°	210°	350°	975°	$\frac{7\pi}{3}$	$\frac{5\pi}{4}$	$\frac{7\pi}{6}$
5.	30°	110°	225°	320°	643°	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{10\pi}{2}$
6.	−25°	−150°	−290°	−250°	−720°	$\frac{11\pi}{6}$	$\frac{5\pi}{3}$	$\frac{7\pi}{8}$
7.	−82°	−120°	−250°	−300°	−1,256°	$-\frac{\pi}{3}$	$\frac{25\pi}{5}$	

8. $-\frac{\pi}{9}, -\frac{5\pi}{3}, -\frac{7\pi}{6}, -\frac{11\pi}{6}, -\frac{27\pi}{6}$

9. $-\frac{5\pi}{6}, -\frac{\pi}{10}, -\frac{\pi}{20}, -\frac{19\pi}{20}, -8\pi$

17·7 Addition and Subtraction of Angles. Addition of angles is illustrated in Fig. 17·8 where $\alpha = \phi + \beta$.

The positive angle ϕ is generated by the counterclockwise rotation of $\overrightarrow{OA}$ from its initial position on the positive side of the X axis to its terminal position, as shown in Fig. 17·8. The positive angle β is

generated by the *counterclockwise* rotation of the vector $\overrightarrow{OB}$ from its initial position coincident with the terminal position of $\overrightarrow{OA}$ to the terminal position of $\overrightarrow{OB}$, as shown in Fig. 17·8.

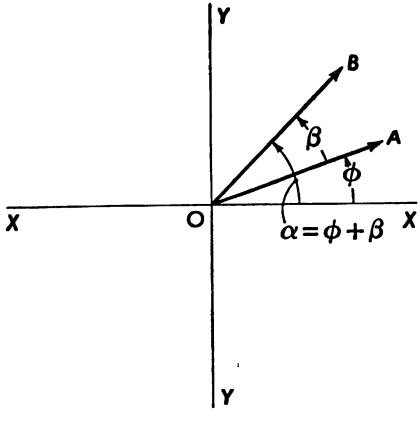


Fig. 17·8

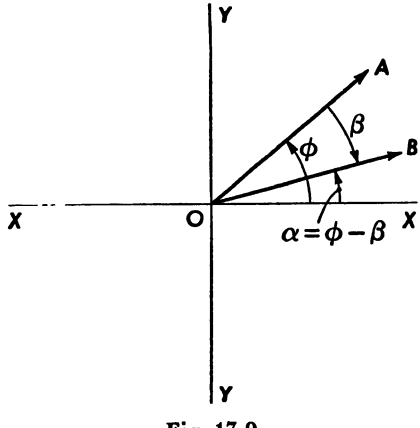


Fig. 17·9

Subtraction of angles is illustrated in Fig. 17·9 where $\alpha = \phi - \beta$. The positive angle ϕ is generated as before by the counterclockwise rotation of $\overrightarrow{OA}$. The angle β is subtracted from ϕ by rotating the vector $\overrightarrow{OB}$ in a *clockwise* direction from its initial position coincident with the terminal position of $\overrightarrow{OA}$ to its terminal position, as shown in Fig. 17·9.

17·8 The Quadrants. The coordinate plane is divided into four parts as shown in Fig. 17·10.

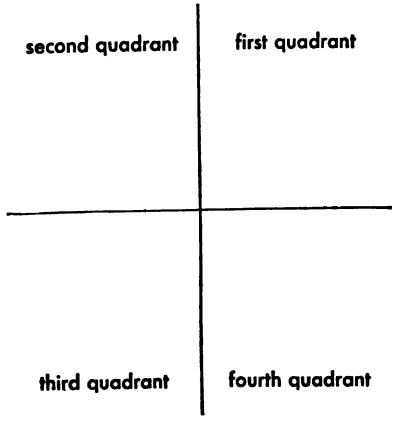


Fig. 17·10

An angle is said to be a first-, second-, third,- or fourth-quadrant angle according as its terminal side lies in the first, second, third, or fourth quadrant (see Fig. 17·11).

An angle whose terminal side lies on an axis is not named in terms of any quadrant. Such angles are 0, 90, 180, 270, 360, 450,° etc.

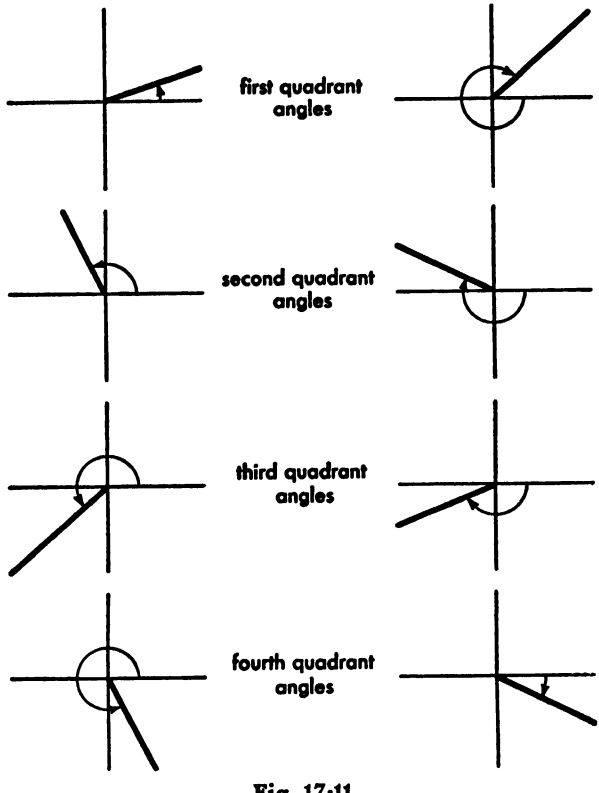


Fig. 17·11

17·9 The Coordinates of P in the Four Quadrants. The vectors in Fig. 17·12 are terminated at points P_1, P_2, P_3, and P_4. The coordinates of P are, in general, $\pm x$ and $\pm y$ (see Sec. 7·4).

In accordance with the conventions of rectangular coordinates, a first-quadrant angle is associated with a positive abscissa and a positive ordinate. This is illustrated in Fig. 17·12*a*. A second-quadrant angle is associated with a negative abscissa and a positive ordinate. For the signs of the coordinates associated with third- and fourth-quadrant angles, see Figs. 17·12*c* and 17·12*d*.

The magnitude of $\overrightarrow{OP}$ is often designated by r, *i.e.*,

$$r = |OP| = \sqrt{x^2 + y^2} \tag{14}$$

The quantity r is sometimes loosely termed the "radius vector" even though only the magnitude of the vector may be under consideration.

It is of extreme importance at this point to note that r is not a signed quantity when used in Eq. (14). Signed numbers in the present connection are used to indicate directions on the axes only. The quantity r is a purely enumerative number used to indicate that the point P is a certain number of linear units from the origin. It is incapable of carrying any information about the sense of $\overrightarrow{OP}$.

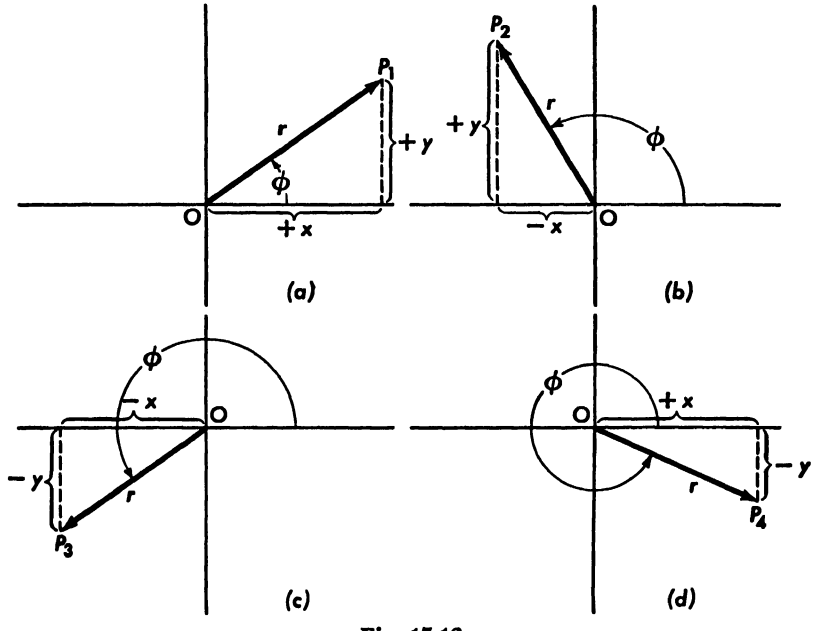


Fig. 17·12

Note that when x and y are given, one and only one angle less than 360° is indicated. If either x or y and r are given, then there are two possible angles less than 360° which will be consistent with the given data.

EXERCISE 5

Use coordinate paper, protractor, and ruler to find r and ϕ when the rectangular coordinates of P are as given below (see Fig. 17·12).

1. 7, 4	**2.** 4, 7	**3.** −3.7, 10
4. −7, 7.2	**5.** 7.8, −1.4	**6.** −8.6, −4.8
7. −3, 5.2	**8.** −9.2, −7.8	**9.** 1.4, 7.9
10. 6.2, −3	**11.** 3, −8	**12.** −2, −3.5
13. 8.5, 3.1	**14.** −6.8, −8	**15.** −10, 8.2

Draw two angles consistent with the data given below and measure the unknown coordinates and the angles.

16. $r = 8$	$x = +7$		**17.** $r = 5$	$y = -3.2$
18. $r = 11$	$x = +8$		**19.** $r = 9$	$x = -7.9$
20. $r = 12$	$x = -8$		**21.** $r = 11$	$y = -3.8$
22. $r = 8$	$y = +4.5$		**23.** $r = 10$	$y = +3.3$

17·10 Polar and Rectangular Coordinates. There are at least two ways by which we may locate a point P in the coordinate plane.

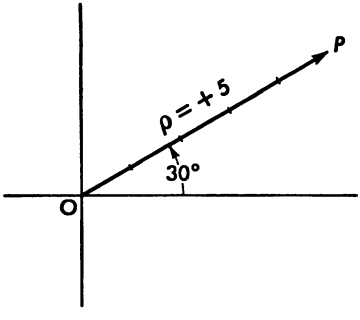


Fig. 17·13

Using *rectangular coordinates*, we may locate it by stating the perpendicular distance between P and each of the coordinate axes, as in Fig. 17·12.

Using *polar coordinates*, we may locate P by defining the vector whose extremities are the point P and the origin.

In Fig. 17·13 the vector $\overrightarrow{OP}$ is 5 units in length, its direction is defined by the reference angle 30°, and its sense is along the terminal side of the reference angle away from the origin. We may indicate this vector by the symbolism $+5\angle 30°$ or in general $\rho\angle\phi$, where ϕ is the reference angle of the vector and defines the direction of the vector. The length and sense of the vector are defined by ρ. The numerical value of ρ is the absolute length of the vector. The sign of

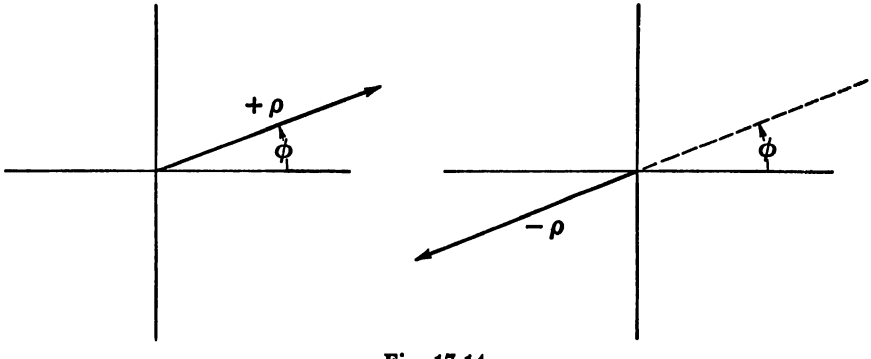


Fig. 17·14

ρ indicates the sense of the vector. If ρ is positive, the sense of the vector is away from the origin along the terminal side of the reference angle. If ρ is negative, the sense of the vector is in the opposite direction (see Fig. 17·14).

EXERCISE 6

Use coordinate paper, protractor, and ruler to find the rectangular coordinates of P when the polar coordinates of P are given below.

1. $\rho = 5$	$\phi = 20°$	**2.** $\rho = 7$	$\phi = 140°$
3. $\rho = -6$	$\phi = 50°$	**4.** $\rho = 10$	$\phi = -40°$
5. $\rho = -10$	$\phi = -120°$	**6.** $\rho = 8$	$\phi = 210°$
7. $\rho = 4.5$	$\phi = 300°$	**8.** $\rho = -5$	$\phi = 270°$
9. $\rho = 10$	$\phi = 360°$	**10.** $\rho = 4$	$\phi = 90°$

Find graphically the polar coordinates of the point P when the rectangular coordinates are given below.

11. 2, 5	**12.** -3, 5	**13.** -4, -5
14. 6, -5	**15.** 3, 7	**16.** -5, 5

17·11 Components and Projections. The quantities x and y, as used in Sec. 17·9, are often called, respectively, the *x projection* of $\overrightarrow{OP}$ and the *y projection* of $\overrightarrow{OP}$. Or they may be called the x and y components of $\overrightarrow{OP}$. Other special names may be attached to similar quantities, depending upon the nature of the practical application.

For example, if we were considering the stresses in a roof or bridge truss, it might be desirable to use $\overrightarrow{OP}$ to define a force and orient the reference axes in such a way that it would be proper to term the quantity x the horizontal component of this force. Similarly, the quantity y would be called the vertical component of the force.

In surveying, when the reference axes are a meridian and a parallel of latitude, x might be called the *departure* and y might be called the *latitude* of the course $\overrightarrow{OP}$.

In electricity, when $\overrightarrow{OP}$ represents an impedance, the quantity x might be called the *resistance* and the quantity y might be called the *reactance*.

This is not the place to define these specialized engineering terms. It is sufficient at this point to indicate that they do exist.

17·12 The Trigonometric Functions of First-quadrant Angles. In Fig. 17·15 the triangles OP_1a, OP_2b, and OP_3c are all similar and

$$\frac{r_1}{x_1} = \frac{r_2}{x_2} = \frac{r_3}{x_3} \tag{15}$$

Also

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3} \tag{16}$$

and

$$\frac{r_1}{y_1} = \frac{r_2}{y_2} = \frac{r_3}{y_3} \tag{17}$$

So long as the angle ϕ does not change, the *ratios* hold constant, regardless of the size of the triangle. The ratios, therefore, depend only on the size of the angle.

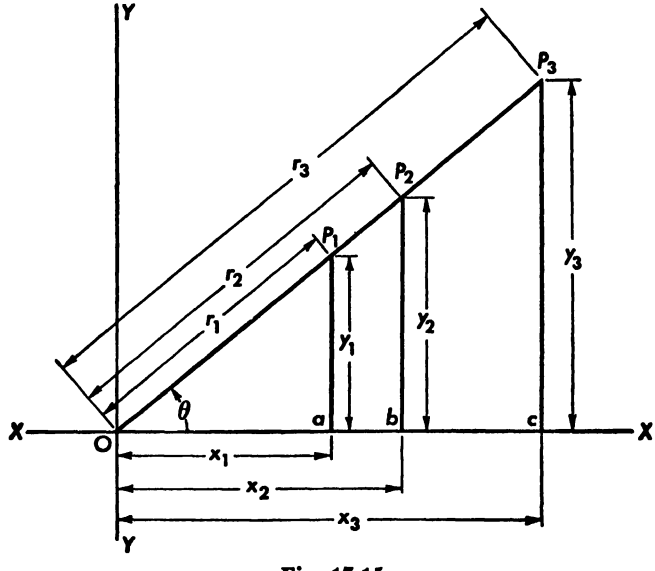


Fig. 17·15

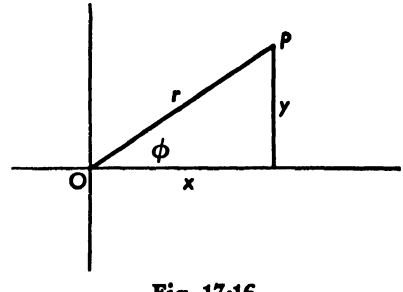


Fig. 17·16

The ratios shown in Eqs. (15), (16), and (17) are called *functions* of the angle ϕ.

There are actually six trigonometric functions of the angle (see Fig. 17·16).

$\frac{y}{r}$ is called the sine of ϕ (18)

$\frac{x}{r}$ is called the cosine of ϕ (19)

$\frac{y}{x}$ is called the tangent of ϕ (20)

$\frac{x}{y}$ is called the cotangent of ϕ (21)

$\frac{r}{x}$ is called the secant of ϕ (22)

$\frac{r}{y}$ is called the cosecant of ϕ (23)

The names of the six functions are usually abbreviated as follows:

$$\frac{y}{r} = \sin \phi \tag{24}$$

$$\frac{x}{r} = \cos \phi \tag{25}$$

$$\frac{y}{x} = \tan \phi \tag{26}$$

$$\frac{x}{y} = \cot \phi \tag{27}$$

$$\frac{r}{x} = \sec \phi \tag{28}$$

$$\frac{r}{y} = \operatorname{cosec} \phi \tag{29}$$

Stated in words,

$$\sin \phi = \frac{\text{vertical projection}}{\text{length of the vector}} \quad \star (30)$$

$$\cos \phi = \frac{\text{horizontal projection}}{\text{length of the vector}} \quad \star (31)$$

$$\tan \phi = \frac{\text{vertical projection}}{\text{horizontal projection}} \quad \star (32)$$

$$\cot \phi = \frac{\text{horizontal projection}}{\text{vertical projection}} \quad \star (33)$$

$$\sec \phi = \frac{\text{length of the vector}}{\text{horizontal projection}} \quad \star (34)$$

$$\operatorname{cosec} \phi = \frac{\text{length of the vector}}{\text{vertical projection}} \quad \star (35)$$

EXAMPLE 4. Find all six functions of the first-quadrant angle in which $x = 3$ and $y = 4$. Leave the functions in ratio form.

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\sin \phi = \frac{y}{r} = \frac{4}{5}$$

$$\cos \phi = \frac{x}{r} = \frac{3}{5}$$

$$\tan \phi = \frac{y}{x} = \frac{4}{3}$$

$$\cot \phi = \frac{x}{y} = \frac{3}{4}$$

$$\sec \phi = \frac{r}{x} = \frac{5}{3}$$

$$\operatorname{cosec} \phi = \frac{r}{y} = \frac{5}{4}$$

EXAMPLE 5. Find the six functions of ϕ when $x = 2y$.

$$r = \sqrt{x^2 + y^2} = \sqrt{(2y)^2 + y^2} = \sqrt{4y^2 + y^2} = y\sqrt{5}$$

$$\sin \phi = \frac{y}{r} = \frac{y}{y\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} = \frac{2.2361}{5} = 0.44722$$

$$\cos \phi = \frac{x}{r} = \frac{2y}{y\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} = 0.89444$$

$$\tan \phi = \frac{y}{x} = \frac{y}{2y} = \frac{1}{2} = 0.50000$$

$$\cot \phi = \frac{x}{y} = \frac{2y}{y} = 2.0000$$

$$\sec \phi = \frac{r}{x} = \frac{y\sqrt{5}}{2y} = \frac{\sqrt{5}}{2} = 1.1180$$

$$\operatorname{cosec} \phi = \frac{r}{y} = \frac{y\sqrt{5}}{y} = \sqrt{5} = 2.2361$$

EXERCISE 7

1. Find all six functions of the first-quadrant angles defined below. Leave answers in ratio form.
   a. $x = 39, y = 80$ **b.** $x = 36, y = 77$
   c. $x = 5, r = 13$ **d.** $r = 37, y = 35$
2. Find an expression for all six functions of the first-quadrant angles defined below. Leave answers in the simplest form.

a. $x = 2a, y = a^2 - 1$ **b.** $x = a, r = \dfrac{a^2 + 1}{2}$

c. $y = a^2 - b^2, r = a^2 + b^2$ **d.** $x = \dfrac{2ab}{a - b}, y = a + b$

3. Find the numerical value of the six trigonometric functions of ϕ when

a. $x = y$ **b.** $x = y\sqrt{3}$

c. $y = x\sqrt{3}$ **d.** $r = 2x$

4. State which has the greater absolute numerical value:

a. $\sin\phi$ or $\tan\phi$ **b.** $\cos\phi$ or $\cot\phi$

c. $\sec\phi$ or $\tan\phi$ **d.** $\operatorname{cosec}\phi$ or $\cot\phi$

17·13 Functions of Complementary Angles in the First Quadrant. One angle is said to be the complement of another if their sum is 90°. In a right triangle the two acute angles are always complementary.

In Fig. 17·17*a* the angles ϕ and θ are complementary angles. The functions of ϕ are

$$\sin\phi = \frac{y}{r} \qquad (36)$$

$$\cos\phi = \frac{x}{r} \qquad (37)$$

$$\tan\phi = \frac{y}{x} \qquad (38)$$

$$\cot\phi = \frac{x}{y} \qquad (39)$$

$$\sec\phi = \frac{r}{x} \qquad (40)$$

$$\operatorname{cosec}\phi = \frac{r}{y} \qquad (41)$$

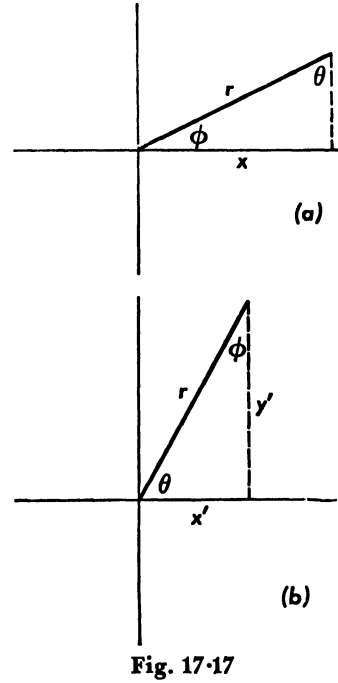


Fig. 17·17

Now let us draw Fig. 17·17*b*, in which r is the same length as r in Fig. 17·17*a*. The projection x' in Fig. 17·17*b* is numerically equal to the projection y in Fig. 17·17*a*. The projection y' in Fig. 17·17*b* is numerically equal to the projection x in Fig. 17·17*a*.

Following the definitions of the functions,

$$\sin\theta = \frac{y'}{r} \qquad (42)$$

$$\cos\theta = \frac{x'}{r} \tag{43}$$

$$\tan\theta = \frac{y'}{x'} \tag{44}$$

$$\cot\theta = \frac{x'}{y'} \tag{45}$$

$$\sec\theta = \frac{r}{x'} \tag{46}$$

$$\operatorname{cosec}\theta = \frac{r}{y'} \tag{47}$$

Remembering that

$$x' = y \tag{48}$$

$$y' = x \tag{49}$$

$$r = r \tag{50}$$

A comparison of Eqs. (36) to (41) with Eqs. (42) to (47), respectively, shows, that if ϕ and θ are complementary angles,

$$\sin\phi = \cos\theta = \cos(90 - \phi) \quad \star(51)$$

$$\cos\phi = \sin\theta = \sin(90 - \phi) \quad \star(52)$$

$$\tan\phi = \cot\theta = \cot(90 - \phi) \quad \star(53)$$

$$\cot\phi = \tan\theta = \tan(90 - \phi) \quad \star(54)$$

$$\sec\phi = \operatorname{cosec}\theta = \operatorname{cosec}(90 - \phi) \quad \star(55)$$

$$\operatorname{cosec}\phi = \sec\theta = \sec(90 - \phi) \quad \star(56)$$

EXERCISE 8

Express as functions of the complementary angle the following:

1. sin 20° **2.** cos 45° **3.** cos 60° **4.** tan 25°
5. cot 17° **6.** sec 84° **7.** cos 38.5° **8.** cosec 80°
9. sec 25°

17·14 The Functions of 30, 45, and 60°. Tables have been computed by which the functions of any angle may be found. Such tables will be treated in the next chapter.

The functions of certain special angles can, however, be calculated by elementary geometry.

In Eq. (6), Appendix B, it is indicated that the triangle shown in Fig. 17·18 is a 45° right triangle. Bearing in mind that any 45° right triangle is similar to all 45° right triangles, it follows that the ratio of any pair of sides in Fig. 17·18 is equal to a corresponding ratio of sides for any 45° right triangle, no matter how large that triangle may be.

Likewise it is suggested in Eq. (31), Appendix B, that the triangle

shown in Fig. 17·19 is a 30° right triangle. Then by similarity the ratio between any pair of sides in Fig. 17·19 will be identical, with a corresponding ratio in any 30-60° right triangle. Thus the functions of 30, 45, and 60° derived from these figures will be valid.

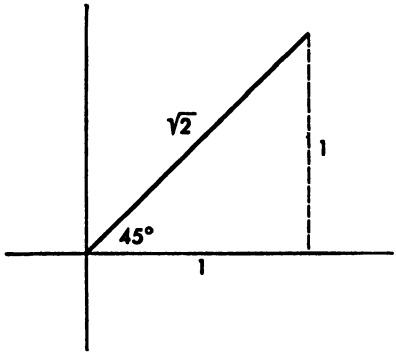


Fig. 17·18

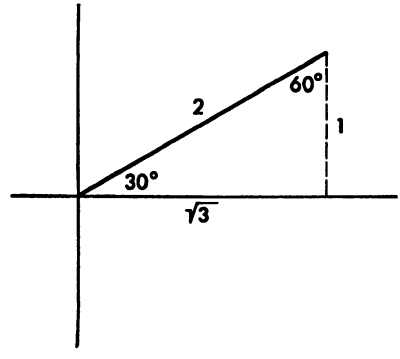


Fig. 17·19

From Fig. 17·18, and consistent with Eqs. (36) to (41), it appears that

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \tag{57}$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \tag{58}$$

$$\tan 45° = \frac{1}{1} = 1 \tag{59}$$

$$\cot 45° = \frac{1}{1} = 1 \tag{60}$$

$$\sec 45° = \frac{\sqrt{2}}{1} = \sqrt{2} \tag{61}$$

$$\operatorname{cosec} 45° = \frac{\sqrt{2}}{1} = \sqrt{2} \tag{62}$$

Referring to Fig. 17·19 and the same basic equations,

$$\sin 30° = \frac{1}{2} \tag{63}$$

$$\cos 30° = \frac{\sqrt{3}}{2} \tag{64}$$

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \tag{65}$$

$$\cot 30° = \frac{\sqrt{3}}{1} = \sqrt{3} \tag{66}$$

$$\sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \tag{67}$$

$$\operatorname{cosec} 30° = \frac{2}{1} = 2 \tag{68}$$

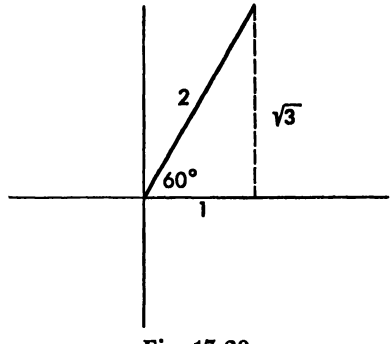


Fig. 17·20

Referring to Fig. 17·20 and again using the same basic equations,

$$\sin 60° = \frac{\sqrt{3}}{2} \tag{69}$$

$$\cos 60° = \frac{1}{2} \tag{70}$$

$$\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3} \tag{71}$$

$$\cot 60° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \tag{72}$$

$$\sec 60° = \frac{2}{1} = 2 \tag{73}$$

$$\operatorname{cosec} 60° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \tag{74}$$

EXERCISE 9

1. Given that $\sqrt{3} = 1.732$ and $\sqrt{2} = 1.414$, compute in decimals the six functions of 30, 45, and 60°.
2. Find the following ratios and simplify:
   a. sin 45 to sin 30°
   b. tan 45 to tan 30°

c. sin 45 to cos 30°
d. sin 30 to sin 60°
e. Is the ratio in **d** equal to the ratio of the angles?

17·15 The Zero Divisor. As far as we are concerned, no number exists which can be expressed as the quotient of a number divided by zero. If $z = x/y$ and y becomes very small compared with x, then z may be an enormous number. However, we shall make no attempt to evaluate x/y when y is precisely zero. We may use the symbol $+\infty$ or $-\infty$ to indicate an enormous positive or negative number, but when the symbol is so used, it is improper to write $x/0 = \pm\infty$.

17·16 Signed and Unsigned Numbers. We do not feel compelled to attach any particular sign to a simple enumerative number. If we wish to express the absolute distance between two points, we may say, for example, that these points are 10 in. apart. Not +10 in. or −10 in. but simply 10 in. apart.

We shall, by definition, attach the following meaning to division problems involving both signed and unsigned numbers:

$$\frac{\text{An unsigned number}}{\text{A positive number}} = \text{a positive number}$$

$$\frac{\text{An unsigned number}}{\text{A negative number}} = \text{a negative number}$$

$$\frac{\text{A positive number}}{\text{An unsigned number}} = \text{a positive number}$$

$$\frac{\text{A negative number}}{\text{An unsigned number}} = \text{a negative number}$$

17·17 The Signs of the Functions. The definitions of the functions given in Sec. 17·12 are not limited to first-quadrant angles. They apply with equal validity to angles in any quadrant, provided we take account of the signs of the projections (see Fig. 17·12). Remember that r is an unsigned number (see Secs. 17·9 and 17·16).

In the first quadrant the sign of all functions is positive, since the sign of both projections is positive.

In the second quadrant, the abscissa is negative and the ordinate is positive. Therefore, only the sine and cosecant are positive in the second quadrant.

In the third quadrant both the ordinate and the abscissa are negative. Therefore, the tangent and the cotangent are positive, since each is the quotient of two negative numbers. All other functions are negative.

In the fourth quadrant the ordinate is negative and the abscissa

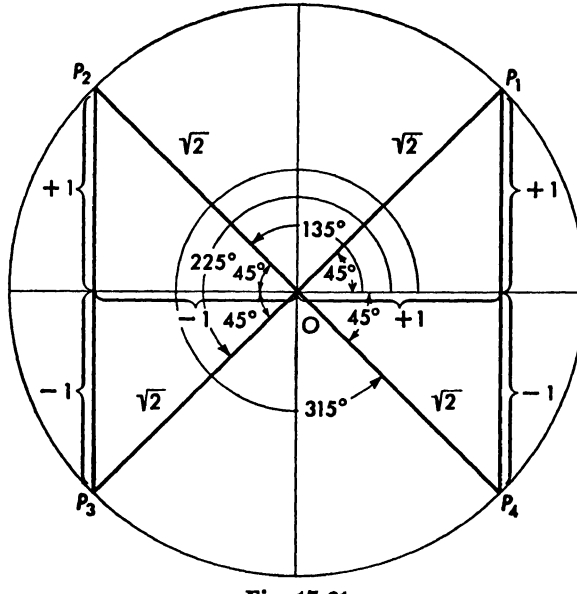


Fig. 17·21

is positive. Therefore, only the cosine and secant are positive. All the other functions are negative.

The functions of 45° given in Sec. 17·14 may be applied to a special series of angles. These angles are illustrated in Fig. 17·21, and their functions are shown in Table 17·1.

Table 17·1

	45°	135°	225°	315°
sin	$\frac{+1}{\sqrt{2}}$	$\frac{+1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$
cos	$\frac{+1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	$\frac{+1}{\sqrt{2}}$
tan	$\frac{+1}{+1}$	$\frac{+1}{-1}$	$\frac{-1}{-1}$	$\frac{-1}{+1}$
cot	$\frac{+1}{+1}$	$\frac{-1}{+1}$	$\frac{-1}{-1}$	$\frac{+1}{-1}$
sec	$\frac{\sqrt{2}}{+1}$	$\frac{\sqrt{2}}{-1}$	$\frac{\sqrt{2}}{-1}$	$\frac{\sqrt{2}}{+1}$
cosec	$\frac{\sqrt{2}}{+1}$	$\frac{\sqrt{2}}{+1}$	$\frac{\sqrt{2}}{-1}$	$\frac{\sqrt{2}}{-1}$

17·18 The Functions of Zero Degrees. When ϕ is zero, as in Fig. 17·22*a*, then $y = 0$ and x is numerically equal to r. Accordingly,

$$\sin 0° = \frac{y}{r} = \frac{0}{r} = 0 \tag{75}$$

$$\cos 0° = \frac{+x}{r} = +1 \tag{76}$$

$$\tan 0° = \frac{y}{x} = \frac{0}{x} = 0 \tag{77}$$

$$\sec 0° = \frac{r}{+x} = +1 \tag{78}$$

The cotangent and the cosecant of zero degrees do not exist, since a calculation of them involves a division by zero. However, if ϕ is a

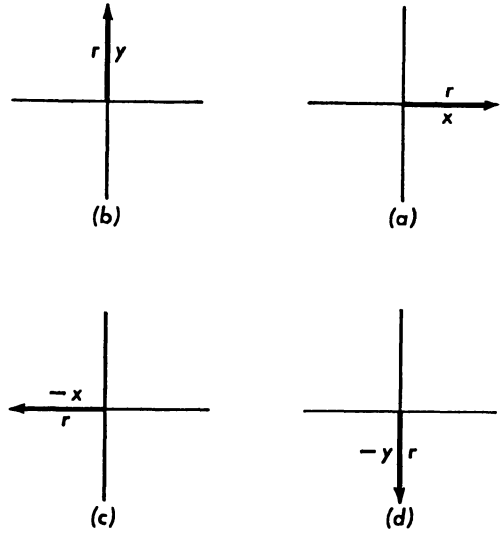


Fig. 17·22

very small positive angle, then y will be a very small linear dimension and both the cotangent and cosecant will be large numbers. The smaller ϕ is made while remaining positive, the larger the cotangent and cosecant become.

17·19 The Functions of 90°. When $\phi = 90°$, as in Fig. 17·22*b*, y is numerically equal to r and $x = 0$. Accordingly,

$$\sin 90° = \frac{+y}{r} = +1 \tag{79}$$

$$\cos 90° = \frac{x}{r} = \frac{0}{r} = 0 \tag{80}$$

$$\cot 90° = \frac{x}{+y} = \frac{0}{+y} = 0 \tag{81}$$

$$\operatorname{cosec} 90° = \frac{r}{+y} = +1 \tag{82}$$

Here the tangent and the secant do not exist. But if ϕ is slightly less than 90°, then the tangent and secant are enormously large numbers. The closer ϕ approaches 90° while remaining in the first quadrant, the larger the tangent and secant become.

It is important to observe that if ϕ is made ever so slightly greater than 90°, then the signs of the tangent and secant change and become negative. The tangent and secant would then be extremely large in absolute value but negative in sign or (what amounts to the same thing) extremely small numbers.

If the angle ϕ approaches 90° from the second quadrant, the tangent decreases without limit. If the angle ϕ approaches 90° from the first quadrant, the tangent increases without limit.

17·20 The Functions of 180°. When $\phi = 180°$, as in Fig. 17·22*c*, y is zero, while x is negative and numerically equal to r. Accordingly,

$$\sin 180° = \frac{y}{r} = 0 \tag{83}$$

$$\cos 180° = \frac{-x}{r} = -1 \tag{84}$$

$$\tan 180° = \frac{y}{-x} = \frac{0}{-x} = 0 \tag{85}$$

$$\sec 180° = \frac{r}{-x} = -1 \tag{86}$$

The cotangent and cosecant 180° do not exist.

17·21 The Functions of 270 and 360°. It is left to the student to verify the following equations, by consulting Figs. 17·22*a* and 17·22*d*.

$\sin 270° = -1$	$\sin 360° = 0$	(87)
$\cos 270° = 0$	$\cos 360° = 1$	(88)
$\tan 270° =$ does not exist	$\tan 360° = 0$	(89)
$\cot 270° = 0$	$\cot 360° =$ does not exist	(90)
$\sec 270° =$ does not exist	$\sec 360° = 1$	(91)
$\operatorname{cosec} 270° = -1$	$\operatorname{cosec} 360° =$ does not exist	(92)

17·22 Reducing the Functions of Any Angle to Functions of a First-quadrant Angle. In Fig. 17·23 the angles ϕ_1, QOb, ROb, and SOa are all numerically equal. Also aP, bQ, bR, and aS are numerically equal, as are Oa and Ob. Furthermore,

$$|OP| = |OQ| = |OR| = |OS| \tag{93}$$

Therefore, without regard to sign, all the functions of ϕ_1 are identical with all the corresponding functions of ϕ_2, ϕ_3, and ϕ_4.

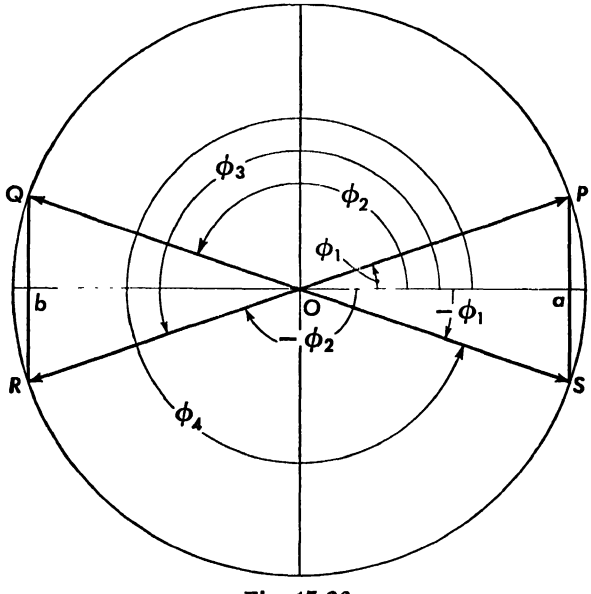


Fig. 17·23

Following the usual conventions of rectangular coordinates, $aP = bQ = -bR = -aS$ and $Ob = -Oa$.

The magnitudes of the angles ϕ_2, ϕ_3, and ϕ_4 are respectively related to the magnitude of ϕ_1 as follows:

$$\phi_1 = 180° - \phi_2 \tag{94}$$
$$\phi_1 = \phi_3 - 180° \tag{95}$$
$$\phi_1 = 360° - \phi_4 \tag{96}$$

For the second-quadrant angle ϕ_2,

$$\sin \phi_2 = +\sin (180° - \phi_2) = +\sin \phi_1 \quad \star (97)$$
$$\cos \phi_2 = -\cos (180° - \phi_2) = -\cos \phi_1 \quad \star (98)$$
$$\tan \phi_2 = -\tan (180° - \phi_2) = -\tan \phi_1 \quad \star (99)$$
$$\cot \phi_2 = -\cot (180° - \phi_2) = -\cot \phi_1 \quad \star (100)$$
$$\sec \phi_2 = -\sec (180° - \phi_2) = -\sec \phi_1 \quad \star (101)$$
$$\operatorname{cosec} \phi_2 = \operatorname{cosec} (180° - \phi_2) = +\operatorname{cosec} \phi_1 \quad \star (102)$$

For the third-quadrant angle ϕ_3,

$$\sin \phi_3 = -\sin (\phi_3 - 180°) = -\sin \phi_1 \quad (103)$$
$$\cos \phi_3 = -\cos (\phi_3 - 180°) = -\cos \phi_1 \quad (104)$$
$$\tan \phi_3 = +\tan (\phi_3 - 180°) = +\tan \phi_1 \quad (105)$$
$$\cot \phi_3 = +\cot (\phi_3 - 180°) = +\cot \phi_1 \quad (106)$$
$$\sec \phi_3 = -\sec (\phi_3 - 180°) = -\sec \phi_1 \quad (107)$$
$$\operatorname{cosec} \phi_3 = -\operatorname{cosec} (\phi_3 - 180°) = -\operatorname{cosec} \phi_1 \quad (108)$$

For the fourth-quadrant angle ϕ_4,

$$\sin \phi_4 = -\sin (360° - \phi_4) = -\sin \phi_1 \quad (109)$$
$$\cos \phi_4 = +\cos (360° - \phi_4) = +\cos \phi_1 \quad (110)$$
$$\tan \phi_4 = -\tan (360° - \phi_4) = -\tan \phi_1 \quad (111)$$
$$\cot \phi_4 = -\cot (360° - \phi_4) = -\cot \phi_1 \quad (112)$$
$$\sec \phi_4 = +\sec (360° - \phi_4) = +\sec \phi_1 \quad (113)$$
$$\operatorname{cosec} \phi_4 = -\operatorname{cosec} (360° - \phi_4) = -\operatorname{cosec} \phi_1 \quad (114)$$

Observe that in Fig. 17·23 the angle aOS is the negative of ϕ_1, or

$$\angle aOS = -\phi_1 \quad (115)$$

Therefore all the functions of ϕ_4 are identical with the functions of $-\phi_1$.

In general, all the functions of $-\phi$ are identical with all the functions of $360° - \phi$.

Also observe that the functions of $360° + \phi$ are the same as the functions of ϕ, or, in general, all of the functions of $(360N + \phi)$ when N is any integer are identical with the functions of ϕ.

17·23 Variations in the Functions. As $\overrightarrow{OP}$ rotates from the initial position, through the first quadrant to the 90° position, the sine of the angle ϕ increases from 0 to 1. In the second quadrant the sine decreases from 1 at the 90° position to 0 at the 180° position. In the third quadrant, the sine decreases from 0 to -1, and in the fourth quadrant it increases from -1 to 0.

In a like manner the variations in the other five functions can be traced.

EXERCISE 10

1. Write all the functions of 0°, 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°, 210°, 225°, 240°, 270°, 300°, 315°, 330°, and 360° in decimals, using proper signs.
2. In which quadrants does the sine increase as the radius vector rotates counterclockwise? In which quadrant does the cosine increase? The tangent? The cotangent? The secant? The cosecant?

3. Within what numerical limits can the sine, the cosine, the tangent, the cotangent, the secant, and the cosecant exist?
4. What are the maximum and minimum values of each of the six functions?
5. Sketch a graph of each function with the value of the function plotted vertically against the angle plotted horizontally.

17·24 Inverse Trigonometric Functions. An equation in the form

$$\phi = \tan^{-1}\frac{a}{b} \tag{116}$$

is frequently encountered in technical literature. By definition, it means "ϕ is the angle whose tangent is a/b." The equation

$$\phi = \sin^{-1}\frac{a}{c} \tag{117}$$

means "ϕ is the angle whose sine is a/c." The equation

$$\phi = \cos^{-1}\tfrac{1}{2} \tag{118}$$

means "ϕ is the angle whose cosine is $\frac{1}{2}$." In this case, of course, ϕ = either 60 or 300°, both of which have $+\frac{1}{2}$ for a cosine.

EXERCISE 11

Find the following angles, 0 to 360°. (Remember there may be two answers.)

1. $\tan^{-1}\dfrac{1}{\sqrt{3}}$
2. $\tan^{-1} 1$
3. $\sin^{-1}\frac{1}{2}$
4. $\sin^{-1}\dfrac{\sqrt{2}}{2}$
5. $\cos^{-1} -\frac{1}{2}$
6. $\tan^{-1} -1$
7. $\cot^{-1}\dfrac{a}{b}$ where $a = b$
8. $\tan^{-1} -\dfrac{\sqrt{3}}{3}$
9. $\tan^{-1}\dfrac{+1}{-\sqrt{3}}$
10. $\sin^{-1} -\dfrac{\sqrt{3}}{2}$

17·25 Some Basic Relations between the Functions. The relationships given below are true for angles in any quadrant. The student should take time to convince himself of this fact.

A comparison between Eqs. (24) and (29), page 353 will show that

$$\sin\phi = \frac{1}{\operatorname{cosec}\phi}, \quad \operatorname{cosec}\phi = \frac{1}{\sin\phi} \qquad \star(119)$$

A similar comparison between Eqs. (25) and (28), page 353, will show that

$$\cos \phi = \frac{1}{\sec \phi}, \quad \sec \phi = \frac{1}{\cos \phi} \qquad \star (120)$$

Also, a comparison between Eqs. (26) and (27), page 353, will show that

$$\tan \phi = \frac{1}{\cot \phi}, \quad \cot \phi = \frac{1}{\tan \phi} \qquad \star (121)$$

Since $\sin \phi = y/r$ and $\cos \phi = x/r$, therefore

$$\frac{\sin \phi}{\cos \phi} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan \phi \qquad \star (122)$$

Applying Eq. (121) to Eq. (122),

$$\frac{\cos \phi}{\sin \phi} = \cot \phi \qquad \star (123)$$

Referring to Fig. 17·16 and remembering the Pythagorean theorem,

$$r^2 = x^2 + y^2 \qquad (124)$$

Dividing both sides of Eq. (124) by r^2, x^2, and y^2 in turn gives

$$\frac{r^2}{r^2} = \frac{x^2}{r^2} + \frac{y^2}{r^2} \qquad (125)$$

$$1 = \cos^2 \phi + \sin^2 \phi \qquad \star (126)$$

$$\frac{r^2}{x^2} = \frac{x^2}{x^2} + \frac{y^2}{x^2} \qquad (127)$$

$$\sec^2 \phi = 1 + \tan^2 \phi \qquad \star (128)$$

$$\frac{r^2}{y^2} = \frac{x^2}{y^2} + \frac{y^2}{y^2} \qquad (129)$$

$$\operatorname{cosec}^2 \phi = \cot^2 \phi + 1 \qquad \star (130)$$

Observe carefully that the symbol $\sin^2 \phi$ means "the square of the sine of the angle ϕ." It does not mean "the sine of the square of the angle ϕ." For example, $\sin 30° = 0.5$ and $\sin^2 30° = 0.25$. On the other hand, $\sin 30^2 = \sin 900° = \sin 180° = 0$. The expressions $(\sin \phi)^2$ and $\sin^2 \phi$ are identical.

17·26 Trigonometric Identities. In the course of analyzing practical problems, it sometimes becomes advisable to check the validity of a trigonometric identity. (The student should at this point review the matter of identities and conditional equations. See page 83.)

For example, suppose that in the process of developing a certain formula we arrive at the mathematical statement

$$\sin \phi = \sqrt{1 - \cot^2 \phi \sin^2 \phi} \tag{131}$$

and we feel that while this equation is probably true, it would be advisable to prove it.

Two somewhat different methods of attack are illustrated in the examples below. Example 7 is, perhaps, somewhat more rigorous, but the authors have no objection to the method illustrated in Example 6. In any case, however, the identity must be proved in the precise form in which it appears. No terms may be shifted across the equal sign.

EXAMPLE 6. Prove that $\sin \phi = \sqrt{1 - \cot^2 \phi \sin^2 \phi}$. (132)

$$\cot \phi = \frac{x}{y} \tag{133}$$

$$\sin \phi = \frac{y}{r} \tag{134}$$

Substituting Eq. (133) in Eq. (132),

$$\sin \phi = \sqrt{1 - \frac{x^2}{y^2} \times \sin^2 \phi} \tag{135}$$

Substituting Eq. (134) in Eq. (135),

$$\sin \phi = \sqrt{1 - \frac{x^2}{y^2} \times \frac{y^2}{r^2}} = \sqrt{1 - \frac{x^2}{r^2}} = \sqrt{1 - \cos^2 \phi} \tag{136}$$

The expression $\sqrt{1 - \cos^2 \phi}$ is recognized from Eq. (126) to be equal to $\sin \phi$. Therefore

$$\sin \phi = \sin \phi \tag{137}$$

and the identity has been proved.

EXAMPLE 7. Prove the identity $\sin \phi = \sqrt{1 - \cot^2 \phi \sin^2 \phi}$. (138)

From Eq. (123) it appears that

$$\cot^2 \phi = \frac{\cos^2 \phi}{\sin^2 \phi} \tag{139}$$

Substituting Eq. (139) in Eq. (138),

$$\sin \phi = \sqrt{1 - \frac{\cos^2 \phi}{\sin^2 \phi} \times \sin^2 \phi} = \sqrt{1 - \cos^2 \phi} \tag{140}$$

But from Eq. (126)

$$\sqrt{1 - \cos^2 \phi} = \sin \phi \tag{141}$$

Therefore

$$\sin \phi = \sin \phi \tag{142}$$

and the identity has been proved.

EXERCISE 12

Prove the following identities:

1. $\sin \phi = \sqrt{1 - \cos^2 \phi}$

2. $\cos \phi = \sqrt{1 - \sin^2 \phi}$

3. $\tan \phi = \dfrac{\sin \phi}{\sqrt{1 - \sin^2 \phi}}$

4. $\tan \phi = \dfrac{\sqrt{1 - \cos^2 \phi}}{\cos \phi}$

5. $\tan \phi = \sin \phi \times \sec \phi$

6. $\cot \phi = \dfrac{\cos \phi}{\sqrt{1 - \cos^2 \phi}}$

7. $\cot \phi = \dfrac{\sqrt{1 - \sin^2 \phi}}{\sin \phi}$

8. $\cot \phi = \sqrt{\operatorname{cosec}^2 \phi - 1}$

9. $\cot \phi = \cos \phi \times \operatorname{cosec} \phi$

10. $\sec \phi = \sqrt{1 + \tan^2 \phi}$

UNIT FIVE

NUMERICAL TRIGONOMETRY OF THE RIGHT TRIANGLE

Classes which have had only a very little algebra can study this unit at any time

Chapter 18 THE RIGHT TRIANGLE

This section of the work will be concerned almost entirely with numerical trigonometry. This branch of trigonometry deals with methods by which certain sides and angles of a triangle may be computed when certain other sides and angles are known. Emphasis in this chapter will be on computation rather than analysis.

18·1 Angles. For the present purpose an angle will be described in terms of Fig. 18·1. The side r was originally coincident with the side OX. The side OX is called the initial position of r, or the *initial side* of the angle. However, r as shown in the diagram has been revolved about O in the direction of the arrow and has become the *terminal side* of the angle.

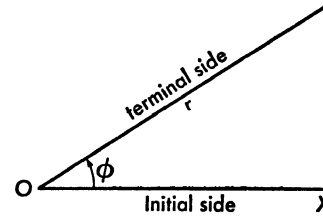


Fig. 18·1

There are two systems of units ordinarily used to define the magnitude of an angle; the *degree system* and the *radian system*. The degree system is the only one of importance in numerical trigonometry.

If the line r had revolved about O for one complete revolution from the initial position, an angle of 360 degrees would have been generated. By definition, 1 degree is equivalent to $\frac{1}{360}$ revolution. An angle of 90 degrees (written 90°) is equivalent to $\frac{90}{360}$ or $\frac{1}{4}$ revolution. An angle of 180° would be equivalent to $\frac{180}{360}$ or $\frac{1}{2}$ revolution

Occasionally angles are measured in degrees and a decimal fraction of a degree, as for example 37.594°. More often they are measured in degrees and sub-units of a degree called minutes and seconds. There are, by definition, 60 minutes in 1 degree and 60 seconds in 1 minute. The angular magnitude 37 degrees 42 minutes and 36 seconds would be written 37°42′36″. Since there are 60 seconds in a minute, it follows that 6″ is equivalent to $\frac{1}{10}$ minute. The angle 37°42′36″ could also have been written 37°42.6′, and indeed angles are frequently measured in degrees, minutes, and tenths of minutes.

18·2 Arithmetical Operations with Degrees, Minutes, and Seconds. Addition, subtraction, multiplication, and division of angles in degrees, minutes, and seconds require somewhat different procedures from those used for similar operations in the decimal system.

EXAMPLE 1. Add 27°42′38″ and 45°32′25″.

$$\begin{array}{r} 27°42'38'' \\ 45°32'25'' \\ \hline 72°74'63'' \end{array}$$

Since 63″ = 1′3″, the above answer could be written

$$72°75'3''$$

Since 75′ = 1°15′, the equivalent form would be

$$73°15'3''$$

EXAMPLE 2. Subtract 15°47′18″ from 25°30′10″.

Notice that while 47′ cannot be subtracted from 30′ and while 18″ cannot be subtracted from 10″, the minuend can be altered to the equivalent form

$$24°89'70''$$

and the subtraction can be carried out as shown below:

$$\begin{array}{r} 24°89'70'' \\ 15°47'18'' \\ \hline 9°42'52'' \end{array}$$

EXAMPLE 3. Divide 25°17′27″ by 2.

$$\frac{25}{2} = 12°$$

with a remainder of 1° or 60′, which we add to 17′ to make 77′.

$$\frac{77}{2} = 38'$$

with a remainder of 1′ or 60″, which we add to 27″ to make 87″.

$$\frac{87''}{2} = 43\tfrac{1}{2}''$$

The answer is then

$$12°38'43\tfrac{1}{2}''$$

EXAMPLE 4. Divide 33°16′46″ by 7.

$$\frac{33°}{7} = 4°$$

with a remainder of 5° or 300′, which we add to 16′ to make 316′.

$$\frac{316'}{7} = 45'$$

with a remainder of 1′ or 60″, which we add to 46″ to make 106″.

$$\frac{106''}{7} = 15'' \text{ to the nearest second}$$

The answer is then

$$4°45'15''$$

EXAMPLE 5. Multiply 35°31′42″ by 2.

$$\begin{array}{r} 35°31'42'' \\ \times\quad\quad 2 \\ \hline 70°62'84'' \text{ or} \\ 71°03'24'' \end{array}$$

EXAMPLE 6. Change 52.379° to degrees, minutes, and seconds.
Since there are 60 minutes in 1 degree, in 0.379° there are

$$0.379 \times 60 = 22.740'$$

Since there are 60 seconds in 1 minute, in 0.74′ there are

$$0.74 \times 60 = 44.40''$$

Therefore

$$52.379° = 52°22'44.4''$$

EXAMPLE 7. Change 18°26′35″ to degrees and a decimal fraction of a degree.

$$26' = \frac{26}{60} \text{ degrees} = 0.4333°$$

$$35'' = \frac{35}{3{,}600} \text{ degrees} = 0.0097°$$

$$18°26'35'' = 18.0000° + 0.4333° + 0.0097° = 18.443°$$

EXERCISE 1

Perform the operations indicated below:

1. 14°23′17″ + 28°19′5″ + 75°41′50″ + 21°13′19″
2. 53°42′39″ + 15°32′45″ + 13°15′28″ + 15°2′25″
3. 25°16′24″ + 27°10′29″ + 26°30′19″ + 10°29′37″
4. 62°3′15″ + 79°1′28″ + 16°59′28″ + 47°17′28″
5. 15°15′30″ + 26°48′13″ + 17°29′15″ + 28°15′42″
6. 15°28′17″ − 10°46′27″
7. 89°59′60″ − 25°36′10″
8. 90° − (27°42′51″)
9. 180° − (132°46′17″)

10. 360° − (42°36′25″)
11. Multiply 26°45′48″ by 2
12. Multiply 16°32′25″ by 5
13. Multiply 3°00′12″ by 6
14. Multiply 19°47′15″ by 8
15. Multiply 22°11′38″ by 4
16. Divide 37°42′31″ by 5
17. Divide 10°13′27″ by 7
18. Divide 72° 19′48″ by 9
19. Divide 52°26′11″ by 10
20. Divide 182°15′27″ by 4
21. Change to degrees, minutes, and seconds:
a. 52.763° **b.** 0.527° **c.** 146.895°
22. Change to decimals:
a. 37°42′17″ **b.** 136°29′52″ **c.** 00°20′13″

18·3 Unique Determination of Triangles. A triangle is *determined* when enough information is at hand so that it can be drawn to scale. (If it can be drawn to scale, it can be solved numerically.) A triangle can be drawn to scale

1. If three sides are known.
2. If one side and two angles are known.
3. If two sides and one angle are known.

If the given angle is less than 90° and one of the two given sides is opposite this angle, there may possibly be two solutions. More will be said of this later in the work.

18·4 Solution of Triangles. A triangle is *solved* when the unknown sides and angles have been found. Triangles may be solved either by drawing them to scale and measuring the unknown sides and angles or by using suitable formulas to calculate the unknown dimensions.

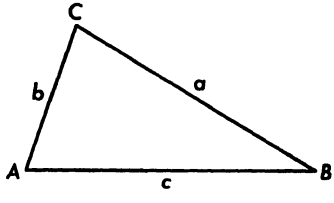


Fig. 18·2

Graphical solutions are common in surveying and mapping. They are also common in certain solutions to bridge- and roof-truss problems.

While computational methods will be the main concern of this chapter, the student should invariably make a scale drawing of the triangle which he is solving. In this way he gets a far more vivid idea of the problem and the successive steps in the solution. Even more important, reference to the scale drawing will call his attention to gross errors in calculation.

Altogether, scale drawings will help him get the right answer more quickly, leaving more time for more pleasant activities.

Figure 18·2 illustrates a standard method of lettering a triangle. Note that capital letters are used to denote the angles and that small letters are used to denote the sides. Furthermore, side a is opposite angle A, side b is opposite angle B, and side c is opposite angle C.

EXERCISE 2

In the following problems, the data are supposed to determine certain triangles. In some cases, these data are incomplete, inconsistent, or ambiguous. In other problems, data are correctly given. Draw these triangles full scale if correct data are given, and measure the unknown sides and angles. If the data are improperly given, tell in what respect they are improper.

1. $a = 3$ in. $b = 4$ in. $C = 90°$
2. $a = 4$ in. $b = 3$ in. $c = 5$ in. $C = 90°$
3. $a = 10$ in. $b = 3$ in. $c = 5$ in.
4. $A = 30°$ $C = 90°$ $c = 5$ in.
5. $A = 30°$ $c = 3$ in. $a = 1$ in.
6. $A = 30°$ $c = 3$ in. $a = 1.5$ in.
7. $A = 30°$ $c = 3$ in. $a = 2$ in.
8. $A = 130°$ $B = 120°$ $c = 5$ in.
9. $a = 4$ in. $c = 6$ in. $B = 30°$
10. $A = 30°$ $B = 120°$ $C = 30°$
11. $a = 4$ in. $b = 3$ in. $c = 3\frac{3}{4}$ in. $C = 90°$
12. $A = 30°$ $B = 45°$ $c = 5$ in.

18·5 Trigonometric Functions. In the early part of the work on numerical trigonometry, we shall be limited to computational methods applied to the right triangle (a triangle containing a 90° angle).

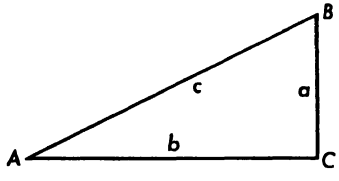


Fig. 18·3

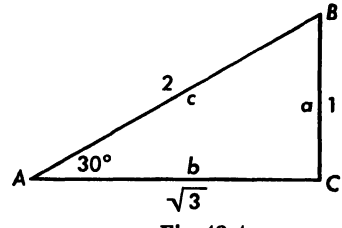


Fig. 18·4

We shall letter right triangles in a way similar to the lettering of the general triangle in Fig. 18·2, with the restriction that the angle C will always be the 90° and side c will always be the hypotenuse (see Fig. 18·3).

In the solution of a right triangle, one angle is always known to be 90°. Therefore, a right triangle is determined when two linear dimensions or one linear dimension and an acute angle are known.

A right triangle having a 30° acute angle can be solved by elementary geometry, provided a linear dimension is given.

It is known from plane geometry that in Fig. 18·4 the following ratios are true because it is a 30° right triangle [see Eq. (31), page 602].

$$\frac{a}{c} = \frac{1}{2} \qquad \frac{b}{c} = \frac{\sqrt{3}}{2} \qquad \frac{a}{b} = \frac{1}{\sqrt{3}}$$

$$\frac{b}{a} = \frac{\sqrt{3}}{1} \qquad \frac{c}{b} = \frac{2}{\sqrt{3}} \qquad \frac{c}{a} = \frac{2}{1}$$

Conversely, if it were stated that a right triangle exists in which the ratio of one side to the hypotenuse is $\frac{1}{2}$, then it would immediately be known that the triangle is a 30° right triangle. If any of the other five ratios were given, the same conclusion would be valid.

In other words, there is a unique correspondence between the six ratios above and an acute angle. The magnitude of the sides or the hypotenuse is not relevant. It is the ratio between them that determines the angles.

Plane geometry is an inadequate tool for solving general right triangles. For solving the *general* right triangle, the usefulness of plane geometry is at an end, and the usefulness of trigonometry begins.

We were concerned in Fig. 18·4 with six ratios a/b, b/c, a/c, b/a, c/b, and c/a. *Any one of these ratios determines the angle.*

EXERCISE 3

Six right triangles are referred to below. Using the ratios as given, plot the right triangles to scale and measure the acute angles.

1. $\frac{a}{c} = \frac{6.5}{9.7}$ **2.** $\frac{b}{c} = \frac{3.6}{4.85}$ **3.** $\frac{a}{b} = \frac{13}{14.4}$

4. $\frac{b}{a} = \frac{18}{16.25}$ **5.** $\frac{c}{b} = \frac{97}{72}$ **6.** $\frac{c}{a} = \frac{19.4}{13}$

7. What statement made in the text does this exercise confirm?

18·6 Trigonometric Tables. All six ratios associated with any angle have been calculated. The theory by which these calculations were made is beyond the scope of the present treatment, and the arithmetical work involved is too cumbersome for everyday use. (Such methods are, however, discussed in Chap. 24.) Instead, the ratios have been published in tabular form for use as needed. Since the ratios are dependent upon the magnitude of the angle in a definite way, they are said to be functions of the angle.

Each function has a name defined below in terms of Fig. 18·3.

$$\text{sine } A = a/c \qquad \text{abbreviated sin } A \tag{1}$$
$$\text{cosine } A = b/c \qquad \text{abbreviated cos } A \tag{2}$$
$$\text{tangent } A = a/b \qquad \text{abbreviated tan } A \tag{3}$$
$$\text{cotangent } A = b/a \qquad \text{abbreviated cot } A \tag{4}$$
$$\text{secant } A = c/b \qquad \text{abbreviated sec } A \tag{5}$$
$$\text{cosecant } A = c/a \qquad \text{abbreviated cosec } A \tag{6}$$

A brief table of the functions is given in Table 18·1.

To find functions of angles between 0 and 45°, look for the angle in the *left-hand* column and look for the name of the desired function at the *top* of the table. The desired function will then be at the intersection of the row containing the angle and the column containing the name of the function.

To find functions of angles between 45 and 90°, look for the angle in the *right-hand* column and the name of the desired function at the *bottom* of the table. The desired function will then be at the intersection of the row containing the angle and the column containing the name of the function.

Table 18·1

Angle	a/c	b/c	a/b	b/a	c/b	c/a	
A	sin	cos	tan	cot	sec	cosec	
0	0.000	1.000	0.000	.....	1.000	.....	90
5	0.087	0.996	0.087	11.430	1.004	11.474	85
10	0.174	0.985	0.176	5.671	1.015	5.759	80
15	0.259	0.966	0.268	3.732	1.035	3.864	75
20	0.342	0.940	0.364	2.747	1.064	2.924	70
25	0.423	0.906	0.466	2 145	1 103	2 366	65
30	0.500	0.866	0 577	1 732	1 155	2.000	60
35	0.574	0.819	0 700	1.428	1.221	1.743	55
40	0.643	0.766	0 839	1.192	1.305	1.556	50
45	0.707	0.707	1.000	1.000	1 414	1.414	45
	cos	sin	cot	tan	cosec	sec	Angle
	b/c	a/c	b/a	a/b	c/a	c/b	A

EXERCISE 4

Verify the following:

1. sin 20° = 0.342 **2.** cos 25° = 0.906 **3.** tan 10° = 0.176
4. cot 15° = 3.732 **5.** sec 40° = 1.305 **6.** cosec 5° = 11.474

EXERCISE 5

1. cosec 45° = 1.414 **2.** cos 60° = 0.500 **3.** sin 60° = 0.866
4. cot 70° = 0.364 **5.** tan 75° = 3.732 **6.** sec 80° = 5.759

18·7 The Use of Table 18·1. Instead of limiting the notion of a function to a ratio, it is often convenient to consider the functions as proportionality constants.

If $\sin A = a/c$, then $a = (\sin A) \times c$. This means that for a given angle the opposite side is directly proportional to the hypotenuse and that $\sin A$ is the proportionality constant. If c were 1 unit in length, then

$$a = \sin A \text{ (numerically)}$$

Formulas equivalent to Eqs. (1) to (6) may be written as

$$a = (\sin A) \times c \tag{7}$$
$$b = (\cos A) \times c \tag{8}$$
$$a = (\tan A) \times b \tag{9}$$
$$b = (\cot A) \times a \tag{10}$$
$$c = (\sec A) \times b \tag{11}$$
$$c = (\operatorname{cosec} A) \times a \tag{12}$$

Figures 18·5 to 18·7 illustrate the situation where certain sides are of unit length.

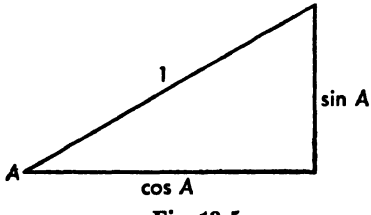


Fig. 18·5

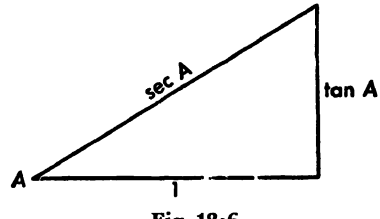


Fig. 18·6

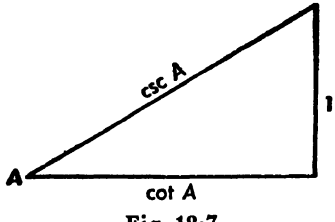


Fig. 18·7

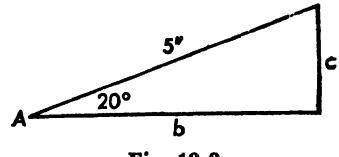


Fig. 18·8

EXAMPLE 8. Solve the right triangle shown in Fig. 18·8. This triangle may be solved by either of two equivalent lines of reasoning. One line of reasoning starts with the functions as proportionality constants, and the other starts with the functions as ratios. Both methods are shown below.

Functions as proportionality constants

If the hypotenuse had been 1 in. long, then

$a = \sin 20° = 0.342$ in.

$b = \cos 20° = 0.940$ in.

However, c is 5 times 1 in.; therefore,

a is 5 times 0.342 in. = 1.71 in.

and

b is 5 times 0.940 in. = 4.70 in.

Functions as ratios

Refer to Eqs. (1) to (6).

$a/c = \sin A$

$a/5$ in. $= \sin 20°$

$a = 5$ in. $\sin 20° = 5$ in. $\times$ $0.342 = 1.71$ in.

$b/c = \cos A$

$b/5$ in. $= 0.940$

$b = 5$ in. $\times$ $.940 = 4.70$ in.

EXERCISE 6

Verify the answers to the following problems, using Table 18·1.

1. $c = 7$ in., $A = 25°$. ANS. $a = 2.961$ in., $b = 6.342$ in., $B = 65°$.
2. $c = 4$ in., $A = 40°$. ANS. $a = 2.572$ in., $b = 3.064$ in., $B = 50°$.
3. $a = 10$ in., $A = 35°$. ANS. $c = 17.43$ in., $b = 14.28$ in., $B = 55°$.
4. $b = 3$ in., $A = 15°$. ANS. $a = 0.804$ in., $c = 3.105$ in., $B = 75°$.
5. $a = 2$ in., $A = 55°$. ANS. $b = 1.400$ in., $c = 2.442$ in., $B = 35°$.
6. $b = 6$ in., $A = 80°$. ANS. $c = 34.554$ in., $a = 34.026$ in., $B = 10°$.

18·8 Trigonometric Functions Defined. The definitions of the functions given in Eqs. (1) to (6) can be extended so as to be independent of any particular lettering system. The more general definitions are given below and refer to Fig. 18·9.

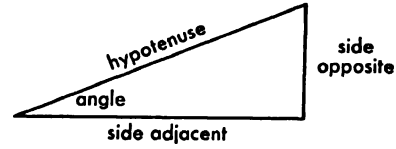


Fig. 18·9

$$\text{Sine of an angle} = \frac{\text{side opposite}}{\text{hypotenuse}} \qquad \star (13)$$

$$\text{Cosine of an angle} = \frac{\text{side adjacent}}{\text{hypotenuse}} \qquad \star (14)$$

$$\text{Tangent of an angle} = \frac{\text{side opposite}}{\text{side adjacent}} \qquad \star (15)$$

$$\text{Cotangent of an angle} = \frac{\text{side adjacent}}{\text{side opposite}} \qquad \star (16)$$

$$\text{Secant of an angle} = \frac{\text{hypotenuse}}{\text{side adjacent}} \qquad \star (17)$$

$$\text{Cosecant of an angle} = \frac{\text{hypotenuse}}{\text{side opposite}} \qquad \star (18)$$

18·9 The Cofunctions. In Fig. 18·3,

$$\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

but

$$\cos B = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{a}{c}$$

Therefore,

$$\sin A = \cos B$$

The student should verify the following relationships:

$$\sin A = \cos B \qquad \star (19)$$
$$\cos A = \sin B \qquad \star (20)$$
$$\tan A = \cot B \qquad \star (21)$$
$$\cot A = \tan B \qquad \star (22)$$
$$\sec A = \operatorname{cosec} B \qquad \star (23)$$
$$\operatorname{cosec} A = \sec B \qquad \star (24)$$

The word *cosine* means the sine of the complementary angle. In general, a cofunction is the function of the complementary angle.

18·10 Reciprocal Functions. Again referring to Fig. 18·3 notice that

$$\sin A = a/c \quad \text{and} \quad \operatorname{cosec} A = c/a$$

Therefore,

$$\sin A \times \operatorname{cosec} A = a/c \times c/a = 1$$

or

$$\sin A = \frac{1}{\operatorname{cosec} A} \quad \text{and} \quad \operatorname{cosec} A = \frac{1}{\sin A} \qquad \star (25)$$

Also,

$$\tan A = a/b \quad \text{and} \quad \cot A = b/a$$

Therefore,

$$\tan A \times \cot A = a/b \times b/a = 1$$

or

$$\tan A = \frac{1}{\cot A} \quad \text{and} \quad \cot A = \frac{1}{\tan A} \qquad \star (26)$$

Also,

$$\cos A = b/c \quad \text{and} \quad \sec A = c/b$$

Therefore,

$$\cos A \times \sec A = b/c \times c/b = 1$$

or

$$\cos A = \frac{1}{\sec A} \quad \text{and} \quad \sec A = \frac{1}{\cos A} \qquad \star (27)$$

It is apparent, then, that

1. The sine of an angle is the reciprocal of the cosecant of the same angle.
2. The cosine of an angle is the reciprocal of the secant of the same angle.
3. The tangent of an angle is the reciprocal of the cotangent of the same angle.

18·11 Use of the Five-place Trigonometric Tables. As a practical compromise between the accuracy of larger tables and the convenience of smaller tables, the five-place table is very commonly used. The values given in it for the various functions are good to about as many places as the data usually encountered in civil, mechanical, and electrical engineering. Yet there are times when a seven-place table is required. Some engineering organizations use them exclusively. On the other hand, slide-rule accuracy is sufficient in many cases. For the present we shall use the five-place table.

In the five-place table, one page is usually devoted to the functions of each degree. When we are concerned with angles less than 45°, we use the column headings at the top of the page and the minutes column on the left-hand margin of the page. When we are concerned with angles between 45 and 90°, we use the column headings at the bottom of the page and the minutes column on the right-hand margin of the page. Tabular entries are made for each minute.

EXERCISE 7

Find the values of the six functions of the following angles, using the five-place tables:

1. 20°30′	**2.** 40°27′	**3.** 44°59′	**4.** 45°01′
5. 25°45′	**6.** 60°42′	**7.** 80°28′	**8.** 42°38′
9. 00°10′	**10.** 89°30′	**11.** 13°25′	**12.** 48°05′
13. 55°55′	**14.** 81°36′	**15.** 01°45′	

Find the angle when the following functions are given:

16. $\sin A = 0.00582$	**17.** $\sin A = 0.18052$
18. $\sin A = 0.68179$	**19.** $\sin A = 0.75414$
20. $\sin A = 0.99986$	**21.** $\cos A = 0.71873$
22. $\cos A = 0.54708$	**23.** $\cos A = 0.93190$
24. $\cos A = 0.16906$	**25.** $\cos A = 0.99854$
26. $\tan A = 0.06993$	**27.** $\tan A = 1.2124$
28. $\tan A = 0.46312$	**29.** $\tan A = 1.4596$
30. $\tan A = 0.90251$	**31.** $\cot A = 0.89883$

32. cot $A = 1.7917$
33. cot $A = 2.1364$
34. cot $A = 0.62487$
35. cot $A = 5.0658$
36. sec $A = 1.0193$
37. sec $A = 2.6040$
38. sec $A = 1.2265$
39. sec $A = 1.5601$
40. sec $A = 1.4755$
41. cosec $A = 1.5212$
42. cosec $A = 1.2215$
43. cosec $A = 2.0466$
44. cosec $A = 6.1880$
45. cosec $A = 11.105$

18·12 Suggestions for Solving Right Triangles. The labor of solving triangles can be reduced considerably by intelligently choosing a method of attack. The student will be well advised to consider the suggestions given below.

As a rule, it is better to use the formula in which the unknown side appears in the numerator. For example, suppose we are to solve the triangle in which $c = 15$ and $A = 42°$. First let us find side a. We write the fraction a/c and then check Eqs. (1) to (6), looking for the fraction a/c. We find that $a/c = \sin A$, and we shall use this formula. There is another formula which would, theoretically, be just as good: $c/a = \text{cosec } A$. However, in solving for a the first equation leads to a multiplication, while the second leads to a division. Obviously the one involving a multiplication is easier to use than the one involving a division.

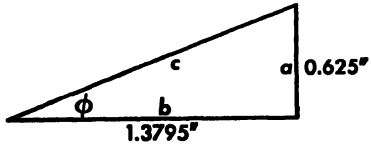


Fig. 18·10

When we have two sides given and wish to find a trigonometric function by division, it is best to choose the formula which places the number with the fewest significant digits in the denominator. This, of course, leads to an easier division.

It is always well to examine the data to see if a given decimal fraction can profitably be converted to a common fraction.

Suppose that in Fig. 18·10 the data are given as shown and the angle ϕ is to be calculated. There are two choices: either the tangent or the cotangent of ϕ may be calculated. If the tangent of ϕ is calculated, the operation involves the division of a three-figure number (0.625) by a five-figure number (1.3795); if the cotangent is calculated, the operation involves the division of a five-figure number by a three-figure number. Obviously the arithmetic will be easier in calculating the cotangent. In this particular problem there is a still easier method if the student recognizes that 0.625 is the decimal equivalent of $\frac{5}{8}$. The arithmetic can now be reduced to a multiplication by 8 and a division by 5, as shown below:

$$\cot \phi = \frac{1.3795}{\frac{5}{8}} = \frac{1.3795 \times 8}{5} = \frac{11.036}{5} = 2.2072$$

EXAMPLE 9. Referring to Fig. 18·3, if $c = 22.000$ and $A = 28°32'$, find B, a, and b.

$$B = 90° - 28°32' = 61°28'$$

According to Eq. (7),

$$a = c \sin A = 22 \sin 28°32'$$
$$= 22 \times 0.47767 = 10.509$$

Following Eq. (8),

$$b = c \cos A = 22 \cos 28°32'$$
$$= 22 \times 0.87854 = 19.328$$

EXAMPLE 10. Referring to Fig. 18·3, if $a = 19$ and $A = 62°27'$, find B, b, and c.

$$B = 90° - 62°27' = 27°33'$$

Following Eq. (12),

$$c = a \operatorname{cosec} A = 19 \operatorname{cosec} 62°27'$$
$$= 19 \times 1.1279 = 21.430$$

Following Eq. (10),

$$b = a \cot A = 19 \cot 62°27'$$
$$= 19 \times 0.52168 = 9.9119$$

EXAMPLE 11. Referring to Fig. 18·3, if $b = 26$ and $A = 15°48'$, find B, a, and c.

$$B = 90° - 15°48' = 74°12'$$

Following Eq. (9),

$$a = b \tan A = 26 \tan 15°48'$$
$$= 26 \times 0.28297 = 7\ 3572$$

Following Eq. (11),

$$c = b \sec A = 26 \sec 15°48'$$
$$= 26 \times 1.0393 = 27.022$$

EXAMPLE 12. Referring to Fig. 18·3, if $a = 45$ and $B = 24°$, find A, b, and c.

$$A = 90° - 24° = 66°$$

Following Eqs. (10) and (22) or Eq. (15),

$$b = a \tan B = 45 \tan 24°$$
$$= 45 \times 0.44523 = 20.035$$

Following Eqs. (12) and (24) or Eq. (17),

$$c = a \sec B = 45 \sec 24°$$
$$= 45 \times 1.0946 = 49.257$$

EXAMPLE 13. Referring to Fig. 18·3, if $b = 75$ and $B = 85°$ find A, a, and c.

$$A = 90° - 85° = 5°$$

Following Eqs. (9) and (21) or Eq. (16),

$$a = \text{b} \cot B = 75 \cot 85°$$
$$= 75 \times 0.08749 = 6.5618$$

Following Eqs. (11) and (23) or Eq. (18),

$$c = b \operatorname{cosec} B = 75 \operatorname{cosec} 85°$$
$$= 75 \times 1.0038 = 75.285$$

EXAMPLE 14. Referring to Fig. 18·3, if $c = 65$ and $B = 8°$, find A, a, and b.

$$A = 90 - 8° = 82°$$

Following Eqs. (7) and (19) or Eq. (14),

$$a = c \cos B = 65 \cos 8°$$
$$= 65 \times 0.99027 = 64.368$$

Following Eqs. (7) and (19) or Eq. (13),

$$b = c \sin B = 65 \sin 8°$$
$$= 65 \times 0.13917 = 9.0460$$

EXAMPLE 15. Referring to Fig. 18·3, if $b = 578$ and $a = 483$, find A, B, and c.

Following Eq. (3),

$$\tan A = a/b = {}^{483}\!/_{578} = 0.83564 \text{ (to five figures)}$$

and

$$A = 39°53'$$

Following Eq. (4),

$$\cot A = b/a = {}^{578}\!/_{483} = 1.1967 \text{ (to five figures)}$$

and

$$A = 39°53'$$
$$B = 90 - 39°53' = 50°07'$$

Following Eq. (11),

$$c = b \sec A = 578 \sec 39°53' = 578 \times 1.3032 = 753.25$$

or following Eq. (12),

$$c = a \operatorname{cosec} A = 483 \operatorname{cosec} 39°53' = 483 \times 1.5595 = 753.24$$

or following Eq. (1),

$$c = \frac{a}{\sin A} = \frac{483}{\sin 39°53'} = \frac{483}{0.64123} = 753.24$$

or following Eq. (2),

$$c = \frac{b}{\cos A} = \frac{578}{\cos 39°53'} = \frac{578}{0.76735} = 753.24$$

or using the Pythagorean theorem,

$$c = \sqrt{578^2 + 483^2} = \sqrt{567373} = 753.24$$

EXAMPLE 16. Referring to Fig. 18·3, if $c = 1{,}237$ and $a = 333$, find A, B, and b.

Following Eq. (1),

$$\sin A = a/c = {}^{333}\!/_{1237} = 0.26920 \text{ (to five figures)}$$

and

$$A = 15°37'$$

or following Eq. (6),

$$\operatorname{cosec} A = c/a = {}^{1237}\!/_{333} = 3.7147 \text{ (to five figures)}$$

and

$$\begin{aligned} A &= 15°37' \\ B &= 90 - 15°37' = 74°23' \\ b &= c \cos A = 1237 \times \cos 15°37' \\ &= 1237 \times 0.96308 = 1{,}191.3 \end{aligned}$$

or

$$b = \sqrt{c^2 - a^2} = \sqrt{1237^2 - 333^2} = \sqrt{1419280} = 1{,}191.3$$

EXAMPLE 17. The following example referring to Fig. 18·3 (when $c = 20$ and $A = 6°$) is given without further comment to illustrate the inherent accuracy of various methods.

Using the sine,

$$a = 20 \sin 6° = 20 \times 0.10453 = 2.0906$$

Using the cosecant,

$$a = \frac{20}{\operatorname{cosec} 6°} = \frac{20}{9.5668} = 2.0906 \text{ (to five figures)}$$

Using the cosine,

$$b = 20 \cos 6° = 20 \times 0.99452 = 19.890 \text{ (to five figures)}$$

Using the secant,

$$b = \frac{20}{\sec 6°} = \frac{20}{1.0055} = 19.891 \text{ (to five figures)}$$

Using seven-place tables of sines and cosines,

$$\begin{aligned} a &= 20 \times \sin 6° = 20 \times 0.1045285 = 2.090570 \\ b &= 20 \times \cos 6° = 20 \times 0.9945219 = 19.89044 \end{aligned}$$

Using a and b as calculated from the five-place tables of sines and cosines,

$$\tan A = \frac{2.0906}{19.890} = 0.105108 \text{ or } 0.10511$$

By interpolation (see Sec. 18·13),

$$A = 6°00'2''$$

Checking side c from the dimensions a and b calculated from five-place tables of sine and cosine,

$$c = \sqrt{2.0906^2 + 19.890^2} = \sqrt{4.3706 + 395.61}$$
$$= \sqrt{399.98} = 19.9997 \text{ approx.}$$

Actually,

$$19.9997^2 = 399.988$$

Checking the side c from the dimension a and b calculated from the seven-place tables of sine and cosine,

$$c = \sqrt{2.090570^2 + 19.89044^2} = \sqrt{4.370482 + 395.6296}$$
$$= \sqrt{400.0001}$$

Actually,

$$20.000003^2 = 400.0001$$

EXERCISE 8

In the following right triangles find the unknown dimensions, using a five-place table. Calculate linear dimensions accurate to five figures and angles accurate to the nearest minute.

1.	$A = 36°14'$	$c = 94.309$ ft
2.	$A = 38°19'$	$c = 8.125$ in.
3.	$B = 62°52'$	$c = 132.00$ ft
4.	$B = 11°10'$	$c = 89.048$ in.
5.	$A = 8° 8'$	$c = 2.1919$ in.
6.	$B = 49°44'$	$c = 355.06$ in.
7.	$A = 68°22'$	$c = 250.00$ in.
8.	$A = 8°25'$	$c = 12.500$ ft
9.	$B = 37°47'$	$c = 64.290$ ft
10.	$B = 39°30'$	$c = 9.6354$ in.
11.	$A = 82°16'$	$a = 40{,}625$ ft
12.	$A = 16° 4'$	$a = 0.9375$ ft
13.	$A = 19°21'$	$a = 47.395$ ft
14.	$A = 19°31'$	$a = 31.25$ in.
15.	$A = 16°18'$	$a = 14.0625$ ft
16.	$B = 74°24'$	$b = 93.75$ in.
17.	$B = 47°35'$	$a = 312.5$ in.
18.	$B = 50°26'$	$b = 15.625$ in.
19.	$A = 48°40'$	$b = 8{,}125$ in.
20.	$A = 72°48'$	$b = 718.75$ ft

21. $a = 347$ ft $b = 167$ ft
22. $a = 199$ ft $b = 160$ ft
23. $a = 67{,}130$ in. $b = 25{,}210$ in.
24. $a = 46.37$ ft $b = 94.72$ ft
25. $a = 141$ in. $b = 203$ in.
26. $c = 3{,}477$ ft $a = 2{,}638$ ft
27. $c = 2{,}691$ ft $a = 839.0$ ft
28. $c = 505.0$ ft $a = 457.0$ ft
29. $c = 43{,}649$ ft $a = 17{,}962$ ft
30. $c = 248.09$ ft $a = 218.54$ ft
31. $c = 11{,}223$ ft $b = 10{,}454$ ft
32. $c = 87.02$ ft $b = 55.43$ ft
33. $c = 2{,}338$ ft $b = 1{,}877$ ft
34. $c = 455$ in. $b = 241$ in.
35. $c = 1{,}029$ ft $b = 985.0$ ft

18·13 Interpolation in the Five-place Tables.[1] Suppose that we wish to find the sine of 26°45′30″. At the moment this is impossible, since this precise angle is not listed in the five-place table. However, this function can be found from the five-place table by a process called *interpolation*. Interpolation is not by any means limited to trigonometric tables; with suitable precautions it can be applied to any table. While the angle required is not listed in the table, the sine of 26°45′ and the sine of 26°46′ are both listed. It seems reasonable to suppose that the sine of 26°45′30″ should be halfway between sin 26°45′ and 26°46′. Although this assumption is only approximately true, the approximation is close enough.

The sine of 26°45′ is 0.45010; the sine of 26°46′ is 0.45036. The difference between them is 0.00026. One-half of this difference is 0.00013, which, added to the sine of 26°45′, gives 0.45023, which is the sine of 26°45′30″.

Suppose we wish to find the sine of 26°45′12″. It would certainly be nearer to the sine of 26°45′ than to the sine of 26°46′; in fact, it should be $\frac{12}{60}$ or 2/10 of the way between sin 26°45′ and sin 26°46′. Two-tenths of this interval is 0.2×0.00026. Therefore, we add 0.00005 to the sine of 26°45′ and find 0.45015 for the sine of 26°45′12″. We could have arrived at the same conclusion by reasoning that the sine of 26°45′12″ is 48/60 or 8/10 of the way between sin 26°46′ and sin 26°45′. In this case, we would have *subtracted* 8/10 of the tabular difference from the sine of 26°46′, thus:

$$\sin 26°45'12'' = 0.45036 - 0.8 \times 0.00026$$
$$= 0.45036 - 0.00021 = 0.45015$$

[1] See also Sec. 3·2.

Suppose we wish to find the cosine of 23°43′18″. We find from the table that

$$\cos 23°43' = 0.91555$$
$$\cos 23°44' = 0.91543$$

Notice that, in this case, as the angle increases the cosine decreases.

The cosine of 23°43′18″ is between 0.91555 and 0.91543, but it is nearer to 0.91555, since the angle is nearer to 23°43′ than it is to 23°44′.

The difference between cos 23°43′ and 23°44′ is 0.00012. Eighteen-sixtieths of this difference is 0.000036, which we *subtract* from 0.91555 to obtain 0.915514. Rounding off to five figures, we find that the cosine of 23°43′18″ is 0.91551.

Here again, we can work from either the larger or the smaller angle. We could have taken $42/60$ of 0.00012 and added this amount to 0.91543.

$$\cos 23°43'18'' = 0.91543 + 42/60 \times 0.00012$$
$$= 0.91543 + 0.00008 = 0.91551$$

EXAMPLE 18. Find the sine of 26°15′39″.

$$\sin 26°16' = 0.44255 \qquad \frac{39}{60} \times 0.00026 = 0.00017$$
$$\sin 26°15' = \underline{0.44229}$$
$$\underline{0.00026}$$

This ratio problem can be solved conveniently on the slide rule, but it is recommended that the student do it mentally if possible. In this particular case, $\frac{39}{60}$ is nearly $2/3$, which should reduce the mental strain considerably.

$$\sin 26°15'39'' = 0.44229 + 0.00017 = 0.44246$$

EXAMPLE 19. Find the sine of 26°15.6′.

$$\sin 26°16' = 0.44255 \qquad \frac{6}{10} \times 0.00026 = 0.000156 \text{ or } 0.00016$$
$$\sin 26°15' = \underline{0.44229}$$
$$\underline{0.00026}$$
$$\sin 26°15.6' = 0.44229 + 0.00016 = 0.44245$$

EXAMPLE 20. Find the cosine 61°13′52″.

$$\cos 61°13' = 0.48150 \qquad \frac{52}{60} \times 0.00026 = 0.00023$$
$$\cos 61°14' = \underline{0.48124}$$
$$\underline{0.00026}$$
$$\cos 61°13'52'' = 0.48150 - 0.00023 = 0.48127$$

EXAMPLE 21. Find the cosine of 43°21.2′.

$$\cos 43°21' = 0.72717 \qquad \frac{2}{10} \times 0.00020 = 0.00004$$
$$\cos 43°22' = \underline{0.72697}$$
$$\underline{0.00020}$$
$$\cos 43°21.2' = 0.72717 - 0.00004 = 0.72713$$

EXAMPLE 22. Find the tangent of 53°27′19″.

$$\tan 53°28' = 1.3498$$
$$\tan 53°27' = \underline{1.3490} \qquad \frac{19}{60} \times 0.0008 = 0.0003$$
$$\underline{0.0008}$$
$$\tan 53°27'19'' = 1.3490 + 0.0003 = 1.3493$$

EXAMPLE 23. Find the cotangent of 38°41′25″.

$$\cot 38°41' = 1.2489$$
$$\cot 38°42' = \underline{1.2482} \qquad \frac{25}{60} \times 0.0007 = 0.0003$$
$$\underline{0.0007}$$
$$\cot 38°41'25'' = 1.2489 - 0.0003 = 1.2486$$

EXAMPLE 24. Find the secant of 20°18′27″.

$$\sec 20°19' = 1.0663$$
$$\sec 20°18' = \underline{1.0662} \qquad \frac{27}{60} \times 0.0001 = 0.0000$$
$$\underline{0.0001}$$
$$\sec 20°18'27'' = 1.0662 + 0.0000 = 1.0662$$

EXAMPLE 25. Find the cosecant of 12°03′45″.

$$\operatorname{cosec} 12°03' = 4.7901$$
$$\operatorname{cosec} 12°04' = \underline{4.7836} \qquad \frac{45}{60} \times 0.0065 = 0.0049$$
$$\underline{0.0065}$$
$$\operatorname{cosec} 12°03'45'' = 4.7901 - 0.0049 = 4.7852$$

EXAMPLE 26. Find ϕ if $\sin \phi = 0.56295$.

$$\sin 34°16' = 0.56305 \qquad \sin \phi = 0.56295$$
$$\sin 34°15' = \underline{0.56280} \qquad \sin 34°15' = \underline{0.56280}$$
$$\underline{0.00025} \qquad \underline{0.00015}$$
$$\phi = 34°15' + \frac{15}{25} \times 60'' = 34°15'36''$$

EXAMPLE 27. Find ϕ if $\cos \phi = 0.81555$.

$$\cos 35°21' = 0.81563 \qquad \cos 35°21' = 0.81563$$
$$\cos 35°22' = \underline{0.81546} \qquad \cos \phi = \underline{0.81555}$$
$$\underline{0.00017} \qquad \underline{0.00008}$$
$$\phi = 35°21' + \frac{8}{17} \times 60'' = 35°21'28''$$

EXAMPLE 28. Find ϕ if $\sec \phi = 2.8811$.

$$\begin{array}{rr} \sec 69°42' = 2.8824 & \sec \phi = 2.8811 \\ \sec 69°41' = \underline{2.8801} & \sec 69°41' = \underline{2.8801} \\ \overline{0.0023} & \overline{0.0010} \end{array}$$

$$\sec \phi = 69°41' + \frac{10}{23} \times 60'' = 69°41'26''$$

EXAMPLE 29. Find ϕ if $\operatorname{cosec} \phi = 1.2809$.

$$\begin{array}{rr} \operatorname{cosec} 51°19' = 1.2810 & \operatorname{cosec} 51°19' = 1.2810 \\ \operatorname{cosec} 51°20' = \underline{1.2807} & \operatorname{cosec} \phi = \underline{1.2809} \\ \overline{0.0003} & \overline{0.0001} \end{array}$$

$$\phi = 51°19' + \frac{1}{3} \times 60'' = 51°19'20''$$

EXAMPLE 30. If $\tan \phi = 0.66052$, find ϕ.

$$\begin{array}{rr} \tan 33°27' = 0.66063 & \tan \phi = 0.66052 \\ \tan 33°26' = \underline{0.66021} & \tan 33°26' = \underline{0.66021} \\ \overline{0.00042} & \overline{0.00031} \end{array}$$

$$\frac{31}{42} \times 60 = 44''$$

or, using tenths of minutes,

$$\frac{31}{42} \times 10 = 0.7'$$

$$\phi = 33°26' + 44'' = 33°26'44''$$

or

$$\phi = 33°26' + 0.7' = 33°26.7'$$

EXAMPLE 31. If $\cot \phi = 3.4147$, find ϕ.

$$\begin{array}{rr} \cot 16°19' = 3.4160 & \cot 16°19' = 3.4160 \\ \cot 16°20' = \underline{3.4124} & \cot \phi = \underline{3.4147} \\ \overline{0.0036} & \overline{0.0013} \end{array}$$

$$\frac{13}{36} \times 60 = 22''$$

or

$$\frac{13}{36} \times 10 = 0.4'$$

$$\phi = 16°19'22''$$

or

$$\phi = 16°19' + 0.4' = 16°19.4'$$

If we are dealing in angular measure to the nearest second, we are in effect recognizing 60 angles between adjacent tabular entries in the angle column. However, there may be less or more than 60 five-

digit numbers between corresponding tabular entries in the function column.

Therefore, if the tabular difference in the function column is small there may be several closely grouped angles measured to the nearest second which will have the same function to five digits. If the tabular difference in the function column is large, a single angle measured to the nearest second may correspond to a range of five-figure numbers in the function column.

Refer to Example 22 above. Here, the difference between tabular entries in the tangent column is 8 in the last decimal place used. The difference in the angle is 60″. Therefore in this range of this table a difference of one unit in the last place in the tangent column corresponds to a difference of $^{60}/_{8} = 7.5''$ in the angle column. Consequently, in this portion of the five-place tables, the tangent may not be sensitive to angular changes of less than 7.5″.

If we were looking up the angle whose tangent is 1.3493 we probably should write the angle as $53°27'22'' \pm 4''$.

While we ordinarily do not go to this extreme in indicating the precision of interpolated angles, the student should certainly be aware of the limitations of the tables he uses.

In Example 22 above we calculated $^{19}/_{60}$ of 0.0008 and rounded off before adding on to 1.3490. We could, with equal reason, have found $^{19}/_{60}$ of 0.0008 to five decimal places and rounded off the tangent after addition. Usually it makes no difference. In the problems to follow it has been the policy to round off after addition.

In the absence of good reason to the contrary all data in the following problems are assumed to have an accuracy consistent with the use of five-place tables with interpolation.

EXERCISE 9

Find the sine, cosine, tangent, cotangent, secant, and cosecant:

1. 2°28′15″	**2.** 6°42′29″	**3.** 14°17′13″	**4.** 23°19′52″
5. 40°12′48″	**6.** 49°36′27″	**7.** 52°47′35″	**8.** 71°28′18″
9. 80°59′40″	**10.** 85°43.4′	**11.** 27°15.3′	**12.** 42°27.6′
13. 72°19.8′	**14.** 85°26.2′	**15.** 17°15.7′	

EXERCISE 10

Find the angle when the following functions are given:

1. $\sin \phi = 0.10572$	**2.** $\sin \phi = 0.32650$	**3.** $\sin \phi = 0.57461$
4. $\sin \phi = 0.81385$	**5.** $\sin \phi = 0.72645$	**6.** $\cos \phi = 0.49695$

7. $\cos \phi = 0.97808$ **8.** $\cos \phi = 0.99981$ **9.** $\cos \phi = 0.02391$
10. $\cos \phi = 0.54980$ **11.** $\tan \phi = 0.86901$ **12.** $\tan \phi = 1.1109$
13. $\tan \phi = 0.46430$ **14.** $\tan \phi = 12.271$ **15.** $\tan \phi = 1.0455$
16. $\cot \phi = 249.88$ **17.** $\cot \phi = 0.06315$ **18.** $\cot \phi = 2.7592$
19. $\cot \phi = 1.5476$ **20.** $\cot \phi = 1.3825$ **21.** $\sec \phi = 23.042$
22. $\sec \phi = 1.0670$ **23.** $\sec \phi = 1.6032$ **24.** $\sec \phi = 1.2130$
25. $\sec \phi = 2.3440$ **26.** $\operatorname{cosec} \phi = 2.7290$
27. $\operatorname{cosec} \phi = 9.3381$ **28.** $\operatorname{cosec} \phi = 1.2880$
29. $\operatorname{cosec} \phi = 1.3834$ **30.** $\operatorname{cosec} \phi = 1.3119$

EXERCISE 11

Solve the following right triangles, using five-place natural functions.

1. $A = 38°50'45''$ $c = 0.87500$ in.
2. $A = 27°12'32''$ $c = 7.9143$ in.
3. $A = 53°35.5'$ $c = 15.453$ in.
4. $A = 37°50'10''$ $c = 98.268$ ft
5. $B = 22°32.1'$ $c = 2726.0$ ft
6. $A = 10°42'47''$ $c = 5.3805$ in.
7. $B = 53°24'34''$ $c = 10.625$ in.
8. $A = 38°14.9'$ $c = 8.1250$ ft
9. $B = 60°34'43''$ $c = 14.000$ ft
10. $A = 32°50'47''$ $c = 15.00$ ft
11. $A = 5°50'31''$ $a = 1250.0$ ft
12. $A = 11°16'44''$ $a = 457.31$ ft
13. $A = 45°15.8'$ $a = 986.91$ ft
14. $B = 36°44'2''$ $a = 12.500$ in.
15. $A = 17°48.3'$ $a = 713.85$ ft
16. $B = 38°56'46''$ $b = 63.275$ ft
17. $B = 86°11.4'$ $b = 756.23$ ft
18. $B = 89°30.6'$ $b = 1.875$ in.
19. $A = 21°42'58''$ $b = 51.387$ ft
20. $A = 10°38'13''$ $b = 46.500$ ft
21. $a = 20$ in. $b = 37.998$ in.
22. $a = 12$ in. $b = 23.828$ in.
23. $a = 4.6397$ ft $b = 17.927$ ft.
24. $a = 389.72$ in. $b = 1303.1$ in.
25. $a = 83.695$ ft $b = 177.70$ ft
26. $a = 1.125$ in. $b = 1.4462$ in.
27. $a = 437.92$ ft $b = 1284.9$ ft
28. $a = 76.392$ ft $b = 191.67$ ft

29. $a = 31.25$ in. $b = 44.679$ in.
30. $a = 2.1875$ ft $b = 3.3878$ ft
31. $a = 147.00$ in. $c = 418.58$ in.
32. $a = 23.458$ in. $c = 236.64$ in.
33. $a = 39.738$ ft $c = 42.973$ ft
34. $a = 358.03$ ft $c = 369.16$ ft
35. $a = 939.52$ in. $c = 1{,}136.6$ in.
36. $b = 0.34297$ ft $c = 0.40083$ ft
37. $b = 125.00$ in. $c = 299.80$ in.
38. $b = 1.625$ in. $c = 2.5898$ in.
39. $b = 13.794$ ft $c = 18.667$ ft
40. $b = 53.187$ in. $c = 65.142$ in.

18·14 Isosceles Triangles. By definition, an isosceles triangle is a triangle in which two sides are equal. From this it follows that two

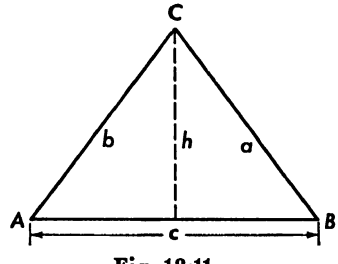


Fig. 18·11

angles must be equal (see Appendix B, Sec. B·1). Thus in Fig. 18·11 if $a = b$, the triangle is isosceles, and $A = B$. The altitude drawn to the base bisects the base and the angle C.

$$h^2 = b^2 - (\tfrac{1}{2}c)^2 \tag{28}$$

$$h = \sqrt{(b - \tfrac{1}{2}c)(b + \tfrac{1}{2}c)} \tag{29}$$

In general, for any triangle,

$$\text{Area} = \tfrac{1}{2} \times \text{base} \times \text{altitude}$$

In the case of the triangle in Fig. 18·11,

$$h = b \sin A$$

or

$$h = b \cos \tfrac{1}{2}C$$

Therefore

$$\text{Area} = \tfrac{1}{2}cb \sin A \tag{30}$$

or

$$\text{Area} = \tfrac{1}{2}bc \cos \tfrac{1}{2}C \tag{31}$$

If angle C and either a or b are known, then

$$c = 2b \sin \tfrac{1}{2}C \tag{32}$$

or

$$c = 2a \sin \tfrac{1}{2}C \tag{33}$$

EXERCISE 12

1. The equal sides of an isosceles triangle are each 5.86 in. long, and each base angle is 23°51′. Find the length of the base and the altitude of the triangle.
2. Sixteen holes in a metal plate are equally spaced on a circle 7⅝ in. in diameter. Find all dimensions for locating centers of holes by spacing off with dividers, measured from one hole.
3. A sheet of metal 15 in. wide is bent along its center line to form a V-shaped gutter. Will the gutter have a greater capacity when it is 6 in. wide at the top, or when 6 in. deep?
4. Find the angle of bend for each case in Prob. 3.
5. A right prism has for its base an equilateral triangle 7.3 in. on each side. It is cut by a plane which makes an angle of 25° with the base; one side of the section includes one side of the base. Find the sides, angles, and area of the section.
6. A sphere 5½ in. in diameter is dropped into a tin cone 8½ in. in diameter and 7½ in. deep. Is the top of the sphere above or below the rim of the cone, and how far?
7. The legs of a tripod are each 4 ft 2¾ in. long, and their feet form an equilateral triangle 2 ft 1½ in. on a side. Find the angle between one leg and a plumb bob hung from its top.
8. Find the area of a segment cut off from a circle 16.82 in. in diameter by a line 2.73 in. from the center.
9. Each leaf of a swinging double door 3 ft 6 in. wide has been opened through an angle of 64°. How far apart are their edges? How far is each edge from the line of closure?

18·15 Regular Polygons.[1] A regular polygon (see Fig. 18·12) is inscribed in a circle of radius R and circumscribed about a circle of radius r. Let

s = length of one side = $\overline{ab}$
n = the number of sides
R = the radius of the circumscribed circle
r = the radius of the inscribed circle
p = the perimeter of the polygon = ns

[1] See also page 604.

A_t = area of a single triangle, as for example the triangle *abc*
A_p = area of the entire polygon

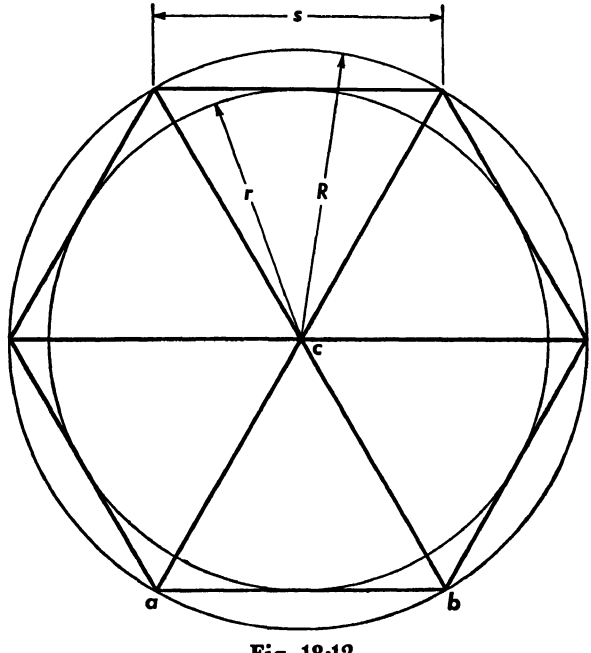


Fig. 18·12

It can be proved that

$$A_t = \frac{s^2}{4} \cot \frac{180^\circ}{n} \tag{34}$$

$$A_t = \frac{R^2}{2} \sin \frac{360^\circ}{n} \tag{35}$$

$$A_t = r^2 \tan \frac{180^\circ}{n} \tag{36}$$

$$r = \frac{s}{2} \cot \frac{180^\circ}{n} \tag{37}$$

$$R = \frac{s}{2} \operatorname{cosec} \frac{180^\circ}{n} \tag{38}$$

$$p = 2nR \sin \frac{180^\circ}{n} \tag{39}$$

$$p = 2nr \tan \frac{180^\circ}{n} \tag{40}$$

EXERCISE 13

1. Prove that in a regular polygon

a. $A_p = \frac{1}{4} ns^2 \cot \frac{180}{n} = \frac{1}{2} nR^2 \sin \frac{360°}{n} = nr^2 \tan \frac{180}{n}$.

b. $r = \frac{s}{2} \cot \frac{180}{n}$.

c. $R = \frac{1}{2} s \operatorname{cosec} \frac{180}{n}$.

d. $p = 2nR \sin \frac{180}{n} = 2nr \tan \frac{180}{n}$.

2. One side of a regular decagon inscribed in a circle is 3.27 in. Find the radius of the circle.

3. Find the area of the decagon of Prob. 2.

4. A ring 15 in. in diameter is suspended from a point by 10 cords, each 9 in. long and equally spaced. Find the angle between two adjacent cords.

5. The length of one side of a regular polygon of six sides is 1 in. Find the area of a regular polygon of 12 sides inscribed in the same circle.

6. Find the difference in area between a regular polygon of eight sides and a regular polygon of nine sides if the perimeter of each is 72 in.

7. The length of a side in a regular polygon of five sides is 2 in. Find the area of the ring bounded by the circumferences of the inscribed and circumscribed circles.

8. Find the length of side of the largest hexagon which can be cut from a square sheet of paper 10 in. on a side.

18·16 Logarithmic Solution of Right Triangles.[1] The multiplication and division necessary in solving right triangles can, of course, be done by logarithms. For this purpose, tables have been published in which the logarithms of the functions are given directly. These tables are organized in exactly the same way as the corresponding tables of natural functions.

The sine or cosine of an angle is always less than 1.00000. Therefore, the logarithm of a sine or cosine always has a negative characteristic. The logarithm of the sine of 17° (log sin 17°) is listed in the table of log functions as 9.46594. It is understood that this logarithm is followed

[1] See Chap. 14.

by a -10, and the complete logarithm would be written

$$\log \sin 17^\circ = 9.46594 - 10$$

or

$$\log \sin 17^\circ = \bar{1}.46594$$

or

$$\log \sin 17^\circ = 0.46594 - 1$$

The log tangent 11° is listed as 9.28865. In this case it is understood that the logarithm is followed by a -10, and the complete logarithm would be written

$$\log \tan 11^\circ = 9.28865 - 10$$

or

$$\log \tan 11^\circ = \bar{1}.28865$$

or

$$\log \tan 11^\circ = 0.28865 - 1$$

Remembering that the tangent of an angle greater than 45° is itself greater than 1, it should be obvious that the log tangent of an angle greater than 45° has a positive or zero characteristic. The log tangent of 76°10′ is listed as 0.60864 and is, of course, not followed by a -10.

$$\log \tan 76^\circ 10' = 0.60864$$

EXAMPLE 32. Find log sin 15°26′15″.

$$\begin{aligned}
\log \sin 15^\circ 27' &= 9.42553 - 10 = 0.42553 - 1 \\
\log \sin 15^\circ 26' &= \underline{9.42507 - 10 = 0.42507 - 1} \\
&\qquad\quad 0.00046 \qquad\quad 0.00046 \\
&\frac{15}{60} \times 46 = 11.5 \\
\log \sin 15^\circ 26' 15'' &= 9.42507 - 10 + 0.000115 \\
&= 9.425185 - 10, \text{ or } 0.425185 - 1
\end{aligned}$$

In rounding off to five places, since the number ends in exactly 5, the last digit in the rounded-off number is left the nearest even digit, or

$$\log \sin 15^\circ 26' 15'' = 9.42518 - 10 = 0.42518 - 1$$

Notice that the difference between adjacent tabular entries is published in the table under the *d* column. This avoids the need for an actual subtraction such as was done in the above example.

EXAMPLE 33. Find ϕ if $\log \tan \phi = 0.42765$.

This number appears directly in the table and is the log tangent of 69°31′. Notice that log tan 14°59′ = 9.42755 − 10, and the log tan 1°32′ = 8.42762 − 10.

Therefore, the student should be particularly careful to make sure that he is using the correct characteristic as well as the correct mantissa.

When solving right triangles, there is some question as to whether it is more efficient to use natural functions or log functions. With oblique triangles it is usually better to use log functions.

EXAMPLE 34. In a certain right triangle, $a = 1.7320$ and $A = 26°30'$. Find b and c, using five-place log functions.

$$c = \frac{1.7320}{\sin 26°30'}$$

$$\log c = \log 1.7320 - \log \sin 26°30'$$

$$\log 1.7320 = 1.23855 - 1$$

$$\log \sin 26°30' = \underline{0.64953 - 1}$$

$$\log c = \overline{0.58902 + 0}$$

$$c = 3.8817$$

$$\log b = \log 1.7320 + \log \cot 26°30'$$

$$\log 1.7320 = 0.23855$$

$$\log \cot 26°30' = \underline{0.30226}$$

$$\log b = \overline{0.54081}$$

$$b = 3.4738$$

EXERCISE 14

Solve the following right triangles, using five-place log functions.

1.	$A = 37°42'16''$	$c = 146.32$ in.
2.	$A = 2°26'\ 5''$	$c = 0.43792$ in.
3.	$B = 72°19'28''$	$c = 157.65$ in.
4.	$B = 36°28'45''$	$c = 29.463$ in.
5.	$A = 28°36'20''$	$c = 1.3752$ in.
6.	$A = 4°13'30''$	$b = 136.48$ in.
7.	$A = 88°25'14''$	$a = 15.358$ in.
8.	$B = 47°28'10''$	$a = 0.03794$ in.
9.	$A = 89°21'38''$	$b = 15279$ in.
10.	$A = 45°13'\ 5''$	$a = 28.365$ in.
11.	$a = 13.625$ in.	$c = 142.98$ in.
12.	$a = 76.5$ ft	$c = 92.8$ ft
13.	$a = 5.436$ in.	$c = 10.83$ in.
14.	$a = 26.932$ in.	$c = 41.867$ in.
15.	$a = .3652$ in.	$c = 0.5847$ in.
16.	$a = 10.932$ in.	$b = 110.36$ in.
17.	$a = 95.632$ in.	$b = 8.7305$ in.
18.	$a = 52.693$ in.	$b = 27.956$ in.
19.	$a = 3.6571$ in.	$b = 7.3058$ in.
20.	$a = 26.328$ in.	$b = 21.497$ in.

18·17 Solution of Oblique Triangles without Special Formulas. So far we have discussed only the solutions of right triangles, but in fact the student has at this point all the information he needs to solve oblique triangles as well.

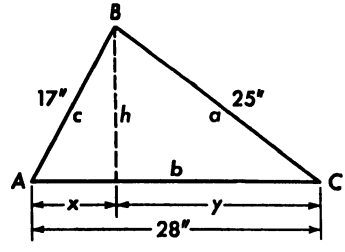


Fig. 18·13

Data determining oblique triangles can be given in any of three ways. Three sides may be given, two angles and one side may be given, or two sides and one angle may be given.

18·18 Solution of an Oblique Triangle When Three Sides Are Given

EXAMPLE 35. Solve the oblique triangle shown in Fig. 18·13.

Draw h perpendicular to b; then

$$h^2 = 17^2 - x^2 \qquad \text{and} \qquad h^2 = 25^2 - y^2$$

Therefore,

$$17^2 - x^2 = 25^2 - y^2$$
$$y^2 - x^2 = 25^2 - 17^2 = 625 - 289 = 336$$
$$(y - x)(y + x) = 336$$

but

$$y + x = b = 28$$
$$(y - x)28 = 336$$
$$y - x = \frac{336}{28} = 12$$

$$\begin{array}{rl} y - x = 12 & \\ \underline{y + x = 28} & \\ 2y = 40 & \text{(by addition)} \\ y = 20 & \end{array} \qquad \begin{array}{rl} y - x = 12 & \\ \underline{y + x = 28} & \\ 2x = 16 & \text{(by subtraction)} \\ x = 8 & \end{array}$$

$$h^2 = 25^2 - y^2 = 25^2 - 20^2 = 625 - 400 = 225$$
$$h = \sqrt{225} = 15$$
$$\tan A = \frac{15}{8} = 1.875$$
$$A = 61°55'39''$$
$$\tan C = \frac{h}{y} = \frac{15}{20} = 0.75000$$
$$C = 36°52'12''$$
$$B = 180° - (61°55'39'' + 36°52'12'') = 81°12'09''$$

18·19 Solution of an Oblique Triangle When Two Angles and a Side Are Given

EXAMPLE 36. Solve the oblique triangle shown in Fig. 18·14.

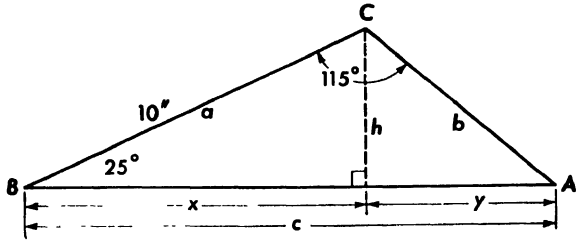


Fig. 18·14

Drop the perpendicular h from C to the side c.

$$h = 10 \times \sin 25° = 10 \times 0.42262 = 4.2262 \text{ in.}$$
$$x = 10 \times \cos 25° = 10 \times 0.90631 = 9.0631$$
$$A = 180° - (25° + 115°) = 40°$$
$$b = h \times \operatorname{cosec} 40° = 4.2262 \times 1.5557 = 6.5747 \text{ in.}$$
$$y = h \cot 40° = 4.2262 \times 1.1918 = 5.0368 \text{ in.}$$
$$c = x + y = 9.0631 \text{ in.} + 5.0368 \text{ in.} = 14.0999 \text{ in. or } 14.100 \text{ in.}$$

18·20 Solution of an Oblique Triangle When Two Sides and an Angle Are Given

EXAMPLE 37. Solve the oblique triangle shown in Fig. 18·15.

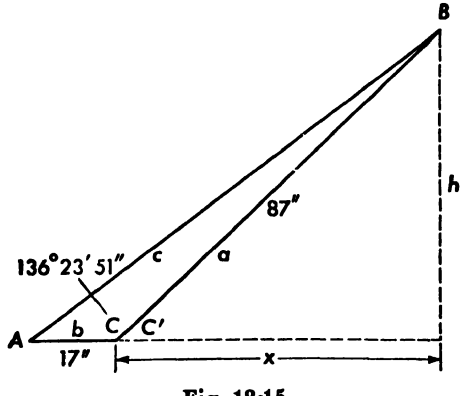


Fig. 18·15

In this example, *two sides and the included angle* are given.

$$C' = 180° - 136°23'51'' = 43°36'9''$$
$$h = 87 \times \sin 43°36'9'' = 87 \times 0.68965 = 60$$
$$x = 87 \times \cos 43°36'9'' = 87 \times 0.72414 = 63$$
$$b + x = 17 + 63 = 80$$

$$\cot A = \frac{b + x}{h} = \frac{80}{60} = 1.3333$$

$$A = 36°52'15''$$

$$c = \frac{h}{\sin 36°52'15''} = \frac{60}{\sin 36°52'15''}$$

$$= \frac{60}{0.60001} = 100$$

EXAMPLE 38. Solve the oblique triangle in which one side is 60.000 in. and the opposite angle is 41°06′44″. Another side is 73.000 in.

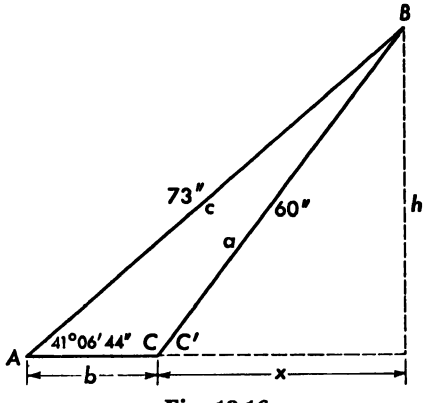


Fig. 18·16

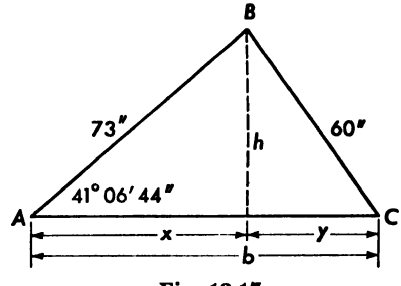


Fig. 18·17

The data here are ambiguous. Observe that both Fig. 18·16 and Fig. 18·17 are consistent with the stated data, while these triangles are by no means equal.

Such ambiguity is possible only when two sides and the angle opposite one of them are given and when the given angle is less than 90°.

We shall solve the triangle in Fig. 18·16 first.

$$h = 73 \times \sin 41°06'44'' = 73 \times 0.65753 = 48 \text{ in.}$$

$$x + b = 73 \times \cos 41°06'44'' = 73 \times 0.75342 = 55 \text{ in.}$$

$$x = \sqrt{60^2 - 48^2} = 36 \text{ in.}$$

$$b = 55 - 36 = 19 \text{ in.}$$

$$\sin C' = \frac{h}{60} = \frac{48}{60} = 0.80000$$

$$C' = 53°07'49''$$

$$C = 180° - 53°07'49'' = 126°52'11''$$

$$B = 180° - (41°06'44'' + 126°52'11'') = 12°01'05''$$

The solution for Fig. 18·17 follows.

$$h = 73 \sin 41°06'44'' = 73 \times 0.65753 = 48 \text{ in.}$$
$$x = 73 \times \cos 41°06'44'' = 73 \times 0.75342 = 55 \text{ in.}$$
$$\sin C = \frac{48}{60} = 0.8000$$
$$C = 53°07'49''$$
$$y = \sqrt{60^2 - 48^2} = 36 \text{ in.}$$
$$b = x + y = 55 + 36 = 91 \text{ in.}$$
$$B = 180° - (53°07'49'' + 41°06'44'') = 85°45'27''$$

EXERCISE 15

Solve the following triangles, using the methods given in Secs. 18·18, 18·19, or 18·20. If the data are inconsistent, state why.

1.	$a = 37$ in.	$b = 91$ in.	$c = 72$ in.
2.	$a = 52$ in.	$b = 25$ in.	$c = 63$ in.
3.	$a = 21$ in.	$b = 31$ in.	$c = 52$ in.
4.	$a = 65$ in.	$b = 89$ in.	$c = 132$ in.
5.	$A = 36°52'12''$	$b = 12$ in.	$C = 118°04'21''$
6.	$A = 73°44'23''$	$b = 75$ in.	$C = 58°20'07''$
7.	$a = 8$ in.	$b = 80$ in.	$C = 36°52'12''$
8.	$b = 53$ in.	$c = 88$ in.	$A = 58°06'34''$
9.	$A = 53°07'49''$	$a = 68$ in.	$b = 13$ in.
10.	$a = 20$ in.	$b = 5$ in.	$A = 22°37'12''$
11.	$b = 86$ in.	$c = 97$ in.	$C = 104°15'$

18·21 The Slide-rule Solution of Right Triangles. Within its inherent limit of accuracy, the slide rule may be used to solve trigonometry problems.

The following table gives a good working idea of the relation between the number of significant digits in the linear dimensions of a triangle and the corresponding precision in angular measure.

Significant digits in linear dimensions	*Precision in angular measure*
2	Nearest degree
3	Nearest 10 minutes
4	Nearest minute
5	Nearest tenth of a minute

On the slide rule, linear dimensions can be used and computed to about one part in a thousand. Angles can be used and computed to about the nearest 0.05°. This is not strictly true over the entire scale, but it is a good working average.

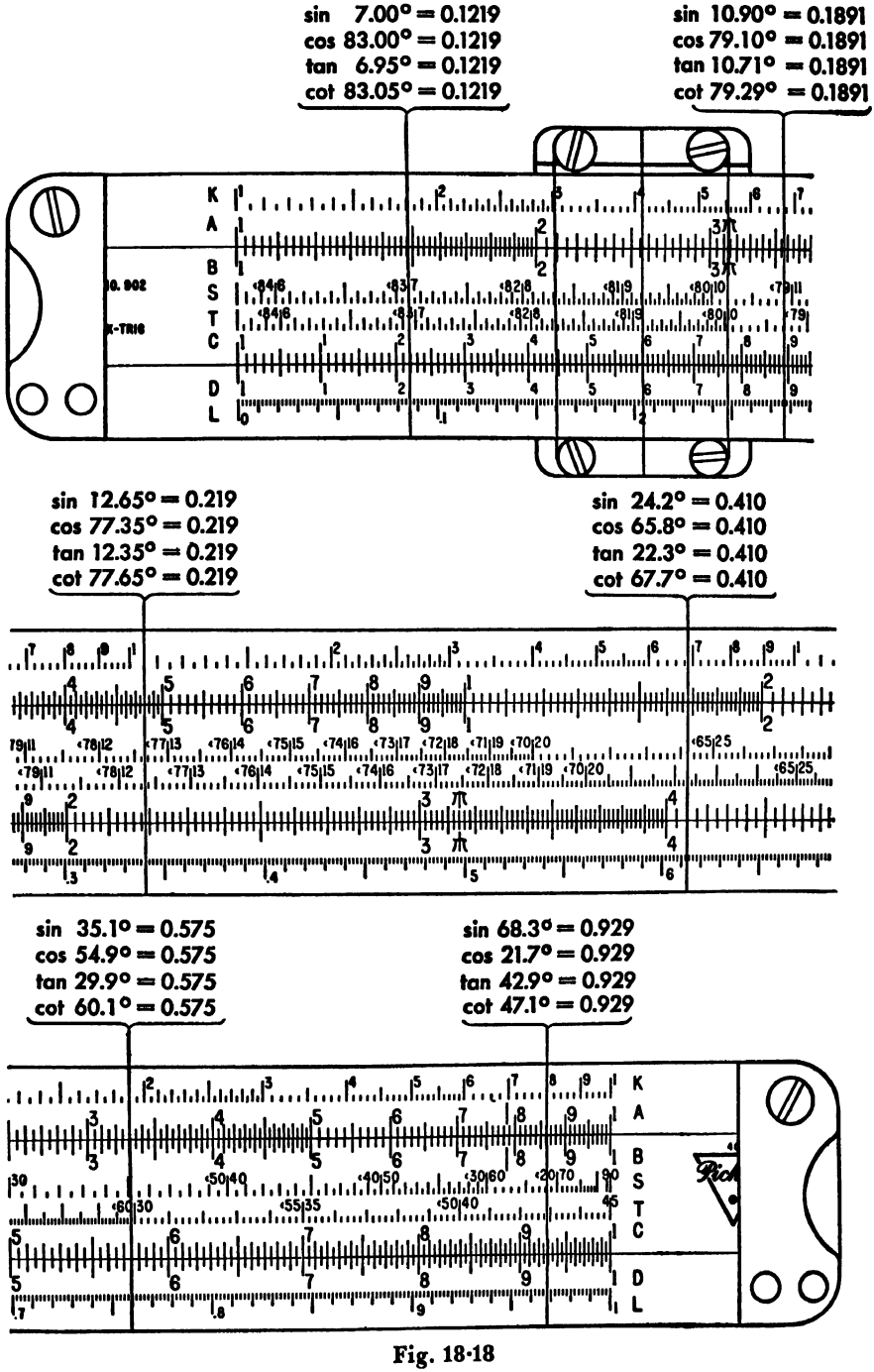

sin 7.00° = 0.1219
cos 83.00° = 0.1219
tan 6.95° = 0.1219
cot 83.05° = 0.1219
sin 10.90° = 0.1891
cos 79.10° = 0.1891
tan 10.71° = 0.1891
cot 79.29° = 0.1891
K
A
B
S
T
C
D
L
sin 12.65° = 0.219
cos 77.35° = 0.219
tan 12.35° = 0.219
cot 77.65° = 0.219
sin 24.2° = 0.410
cos 65.8° = 0.410
tan 22.3° = 0.410
cot 67.7° = 0.410
sin 35.1° = 0.575
cos 54.9° = 0.575
tan 29.9° = 0.575
cot 60.1° = 0.575
sin 68.3° = 0.929
cos 21.7° = 0.929
tan 42.9° = 0.929
cot 47.1° = 0.929

Fig. 18·18

On the particular rule used here for illustration, the S scale (see Fig. 18·18) is calibrated in degrees and decimal parts of a degree. The long markers are double-numbered in pairs of complementary angles, *e.g.*, 82/8, 70/20, 60/30.

When the *right-hand* numerals are used, the hairline simultaneously indicates an angle on the S scale and the *sine* of that angle on the C scale.

When the *left-hand* numerals are used, the hairline simultaneously indicates an angle on the S scale and the cosine of that angle on the C scale.

EXERCISE 16

1. Using a slide rule, find the sine and cosine of the following angles:

a. 28°	**b.** 8°	**c.** 17°	**d.** 13°	**e.** 20°	**f.** 30°
g. 7.2°	**h.** 12.2°	**i.** 26.2°	**j.** 7.63°	**k.** 16.75°	**l.** 41.4°

The slide-rule setting illustrated in Fig. 18·19 is adequate for solving Examples 39 to 42.

EXAMPLE 39. The hypotenuse of a right triangle is 8 in., and one angle is 30°. Find the side opposite the 30° angle. Here the slide rule is set to multiply 8 in. by the sine of 30°, giving 4 in. as the length of the opposite side.

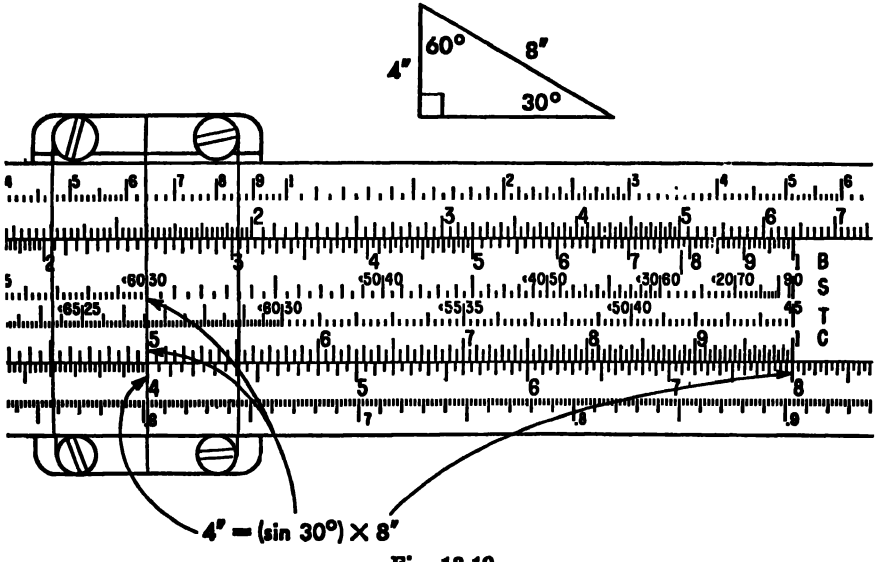


Fig. 18·19

EXAMPLE 40. The hypotenuse of a right triangle is 8 in., and one angle is 60°. Find the side adjacent to the 60° angle. In Fig. 18·19 the slide rule is set to multiply 8 in. by the cosine of 60°.

EXAMPLE 41. One side of a right triangle is 4 in., and the hypotenuse is 8 in. Find the angle opposite the 4-in. side. Instead of dividing 4 by 8 to obtain the sine of the required angle, it is more convenient to answer the question, "Eight times the sine of what angle equals 4?" Therefore the slide rule is set to multiply by 8, and the hairline is set over the product 4. The angle whose sine gives this product is read under the hairline on the S scale. A division of 4 by 8 would give a quotient of 0.5 on the D scale, and a new setting would have to be made to find the angle whose sine is this quotient.

EXAMPLE 42. One side of a right triangle is 4 in., and the hypotenuse is 8. Find the angle adjacent to the 4-in. side. In this problem we answer the question, "Eight times the cosine of what angle yields 4 as the product?" (see Fig. 18·19).

EXERCISE 17

1. Verify the statement that the slide-rule setting shown in Fig. 2·5 is consistent with the following relations:
   - **a.** $\frac{2.53}{14.28} = \sin 10.21°$
   - **b.** $\frac{2.53}{14.28} = \cos 79.79°$
   - **c.** $14.28 \cos 79.79° = 2.53$
   - **d.** $14.28 \sin 10.21° = 2.53$
2. Verify the statement that the slide-rule setting in Fig. 2·7 is consistent with the following relations:
   - **a.** $\frac{8}{9.03} = \sin 62.4°$
   - **b.** $\frac{8}{9.03} = \cos 27.6°$
   - **c.** $9.03 \sin 62.4° = 8$
   - **d.** $9.03 \cos 27.6° = 8$
3. Write a series of four equations similar to those given in Probs. 1 and 2 above, but consistent with Figs. 2·9, 2·11, and 2·13.
4. Using a slide rule, solve Probs. 1 to 10 and 31 to 40, page 390.

The T scale is calibrated in degrees and decimal parts of a degree. The long markers are double-numbered in pairs of complementary angles in a way similar to the S scale.

When the *right-hand* numerals are used, the hairline simultaneously indicates an angle on the T scale and the *tangent* of that angle on the C scale. When the *left-hand* numerals are used, the hairline simultaneously indicates an angle on the T scale and the *cotangent* of that angle on the C scale.

The C scale is used for sines, cosines, tangents, and cotangents. The left-hand index of the C scale is used as 0.1, and the right-hand index is used as 1.0. Therefore on this particular rule we are limited to angles whose sine is between 0.1 and 1.0, angles whose cosine is between 0.1 and 1.0, and angles whose tangent or cotangent is between 0.1 and 1.0. The smallest angle we can use directly is about 5.7°. When using sines, we can process angles up to 90°, although the scale is crowded near 90°. When using cosines, we can use angles up to about 84.3°. The upper limit of the tangent scale and the lower limit of the cotangent scale is 45°. However, as will be illustrated in subsequent examples, this does not put any additional limitations on the usefulness of the slide rule.

If it should be necessary to find the tangent of an angle greater than 45°, the reciprocal of its cotangent may be found.

EXERCISE 18

1. Using a slide rule, find

a. tan 20°	**b.** tan 35.76°	**c.** tan 44.2°
d. tan 7.9°	**e.** tan 15.82°	**f.** tan 29.62°
g. cot 48.2°	**h.** cot 82.1°	**i.** cot 52.7°

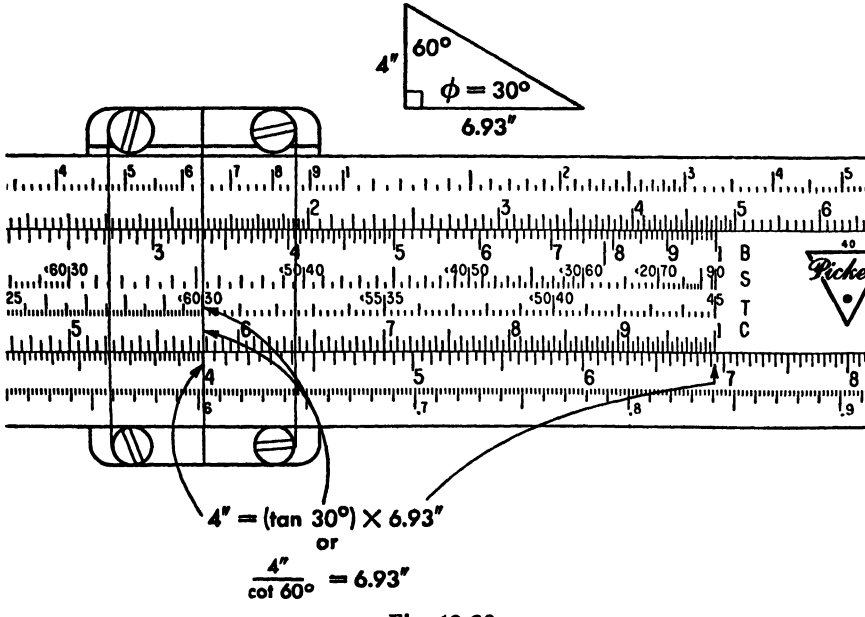


Fig. 18·20

The slide-rule setting illustrated in Fig. 18·20 is adequate for solving the following examples:

EXAMPLE 43. One side of a right triangle is 6.93 in., and the adjacent angle is 30°. Find the side opposite the 30° angle.

Use the equation

$$\text{Opposite side} = \text{adjacent side} \times \text{tangent } \phi$$

The slide-rule setting in Fig. 18·20 accomplishes this multiplication.

EXAMPLE 44. One side of a right triangle is 4 in., and the adjacent angle is 60°. Find the side opposite the 60° angle.

Use the equation

$$\text{Opposite side} = \frac{\text{adjacent side}}{\cot \phi}$$

The slide-rule setting shown in Fig. 18·20 accomplishes this division. Thus we avoid using tangents of angles greater than 45°.

EXAMPLE 45. One side of a right triangle is 6.93 in., and the other side is 4 in. Find the angle opposite the 4-in. side. Again refer to Fig. 18·20. If we divided 4 in. by 6.93 in. directly, the quotient would appear on the C scale and the angle could not be read directly. Instead we find the angle whose tangent multiplied by 6.93 in. equals 4 in. This angle is read directly on the T scale.

EXAMPLE 46. One side of a right triangle is 6.93 in., and the other side is 4 in. Find the angle opposite the 6.93-in. side. Here we find the angle whose cotangent divided into 4 in. yields 6.93 in. as a quotient (see Fig. 18·20).

Observe that in the preceding examples, two linear dimensions appear on the D scale. The longest linear dimension appears under a C index and the shorter appears under the hairline. The angle opposite the shorter side is under the hairline on the S scale if the hypotenuse is involved and on the T scale if the hypotenuse is *not* involved.

EXERCISE 19

1. Verify the statement that the slide-rule setting shown in Fig. 2·5 is consistent with the following relations:

   a. $\dfrac{2.53}{14.28} = \tan 10.05°$

   b. $\dfrac{2.53}{14.28} = \cot 79.95°$

   c. $2.53 = 14.28 \tan 10.05°$

   d. $2.53 = 14.28 \cot 79.95°$

2. Verify the statement that the slide-rule setting shown in Fig. 2·7 is consistent with the following relations:

   a. $\frac{8}{9.03} = \tan 41.55°$

   b. $\frac{8}{9.03} = \cot 48.45°$

   c. $8 = 9.03 \tan 41.55°$

   d. $8 = 9.03 \cot 48.45°$

3. Write a series of four equations similar to those given in Probs. 1 and 2 above but consistent with Figs. 2·9, 2·11, and 2·13.
4. Using a slide rule, solve Probs. 11 to 30, page 390.

Section 2·12 describes a method of solving Pythagorean theorem problems on the slide rule. The method illustrated in the following example is somewhat more convenient for those with a knowledge of trigonometry.

EXAMPLE 47. The hypotenuse of a right triangle is 8 in., and one side is 4 in. Find the other side. First find the smaller angle, as in Fig. 18·19 and Example 41. Then, having found the smaller angle (in this case 30°), divide the smaller side by the tangent of the smaller angle, as in Fig. 18·20.

EXAMPLE 48. The two sides of a right triangle are 4 and 6.93 in., respectively. Find the hypotenuse. First find the smaller angle, as in Example 45 and Fig. 18·20. Then, having found the smaller angle, divide the sine of this angle into the shorter side to find the hypotenuse, as in Fig. 18·19.

EXERCISE 20

1. Using a slide rule, solve Probs. 21 to 30 in Exercise 11, page 390.
2. Using a slide rule, solve Probs. 31 to 40 in Exercise 11, page 391.

18·22 The Slide-rule Solution of Oblique Triangles. The slide rule can be used to solve oblique triangles, using methods similar to those described in Secs. 18·18 to 18·20.

EXAMPLE 49. Given the triangle shown in Fig. 18·21, where $b = 8$ in., $A = 30°$, and $B = 34°$, find a, c, and C.

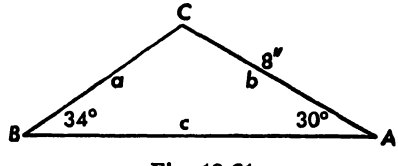


Fig. 18·21

1. Set the rule to indicate h, as in Fig. 18·22.

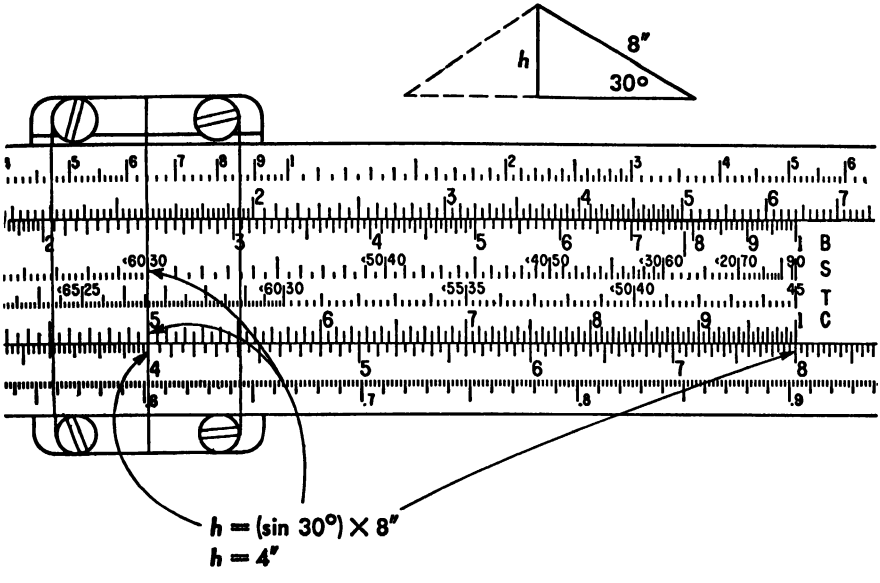


Fig. 18·22

2. Set the rule to indicate a, as in Fig. 18·23*a*.

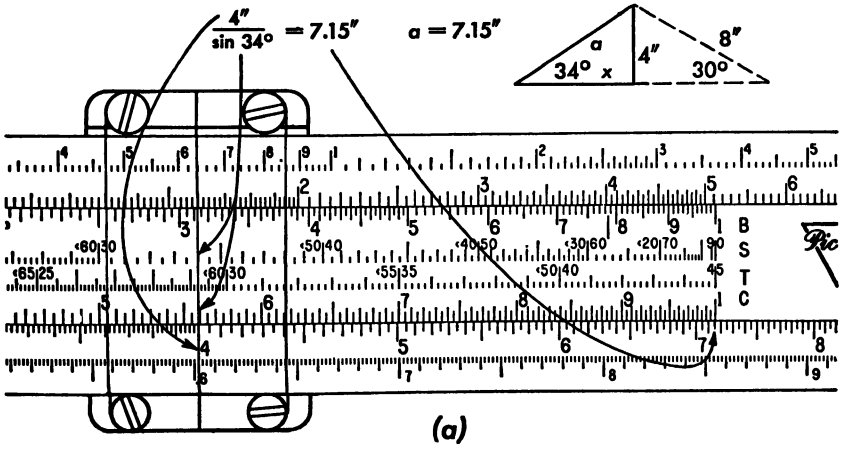


Fig. 18·23*a*

3. Set the rule to indicate x, as in Fig. 18·23*b*.

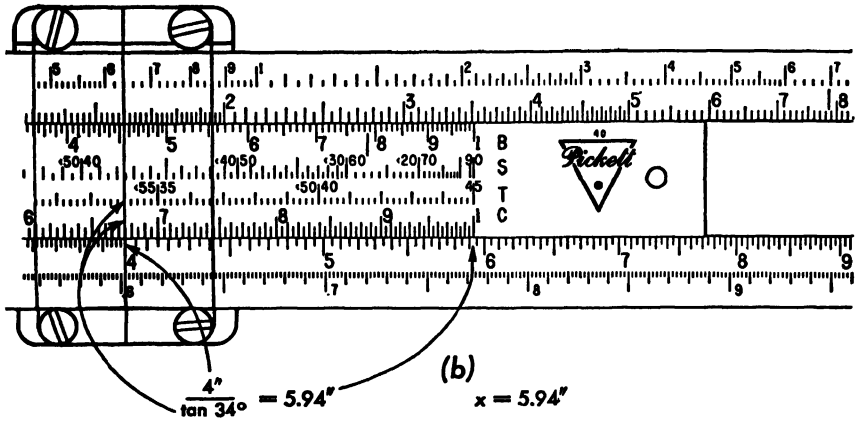


Fig. 18·23*b*

4. Set the rule to indicate y, as in Fig. 18·24.

Side $c = x + y$, angle $C = 180° - (A + B)$

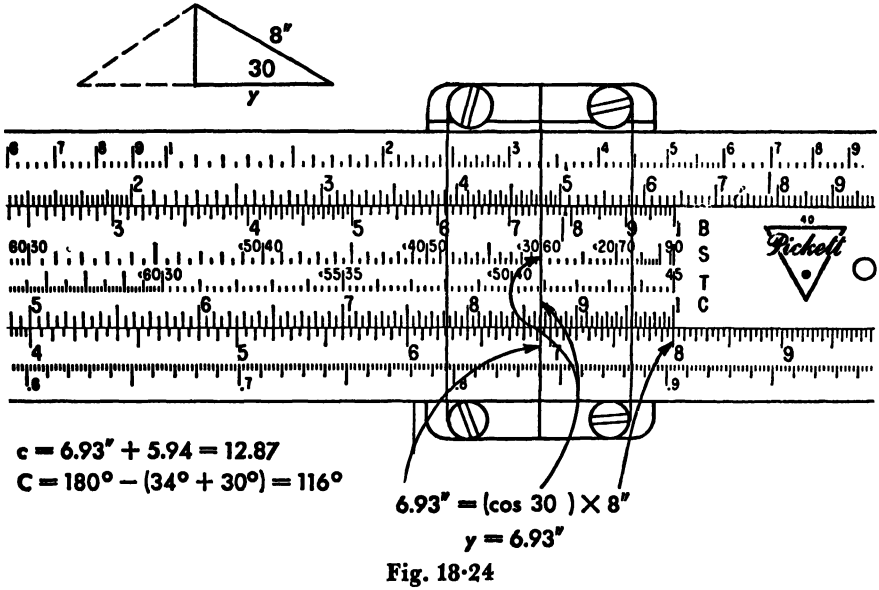


Fig. 18·24

EXAMPLE 50. Given the triangle shown in Fig. 18·25, if $b = 8$ in., $c = 12.87$ in., and $A = 30°$, find a, B, and C.

1. Find h and y, as in Figs. 18·22 and 18·24, respectively.
2. Since $x = 12.87 - y$, the length x may be found by subtraction.

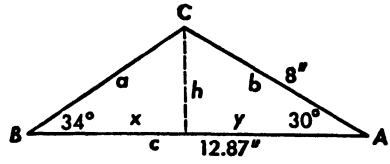


Fig. 18·25

3. Knowing h and x, find B, as in Fig. 18·23b.
4. Knowing h and B, find a, as in Fig. 18·23a.

EXERCISE 21

Solve Probs. 1 to 11 in Exercise 15 on a slide rule.

18·23 The Solution of Right Triangles Involving Small Angles. The slide rule illustrated in this book does not indicate angles less than about 5.7°. However, with one exception the slide rule can be used to solve triangles involving smaller angles by using special methods described below.

The exception occurs when the hypotenuse and long side are known and are nearly equal. In this case, the method required for solution is inherently quite inaccurate. It is inherently inaccurate regardless of whether a table of trigonometric functions or the slide rule is used. However, since tables are in general more precise than slide rules, the inherent inaccuracies in the method of solution are not so noticeable when using tables as when using a slide rule.

EXAMPLE 51. Find the short side of a right triangle in which the hypotenuse is 134.4 in. and the long side is 133.6 in.

Let the hypotenuse be c, the long side be b, and the short side be a.

$$a = \sqrt{c^2 - b^2} = \sqrt{(c - b)(c + b)} \tag{41}$$
$$c - b = 134.4 - 133.6 = 0.8 \text{ in.}$$

Notice that the difference (0.8 in.) is expressed to only one significant figure, whereas the c and b are given to four significant figures.

$$c + b = 134.4 + 133.6 = 268.0 \text{ in.}$$
$$a = \sqrt{0.8 \times 268} = \sqrt{214.4} = 14.64 \text{ in.}$$

If the length of the side b is changed by 0.1 in. to make $b = 133.7$, then

$$c = \sqrt{(134.4 - 133.7)(134.4 + 133.7)} = 13.70 \text{ in.}$$

Thus a change of 0.1 in. in b makes a change of 0.94 in. in a. Consequently the slide rule would hardly be an appropriate instrument for this type of problem.

The examples given below illustrate problems in which the use of the slide rule is appropriate.

EXAMPLE 52. Solve the right triangle shown in Fig. 18·26.

The side a in Fig. 18·26 is nearly equal to the arc length BX in Fig. 18·27. Of course this approximation will be close only when A is small. The arc

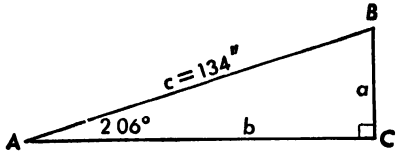


Fig. 18·26

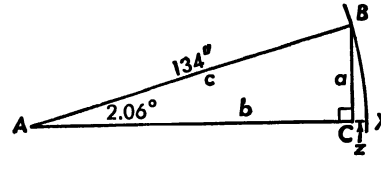


Fig. 18·27

length BX is a fraction of the circumference of a circle of radius 134 in. whose center is at A.

This fraction is 2.06°/360°.

In other words,

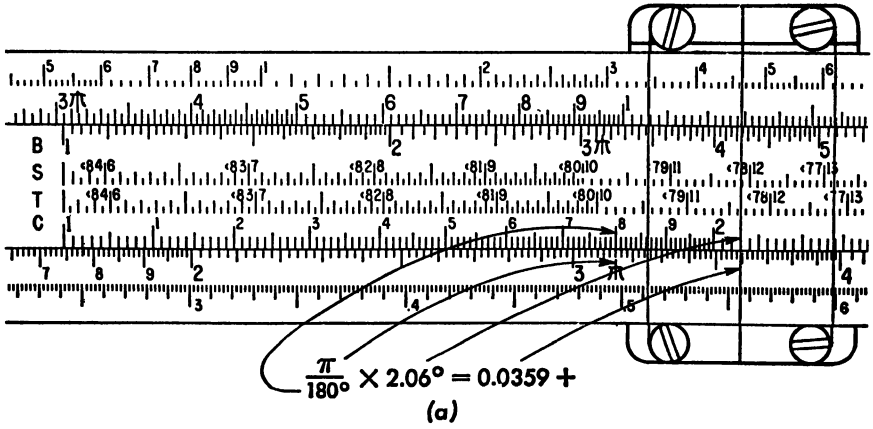
$$\frac{2.06^\circ}{360^\circ} \times 2\pi \times 134 = \text{arc length } BX = \frac{\pi}{180^\circ} \times 2.06^\circ \times 134 \qquad (42)$$

Or the side a is given by the approximate formula

$$a \approx \frac{2\pi}{360^\circ} \times 2.06^\circ \times 134 = \frac{\pi}{180^\circ} \times 2.06^\circ \times 134 \qquad (43)$$

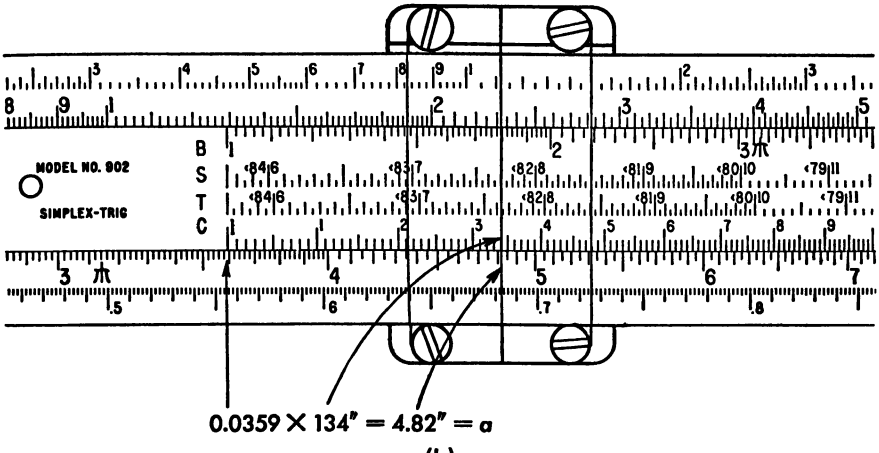
$$a \approx 0.0359 \times 134 = 4.82 \text{ in.}$$

See Fig. 18·28 for the appropriate settings to solve Eq. (43).



(a)

Fig. 18·28a



(b)

Fig. 18·28*b*

The general formula for a is

$$a \approx \frac{\pi}{180} \times A \times c \tag{44}$$

To find b we first find the length z (Fig. 18·27). In Fig. 18·27,

$$c = b + z \tag{45}$$

and

$$c - b = z \tag{46}$$

since both c and $b + z$ are radii of the same circle. But, by the Pythagorean theorem,

$$a^2 = c^2 - b^2 = (c - b)(c + b) \tag{47}$$

$$a^2 = z(c + b) \tag{48}$$

or

$$z = \frac{a^2}{c + b} \tag{49}$$

If c and b are nearly equal, we may unite

$$z \approx \frac{a^2}{2c} \qquad \text{(See Table A·2, Appendix A)} \tag{50}$$

In the present example

$$a = 4.82 \text{ in.}$$
$$c = 134 \text{ in.}$$

Therefore

$$z \approx \frac{4.82^2}{2 \times 134} = 0.0867 \text{ in.} \tag{51}$$

and

$$b \approx 134.0 \text{ in.} - 0.0867 \text{ in.} \tag{52}$$

or, within slide-rule precision,

$$b \approx 134.0 \text{ in.} - 0.1 \text{ in.} = 133.9 \text{ in.}$$

The slide-rule settings used to solve Eq. (51) are illustrated in Fig. 18·29.

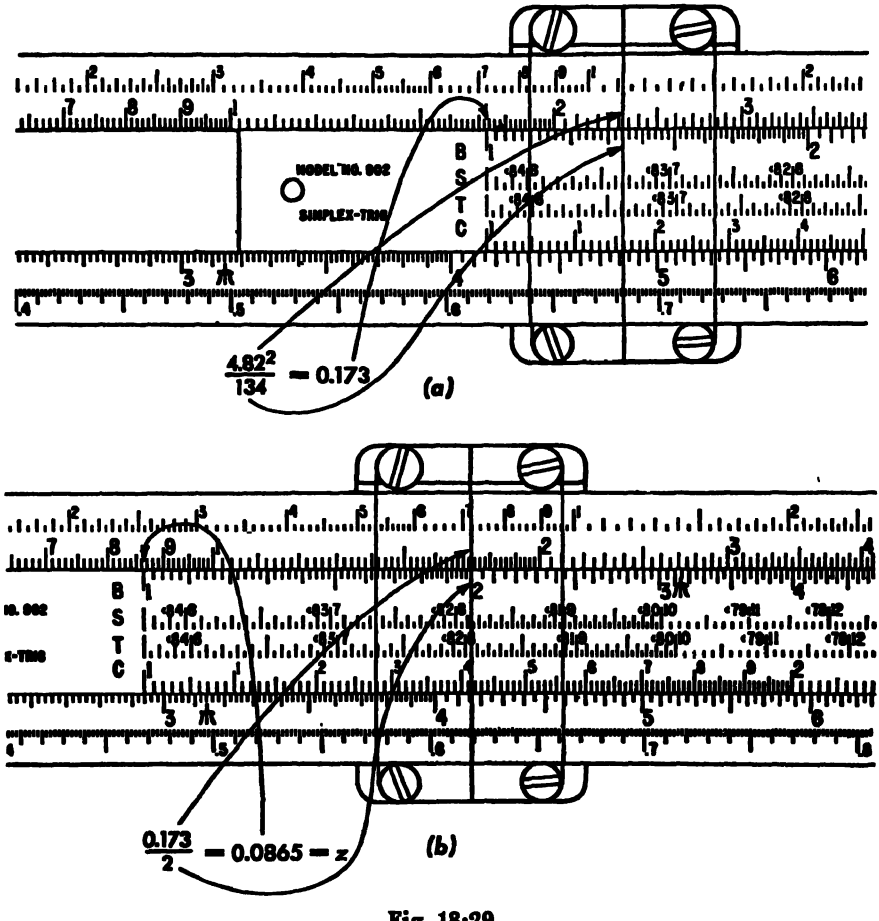


Fig. 18·29

EXAMPLE 53. Solve the right triangle shown in Fig. 18·30.

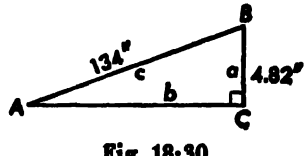


Fig. 18·30

Following Eq. (44),

$$\frac{a}{c} \approx \frac{\pi}{180} \times A \tag{53}$$

The ratio a to c is set on the slide rule shown in Fig. 18·28*b*. To this ratio we match the ratio $\frac{\pi}{180}$ multiplied by the value of A (as yet unknown) (see Fig. 18·28*a*). This requires that A be 2.06°.

The side b is calculated as in Example 52.

EXAMPLE 54. Solve the triangle shown in Fig. 18·31.

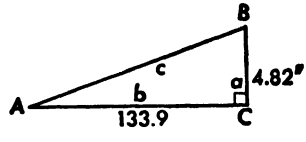


Fig. 18·31

Since b and c are nearly equal, Eq. (44) may be altered to read

$$\frac{a}{b} \approx \frac{\pi}{180^\circ} \times A \tag{54}$$

To solve for c, we alter Eq. (50) as follows:

$$z \approx \frac{a^2}{2b} \tag{55}$$

Therefore

$$c \approx b + \frac{a^2}{2b} \qquad \text{(See Table A·2, Appendix A)} \tag{56}$$

UNIT SIX

OBLIQUE TRIANGLES AND APPLICATIONS OF NUMERICAL TRIGONOMETRY

This is a continuation of Unit Five but requires a somewhat greater knowledge of algebra

Chapter 19 OBLIQUE TRIANGLES

For efficiency and convenience, the solution of oblique triangles is often accomplished by the use of special formulas rather than by the methods outlined in Secs. 18·18 to 18·20. We shall emphasize four of these formulas. All four formulas refer to the general triangle shown in Fig. 19·1.

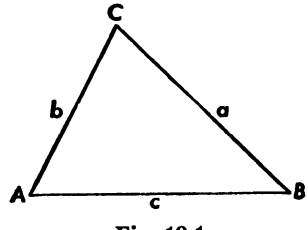


Fig. 19·1

19·1 The Sine Law. This formula is useful when we know two angles and a side or when we know two sides and the angle opposite one of them. The formula is usually stated in the form of Eq. (1):

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \star (1)$$

Equation (1) is equivalent to Eqs. (2) to (4) below.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad (2)$$

$$\frac{a}{\sin A} = \frac{c}{\sin C} \qquad (3)$$

$$\frac{b}{\sin B} = \frac{c}{\sin C} \qquad (4)$$

If one of the angles in the triangle is obtuse, its sine may be found by the rule below.

To find the sine of an obtuse angle, subtract the obtuse angle from 180°, *and use the sine of the acute angle so found.*[1]

In general, when the angle A is an obtuse angle,

$$\sin A = \sin (180° - A) \tag{5}$$

EXAMPLE 1. Find the sine of 126°.

$$\sin 126° = \sin (180° - 126°) = \sin 54°$$

From the five-place table,

$$\sin 54° = 0.80902$$

Therefore

$$\sin 126° = 0.80902$$

EXERCISE 1

Find the sine of each of the following angles:

1. 140°	**2.** 170°	**3.** 95°30′	**4.** 125°46′17″
5. 113°29′50″	**6.** 165°27′50″	**7.** 175°09′55″	**8.** 100°10′20″
9. 179°40′30″			

Find each of two angles less than 180° which have their sine listed below.

10. 0.05814	**11.** 0.33244	**12.** 0.57715
13. 0.72837	**14.** 0.90766	**15.** 0.99824

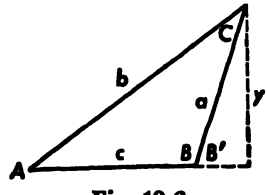


Fig. 19·2

19·2 Derivation of the Sine Law. Referring to Fig. 19·2,

$$y = b \sin A \tag{6}$$

and

$$y = a \sin B' \tag{7}$$

But from the geometry of the figure the angles B and B' are supplementary angles; therefore, from Eq. (5), their sines are equal, and Eq. (7) may be written

$$y = a \sin B \tag{8}$$

[1] See also Eq. (97), page 363.

Combining Eqs. (6) and (8),

$$a \sin B = b \sin A \tag{9}$$

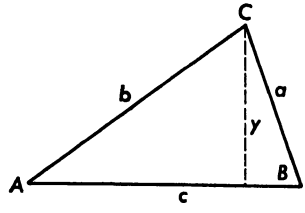


Fig. 19·3

Similarly, in Fig. 19·3

$$y = a \sin B \tag{10}$$

and

$$y = b \sin A \tag{11}$$

Therefore

$$a \sin B = b \sin A \tag{12}$$

Equation (9) or (12) may be written

$$\frac{a}{\sin A} = \frac{b}{\sin B} \tag{13}$$

The same equation applies whether B is obtuse or acute. By symmetry,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad * \tag{14}$$

Since Eq. (14) involves multiplication and division, logarithms may be used to good advantage in the solution.

EXAMPLE 2. Solve the triangle $A = 42°10'$, $B = 78°40'$, $c = 150$ in.

$$C = 180° - (42°10' + 78°40') = 59°10'$$

$$\frac{a}{\sin A} = \frac{c}{\sin C} \qquad \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 42°10'} = \frac{150}{\sin 59°10'} \qquad \frac{b}{\sin 78°40'} = \frac{150}{\sin 59°10'}$$

$$a = \frac{150 \times \sin 42°10'}{\sin 59°10'} \qquad b = \frac{150 \times \sin 78°40'}{\sin 59°10'}$$

log 150 in. =	2.17609	log 150 in. =	2.17609
log sin 42°10′ =	9.82691 − 10	log sin 78°40′ =	9.99145 − 10
	12.00300 − 10		12.16754 − 10
log sin 59°10′ =	9.93382 − 10	log sin 59°10′ =	9.93382 − 10
log a =	2.06918	log b =	2.23372
a =	117.27 in.	b =	171.28 in.

The labor of calculation can be reduced somewhat by the use of cologarithms (see Sec. 14.24).

$$\begin{aligned} \log 150 &= 2.17609 \\ \log \sin 42°10' &= 9.82691 - 10 \\ \operatorname{colog} \sin 59°10' &= 0.06618 \\ \log a &= 12.06918 - 10 \\ a &= 117.27'' \end{aligned} \qquad \begin{aligned} \log 150 &= 2.17609 \\ \log \sin 78°40' &= 9.99145 - 10 \\ \operatorname{colog} \sin 59°10' &= 0.06618 \\ \log b &= 12.23372 - 10 \\ b &= 171.28'' \end{aligned}$$

If the basic equations are written in the form

$$\frac{\sin 42°10'}{a} = \frac{\sin 59°10'}{150}$$

and if the above equation is treated as a proportion, the problem may be checked on the slide rule (see Sec. 2·8 and Secs. 18·21 and 18·22).

Here, if we match 59°10′ (59.2°) on the S scale to 150 on the D scale we shall find 78°40′ (78.7°) opposite 171.3 and 42°10′ (42.2°) opposite 117.3.

EXAMPLE 3. Solve the oblique triangle $B = 40°$, $C = 25°$, and $a = 23.529$ in.

$$A = 180° - (40° + 25°) = 180 - 65° = 115°$$

$$\frac{b}{\sin B} = \frac{a}{\sin A} \qquad \frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\log \sin 115° = \log \sin (180° - 115°) = \log \sin 65°$$

$$\begin{aligned} \log \sin 65° &= 9.95728 - 10 \\ \log \sin 115° &= 9.95728 - 10 \end{aligned}$$

$$\begin{aligned} \log 23.529 \text{ in.} &= 1.37160 \\ \log \sin 40° &= 9.80807 - 10 \\ \operatorname{colog} \sin 115° &= 0.04272 \\ \log b &= 11.22239 - 10 \\ b &= 16.687 \text{ in.} \end{aligned} \qquad \begin{aligned} \log 23.529 \text{ in.} &= 1.37160 \\ \log \sin 25° &= 9.62595 - 10 \\ \operatorname{colog} \sin 115° &= 0.04272 \\ \log c &= 11.04027 - 10 \\ c &= 10.972 \text{ in.} \end{aligned}$$

EXAMPLE 4. Solve the oblique triangle $A = 25°$, $b = 125$ in., and $a = 80$ in.

Here, two sides and the angle opposite one of them are given. In Sec. 18·20 it was indicated that under these conditions there is a possibility of two solutions consistent with the given data.

Figures 19·4*a* and 19·4*b* are both drawn approximately to scale.

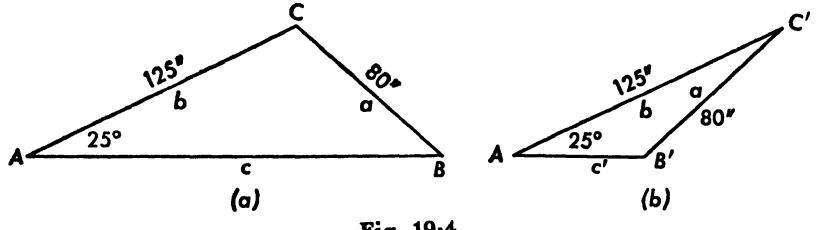


Fig. 19·4

We shall solve both triangles.

$$\frac{b}{\sin B} = \frac{a}{\sin A} \quad \text{or} \quad \frac{\sin B}{b} = \frac{\sin A}{a} \quad \text{or} \quad \sin B = \frac{b \times \sin A}{a}$$

$$\sin B = \frac{125 \times \sin 25°}{80}$$

$$\begin{aligned} \log 125 &= 2.09691 \\ \log \sin 25° &= 9.62595 - 10 \\ \operatorname{colog} 80 &= 8.09691 - 10 \\ \log \sin B &= 19.81977 - 20 \\ B &= 41°19'34'' \end{aligned}$$

$$C = 180° - (25° + 41°19'34'') = 180° - 66°19'34'' = 113°40'26''$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{c}{\sin 113°40'26''} = \frac{80}{\sin 25°}$$

$$c = \frac{80 \times \sin 113°40'26''}{\sin 25°}$$

$$\log \sin 113°40'26'' = \log \sin (180° - 113°40'26'') = \log \sin 66°19'34''$$

$$\log \sin 66°19'34'' = 9.96182 - 10$$

$$\log \sin 113°40'26'' = 9.96182 - 10$$

$$\begin{aligned} \log 80 &= 1.90309 \\ \log \sin 113°40'26'' &= 9.96182 - 10 \\ \operatorname{colog} \sin 25° &= 0.37405 \\ \log c &= 12.23896 - 10 \\ c &= 173.36 \end{aligned}$$

$$B' = 180° - B = 180° - 41°19'34'' = 138°40'26''$$

$$C' = 180° - (138°40'26'' + 25°) = 16°19'34''$$

$$c' = \frac{a \times \sin C'}{\sin A} = \frac{80 \times \sin 16°19'34''}{\sin 25°}$$

$$\begin{aligned} \log 80 &= 1.90309 \\ \log \sin 16°19'34'' &= 9.44886 - 10 \\ \operatorname{colog} \sin 25° &= 0.37405 \\ \log c' &= 11.72600 - 10 \\ c' &= 53.211 \end{aligned}$$

EXERCISE 2

1. Redraw Figs. 19·2 and 19·3 and by the use of suitable construction lines prove that

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

2. In Fig. 19·5 prove that

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

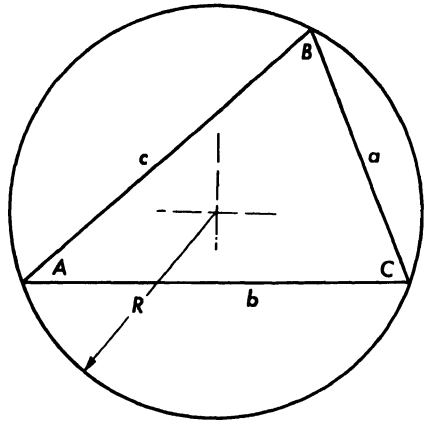


Fig. 19·5

3. Rewrite Eq. (13) for the case where $B = 90°$.

4. In a certain triangle, the angles A, B, and C are, respectively, as 3:4:5. Side a is 10.000 in. Find sides b and c.

5. The sides of a triangle a, b, c are, respectively, 25, 38, and 51 in. The angles (not listed in the same order as the sides) are approximately 28°04′, 106°15′, and 45°40′. Identify the angles A, B, and C.

6. Given $a = 10\sqrt{2}$, $A = 30°$, $C = 105°$, find b and c.

7. If $A = 45°$ and $B = 30°$, find the sides a and b when $c = 10$ in.

8. In Fig. 19·6 prove by the law of sines that

$$\frac{ab}{bc} = \frac{oa}{oc}$$

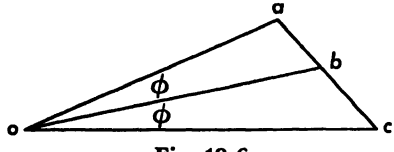


Fig. 19·6

9. If $A = 30°$, $B = 120°$, and $c = 1{,}000$, find a, b, and the altitude drawn to c.

10. Referring to Fig. 19·7,

a. Find h and x and $\angle ACO$.

b. Knowing h, find a and y and $\angle BCO$.

c. Knowing x, y, $\angle ACO$, and $\angle OCB$, find C and side c.

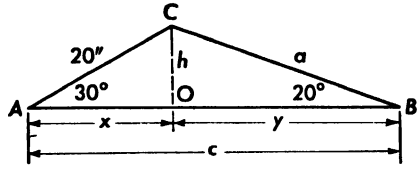


Fig. 19·7

11. a. Angle A of a certain triangle is 17°27.7′, side a is 30 in., and angle B is 42°. Draw the triangle to scale and calculate side b.

b. If angle A is 162°32.3′ and a = 30 in. and B = 10°, draw the triangle to scale and solve for side b.

12. The base of a triangle is 4,500 ft, and the angles at the base are 10°20.4′ and 15°34′. Find the unknown sides.

In the problems below solve for unknown sides to five figures and unknown angles to the nearest minute.

13. b = 1.557 ft	A = 38°19′	C = 88°10′
14. b = 3.3492 ft	B = 144°10′	C = 13°02′
15. b = 4.1759 in.	A = 31°20′	C = 18°27′
16. b = 4.2997 in.	A = 7°36′	C = 10°29′
17. b = 6.8926 ft	A = 9°36′	C = 47°38′
18. a = 1.1831 in.	B = 8°41′	C = 29°54′
19. c = 11.863 in.	A = 5°24′	B = 83°50′
20. b = 3.8372 in.	A = 25°26′	B = 124°36′
21. a = 6.3397 in.	C = 26°23′	B = 49°07′
22. c = 10.165 in.	B = 20°12′	C = 75°50′
23. c = 39.862 in.	B = 39°11′	A = 12°37′
24. a = 2.757 ft	B = 43° 7′	C = 35°14′
25. b = 1.7089 in.	c = 1.2788 in.	C = 48°21′
26. b = 4.860 ft	a = 3.5683 ft	B = 68°01′
27. b = 7.2199 ft	A = 57°37′	a = 6.5867 ft
28. a = 1.1690 in.	c = .63966 in.	C = 28°04′
29. b = 4.4682 ft	A = 61°22′	a = 4.1244 ft
30. b = 1.0226 ft	c = 1.3378 ft	C = 70°14′
31. c = 3.7198 in.	A = 62°31′	a = 3.3145 in.
32. b = 5.2979 ft	A = 74° 3′	C = 11°17′
33. a = 3.7956 ft	B = 38° 7′	C = 82°41′
34. b = 0.43972 in.	A = 51° 8′	C = 36°28′

Find the unknown sides to five figures and the unknown angles to the nearest second, using the sine law.

35. $a = 17.230$ in. $A = 56°22'13''$ $C = 35°53'16''$
36. $c = 11.855$ in. $A = 7°30'47''$ $C = 47°36'12''$
37. $c = 133.7$ in. $A = 4°17'22''$ $B = 165°29'23''$
38. $b = 105.46$ in. $B = 85°10'31''$ $c = 105.09$ in.
39. $b = 145.70$ in. $a = 145.10$ in. $B = 85°09'55''$
40. $b = 16.683$ in. $A = 79°50'30''$ $a = 17.938$ in.
41. $c = 16.481$ in. $b = 32.675$ in. $B = 109°13'16''$
42. $c = 12.781$ in. $a = 12.412$ in. $C = 46°28'50''$
43. $a = 17.219$ in. $c = 19.751$ in. $C = 88°56'\ 3''$
44. $a = 6.6435$ in. $B = 53°21'\ 9''$ $C = 48°48'37''$
45. $a = 10.959$ in. $A = 80°43'53''$ $B = 33°24'58''$
46. $c = 30.361$ in. $A = 21°14'28''$ $B = 146°40'26''$
47. $a = 23.293$ in. $A = 24°15'35''$ $C = 25°25'25''$
48. $b = 10.878$ in. $A = 44°40'38''$ $B = 49°41'43''$
49. $b = 36.234$ in. $A = 20°\ 6'20''$ $C = 48°16'42''$
50. $a = 11.306$ in. $A = 25°24'44''$ $B = 92°24'\ 3''$
51. $a = 6527.6$ in. $B = 70°55'29''$ $C = 52°\ 9'43''$
52. $c = 1004$ in. $A = 79°19'25''$ $B = 53°27'10''$
53. $b = 14.752$ in. $B = 13°19.7'$ $C = 59°13.6'$
54. $b = 999.9$ in. $A = 37°58.7'$ $C = 65°2.9'$
55. $a = 497.32$ in. $A = 10°36.4'$ $B = 46°37.9'$
56. $a = 832.76$ in. $A = 82°36'42''$ $B = 45°32'10''$
57. $a = 796.38$ in. $A = 99°36'24''$ $C = 49°37'45''$
58. $a = 827.56$ in. $C = 12°48.3'$ $B = 140°59.7'$
59. $a = 143.62$ in. $B = 37°42.7'$ $C = 28°26.5'$

19·3 The Cosine Law. This law is useful when two sides and the included angle are known or when three sides are known.

Referring to Fig. 19·1,

$$a^2 = b^2 + c^2 - 2 \times b \times c \times \cos A \qquad \star (15)$$
$$b^2 = a^2 + c^2 - 2 \times a \times c \times \cos B \qquad \star (16)$$
$$c^2 = a^2 + b^2 - 2 \times a \times b \times \cos C \qquad \star (17)$$

If one of the angles in the triangle is obtuse, its cosine may be found by the rule below:

To find the cosine of an obtuse angle, subtract the obtuse angle from 180° *and use the* negative *of the cosine of the acute angle so found.*[1]

In general, when the angle A is obtuse,

$$\cos A = -\cos (180° - A) \qquad (18)$$

EXAMPLE 5. Find the cosine of 126°.

$$\cos 126° = -\cos (180° - 126°) = -\cos 54°$$

[1] See Eq. (98), page 363.

From the five-place tables

$$\cos 54° = 0.58779$$

Therefore

$$\cos 126° = -0.58779$$

If the angles A, B, or C are obtuse, then Eqs. (15), (16), and (17) may be written

$$a^2 = b^2 + c^2 + 2 \times b \times c \times \cos (180° - A) \tag{19}$$
$$b^2 = a^2 + c^2 + 2 \times a \times c \times \cos (180° - B) \tag{20}$$
$$c^2 = a^2 + b^2 + 2 \times a \times b \times \cos (180° - C) \tag{21}$$

19·4 Derivation of the Cosine Law. Given the oblique triangle ABC in Fig. 19·8, where A, b, and c are known and where A is an acute angle, find a formula for a in terms of A, b, and c.

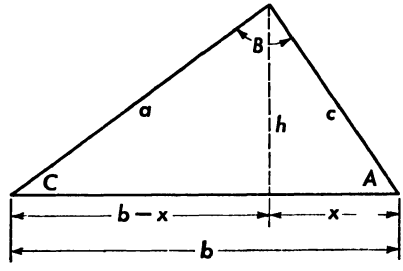


Fig. 19·8

Drop the perpendicular h from B to side b.

$$x = c \times \cos A \tag{22}$$
$$h = c \times \sin A \tag{23}$$
$$a^2 = h^2 + (b - x)^2 \tag{24}$$

Substituting Eqs. (22) and (23) in Eq. (24)

$$a^2 = c^2 \sin^2 A + (b - c \cos A)^2 \tag{25}$$
$$a^2 = c^2 \sin^2 A + b^2 - 2bc \cos A + c^2 \cos^2 A \tag{26}$$
$$a^2 = c^2(\sin^2 A + \cos^2 A) + b^2 - 2bc \cos A \tag{27}$$

From the geometry of the figure

$$\sin^2 A = \frac{h^2}{c^2} \tag{28}$$

$$\cos^2 A = \frac{x^2}{c^2} \tag{29}$$

$$\sin^2 A + \cos^2 A = \frac{h^2}{c^2} + \frac{x^2}{c^2} = \frac{h^2 + x^2}{c^2} = \frac{c^2}{c^2} = 1 \tag{30}$$

Substituting Eq. (30) in Eq. (27),

$$a^2 = c^2 + b^2 - 2 \times b \times c \times \cos A \tag{31}$$

By symmetry, Eqs. (16) and (17) follow:

If the known angle is obtuse, the cosine law can be derived from Fig. 19·9. Here the angle A and the sides b and c are given. Angle A is obtuse.

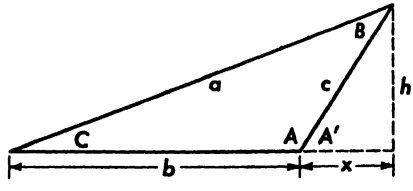


Fig. 19·9

Drop the perpendicular h from B to the side b extended.

$$h = c \times \sin A' \tag{32}$$
$$x = c \times \cos A' \tag{33}$$
$$a^2 = h^2 + (b + x)^2 = c^2 \sin^2 A' + (b + c \cdot \cos A')^2 \tag{34}$$
$$a^2 = c^2 \sin^2 A' + b^2 + 2bc \cos A' + c^2 \cos^2 A' \tag{35}$$
$$a^2 = c^2(\sin^2 A' + \cos^2 A') + b^2 + 2bc \cos A' \tag{36}$$

since

$$\sin^2 A' + \cos^2 A' = 1 \qquad \text{[See Eq. (30) above]}$$

Therefore

$$a^2 = b^2 + c^2 + 2 \times b \times c \times \cos A' \tag{37}$$

From the geometry of Fig. 19·9,

$$A' = 180° - A \tag{38}$$

Therefore

$$a^2 = b^2 + c^2 - 2 \times b \times c \times \cos A \tag{39}$$

Therefore the cosine law as stated in Eqs. (15) to (17) is valid for both acute and obtuse angles.

It is not at all unusual to encounter situations where it is more convenient to use the acute exterior angle A' rather than the obtuse interior angle A (see Fig. 19·9). Such cases frequently occur in alternating-current problems and concurrent-force problems. In these problems the side b, the side c, and the angle A' would naturally appear in the data.

If the angles A', B', and C' are exterior angles corresponding to the interior angles A, B, and C, then the cosine law may be written

$$a^2 = b^2 + c^2 + 2 \times b \times c \times \cos A' \quad (40)$$
$$b^2 = a^2 + c^2 + 2 \times a \times c \times \cos B' \quad (41)$$
$$c^2 = a^2 + b^2 + 2 \times a \times b \times \cos C' \quad (42)$$

As a special case, if $A' = 90°$, Eq. (40) becomes

$$a^2 = b^2 + c^2 + 2 \times b \times c \times \cos 90° \quad (43)$$
$$a^2 = b^2 + c^2 + 2 \times b \times c \times 0 \quad (44)$$
$$a^2 = b^2 + c^2 \quad (45)$$

which is the Pythagorean theorem. The cosine law is sometimes called the *generalized Pythagorean theorem* because, with the $(\pm 2bc \cos A)$ term included, the traditional Pythagorean theorem applies to all triangles.

EXAMPLE 6. Solve the triangle $a = 25$, $b = 56$, $C = 36°52'12''$.

$$c^2 = a^2 + b^2 - 2ab \cos 36°52'12''$$
$$c^2 = 25^2 + 56^2 - 2 \times 25 \times 56 \times 0.8$$
$$c^2 = 1{,}521$$
$$c = \sqrt{1{,}521} = 39$$

One of the unknown angles can now be computed by use of the sine law.

$$\frac{c}{\sin C} = \frac{b}{\sin B} \qquad \frac{39}{\sin 36°52'12''} = \frac{56}{\sin B}$$
$$\sin B = \frac{56 \times \sin 36°52'12''}{39}$$
$$\log 56 = 1.74819$$
$$\log \sin 36°52'12'' = 0.77815 - 1$$
$$\operatorname{colog} 39 = 0.40894 - 2$$
$$\log \sin B = \overline{0.93528} - 1$$

B would equal $59°29'26''$ *if B were acute*. However, from the scale drawing of the problem, B is obviously obtuse. Therefore

$$B = 180° - 59°29'26'' = 120°30'34''$$

EXAMPLE 7. Solve the triangle $a = 25$, $b = 16$, $C = 143°7'48''$.

Here the known angle is obtuse, and there are two ways of thinking when attacking the problem. Equation (42) can be used and

$$c^2 = a^2 + b^2 + 2ab \cos (180° - 143°7'48'')$$
$$c^2 = a^2 + b^2 + 2ab \cos 36°52'12''$$
$$c^2 = 25^2 + 16^2 + 2 \times 25 \times 16 \times 0.8 = 1{,}521$$
$$c = \sqrt{1{,}521} = 39$$

Or the fact stated in Eq. (18) may be used and therefore

$$\cos 143°7'48'' = -\cos (180° - 143°7'48'') = -\cos 36°52'12'' = -0.8$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$c^2 = 25^2 + 16^2 - 2 \times 25 \times 16(-0.8)$$
$$c^2 = 25^2 + 16^2 + 2 \times 25 \times 16 \times 0.8 = 1{,}521$$
$$c = \sqrt{1{,}521} = 39$$

Angle B can be calculated by the sine law, which is here illustrated by the use of natural functions rather than log functions.

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$
$$\frac{\sin B}{16} = \frac{\sin 143°7'48''}{39}$$

Referring to Eq. (5),

$$\sin 143°7'48'' = \sin (180° - 143°7'48'') = \sin 36°52'12'' = 0.6$$
$$\sin B = \frac{16 \times 0.6}{39} = 0.24615$$
$$B = 14°15'$$
$$A = 180° - (14°15' + 143°7'48'') = 22°37'12''$$

The cosine law is also useful when three sides are known.

EXAMPLE 8. Solve for all the angles in the triangle $a = 13$, $b = 21$, $c = 20$. Use Eqs. (15) to (17).

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$13^2 = 21^2 + 20^2 - 2 \times 21 \times 20 \cos A$$
$$169 = 441 + 400 - 840 \cos A$$
$$169 = 841 - 840 \cos A^*$$
$$\cos A = \frac{672}{840} = 0.80000 \qquad \text{ANS.: } A = 36°52'11''$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$21^2 = 13^2 + 20^2 - 2 \times 13 \times 20 \cos B$$
$$441 = 169 + 400 - 520 \cos B$$
$$441 = 569 - 520 \cos B$$
$$\cos B = \frac{128}{520} = 0.24615 \qquad \text{ANS.: } B = 75°45'00''$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$20^2 = 13^2 + 21^2 - 2 \times 13 \times 21 \cos C$$
$$400 = 169 + 411 - 546 \cos C$$
$$400 = 610 - 546 \cos C$$
$$\cos C = \frac{210}{546} = 0.38462 \qquad \text{ANS.: } C = 67°22'47''$$
$$\text{CHECK: } = \overline{179°59'58''}$$

* Do not try to combine 841 and 840 by subtraction, obtaining $169 = \cos A$.

EXAMPLE 9. Find the angle C in the triangle $a = 78$, $b = 35$, $c = 97$.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cos C \\ 97^2 &= 78^2 + 35^2 - 2 \times 78 \times 35 \cos C \\ 9{,}409 &= 6{,}084 + 1{,}225 - 5{,}460 \cos C \\ 9{,}409 &= 7{,}309 - 5{,}460 \cos C \end{aligned}$$

Here C is obviously obtuse, since c^2 actually is larger than $a^2 + b^2$; in other words, c^2 actually is larger than it would be if C were 90°. A scale drawing would indicate the same fact. Also

$$-\cos C = \frac{2{,}100}{5{,}460}$$

Therefore from Eq. (18), page 422, C must be obtuse.

$$\begin{aligned} \cos C &= -\frac{2{,}100}{5{,}460} = -0.38462 \\ C &= 180° - 67°22'47'' = 112°37'13'' \end{aligned}$$

EXERCISE 3

1. Verify the following:
   a. $\cos 42° = 0.74314$ **b.** $\cos 138° = -0.74314$
   c. $\cos 130° = -0.64279$ **d.** $\cos 179° = -0.99985$
2. Given the cosine of the angle, verify the following:
   a. -0.89101 is the cosine of 153°.
   b. 0.89101 is the cosine of 27°.
   c. -0.96126 is the cosine of 164°.
   d. 0.96126 is the cosine of 16°.
3. Show that with proper regard to signs

$$a = b \cos C + c \cos B$$

   whether B is acute or obtuse.
4. Prove that in a triangle with sides a, b, and c

$$a^2 + b^2 + c^2 = 2(ab \cos C + bc \cos A + ca \cos B)$$

5. Show that if

$$\frac{\cos A}{b} = \frac{\cos B}{a}$$

   the triangle is either an isosceles triangle or a right triangle.
6. Prove that

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

7. Prove that

$$\frac{c^2}{b}\cos B + \frac{b^2}{a}\cos A + \frac{a^2}{c}\cos C = \frac{a^4 + b^4 + c^4}{2abc}$$

Also show that the equation is dimensionally correct.

Find the side opposite the given angle, using the cosine law.

8. $a = 4 \quad b = 5 \quad C = 66°25'18''$
9. $b = 7 \quad c = 10 \quad A = 45°34'23''$
10. $a = 20 \quad b = 35 \quad C = 60°$
11. $a = 7 \quad b = 12 \quad C = 123°22'02''$
12. $c = 11 \quad b = 9 \quad A = 31°47'19''$
13. $a = 13 \quad b = 18 \quad C = 36°52'11''$
14. $a = 5 \quad b = 10 \quad C = 126°52'13''$
15. $a = 15 \quad b = 22 \quad C = 130°32'30''$
16. $a = 17 \quad b = 15 \quad C = 154°\ 9'28''$
17. $b = 10 \quad c = 20 \quad A = 36°52.2'$

Find all the angles. Make three separate calculations, one calculation for each angle. Find angles to seconds.

18. $a = 61 \quad b = 87 \quad c = 74$
19. $a = 65 \quad b = 87 \quad c = 44$
20. $a = 78 \quad b = 95 \quad c = 97$
21. $a = 25 \quad b = 28 \quad c = 17$
22. $a = 3 \quad b = 4 \quad c = 6$
23. $a = 4 \quad b = 5 \quad c = 7$
24. $a = 5 \quad b = 7 \quad c = 9$
25. $a = 5 \quad b = 6 \quad c = 7$
26. $a = 6 \quad b = 9 \quad c = 11$
27. $a = 13 \quad b = 16 \quad c = 19$
28.* $a = 5\frac{7}{8}$ in. $\quad b = 3\frac{1}{4}$ in. $\quad c = 8\frac{1}{2}$ in.
29. $a = 5$ in. $\quad b = 8$ in. $\quad c = 12$ in.

19·5 The Tangent Law. The tangent law is well adapted for the use of logarithms and is useful when two sides and the included angle are known. It may be stated as follows:

$$\frac{a-b}{a+b} = \frac{\tan \tfrac{1}{2}(A-B)}{\tan \tfrac{1}{2}(A+B)} \quad \text{or} \quad \frac{b-a}{b+a} = \frac{\tan \tfrac{1}{2}(B-A)}{\tan \tfrac{1}{2}(B+A)} \qquad \star (46)$$

$$\frac{a-c}{a+c} = \frac{\tan \tfrac{1}{2}(A-C)}{\tan \tfrac{1}{2}(A+C)} \quad \text{or} \quad \frac{c-a}{c+a} = \frac{\tan \tfrac{1}{2}(C-A)}{\tan \tfrac{1}{2}(C+A)} \qquad \star (47)$$

$$\frac{b-c}{b+c} = \frac{\tan \tfrac{1}{2}(B-C)}{\tan \tfrac{1}{2}(B+C)} \quad \text{or} \quad \frac{c-b}{c+b} = \frac{\tan \tfrac{1}{2}(C-B)}{\tan \tfrac{1}{2}(C+B)} \qquad \star (48)$$

* See Appendix A, Sec. A·11.

Observe that the numerator of the left-hand side of each of the above equations includes a subtraction of linear dimensions. If these linear dimensions are large and nearly equal, a considerable loss of accuracy may result from the use of these formulas. For example, suppose $b = 14{,}627$ and $a = 15{,}164$, both dimensions being accurate to five figures.

$$a - b = 15{,}164 - 14{,}627 = 537$$

which is accurate to only three figures. This suggests one rather serious disadvantage of these formulas under certain conditions.

19·6 Derivation of the Tangent Law. Given the triangle ABC in Fig. 19·10 let it be required to develop Eq. (46).

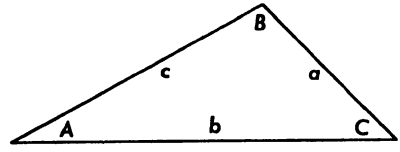


Fig. 19·10

Draw the construction lines shown in Fig. 19·11 as follows:
Draw a semicircle of radius a with the center at z.
Draw the line uw through y.
Draw the line vw perpendicular to b at v.
Draw the chord xy.

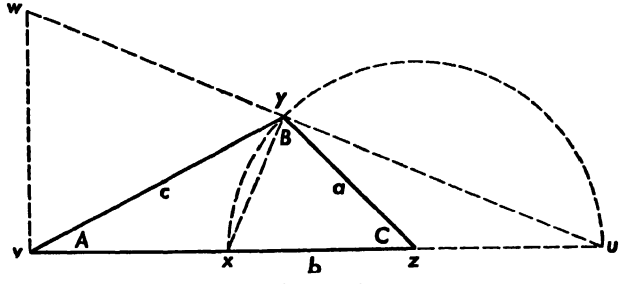


Fig. 19·11

As a result of these constructions,

$$\angle wuv = \tfrac{1}{2}C \qquad \text{(See geometric fact 41, Appendix B)} \qquad (49)$$

$$\angle xyw = 90° \qquad \text{(See geometric fact 42, Appendix B)} \qquad (50)$$

In the right triangle vwu,

$$\angle vwu = 90° - \angle wuv = 90° - \tfrac{1}{2}C \qquad (51)$$

In triangle ABC,

$$\angle C = 180° - (A + B) \qquad (52)$$

$$\tfrac{1}{2}C = 90° - \tfrac{1}{2}(A + B) \qquad (53)$$

Substituting Eq. (53) in Eq. (51),

$$\angle vwu = 90° - [90° - \tfrac{1}{2}(A + B)] \tag{54}$$

$$\angle vwu = 90° - 90° + \tfrac{1}{2}(A + B) \tag{55}$$

$$\angle vwu = \tfrac{1}{2}(A + B) \tag{56}$$

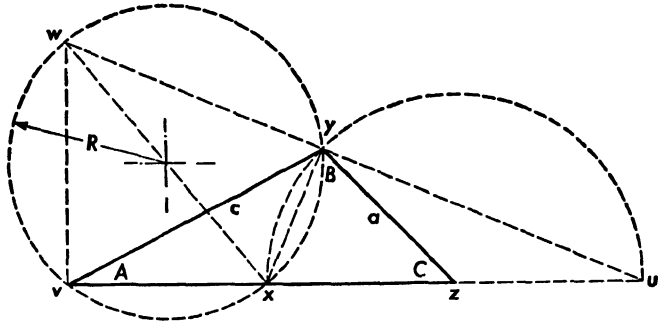


Fig. 19·12

Draw a circle of radius R, as in Fig. 19·12, on the line wx as a diameter. This diameter is the hypotenuse of two right triangles wyx and wvx. Therefore, this circle will pass through the points w, v, x, and y.

$$\angle xwu = \angle A \tag{57}$$

$$\angle vwx = \angle vwu - \angle xwu \tag{58}$$

Substituting Eqs. (56) and (57) in Eq. (58),

$$\angle vwx = \tfrac{1}{2}A + \tfrac{1}{2}B - A = \tfrac{1}{2}(B - A) \tag{59}$$

$$\tan \angle vwu = \tan \tfrac{1}{2}(B + A) = \frac{b + a}{wv} \tag{60}$$

$$\tan \angle vwx = \tan \tfrac{1}{2}(B - A) = \frac{b - a}{wv} \tag{61}$$

Dividing Eq. (61) by Eq. (60),

$$\frac{b - a}{b + a} = \frac{\tan \tfrac{1}{2}(B - A)}{\tan \tfrac{1}{2}(B + A)} \tag{62}$$

Equations (47) and (48) follow by symmetry.

A general statement of this formula is, *The difference between two sides of a triangle is to their sum as the tangent of one-half the difference between their opposite angles is to the tangent of one-half the sum of their opposite angles.*

If A were greater than B, then the formula would have been

$$\frac{a - b}{a + b} = \frac{\tan \tfrac{1}{2}(A - B)}{\tan \tfrac{1}{2}(A + B)} \tag{63}$$

By symmetry,

$$\frac{a-c}{a+c} = \frac{\tan \tfrac{1}{2}(A-C)}{\tan \tfrac{1}{2}(A+C)} \tag{64}$$

$$\frac{b-c}{b+c} = \frac{\tan \tfrac{1}{2}(B-C)}{\tan \tfrac{1}{2}(B+C)} \tag{65}$$

EXAMPLE 10. Find the unknown angles in the triangle $a = 75.000$ in., $b = 136.00$ in., $C = 53°7'47''$.

$$\begin{aligned} b + a &= 136 + 75 = 211 \\ b - a &= 136 - 75 = 61 \\ A + B + C &= 180° \\ A + B &= 180° - 53°7'47'' = 126°52'13'' \\ \tfrac{1}{2}(A + B) &= 63°26'6'' \text{ (to the nearest second)} \end{aligned}$$

Using Eq. (46), altered to the situation where B is greater than A,

$$\frac{b-a}{b+a} = \frac{\tan \tfrac{1}{2}(B-A)}{\tan \tfrac{1}{2}(B+A)}$$

$$\frac{61}{211} = \frac{\tan \tfrac{1}{2}(B-A)}{\tan 63°26'6''}$$

$$\tan \tfrac{1}{2}(B-A) = \frac{61 \tan 63°26'6''}{211}$$

Now,

$$\begin{aligned} \log 61 &= 1.78533 \\ \log \tan 63°26'6'' &= 0.30103 \\ \operatorname{colog} 211 &= \underline{7.67572 - 10} \end{aligned}$$

Therefore

$$\begin{aligned} \log \tan \tfrac{1}{2}(B-A) &= 9.76208 - 10 \\ \tfrac{1}{2}(B-A) &= 30°2'12'' \\ \tfrac{1}{2}B - \tfrac{1}{2}A &= 30°2'12'' \end{aligned}$$

However, we know that

$$\tfrac{1}{2}B + \tfrac{1}{2}A = 63°26'6''$$

Here we have two simultaneous equations which can be solved for B by addition and solved for A by subtraction,

$$\begin{aligned} \tfrac{1}{2}B + \tfrac{1}{2}A &= 63°26'6'' \\ \underline{\tfrac{1}{2}B - \tfrac{1}{2}A} &= \underline{30°2'12''} \\ B &= 93°28'18'' \text{ (by addition)} \\ A &= 33°23'54'' \text{ (by subtraction)} \end{aligned}$$

The sum of the angles A, B, and C should be 180°, as below:

$$\begin{aligned} A &= 33°23'54'' \\ B &= 93°28'18'' \\ C &= \underline{53°59'59''} \\ & 179°59'59'' \end{aligned}$$

This, however, does not constitute a check of the numerical computation.

Let

$$\tfrac{1}{2}A + \tfrac{1}{2}B = x$$

and

$$\tfrac{1}{2}A - \tfrac{1}{2}B = y$$

$$A = x + y \text{ (by addition)}$$
$$B = x - y \text{ (by subtraction)}$$
$$C = 180° - A - B$$
$$C = 180° - x - y - x + y = 180° - 2x$$

Checking now by adding A, B, and C,

$$\begin{array}{r} x + y \\ x - y \\ \underline{180° - 2x} \\ 180° \end{array}$$

Therefore the check will prove for any values of A and B, regardless of whether or not they are consistent with the given triangle. The unknown side may be calculated by the sine law.

EXERCISE 4

Find the unknown side to five figures and the unknown angles to seconds, using the tangent law.

1.	$B = 110°44'$	$c = 18.1419$	$a = 26.2635$
2.	$B = 74°30'$	$c = 113.031$	$a = 145.856$
3.	$A = 148°40'$	$b = 1.35482$	$c = 1.92802$
4.	$A = 7°36'$	$b = 21.1113$	$c = 76.3668$
5.	$A = 9°36'$	$b = 110.007$	$c = 131.919$
6.	$A = 25°26'$	$b = 12.1591$	$c = 76.8406$
7.	$A = 5°24'$	$b = 3.49028$	$c = 3.54202$
8.	$A = 26°23'$	$b = 19.1707$	$c = 37.506$
9.	$A = 9°18'$	$b = 7.63674$	$c = 6.18797$
10.	$A = 27°12'46''$	$b = 378.95$	$c = 472.63$
11.	$A = 11°25'16''$	$b = 114$	$c = 101$
12.	$A = 14°15'$	$b = 93$	$c = 65$
13.	$A = 12°40'50''$	$b = 162.36$	$c = 285.88$
14.	$B = 143°7'48''$	$a = 0.61087$	$c = 0.76794$
15.	$A = 22°37'12''$	$b = 90.757$	$c = 57.596$
16.	$A = 25°3'27''$	$b = 29.671$	$c = 17.453$
17.	$A = 28°4'21''$	$b = 8.9012$	$c = 2.2689$
18.	$B = 126°52'12''$	$a = 26.180$	$c = 12.217$
19.	$A = 33°24'56''$	$b = 109$	$c = 80$
20.	$A = 36°52'12''$	$b = 20$	$c = 11$

19·7 The Half-angle Laws. The half-angle tangent law may be used when three sides of an oblique triangle are known. It is well adapted for the use of logarithms. It may be stated as follows:

$$\tan \tfrac{1}{2}A = \frac{r}{s - a} \qquad \star (66)$$

$$\tan \tfrac{1}{2}B = \frac{r}{s - b} \qquad \star (67)$$

$$\tan \tfrac{1}{2}C = \frac{r}{s - c} \qquad \star (68)$$

where

$$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}$$

$$s = \frac{a + b + c}{2} \qquad \star (69)$$

19·8 Derivation of the Half-angle Laws. Given the triangle shown in Fig. 19·13, develop Eq. (66).

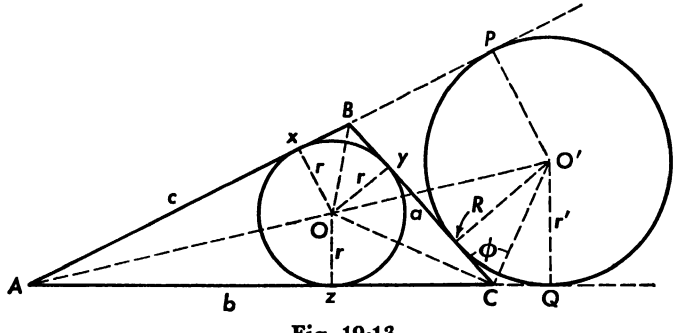


Fig. 19·13

Extend the sides b and c, and draw the circle with center at O' and tangent to side a and the extensions of sides b and c. Draw the inscribed circle with center at O.

$$BP = BR \qquad \text{(See geometric fact 37, Appendix B)} \qquad (70)$$

$$CQ = CR \qquad \text{(See geometric fact 37, Appendix B)} \qquad (71)$$

Adding Eqs. (70) and (71),

$$BP + CQ = BR + CR = a \qquad (72)$$

$$2s = c + BR + CR + b \qquad (73)$$

Substituting Eqs. (70) and (71) in Eq. (73),

$$2s = c + BP + CQ + b \qquad (74)$$

From geometric fact 37, Appendix B,

$$c + BP = b + CQ \tag{75}$$

Therefore

$$s = b + CQ = AQ \tag{76}$$

and

$$CQ = s - b \tag{77}$$
$$2\phi + C = 180° \tag{78}$$
$$\phi + \tfrac{1}{2}C = 90° \tag{79}$$

Therefore OC and CO' are perpendicular and

$$\angle CO'Q = \tfrac{1}{2}C \tag{80}$$

The right triangles ZOC and $CO'Q$ are similar and

$$\frac{r}{ZC} = \frac{CQ}{r'} \tag{81}$$
$$2s = AX + XB + BY + YC + CZ + ZA \tag{82}$$
$$AX = ZA \tag{83}$$
$$XB = BY \tag{84}$$
$$YC = CZ \tag{85}$$
$$2s = 2AX + 2XB + 2ZC \tag{86}$$
$$s = AX + XB + ZC \tag{87}$$
$$c = AX + XB \tag{88}$$

Substituting Eq. (88) in Eq. (87),

$$s = c + ZC \tag{89}$$
$$ZC = s - c \tag{90}$$

Substituting Eqs. (77) and (90) in Eq. (81),

$$\frac{r}{s - c} = \frac{s - b}{r'} \tag{91}$$

Triangles $AO'Q$ and AOZ are similar; therefore

$$\frac{AQ}{r'} = \frac{AZ}{r} \tag{92}$$

Following Eqs. (82) to (85),

$$s = AZ + YC + BY \tag{93}$$
$$a = BY + YC \tag{94}$$

Substituting Eq. (94) in Eq. (93),

$$s = a + AZ \tag{95}$$
$$AZ = s - a \tag{96}$$

Substituting Eqs. (76) and (96) in Eq. (92),

$$\frac{s}{r'} = \frac{s - a}{r} \tag{97}$$

Dividing Eq. (91) by Eq. (97),

$$\frac{s - b}{s} = \frac{r^2}{(s - c)(s - a)} \tag{98}$$

Solving Eq. (98) for r,

$$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}} \tag{99}$$

Following Eqs. (82) to (85),

$$s = AZ + CZ + XB \tag{100}$$

$$b = AZ + CZ \tag{101}$$

Substituting Eq. (101) in Eq. (100),

$$XB = s - b \tag{102}$$

From the geometry of Fig. 19·13,

$$\tan \tfrac{1}{2}A = \frac{r}{AZ} \tag{103}$$

$$\tan \tfrac{1}{2}B = \frac{r}{XB} \tag{104}$$

$$\tan \tfrac{1}{2}C = \frac{r}{ZC} \tag{105}$$

Substituting Eqs. (96), (102), and (90), respectively, in Eqs. (103), (104), and (105),

$$\tan \tfrac{1}{2}A = \frac{r}{s - a} \tag{106}$$

$$\tan \tfrac{1}{2}B = \frac{r}{s - b} \tag{107}$$

$$\tan \tfrac{1}{2}C = \frac{r}{s - c} \tag{108}$$

Somewhat similar equations can be developed for the sine and cosine of the half angles in terms of the radius of the inscribed circle and the sides.

Following Eq. (128), page 366,

$$\sec^2 \tfrac{1}{2}A = \tan^2 \tfrac{1}{2}A + 1 \tag{109}$$

Following Eq. (120), page 366,

$$\cos^2 \tfrac{1}{2}A = \frac{1}{\sec^2 \tfrac{1}{2}A} \tag{110}$$

Substituting Eq. (109) in Eq. (110),

$$\cos^2 \tfrac{1}{2}A = \frac{1}{\tan^2 \tfrac{1}{2}A + 1} \tag{111}$$

Substituting Eq. (106) in Eq. (111),

$$\cos^2 \tfrac{1}{2}A = \frac{1}{\dfrac{r^2}{(s-a)^2} + 1} \tag{112}$$

$$\cos^2 \tfrac{1}{2}A = \frac{(s-a)^2}{(s-a)^2 + r^2} \tag{113}$$

Substituting Eq. (99) in Eq. (113),

$$\cos^2 \tfrac{1}{2}A = \frac{(s-a)^2}{(s-a)^2 + \dfrac{(s-a)(s-b)(s-c)}{s}} \tag{114}$$

$$\cos^2 \tfrac{1}{2}A = \frac{s(s-a)}{s(s-a) + (s-b)(s-c)} \tag{115}$$

$$\cos^2 \tfrac{1}{2}A = \frac{s(s-a)}{s(2s-a-b-c) + bc} \tag{116}$$

Since $2s = a + b + c$, Eq. (116) becomes

$$\cos^2 \tfrac{1}{2}A = \frac{s(s-a)}{bc} \tag{117}$$

$$\cos \tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}} \quad \star \tag{118}$$

By symmetry,

$$\cos \tfrac{1}{2}B = \sqrt{\frac{s(s-b)}{ac}} \quad \star \tag{119}$$

$$\cos \tfrac{1}{2}C = \sqrt{\frac{s(s-c)}{ab}} \quad \star \tag{120}$$

Following Eq. (122), page 366,

$$\tan \tfrac{1}{2}A \times \cos \tfrac{1}{2}A = \sin \tfrac{1}{2}A \tag{121}$$

Substituting Eqs. (106) and (118) in Eq. (121),

$$\sin \tfrac{1}{2}A = \frac{r}{s-a}\sqrt{\frac{s(s-a)}{bc}} \tag{122}$$

Substituting Eq. (99) in Eq. (122),

$$\sin \tfrac{1}{2}A = \sqrt{\frac{s(s-a)(s-b)(s-c)(s-a)}{s(s-a)^2 \times bc}} \tag{123}$$

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}} \quad \star \tag{124}$$

By symmetry,

$$\sin \tfrac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{ac}} \quad \star \tag{125}$$

$$\sin \tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{ab}} \quad \star \tag{126}$$

When the three sides of a triangle are given and we wish to find the angles, the formulas for the sine, for the tangent, or for the cosine of the half angles may be used. As a rule, the tangent formulas are the best to use, since the tangent is more sensitive to changes in the angle than are either the sine or the cosine.

The solution of a triangle using the half-angle tangent law can be rather complicated unless the calculations are carefully tabulated. The tabulation shown below is suggested, and its advantages should be apparent.

EXAMPLE 11. Find the angles in the triangle $a = 75.000$, $b = 86.000$, $c = 97.000$. To find s (*i.e.*, one-half the perimeter), add the three sides and divide by two, as shown below. Here the sum is written at the top of the rows of figures, rather than at the bottom as is the usual practice in addition.

$$\begin{array}{rr} & /129 = s \\ & 2/258 \\ a = & 75 \\ b = & 86 \\ c = & 97 \end{array}$$

The arrangement of the quantities is now convenient for subtraction, *i.e.*, for finding $s - a$, $s - b$, and $s - c$.

$$
\begin{array}{rr}
 & 129 = s \\
 & 2/\overline{258} \\
a = & 75 \uparrow \\
b = & 86 \mid \text{add up} \\
c = & 97 \\
\hline
s - a = & 54 \\
s - b = & 43 \mid \text{add down} \\
s - c = & 32 \downarrow \\
\hline
\text{CHECK:} & 129^{*}
\end{array}
$$

The formula for r requires that $(s - a)$, $(s - b)$, and $(s - c)$ be multiplied and the product be divided by s. Then the square root is obtained. A suggested tabulation of this operation is shown below.

$$
\begin{array}{rl}
 & 1.38021 = \log r \\
 & 2/\overline{2.76042} \\
\operatorname{colog} 129 = & \bar{3}.88941 \\
\log 54 = & 1.73239 \\
\log 43 = & 1.63347 \\
\log 32 = & 1.50515
\end{array}
$$

The next step in the calculation is to subtract the log of $s - a$, $s - b$, and $s - c$ in turn from the log of r. The complete tabulation appears below:

$$
\begin{array}{rrrr}
 & & 1.38021 = \log r & \\
 & & 2/\overline{2.76042} & \\
 & 129 = s \operatorname{colog} = & \bar{3}.88941 & \\
 & 2/\overline{258} & & \\
a = & 75 & & \\
b = & 86 & & \\
c = & 97 & & \\
s - a = & 54 & \log = 1.73239 & \\
s - b = & 43 & \log = 1.63347 & \\
s - c = & 32 & \log = 1.50515 & \\
\text{CHECK:} & 129 & & \\
\log \tan \tfrac{1}{2}A = & & \bar{1}.64782 & \tfrac{1}{2}A = 23°57'46'' \quad A = 47°55'32'' \\
\log \tan \tfrac{1}{2}B = & & \bar{1}.74674 & \tfrac{1}{2}B = 29°10'02'' \quad B = 58°20'04'' \\
\log \tan \tfrac{1}{2}C = & & \bar{1}.87506 & \tfrac{1}{2}C = 36°52'12'' \quad C = 73°44'24'' \\
 & & & \text{CHECK: } 178°119'60''
\end{array}
$$

The check would have been satisfactory had the sum of the angles differed from 180° by one or two seconds.

* Since $(s - a) + (s - b) + (s - c) = 3s - (a + b + c) = s$.

EXERCISE 5

Do not round off linear dimensions to five significant figures until you are ready to look up logarithms. Find all angles to nearest minute.

1. $a = 410.95$, $b = 469.23$, $c = 389.76$

2. $a = 185{,}692$, $b = 148{,}019$, $c = 265{,}597$

3. $a = 0.417744$, $b = 0.313249$, $c = 0.702457$

Find all angles to degrees, minutes, and seconds.

4. $a = 13.492$, $b = 15.786$, $c = 10.387$

5. $a = 431.76$, $b = 609.05$, $c = 710.89$

6. $a = 6.6168$, $b = 9.71610$, $c = 15.6876$

7. $a = 1.3701$, $b = 4.75026$, $c = 5.48676$

8. $a = 3.72726$, $b = 22.1369$, $c = 22.1389$

9. $a = 9.8253$, $b = 12.6837$, $c = 21.9695$

10. $a = 303.167$, $b = 394.942$, $c = 147.732$

11. $a = 2.38751$, $b = 3.86670$, $c = 2.07480$

12. $a = 3.0345$, $b = 3.8889$, $c = 0.95486$

13. $a = 79.1937$, $b = 68.2491$, $c = 20.8955$

14. $a = 4.27287$, $b = 2.28998$, $c = 4.34193$

15. $a = 4.89794$, $b = 9.37680$, $c = 13.3959$

16. $a = 34.1524$, $b = 180.210$, $c = 179.447$

17. $a = 3.15979$, $b = 4.63993$, $c = 1.92302$

Find all angles to degrees, minutes, and tenths of minutes.

18. $a = 1.38952$, $b = 1.94024$, $c = 3.00757$

19. $a = 3.9824$, $b = 5.83898$, $c = 8.51204$

20. $a = 26.9528$, $b = 36.4792$, $c = 20.5972$

19·9 Area of Triangles[1]

Given the Base and Altitude

From geometry [see Eq. (3), Appendix B, Sec. B·3],

$$A_t = \tfrac{1}{2}bh \tag{127}$$

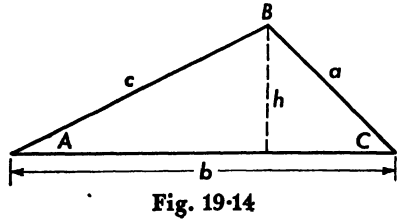


Fig. 19·14

[1] See Fig. 19·14.

where A_t = area of the triangle
b = base of the triangle
h = altitude of the triangle

Given Two Sides and the Included Angle

Let the two known sides be c and b. Let the known angle be A.

$$h = c \times \sin A \tag{128}$$

Substituting Eq. (128) in Eq. (127),

$$A_t = \tfrac{1}{2}b \times c \times \sin A \quad \star \text{(129)}$$

By symmetry,

$$A_t = \tfrac{1}{2}a \times b \times \sin C \quad \star \text{(130)}$$
$$A_t = \tfrac{1}{2}a \times c \times \sin B \quad \star \text{(131)}$$

Given Two Angles and Any Side

$$\frac{\sin A}{\sin B} = \frac{a}{b} \qquad \text{[See Eq. (2), page 415]} \tag{132}$$

$$b = \frac{a \times \sin B}{\sin A} \tag{133}$$

Substituting Eq. (133) in Eq. (130),

$$A_t = \frac{a^2 \times \sin B \times \sin C}{2 \times \sin A} \quad \star \text{(134)}$$

By symmetry,

$$A_t = \frac{b^2 \times \sin A \times \sin C}{2 \times \sin B} \quad \star \text{(135)}$$

$$A_t = \frac{c^2 \times \sin A \times \sin B}{2 \times \sin C} \quad \star \text{(136)}$$

Given Three Sides

The area of the triangle ABC (Fig. 19·15) is the sum of the areas of the triangles AOB, BOC, and COA. Therefore, the total area A_t of the

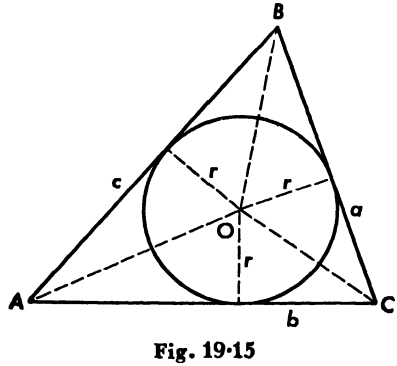


Fig. 19·15

triangle ABC is

$$A_t = \tfrac{1}{2}ra + \tfrac{1}{2}rb + \tfrac{1}{2}rc \quad (137)$$

$$A_t = \tfrac{1}{2}r(a + b + c) \quad (138)$$

Since

$$s = \tfrac{1}{2}(a + b + c)$$

$$A_t = rs \quad (139)$$

Substituting Eq. (99) in Eq. (139),

$$A_t = \sqrt{s(s - a)(s - b)(s - c)} \quad \star (140)$$

EXERCISE 6

Find the areas of the following triangles:

	a	b	c	A	B	C
1.	38		56		49°	
2.	15	12	13			
3.		65		36°	75°	
4.		48	84	73°		
5.	124				115°	26°
6.	34	36	38			
7.			682	56°32′	65°10′	
8.	26.8	9.7				102°18′
9.	364	812	526			
10.	3.142				31°9.7′	62°54.5′
11.	42.7	5.6	38.5			
12.		103	264	18°42′16″		
13.			4,122		154°18′27″	12°12′12″
14.	56.24	43.29	27.57			
15.	3,721		4,609		27°8.7′	

19·10 The Ratio of Sin ϕ to ϕ. At this point we can apply some of the formulas for the areas of triangles to develop an important and basic theorem. This theorem states that the ratio $\frac{\sin \phi}{\phi}$ approaches 1 as ϕ approaches zero. The theorem can be proved as follows:

Let ϕ in Fig. 19·16 be measured in radians. The area of the triangle oac is [from Eq. (3), Appendix B, Sec. B·3]

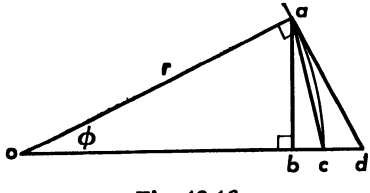


Fig. 19·16

$$A_{oac} = \tfrac{1}{2}(oc)(ab) = \tfrac{1}{2}r \sin \phi \times r = \tfrac{1}{2}r^2 \sin \phi \tag{141}$$

The area of the sector *oac* is $\frac{\phi}{2\pi} \times \pi r^2$; therefore

$$A_{\text{sector } oac} = \tfrac{1}{2}r^2 \phi \tag{142}$$

The area of the triangle *oad* is

$$\begin{aligned} A_{oad} &= \tfrac{1}{2}r(ad) = \tfrac{1}{2}r \times r \times \tan \phi \\ &= \tfrac{1}{2}r^2 \tan \phi \end{aligned} \tag{143}$$

From the geometry of the figure, area of triangle *oad* > area of sector *oac* > area of triangle *oac*.

$$\tfrac{1}{2}r^2 \tan \phi > \tfrac{1}{2}r^2 \phi > \tfrac{1}{2}r^2 \sin \phi \tag{144}$$

Dividing by $\frac{r^2}{2}$,

$$\tan \phi > \phi > \sin \phi \tag{145}$$

but

$$\tan \phi = \frac{\sin \phi}{\cos \phi}$$

$$\frac{\sin \phi}{\cos \phi} > \phi > \sin \phi \tag{146}$$

Dividing by $\sin \phi$,

$$\frac{1}{\cos \phi} > \frac{\phi}{\sin \phi} > 1 \tag{147}$$

As ϕ approaches zero, cosine ϕ approaches 1. Therefore,

$$\frac{1}{\cos \phi} \text{ approaches 1 as } \phi \text{ approaches zero} \tag{148}$$

but $\frac{\phi}{\sin \phi}$ remains between $\frac{1}{\cos \phi}$ and 1. Therefore $\frac{\phi}{\sin \phi}$ approaches 1 and $\frac{\sin \phi}{\phi}$ approaches 1 as ϕ approaches zero.

From Sec. 17·5,

$$\phi \text{ (in radians)} = \frac{\text{arc length } ac}{r} \tag{149}$$

$$\phi \text{ (in degrees)} = \frac{\text{arc length } ac}{r} \times 57.296 \tag{150}$$

By definition,

$$\sin \phi = \frac{\overline{ab}}{r} \tag{151}$$

When ϕ is small the arc length ac and the line ab are nearly equal. Therefore

$$\frac{\overline{ab}}{r} \approx \frac{\overset{\frown}{ac}}{r} \tag{152}$$

or

$$\sin \phi \approx \frac{\overline{ac}}{r} \approx \phi \text{ (in radians)} \tag{153}$$

$$\sin \phi \approx \phi \times 0.01745 \text{ (in degrees)} \tag{154}$$

Chapter 20 TRIGONOMETRIC APPLICATIONS AND REVIEW

In this chapter we shall illustrate some of the applications of numerical trigonometry to civil- and mechanical-engineering problems.

20·1 Plane Surveying. We shall limit any discussion of surveying to *plane surveying*, *i.e.*, to surveys conducted on the assumption that the earth is flat over the area under consideration. The errors involved in this approximation are small, provided the area covered by the survey is small.

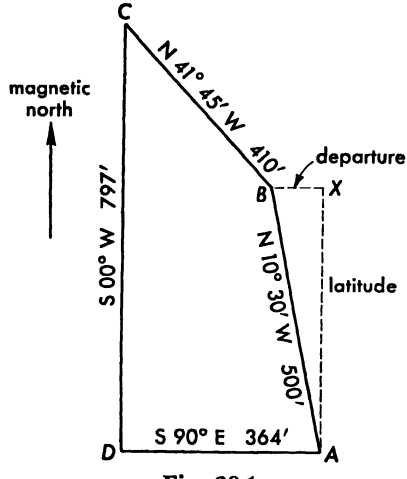


Fig. 20·1

20·2 Measurement of Angles. Angles are measured directly in the field with either one of two instruments, the engineer's transit or the surveyor's compass. For the construction and operation of these instruments the student is referred to any standard surveying text.

20·3 Mapping. A typical problem of the land surveyor is to measure the length and direction of the boundary lines of a parcel of land in order to draw a scale map of the parcel and calculate the area.

Figure 20·1 shows the map of a small parcel of land plotted from the field notes shown in Table 20·1.

Table 20·1

Course		
AB	N 10°30′W	500.00′
BC	N 41°45′W	410.00′
CD	S 00°00′W	797.00′
DA	S 90°E	364.00′

All bearings from magnetic north.

The surveyor started from point A and found by observation that the line AB makes an angle of 10°30′ with the magnetic north. Furthermore, in moving along the line AB, he moves north and also west. Hence the bearing is recorded North 10°30′ West. A property line is not completely described unless both *direction* and *length* are given. This particular property line was measured (probably with a tape) and found to be 500 ft long.

As the notes read, we may assume that the surveyor traversed this survey in a counterclockwise direction. Therefore, he moved west and north along AB and west and north also along BC. Along CD he moved directly south, and along DA he moved directly east, returning to the starting point.

We would be justified in assuming that he measured linear dimensions to the nearest 0.01 ft. For our purpose, we shall assume that bearings were measured to the nearest minute.

20·4 Latitudes and Departures. In Fig. 20·1 the line XB is called the *departure* of the course AB. Since the point B is west of the point X, the line XB is a westerly departure.

Departure is defined as the distance by which the second extremity of a course is east or west of the first extremity.

The departure of the course AB is

$$500 \sin 10°30' = 500 \times 0.18224 = 91.12 \text{ ft}$$

The departure of the course BC is

$$410 \sin 41°45' = 410 \times 0.66588 = 273.01 \text{ ft}$$

The departure of the course CD is zero, since in moving along this line the surveyor moves neither east nor west.

The departure of the course DA is 364 ft, since in moving along this line he moves directly east.

Adding the departures in a westerly direction, we find that the total westerly departure is

$$91.12 + 273.01 = 364.13 \text{ ft}$$

In other words, if we take the survey notes at their face value, they indicate that the surveyor moved 364.13 ft westward and 364.00 ft eastward and arrived back at the starting point. This discrepancy may be explained by the fact that it is impossible to measure with absolute precision. Yet the departures check reasonably well.

If the survey had proceeded in a clockwise direction, then the departure of the course CB would have been an easterly departure,

and the departure of the course BA would have been an easterly departure also. Their numerical values would have been the same.

In Fig. 20·1 the line AX is called the *latitude* of the course AB. Since the point X is north of the point A, the line AX is a northerly latitude.

Latitude is defined as the distance by which the second extremity of a course is north or south of the first extremity.

The latitude of the course AB is

$$500 \text{ ft} \cos 10°30' = 500 \text{ ft} \times 0.98325 = 491.62 \text{ ft}$$

The latitude of the course BC is

$$410 \text{ ft} \cos 41°45' = 410' \times 0.74606 = 305.88 \text{ ft}$$

The latitude of the course CD is 797 ft, since in moving along this line the surveyor moves directly south.

The latitude of the course DA is zero, since in moving along this line the surveyor moves neither north nor south.

Adding the northerly latitudes, we find that the total northerly latitude is

$$491.62 \text{ ft} + 305.88 \text{ ft} = 796.50 \text{ ft}$$

Again, if we take the survey notes at their face value, they indicate that the surveyor moved 796.5 ft north and 797.0 ft south. As before, we assume that this discrepancy occurs because we cannot measure bearings and lengths with absolute precision.

This suggests a method of checking the accuracy of the field work. We know that if the surveyor ended the survey at the starting point, he must have moved south exactly the same distance that he moved north. Also he must have moved east exactly the same distance as he moved west. Stated more formally, *If northerly latitudes and easterly departures are considered positive and southerly latitudes and westerly departures are considered negative, the algebraic sum of the departures must be zero and the algebraic sum of the latitudes must be zero if the traverse is to close.*

The linear error of closure is, by definition,

$$e = \sqrt{(\text{error in latitude})^2 + (\text{error in departure})^2} \qquad \star (1)$$

The calculations for latitudes and departure are repeated below in tabular form. The linear error is also calculated. While the latitudes and departures are calculated here to the nearest 1/100 ft, a greater number of places may be retained in practical problems, depending on circumstances.

Course	*Bearing*	*Distance, ft*	*N lat, ft*	*S lat, ft*	*E dep, ft*	*W dep, ft*
AB	N 10°30′W	500	491.62			91.12
BC	N 41°45′W	410	305.88			273 01
CD	S 00°	797		797.00		
DA	S 90°00′E	364			364 00	
			797.50	797.00	364.00	364 13
			797.00			364.00
		Error in lat	0 50		Error in dep	0.13

$$\text{Linear error} = \sqrt{0.50^2 + 0.13^2} = 0.52$$

There are statistical methods of distributing the error of closure among the several courses. For our purpose such procedures would be of little importance.

20·5 Azimuth Angles. The bearings of courses can be recorded in terms of *azimuth angles*. Azimuth angles are measured from some reference direction to the line in question. Usually zero degrees azimuth is taken to be true south. Azimuth angles are measured in a clockwise direction from true south.

On maps and in field notes this fact should be recorded in some such fashion as "00°00′ azimuth true south." In terms of azimuth angles the notes in Table 20·1 would be as follows:

	Azimuth	
Course	*0° true south*	*Distance, ft*
AB	169°30′	500
BC	138°15′	410
CD	00°00′	797
DA	270°00′	364

20·6 Interior Angles. The sum of the interior angles of any polygon is 180° $(n - 2)$. Therefore, if we subtract 2 from the number of courses and multiply by 180°, we obtain precisely the sum of the interior angles. This total can be compared with a similar total obtained from the field notes.

In Fig. 20·1 the interior angles are checked as follows:

$$\begin{aligned}
\angle DAB &= 90° - 10°30′ = 79°30′ \\
\angle CBA &= 180° - 41°45′ + 10°30′ = 148°45′ \\
\angle DCB &= 41°45′ \\
\angle CDA &= 90° \\
&\overline{358°120′} = 360°
\end{aligned}$$

The figure is a four-sided polygon, the sum of whose interior angles is (4-2) 180 = 360°. Thus the check is perfect.

EXERCISE 1

Plot the following traverses to a scale. Compute the error in latitudes, the error in departures, and the linear error, using five-place tables.

1.

Course	*Length, ft*	*Bearing*
AB	683.47	N 11°49′E
BC	221.63	N 53°40′W
CD	412.90	S 88°36′W
DE	513.27	S 1°00′E
EF	225.41	S 33°17′E
FA	330.79	S 74°30′E

2.

Course	*Length, ft*	*Bearing*
AB	4,181.2	N 10°08′E
BC	3,872.1	N 58°28′W
CD	4,746.9	S 41°15′W
DE	3,751.7	S 16°10′E
EA	4,662.8	N 77°30′E

3. Find the bearing and the length of the line *DA* in Prob. 1.

4. Plot the following traverse to scale. Calculate the error in the latitudes and departures and the linear error.

Course	*Length, ft*	*Bearing*
AB	885.84	N 10°18′W
BC	542.67	S 69°45′W
CD	875.04	S 12°15′E
DE	230.77	S 84°00′E
EF	210.03	N 14°00′*E*
FG	162.70	N 86°20′E
GH	249.95	S 14°00′W
HA	245.00	N 24°00′E

5. Plot the following traverse to scale, and calculate the errors in latitude and departure and the linear error.

Course	*Length, ft*	*Azimuth* 0° *true south*
AB	578.63	265°45′
BC	206.30	174°36′
CD	250.22	106°20′
DE	452.51	226°30′
EF	137.00	180°00′
FG	524.86	91°45′
GA	792.18	8°45′

6. The owner of the tract of land described in Prob. 5 wishes to buy the triangular piece of land *CDE*. What is the length of the line *CE*, and what is its bearing? How much area would this purchase add to his land?
7. The owner of the tract of land described in Prob. 4 wishes to sell his holdings south of the line *FG*. Two new bounds *J* and *K* will have to be established. *J* will be at the intersection of the line *CD* and the westward extension of the line *FG*. *K* will be at the intersection of the line *BA* and the eastward extension of the line *FG*. Find the distance *CJ* and *BK*.

20·7 Calculation of Areas. A useful method of finding the area enclosed by a traverse consisting of straight lines is illustrated below. Ordinarily, it is more convenient to establish reference coordinates so that the entire survey is placed in one quadrant, say the northeast quadrant, as shown in Fig. 20·2. The corner points are numbered

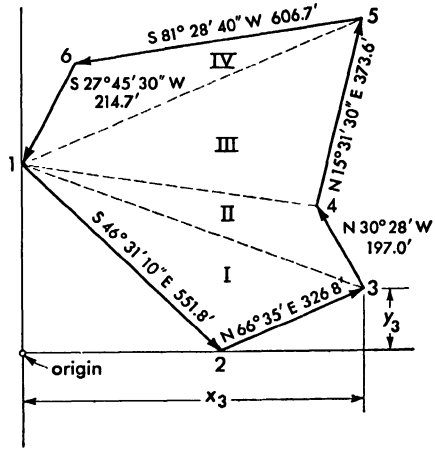


Fig. 20·2

consecutively around the traverse in a counterclockwise direction. The area is given by Eq. (2) below (see Sec. 7.14).

$$A_t = \tfrac{1}{2}[x_1(y_2 - y_6) + x_2(y_3 - y_1) + x_3(y_4 - y_2) + x_4(y_5 - y_3) + x_5(y_6 - y_4) + x_6(y_1 - y_5)] \quad (2)$$

To find the area enclosed by a traverse, multiply the x *distance of each corner by the difference between the* y *distance of the following corner and the* y *distance of the preceding corner. The sum of these products is twice the area.*

The following convention is sometimes used:

$$\text{Area} = \tfrac{1}{2}\left[\frac{x_1}{y_1}\diagdown\frac{x_2}{y_2}\diagdown\frac{x_3}{y_3}\diagdown\frac{x_4}{y_4}\diagdown\frac{x_5}{y_5}\diagdown\frac{x_6}{y_6}\diagdown\frac{x_1}{y_1}\right] \quad (3)$$

Here we multiply the coordinates connected by solid lines and separately multiply the coordinates connected by dotted lines. Then we find the sum of the solid-line products and the sum of the dotted-line products. The difference between these two sums is twice the area.

EXAMPLE 1. Find the area enclosed by the traverse shown in Fig. 20·2.

First we shall find the latitude and departure of each course by the methods already discussed in Sec. 20·2. A tabulation of the results is shown below:

Course	*Bearing*	*Distance, ft*	*N lat, ft*	*S. lat, ft*	*E dep, ft*	*W dep, ft*
1–2	S 46°31′10″E	551.8		379.7	400.4	
2–3	N 66°35′E	326.8	129.8		298.5	
3–4	N 30°28′W	197.0	169.8			99.9
4–5	N 15°31′30″E	373.6	360.0		101.0	
5–6	S 81°28′40″W	606.7		89.9		600.1
6–1	S 27°45′30″W	214.7		190.0		100.0
			659.6	659.6	799.9	800.0

The latitudes check perfectly within the limits of the precision to which the data were obtained. The departures check within 0.1 ft. The coordinates of the several corners are found by reference to Fig. 20·2 and the tabulation above.

$$\begin{aligned} x_1 &= 000.0 & y_1 &= 379.7 \\ x_2 &= 400.4 & y_2 &= 000.0 \\ x_3 &= 698.9 & y_3 &= 129.8 \\ x_4 &= 599.1 & y_4 &= 299.6 \\ x_5 &= 700.1 & y_5 &= 659.6 \\ x_6 &= 100.0 & y_6 &= 569.7 \end{aligned}$$

Substituting the x and y values just found in Eq. (2).

$$\begin{aligned} A_t &= \tfrac{1}{2}[0(0 - 569.7) + 400.4(129.8 - 379.7) + 698.9(299.6 - 0) \\ &\qquad + 599.1(659.6 - 129.8) + 700.1(569.7 - 299.6) \\ &\qquad + 100(379.7 - 659.6)] \\ &= \tfrac{1}{2}(-400.4 \times 249.9 + 698.9 \times 299.6 + 599.1 \times 529.8 \\ &\qquad + 700.1 \times 270.1 - 100 \times 279.9) \\ A_t &= \tfrac{1}{2}(-100{,}060 + 209{,}390 + 317{,}403 + 189{,}097 - 27{,}990) \\ &= \tfrac{1}{2}(587{,}840) = 293{,}900 \text{ sq ft to four significant figures} \end{aligned}$$

Using Eq. (3),

$$A_t = \tfrac{1}{2}\left[\frac{0}{379.7}\times\frac{100}{569.7}\times\frac{700.1}{659.6}\times\frac{599.1}{299.6}\times\frac{698.9}{129.8}\times\frac{400.4}{0}\times\frac{0}{379.7}\right]$$

Carrying out the calculations indicated above,

$0 \times 569.7 =$	00,000	100×379.7	37,970
$659.6 \times 100.0 =$	65,960	$700.1 \times 569.7 =$	398,847
$700.1 \times 299.6 =$	209,750	$599.1 \times 659.6 =$	395,166
$599.1 \times 129.8 =$	77,763	$698.9 \times 299.6 =$	209,390
$698.9 \times 0 =$	00,000	$129.8 \times 400.4 =$	51,972
$400.4 \times 379.7 =$	152,032	$0 \times 0 =$	00,000
	505,505		1,093,345

$$A_t = \frac{1{,}093{,}345 - 505{,}505}{2}$$

$$= \frac{587{,}840}{2} = 293{,}900 \text{ sq ft to four significant figures}$$

EXERCISE 2

1. Find the area enclosed by the traverse shown in Fig. 20·1.
2. Find the area enclosed by the traverse described in Prob. 1 in Exercise 1.
3. Find the area enclosed by the traverse described in Prob. 2 in Exercise 1.

20·8 Measurement of Distances in a Vertical Plane. Practically all the distances and angles used in surveying are measured in either a vertical or a horizontal plane.

The property line *ab* in Fig. 20·3 slopes uphill. However, the distance recorded for the length of this property line is not the actual length *ab* but the horizontal projection of the line *ab*, which is *ax*.

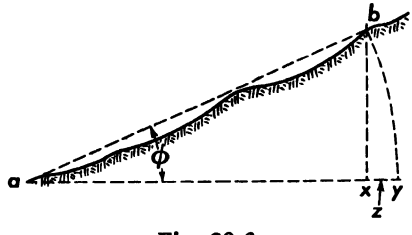


Fig. 20·3

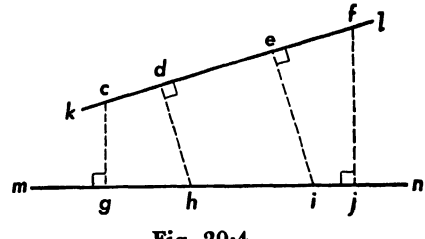


Fig. 20·4

In general, the projection of one line on another is defined in Fig. 20·4. Here *gj* is the projection of the line *cf* on the line *mn*, and *de* is the projection of the line *hi* on the line *kl*. Notice the positions of the right angles.

Ordinarily in the field it is not difficult to measure the distance *ab* in Fig. 20·3; neither is it difficult to measure the difference in elevation

between a and b. As a rule, it is a relatively simple matter to measure the vertical angle ϕ. It may, however, be somewhat more difficult to arrive at the distance ax by direct measurement in the field.

The distance ax in Fig. 20·3 can be calculated from field notes, provided that the length ab and the difference in elevation between a and b are known.

Following the Pythagorean theorem [see Eq. (2), Appendix B, Sec. B·3],

$$ax = \sqrt{(ab)^2 - (bx)^2} \tag{4}$$

If the angle ϕ is small, the approximation developed below may be perfectly satisfactory and will effect a considerable saving of time, particularly when there are many data to be reduced. Referring to Fig. 20·3,

$$xy = ab - ax \tag{5}$$

$$(bx)^2 = (ab)^2 - (ax)^2 = (ab - ax)(ab + ax)^*$$

$$ab - ax = \frac{(bx)^2}{ab + ax} \tag{6}$$

Substituting Eq. (6) in Eq. (5),

$$xy = \frac{(bx)^2}{ab + ax}$$

or if ab and ax are nearly equal, as they will be if ϕ is small,

$$xy \approx \frac{(bx)^2}{2(ab)} \tag{7}$$

The distance xy is the correction which when subtracted from ab gives the horizontal projection ax. Therefore,

$$ax \approx ab - \frac{(bx)^2}{2(ab)} \qquad \text{(See Table A·2, Appendix A)} \tag{8}$$

If the field notes give the angle ϕ instead of the difference in elevation between a and b and if ϕ is small, the correction bx can be found by the approximation developed below.

For small angles, the sine of the angle is very nearly proportional to the angle [see Eq. (154), page 443]. That is,

$$\sin \phi \approx k\phi \tag{9}$$

where k is the proportionality constant.

* See Eq. (2), Sec. 5·17.

Up to almost 8° in the five-place tables, each increase of 1 min increases the sine of the angle 0.00029. That is, within the limits of the five-place table and up to 7°48′,

$$\sin \phi \approx 0.00029\phi \tag{10}$$

where ϕ is measured in minutes. Or

$$\sin \phi \approx 0.00029 \times 60 \times \phi = 0.0174\phi \tag{11}$$

where ϕ is measured in degrees.

In Fig. 20·3,

$$bx = (ab) \sin \phi \tag{12}$$

or, substituting Eq. (11) in Eq. (12),

$$bx \approx 0.0174 \times ab \times \phi \tag{13}$$

where ϕ is measured in degrees.

Following Eq. (7),

$$xy \approx \frac{(bx)^2}{2(ab)} \approx \frac{0.0174^2 \times (ab)^2 \times \phi^2}{2(ab)} \tag{14}$$

$$xy \approx \frac{0.00030275 \times (ab) \times \phi^2}{2} \tag{15}$$

Within about 1 per cent, Eq. (16) is valid.

$$xy \approx 3/2 \times \phi^2 \times 10^{-1}(ab) \tag{16}$$

Another method may be used if a table of versines is available. By definition,

$$\text{vers } \phi = 1 - \cos \phi \tag{17}$$

In Fig. 20·3

$$xy = ab - (ab) \cos \phi = (ab)(1 - \cos \phi) \tag{18}$$

$$xy = (ab) \text{ vers } \phi \tag{19}$$

EXAMPLE 2. In Fig. 20·3, if $ab = 375$ and $bx = 30$, calculate ax, using both formulas

$$ax = \sqrt{(ab)^2 - (bx)^2}$$

and

$$ax \approx ab - \frac{(bx)^2}{2ab}$$

Using the Pythagorean theorem,

$$ax = \sqrt{375^2 - 30^2} = 373.80 \text{ (to five figures)}$$

Using the approximate formula above,

$$ax \approx 375 - \frac{30^2}{2 \times 375} = 375 - 1.20 = 373.80 \text{ (to five figures)}$$

Note that the accuracy required of this correction factor 1.20 is not beyond the capabilities of the slide rule.

20·9 Surveying around Obstructions. Equation (11) is convenient in another connection (see Fig. 20·5). If ϕ is small and is measured in degrees,

$$a \approx 0.0174 \times c \times \phi \tag{20}$$

to a close approximation.

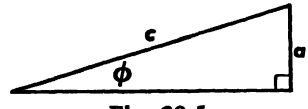


Fig. 20·5

A problem sometimes arises where it is desired to locate a point x (Fig. 20·6) on an extension of the line ab. The exact distance ax is not important, but the point x must be located so that the points a, b, and x are on the same straight line. Between b and x there are obstructions such as houses, trees, and boulders, so that a direct line of sight cannot be used. The distances bc, cd, de, and ex can be measured. Also the angles u, v, w, and y can be measured. The surveyor runs a traverse bc, cd, and de, and at the point e he sets the angle y at some convenient value. He wishes to find the distance ex such that x will fall on the continuation of the line ab. The offsets fc, gd, he, and ix can be calculated to a close approximation, as follows:

$$fc \approx 0.0174(bc)u \tag{21}$$
$$gd \approx 0.0174(cd)v \tag{22}$$
$$he \approx 0.0174(de)w \tag{23}$$
$$ix \approx 0.0174(ex)y \tag{24}$$

If the offsets downward are considered to be negative and the offsets upward are considered to be positive, then from the geometry of the figure

$$fc + he = gd + ix \tag{25}$$

Substituting Eqs. (21), (22), (23), and (24) in Eq. (25) and dividing out the 0.0174,

$$(bc)u + (de)w = (cd)v + (ex)y \tag{26}$$

In Eq. (26) the angles u, w, v, and y may be in any units, but we shall use minutes. To find ex,

$$ex = \frac{(bc)u + (de)w - (cd)v}{y} \tag{27}$$

EXAMPLE 3. Find ex in Fig. 20·6 if $bc = 120$ ft, $cd = 375$ ft, $de = 500$ ft, $\angle u = 50'$, $\angle v = 200'$, $\angle w = 240'$, and $\angle y = 80'$.

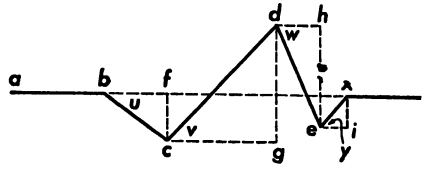


Fig. 20·6

Substituting in Eq. (27),

$$ex = \frac{50 \times 120 + 240 \times 500 - 200 \times 375}{80}$$

$$= 637.5 \text{ ft}$$

EXERCISE 3

1. Plot a graph of $\sin \phi$ vertically and ϕ in minutes horizontally for angles between $0'$ and $500'$.
   - **a.** What is the shape of this graph?
   - **b.** What is the slope?
   - **c.** Over this range about how much does $\sin \phi$ change for a change of each $60'$, in ϕ?
2. In Fig. 20·3, if $ab = 300$ ft and the difference in elevation between points a and b is 15 ft, find the distance ax, using Eq. (8). Find the percentage of error that results from using this approximation.
3. In Fig. 20·3 if $ab = 650$ ft and $\phi = 5°$, find ax, using Eq. (16).
4. Solve Prob. 3, using a table of versines (if possible) and Eq. (19).
5. In Fig. 20·6 find the distance ex if $bc = 100$ ft, $cd = 200$ ft, $de = 150$ ft, $\angle u = 1°$, $\angle v = 1°30'$, $\angle w = 2°$, and $\angle y = 45'$.

EXERCISE 4

The problems in this exercise have their application in measuring inaccessible distances, highway curves, etc. It is beyond the scope of this book to dwell in detail on the engineering principles involved, but the student should have no particular difficulty in sensing the general nature of the applications.

1. Referring to Fig. 20·7, prove that

$$h = \frac{b}{\cot A - \cot B}$$

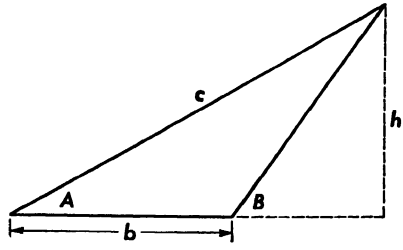


Fig. 20·7

2. If b in Prob. 1 is 362′, $A = 26°48'$, and $B = 48°32'$, find h.

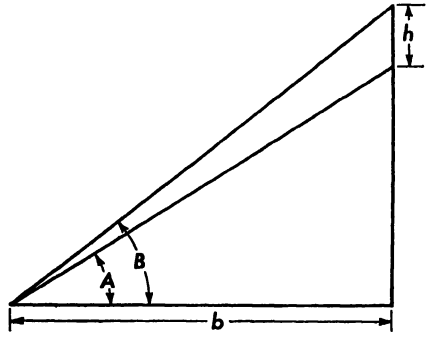


Fig. 20·8

3. In Fig. 20·8 show that

$$h = b(\tan B - \tan A)$$

4. If in Prob. 3 the side $b = 986'$, angle $A = 15°46'$, and angle $B = 23°18'$, find h.

5. In Fig. 20·9 find an expression for h in terms of b, A, B, C, and D.

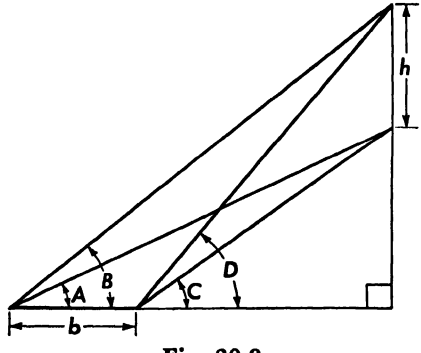


Fig. 20·9

6. Find ϕ and x in Fig. 20·10.

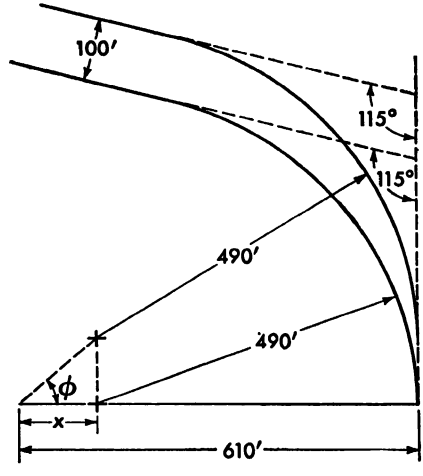


Fig. 20·10

7. In Fig. 20·11 find an expression for the length of the line *ab* in terms of the line *ac* and the angle ϕ.

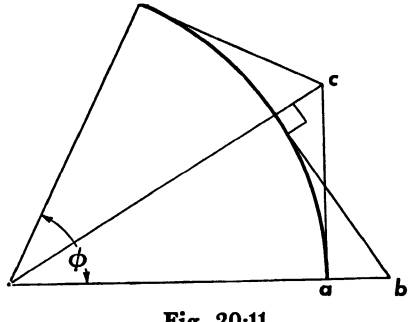


Fig. 20·11

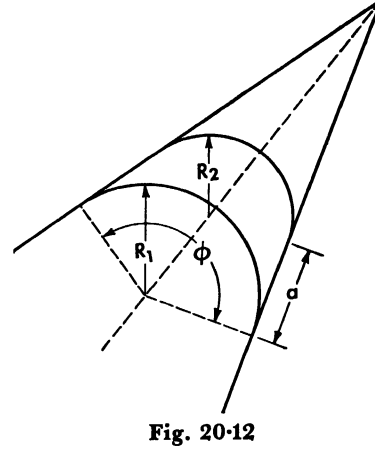


Fig. 20·12

8. Referring to Fig. 20·12, show that $R_1 - R_2 = a \cot \frac{\phi}{2}$.

The following problems may be done more conveniently by the use of the oblique-triangle formulas.

9. The inaccessible distance AB across the pond is to be found from the survey data shown in Fig. 20·13. The given distances were measured from point O by means of stadia.

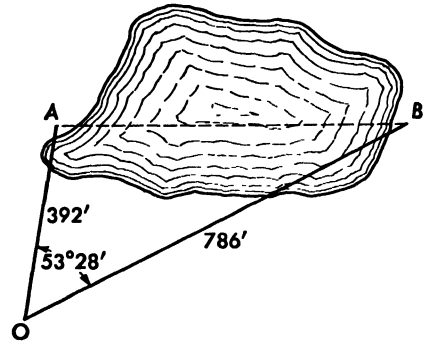


Fig. 20·13

10. The points A and B in Fig. 20·14 are on opposite sides of the river and inaccessible from points x and y. Find the distance AB from the survey notes below:

$$\begin{aligned} xy &= 450 \text{ ft} \\ \angle Ayx &= 32° \\ \angle Axy &= 129° \\ \angle Bxy &= 43° \\ \angle Byx &= 113° \end{aligned}$$

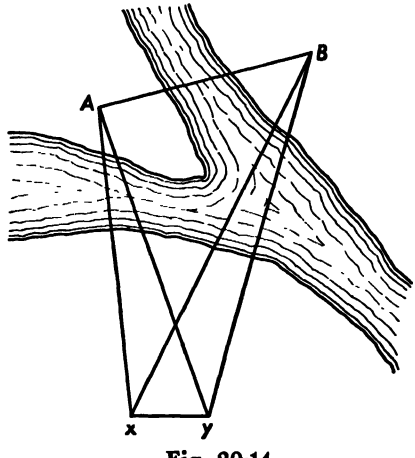


Fig. 20·14

11. An aircraft at d (Fig. 20·15) is located from two ground stations a and b by the following data:

$$\begin{aligned} \angle dac &= 25°37' \\ \angle dbc &= 30°40' \\ \angle cab &= 40°15' \\ ab &= 5{,}676 \text{ yd} \end{aligned}$$

Find *da*, *ac*, and *bc*.

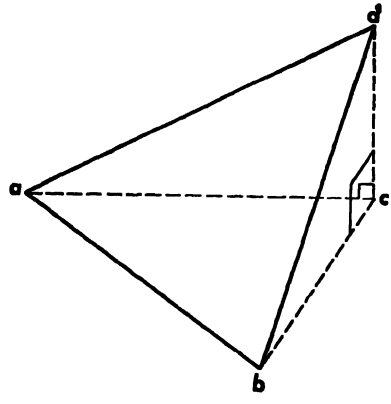


Fig. 20·15

20·10 Applications to Machine- and Tool-design Problems. The problems in the following sections will be of particular interest to students of machine and tool design. Unfortunately, no definite rules can be given for solving this type of problem. The student will have to draw on his knowledge of basic mathematics and his own native ingenuity, building up experience as he goes along. However, the few general suggestions given below may be of some help.

In practically all the problems in Exercise 5 the student will have to draw certain construction lines. In general, it is suggested that he consider drawing these lines perpendicular or parallel to known dimensions. If there are arcs of circles involved, it may be helpful to connect the centers. Often it is helpful to draw construction lines from the center of one arc and tangent to another. Figure 20·16 is an idealized drawing

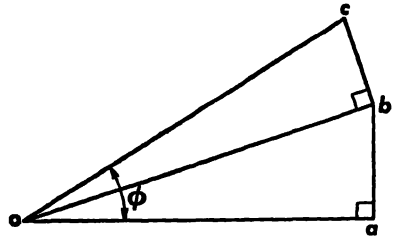


Fig. 20·16

of a situation frequently encountered, in which we are given *oa*, *ab*, and *bc*. It is required to find ϕ. The procedure in solving this problem is first to find the angle *boa*, using the side *ab* and the side *oa*. Second, the side *ob* is calculated and used in connection with the side *bc* to find the angle *boc*. The angle ϕ will then be the sum of *aob* and *boc*.

Illustrated below is a somewhat more direct method, in which the hypotenuse *ob* is not actually calculated.

$$\tan boa = \frac{ba}{oa}$$

$$ob = \frac{oa}{\cos boa}$$

$$\tan cob = \frac{cb}{oa/\cos boa} = \frac{cb \times \cos boa}{oa}$$

$$\angle boa + \angle cob = \angle coa = \phi$$

This method is particularly useful when *cb* and *oa* have common factors.

In Fig. 20·17 it is desired to calculate the angle by use of the sine. Decimal equivalents could be used, but common fractions make the problem much easier.

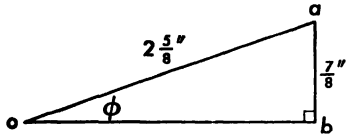


Fig. 20·17

$$\sin \phi = \frac{\frac{7}{8}}{2\frac{5}{8}} = \frac{7}{8} \times \frac{8}{21} = \frac{1}{3} = 0.33333$$

If the side *ob* is to be calculated by means of the Pythagorean theorem, again decimal equivalents could be used, but it is much better to reason as follows: Use $\frac{1}{8}$ in. as the basic unit in the problem (see also example 32, Appendix A, Sec. A·11). Hence the hypotenuse will be 21 units long, and the side *ab* will be 7 units long; therefore,

$$ob = \tfrac{1}{8}\sqrt{21^2 - 7^2} = \tfrac{1}{8}\sqrt{441 - 49} = \frac{19.799}{8} = 2.475 \text{ in.}$$

Figure 20·18 illustrates a type of problem which is very frequently done incorrectly. A solution is often attempted assuming that the figure

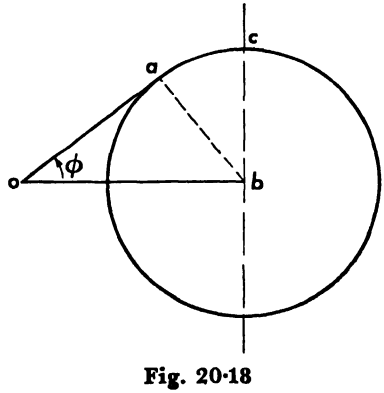


Fig. 20·18

obc is a right triangle. As a matter of fact, it is not even any kind of a triangle. A triangle is a figure formed by three intersecting straight lines. The line *oc* is not a straight line. It is a combination of straight lines *oa* and the arc of a circle *ac*. Sin ϕ equals *ab*/*ob* and not *bc*/*oc*.

As far as tolerances and limits are concerned, no attempt is made to follow commercial shop practices in Exercise 5. Nominal dimensions are given on the drawings, and it will be assumed that their accuracy is consistent with the use of five-place tables. Angles will be computed either to tenths of minutes or to seconds at the discretion of the instructor. A deviation from the answer of three or four seconds is not usually significant.

EXERCISE 5

1. Find the angle ϕ in Fig. 20·19.

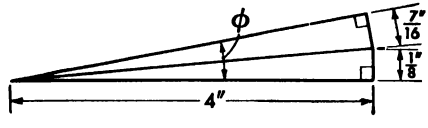


Fig. 20·19

2. Find the angle ϕ in Fig. 20·20.

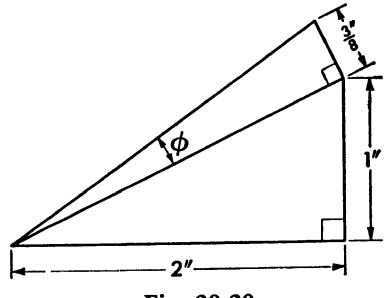


Fig. 20·20

3. Find the diameter *d* in Fig. 20·21.

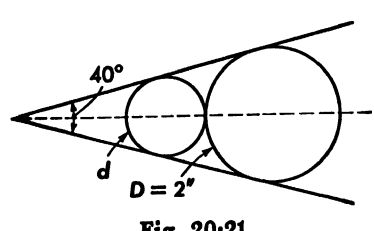


Fig. 20·21

4. Find ϕ in Fig. 20·22.
5. Find ϕ in Fig. 20·23.

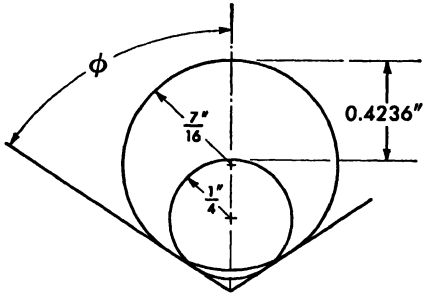


Fig. 20·22

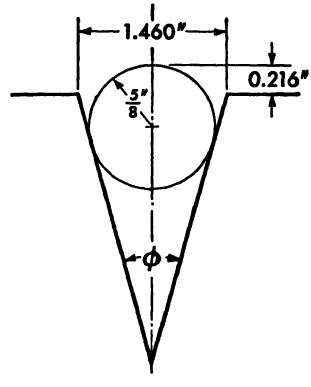


Fig. 20·23

6. Find ϕ in Fig. 20·24.

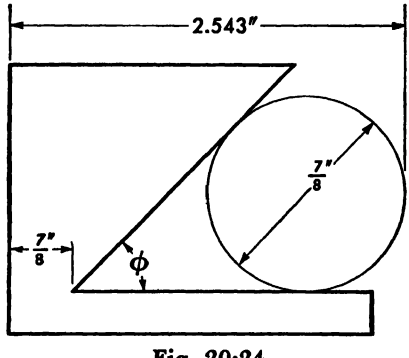


Fig. 20·24

7. Find ϕ in Fig. 20·25.

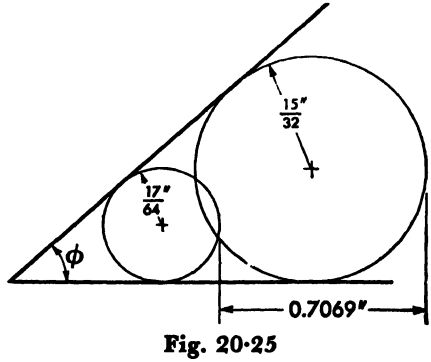


Fig. 20·25

8. Find ϕ in Fig. 20·26.

9. Find the quantity D-d in Fig. 20·27.

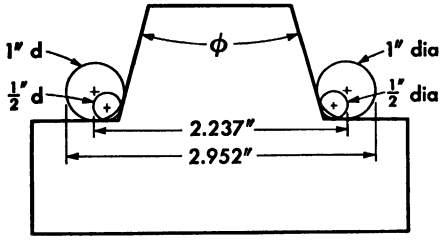


Fig. 20·26

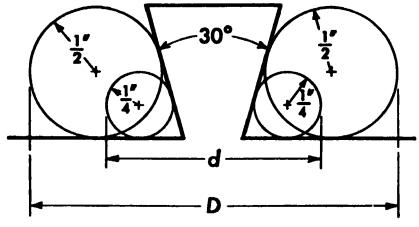


Fig. 20·27

10. Find ϕ in Fig. 20·28.

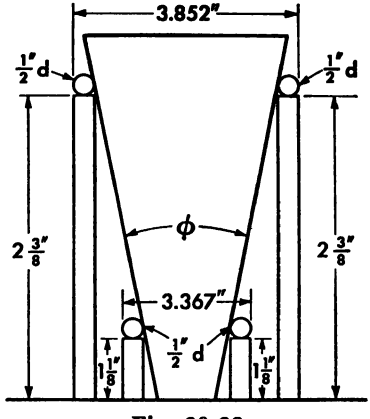


Fig. 20·28

11. Find ϕ in Fig. 20·29.

12. Find R in Fig. 20·30.

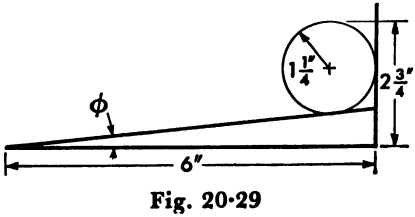


Fig. 20·29

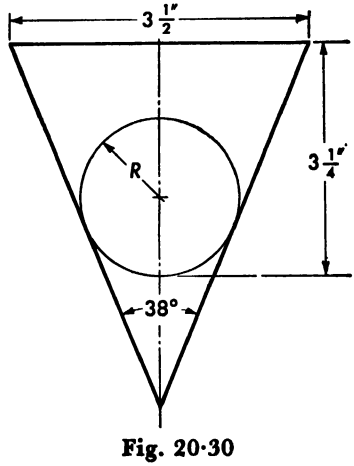


Fig. 20·30

13. Find R in Fig. 20·31.

14. Find R in Fig. 20·32.

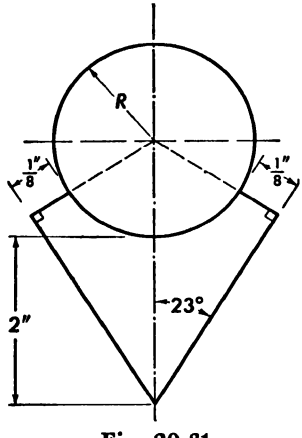


Fig. 20·31

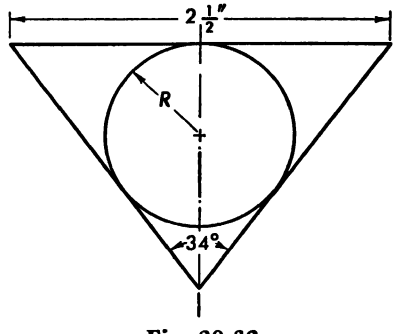


Fig. 20·32

15. Find ϕ in Fig. 20·33.

16. Find ϕ in Fig. 20·34.

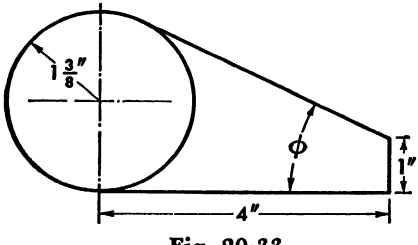


Fig. 20·33

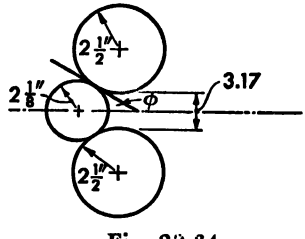


Fig. 20·34

17. Find x in Fig. 20·35.

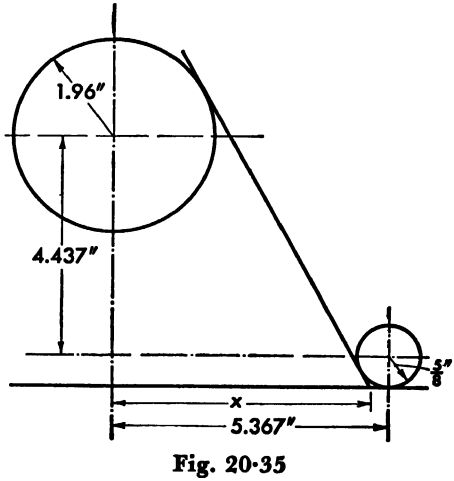


Fig. 20·35

18. Find ϕ in Fig. 20·36.

19. Find ϕ in Fig. 20·37.

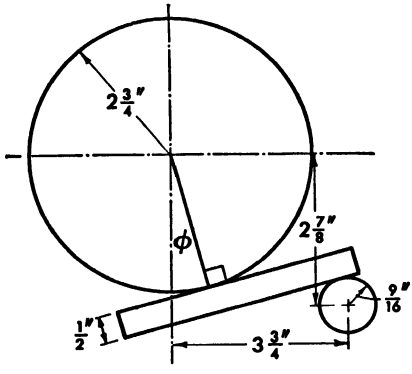


Fig. 20·36

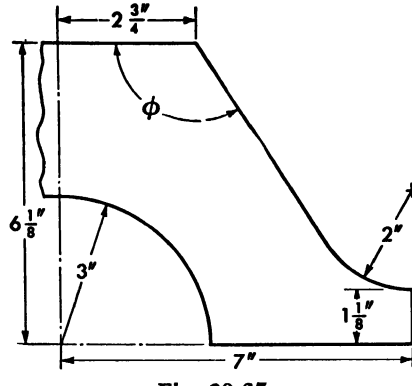


Fig. 20·37

20. Find the angle ϕ and the lengths x and y in Fig. 20·38.

21. Find x in Fig. 20·39. The five holes in the $2\frac{9}{16}$-in. circle are evenly spaced.

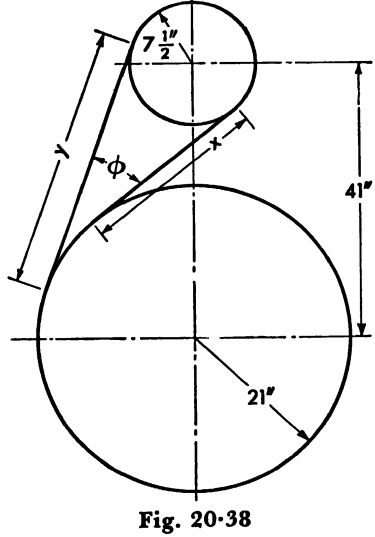


Fig. 20·38

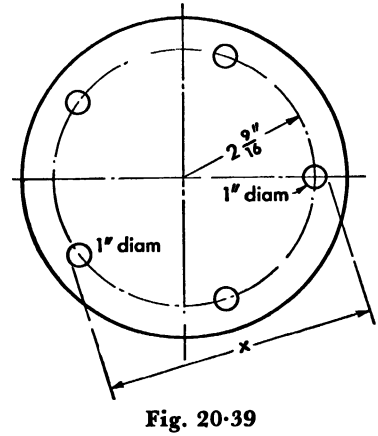


Fig. 20·39

22. Find R in Fig. 20·40.

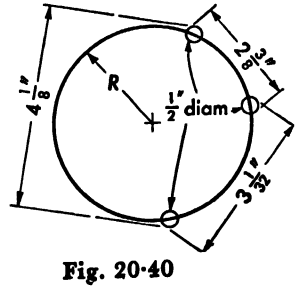


Fig. 20·40

23. Find R in Fig. 20·41 if $oa = 2.758$ in., $ob = 3.265$ in., $oc = 2.932$ in. and $\angle aob = \angle boc$.

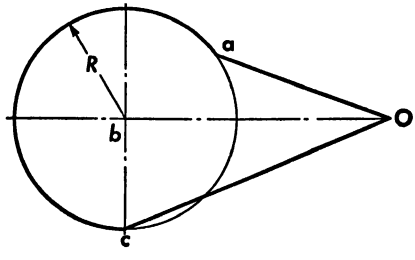


Fig. 20·41

24. Find x and y in Fig. 20·42.

25. Find x and y in Fig. 20·43.

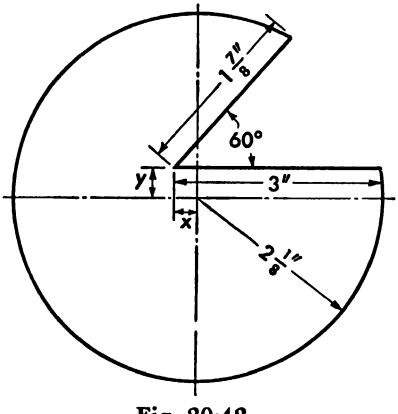


Fig. 20·42

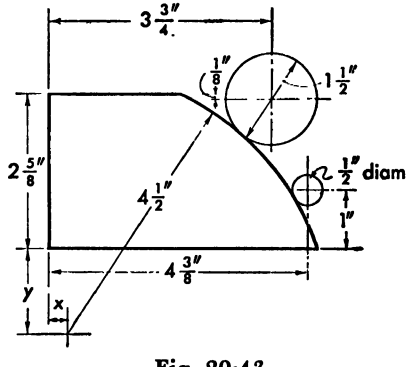


Fig. 20·43

26. Find x in Fig. 20·44.

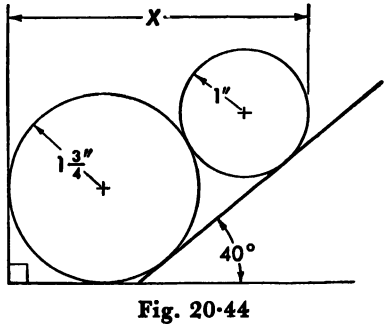


Fig. 20·44

20·11 Solid Figures. Many of the problems in this chapter have involved more than one triangle, but these triangles have usually been in the same plane.

There is a wide range of practical applications in which related triangles lie in different planes. The authors feel that a thorough treatment of this type of problem would consume an unwarranted amount of textbook space and classroom time. Therefore, the few problems which are given in this section are intended only to introduce the student to this highly specialized and restricted field of applications. The student who has reason to be interested in this particular type of problem is referred to Wolfe and Phelp's "Practical Shop Mathematics," Vol. 2 (3d ed., McGraw-Hill Book Company Inc., 1949).

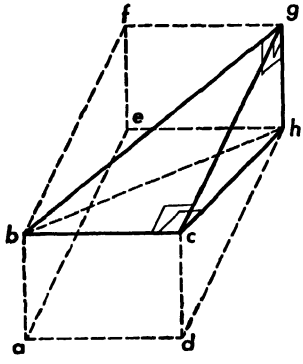


Fig. 20·45

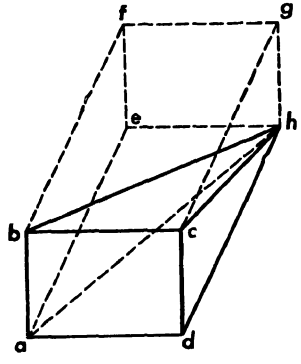


Fig. 20·46

Figures 20·45 and 20·46 illustrate two of several figures which for this purpose might be called basic. A typical problem in this type of work requires that one of the angles be calculated when certain other angles are given.

EXAMPLE 4. In Fig. 20·45 $\angle gch = 20°$ and $\angle bgc = 30°$ (note the positions of the 90° angles); find the angle *gbh*. The data as given determine the *shape* of the figure, but since no linear dimensions are given, the *size* of the figure is not determined. However, this is beside the point, since the required answer is an angle and not a side.

The general procedure in the solution is as follows:

1. Locate the known triangles. In this problem they are the triangles *gch* and *bgc*.

2. Locate the side common to both these known triangles. In this problem this is the side *cg*. Let the length at this side be 1 unit.

3. Locate the unknown triangle. In this problem it is the triangle *bgh*.

4. Locate the common sides between the unknown triangle and each of the known triangles.

5. Solve for the common sides found in step 4 in terms of functions of the known angles.

6. Solve for a function of the unknown angle in terms of the sides found in step 5.

7. Find the angle whose function was found in step 6. The arithmetical work involved in steps 5, 6, and 7 follows:

8. In triangle *gch*,

$$gh = 1 \times \tan gch = \tan gch = \tan 20° = 0.36397$$

9. In triangle *bgc*,

$$bg = 1 \times \sec bgc = \sec bgc = \sec 30° = 1.1547$$

10. In triangle *gbh*,

$$\tan gbh = \frac{gh}{bg} = \frac{\tan 20°}{\sec 30°} = \frac{0.36397}{1.1547} = 0.31521$$
$$gbh = 17°29'43''$$

Since in general it is easier to multiply than it is to divide, the last equation probably should be altered to its equivalent:

$$\tan gbh = \tan 20° \times \cos 30° = 0.36397 \times 0.86603 = 0.31521$$
$$\angle gbh = 17°29'43''$$

EXERCISE 6

1. Draw Figs. 20·45 and 20·46 in orthographic projection. Draw enough views including auxiliaries so that the triangles *bcg*, *bch*, *cgh*, and *bgh* may be seen in their true dimensions. Let $ad = 1$, $ab = 1\frac{1}{2}$, and $dh = 2$ in.
2. Referring to Fig. 20·45, write equations, involving multiplication only, which can be used as follows:
   - **a.** Solve for $\angle gbh$ in terms of $\angle bgc$ and $\angle gch$.
   - **b.** Solve for $\angle gbh$ in terms of $\angle bgc$ and $\angle hbc$.
   - **c.** Solve for $\angle gbh$ in terms of $\angle gch$ and $\angle hbc$.
   - **d.** Solve for $\angle bgc$ in terms of $\angle gch$ and $\angle hbc$.
   - **e.** Solve for $\angle bgc$ in terms of $\angle gch$ and $\angle gbh$.
   - **f.** Solve for $\angle bgc$ in terms of $\angle gbh$ and $\angle hbc$.
   - **g.** Solve for $\angle gch$ in terms of $\angle gbh$ and $\angle bgc$.
   - **h.** Solve for $\angle gch$ in terms of $\angle gbh$ and $\angle hbc$.
   - **i.** Solve for $\angle gch$ in terms of $\angle bgc$ and $\angle hbc$.
   - **j.** Solve for $\angle hbc$ in terms of $\angle gbh$ and $\angle bgc$.
   - **k.** Solve for $\angle hbc$ in terms of $\angle gbh$ and $\angle ghc$.
   - **l.** Solve for $\angle hbc$ in terms of $\angle bgc$ and $\angle ghc$.

3. Prove that in Fig. 20·46 angle *bha* is less than angle *chd*.
4. Prove that in Fig. 20·46 angle *bhc* is less than angle *ahd*.
5. If in Fig. 20·46 $ad = 2$ in., $dh = 4$ in., and $ab = 3$ in. find $\angle bha$.
6. Referring to Fig 20·46, write equations which can be used as follows:
   - **a.** Solve for $\angle bhc$ in terms of $\angle chd$ and $\angle ahd$.
   - **b.** Solve for $\angle bha$ in terms of $\angle chd$ and $\angle ahd$.
   - **c.** Solve for $\angle chd$ in terms of $\angle bha$ and $\angle bhc$.
   - **d.** Solve for $\angle ahd$ in terms of $\angle bhc$ and $\angle bha$.
7. Find the angle ϕ in Fig. 20·47.
8. Find the angles ϕ and θ in Fig. 20·48.

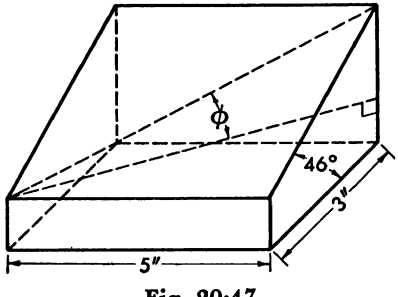


Fig. 20·47

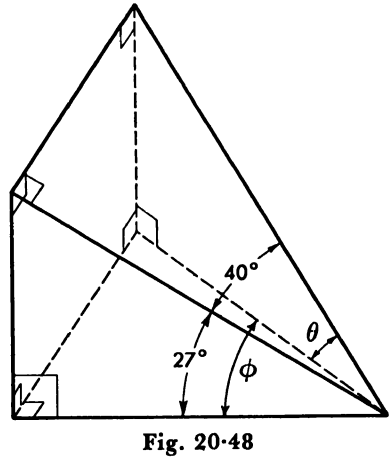


Fig. 20·48

9. In Fig. 20·49 find the angle which the center line *ab* makes with the plane *cdef*.
10. Find ϕ in Fig. 20·50.

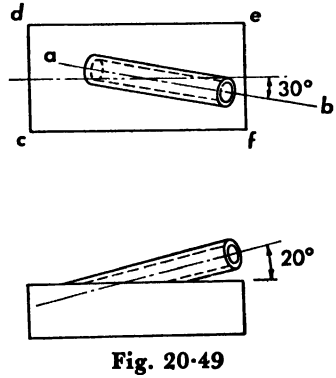


Fig. 20·49

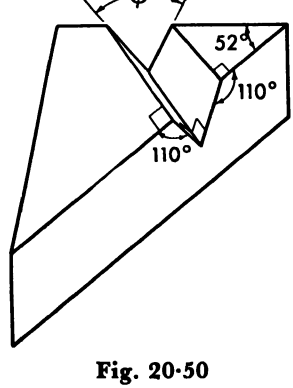


Fig. 20·50

11. Find the angle ϕ in Fig. 20·51.

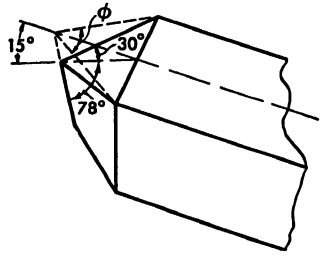


Fig. 20·51

12. In Fig. 20·52 $gb = 3$, $ah = 1$, $ci = 1.3$, $gi = 4$, $gh = 3$, and $hi = 2$. Find $\angle bfg$.

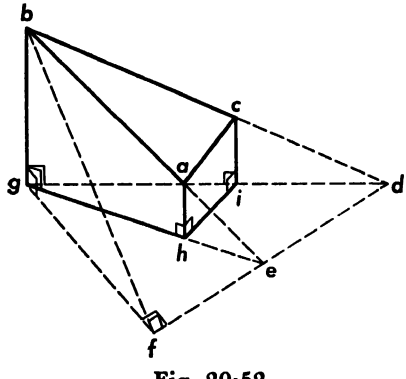


Fig. 20·52

UNIT SEVEN

ANALYTICAL TRIGONOMETRY

This unit will be of particular interest to electrical and electronics students. It presupposes a study of Units Three to Six.

Chapter 21 VECTORS

In engineering work we frequently have occasion to solve problems involving vector quantities.

There are several recognized vector operations which are so well defined and so frequently used that they have been given names. These names are derived, by analogy, from the names of the fundamental algebraic operations, addition, subtraction, multiplication, and division. The analogy is by no means exact, however, since the nature of a vector is quite different from the nature of an algebraic number. Therefore *vector* addition, subtraction, multiplication, and division are quite different processes from *algebraic* addition, subtraction, multiplication, and division.

In this work we shall develop the consideration of vector operations along rather restricted lines. We shall name and define operations for which we have an immediate use, but we shall have to omit consideration of those vector processes appropriate to the more advanced fields of engineering.

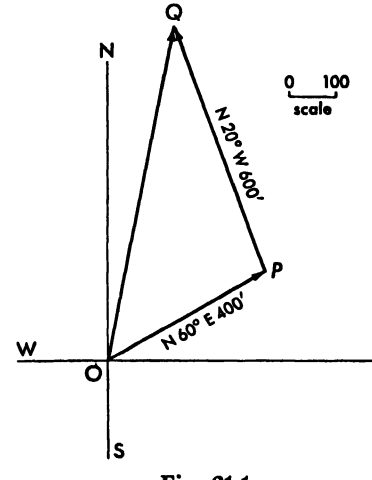


Fig. 21·1

21·1 Traverse Courses as Vectors. Traverse problems such as those illustrated in Secs. 20·1 to 20·6 are vector problems, since each course is distinguished by length, direction, and sense.

The location of point Q (Fig. 21·1) may be defined by describing the courses traversed in reaching it. The reference system is a system of rectangular coordinates in which one axis is a meridian and the other axis is a parallel of latitude.

The bearing and length of each of two courses are shown in Fig. 21·1. Therefore sufficient data are given on the sketch so that the entire

traverse may be drawn to scale. After the figure has been drawn to scale, the bearing of $\overrightarrow{OQ}$ may be found by using a protractor. The length of $\overrightarrow{OQ}$ may be found by using a ruler.

In moving from point O to point Q, the surveyor may traverse the courses $\overrightarrow{OP}$ and $\overrightarrow{PQ}$ in succession, or he may traverse the course $\overrightarrow{OQ}$ directly. Since there is an equivalence between these two traverses and since one of them is additive, the course $\overrightarrow{OQ}$ might be called the *vector sum* of $\overrightarrow{OP}$ and $\overrightarrow{PQ}$, written

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ} \tag{1}$$

Ordinarily the professional language of the surveyor does not include this particular terminology, but there is nothing inconsistent in it and it is characteristically descriptive.

A traverse may, of course, include more than two courses, as in

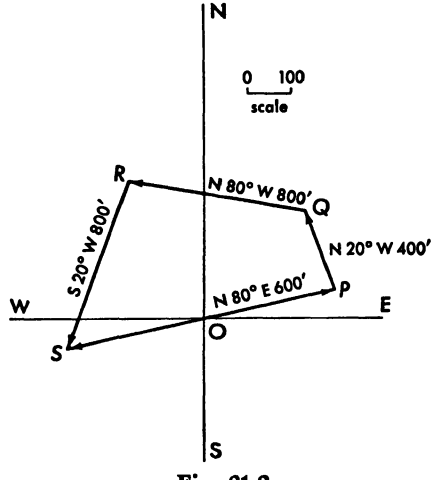


Fig. 21·2

Fig. 21·2. Here we may say that $\overrightarrow{OS}$ is the vector sum of all the other courses; then

$$\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} \tag{2}$$

By definition, we shall consider the points O, P, Q, and R to be the *initial* point of the vectors $\overrightarrow{OP}$, $\overrightarrow{PQ}$, $\overrightarrow{QR}$, and $\overrightarrow{RS}$, respectively. Also by

definition, we shall consider the points P, Q, R, and S to be the terminal points of the vectors $\overrightarrow{OP}$, $\overrightarrow{PQ}$, $\overrightarrow{QR}$, and $\overrightarrow{RS}$, respectively.

Therefore, to find the vector sum of a series of vectors we first establish a reference point; in this case, it is the intersection of the meridian and the parallel of latitude. Then, beginning at this reference point, we draw the vectors to scale correct for both magnitude and direction, first placing the initial point of the second vector on the terminal point of the first vector. Then we place the initial point of the third vector on the terminal point of the second vector, and so on until the last vector in the traverse is plotted. The vector whose initial point is at the origin and whose terminal point coincides with the terminal point of the last vector in the traverse is the sum of all the vectors in the traverse.

An equivalent definition of vector addition has reference to surveying. *The sum of the several courses in a traverse is a course whose latitude and departure are equal to the algebraic sum of the latitudes and departures of the other courses in the traverse. Northerly latitudes and easterly departures are considered to be positive, and southerly latitudes and westerly departures negative.*

If the algebraic sum of the latitudes and departures is zero, it means that the last course in the traverse took the surveyor back to the starting point, and the vector sum of all the traverses is zero (see Sec. 20·4).

21·2 Vector Addition. Since plane vectors are so widely used in so many fields of engineering, each with its own professional nomenclature, we shall for reference adopt the conventional system of rectangular coordinates used in Chap. 17.

A general definition of vector addition is, *The sum of several vectors is a vector, each of whose projections is the algebraic sum of the corresponding projections of the several vectors.* The term composition of vectors is sometimes used synonymously with vector addition, particularly if the vectors represent forces.

Vector addition can be accomplished in either of two ways, depending entirely on convenience. If, as in Examples 1 and 2 below, the projections are given, then it is easier to add the respective projections to find the projections of the vector sum. If the vectors are given in polar form and the sum is to be found graphically, then it is easier to plot each vector with its initial point on the terminal point of its predecessor. This is illustrated in Examples 3 and 4.

EXAMPLE 1. The latitude and departure of each of the courses in a traverse are given below. Sketch the traverse on coordinate paper, and find the latitude and departure of the course which leads from the initial point of the traverse to the terminal point of the traverse.

Course	*Departure*	*Latitude*
$\overrightarrow{OA}$	320′E	200′N
$\overrightarrow{AB}$	250′W	100′N
$\overrightarrow{BC}$	50′W	500′S

The course which leads from the initial point of the traverse to the terminal point of the traverse is the course whose departure is $(+320' - 250' - 50') = +20'$ and whose latitude is $(+200' + 100' - 500') = -200'$. This course is identified as the course $\overrightarrow{OC}$ in Fig. 21·3.

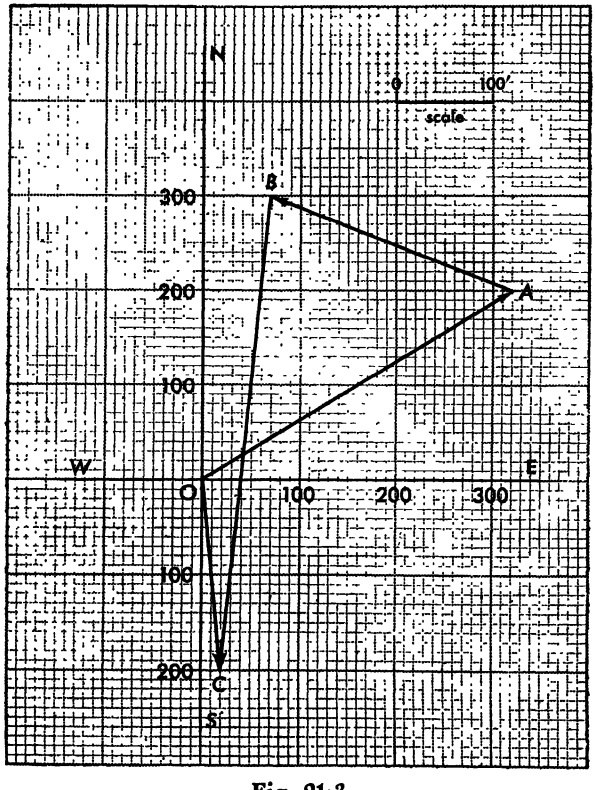


Fig. 21·3

EXAMPLE 2. Find the x and y projections of the sum of the following vectors:

Vector	*x projection*	*y projection*
$\overrightarrow{OA}$	−4.7	−2.1
$\overrightarrow{AB}$	+1.0	+5.0
$\overrightarrow{BC}$	+4.5	+2.0

The sum of the vectors $\overrightarrow{OA}$, $\overrightarrow{AB}$, and $\overrightarrow{BC}$ is the vector whose x projection is $(-4.7 + 1.0 + 4.5) = +0.8$ and whose y projection is $(-2.1 + 5.0 + 2.0) = +4.9$. This vector is designated as the vector $\overrightarrow{OC}$ in Fig. 21·4.

EXAMPLE 3. Plot the following traverse to scale, using a protractor and ruler. Find the bearing and length of the course leading from the initial point of the traverse to the terminal point of the traverse.

N 60°W	500′
N 80°E	300′
N 00°E	200′

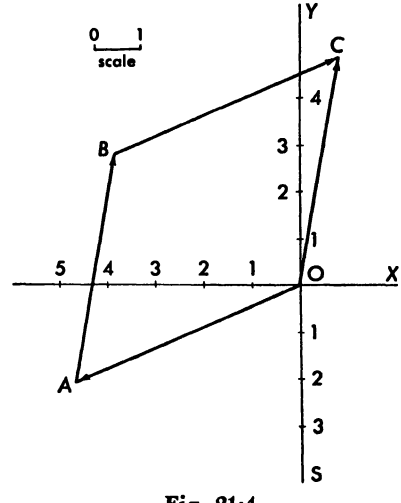


Fig. 21·4

The three courses are plotted in Fig. 21·5, where point C is the terminal point of the traverse. The course $\overrightarrow{OC}$ which connects the initial point of the traverse with the terminal point of the traverse has a length of 502′, as determined by measurement. By measurement with a protractor, the bearing is found to be N 15°W.

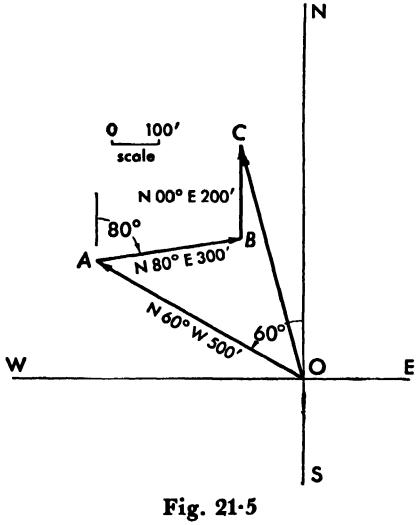


Fig. 21·5

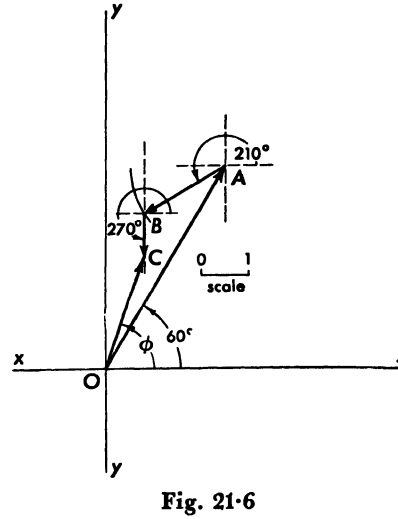


Fig. 21·6

EXAMPLE 4. Find the sum of the following vectors graphically:

$$5/\underline{60°} + 2/\underline{210°} + 1/\underline{270°}$$

The three given vectors are plotted as $\overrightarrow{OA}$, $\overrightarrow{AB}$, and $\overrightarrow{BC}$ in Fig. 21·6. The vector $\overrightarrow{OC}$ is the sum of the given vectors. By scaling with a ruler, $|OC| =$

$= 2.45$. By scaling with a protractor, $\phi = 72°$. Therefore

$$\overrightarrow{OC} = 2.45\underline{/72°}$$

EXERCISE 1

1. The departure and latitude of each of the courses in certain traverses are given below. Sketch each traverse on coordinate paper, and find the departure and latitude of the course which will lead from the initial point of the traverse to the terminal point of the traverse.

a.

Course	*Departure*	*Latitude*
$\overrightarrow{OA}$	100′E	600′N
$\overrightarrow{AB}$	700′E	200′S
$\overrightarrow{BC}$	100′E	600′S
$\overrightarrow{CD}$	1000′W	200′S

b.

Course	*Departure*	*Latitude*
$\overrightarrow{OA}$	200′E	200′S
$\overrightarrow{AB}$	100′E	200′N
$\overrightarrow{BC}$	000′ E	400′N
$\overrightarrow{CD}$	100′W	200′S

c.

Course	*Departure*	*Latitude*
$\overrightarrow{OA}$	100′E	200′S
$\overrightarrow{AB}$	100′E	400′N
$\overrightarrow{BC}$	200′W	300′N
$\overrightarrow{CD}$	100′W	300′S
$\overrightarrow{DE}$	300′W	200′N
$\overrightarrow{EF}$	100′E	200′S
$\overrightarrow{FG}$	500′E	800′S

2. Plot the following traverses to scale, using a protractor and ruler. Find the bearing and length of the course leading from the initial point of the traverse to the terminal point of the traverse.

a.

N 73°E	250′
N 48°W	175′
N 20°E	160′
S 75°W	400′

b.

N 45°W	175′
N 30°E	240′
N 73°E	250′
S 08°E	180′

c. S 60°E 260′
S 79°W 210′
S 45°E 230′
N 75°W 450′

d. N 90°E 410′
S 00°E 175′
N 63°W 230′
S 65°W 360′

e. N 82°W 200′
S 12°W 240′
N 79°E 250′
S 74°E 150′
N 35°E 100′

3. Find the x and y projections of the sum of the following sets of vectors:

	Vector	x *projection*	y *projection*
a.	$\overrightarrow{OA}$	3.6	−4.5
	$\overrightarrow{AB}$	−2.9	−8.4
	$\overrightarrow{BC}$	5.6	10.1
b.	$\overrightarrow{OA}$	−9.8	3.5
	$\overrightarrow{AB}$	2.6	6.8
	$\overrightarrow{BC}$	5.8	−8.1
c.	$\overrightarrow{OA}$	5.6	0.0
	$\overrightarrow{AB}$	0.0	5.6
	$\overrightarrow{BC}$	−3.0	3.0
	$\overrightarrow{CD}$	−5.8	7.2
d.	$\overrightarrow{OA}$	−7.0	4.0
	$\overrightarrow{AB}$	0.0	2.0
	$\overrightarrow{BC}$	5.0	−2.0
e.	$\overrightarrow{OA}$	5.0	0.0
	$\overrightarrow{AB}$	2.0	2.0
	$\overrightarrow{BC}$	−1.0	−5.0

4. Find the sum of the following vectors graphically. The reference angle of each vector is measured from the positive side of the X axis in a counterclockwise direction to the vector.

a. $15\underline{/25°}$
$20\underline{/140°}$
$10\underline{/180°}$

b. $25\underline{/120°}$
$10\underline{/-90°}$
$10\underline{/300°}$

c. $3/\underline{0°}$
$4/\underline{90°}$
$3/\underline{180°}$

d. $2/\underline{-30°}$
$10/\underline{-90°}$
$5/\underline{120°}$

21·3 Polar and Rectangular Coordinates.[1] A vector drawn in the coordinate plane may be defined in two ways: (1) by giving the polar coordinates of the terminal point or (2) by giving the rectangular coordinates of the terminal point.

In Fig. 21·7 the vector $\overrightarrow{OP}$ is a vector 10 in. long whose reference angle is 30°.

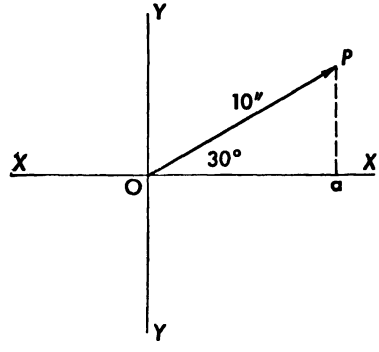


Fig. 21·7

Notation describing the vector in this way is called *polar notation*, and the polar coordinates of P are 10″ and 30°, often written

$$10''/\underline{30°}$$

Following Sec. 17·14, it is apparent that

$$Oa = 10 \cos 30°$$
$$Oa = 10\,\frac{\sqrt{3}}{2}$$

and

$$aP = 10 \sin 30°$$
$$aP = 5$$

Therefore the rectangular coordinates of P are

$$10\,\frac{\sqrt{3}}{2},\ 5$$

[1] Refer to Sec. 17.10.

In general, by reference to Fig. 21·8,

$$|OP| = \sqrt{x^2 + y^2} = r \tag{3}$$

$$\phi = \tan^{-1}\frac{y}{x} \qquad \text{(See Sec. 17·24)} \tag{4}$$

The polar coordinates of P in terms of its rectangular coordinates are

$$\sqrt{x^2 + y^2} \,/\, \tan^{-1}\frac{y}{x} \qquad \star\ (5)$$

The rectangular coordinates of P in terms of its polar coordinates are

$$x = |OP| \cos \phi \qquad \star\ (6)$$

$$y = |OP| \sin \phi \qquad \star\ (7)$$

Thus polar notation may be converted to rectangular notation, and vice versa, by Eqs. (3) to (7).

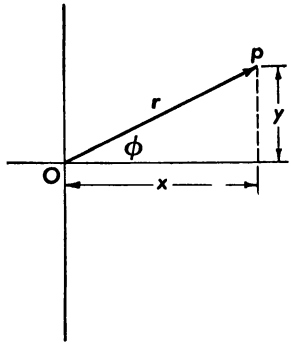


Fig. 21·8

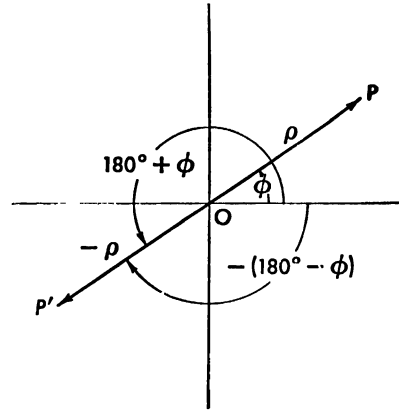


Fig. 21·9

Ordinarily the symbol r is used to denote the magnitude of $\overrightarrow{OP}$ when we are using rectangular coordinates thus.

$$r = \sqrt{x^2 + y^2} \tag{8}$$

Remember that from Sec. 17·9 r is an unsigned number. In polar notation it is usual to denote the magnitude and sense of $\overrightarrow{OP}$ by the symbol ρ, where ρ is a signed number. The significance of this sign is given in Fig. 21·9 (see also Sec. 17·10), where the magnitudes $|OP|$ and $|OP'|$ are equal. The polar coordinates of P' are

$$-\rho/\underline{\phi} \tag{9}$$

but they could equally well have been written

$$+\rho/\underline{180^\circ + \phi} \tag{10}$$

or even

$$+\rho/\underline{-(180^\circ - \phi)} \tag{11}$$

In Fig. 21·10, if $\overrightarrow{OP'} = -\overrightarrow{OP}$ and the rectangular coordinates of P are x and y, the rectangular coordinates of P' are $-x$ and $-y$. The

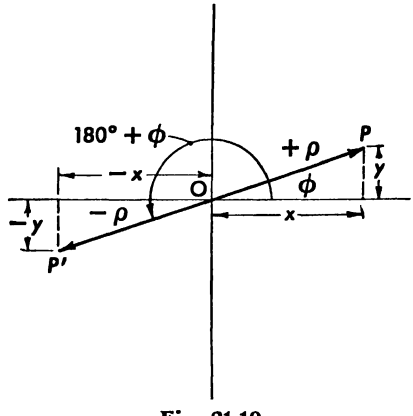


Fig. 21·10

magnitude of both $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ in rectangular notation is

$$r = \sqrt{x^2 + y^2} \tag{12}$$

The magnitude and sense of $\overrightarrow{OP}$ in polar coordinates is $+\rho$; the magnitude, and sense $\overrightarrow{OP'}$ in polar coordinates is $-\rho$. The polar coordinates of P are (see Fig. 21·10)

$$+\rho/\underline{\phi} \tag{13}$$

The polar coordinates of P' are (see Fig. 21·10)

$$-\rho/\underline{\phi} \tag{14}$$

$$+\rho/\underline{\phi + 180^\circ} \tag{15}$$

$$+\rho/\underline{-(180^\circ - \phi)} \tag{16}$$

That is, in polar notation the size of the angle shows how much the terminal side of the angle has been rotated, and the sign before ρ denotes the sense of the vector along that terminal side.

EXAMPLE 5. If the polar coordinates of a point are $6/\underline{40°}$, find the rectangular coordinates of the point.

$$x = 6 \cos 40° = 6 \times 0.7660 = +4.596$$
$$y = 6 \sin 40° = 6 \times 0.6428 = +3.857$$

EXAMPLE 6. If the polar coordinates of a point are $5/\underline{120°}$, find the rectangular coordinates of the point.

$$x = 5 \cos 120° = 5 \times (-0.5) = -2.5$$
$$y = 5 \sin 120° = 5 \times (+0.8660) = +4.330$$

EXAMPLE 7. If the polar coordinates of a point are $3/\underline{260°}$, find the rectangular coordinates of the point.

$$x = 3 \cos 260° = 3 \times (-0.1737) = -0.5211$$
$$y = 3 \sin 260° = 3 \times (-0.9848) = -2.9544$$

EXAMPLE 8. If the polar coordinates of a point are $2/\underline{-30°}$, find the rectangular coordinates of the point.

$$x = 2 \cos (-30°) = 2 \times 0.8660 = +1.732$$
$$y = 2 \sin (-30°) = 2 \times -0.5000 = -1.000$$

EXAMPLE 9. If the polar coordinates of a point are $-4/\underline{120°}$, find the rectangular coordinates of the point.

$$x = -4 \cos 120° = -4 \times (-0.5) = +2.000$$
$$y = -4 \sin 120° = -4 \times (+0.8660) = -3.464$$

EXAMPLE 10. If the rectangular coordinates of a point are 2,5, find the polar coordinates.

Following Eq. (5), page 479,

$$\text{Polar coordinates} = \sqrt{2^2 + 5^2} \Big/ \tan^{-1} \frac{y}{x}$$
$$= \sqrt{29} \Big/ \tan^{-1} \frac{y}{x}$$
$$= 5.385/\underline{\tan^{-1} 5/2}$$
$$= 5.385/\underline{68°12'}$$

EXAMPLE 11. If the rectangular coordinates of a point are $-5,4$, find the polar coordinates.

$$r = \sqrt{(-5)^2 + 4^2} = \sqrt{41} = 6.403$$
$$\tan \phi = \frac{4}{-5} = -0.80000$$
$$\phi = 141°20'$$

The polar coordinates are then $6.403/\underline{141°20'}$.

EXERCISE 2

1. The polar coordinates of the terminal point of certain vectors are given below. Calculate the corresponding rectangular coordinates. Use four-place tables or the slide rule.

a. $3/27°$	**b.** $4/320°$	**c.** $-10/-30°$
d. $3/-270°$	**e.** $7/90°$	**f.** $5/160°$
g. $15/29°$	**h.** $15/-250°$	**i.** $20/-135°$
j. $-7/-90°$	**k.** $2/220°$	**l.** $10/-140°$
m. $3/270°$	**n.** $4/225°$	**o.** $5/1{,}470°$

2. The rectangular coordinates of the terminal point of certain vectors are given below. Calculate the corresponding polar coordinates.

a. 3, 6	**b.** 2, -3	**c.** -2, $+2$
d. -4, 4	**e.** 5, -6	**f.** $\sqrt{3}$, 1
g. -3, -2	**h.** -5, 6	**i.** $\frac{\sqrt{2}}{2}$, $\frac{-\sqrt{2}}{2}$
j. $(a^2 - b^2)$, $2ab$	**k.** 6, -6	**l.** 5, 0

21·4 Topographic and Polar Diagrams. Figure 21·11 is a diagram containing information similar to that contained in the diagram in Fig. 21·1.

In both diagrams we propose to find the vector sum of two vectors. The equivalence of the two diagrams is indicated below.

$\overrightarrow{OR}$ in Fig. 21·11 equals $\overrightarrow{OP}$ in Fig. 21·1 (17)

$\overrightarrow{OS}$ in Fig. 21·11 equals $\overrightarrow{PQ}$ in Fig. 21·1 (18)

$\overrightarrow{OT}$ in Fig. 21·11 equals $\overrightarrow{OQ}$ in Fig. 21·1 (19)

These diagrams are therefore consistent with the same data.

Figure 21·1 is called a topographic diagram, and Fig. 21·11 is called a polar diagram. In finding a vector sum, either type of diagram may be used, depending entirely on convenience. In Fig. 21·2 the topographic diagram is called a vector polygon; in Fig. 21·1 it is called a vector triangle. The polar diagram in Fig. 21·11 always leads to a vector parallelogram.

In actual surveying problems, the order in which we add courses is important; in abstract vector problems, it is not important. In our general treatment of vectors we shall assert that the order in which we add vectors is immaterial; *i.e*, if

$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OS} \tag{20}$$

then

$$\overrightarrow{OR} = \overrightarrow{OQ} + \overrightarrow{OP} + \overrightarrow{OS} \tag{21}$$

and so on.

Several vectors can be added by means of polar diagrams. Two vectors are added to obtain a vector sum, and then this vector sum

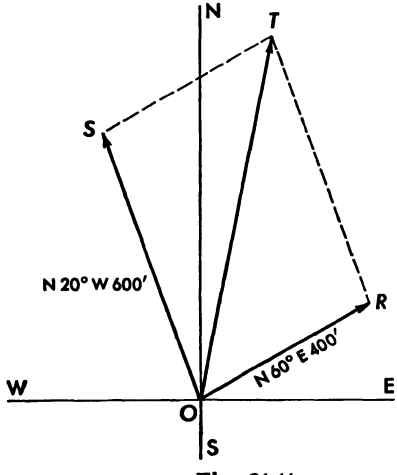


Fig. 21·11

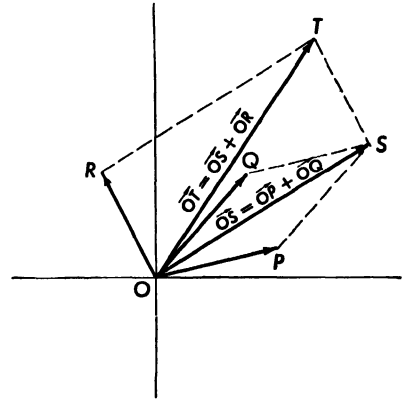


Fig. 21·12

is combined with a third vector to obtain a second vector sum, etc. This is illustrated in Fig. 21·12.

Here we wish to find the vector sum of $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$. First we add $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ to get $\overrightarrow{OS}$; then we add $\overrightarrow{OS}$ and $\overrightarrow{OR}$ to get $\overrightarrow{OT}$. The same result is accomplished by the topographic diagram shown in Fig. 21·13.

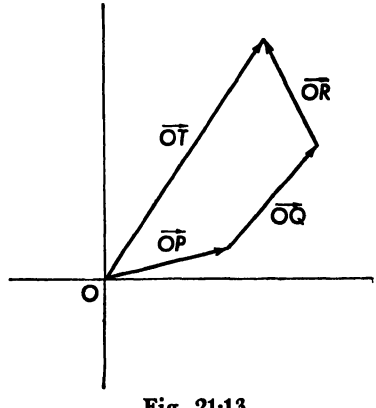


Fig. 21·13

21·5 Vector Subtraction. The process of vector subtraction is defined as follows: "The vector difference between two vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ is a vector $\overrightarrow{OR}$ such that the projections of $\overrightarrow{OR}$ are the differences between the corresponding projections of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$" (see Fig. 21·14).

$$\overrightarrow{OR} = \overrightarrow{OP} - \overrightarrow{OQ} \tag{22}$$

$$x_r = x_p - x_q \tag{23}$$

and

$$y_r = y_p - y_q \tag{24}$$

If, as in Fig. 21·15, $\overrightarrow{OQ'} = -\overrightarrow{OQ}$ and $\overrightarrow{OR} = \overrightarrow{OP} - \overrightarrow{OQ}$, then $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ'}$.

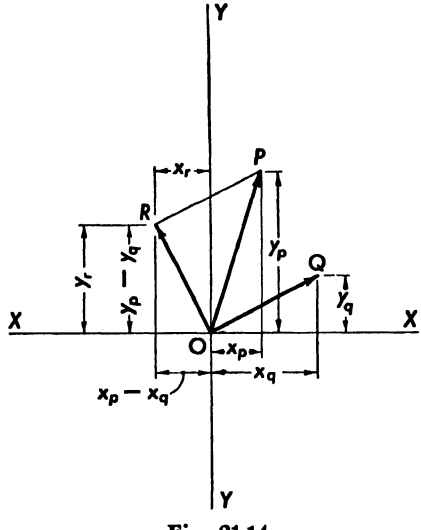


Fig. 21·14

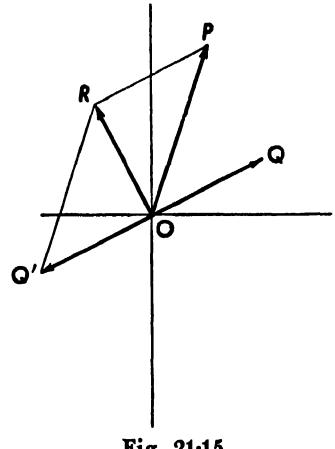


Fig. 21·15

The processes of vector addition and subtraction can be accomplished analytically only when the data are in rectangular notation. If the data are given in polar notation, they must be changed to rectangular form before adding.

EXAMPLE 12. Perform the following vector addition:

$$2/\underline{20°} + 3/\underline{120°} + 4/\underline{250°} + 5/\underline{310°}$$

A topographic diagram illustrating this problem is shown in Fig. 21·16.

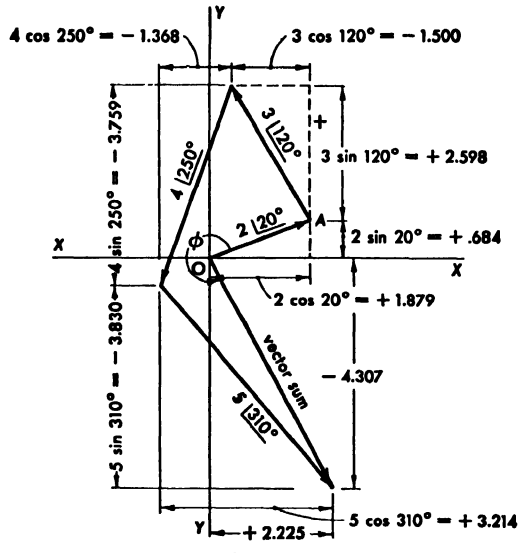


Fig. 21·16

The work will be easier and the correct answer more certain if the solution is organized according to some orderly scheme, such as the arrangement below.

Vector	*cos*	*x projection* +	*x projection* −	*sin*	*y projection* +	*y projection* −
$2/\underline{20°}$	+0.9397	1.879		+0.3420	0.684	
$3/\underline{120°}$	−0.5000		1.500	+0.8660	2.598	
$4/\underline{250°}$	−0.3420		1.368	−0.9397		3.759
$5/\underline{310°}$	+0.6428	3.214		−0.7660		3.830
$+x$ projection		5.093				
$-x$ projection			2.868			
$+y$ projection					3.282	
$-y$ projection						7.589

Total x projection = 5.093 − 2.868 = 2.225
Total y projection = 3.282 − 7.589 = −4.307

The magnitude of the vector sum is

$$r = \sqrt{2.225^2 + (-4.307)^2} = \sqrt{23.50} = 4.85$$

If ϕ is the reference angle of the vector sum,

$$\tan \phi = \frac{-4.307}{+2.225} = -1.936$$

$$\phi = 297°19'$$

The polar notation of the vector sum is $4.85/\underline{297°19'}$.

EXAMPLE 13. A vector $\overrightarrow{OA}$ has an x projection of 3 units and a y projection of 4 units. Another vector $\overrightarrow{OB}$ has an x projection of 6 units and a y projection of 2 units. Find the polar coordinates of the terminal point of the vector $\overrightarrow{OR}$ when $\overrightarrow{OR} = \overrightarrow{OA} - \overrightarrow{OB}$.

Following Eq. (23),

$$\text{the } x \text{ projection of } \overrightarrow{OR} = 3 - 6 = -3$$

Following Eq. (24),

$$\text{the } y \text{ projection of } \overrightarrow{OR} = 4 - 2 = +2$$

The magnitude of $\overrightarrow{OR}$ is

$$|OR| = \sqrt{(-3)^2 + 2^2} = \sqrt{13} = 3.61$$

$$\tan \phi_r = \frac{+2}{-3} = -0.6667$$

where

$$\phi_r = \text{the reference angle of } \overrightarrow{OR}$$
$$\phi_r = 146°19'$$

In polar notation the value of $\overrightarrow{OR}$ is $3.61/\underline{146°19'}$.

EXAMPLE 14. Solve for the vector $\overrightarrow{OR}$ in polar form if

$$\overrightarrow{OR} = 2/\underline{45°} + 3/\underline{30°} - 5/\underline{120°}$$

This example is similar to Example 12 except that it involves a vector subtraction. However, by Eqs. (9) and (10), the above equation can be altered to show straight addition.

$$\overrightarrow{OR} = 2/\underline{45°} + 3/\underline{30°} + 5/\underline{300°}$$

The method of solution is now identical with the method outlined in Example 12.

EXERCISE 3

1. Add the following vectors. Draw a topographic diagram to illustrate the vector addition. Calculate the polar and the rectangular coordinates of the terminal point of the vector sum.
   - **a.** $10/\underline{42°} + 20/\underline{140°} + 30/\underline{240°}$
   - **b.** $5/\underline{180°} + (-10)/\underline{30°} + 20/\underline{-30°}$
   - **c.** $2/\underline{150°} + 6/\underline{90°} + 12/\underline{330°} + 10/\underline{225°}$
   - **d.** $8/\underline{300°} + 5/\underline{20°} + 6/\underline{270°} + 15/\underline{220°}$
   - **e.** $20/\underline{360°} + 20/\underline{180°} + 40/\underline{90°}$

2. In the following problems, subtract the vector $\overrightarrow{OA}$ from the vector $\overrightarrow{OB}$. The x and y projections of each vector are given below. Record the polar coordinates of the terminal point of the vector difference.

	Vector	x *projection*	y *projection*
a.	$\overrightarrow{OA}$	+ 7	+ 5
	$\overrightarrow{OB}$	+ 2	+ 6
b.	$\overrightarrow{OA}$	− 3	− 5
	$\overrightarrow{OB}$	+ 2	+ 8
c.	$\overrightarrow{OA}$	+15	+20
	$\overrightarrow{OB}$	−20	+15
d.	$\overrightarrow{OA}$	−20	−30
	$\overrightarrow{OB}$	−10	−10
e.	$\overrightarrow{OA}$	−32	+42
	$\overrightarrow{OB}$	+26	−20

3. Perform the following vector operations:

a. $30/45° - 20/90°$
b. $20/36° - 5/80°$
c. $15/160° - 20/330°$
d. $90/220° - 45/20°$
e. $25/110° - 60/220°$
f. $20/45° - 30/30° + 10/225°$
g. $100/90° + 200/180° - 100/0°$
h. $-25/330° + 40/160° - 20/260°$
i. $15/60° - 30/120° + 50/330°$
j. $20/115° - 15/250° - 30/25°$

21·6 The Product of a Vector and a Scalar Quantity. The product of a scalar quantity U and a vector $\overrightarrow{OP}$ is, by definition, a vector whose magnitude is U times the magnitude $\overrightarrow{OP}$, whose direction is the same as the direction of $\overrightarrow{OP}$, and whose sense is either the same as or exactly opposite to the sense of $\overrightarrow{OP}$ according as U is positive or negative.

The quotient of a vector $\overrightarrow{OP}$ divided by a scalar quantity U is,

by definition, a vector whose magnitude is $|\overrightarrow{OP}|$ divided by U, whose direction is the same as the direction of $\overrightarrow{OP}$, and whose sense is either the same as or exactly opposite to the sense of $\overrightarrow{OP}$ according as U is positive or negative.

21·7 Velocity as a Vector. The linear speed of a moving point is the straight-line distance over which the point moves per unit time, without regard to the sense of motion along that straight line. Velocity,

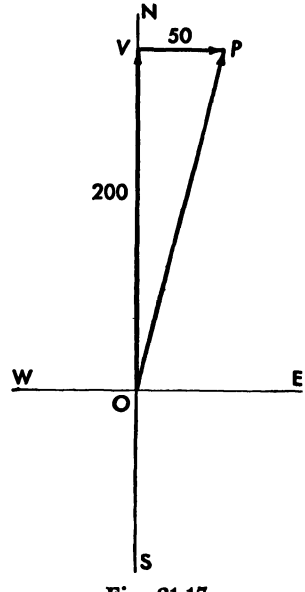


Fig. 21·17

on the other hand, is completely described only when speed, direction, and sense of motion are known. Velocity is a vector quantity; speed is a scalar quantity.

Suppose the propeller of an airplane is imparting a 200-mph northerly velocity to the plane while the plane is encountering a 50-mph cross wind blowing directly toward the east. Assuming that these velocities remain constant for 1 hr, then at the end of 1 hr the plane will be 200 miles north and 50 miles east of the starting point.

A vector diagram of the distances will appear as in Fig. 21·17.

The vector $\overrightarrow{OP}$ then represents the course of the plane. The magnitude of $\overrightarrow{OP}$ represents the actual distance traveled in this hour.

The vectors $\overrightarrow{OV}$, $\overrightarrow{VP}$, and $\overrightarrow{OP}$ in Fig. 21·17 are distances. However, each of these vectors may be divided by the scalar quantity, time (in

this case, 1 hr), so as to construct a similar diagram in which the vectors represent velocities. With this interpretation, Fig. 21·17 is identical in appearance with a corresponding velocity diagram.

21·8 Forces as Vectors. When two forces act through a single point, as in Fig. 21·18, these forces are called *concurrent forces.*

In Fig. 21·18 the strings indicated as F_1 and F_2 will presently carry the forces F_1 and F_2, but we shall apply these forces one at a time.

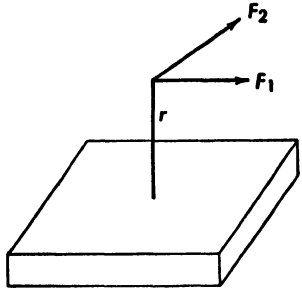


Fig. 21·18

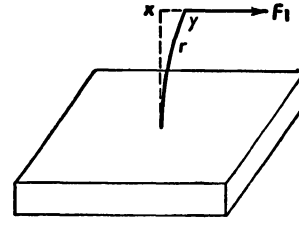


Fig. 21·19

First let the elastic rod r be pulled a little by the force F_1; then the top of the rod will be displaced by a certain distance, say xy, as in Fig. 21·19. Now, while holding F_1 constant both as regards magnitude and direction, let us apply force F_2, as shown in Fig. 21·20. Now the rod r will be deflected by the further amount yz.

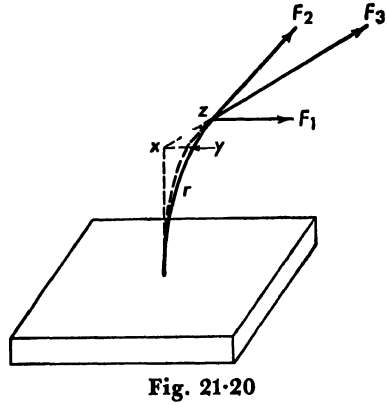


Fig. 21·20

The top of the rod could have been made to occupy the position z by the action of a single force F_3 of some as yet undetermined magnitude acting along the line xz. This force F_3 is called the resultant of the forces F_1 and F_2.

If the deflections do not exceed the elastic limit of the rod, then it can be proved experimentally that the forces are proportioned to the deflections that they cause.

In Fig. 21·21 the vectors $\overrightarrow{xy}$, $\overrightarrow{yz}$, and $\overrightarrow{xz}$ are deflections, but if we divide all three vector deflections by the spring constant, which is a scalar, we arrive at a vector diagram shown in Fig. 21·22, in which the vectors represent forces.

While the amounts of the deflections $\overrightarrow{xy}$, $\overrightarrow{yz}$, and $\overrightarrow{xz}$ have not been stated, the student may be thinking of deflections in the order of an inch or two or perhaps more. However, the discussion would be just

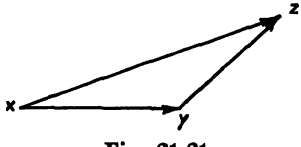


Fig. 21·21

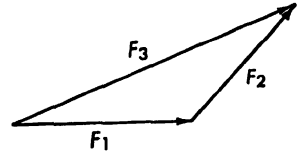


Fig. 21·22

as valid if the deflections were a few thousandths of an inch or, for that matter, even less.

Therefore, in the vector solutions of force problems we ordinarily pay no attention to deflections but instead deal directly with forces.

The process just described illustrates vector addition or a problem in the *composition of forces*. In vector nomenclature we may write

$$\overrightarrow{F_1} + \overrightarrow{F_2} = \overrightarrow{F_3} \tag{25}$$

Problems dealing with concurrent forces can be solved either graphically or by computation. When solved graphically, the lengths of the vectors $\overrightarrow{xy}$, $\overrightarrow{yz}$, and $\overrightarrow{xz}$ are plotted proportional to the magnitude of F_1, F_2, and F_3, respectively. The angles on the scale drawing are the same as the angles between the respective forces.

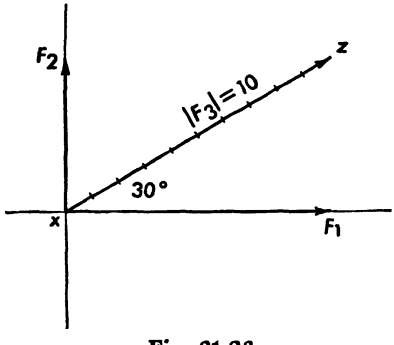


Fig. 21·23

The force F_3 in Fig. 21·22 is called the *resultant* of forces F_1 and F_2. A force which has the same magnitude and direction as F_3 but which is opposite in sense is called the *equilibrant* of F_1 and F_2.

21·9 Resolution of Forces. In Fig. 21·23 it is known that $|F_3| = 10$ lb and its line of action makes an angle of 30° with the line of action of $\overrightarrow{F_1}$. Also, it is known that $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ are at right angles

and that $\vec{F}_3$ is the resultant of $\vec{F}_1$ and $\vec{F}_2$. The problem is to find the magnitude of $\vec{F}_1$ and $\vec{F}_2$.

Graphically, this can be done by drawing the figure to scale, as in Fig. 21·23. Now $|F_1|$ and $|F_2|$ can be measured. By calculation,

$$|F_2| = 10 \sin 30 = 10 \times 0.5 = 5$$
$$\vec{F}_2 = 5\underline{/90°}$$
$$|F_1| = 10 \cos 30 = 10 \times 0.866 = 8.66$$
$$\vec{F}_1 = 8.66\underline{/0°}$$

This process is called the *resolution* of the force $\vec{F}_3$ into two rectangular *components* $\vec{F}_1$ and $\vec{F}_2$.

If it is assumed that $\vec{F}_2$ is vertical and $\vec{F}_1$ is horizontal, then $\vec{F}_2$ is called the vertical component of $\vec{F}_3$ and $\vec{F}_1$ is called the horizontal component of $\vec{F}_3$.

21·10 Resolution of General Vectors. Having been given the vector $\vec{OP}$ in Fig. 21·24, we may under certain conditions find it

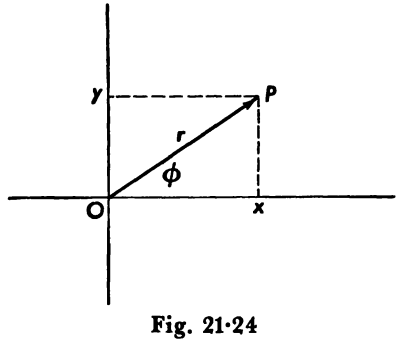


Fig. 21·24

convenient to *resolve* the vector $\vec{OP}$ into its rectangular *components; i.e.*, to find two vectors $\vec{Ox}$ and $\vec{Oy}$ both of which are mutually perpendicular and lie on the axes such that

$$\vec{Oy} + \vec{Ox} = \vec{OP} \tag{26}$$

Here the sum of the vectors $\vec{Oy}$ and $\vec{Ox}$ is identical with the single vector $\vec{OP}$.

In general, we can resolve any vector into its mutually perpendicular components if we know the direction and sense of the vector relative to the direction of the components. By reference to Fig. 21·24,

$$|OP| = r = \sqrt{|Ox|^2 + |Oy|^2} \tag{27}$$

$$|Ox| = r \cos \phi \tag{28}$$

$$|Oy| = r \sin \phi \tag{29}$$

EXAMPLE 15. Find the resultant and the equilibrant of a force of 3 lb and a force of 4 lb acting at right angles.

Draw the vector forces to scale (see Fig. 21·25). The resultant appears to be 5 lb, as determined by measurement.

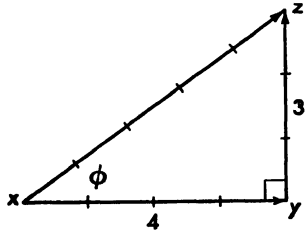


Fig. 21·25

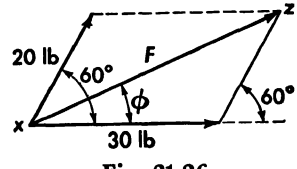


Fig. 21·26

By calculation, since the angle $xyz = 90°$,

$$|xz| = \sqrt{3^2 + 4^2} = 5$$

$$\tan \phi = \tfrac{3}{4} = 0.75000$$

$$\phi = 36°52'$$

The resultant is, then, the vector force $5/\underline{36°52'}$, and the equilibrant is the vector force $-5/\underline{36°52'}$.

EXAMPLE 16. Two forces act on a single point. The angle between them is 60°, and the forces are 20 and 30 lb, respectively. Find the magnitude of the resultant and the equilibrant and the angles they make with the 30-lb force.

The problem can be diagrammed as shown in Fig. 21·26. Following Eq. (40), page 425,

$$|F| = \sqrt{30^2 + 20^2 + 2 \times 20 \times 30 \cos 60°}$$

$$= 43.6 \text{ lb}$$

$$\sin \phi = \frac{20 \times \sin 60°}{43.6} = \frac{20 \times 0.866}{43.6} = 0.397$$

$$\phi = 23.4°$$

The resultant is a force whose magnitude is 43.6 lb in the direction xz. The equilibrant is a force whose magnitude is also 43.6 lb but which acts in the direction zx.

EXERCISE 4

1. Find the resultant of two concurrent forces, one of 253 and the other of 578 lb, acting at an angle of 18°27′20″ with each other.
2. Two forces acting at right angles with each other have a resultant of 350 lb, and one of the forces is 270 lb. Find the other force and the angle which it makes with the resultant.
3. Two forces of 70 and 85 lb, respectively, have a resultant of 125 lb. Find the angle which each of the forces makes with the resultant.
4. Find the horizontal and vertical components of a force of 5,680 lb acting at an angle of 23°46′25″ with the horizontal.
5. A horse pulls a canalboat with a force of 425 lb on the end of a towline which makes an angle of 6° with the side of the canal. Find the force tending to urge the boat toward shore and the effective force which pulls the boat forward.
6. Two forces of 3,600 and 5,300 lb, respectively, have a resultant of 7,500 lb. Find the angle which the resultant makes with each force.
7. A man pushes a floating plank with a force of 25 lb on the end of a pole which makes an angle of 53° with the surface of the water. What force tends to submerge the plank, and what to move it horizontally?
8. Three forces of 125, 150, and 175 lb, respectively, are concurrent and in equilibrium. The first pulls toward the north and the second in a general direction south of east. Find the direction of the third force.
9. In the derrick shown in Fig. 21·27 the mast DC is 45 ft high, the boom AB is 42 ft long, and pin A is 2 ft above D. W is a weight of 1,000 lb and angle CAB is 44°. Find the length of the line BC and the force in BC and in AB.

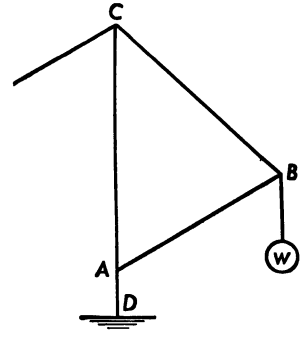


Fig. 21·27

21·11 The Product of Two Vectors. We have discussed one aspect of vector multiplication and division in Sec. 21·6, in which we multiplied a vector by scalar quantity and divided a vector by a scalar quantity.

We shall have occasion to multiply vectors by vectors and to divide vectors by vectors. We shall define vector multiplication as follows: *The product of two vectors* $\overrightarrow{OP}$ *and* $\overrightarrow{OQ}$ *is a vector* $\overrightarrow{OR}$ *such that the magnitude of* $\overrightarrow{OR}$ *is the product of the magnitudes of* $\overrightarrow{OP}$ *and* $\overrightarrow{OQ}$ *and such that the*

reference angle of $\overrightarrow{OR}$ *is the sum of the reference angles of* $\overrightarrow{OP}$ *and* $\overrightarrow{OQ}$. (See Fig. 21·28.)

If

$$\overrightarrow{OP} = |OP| \angle \phi_p \tag{30}$$

$$\overrightarrow{OQ} = |OQ| \angle \phi_q \tag{31}$$

$$\overrightarrow{OR} = |OR| \angle \phi_r \tag{32}$$

and if

$$\overrightarrow{OR} = \overrightarrow{OP} \times \overrightarrow{OQ} \tag{33}$$

then

$$|OR| \angle \phi_r = |OP| \times |OQ| \angle \phi_p + \phi_q \tag{34}$$

$$|OR| = |OP| \times |OQ| \tag{35}$$

and

$$\phi_r = \phi_p + \phi_q \tag{36}$$

Vector division is defined as follows: *The quotient of a vector* $\overrightarrow{OP}$ *divided by a vector* $\overrightarrow{OQ}$ *is a vector* $\overrightarrow{OR}$ *such that the magnitude of* $\overrightarrow{OR}$ *is the quotient*

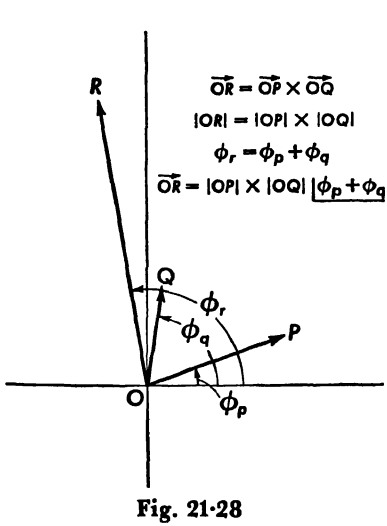


Fig. 21·28

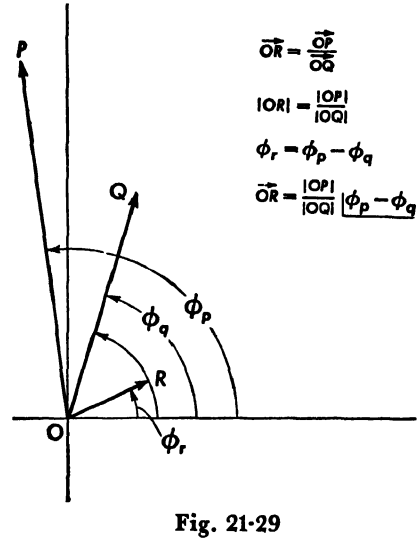


Fig. 21·29

of the magnitude of $\overrightarrow{OP}$ *divided by the magnitude of* $\overrightarrow{OQ}$ *and such that the reference angle of* $\overrightarrow{OR}$ *is the difference between the reference angles of* $\overrightarrow{OP}$ *and* $\overrightarrow{OQ}$. (See Fig. 21·29.)

For example, if

$$\overrightarrow{OP} = |OP| \underline{/\phi_p} \tag{37}$$

$$\overrightarrow{OQ} = |OQ| \underline{/\phi_q} \tag{38}$$

$$\overrightarrow{OR} = |OR| \underline{/\phi_r} \tag{39}$$

and if

$$\overrightarrow{OR} = \overrightarrow{OP} \div \overrightarrow{OQ} \tag{40}$$

then

$$|OR| \underline{/\phi_r} = \frac{|OP|}{|OQ|} \underline{/\phi_p - \phi_q} \tag{41}$$

At this stage of the work, vector multiplication and division can be accomplished directly when the data are given in polar notation. If the data are given in rectangular notation, they must be converted to polar notation before multiplying or dividing.

EXAMPLE 17. If the vector $\overrightarrow{OR} = 2\underline{/17°} \times 5\underline{/180°}$, find the magnitude and reference angle of $\overrightarrow{OR}$.

Following Eq. (35),

$$|OR| = 2 \times 5 = 10$$

Following Eq. (36),

$$\phi_r = 17° + 180° = 197°$$

then

$$\overrightarrow{OR} = 10\underline{/197°}$$

EXAMPLE 18. If the vector $\overrightarrow{OR} = \dfrac{16\underline{/40°}}{2\underline{/-20°}}$, find the magnitude of $\overrightarrow{OR}$ and the reference angle of $\overrightarrow{OR}$.

Following Eq. (41),

$$\overrightarrow{OR} = 16/2\underline{/40° - (-20°)}$$

$$\overrightarrow{OR} = 8\underline{/60°}$$

EXAMPLE 19. Find $\overrightarrow{OR}$ in polar notation if $\overrightarrow{OR} = \overrightarrow{OA} \times \overrightarrow{OB}$, where the x projection of $\overrightarrow{OA}$ is 2, the y projection of $\overrightarrow{OA}$ is 3, the x projection of $\overrightarrow{OB}$ is 5, and the y projection of $\overrightarrow{OB}$ is -4.

$$
\begin{aligned}
|OA| &= \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13} = 3.606 \\
\tan \phi_a &= \tfrac{3}{2} = 1.5 \\
\phi_a &= 56°19' \\
|OB| &= \sqrt{5^2 + 4^2} = \sqrt{25 + 16} = \sqrt{41} = 6.403 \\
\tan \phi_b &= \frac{-4}{5} = -0.8 \\
\phi_b &= -38°40' \\
|OR| &= 3.606 \times 6.403 = 23.09 \\
\phi_r &= 56°19' + (-38°40') = 17°39' \\
\overrightarrow{OR} &= \underline{23.09/17°39'}
\end{aligned}
$$

EXAMPLE 20. Find $\overrightarrow{OR}$ in polar notation if $\overrightarrow{OR} = \overrightarrow{OA}/\overrightarrow{OB}$, where the x projection of $\overrightarrow{OA}$ is 2, the y projection of $\overrightarrow{OA}$ is 3, the x projection of $\overrightarrow{OB}$ is 5, and the y projection of $\overrightarrow{OB}$ is -4.

$$
\begin{aligned}
|OA| &= \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13} = 3.606 \\
\tan \phi_a &= \tfrac{3}{2} = 1.5 \\
\phi &= 56°19' \\
|OB| &= \sqrt{5^2 + (-4)^2} = \sqrt{25 + 16} = \sqrt{41} = 6.403 \\
\tan \phi_b &= \frac{-4}{5} = -0.8 \\
\phi_b &= -38°40' \\
|OR| &= \frac{3.606}{6.403} = 0.5632 \\
\phi &= 56°19' - (-38°40') = 56°19' + 38°40' \\
&= 94°59' \\
\overrightarrow{OR} &= \underline{0.5632/94°59'}
\end{aligned}
$$

EXAMPLE 21. Solve for ρ and ϕ in the equation

$$\underline{\rho/\phi} = \frac{\underline{3/45°} + \underline{2/30°}}{\underline{5/60°} - \underline{4/30°}}$$

The addition and subtraction must be performed in rectangular notation.

We shall simplify the numerator of the given equation first and express this numerator as a single vector in polar form. The x projection of the vector sum of the numerator is

$$
\begin{aligned}
x_n &= 3 \cos 45° + 2 \cos 30° \\
&= 3 \times 0.7071 + 2 \times 0.8660 \\
&= 2.121 + 1.732 = +3.853
\end{aligned}
$$

The y projection of the vector sum of the numerator is

$$\begin{aligned} y_n &= 3 \sin 45° + 2 \sin 30° \\ &= 3 \times 0.7071 + 2 \times 0.5 \\ &= 2.121 + 1.000 = 3.121 \end{aligned}$$

The magnitude of the vector sum of the numerator is

$$\rho_n = \sqrt{3.853^2 + 3.121^2} = 4.958$$

The tangent of the numerator reference angle ϕ_n is

$$\begin{aligned} \tan \phi_n &= \frac{3.121}{3.853} = 0.8100 \\ \phi_n &= 39° \end{aligned}$$

We shall now simplify the denominator of the given equation and express this denominator as a single vector in polar form.

The x projection of the vector sum of the denominator is

$$\begin{aligned} x_d &= 5 \cos 60° - 4 \cos 30° \\ &= 5 \times 0.5000 - 4 \times 0.8660 \\ &= 2.500 - 3.464 = -0.964 \end{aligned}$$

The y projection of the vector sum of the denominator is

$$\begin{aligned} y_d &= 5 \sin 60° - 4 \sin 30° \\ &= 5 \times 0.8660 - 4 \times 0.500 \\ &= 4.330 - 2.000 = +2.330 \end{aligned}$$

The magnitude of the vector sum of the denominator is

$$\rho_d = \sqrt{(-0.964)^2 + 2.330^2} = \sqrt{6.3582} = 2.522$$

The tangent of the denominator reference angle ϕ_d is

$$\begin{aligned} \tan \phi_d &= \frac{2.330}{-0.964} = -2.417 \\ \phi_d &= 180 - 67°32' = 112°28' \end{aligned}$$

The original equation may now be written

$$\underline{\rho/\phi} = \frac{4.958/39°}{2.522/112°28'}$$

Following Eq. (41),

$$\underline{\rho/\phi} = \frac{4.958}{2.522} \underline{/39° - 112°28'} = 1.966\underline{/-73°28'}$$

or

$$\underline{\rho/\phi} = 1.966\underline{/286°32'}$$

EXERCISE 5

Solve the following vector equations for ρ and ϕ. Draw a scale diagram to illustrate each problem.

1. $\rho/\underline{\phi} = 2/\underline{20^\circ} \times 3/\underline{-30^\circ}$
2. $\rho/\underline{\phi} = 4/\underline{300^\circ} \times 5/\underline{60^\circ}$
3. $\rho/\underline{\phi} = 5/\underline{160^\circ} \times 4/\underline{-90^\circ}$
4. $\rho/\underline{\phi} = 3/\underline{225^\circ} \times 2/\underline{40^\circ}$
5. $\rho/\underline{\phi} = 6/\underline{159^\circ} \times 4/\underline{340^\circ}$
6. $\rho/\underline{\phi} = 25/\underline{-120^\circ} \times 50/\underline{120^\circ}$

Find the vector product of each set of vectors given below. Express the product in polar form.

	Vector	*x projection*	*y projection*
7.	$\overrightarrow{OA}$	+3	+2
	$\overrightarrow{OB}$	−2	+7
8.	$\overrightarrow{OA}$	+4	+5
	$\overrightarrow{OB}$	+1	+2
9.	$\overrightarrow{OA}$	−6	−5
	$\overrightarrow{OB}$	−2	−3
10.	$\overrightarrow{OA}$	+5	−1
	$\overrightarrow{OB}$	−3	+2

Solve for ρ and ϕ:

11. $\rho/\underline{\phi} = 6/\underline{75^\circ} \div 2/\underline{15^\circ}$
12. $\rho/\underline{\phi} = 15/\underline{150^\circ} \div 3/\underline{50^\circ}$
13. $\rho/\underline{\phi} = 25/\underline{300^\circ} \div 10/\underline{-20^\circ}$
14. $\rho/\underline{\phi} = 90/\underline{-27^\circ} \div 10/\underline{-50^\circ}$
15. $\rho/\underline{\phi} = -100/\underline{300^\circ} \div 20/\underline{-50^\circ}$

16. Find the polar form of the vector quotient of the vectors $\dfrac{\overrightarrow{OA}}{\overrightarrow{OB}}$ for Prob. 7.

17. Find the polar form of the vector quotient of the vectors $\dfrac{\overrightarrow{OA}}{\overrightarrow{OB}}$ for Prob. 8.

18. Find the polar form of the vector quotient of the vectors $\frac{\vec{OA}}{\vec{OB}}$ for Prob. 9.

19. Find the polar form of the vector quotient of the vectors $\frac{\vec{OA}}{\vec{OB}}$ for Prob. 10.

Solve for ρ and ϕ

20. $\rho\underline{/\phi} = \dfrac{3\underline{/60°} \times 7\underline{/50°}}{3\underline{/20°}}$

21. $\rho\underline{/\phi} = \dfrac{4\underline{/76°} \times 10\underline{/120°}}{3\underline{/40°} + 7\underline{/82°}}$

22. $\rho\underline{/\phi} = \dfrac{6\underline{/200°} \times 3\underline{/-20°}}{5\underline{/45°} + 7\underline{/30°}}$

23. $\rho\underline{/\phi} = \dfrac{10\underline{/20°} \times 20\underline{/50°}}{10\underline{/20°} + 20\underline{/50°}}$

24. $\rho\underline{/\phi} = \dfrac{32\underline{/150°} \times 20\underline{/320°}}{32\underline{/150°} + 20\underline{/320°}}$

25. $\rho\underline{/\phi} = \dfrac{50\underline{/75°} \times 10\underline{/25°}}{50\underline{/75°} + 10\underline{/25°}}$

26. $\rho\underline{/\phi} = \dfrac{30\underline{/60°} \times 30\underline{/30°}}{30\underline{/60°} + 30\underline{/30°}}$

27. $\rho\underline{/\phi} = \dfrac{25\underline{/-30°} \times -50\underline{/120°}}{25\underline{/-30°} + -50\underline{/120°}}$

21·12 Reciprocal Relations. An equation in the form

$$\frac{1}{|Z_t|}\underline{/-\phi_{z_t}} = \frac{1}{|Z_1|}\underline{/-\phi_{z_1}} + \frac{1}{|Z_2|}\underline{/-\phi_{z_2}} \tag{42}$$

is often encountered in the elementary study of parallel alternating-current circuits. Ordinarily the quantities $|Z_1|$, $|Z_2|$, ϕ_{z_1}, and ϕ_{z_2} are given, and $|Z_t|$ and ϕ_{z_t} are to be found.

By definition of vector division,

$$\frac{1\ /0^\circ}{|Z|\ \underline{/\phi_z}} = \frac{1}{|Z|}\ \underline{/0^\circ - \phi_z} = \frac{1}{|Z|}\ \underline{/-\phi_z} \tag{43}$$

Therefore Eq. (42) may be written

$$\frac{1}{|Z_t|\ \underline{/\phi_{z_t}}} = \frac{1}{|Z_1|\ \underline{/\phi_{z_1}}} + \frac{1}{|Z_2|\ \underline{/\phi_{z_2}}} \tag{44}$$

or

$$\frac{1}{\vec{Z}_t} = \frac{1}{\vec{Z}_1} + \frac{1}{\vec{Z}_2} \tag{45}$$

$$\frac{1}{\vec{Z}_t} = \frac{\vec{Z}_1 + \vec{Z}_2}{\vec{Z}_1 \times \vec{Z}_2} \tag{46}$$

Therefore

$$\vec{Z}_t = \frac{\vec{Z}_1 \times \vec{Z}_2}{\vec{Z}_1 + \vec{Z}_2} \tag{47}$$

EXERCISE 6

In the following problems, solve for $|Z_t|$ and ϕ_{z_t} using Eq. (42) or (47) as directed by the instructor:

1.	$\|Z_1\| = 3$	$\phi_{z_1} = 20^\circ$	**2.**	$\|Z_1\| = 10$	$\phi_{z_1} = -50^\circ$
	$\|Z_2\| = 6$	$\phi_{z_2} = 70^\circ$		$\|Z_2\| = 20$	$\phi_{z_2} = +60^\circ$
3.	$\|Z_1\| = 15$	$\phi_{z_1} = 15^\circ$	**4.**	$\|Z_1\| = 7$	$\phi_{z_1} = 45^\circ$
	$\|Z_2\| = 20$	$\phi_{z_2} = 80^\circ$		$\|Z_2\| = 11$	$\phi_{z_2} = -40^\circ$
5.	$\|Z_1\| = 25$	$\phi_{z_1} = 90^\circ$	**6.**	$\|Z_1\| = 5$	$\phi_{z_1} = -16^\circ$
	$\|Z_2\| = 50$	$\phi_{z_2} = -45^\circ$		$\|Z_2\| = 8$	$\phi_{z_2} = -36^\circ$

Chapter 22 TRIGONOMETRIC FORMULAS, IDENTITIES, AND EQUATIONS

In this chapter we shall derive some of the basic trigonometric identities not covered in Sec. 17·25.

We shall also discuss methods of checking the validity of expressions suspected of being identities and methods of solving trigonometric equations.

22·1 Functions of the Sum of Two Angles

(*a*) $\sin(\phi + \theta)$

Referring to Fig. 22·1, where $AB \perp OB$ and $AD \perp OD$,

$$
\begin{aligned}
AB &= OA \sin\theta \\
\angle BAC &= \phi \qquad \text{(See geometric fact 13, Appendix B)} \\
AC &= AB \cos\phi \\
AC &= OA \sin\theta \cos\phi \\
OB &= OA \cos\theta \\
BE &= OB \sin\phi \\
BE &= OA \cos\theta \sin\phi \\
BE &= CD \\
CD &= OA \cos\theta \sin\phi \\
\sin(\phi + \theta) &= \frac{AC + CD}{OA} \\
\sin(\phi + \theta) &= \frac{OA \sin\theta \cos\phi + OA \cos\theta \sin\phi}{OA} \\
\sin(\phi + \theta) &= \sin\phi \cos\theta + \cos\phi \sin\theta \qquad \star(1)
\end{aligned}
$$

(*b*) *cos* $(\phi + \theta)$

Again referring to Fig. 22·1,

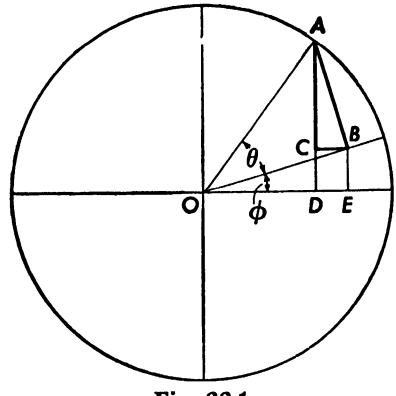


Fig. 22·1

$$OB = OA \cos \theta$$
$$OE = OB \cos \phi$$
$$OE = OA \cos \theta \cos \phi$$
$$AB = OA \sin \theta$$
$$CB = AB \sin \phi$$
$$DE = CB = OA \sin \phi \sin \theta$$
$$\cos (\phi + \theta) = \frac{OE - DE}{OA}$$
$$\cos (\phi + \theta) = \frac{OA \cos \phi \cos \theta - OA \sin \phi \sin \theta}{OA}$$
$$\cos (\phi + \theta) = \cos \phi \cos \theta - \sin \phi \sin \theta \qquad \star (2)$$

(*c*) *tan* $(\phi + \theta)$

Following Eq. (122), page 366,

$$\tan (\phi + \theta) = \frac{\sin (\phi + \theta)}{\cos (\phi + \theta)}$$

Substituting Eqs. (1) and (2) in the above equation,

$$\tan (\phi + \theta) = \frac{\sin \phi \cos \theta + \cos \phi \sin \theta}{\cos \phi \cos \theta - \sin \phi \sin \theta}$$

Dividing numerator and denominator by $\cos \phi \cos \theta$,

$$\tan (\phi + \theta) = \frac{\dfrac{\sin \phi}{\cos \phi} + \dfrac{\sin \theta}{\cos \theta}}{1 - \dfrac{\sin \phi}{\cos \phi} \times \dfrac{\sin \theta}{\cos \theta}}$$

$$\tan (\phi + \theta) = \frac{\tan \phi + \tan \theta}{1 - \tan \phi \tan \theta} \qquad \star (3)$$

(*d*) *cot* $(\phi + \theta)$

Since the cotangent of an angle is the reciprocal of the tangent of that angle,

$$\cot(\phi + \theta) = \frac{1 - \tan\phi \tan\theta}{\tan\phi + \tan\theta}$$

and

$$\cot(\phi + \theta) = \frac{1 - \dfrac{1}{\cot\phi \cot\theta}}{\dfrac{1}{\cot\phi} + \dfrac{1}{\cot\theta}}$$

$$\cot(\phi + \theta) = \frac{\cot\phi \cot\theta - 1}{\cot\phi + \cot\theta} \qquad \star\ (4)$$

22·2 Functions of the Difference between Two Angles.

(*a*) *sin* $(\phi - \theta)$

Referring to Fig. 22·2, where $OA \perp AB$ and $AD \perp OD$,

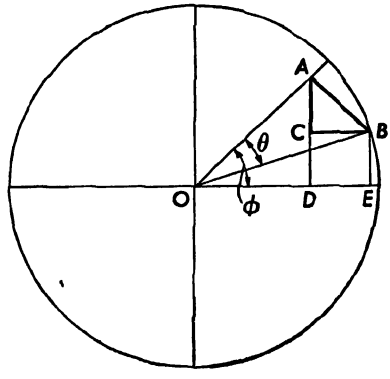


Fig. 22·2

$$AB = OB \sin\theta$$
$$AC = AB \cos\phi$$
$$AC = OB \cos\phi \sin\theta$$
$$AD = OA \sin\phi$$
$$OA = OB \cos\theta$$
$$AD = OB \sin\phi \cos\theta$$

$$\sin(\phi - \theta) = \frac{AD - AC}{OB}$$

$$\sin(\phi - \theta) = \frac{OB \sin\phi \cos\theta - OB \cos\phi \sin\theta}{OB}$$

$$\sin(\phi - \theta) = \sin\phi \cos\theta - \cos\phi \sin\theta \qquad \star\ (5)$$

(*b*) *cos* $(\phi - \theta)$

Again referring to Fig. 22·2,

$$OA = OB \cos \theta$$
$$OD = OA \cos \phi$$
$$OD = OB \cos \phi \cos \theta$$
$$AB = OB \sin \theta$$
$$CB = AB \sin \phi$$
$$CB = OB \sin \phi \sin \theta$$
$$\cos (\phi - \theta) = \frac{OD + CB}{OB}$$
$$\cos (\phi - \theta) = \frac{OB \cos \phi \cos \theta + OB \sin \phi \sin \theta}{OB}$$
$$\cos (\phi - \theta) = \cos \phi \cos \theta + \sin \phi \sin \theta \qquad \star (6)$$

(*c*) *tan* $(\phi - \theta)$

$$\tan (\phi - \theta) = \frac{\sin (\phi - \theta)}{\cos (\phi - \theta)}$$

Substituting Eqs. (5) and (6) in the above equation,

$$\tan (\phi - \theta) = \frac{\sin \phi \cos \theta - \cos \phi \sin \theta}{\cos \phi \cos \theta + \sin \phi \sin \theta}$$

Dividing numerator and denominator by $\cos \phi \cos \theta$ gives

$$\tan (\phi - \theta) = \frac{\dfrac{\sin \phi \cos \theta}{\cos \phi \cos \theta} - \dfrac{\cos \phi \sin \theta}{\cos \phi \cos \theta}}{\dfrac{\cos \phi \cos \theta}{\cos \phi \cos \theta} + \dfrac{\sin \phi \sin \theta}{\cos \phi \cos \theta}}$$
$$= \frac{\dfrac{\sin \phi}{\cos \phi} - \dfrac{\sin \theta}{\cos \theta}}{1 + \dfrac{\sin \phi}{\cos \phi} \times \dfrac{\sin \theta}{\cos \theta}}$$
$$\tan (\phi - \theta) = \frac{\tan \phi - \tan \theta}{1 + \tan \phi \tan \theta} \qquad \star (7)$$

(*d*) *cot* $(\phi - \theta)$

$$\cot (\phi - \theta) = \frac{1}{\tan (\phi - \theta)}$$

Following Eq. (7),

$$\cot(\phi - \theta) = \frac{1 + \tan\phi \tan\theta}{\tan\phi - \tan\theta}$$

$$= \frac{1 + \dfrac{1}{\cot\phi \cot\theta}}{\dfrac{1}{\cot\phi} - \dfrac{1}{\cot\theta}}$$

$$\cot(\phi - \theta) = \frac{\cot\phi \cot\theta + 1}{\cot\theta - \cot\phi} \qquad * \text{ (8)}$$

22·3 The Functions of Twice an Angle

(*a*) *sin* 2ϕ

In Eq. (1) let $\phi = \theta$; then

$$\sin(\phi + \phi) = \sin 2\phi = \sin\phi\cos\phi + \sin\phi\cos\phi$$

$$\sin 2\phi = 2\sin\phi\cos\phi \qquad * \text{ (9)}$$

(*b*) *cos* 2ϕ

In Eq. (2) let $\phi = \theta$; then

$$\cos(\phi + \phi) = \cos 2\phi = \cos\phi\cos\phi - \sin\phi\sin\phi$$

$$\cos 2\phi = \cos^2\phi - \sin^2\phi \qquad * \text{ (10)}$$

(*c*) *tan* 2ϕ

In Eq. (3) let $\phi = \theta$; then

$$\tan(\phi + \phi) = \tan 2\phi = \frac{\tan\phi + \tan\phi}{1 - \tan\phi \times \tan\phi}$$

$$\tan 2\phi = \frac{2\tan\phi}{1 - \tan^2\phi} \qquad * \text{ (11)}$$

(*d*) *cot* 2ϕ

In Eq. (4) let $\phi = \theta$; then

$$\cot(\phi + \phi) = \cot 2\phi = \frac{\cot\phi \cot\phi - 1}{\cot\phi + \cot\phi}$$

$$\cot 2\phi = \frac{\cot^2\phi - 1}{2\cot\phi} \qquad * \text{ (12)}$$

22·4 The Functions of Half an Angle

(*a*) $\sin\frac{\phi}{2}$

Following Eq. (126), page 366, and Eq. (10),

$$\sin^2\frac{\phi}{2} + \cos^2\frac{\phi}{2} = 1 \tag{13}$$

$$-\sin^2\frac{\phi}{2} + \cos^2\frac{\phi}{2} = \cos\phi \tag{14}$$

Solving the above equations by subtraction,

$$2\sin^2\frac{\phi}{2} = 1 - \cos\phi$$

$$\sin^2\frac{\phi}{2} = \frac{1-\cos\phi}{2}$$

$$\sin\frac{\phi}{2} = \pm\sqrt{\frac{1-\cos\phi}{2}} \qquad \star\ (15)$$

(*b*) $\cos\frac{\phi}{2}$

Solving Eqs. (13) and (14) simultaneously by addition gives

$$2\cos^2\frac{\phi}{2} = 1 + \cos\phi$$

$$\cos^2\frac{\phi}{2} = \frac{1+\cos\phi}{2}$$

$$\cos\frac{\phi}{2} = \pm\sqrt{\frac{1+\cos\phi}{2}} \qquad \star\ (16)$$

(*c*) $\tan\frac{\phi}{2}$

$$\tan\frac{\phi}{2} = \frac{\sin\frac{\phi}{2}}{\cos\frac{\phi}{2}}$$

$$\tan\frac{\phi}{2} = \frac{\pm\sqrt{\frac{1-\cos\phi}{2}}}{\pm\sqrt{\frac{1+\cos\phi}{2}}}$$

$$\tan\frac{\phi}{2} = \pm\sqrt{\frac{1-\cos\phi}{1+\cos\phi}} \qquad \star\ (17)$$

(*d*) $\cot\frac{\phi}{2}$

$$\cot\frac{\phi}{2} = \frac{1}{\tan \phi/2}$$

$$\cot\frac{\phi}{2} = \pm\sqrt{\frac{1+\cos\phi}{1-\cos\phi}} \qquad * \text{ (18)}$$

22·5 Trigonometric Identities. A trigonometric equation which is true for all values of the angle is called a trigonometric identity (see discussion of identities and conditional equations on page 82).

In the course of analyzing practical problems mathematically, we sometimes encounter a trigonometric equation that we suspect—without being certain—may be an identity. In such cases it is important to verify the equation and determine whether or not it is an identity.

Before attempting a formal verification, it is often more efficient to see if the suspected identity is true for a special angle. If it is not true for a special angle, it is certainly not true generally, and there would be no point in carrying the investigation any further.

For example, suppose we suspect that the equation

$$\frac{1+\cos\phi}{1-\cos\phi} = \frac{1-\cos\frac{\phi}{2}}{\sin\frac{\phi}{2}}$$

is an identity. Let us substitute $\phi = 60°$. Then

$$\frac{1+\frac{1}{2}}{1-\frac{1}{2}} \stackrel{?}{=} \frac{1-\sqrt{3}/2}{\frac{1}{2}}$$

$$\frac{\frac{3}{2}}{\frac{1}{2}} \stackrel{?}{=} 2-\sqrt{3}$$

$$3 \stackrel{?}{=} 2-\sqrt{3}$$

Obviously the given equation is not true when $\phi = 60°$, therefore it is not an identity. There may, however, be other values which if substituted for ϕ would check.

There are three formal methods by which we may prove an identity. We can substitute the proper ratios for the functions themselves, as was done in Example 6, page 367. Sometimes we can prove a suspected identity by referring to an identity of which we are already sure. Again, we may resort to a geometrical proof, as was done in Sec. 22·1.

When we use algebraic methods in proving an identity, the identity should be verified in the precise form in which it appears. We can alter either or both sides by appropriate substitutions, but it is unwise to transpose any quantities across the equals sign.

EXAMPLE 1. Verify the identity

$$\tan \phi = \frac{2 \tan \frac{\phi}{2}}{1 - \tan^2 \frac{\phi}{2}}$$

First let us see if the identity is true when $\phi = 60°$:

$$\sqrt{3} \stackrel{?}{=} \frac{2 \times \frac{1}{\sqrt{3}}}{1 - \frac{1}{3}}$$

$$\sqrt{3} \stackrel{?}{=} \frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}} = \frac{2}{\sqrt{3}} \times \frac{3}{2} = \frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

We are now sure that the equation is true for at least one angle. Let us proceed with a more general check. A recommended arrangement of work is shown below:

$\tan \phi$	$\dfrac{2 \tan \frac{1}{2}\phi}{1 - \tan^2 \frac{1}{2}\phi}$
	By Eq. (17),
$\dfrac{\sin \phi}{\cos \phi}$	$\dfrac{2\sqrt{\dfrac{1 - \cos \phi}{1 + \cos \phi}}}{1 - \dfrac{1 - \cos \phi}{1 + \cos \phi}}$
	Rationalizing the numerator and simplifying the denominator,
	$\dfrac{2\sqrt{\dfrac{(1 - \cos \phi)(1 + \cos \phi)}{(1 + \cos \phi)^2}}}{\dfrac{1 + \cos \phi - 1 + \cos \phi}{1 + \cos \phi}}$
	$\dfrac{2\sqrt{1 - \cos^2 \phi}}{1 + \cos \phi} \times \dfrac{1 + \cos \phi}{2 \cos \phi}$
	By Eq. (126), page 366,
	$\dfrac{2 \sin \phi}{1 + \cos \phi} \times \dfrac{1 + \cos \phi}{2 \cos \phi}$
	$\dfrac{\sin \phi}{\cos \phi}$

EXAMPLE 2. Verify the identity

$$\sqrt{\frac{1-\sin\beta}{1+\sin\beta}} = \frac{1-\sin\beta}{\cos\beta}$$

$\sqrt{\frac{1-\sin\beta}{1+\sin\beta}}$	$\frac{1-\sin\beta}{\cos\beta}$
Rationalize by multiplying numerator and denominator by $1-\sin\beta$:	
$\sqrt{\frac{(1-\sin\beta)(1-\sin\beta)}{(1+\sin\beta)(1-\sin\beta)}}$	
$\sqrt{\frac{(1-\sin\beta)^2}{1-\sin^2\beta}}$	
$\frac{1-\sin\beta}{\cos\beta}$ ←	

EXAMPLE 3. Verify the identity

$$\sqrt{\frac{1-\sin\omega}{1+\sin\omega}} = \frac{\cos\omega}{1+\sin\omega}$$

$\sqrt{\frac{1-\sin\omega}{1+\sin\omega}}$	$\frac{\cos\omega}{1+\sin\omega}$
Rationalize by multiplying numerator and denominator by $1+\sin\omega$:	
$\sqrt{\frac{(1-\sin\omega)(1+\sin\omega)}{(1+\sin\omega)^2}}$	
$\sqrt{\frac{1-\sin^2\omega}{(1+\sin\omega)^2}}$	
$\frac{\cos\omega}{1+\sin\omega}$ ←	

EXERCISE 1

Verify the following suspected identities. Some of the "equations" given in this exercise may be incorrect. It is left to the student to determine which are true and which are not.

1. $\dfrac{1+\cot\phi\cot\theta}{1+\tan\phi\tan\theta} = \dfrac{1+\cos\phi\cos\theta}{1+\sin\phi\sin\theta}$

2. $1-\sin\phi = \dfrac{\cos^2\phi}{1+\sin\phi}$

3. $\tan\phi = \dfrac{\sin\phi+\tan\phi}{\sin\phi\cos\phi+1}$

4. $\sin^2 \phi + \operatorname{cosec}^2 \phi = \dfrac{1 + \sin^4 \phi}{\sin^2 \phi}$

5. $\dfrac{\sin \phi}{1 - \cos \phi} + \dfrac{\sin \phi}{1 + \cos \phi} = 2 \cot \phi$

6. $\sec \phi \operatorname{cosec} \phi = \dfrac{\tan \phi - \cot \phi}{\sin^2 \phi - \cos^2 \phi}$

7. $\sin \phi = \dfrac{2 \tan \frac{1}{2}\phi}{1 + \tan^2 \frac{1}{2}\phi}$

8. $\sin 2\phi = \dfrac{2 \tan \phi}{1 + \tan^2 \phi}$

9. $\sec \phi = \tan \phi \sin \phi + \sin \phi$

10. $\dfrac{\tan^2 \phi - \sin^2 \phi}{\tan^2 \phi} = \dfrac{\sec^2 \phi - 1}{1 + \tan^2 \phi}$

11. $4 \sin^3 \phi + \sin 3\phi = 3 \sin \phi$

12. $\dfrac{\cot \phi + \sin \phi}{\cos \phi \cot \phi} = \tan \phi + \sec \phi$

13. $\sin \phi = \dfrac{\cos \phi - \sin \phi \tan \phi}{2 \cot 2\phi}$

22·6 Trigonometric Equations. A conditional trigonometric equation or simply a trigonometric equation is an equation which is valid for some but not all of the possible values of the angles involved.

A solution of a trigonometric equation is a value of the angle, in degrees or radians, which satisfies the equation.

The three suggestions given below will take care of most cases but not all. Unfortunately no rules exist which will apply universally.

First. If only one function is involved, the equation can usually be solved algebraically (see Example 4).

Second. If several functions of the same angle are involved, it is best to use some of the fundamental identities to express the entire equation in terms of a single function (see Example 5).

Third. If several angles are involved, it is usually best to express the entire equation in terms of functions of a single angle (see Example 6).

EXAMPLE 4. Solve the equation

$$15 \cos^2 \phi - 7 \cos \phi - 2 = 0$$

This equation can be factored into

$$(3 \cos \phi - 2)(5 \cos \phi + 1) = 0$$

Therefore

$$3 \cos \phi = 2$$
$$\cos \phi = \tfrac{2}{3}$$

and

$$5 \cos \phi = -1$$
$$\cos \phi = -\tfrac{1}{5}$$

The quadratic formula could also have been used and

$$\cos \phi = \frac{7 \pm \sqrt{49 + 120}}{30} = \frac{7 \pm \sqrt{169}}{30} = \frac{7 \pm 13}{30}$$
$$\cos \phi = \tfrac{2}{3} \text{ and } -\tfrac{1}{5}$$
$$\cos \phi = \tfrac{2}{3}$$
$$\phi = 48°11' \text{ (to the nearest minute)}$$

or

$$\phi = 311°49' \text{ (to the nearest minute)}$$

when

$$\cos \phi = -\tfrac{1}{5}$$
$$\phi = 101°32'$$

or

$$\phi = 258°28'$$

To complete the problem the results should be substituted in the original equation to eliminate any extraneous roots which may have been introduced in the process of solving.

Substituting $\cos \phi = \tfrac{2}{3}$,

$$15 \times \tfrac{4}{9} - 7 \times \tfrac{2}{3} - 2 \stackrel{?}{=} 0$$
$$\tfrac{60}{9} - \tfrac{14}{3} - 2$$
$$\tfrac{60}{9} - \tfrac{42}{9} - 2$$
$$\tfrac{18}{9} - 2$$
$$2 - 2 = 0 \qquad \text{(check)}$$

Substituting $\cos \phi = -\tfrac{1}{5}$

$$15 \times \tfrac{1}{25} + 7 \times \tfrac{1}{5} - 2 \stackrel{?}{=} 0$$
$$\tfrac{3}{5} + \tfrac{7}{5} - 2 = 0$$
$$\tfrac{10}{5} - 2 = 0 \qquad \text{(check)}$$

EXAMPLE 5. Solve the equation

$$2 \sin \phi = \cos \phi$$

Since

$$\cos \phi = \sqrt{1 - \sin^2 \phi}$$
$$2 \sin \phi = \sqrt{1 - \sin^2 \phi}$$

Squaring both sides,

$$4 \sin^2 \phi = 1 - \sin^2 \phi$$

or

$$5 \sin^2 \phi = 1$$
$$\sin^2 \phi = \tfrac{1}{5}$$
$$\sin \phi = \pm \frac{1}{\sqrt{5}} = \pm \frac{\sqrt{5}}{5} = \frac{2.236068}{5} = \pm 0.44721$$
$$\phi = 26°34' \text{ and } 206°34'$$

EXAMPLE 6. Solve the equation

$$\sin 2\phi = \tan \phi$$
$$2 \sin \phi \cos \phi = \frac{\sin \phi}{\cos \phi}$$

Multiply both sides by $\cos \phi$:

$$2 \sin \phi \cos^2 \phi = \sin \phi$$
$$2 \sin \phi \cos^2 \phi - \sin \phi = 0$$
$$\sin \phi(2 \cos^2 \phi - 1) = 0$$
$$\sin \phi = 0$$
$$2 \cos^2 \phi = 1$$
$$\cos^2 \phi = \tfrac{1}{2}$$
$$\cos \phi = \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$$

When $\sin \phi = 0$,

$\phi = 0°$ and $180°$

When $\cos \phi = + \frac{\sqrt{2}}{2}$,

$\phi = 45°$ and $315°$

When $\cos \phi = - \frac{\sqrt{2}}{2}$,

$\phi = 135°$ and $225°$

Checking in the original problem:

When $\phi = 0$,

$$\sin (2 \times 0) \stackrel{?}{=} \tan 0$$
$$0 = 0 \qquad \text{(check)}$$

when $\phi = 180$,

$$\sin (2 \times 180) \stackrel{?}{=} \tan 180$$
$$0 = 0 \qquad \text{(check)}$$

When $\phi = 45$,

$$\sin (2 \times 45) \stackrel{?}{=} \tan 45$$
$$\sin 90 \stackrel{?}{=} \tan 45$$
$$1 = 1 \qquad \text{(check)}$$

When $\phi = 315$,

$$\begin{aligned} \sin (2 \times 315) &\stackrel{?}{=} \tan 315 \\ \sin 630 &\stackrel{?}{=} \tan 315 \\ \sin 270 &= \tan 315 \\ -1 &= -1 \qquad \text{(check)} \end{aligned}$$

When $\phi = 135$,

$$\begin{aligned} \sin (2 \times 135) &\stackrel{?}{=} \tan 135 \\ \sin 270 &\stackrel{?}{=} \tan 135 \\ -1 &= -1 \qquad \text{(check)} \end{aligned}$$

When $\phi = 225$,

$$\begin{aligned} \sin (2 \times 225) &\stackrel{?}{=} \tan 225 \\ \sin 450 &\stackrel{?}{=} \tan 225 \\ \sin 90 &= \tan 225 \\ +1 &= +1 \qquad \text{(check)} \end{aligned}$$

EXERCISE 2

Find the solutions to each of the following equations for which solutions exist.

1. $\sin \phi = \cos 2\phi$

2. $\sin \phi = \dfrac{\tan \phi}{2}$

3. $\sin \phi = \dfrac{\sec \phi}{2}$

4. $\cos \phi - \cos 2\phi = 1$

5. $\tan \phi + \cot \phi = 1$

6. $\cos^2 \phi + \cos 2\phi = 1$

7. $\dfrac{\tan \phi}{\sin \phi} = 5$

8. $\cos^2 (\phi - 30°) = 1$

9. $3 \operatorname{cosec}^2 \phi = 9$

Chapter 23 GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

In this chapter we shall graph each of the six trigonometric functions. By observing the nature of the graphs, we can deduce many of the properties of the functions themselves.

In addition to such theoretical uses, the graphs of the functions, particularly those of the sine and cosine, are very convenient devices to use in the analysis and solution of certain engineering problems.

Figure 23·1 shows the graphs of the six trigonometric equations:

$$y = \sin \phi \tag{1}$$
$$y = \cos \phi \tag{2}$$
$$y = \tan \phi \tag{3}$$
$$y = \cot \phi \tag{4}$$
$$y = \sec \phi \tag{5}$$
$$y = \operatorname{cosec} \phi \tag{6}$$

While it is true that the choice of scale is to a certain extent arbitrary, we do obtain a somewhat more characteristic shape of the graphs if we use the same scale on both axes and plot the angles in radians rather than in degrees. Figure 23·1 is plotted in this way.

However the theoretical advantage gained by using radian measure rather than degree measure does not ordinarily outweigh the disadvantage of using the irrational number π.

23·1 Some Periodic Functions. One of the most important characteristics of the trigonometric functions is their repetitive nature.

A study of Fig. 23·1 will show that as the angle increases, the six functions undergo certain characteristic changes. As the angle indefinitely increases, these changes repeat periodically. The trigonometric functions are therefore called *periodic functions*. One *cycle* has elapsed when a function has traversed once all the variations of which that particular function is capable.

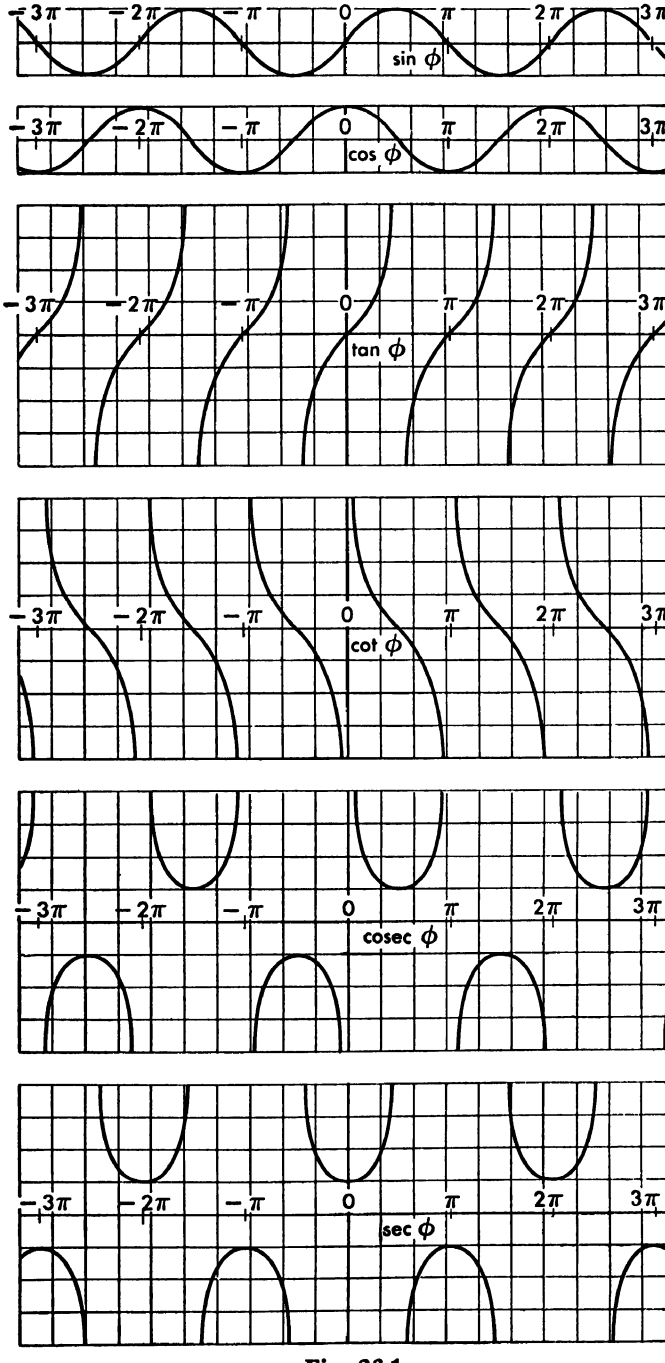

−3π −2π −π 0 π 2π 3π
sin φ
−3π −2π −π 0 π 2π 3π
cos φ
−3π −2π −π 0 π 2π 3π
tan φ
−3π −2π −π 0 π 2π 3π
cot φ
−3π −2π −π 0 π 2π 3π
cosec φ
−3π −2π −π 0 π 2π 3π
sec φ

Fig. 23·1

The *angular period* of any of the trigonometric functions is the angular interval required for the completion of one cycle.

Beginning at $\phi = 0$, let us trace the characteristic changes of the sine function. From $\phi = 0$ to $\phi = \pi/2$ radians, the sine increases from 0 to 1; from $\pi/2$ radians to π radians, the sine decreases from 1 to 0; from π radians to $3\pi/2$ radians, the sine decreases from 0 to -1; from $3\pi/2$ radians to 2π radians, the sine increases from -1 to 0. As ϕ begins to increase beyond 2π radians, the cycle begins to repeat. Thus the angular period of the sine function is 2π radians, or 360°.

The period of the tangent function is π radians. Over the interval $\phi = -\pi/2$ to $\phi = 0$, the tangent increases from $-\infty$ to 0 (see Secs. 17·23 and 17·15). As the angle increases from 0 to $+\pi/2$ radians, the tangent increases from 0 to $+\infty$. If ϕ increases beyond $\pi/2$ radians, the tangent again traverses the complete array of values from $-\infty$ to $+\infty$, repeating the variation which took place in the previous angular period of π radians.

EXERCISE 1

1. In the equation $y = 5 \sin \beta$, find y from the tables when β is 10°, 50°, 160°, 250°, and 300°.
2. Plot the curve of the equation $y = \sin \phi$ between $\phi = 0$ and $\phi = 360°$.
3. Plot the curve of the equation $y = \cos \phi$ between $\phi = 0$ and $\phi = 360°$.
4. Plot the curve of the equation $y = \tan \phi$ between $\phi = 0$ and $\phi = 360°$.
5. Plot the curve of the equation $y = \cot \phi$ between $\phi = 0$ and $\phi = 360°$.
6. Plot the curve of the equation $y = \sec \phi$ between $\phi = 0$ and $\phi = 360°$.
7. Plot the curve of the equation $y = \operatorname{cosec} \phi$ between $\phi = 0$ and $\phi = 360°$.
8. On the same axes plot the curves of $y = \sin \beta$ and $y = \tan \beta$. Use radian measure rather than degree measure, and plot over the interval $\beta = 0$ radians to $\beta = 0.02$ radian. What is the approximate slope of the graphs in this region? What would you expect the slope to be when $\beta = 0$? For any given angle in this region, which is greater, the sine or the tangent?

23·2 Sine and Cosine Graphs. For our immediate purpose, the graphs of the sine and cosine are by far the most important.

Since the period of a sine or cosine curve is 2π radians or 360°, we

usually plot only in the interval between $\phi = 0$ and $\phi = 2\pi$ radians. All the characteristic changes of these functions appear in this region.

Figure 23·2 shows a plot of the equations

$$y = \sin \phi \tag{7}$$
$$y = \cos \phi \tag{8}$$

The horizontal axis is calibrated in degrees rather than in radians. A comparison between the shape of the graphs of these functions in

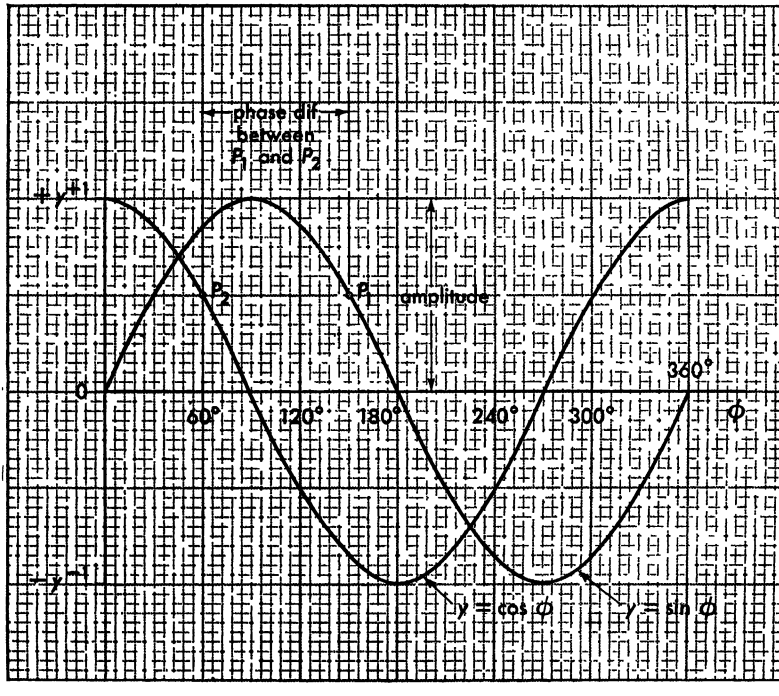


Fig. 23·2

Figs. 23·1 and 23·2 will show the practical equivalence of the two plots.

Notice that the sine and cosine curves in Fig. 23·2 have exactly the same shape, but the cosine curve is displaced 90° to the left with respect to the sine curve.

The *amplitude* of a sine or cosine curve is the absolute value of the maximum and minimum ordinates. In Fig. 23·2 the amplitude of both curves is 1.

The *phase* of a point on the graph is defined as the abscissa of that point, and it may be measured in degrees or radians.

The phase of point P_1 on the sine curve is 150°.

The phase of the corresponding point P_2 on the cosine curve is 60°.

The *phase difference* between two points on a sine or cosine curve is the difference between their abscissas measured in degrees or radians. The term *phase difference* may be applied to points on different sine or cosine curves, provided both curves are referred to the same zero reference angle.

The phase difference between P_1 and P_2 in Fig. 23·2 is $150° - 60° = 90°$. The same phase difference exists between any other pair of corresponding points. In fact, all the characteristic changes in the cosine functions of ϕ occur at angles differing by 90° from the corresponding changes in the sine functions of ϕ. Therefore, a cosine function of ϕ is said to have a 90° phase difference when compared to a sine function of ϕ. Since, in scanning the curve from left to right in the direction of increasing angles, we observe points on the cosine curve before we observe the corresponding points on the sine curve, we say that the cosine function *leads* the sine function.

The above facts may be written in the form of equations:

$$\cos \phi = \sin (\phi + 90°) \tag{9}$$
$$\sin \phi = \cos (\phi - 90°) \tag{10}$$

The equations for the sine and cosine functions may be generalized as

$$y = k \sin (\phi + \psi) \tag{11}$$

and

$$y = k \cos (\phi + \psi) \tag{12}$$

where k and ψ are constants.

When $(\phi + \psi) = 90°$ in Eq. (11), the ordinate of the sine curve is a maximum and equal to k. When $(\phi + \psi) = 0°$ in Eq. (12), the ordinate of the cosine curve is a maximum and also equal to k. The factor k is called the *amplitude* of the curve and is numerically equal to the maximum value of the function.

The constant angle ψ is known as the *phase constant* and has the effect of shifting the graphs of the equations

$$y = k \sin \phi$$

or

$$y = k \cos \phi$$

to the right or left along the ϕ axis. The curve is shifted to the left when ψ is positive and to the right when ψ is negative. The effect of k and ψ is illustrated in Fig. 23·3.

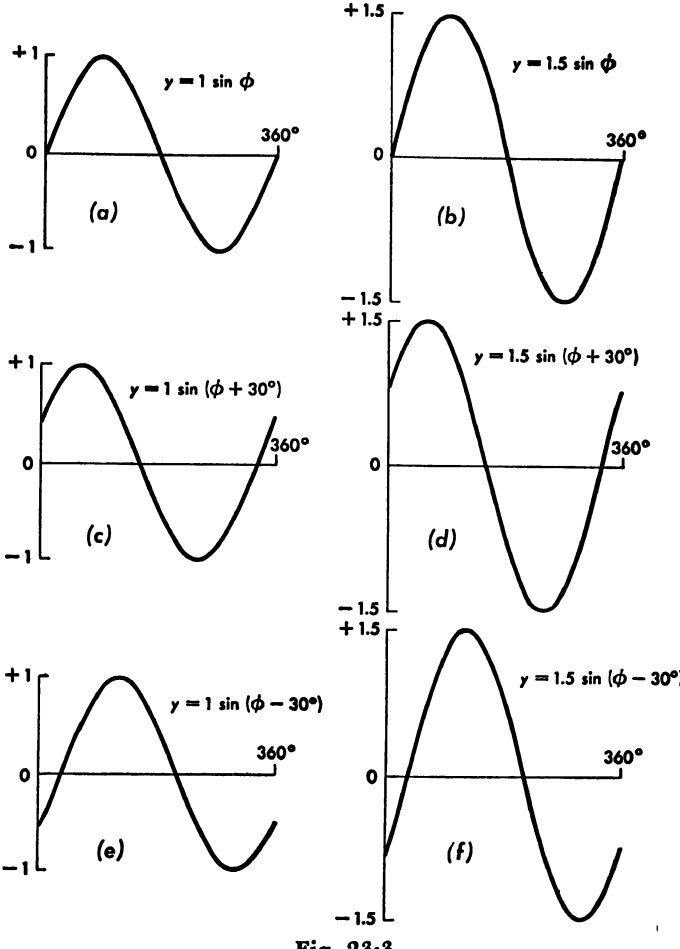


Fig. 23·3

EXERCISE 2

1. The amplitude of a sine curve is 25. What is the ordinate of the sine curve at the following positions in its cycle?
   a. 20° **b.** 90° **c.** 130° **d.** 180°
   e. 210° **f.** 270° **g.** 300° **h.** 360°
2. Write an equation for a sine function in which the amplitude is 23 and the phase difference is +56°.
3. Plot the following curves on the same axes:
   a. $y = \sin \phi$
   b. $y = \sin (\phi - 90°)$

c. $y = \sin(\phi - 60°)$
d. $y = \sin(\phi - 30°)$
e. $y = \sin(\phi + 30°)$
f. $y = \sin(\phi + 60°)$
g. $y = \sin(\phi + 90°)$

4. Plot the following curves on the same axes:
a. $y = \cos \phi$
b. $y = \cos(\phi - 90°)$
c. $y = \cos(\phi - 60°)$
d. $y = \cos(\phi - 30°)$
e. $y = \cos(\phi + 30°)$
f. $y = \cos(\phi + 60°)$
g. $y = \cos(\phi + 90°)$

5. The ordinate of a sine curve is +25 when 100° of its cycle has been completed. Find the amplitude.

6. The ordinate of a sine curve is 15 when 80° of its cycle has been completed. What is the ordinate when 300° has been completed?

7. At what angles less than 360° will the ordinate of a cosine curve be 70 per cent of its amplitude?

8. The amplitude of a sine curve is 15. Find the ordinate at a phase 20° after its minimum.

9. What is the phase difference between the curves of the functions $y = 3 \sin \phi$ and $y' = 7 \sin(\phi + 40°)$?

10. Find the phase difference between the sine curve $y = \sin \phi$ and a sine curve whose amplitude is 100 and whose ordinate is 75 when 50° of the cycle is completed. (Two answers.)

11. One sine curve has an amplitude of 25 and an ordinate of 18 when 30° of its cycle is completed. A second sine curve has an amplitude of 50 and an ordinate of 25 when 40° of its cycle has been completed. Find the phase difference between these two sine curves. (Four answers.)

23·3 Average Ordinates of the Curves of Sin ϕ and Sin^2 ϕ. Figure 23·4 shows the plot of two curves on the same axes. The equations of these curves are

$$y = \sin \phi \tag{13}$$

and

$$z = y^2 = \sin^2 \phi \tag{14}$$

The mean or average ordinates of these curves can be calculated by the method discussed on Appendix B, Sec. B·6. Notice that the same scale is used on both axes and that the ϕ axis is calibrated in radians. While it is not entirely necessary to employ this particular scale, it does tend to emphasize some important theoretical aspects.

Between $\phi = 0$ and $\phi = \pi$ radians it was determined by laborious counting that there are 804 square divisions under the plot of Eq. (13). From the scale of the drawing, each square division is $\frac{1}{400}$ of a square unit. Thus the area under this curve appears to be 2.01 square units. By the more precise methods of calculus this area is found to be exactly 2 square units. The average ordinate of the graph of Eq. (13) is therefore

$$\bar{y} = 2/\pi = 0.637 \text{ (approx)} \tag{15}$$

where $\bar{y}$ means "average value of y."

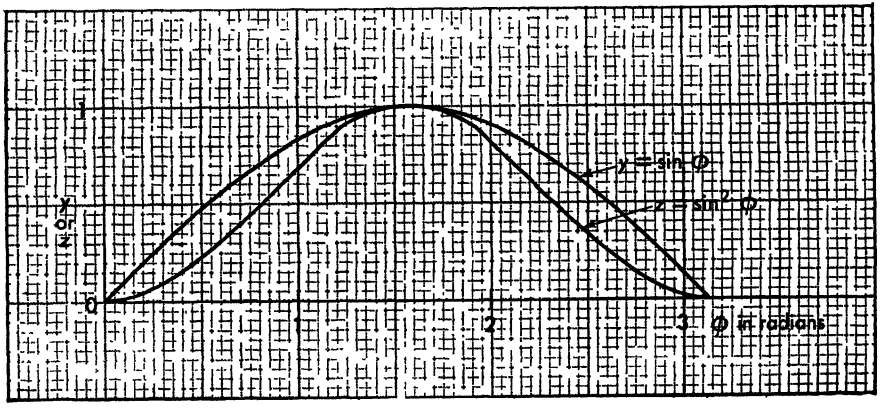


Fig. 23·4

If the amplitude had been something other than unity, say in general k, Eq. (13) would be written

$$y = k \sin \phi \tag{16}$$

and the average ordinate would be

$$\bar{y} = 2/\pi \times k \tag{17}$$

Since k is the maximum ordinate of Eq. (16) and since $\bar{y}$ signifies the average ordinate, Eq. (17) is often written

$$y_{av} = \frac{2}{\pi} y_{max} \tag{18}$$

The plot of the curve of Eq. (14) is also shown in Fig. 23·4. The area under this curve is found, by counting, to be 624 square divisions. This corresponds to 1.56 square units. By the more precise methods

of calculus this area is found to be exactly $\pi/2$ square units. The mean ordinate ($\bar{z}$) of the graph of Eq. (14) is therefore

$$\bar{z} = \frac{\pi/2}{\pi} = \frac{1}{2} \tag{19}$$

For some applications we are particularly interested in $\sqrt{\bar{z}}$, which is $\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$; that is

$$\sqrt{\bar{z}} = \frac{\sqrt{2}}{2} = 0.707 \text{ (approx)} \tag{20}$$

When

$$y = k \sin \phi$$

and

$$z = y^2$$
$$z = k^2 \sin^2 \phi \tag{21}$$

and

$$\bar{z} = \tfrac{1}{2}k^2$$

Then

$$\sqrt{\bar{z}} = \frac{\sqrt{2}}{2} k \tag{22}$$

Since $\sqrt{\bar{z}}$ signifies the square root of the mean of the squared ordinates of the function $k \sin \phi$, this $\sqrt{\bar{z}}$ is sometimes called the *root mean square* of $k \sin \phi$. This value is also called the *effective* value of the function $y = k \sin \phi$. Remembering that k is the maximum ordinate of the curve plotted from Eq. (16), it is common to write Eq. (22) as

$$y_{\text{eff}} = \frac{\sqrt{2}}{2} y_{\max} \tag{23}$$

EXERCISE 3

1. By graphical means similar to those described in Sec. 23·3, find the average ordinate of the cosine curve whose equation is $y = 5 \cos \phi$.
2. Find by graphical methods the effective value of the sine curve whose equation is $y = 5 \sin \phi$.
3. What is the average value of a sine curve whose amplitude is 45?
4. What is the effective value of a sine curve whose amplitude is 45?
5. If the average ordinate of a sine curve is 20, find the amplitude.
6. If the effective ordinate of a sine curve is 25, find the amplitude.

7. If the effective value of a cosine curve is 35, find the average ordinate.
8. If the average ordinate of a sine curve is 50, find the effective value.
9. If the effective value of a sine curve is 84, find the ordinate when 230° of its cycle has been completed.
10. How do the effective values of the following curves compare?

$$y = 10 \sin (\phi + 20°)$$
$$y_1 = 10 \sin (\phi + 90°)$$
$$y_2 = 10 \sin (\phi - 20°)$$
$$y_3 = 10 \cos \phi$$
$$y_4 = 10 \cos (\phi + 60°)$$

23·4 Plotting the Sine and Cosine Functions by Geometrical Methods. Thus far we have plotted the sine and cosine curves by the use of tables. There is a geometrical method which is extremely useful in the analysis of certain problems. This method is illustrated in Fig. 23·5. First a base line ox of arbitrary length is established.

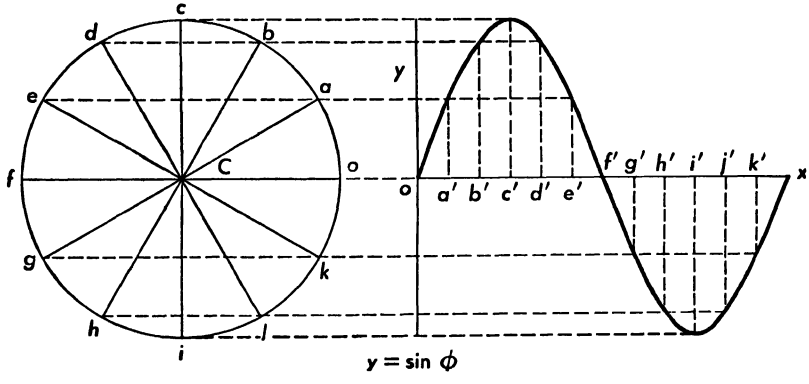


Fig. 23·5

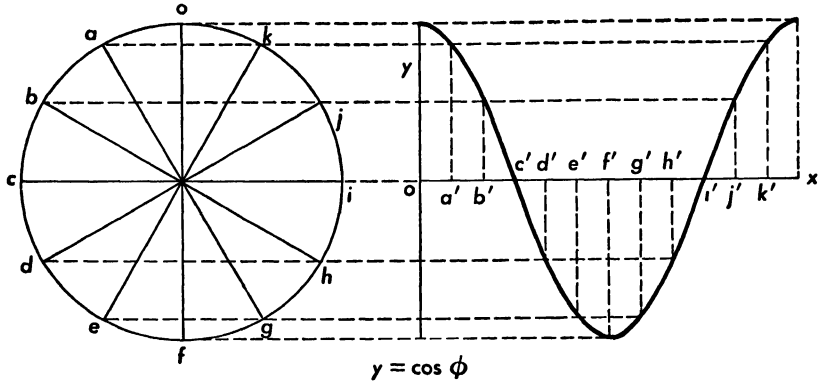
Then a circle is drawn with its center (C) on an extension of ox. The radius of this circle is the amplitude of the sine curve to be plotted. The length ox is to scale numerically equal to the angular measure of one period.

The circle is divided into a convenient number of equal arcs, and the line ox is calibrated with the same number of equally spaced angle markers.

The points a, b, c, d, etc., are projected from the circle parallel to ox. The points a', b', c', d', etc., are projected vertically from the ox axis. Points on the sine curve are found at the intersection of corresponding projections.

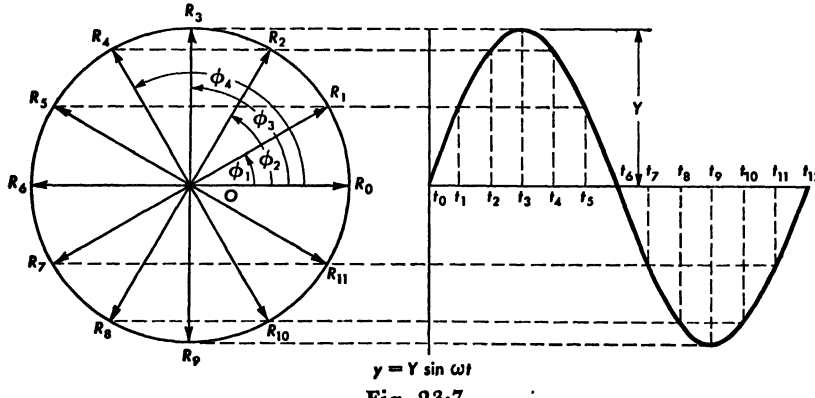
Figure 23·6 is a similar construction of a cosine curve. Note that the reference point o on the circle has been advanced 90° counterclockwise.

23·5 Rotating Vectors. The geometrical method of plotting the curves illustrated in Figs. 23·5 and 23·6 can be extended into a much more practical and useful form.



$y = \cos \phi$

Fig. 23·6



$y = Y \sin \omega t$

Fig. 23·7

Instead of considering the radii of the circle to be stationary lines, we shall consider them to represent successive, momentary positions of a rotating vector (see Fig. 23·7). Here we have a diagram similar to Figs. 23·5 and 23·6. However, very important differences do exist. The vectors $\overrightarrow{OR_0}$, $\overrightarrow{OR_1}$, $\overrightarrow{OR_2}$, etc., represent successive positions of the rotating vector $\overrightarrow{OR}$ at times t_0, t_1, t_2, etc. The horizontal axis of the sine curve is calibrated with equally spaced *time* markers rather than with equally spaced angle markers. The ordinate of the sine curve at

t_0, t_1, t_2, t_3, etc., is the momentary vertical projection of the vector $\overrightarrow{OR}$ at that time.

During each revolution the vector $\overrightarrow{OR}$ rotates through 360°, or 2π radians. The sequence of events which takes place during one revolution of the vector is called a *cycle*.

The number of cycles taking place in 1 sec is called the *frequency*. The time in seconds required for the completion of 1 cycle is called the *time period* or simply the *period*.

From these definitions the following relations appear:

$$\phi = 360° \times f \times t \qquad \text{(in degrees)} \tag{24}$$

$$\phi = 2\pi \times f \times t \qquad \text{(in radians)} \tag{25}$$

$$T = \frac{1}{f} \tag{26}$$

where ϕ = momentary reference angle of the rotating vector, referred to the position of the rotating vector when we choose to begin to count time

f = frequency in cycles per second

t = time in seconds during which the vector has been rotating from some arbitrary zero time-reference position

T = period in seconds

The zero reference position R_0 can be rotated either clockwise or counterclockwise, as shown in Fig. 23·8. In such cases we introduce the

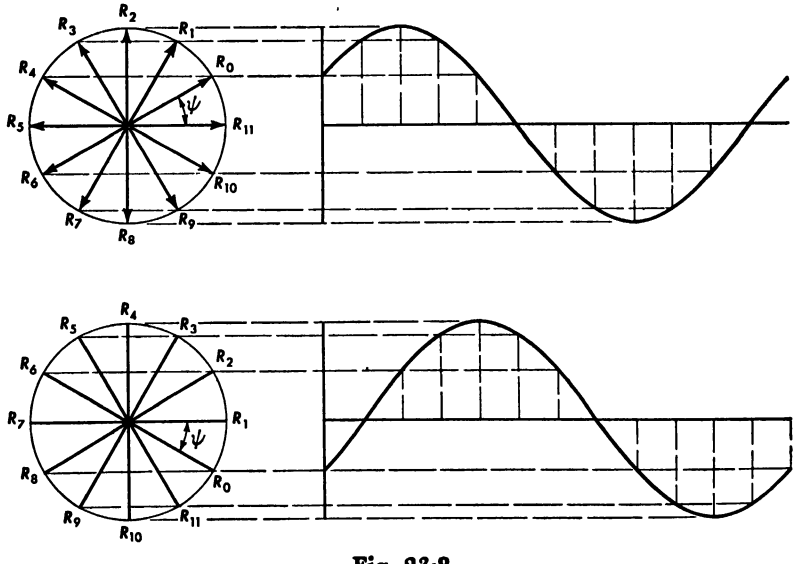


Fig. 23·8

phase constant ψ. The momentary reference angle of the rotating vector now becomes $(\phi - \psi)$ or $(\phi + \psi)$, according as R_0 has been rotated clockwise or counterclockwise.

The general equation for the ordinate of the sine curve as a function of time is

$$y = Y \sin (2\pi ft \pm \psi) \tag{27}$$

or

$$y = Y \sin (360°ft \pm \psi) \tag{28}$$

The term $2\pi f$ or $360°f$ is in the nature of angular velocity.

Ordinarily we let

$$\omega = 2\pi f \text{ (when measuring } \omega \text{ in radians per second)} \tag{29}$$

or

$$\omega = 360°f \text{ (when measuring } \omega \text{ in degrees per second)} \tag{30}$$

Equation (27) then becomes

$$y = Y \sin (\omega t \pm \psi) \tag{31}$$

where Y = a constant amplitude
ω = the angular velocity in either radians or degrees per second
t = the time in seconds
ψ = a phase constant measured in the same angular units as ωt
$(\omega t \pm \psi)$ = the momentary reference angle of the rotating vector referred to the conventional reference position of a system of rectangular coordinates

As a matter of expediency, we usually plot y against the angle ωt rather than against t itself. However, if after plotting y against ωt it seems worth while to show the plot of y against t, the horizontal scale can be altered to read directly in time units.

As an example, let

$$\omega = 8 \text{ radians per sec}$$
$$2\pi f = 8$$
$$f = \frac{8}{2\pi}$$
$$T = \frac{2\pi}{8}$$

Evidently it takes $2\pi/8$ sec for one complete cycle. The distance ox (Fig. 23·9) is to scale $2\pi/8$ sec or about 0.785 sec. If the distance ox is, on the drawing, say 10.2 in., then 1 sec corresponds to $\frac{10.2}{0.785} =$ 13 in.

Suppose we wish to mark on the *ox* scale each tenth of a second (see Fig. 23·9). Extend the line *ox* to *a*, making *oa* = 13 in. Draw the line *ob* at any convenient angle with *oa*. With dividers mark off 10 equal spaces of any convenient length on *ob*.

Draw the line *ba*, and from each division on *ob* project back to line *ox* parallel to line *ba*. The intersections with *ox* will now be to scale 0.1 sec apart.

The terms *amplitude*, *frequency*, *angular velocity*, *period*, and *phase constant* have been defined. Now let us interpret these terms in relation to specific quantities appearing in the general equation for a sine function.

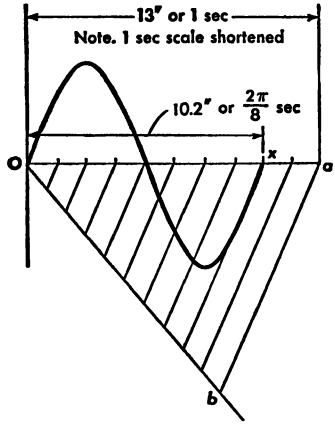


Fig. 23·9

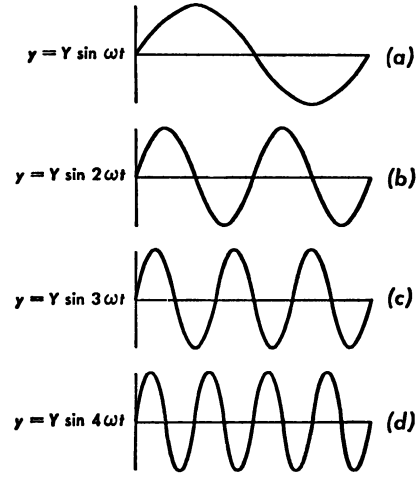


Fig. 23·10

With reference to Eq. (31) the constant Y is the amplitude. A change in Y alters each ordinate of the sine curve but does not destroy the characteristic shape of the sine curve. A comparison of Figs. 23·3*a*, *c*, and *e* with Figs. 23·3*b*, *d*, and *f*, respectively, should illustrate this.

The phase constant ψ has no effect on the amplitude or the shape of the curve but does have the effect of shifting the whole curve to the right or left along the time axis. A comparison among Figs. 23·3*a*, *c*, and *e* should illustrate this fact.

The constant ω or $(2\pi f)$ has no effect on either the amplitude or the relative position of the curve along the time axis. The constant ω or $(2\pi f)$ determines the number of cycles occurring in a given time interval. Figure 23·10 shows four curves all having the same amplitude and phase constant. In this case the phase constant is zero. The ω in curve *a* has been replaced by 2ω, 3ω, and 4ω in curves *b*, *c*, and *d*,

respectively. This means that, referred to curve a, curve b has twice the frequency, curve c has 3 times the frequency, and curve d has 4 times the frequency.

When the frequency of one sine function occurs as an integral multiple of the frequency of another sine function, these two functions are said to be *harmonically* related.

Referred to Fig. 23·10, the function $y = \sin \omega t$ is called the *fundamental* frequency. The function $y = \sin 2\omega t$ is called the *second harmonic* of the fundamental frequency.

The third and fourth harmonics of the function $y = \sin \omega t$ are illustrated in Figs. 23·10c and d.

EXERCISE 4

1. On the same axes plot the following equations. Use y as the ordinate and ωt measured in degrees as the abscissa. Plot over an interval of one cycle.
   a. $y = \sin \omega t$
   b. $y = 2 \sin \omega t$
   c. $y = 3 \sin \omega t$
   d. $y = 4 \sin \omega t$
2. Plot the following equations as instructed in Prob. 1.
   a. $y = \cos \omega t$
   b. $y = 2 \cos \omega t$
   c. $y = 3 \cos \omega t$
   d. $y = 4 \cos \omega t$
3. Plot the following equations as instructed in Prob. 1.
   a. $y = \sin \omega t$
   b. $y = \sin 2\omega t$
   c. $y = \sin 3\omega t$
   d. $y = \sin 4\omega t$
4. Plot the following curves as instructed in Prob. 1.
   a. $y = \cos \omega t$
   b. $y = \cos 2\omega t$
   c. $y = \cos 3\omega t$
   d. $y = \cos 4\omega t$
5. Plot as instructed in Prob. 1.
   a. $y = \sin (\omega t + 30°)$
   b. $y = 3 \sin (\omega t + 60°)$
   c. $y = 3 \sin \omega t$
   d. $y = \sin (\omega t - 30°)$
   e. $y = \sin (\omega t - 60°)$
6. In the following equations find the amplitude, frequency, period, angular velocity, and phase constant. Express angular velocity in

both radians per second and degrees per second; also express the phase constant in both radians and degrees. Note carefully when the data are given in radians and when in degrees.

a. $y = 23 \sin (314.16t + \pi/6)$
b. $y = 144 \sin (157.08t - \pi)$
c. $y = 77 \sin (251.327t + 2\pi/7)$
d. $y = 100 \sin (9 \times 10^5 t + 45°)$
e. $y = 15 \sin (3.76992 \times 10^8 t - 3\pi/5)$
f. $y = 147 \sin (21600t + 60°)$
g. $y = 200 \sin (6.2832t - 2\pi/5)$
h. $y = 300 \sin (1.6588 \times 10^{12} t + \pi/2)$

7. Using radian measure, write the equation for each of the sine curves described below.

	Amplitude	*Frequency*	*Phase constant*
a.	200	30	50°
b.	150	3.0×10^8	00°
c.	5	10^{12}	$\pi/2$
d.	9	60	00°
e.	10	1,000	45°
f.	20	500	30°
g.	20	25	00°
h.	115	40	2π
i.	100	50	90°

23·6 Adding Sine Waves of the Same Frequency. In dealing with alternating-current problems we have occasion to consider rotating vectors. We impose four conditions on these vectors:

1. They rotate at a constant angular velocity.
2. They are of constant length.
3. They rotate in a coordinate plane about the origin of the coordinate axes.
4. They rotate in a counterclockwise direction.

We shall be particularly interested in the projection of these vectors on one of the coordinate axes. For the present, let this axis of particular interest be the y axis. Let

$$y_3 = y_1 + y_2 \tag{32}$$
$$y_1 = Y_1 \times \sin \omega t \tag{33}$$
$$y_2 = Y_2 \sin (\omega t + \psi) \tag{34}$$

Y_1 = the magnitude of the rotating vector $\overrightarrow{Y_1}$

Y_2 = the magnitude of the rotating vector $\overrightarrow{Y_2}$

$\omega/2\pi$ = the frequency of both sine functions

It is required to plot y_3 against ωt. If Y_1, Y_2, and ψ are known, Eqs. (33) and (34) can be plotted against ωt on the same axes. Then successive ordinates of these equations may be added to yield successive points on the graph of y_3. As an alternative method, the ordinates of y_1 and y_2 may be calculated and added algebraically to give the corresponding ordinates of y_3.

Figure 23·11 shows a plot of the equations.

$$y_1 = 2 \sin \omega t \tag{35}$$
$$y_2 = 3 \sin (\omega t + 60°) \tag{36}$$

From the graph, the amplitude of y_3 appears to be about 4.4 and the phase constant of y_3 appears to be about 36°, referred to the phase of y_1.

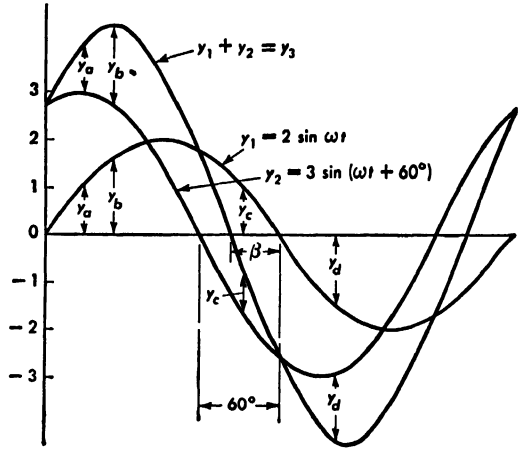


Fig. 23·11

The above method of finding the amplitude and phase of y_3 is useful and instructive. It will also be useful and instructive to investigate algebraic methods by which the equation for y_3 may be calculated when the equations of y_1 and y_2 are given.

From Eqs. (32), (33), and (34),

$$y_3 = Y_1 \sin \omega t + Y_2 \sin (\omega t + \psi) \tag{37}$$

From Eq. (1), page 501, applied to Eq. (37),

$$y_3 = Y_1 \sin \omega t + Y_2(\sin \omega t \times \cos \psi + \cos \omega t \times \sin \psi)$$
$$y_3 = Y_1 \sin \omega t + Y_2 \sin \omega t \times \cos \psi + Y_2 \cos \omega t \sin \psi \tag{38}$$
$$= (Y_1 + Y_2 \cos \psi) \sin \omega t + Y_2 \sin \psi \cos \omega t \tag{39}$$

Let us multiply and divide the right-hand side of Eq. (39) by the quantity

$$\sqrt{(Y_2 \sin \psi)^2 + (Y_1 + Y_2 \cos \psi)^2}$$

Then

$$y_3 = \sqrt{(Y_2 \sin \psi)^2 + (Y_1 + Y_2 \cos \psi)^2} \left[\frac{(Y_1 + Y_2 \cos \psi) \sin \omega t + Y_2 \sin \psi \cos \omega t}{\sqrt{(Y_2 \sin \psi)^2 + (Y_1 + Y_2 \cos \psi)^2}} \right] \quad (40)$$

Now let us introduce a new angle β such that

$$\tan \beta = \frac{Y_2 \sin \psi}{Y_1 + Y_2 \cos \psi} \quad (41)$$

This angle β is a constant since Y_1, Y_2, and ψ are constants. The angle β may be illustrated graphically, as in Figs. 23·11 and 23·12.

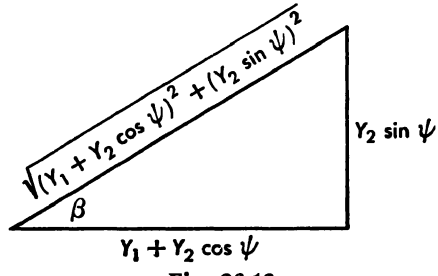


Fig. 23·12

Referring to Fig. 23·12, Eqs. (42) and (43) follow:

$$\sin \beta = \frac{Y_2 \sin \psi}{\sqrt{Y_2^2 \sin^2 \psi + (Y_1 + Y_2 \cos \psi)^2}} \quad (42)$$

$$\cos \beta = \frac{Y_1 + Y_2 \cos \psi}{\sqrt{Y_2^2 \sin^2 \psi + (Y_1 + Y_2 \cos \psi)^2}} \quad (43)$$

Substituting Eqs. (42) and (43) in Eq. (40),

$$y_3 = \sqrt{(Y_2 \sin \psi)^2 + (Y_1 + Y_2 \cos \psi)^2} \, [\sin \beta \cos \omega t + \cos \beta \sin \omega t]$$

or following Eq. (1), page 501,

$$y_3 = \sqrt{Y_2^2 \sin^2 \psi + (Y_1 + Y_2 \cos \psi)^2} \, [\sin (\omega t + \beta)] \quad (44)$$

Equation (44) is the characteristic equation for a sine curve whose frequency is $\frac{\omega}{2\pi}$. It represents the sum of two sine functions whose frequencies are also $\frac{\omega}{2\pi}$. Therefore we have demonstrated that the sum

of two sine curves of a given frequency is a sine curve of that same frequency.

Observe that the factor $\sqrt{Y_2^2 \sin^2 \phi + (Y_1 + Y_2 \cos \psi)^2}$ is a constant and that from Sec. 23·2 this factor is the amplitude of y_3. The angle β is the phase constant of y_3 referred to y_1.

23·7 Amplitude and Phase Relationships. Let Y_3 be the amplitude of Eq. (44); then

$$Y_3 = \sqrt{Y_2^2 \sin^2 \psi + (Y_1 + Y_2 \cos \psi)^2} \tag{45}$$

$$Y_3 = \sqrt{Y_2^2 \sin^2 \psi + Y_1^2 + 2Y_1Y_2 \cos \psi + Y_2^2 \cos^2 \psi} \tag{46}$$

$$Y_3 = \sqrt{Y_2^2(\sin^2 \psi + \cos^2 \psi) + Y_1^2 + 2Y_1Y_2 \cos \psi} \tag{47}$$

Following Eq. (126), page 366,

$$\sin^2 \psi + \cos^2 \psi = 1$$

Therefore

$$Y_3 = \sqrt{Y_2^2 + Y_1^2 + 2Y_1Y_2 \cos \psi} \tag{48}$$

The phase constant of y_3 is

$$\beta = \tan^{-1} \frac{Y_2 \sin \psi}{Y_1 + Y_2 \cos \psi} \tag{49}$$

Equations (48) and (49) give the amplitude and phase, respectively, of y_3 in terms of the amplitude of y_1 and y_2 and the phase of y_2, all relative to the phase of y_1.

23·8 The Equivalence of Graphical and Algebraic Methods. The curves of Fig. 23·11 are repeated in Fig. 23·13.

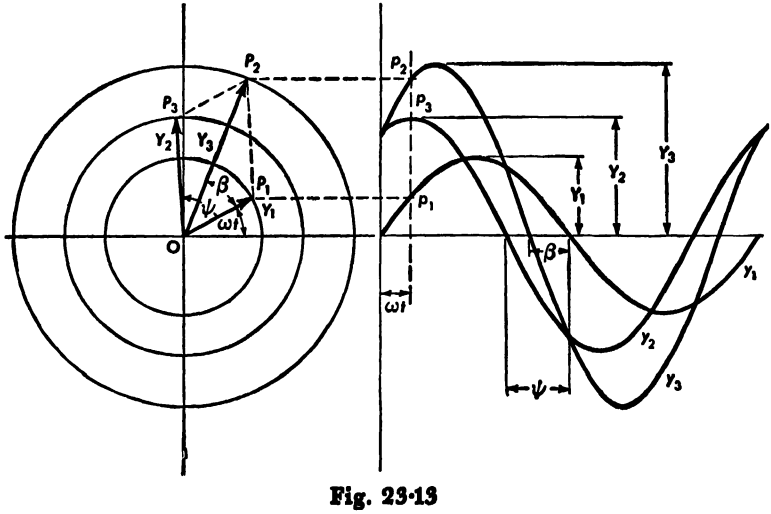


Fig. 23·13

Following the graphical methods similar to those discussed in Sec. 23·4, the curve of y_1 could have been plotted from the successive vertical projections of the rotating vector Y_1. The curve of y_2 could have been plotted from the successive vertical projections of the rotating vector Y_2. The curve of y_3 could have been plotted from successive positions of the rotating vector Y_3.

We shall choose any phase angle ωt and locate on each curve the point corresponding to ωt. These points are p_1, p_2, and p_3 in Fig. 23·13. Now observe the momentary positions of Y_1, Y_2, and Y_3 which correspond to the points p_1, p_2, and p_3. The figure $OP_1P_2P_3$ appears to be

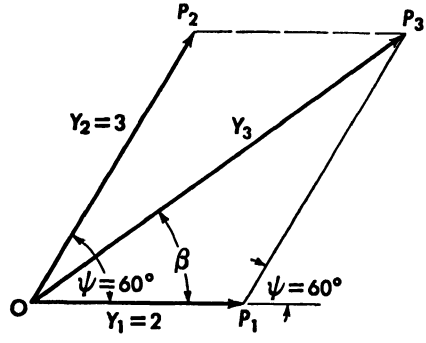


Fig. 23·14

a parallelogram. If we assume that it is a parallelogram, then by Eq. (40), page 425, the following relation exists:

$$Y_3^2 = Y_1^2 + Y_2^2 + 2Y_1Y_2 \cos \psi \qquad (50)$$

where Y_1, Y_2, and Y_3 are the magnitudes of their respective vectors. Since Eqs. (50) and (48) are identical, the figure $OP_1P_2P_3$ must be a parallelogram.

Therefore, if we have given the amplitude and phase of each of two sine waves of the same frequency, we can calculate the amplitude and phase of their sum by using a parallelogram similar to the one shown in Figs. 23·13 and 23·14.

The parallelogram can be drawn to scale and the solution found graphically. The unknown amplitude and phase constant can also be found by use of the cosine law.

Note that the angle ψ in Fig. 23·14 is an exterior angle referred to the triangle OP_1P_3 and hence is the supplement of the adjacent interior angle. Consequently the positive sign is used before the term $2Y_1Y_2 \cos \psi$ when using the cosine law (see page 424).

The trigonometric solution of the problem illustrated in Fig. 23·14 follows:

$$Y_3^2 = 2^2 + 3^2 + 2 \times 2 \times 3 \cos 60°$$
$$= 4 + 9 + 6 = 19$$
$$Y_3 = \sqrt{19} = 4.359 \qquad \text{(ans.)}$$
$$\frac{\sin \beta}{3} = \frac{\sin 120°}{4.359}$$
$$\sin \beta = 3 \times \frac{0.86603}{4.359} = 0.5960$$
$$\beta = 36°35' \qquad \text{(ans.)}$$

23·9 Adding Harmonically Related Sine Functions. Figures 23·15 to 23·17 show the plots of the following equations:

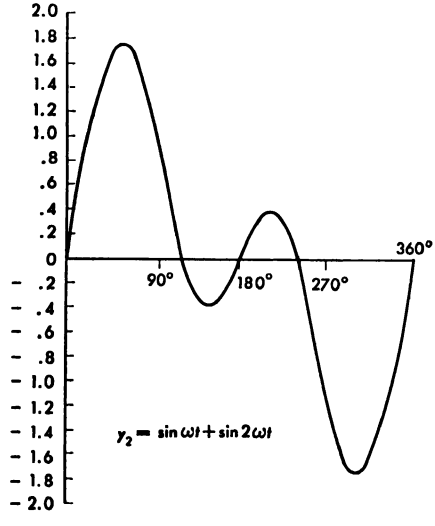


Fig. 23·15

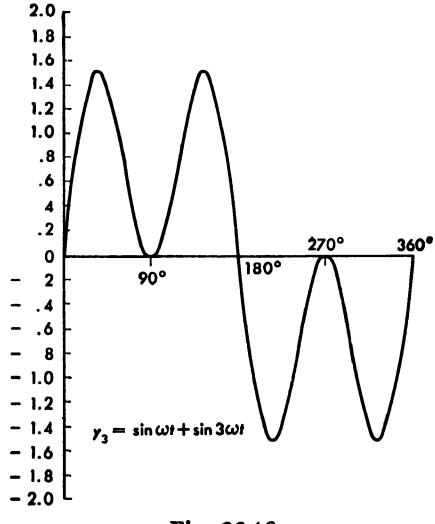


Fig. 23·16

$$y_2 = \sin \omega t + \sin 2\omega t \tag{51}$$
$$y_3 = \sin \omega t + \sin 3\omega t \tag{52}$$
$$y_4 = \sin \omega t + \sin 4\omega t \tag{53}$$

It is important to observe that while y_2, y_3, and y_4 are periodic functions, they are not sine functions.

Such curves can best be plotted from a calculated table. The parallelogram method of finding amplitudes and phase constants does not apply in this case, since each vector would rotate at a different angular velocity and therefore a different parallelogram would apply for each moment.

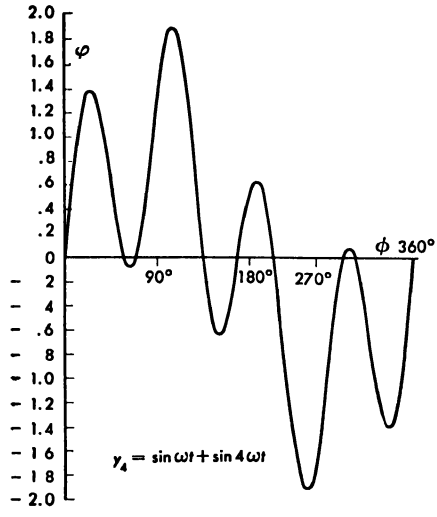


Fig. 23·17

EXERCISE 5

1. In the following problems, $y_3 = y_1 + y_2$. Plot the curves for y_1 and y_2 on the same axes; and, by adding the ordinates of y_1 and y_2 for various values of ωt, find successive points on the curve of y_3. By reading the graph, find the amplitude of y_3 and the phase constant of y_3 relative to y_1.
   - **a.** $y_1 = 2 \sin \omega t$ $\quad y_2 = 3 \sin (\omega t + 30°)$
   - **b.** $y_1 = 3 \sin \omega t$ $\quad y_2 = 2 \sin (\omega t + 30°)$
   - **c.** $y_1 = \sin \omega t$ $\quad y_2 = \cos \omega t$
   - **d.** $y_1 = 2 \sin \omega t$ $\quad y_2 = 3 \sin (\omega t - 30°)$
   - **e.** $y_1 = \sin \omega t$ $\quad y_2 = \sin (\omega t + 180°)$
2. In the following problems, $y_3 = y_1 + y_2$. Calculate the amplitude of y_3 and the phase constant of y_3 relative to y_1.
   - **a.** $y_1 = 256 \sin \omega t$ $\quad y_2 = 164 \sin (\omega t + 24°)$
   - **b.** $y_1 = 59 \sin \omega t$ $\quad y_2 = 78 \sin (\omega t - 150°)$
   - **c.** $y_1 = 20 \sin (\omega t + 30°)$ $\quad y_2 = 40 \sin (\omega t - 30°)$
3. If the effective value of one sine curve is 110 with a phase constant of zero and the effective value of a second sine curve is 220 with a phase constant of 45°, find the effective value of the sum of these two curves.

23·10 Simple Harmonic Motion. We shall now enlarge upon the principles already discussed in this chapter in order to bring them to bear on one particularly important type of periodic motion.

A periodic motion is one that repeats itself in successive intervals of time. Examples of periodic motion are the swing of a pendulum, the rotation of the moon about the earth, the motion of the ends of a vibrating tuning fork, the reciprocating motion of a piston, a point on a vibrating violin string, etc. A knowledge of periodic motions is necessary for an understanding of mechanical vibration, alternating-current electricity, electromagnetic waves, etc.

One of the simplest and by far the most important type of periodic motion is simple harmonic motion.

23·11 Simple Harmonic Motion Defined. When a point P moves in a circular path at a constant angular velocity, the projection of P on any diameter will oscillate with a type of motion called *simple harmonic motion.*

23·12 Equations for Coordinates of P. The definition of simple harmonic motion given above can be illustrated in terms of Fig. 23·18.

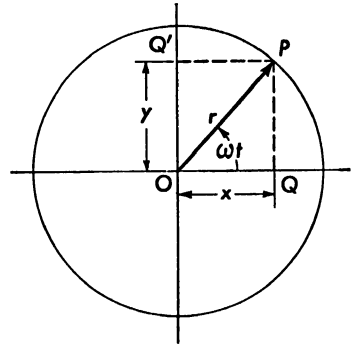


Fig. 23·18

Here the vector $\overrightarrow{OP}$ is revolving about O with the constant angular velocity ω. The projections of P (that is Q and Q') will oscillate back and forth along their respective axes. When $\overrightarrow{OP}$ has rotated through 360° or 2π radians, then both Q and Q' will have traversed one complete cycle of motion.

The coordinates of P are given by the following equations:

$$x = |OP| \times \cos \omega t = r \cos \omega t \tag{54}$$

$$y = |OP| \times \sin \omega t = r \sin \omega t \tag{55}$$

where $r = |OP|$ = amplitude of the oscillatory motion
ω = constant angular velocity of r, often measured in radians per second, but sometimes measured in degrees per second
t = time in seconds

Note that r is the maximum displacement of Q and Q'. The displacements of Q and Q' will be positive or negative in accordance with the convention of rectangular coordinates.

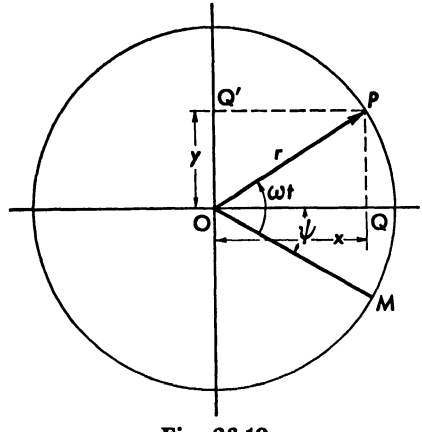


Fig. 23·19

23·13 Phase. The angle of rotation can be referred to any stationary radius, such as OM in Fig. 23·19. In this case

$$x = r \cos (\omega t - \psi) \tag{56}$$
$$y = r \sin (\omega t - \psi) \tag{57}$$

Referred to Fig. 23·20,

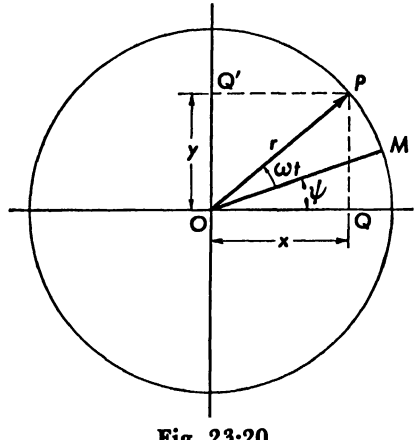


Fig. 23·20

$$x = r \cos (\omega t + \psi) \tag{58}$$
$$y = r \sin (\omega t + \psi) \tag{59}$$

The angle ψ is called the *phase constant.*

23·14 Parametric Equation. When the variable coordinates of a point are each expressed as a function of a third variable, the equations defining these coordinates are called *parametric equations.* The third variable is called the *parameter.*

In the present case the equation

$$x = r \cos (\omega t \pm \psi) \tag{60}$$

expresses the x coordinate of P. The equation

$$y = r \sin (\omega t \pm \psi) \tag{61}$$

expresses the y coordinate of P. Therefore, Eqs. (60) and (61) are parametric equations, and t is the parameter. Both Eqs. (60) and (61)

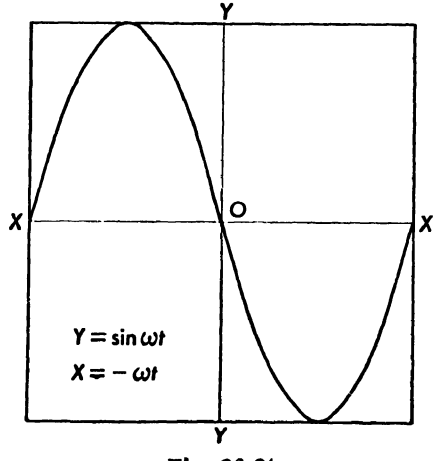


Fig. 23·21

are equations for simple harmonic motion. Equation (60) is the equation of motion for the point Q, and Eq. (61) is the equation of motion for the point Q'.

23·15 Parametric Equations for a Sine Curve. It is perfectly possible for a point to have a motion in one direction which is simple harmonic motion and simultaneously have a motion in another direction which is another kind of motion.

For example, a point can move parallel to the y axis in simple harmonic motion and parallel to the x axis in uniform linear motion.

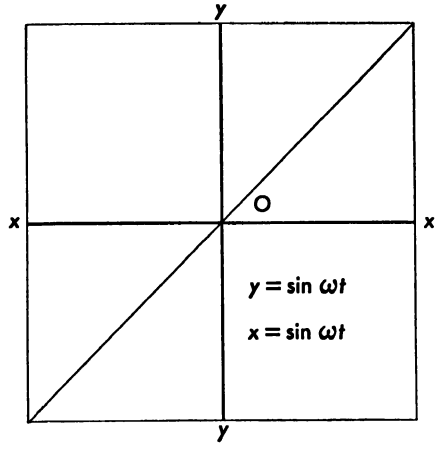


Fig. 23·22

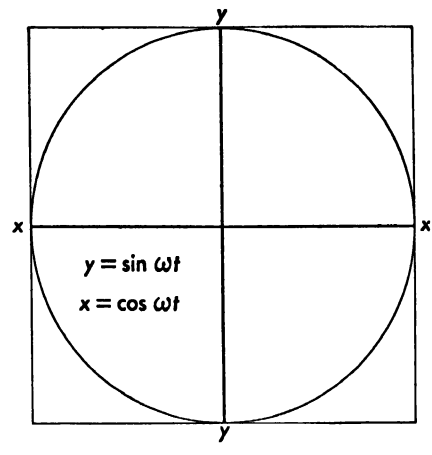


Fig. 23·23

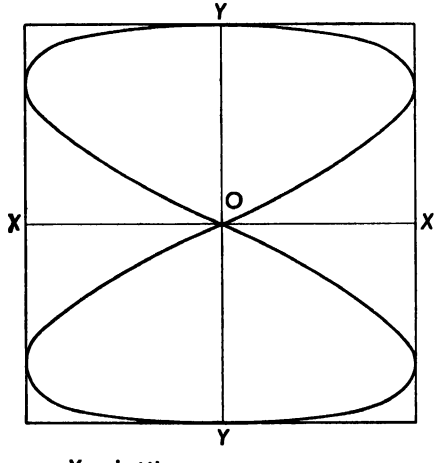


Fig. 23·24

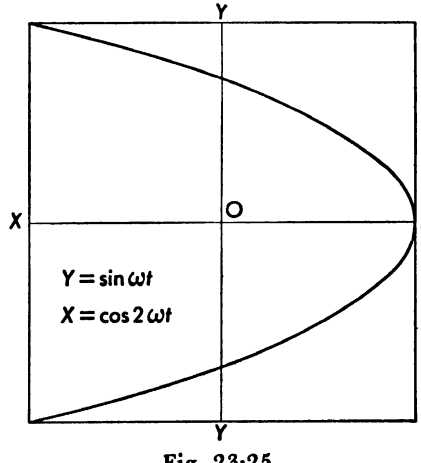


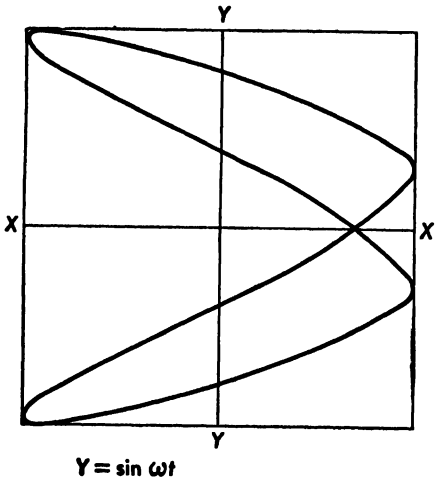
Fig. 23·25

The parametric equations for this situation could be, for example,

$$y = r \sin (\omega t + \psi) \tag{62}$$
$$x = -\omega t \tag{63}$$

23·16 Lissajous Figures. Consider the parametric equations for the motions of a point P.

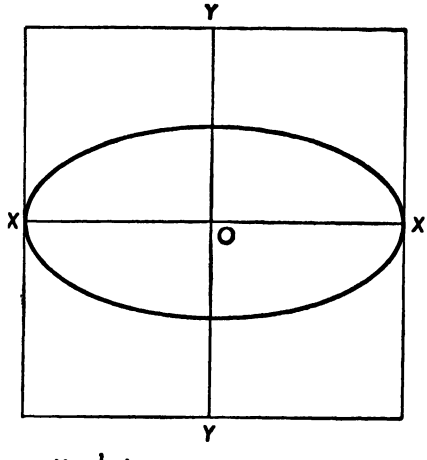
$$y = r_1 \sin (k_1 \omega t + \psi_1)$$
$$x = r_2 \sin (k_2 \omega t + \psi_2)$$



$Y = \sin \omega t$

$X = \cos (2\omega t + 45°)$

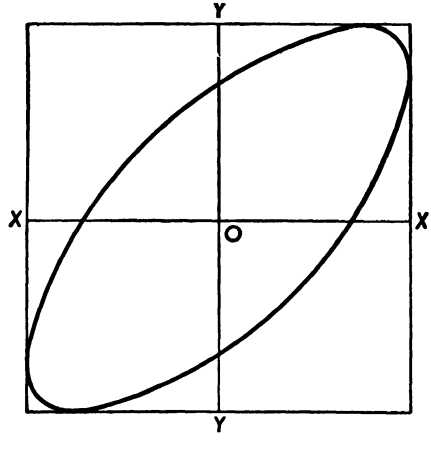
Fig. 23·26



$Y = \frac{1}{2} \sin \omega t$

$X = \cos \omega t$

Fig. 23·27



$Y = \sin (\omega t + 45°)$

$X = \cos \omega t$

Fig. 23·28

The plot of such a pair of equations is called a Lissajous figure. Figures 23·22 to 23·28 illustrate several Lissajous figures.

Table 23·1 is offered as a convenience to the student in working the problems in Exercise 6.

Table 23·1

$\phi°$	$\sin\phi$	$\sin 2\phi$	$\sin 3\phi$	$\sin 4\phi$	$\cos\phi$	$\cos 2\phi$	$\cos 3\phi$	$\cos 4\phi$
0	0	0	0	0	1.000	1.000	1.000	1.000
5	0.087	0.174	0.259	0.342	0.996	0.985	0.966	0.940
10	0.174	0.342	0.500	0.643	0.985	0.940	0.866	0.766
15	0.259	0.500	0.707	0.866	0.966	0.866	0.707	0.500
20	0.342	0.643	0.866	0.985	0.940	0.766	0.500	0.174
25	0.423	0.766	0.966	0.985	0.906	0.643	0.259	−0.174
30	0.500	0.866	1.000	0.866	0.866	0.500	0	−0.500
35	0.574	0.940	0.966	0.643	0.819	0.342	−0.259	−0.766
40	0.643	0.985	0.866	0.342	0.766	0.174	−0.500	−0.940
45	0.707	1.000	0.707	0	0.707	0	−0.707	−1.000
50	0.766	0.985	0.500	−0.342	0.643	−0.174	−0.866	−0.940
55	0.819	0.940	0.259	−0.643	0.574	−0.342	−0.966	−0.766
60	0.866	0.866	0	−0.866	0.500	−0.500	−1.000	−0.500
65	0.906	0.766	−0.259	−0.985	0.423	−0.643	−0.966	−0.174
70	0.940	0.643	−0.500	−0.985	0.342	−0.766	−0.866	+0.174
75	0.966	0.500	−0.707	−0.866	0.259	−0.866	−0.707	+0.500
80	0.985	0.342	−0.866	−0.643	0.174	−0.940	−0.500	+0.766
85	0.996	0.174	−0.966	−0.342	0.087	−0.985	−0.259	+0.940
90	1.000	0	−1.000	0	0	−1.000	0	+1.000

EXERCISE 6

Plot the Lissajous figures defined by the following pairs of equations:

1. $y = \sin \omega t$
$x = \omega t$

2. $y = \sin \omega t$
$x = \cos \omega t$

3. $y = \sin \omega t$
$x = \sin 2\omega t$

4. $y = \cos 2\omega t$
$x = \sin \omega t$

5. $y = \sin \omega t$
$x = \sin 3\omega t$

6. $y = \sin \omega t$
$x = \sin 4\omega t$

7. $y = \sin \omega t$
$x = \sin 5\omega t$

8. $y = \sin 2\omega t$
$x = \sin 3\omega t$

9.* $y = \sin 2\omega t$
$x = \sin 4\omega t$

10. $y = \sin 3\omega t$
$x = \sin 4\omega t$

11. $y = \cos (2\omega t + 30°)$
$x = \sin \omega t$

12. $y = \cos 2\omega t$
$x = \sin (\omega t + 30°)$

*Compare with Prob. 3.

Chapter 24 COMPLEX NOTATION AND VECTOR ALGEBRA

It has been discovered that complex numbers (see Sec. 9·16) can be used uniquely to represent vectors. It has also been discovered that the vector operations can be accomplished through the corresponding algebraic operations with complex numbers (see Secs. 21·2, 21·5, and

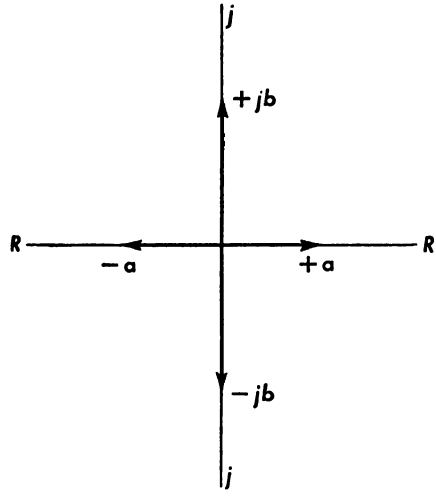


Fig. 24·1

21·11). These were actual discoveries and not logical but uninspired extensions of classical algebra.

In this chapter we shall make no pretense of reasoning from the basic principles of classical algebra into the applications of complex numbers to the vector operations. We shall assert that if we do represent vectors by complex numbers, the *algebraic* addition, subtraction, multiplication, or division of these complex numbers gives rise to a complex number which represents the *vector* sum, difference, product, or quotient. Then we shall proceed to demonstrate that the assertion

is correct. We shall also explore some of the contiguous areas of mathematical thought and develop some alternative vector expressions.

24·1 The Reference Vectors. The axes in the rectangular coordinate system of Fig. 24·1 have been renamed for convenience the j axis and the R axis. Vectors drawn on these axes are defined below:

$+a$ = the vector a units long drawn to the right on the R axis
$-a$ = the vector a units long drawn to the left on the R axis
$+jb$ = the vector b units long drawn upward on the j axis
$-jb$ = the vector b units long drawn downward on the j axis

Therefore vector quantities such as $+2$, $+j3$, -4, $-j5$, etc., have a unique significance illustrated in Fig. 24·2.

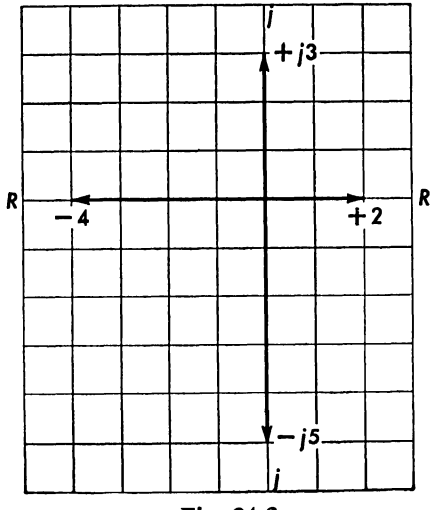


Fig. 24·2

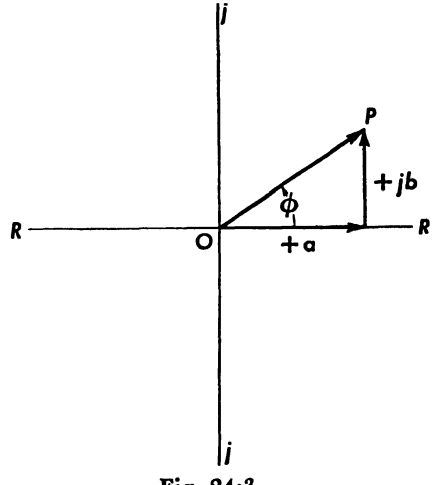


Fig. 24·3

As in Fig. 24·2, "imaginary" numbers are plotted along the j axis and "real" numbers are plotted along the R axis.

It is important to notice three aspects of this notation.

1. The presence or absence of the "imaginary" unit j indicates the direction of the vector.
2. The sign before the "real" or "imaginary" number indicates the sense of the vector.
3. In all cases an unsigned arithmetical number denotes magnitude only and not direction.

24·2 Addition of the Reference Vectors. The point P in Fig. 24·3 may be reached from O by moving a units in the positive direction along the R axis and *in addition* by moving b units in the positive direc-

tion referred to the j axis. The point P could also have been reached by moving along the vector $\overrightarrow{OP}$. The equivalence between these two traverses is indicated by the equation

$$\overrightarrow{OP} = a + jb \tag{1}$$

Hereafter in this chapter, when we encounter an expression in the form $a + jb$, we shall understand that a vector is indicated. If the

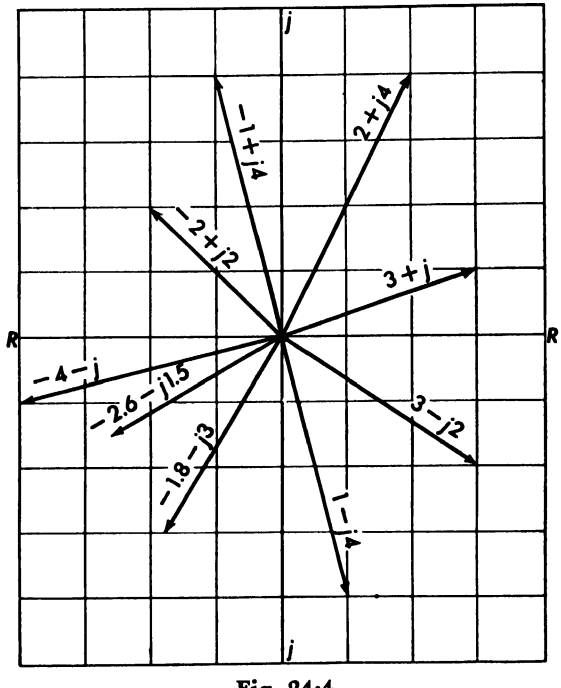


Fig. 24·4

vector a in Eq. (1) is drawn in a positive sense, it is customary to omit the plus sign before it. Several vectors are shown in Fig. 24·4 together with the complex numbers which represent them.

From the geometry of Fig. 24·3 it has become conventional to write

$$|OP| = \sqrt{a^2 + b^2} \tag{2}$$

and

$$\tan \phi = \frac{b}{a} \tag{3}$$

However, as used in Eqs. (2) and (3), the quantities a and b are algebraic numbers and not vectors. Observe that the characteristic

vector symbol j is missing from both equations. Furthermore, the multiplication involved in obtaining a^2 and b^2 in Eqs. (2) and (3) is not a vector multiplication. Neither is the division in Eq. (3) a vector division.

Equation (1) expresses the vector $\overrightarrow{OP}$ in terms of its rectangular components. The corresponding polar form is shown in Eq. (4):

$$\overrightarrow{OP} = \sqrt{a^2 + b^2} \angle \tan^{-1} \frac{b}{a} \tag{4}$$

EXAMPLE 1. Plot the vector $2 + j9$, and calculate its magnitude and reference angle. Also express the vector in polar form.

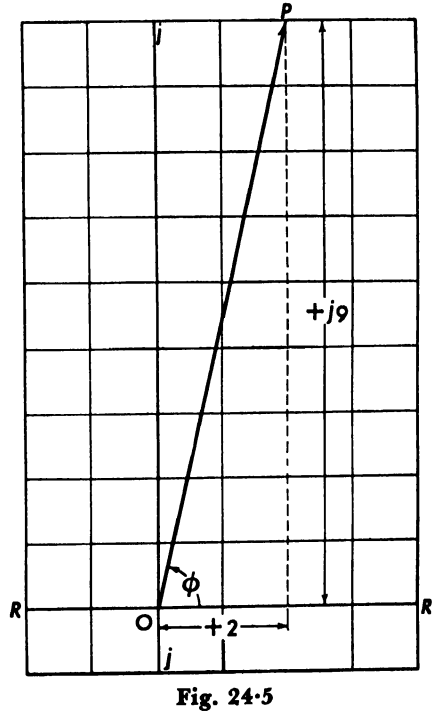


Fig. 24·5

The vector $2 + j9$ is plotted in Fig. 24·5.

$$|OP| = \sqrt{2^2 + 9^2} = \sqrt{4 + 81} = \sqrt{85} = 9.22$$
$$\tan \phi = \tfrac{9}{2} = 4.5$$
$$\phi = 77.5°$$
$$\overrightarrow{OP} = 9.22 \angle 77.5°$$

EXERCISE 1

Plot the following vectors, and calculate the magnitude and reference angle of each:

1. $3 + j4$	**2.** $6 - j2$	**3.** $-3 + j7$	**4.** $-2 - j6$
5. $3 - j4$	**6.** $4 + j2$	**7.** $-5 - j3$	**8.** $6 + j7$
9. $-2 + j2$	**10.** $3 - j2$	**11.** $-2 - j2$	**12.** $7 - j6$

24·3 Vector Addition. In this section we shall demonstrate that if $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{OQ}$ and if $\overrightarrow{OP} = a + jb$ and $\overrightarrow{OQ} = c + jd$, then

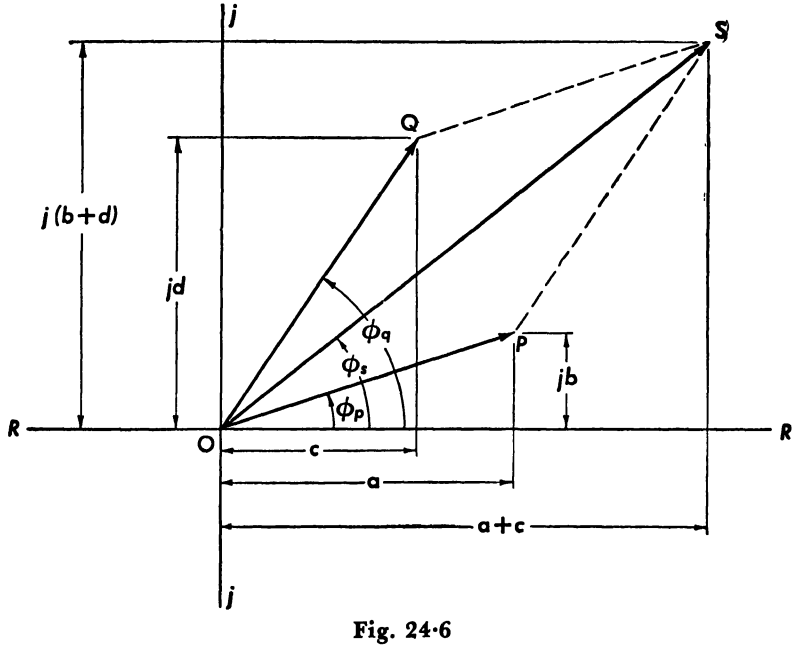


Fig. 24·6

the *algebraic* sum of $(a + jb)$ and $(c + jd)$ is a complex number defining the *vector* sum of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

1. By definition of vector addition (referred to Fig. 24·6), the R projection of $\overrightarrow{OS}$, $(a + c)$, is the algebraic sum of the R projections of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. The j projection of $\overrightarrow{OS}$, $(b + d)$, is the algebraic sum of the j projections of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.
2. If we let

$$\overrightarrow{OP} = a + jb$$

and

$$\overrightarrow{OQ} = c + jd$$

then

$$\overrightarrow{OS} = (a + jb) + (c + jd)$$
$$\overrightarrow{OS} = (a + c) + j(b + d) \tag{5}$$

Part 1 above is a definitive statement for the *vector* sum of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Equation (5) is the result of the *algebraic* addition of $(a + jb)$ and $(c + jd)$. The definitive statement above and Eq. (5) correspond. Therefore, we may conclude that *vector* addition may be accomplished by the *algebraic* addition of the complex numbers representing the vectors.

EXAMPLE 2. Add the vectors $(3 + j4)$ and $(2 + j5)$ and express the sum in polar form. Let

$$\overrightarrow{OS} = (3 + j4) + (2 + j5)$$
$$\overrightarrow{OS} = 5 + j9$$

Then

$$|OS| = \sqrt{5^2 + 9^2} = \sqrt{106} = 10.30$$
$$\tan \psi = \tfrac{9}{5} = 1.8$$

where

$$\psi = \text{the reference angle of } \overrightarrow{OS}$$
$$\psi = 60.9°$$
$$\overrightarrow{OS} = 10.30\underline{/60.9°}$$

EXAMPLE 3. Add the vectors $(2 + j5)$ and $(-6 - j7)$. Express the sum in polar form. Let

$$\overrightarrow{OS} = (2 + j5) + (-6 - j7)$$
$$\overrightarrow{OS} = -4 - j2$$
$$|OS| = \sqrt{4^2 + 2^2} = \sqrt{16 + 4} = \sqrt{20} = 4.47$$
$$\tan \psi = \frac{-2}{-4} = +0.5$$

where ψ = reference angle of $\overrightarrow{OS}$

$$\psi = 26.6° + 180° = 206.6°$$
$$\overrightarrow{OS} = 4.47\underline{/206.6°}$$

EXAMPLE 4. Express the vector $10/\underline{30°}$ in complex notation.

$$\text{The } R \text{ projection} = 10 \cos 30°$$
$$\text{The } j \text{ projection} = 10 \sin 30°$$

Therefore we may write (also see Sec. 24·7)

$$\begin{aligned} 10/\underline{30°} &= 10 \cos 30 + j10 \sin 30 \\ &= 10 \times 0.866 + j10 \times 0.5 \\ &= 8.66 + j5 \end{aligned}$$

EXAMPLE 5. Add the vectors $5/\underline{150°}$ and $10/\underline{45°}$. Express the sum in polar form. Let

$$\vec{OS} = 5/\underline{150°} + 10/\underline{45°}$$

$$\begin{aligned} \vec{OS} &= (5 \cos 150° + j5 \sin 150°) + (10 \cos 45° + j10 \sin 45°) \\ &= 5(-0.866) + j5 \times 0.5 + 10 \times 0.707 + j10 \times 0.707 \\ &= -4.33 + j2.5 + 7.07 + j7.07 \\ &= 2.74 + j9.57 \end{aligned}$$

$$|OS| = \sqrt{2.74^2 + 9.57^2} = \sqrt{7.5076 + 91.5849} = \sqrt{99.0925} = 9.95$$

$$\tan \psi = \frac{9.57}{2.74} = 3.4927$$

$$\vec{OS} = 9.95/\underline{74°01'}$$

EXERCISE 2

Perform the following vector additions, and express the sum in polar form:

1. $(2 + j3) + (4 + j5)$
2. $(5 + j7) + (-6 + j5)$
3. $(9 + j2) + (-3 + j2)$
4. $(8 + j7) + (-2 - j3)$
5. $(6 + j10) + (-9 - j5)$
6. $(4 + j3) + (-8 - j6)$
7. $(7 + j9) + (4 - j5)$
8. $(2 + j3) + (5 - j8)$
9. $(-5 + j2) + (-2 + j6)$
10. $(-2 + j5) + (-4 - j1)$
11. $(-3 + j8) + (-4 - j12)$
12. $(-5 + j12) + (3 - j4)$
13. $(-2 + j10) + (1 - j15)$
14. $(-2 + j2) + (6 - j5)$
15. $(-5 - j9) + (-2 - j6)$
16. $(-2 - j8) + (1 - j2)$
17. $(-5 - j10) + (10 - j5)$
18. $(3 - j2) + (7 - j5)$

Perform the following vector additions, and express the sum in rectangular form:

19. $6/\underline{40°} + 10/\underline{70°}$
20. $3/\underline{30°} + 10/\underline{160°}$
21. $15/\underline{20°} + 20/\underline{100°}$
22. $20/\underline{50°} + 2/\underline{200°}$
23. $20/\underline{20°} + 10/\underline{150°}$
24. $5/\underline{80°} + 2/\underline{200°}$

25. $5/40° + 2/205°$ **26.** $13/15° + 2/340°$

27. $25/10° + 25/300°$ **28.** $15/140° + 20/120°$

29. $10/170° + 5/260°$ **30.** $8/160° + 2/240°$

24·4 Vector Subtraction. In this section we shall demonstrate that if $\overrightarrow{OS} = \overrightarrow{OP} - \overrightarrow{OQ}$ and if $\overrightarrow{OP} = a + jb$ and $\overrightarrow{OQ} = c + jd$, then the *algebraic* difference between $a + jb$ and $c + jd$ is a complex number defining the *vector* difference between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ (see Fig. 24·7).

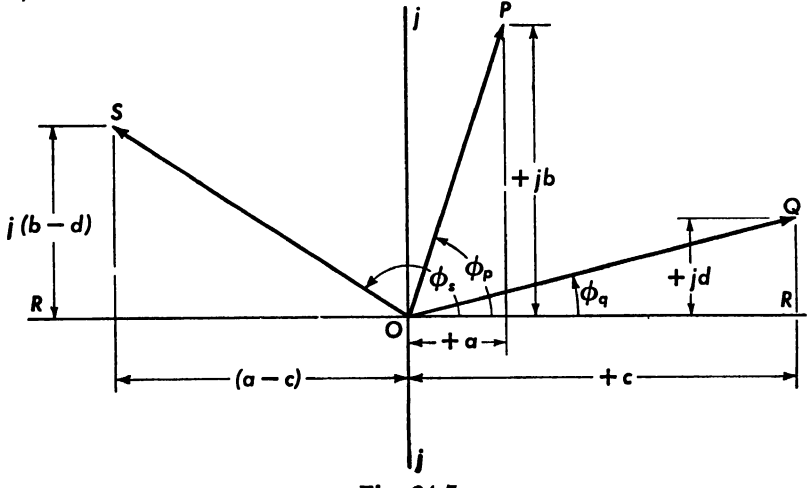


Fig. 24·7

1. By definition of vector subtraction (referred to Fig. 24·7), the R projection of $\overrightarrow{OS}$, $(a - c)$, is the algebraic difference between the R projections of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. The j projection of $\overrightarrow{OS}$ is the algebraic difference between the j projections of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

2. If we let

$$\overrightarrow{OP} = a + jb$$

and

$$\overrightarrow{OQ} = c + jd$$

then

$$\overrightarrow{OS} = (a + jb) - (c + jd)$$

$$\overrightarrow{OS} = a + jb - c - jd = a - c + jb - jd$$

$$\overrightarrow{OS} = (a - c) + j(b - d) \qquad (6)$$

Part 1 above is a definitive statement of the *vector* difference between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Equation (6) is the result of the *algebraic* subtraction of $(c + jd)$ from $(a + jb)$. The definitive statement above and Eq. (6) correspond. Therefore, we may conclude that vector subtraction may be accomplished by the algebraic subtraction of the complex numbers representing the vectors.

EXAMPLE 6. Subtract the vector $5 + 3j$ from the vector $2 - 6j$. Let

$$\begin{aligned}
\overrightarrow{OS} &= (2 - 6j) - (5 + 3j) \\
&= -3 - 9j \\
|OS| &= \sqrt{3^2 + 9^2} = \sqrt{9 + 81} = \sqrt{90} = 9.49 \\
\tan \psi &= \frac{-9}{-3} = +3
\end{aligned}$$

When

$$\begin{aligned}
\psi &= \text{reference angle of } \overrightarrow{OS} \\
&= 251.6^\circ \\
\overrightarrow{OS} &= 9.49\underline{/251.6^\circ}
\end{aligned}$$

EXAMPLE 7. Subtract the vector $30\underline{/60^\circ}$ from the vector $10\underline{/30^\circ}$. Let

$$\begin{aligned}
\overrightarrow{OS} &= 10\underline{/30^\circ} - 30\underline{/60^\circ} \\
30\underline{/60^\circ} &= 30 \cos 60 + j30 \sin 60^\circ \\
&= 30 \times 0.5 + j30 \times 0.866 \\
&= 15 + j25.980 \\
10\underline{/30^\circ} &= 10 \cos 30^\circ + j10 \sin 30^\circ \\
&= 10 \times 0.866 + j10 \times 0.5 \\
&= 8.66 + j5 \\
\overrightarrow{OS} &= (8.66 + j5) - (15 + j25.98) \\
&= -6.34 - j20.98 \\
|OS| &= \sqrt{6.34^2 + 20.98^2} = \sqrt{480.3560} = 21.92 \\
\tan \psi &= \frac{-20.98}{-6.34} = +3.309 \\
\psi &= 253.2^\circ \\
\overrightarrow{OS} &= 21.92\underline{/253.2^\circ}
\end{aligned}$$

EXERCISE 3

Perform the following vector subtractions, and record the vector difference in both polar and complex form:

1. $(3 + j5) - (2 - j3)$
2. $(4 + j6) - (10 - j9)$
3. $(-2 + j5) - (3 + j4)$
4. $(10 - j3) - (6 + j8)$
5. $(-1 + j2) - (-4 + j2)$
6. $10/\underline{42°} - 30/\underline{160°}$
7. $3/\underline{100°} - 4/\underline{340°}$
8. $2/\underline{250°} - 3/\underline{40°}$
9. $6/\underline{300°} - 8/\underline{125°}$
10. $3/\underline{270°} - 2/\underline{180°}$

24·5 Vector Multiplication. In this section we shall demonstrate that if $\overrightarrow{OS} = \overrightarrow{OP} \times \overrightarrow{OQ}$ and if $\overrightarrow{OP} = a + jb$ and $\overrightarrow{OQ} = c + jd$,

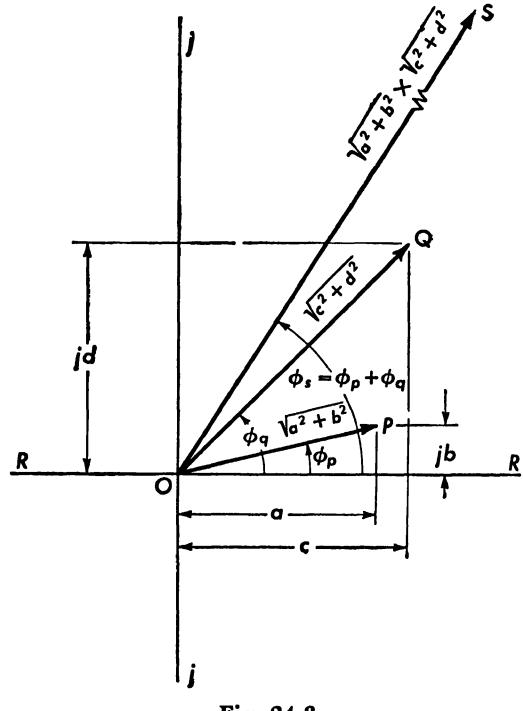


Fig. 24·8

then the *algebraic* product of $a + jb$ and $c + jd$ is a complex number defining the *vector* product of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ (see Fig. 24·8).

1. By definition,

$$\overrightarrow{OS} = |OP| \times |OQ| / \phi_p + \phi_q \tag{7}$$

where

ϕ_p = reference angle of $\overrightarrow{OP}$

and

ϕ_q = reference angle of $\overrightarrow{OQ}$

2. If we let

$$\overrightarrow{OP} = a + jb$$

and

$$\overrightarrow{OQ} = c + jd$$

then

$$\begin{aligned}\overrightarrow{OS} &= (a + jb) \times (c + jd) \\ &= (ac - bd) + j(bc + ad) \qquad \text{(see Example 55, page 176)} \qquad (8)\\ |OS| &= \sqrt{(ac - bd)^2 + (bc + ad)^2}\end{aligned}$$

$$\begin{aligned}&= \sqrt{a^2c^2 - 2abcd + b^2d^2 + b^2c^2 + 2abcd + a^2d^2} \\ &= \sqrt{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2 + b^2} \times \sqrt{c^2 + d^2}\end{aligned}$$

but

$$\sqrt{a^2 + b^2} = |OP|$$

and

$$\sqrt{c^2 + d^2} = |OQ|$$

Therefore

$$|OS| = |OP| \times |OQ| \qquad (9)$$

From Eq. (8)

$$\tan \phi_s = \frac{bc + ad}{ac - bd} \qquad (10)$$

where

$$\phi_s = \text{reference angle of } \overrightarrow{OS}$$

Dividing numerator and denominator of Eq. (10) by ac,

$$\tan \phi_s = \frac{\frac{bc}{ac} + \frac{ad}{ac}}{\frac{ac}{ac} - \frac{bd}{ac}} = \frac{\frac{b}{a} + \frac{d}{c}}{1 - \frac{b}{a} \times \frac{d}{c}}$$

but

$$\frac{b}{a} = \tan \phi_p$$

$$\frac{d}{c} = \tan \phi_q$$

Therefore

$$\tan \phi_s = \frac{\tan \phi_p + \tan \phi_q}{1 - \tan \phi_p \tan \phi_q} \tag{11}$$

From Eq. (3), page 502, we recognize that if Eq. (11) is valid, then

$$\phi_s = \phi_p + \phi_q \tag{12}$$

Therefore from Eqs. (9) and (12)

$$\overrightarrow{OS} = |OP| \times |OQ| / \underline{\phi_p + \phi_q} \tag{13}$$

Equation (7) is a definitive statement of the *vector* product of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Equation (13) is the result of the *algebraic* multiplication of $(a + jb)$ and $(c + jd)$. These two equations are identical. Therefore, we may conclude that *vector* multiplication may be accomplished by the *algebraic* multiplication of the complex numbers representing the vectors.

EXAMPLE 8. Solve for $\overrightarrow{OR}$ in polar notation if

$$\begin{aligned}
\overrightarrow{OR} &= (2 + j3) \times (5 - j4) \\
&\quad 2 + j3 \\
&\quad \underline{5 - j4} \\
&\quad 10 + j15 - j8 + 12 \\
\overrightarrow{OR} &= 22 + j7 \\
\overrightarrow{OR} &= \sqrt{22^2 + 7^2} / \underline{\tan^{-1} \tfrac{7}{22}} \\
|OR| &= \sqrt{484 + 49} \\
|OR| &= \sqrt{533} = 23.09 \\
\overrightarrow{OR} &= 23.09 / \underline{17°39'}
\end{aligned}$$

Compare with Example 19, page 495.

EXERCISE 4

Using complex notation, perform the following vector multiplications and compare with Probs. 7 to 10, page 498. Express the answer in polar form.

1. $(3 + j2) \times (-2 + j7)$
2. $(4 + j5) \times (1 + j2)$
3. $(-6 - j5) \times (-2 - j3)$
4. $(5 - j1) \times (-3 + j2)$

Multiply the following vectors and leave answers in rectangular form. Sketch each vector, including the product, approximately to scale.

5. $(2 + j5)(3 + j7)$
6. $(-4 + j2)(3 + j6)$
7. $(-3 - j4)(3 + j5)$
8. $(7 - j3)(3 + j2)$
9. $(3 + j7)(-2 + j5)$
10. $(-4 + j2)(-3 + j6)$
11. $(-3 - j4)(-3 + j5)$
12. $(7 - j3)(-3 + j2)$
13. $(3 + j7)(2 - j5)$
14. $(-4 + j2)(3 - j6)$
15. $(-3 - j4)(3 - j5)$
16. $(7 - j3)(3 - j2)$
17. $(3 + j7)(-2 - j5)$
18. $(-4 + j2)(-3 - j6)$
19. $(-3 - j4)(-3 - j5)$
20. $(7 - j3)(-3 - j2)$

24·6 Vector Division. In this section we shall demonstrate that if $\overrightarrow{OS} = \overrightarrow{OP} \div \overrightarrow{OQ}$ and if $\overrightarrow{OP} = a + jb$ and $\overrightarrow{OQ} = c + jd$, then the

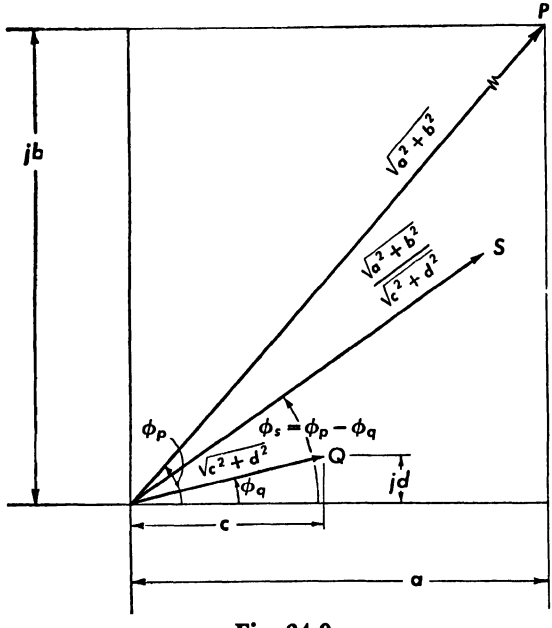


Fig. 24·9

algebraic quotient of $a + jb$ divided by $c + jd$ is a complex number defining the *vector* quotient of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ (see Fig. 24·9).

1. By definition,

$$\overrightarrow{OS} = \frac{|OP|}{|OQ|} \underline{/\phi_p - \phi_q} \qquad (14)$$

where

$$\phi_p = \text{reference angle of } \overrightarrow{OP}$$

$$\phi_q = \text{reference angle of } \overrightarrow{OQ}$$

2. If we let

$$\overrightarrow{OP} = a + jb$$

and

$$\overrightarrow{OQ} = c + jd$$

then

$$\overrightarrow{OS} = (a + jb) \div (c + jd)$$

$$\overrightarrow{OS} = \frac{(a + jb)}{(c + jd)}$$

Rationalizing the denominator (see Sec. 9·16 and Example 57, page 177),

$$\overrightarrow{OS} = \frac{(a + jb)(c - jd)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + j\frac{(bc - ad)}{c^2 + d^2} \tag{15}$$

$$|OS| = \sqrt{\left(\frac{ac + bd}{c^2 + d^2}\right)^2 + \left(\frac{bc - ad}{c^2 + d^2}\right)^2}$$

$$|OS| = \sqrt{\frac{a^2c^2 + 2abcd + b^2d^2 + b^2c^2 - 2abcd + a^2d^2}{(c^2 + d^2)^2}}$$

$$|OS| = \frac{1}{c^2 + d^2}\sqrt{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2}$$

$$|OS| = \frac{1}{c^2 + d^2}\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}$$

$$|OS| = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}}$$

But

$$\sqrt{a^2 + b^2} = |OP|$$

and

$$\sqrt{c^2 + d^2} = |OQ|$$

Therefore

$$|OS| = \frac{|OP|}{|OQ|} \tag{16}$$

From Eq. (15)

$$\tan \phi_s = \frac{\dfrac{bc - ad}{c^2 + d^2}}{\dfrac{ac + bd}{c^2 + d^2}} = \frac{bc - ad}{ac + bd} \tag{17}$$

where

$$\phi_s = \text{reference angle of } \overrightarrow{OS}$$

Dividing numerator and denominator of Eq. (17) by ac,

$$\tan \phi_s = \frac{\frac{bc}{ac} - \frac{ad}{ac}}{\frac{ac}{ac} + \frac{bd}{ac}} = \frac{\frac{b}{a} - \frac{d}{c}}{1 + \frac{b}{a} \times \frac{d}{c}}$$

But

$$\frac{b}{a} = \tan \phi_p$$

and

$$\frac{d}{c} = \tan \phi_q$$

Therefore

$$\tan \phi_s = \frac{\tan \phi_p - \tan \phi_q}{1 + \tan \phi_p \tan \phi_q} \tag{18}$$

From Eq. (7), page 504, we recognize that if Eq. (18) is valid, then

$$\phi_s = \phi_p - \phi_q \tag{19}$$

Therefore from Eqs. (16) and (19)

$$\overrightarrow{OS} = \frac{|OP|}{|OQ|} \underline{/\phi_p - \phi_q} \tag{20}$$

Equation (14) is a definitive statement of the *vector* quotient of $\overrightarrow{OP}$ divided by $\overrightarrow{OQ}$. Equation (20) is the result of the *algebraic* division of $(a + jb)$ by $(c + jd)$. These two equations are identical. Therefore, we may conclude that the *vector* division may be accomplished by the *algebraic* division of the complex numbers representing the vectors.

EXAMPLE 9. Find $\overrightarrow{OR}$ in polar form if $\overrightarrow{OR} = \dfrac{2 + j3}{5 - j4}$.

$$\overrightarrow{OR} = \frac{(2 + j3)(5 + j4)}{(5 - j4)(5 + j4)}$$

Here we have multiplied both numerator and denominator of the given ratio by the complex conjugate of the denominator (see Example 57, page 177). The complex conjugate of $(a + jb)$ is, by definition, $(a - jb)$. The

product of a complex number and the conjugate is always a real number. The multiplications are written out below:

$$\begin{array}{r} 2 + j3 \\ \underline{5 + j4} \\ 10 + j15 + j8 + j^2 12 \\ 10 + j15 + j8 - 12 \\ -2 + j23 = \text{numerator} \end{array}$$

$$\begin{array}{r} 5 - j4 \\ \underline{5 + j4} \\ 25 - j20 + j20 - j^2 16 \\ 25 - j20 + j20 + 16 \\ 25 + 16 \\ 41 = \text{denominator} \end{array}$$

Therefore

$$\overrightarrow{OR} = \frac{2 + j3}{5 - j4} = \frac{-2 + j23}{41}$$

$$\overrightarrow{OR} = \frac{-2}{41} + j\frac{23}{41}$$

$$\overrightarrow{OR} = -0.04878 + j0.56098$$

The magnitude of the vector quotient will be

$$|OR| = \sqrt{(-0.04878)^2 + 0.56098^2} = 0.5631$$

$$\overrightarrow{OR} = 0.5631 \angle \tan^{-1} \frac{0.56098}{-0.04878}$$

$$\tan \phi_r = \frac{0.56098}{-0.04878} = -11.500$$

When ϕ_r = reference angle of $\overrightarrow{OR}$

$$\phi_r = 94°58'$$

$$\overrightarrow{OR} = 0.5631 \angle 94°58'$$

Compare with Example 20, page 496.

EXERCISE 5

1. Using complex notation, find the quotient of the following vectors. Express the answer in both polar and rectangular form. Compare these problems with Probs. 16 to 19, page 498.

a. $\dfrac{3 + j2}{-2 + j7}$ **b.** $\dfrac{4 + j5}{1 + j2}$ **c.** $\dfrac{-6 - j5}{-2 - j3}$ **d.** $\dfrac{5 - j1}{-3 + j2}$

In the following problems, divide as indicated.

2. $\dfrac{3+j7}{2+j5}$	**3.** $\dfrac{-4+j2}{3+j6}$	**4.** $\dfrac{-3-j4}{3+j5}$	**5.** $\dfrac{7-j3}{3+j2}$
6. $\dfrac{3+j7}{-2+j5}$	**7.** $\dfrac{-4+j2}{-3+j6}$	**8.** $\dfrac{-3-j4}{-3+j5}$	**9.** $\dfrac{7-j3}{-3+j2}$
10. $\dfrac{3+j7}{2-j5}$	**11.** $\dfrac{-4+j2}{3-j6}$	**12.** $\dfrac{-3-j4}{3-j5}$	**13.** $\dfrac{7-j3}{3-j2}$
14. $\dfrac{3+j7}{-2-j5}$	**15.** $\dfrac{-4+j2}{-3-j6}$	**16.** $\dfrac{-3-j4}{-3-j5}$	**17.** $\dfrac{7-j3}{-3-j2}$

EXERCISE 6

Using complex notation exclusively, solve the following vector equations. Leave the answer in rectangular form.

1. $\overrightarrow{OP} = \dfrac{(4+j7)(3-j6)}{(4+j7)+(3-j6)}$

2. $\overrightarrow{OP} = \dfrac{(146-j248)(320-j480)}{(146-j248)+(320-j480)}$

3. $\overrightarrow{OP} = \dfrac{(3.6-j4.7)(7.9+j5.6)}{(3.6-j4.7)+(7.9+j5.6)}$

4. $\overrightarrow{OP} = \dfrac{(27.6-j38.5)(27.9+j35.8)}{(27.6-j38.5)+(27.9+j35.8)}$

5. $\overrightarrow{OP} = \dfrac{(7.8+j3.8)(4.6+j3.5)}{(7.8+j3.8)+(4.6+j3.5)}$

6. $\overrightarrow{OP} = \dfrac{(2.4+j3.7)(2.5+j7.6)}{(2.4+j3.7)+(2.5+j7.6)}$

7. $\overrightarrow{OP} = \dfrac{(25.8-j36.4)(48.7+j32.9)}{(25.8-j36.4)+(48.7+j32.9)}$

8. $\overrightarrow{OP} = \dfrac{(-20+j27)(30-j20)}{(-20+j27)+(30-j20)}$

24·7 An Alternative Vector Expression. We have shown that the vector $\overrightarrow{OP}$ may be represented by any of the following expressions (see Fig. 24·10):

$$\overrightarrow{OP} = |OP|/\underline{\phi_p} \tag{21}$$

$$\overrightarrow{OP} = a + jb \tag{22}$$

$$\overrightarrow{OP} = \sqrt{a^2+b^2}/\underline{\phi_p} \tag{23}$$

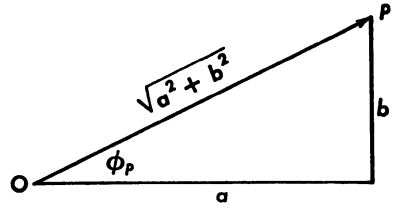


Fig. 24·10

Where ϕ_p = the reference angle of $\overrightarrow{OP}$,

$$\tan \phi_p = \frac{b}{a}$$

$$|OP| = \sqrt{a^2 + b^2}$$

We may multiply and divide the right-hand side of Eq. (22) by $\sqrt{a^2 + b^2}$ and obtain

$$\overrightarrow{OP} = \sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}} + j\frac{b}{\sqrt{a^2 + b^2}}\right) \tag{24}$$

Referring to Fig. 24·10,

$$\frac{a}{\sqrt{a^2 + b^2}} = \cos \phi_p$$

$$\frac{b}{\sqrt{a^2 + b^2}} = \sin \phi_p$$

Therefore

$$\overrightarrow{OP} = |OP|(\cos \phi_p + j \sin \phi_p) = |OP|\underline{/\phi_p} \tag{25}$$

EXERCISE 7

1. If $\overrightarrow{OP} = 3 + j5$, express $\overrightarrow{OP}$ in the form

$$\overrightarrow{OP} = |OP|(\cos \phi + j \sin \phi)$$

2. If $\overrightarrow{OP} = 15(\cos 20° + j \sin 20°)$, find a and b in the equation

$$\overrightarrow{OP} = a + jb$$

3. Using Eq. (25), write an expression describing the vector whose magnitude is 150 and whose reference angle is 130°.
4. Show that the magnitude of the vector

$$(\cos \phi + j \sin \phi)$$

is unity.

24·8 The *n*th Power of a Vector.[1] We shall define the *n*th power of the vector $\overrightarrow{OP}$ as being the vector whose magnitude is the arithmetic *n*th power of the magnitude of $\overrightarrow{OP}$ and whose reference angle is *n* times the reference angle of $\overrightarrow{OP}$. That is,

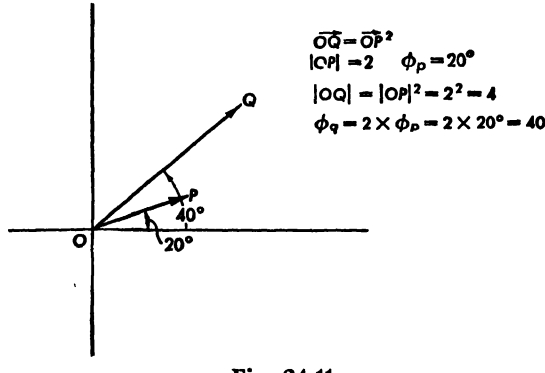


Fig. 24·11

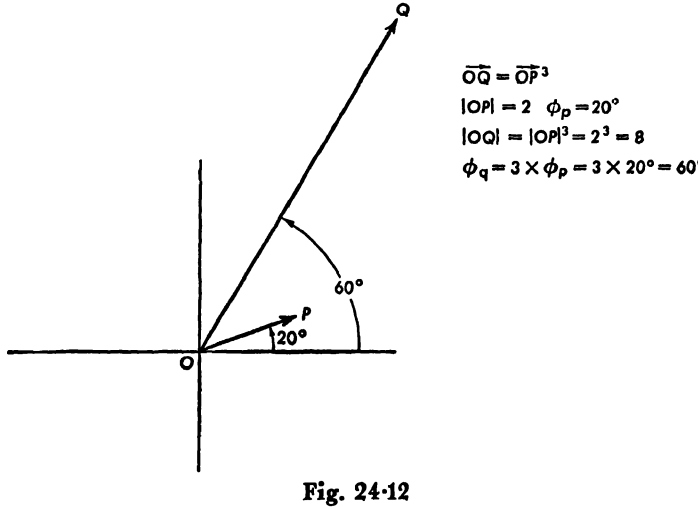


Fig. 24·12

$$\overrightarrow{OP}^n = |OP|^n \underline{/n\phi_p} \tag{26}$$

where ϕ_p = the reference angle of $\overrightarrow{OP}$
n = any real number

or, following the principle illustrated in Eq. (25),

$$\overrightarrow{OP}^n = |OP|^n(\cos n\phi_p + j \sin n\phi_p) \tag{27}$$

[1] See Figs. 24·11 to 24·13.

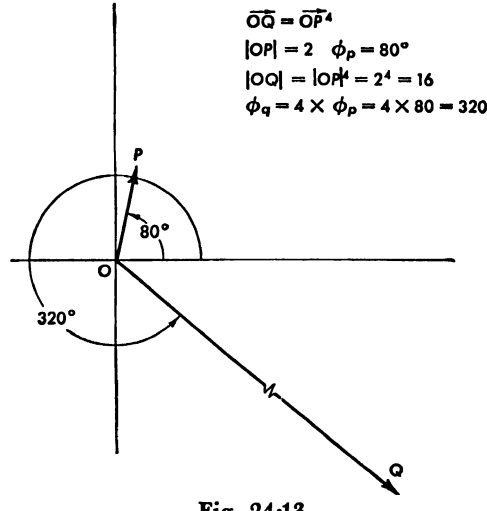


Fig. 24·13

EXERCISE 8

1. If $\overrightarrow{OP} = 5/20°$, find $\overrightarrow{OP}^3$.
2. If $\overrightarrow{OP} = 3/120°$, find $\overrightarrow{OP}^2$.
3. If $\overrightarrow{OP} = 2/150°$, find $\overrightarrow{OP}^4$.
4. If $\overrightarrow{OP} = 22/360°$, find $\overrightarrow{OP}^2$.
5. If $\overrightarrow{OP} = 5/90°$, find $\overrightarrow{OP}^1$.

24·9 The *n*th Roots of a Vector. We shall define an *n*th root of the vector $\overrightarrow{OP}$ as being a vector whose magnitude is the arithmetic *n*th root of $|OP|$ and whose reference angle is $1/n$ times the reference angle of $\overrightarrow{OP}$.

Usually we express the direction of a vector in terms of an angle

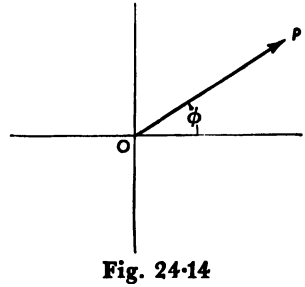


Fig. 24·14

less than 360°, or 2π radians, as in Fig. 24·14. However, we can express the same direction as $(\phi + 2\pi k)$ in radians or as $(\phi + 360°k)$ in degrees, where k is any integer including zero (see Fig. 24·15).

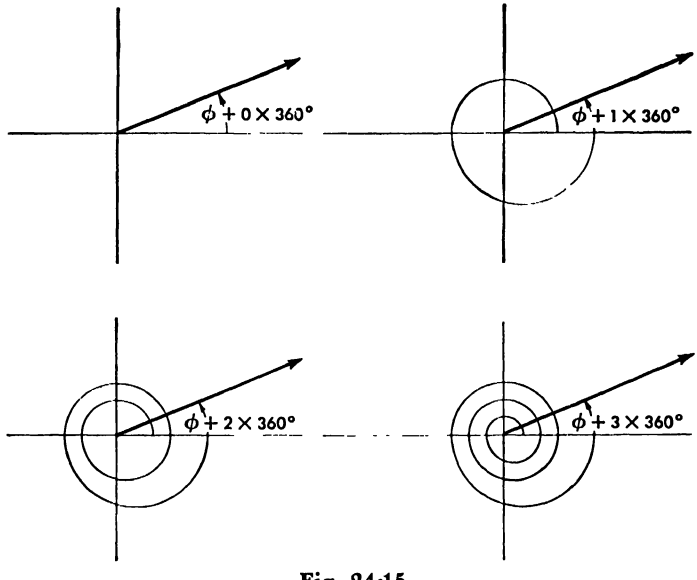


Fig. 24·15

Such nomenclature ordinarily serves no useful purpose, but in the present instance we shall have need for it. We shall indicate the nth roots of $\overrightarrow{OP}$ as follows:

$$\sqrt[n]{\overrightarrow{OP}} = \sqrt[n]{|OP|} \Big/ \frac{\phi_p + 360°k}{n} \tag{28}$$

Now let us tabulate the angle $\dfrac{\phi_p + 360°k}{n}$ for various values of k:

k	$\dfrac{\phi_p + 360°k}{n}$
0	$\dfrac{\phi_p}{n}$
1	$\dfrac{\phi_p}{n} + \dfrac{360°}{n}$
2	$\dfrac{\phi_p}{n} + \dfrac{2 \times 360°}{n}$

$$3 \qquad \frac{\phi_p}{n} + \frac{3 \times 360^\circ}{n}$$

$$4 \qquad \frac{\phi_p}{n} + \frac{4 \times 360^\circ}{n}$$

$$5 \qquad \frac{\phi_p}{n} + \frac{5 \times 360^\circ}{n}$$

$$n - 1 \qquad \frac{\phi_p}{n} + \frac{(n-1)360^\circ}{n} = \frac{\phi_p}{n} - \frac{360^\circ}{n} + 360^\circ$$

$$n \qquad \frac{\phi_p}{n} + \frac{n360^\circ}{n} = \frac{\phi_p}{n} + 360^\circ$$

$$n + 1 \qquad \frac{\phi_p}{n} + \frac{360^\circ}{n} + 360^\circ$$

$$n + 2 \qquad \frac{\phi_p}{n} + \frac{720^\circ}{n} + 360^\circ$$

$$n + 3 \qquad \frac{\phi_p}{n} + \frac{1{,}080^\circ}{n} + 360^\circ$$

$$n + 4 \qquad \frac{\phi_p}{n} + \frac{1{,}440^\circ}{n} + 360^\circ$$

$$n + 5 \qquad \frac{\phi_p}{n} + \frac{1{,}800^\circ}{n} + 360^\circ$$

In the above series of roots observe that the respective reference angles increase by steps of $\frac{360^\circ}{n}$. When n steps have been completed, the reference angle has increased by

$$\frac{n \times 360^\circ}{n} = 360^\circ$$

or one complete revolution. At this point a sequence of vectors begins which will be superimposed on those already found. Thus, when $k > (n - 1)$, no new vector roots are discovered.

In general, there are n and only n nth roots of a vector.

Following the principle illustrated in Eq. (25), Eq. (28) may be written

$$\sqrt[n]{\overrightarrow{OP}} = \sqrt[n]{|OP|/\underline{\phi_p}} = \sqrt[n]{|OP|}\left(\cos\frac{\phi_p + 360^\circ k}{n} + j\sin\frac{\phi_p + 360^\circ k}{n}\right) \tag{29}$$

EXAMPLE 10. Find the five fifth roots of the vector $32\underline{/30°}$ (see Fig. 24·16). Applying Eq. (28),

$$\sqrt[5]{32\underline{/30°}} = \sqrt[5]{32}\,\underline{/\frac{30° + 360°k}{5}}$$

$$= 2\,\underline{/\frac{30° + 360°k}{5}}$$

where $k = 0, 1, 2, 3, 4$.

$\overrightarrow{OQ_1} = \overrightarrow{OQ_2} = \overrightarrow{OQ_3} = \overrightarrow{OQ_4} = \overrightarrow{OQ_5} = \sqrt[5]{\overrightarrow{OP}}$

$|OP| = 32$

$\phi_P = 30°$

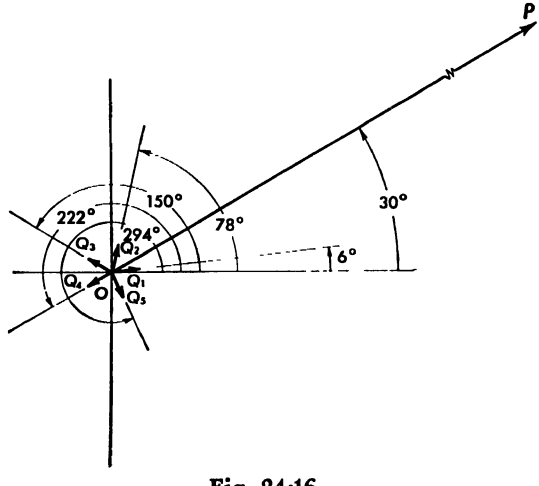


Fig. 24·16

Let us tabulate the angles $\dfrac{30° + 360°k}{5}$ for a succession of values of k:

k	$\dfrac{30° + 360°k}{5}$
0	$\dfrac{30°}{5} = 6°$
1	$\dfrac{30° + 360°}{5} = 78°$
2	$\dfrac{30° + 720°}{5} = 150°$
3	$\dfrac{30° + 1{,}080°}{5} = 222°$
4	$\dfrac{30° + 1{,}440°}{5} = 294°$

5 $\dfrac{30° + 1{,}800°}{5} = 366°$, or a reference angle of 6°

6 $\dfrac{30° + 2{,}160°}{5} = 438°$, or a reference angle of 78°

7 $\dfrac{30° + 2{,}520°}{5} = 510°$, or a reference angle of 150°

8 $\dfrac{30° + 2{,}880°}{5} = 582°$, or a reference angle of 222°

From this series of reference angles we can distinguish five and only five fifth roots of the vector $32\angle 30°$. These roots are referred to Fig. 24·16.

$$\overrightarrow{OQ}_1 = 2\angle 6°$$
$$\overrightarrow{OQ}_2 = 2\angle 78°$$
$$\overrightarrow{OQ}_3 = 2\angle 150°$$
$$\overrightarrow{OQ}_4 = 2\angle 222°$$
$$\overrightarrow{OQ}_5 = 2\angle 294°$$

Subsequent roots found by making $k > (n - 1)$ (in this case, making $k > 4$) merely duplicate those already found.

EXERCISE 9

Find all the indicated roots of the following vectors. Plot the vector roots to scale.

1. $\sqrt[3]{8\angle 30°}$ **2.** $\sqrt[3]{27\angle 180°}$ **3.** $\sqrt[2]{4\angle 0°}$

4. $\sqrt[2]{9\angle 210°}$ **5.** $\sqrt[4]{81\angle 280°}$ **6.** $\sqrt[10]{1{,}024\angle 330°}$

24·10 Demoivre's Theorem. According to Eq. (25), the vector $\overrightarrow{OP}$ may be expressed as

$$\overrightarrow{OP} = |OP|(\cos \phi + j \sin \phi) \tag{30}$$

We may raise both sides of Eq. (30) to the nth power, obtaining

$$\overrightarrow{OP}^n = |OP|^n(\cos \phi + j \sin \phi)^n \tag{31}$$

We may now equate the right-hand sides of Eqs. (27) and (31) to obtain

$$|OP|^n(\cos \phi + j \sin \phi)^n = |OP|^n(\cos n\phi + j \sin n\phi)$$

Dividing both sides of the above equation by $|OP|^n$ gives

$$(\cos \phi + j \sin \phi)^n = \cos n\phi + j \sin n\phi \tag{32}$$

If we take the nth root of Eq. (30), we obtain

$$\sqrt[n]{\overrightarrow{OP}} = \sqrt[n]{|OP|}\ \sqrt[n]{\cos \phi + j \sin \phi} \tag{33}$$

We may now equate the right-hand sides of Eqs. (29) and (33) to obtain

$$\sqrt[n]{|OP|}\ \sqrt[n]{\cos \phi + j \sin \phi} = \sqrt[n]{|OP|}\left(\cos \frac{\phi + 360°k}{n} + j \sin \frac{\phi + 360°k}{n}\right) \tag{34}$$

Dividing both sides of Eq. (34) by $\sqrt[n]{|OP|}$ gives

$$\sqrt[n]{\cos \phi + j \sin \phi} = \cos \frac{\phi + 360°k}{n} + j \sin \frac{\phi + 360°k}{n} \tag{35}$$

Equations (32) and (35) are forms of Demoivre's theorem. As a special case of Demoivre's theorem, let us investigate the roots of the vector $|OP|(\cos \phi + j \sin \phi)$ when $\phi = 0$. That is to say, let us find the roots of the vector

$$\overrightarrow{OP} = (\cos 0 + j \sin 0)$$

or

$$\overrightarrow{OP} = \cos 0$$

or

$$\overrightarrow{OP} = 1$$

Here $\overrightarrow{OP}$ is a real number, unity, and we are about to find a method of calculating the n nth roots of unity. Following Eq. (29),

$$\sqrt[n]{\overrightarrow{OP}} = \sqrt[n]{|OP|}\left(\cos \frac{0 + 360°k}{n} + j \sin \frac{0 + 360°k}{n}\right)$$

$$\sqrt[n]{\overrightarrow{OP}} = \sqrt[n]{|OP|}\left(\cos \frac{360°k}{n} + j \sin \frac{360°k}{n}\right)$$

where

$$0 \leq k < n$$

EXAMPLE 11. Find the three cube roots of 1.

When $k = 0$,

$$\sqrt[3]{1} = \sqrt[3]{1}\left(\cos \frac{360° \times 0}{3} + j \sin \frac{360° \times 0}{3}\right)$$

$$= \cos 0 + j \sin 0 = 1 + 0 = +1$$

When $k = 1$,

$$\sqrt[3]{1} = \sqrt[3]{1}\left(\cos\frac{360°}{3} + j\sin\frac{360°}{3}\right)$$
$$= (\cos 120 + j\sin 120)$$
$$= 1\left(-\frac{1}{2} + j\cdot\frac{\sqrt{3}}{2}\right) = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

When $k = 2$,

$$\sqrt[3]{1} = 1\left(\cos\frac{360° \times 2}{3} + j\sin\frac{360° \times 2}{3}\right)$$
$$= 1(\cos 240° + j\sin 240°)$$
$$= 1\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) = -\tfrac{1}{2} - j\frac{\sqrt{3}}{2}$$

These roots can also be found by the algebraic solution of the equation

$$x^3 - 1 = 0$$

Factoring,

$$(x - 1)(x^2 + x + 1) = 0$$

When

$$x - 1 = 0$$
$$x = 1$$

When

$$x^2 + x + 1 = 0$$
$$x = \frac{-1 \pm \sqrt{1 - 4}}{2}$$
$$x = -\frac{1}{2} \pm \frac{j\sqrt{3}}{2}$$

That these roots are in fact cube roots of 1 can be verified by multiplication. By elementary algebra,

$$\sqrt[3]{1} \times \sqrt[3]{1} \times \sqrt[3]{1} = 1$$

Also

$$\begin{array}{l} -\frac{1}{2} + j\frac{\sqrt{3}}{2} \\ -\frac{1}{2} + j\frac{\sqrt{3}}{2} \\ \hline +\frac{1}{4} - j\frac{\sqrt{3}}{4} - j\frac{\sqrt{3}}{4} + j^2\frac{3}{4} = -\frac{1}{2} - j\frac{\sqrt{3}}{2} \\ \qquad\qquad -\frac{1}{2} + j\frac{\sqrt{3}}{2} \\ \hline \qquad\qquad \frac{1}{4} + j\frac{\sqrt{3}}{4} - j\frac{\sqrt{3}}{4} - j^2\frac{3}{4} = 1 \end{array}$$

and

$$\begin{array}{r} -\frac{1}{2}-j\frac{\sqrt{3}}{2} \\ -\frac{1}{2}-j\frac{\sqrt{3}}{2} \\ \hline +\frac{1}{4}+j\frac{\sqrt{3}}{4}+j\frac{\sqrt{3}}{4}+j^2\frac{3}{4}= \end{array} \quad \begin{array}{r} -\frac{1}{2}+j\frac{\sqrt{3}}{2} \\ -\frac{1}{2}-j\frac{\sqrt{3}}{2} \\ \hline +\frac{1}{4}-j\frac{\sqrt{3}}{4}+j\frac{\sqrt{3}}{4}-j^2\frac{3}{4}=1 \end{array}$$

EXERCISE 10

1. Find the two square roots of 1.
2. Find the four fourth roots of 1.
3. Find the five fifth roots of 1.
4. Show that the sum of the five fifth roots of 1 is zero.
5. Prove that, in general, the sum of the n nth roots of 1 is zero.
6. Draw a scale vector diagram to illustrate Probs. 1, 2, 3, and 4.

EXAMPLE 12. Find the five fifth roots of 32.

$$\sqrt[5]{32} = \sqrt[5]{32}\left(\cos\frac{360°k}{5}+j\sin\frac{360°k}{5}\right)$$

When $k = 0$,

$$\begin{aligned}\sqrt[5]{32} &= 2(\cos 0° + j\sin 0°)\\ &= 2(1+0) = 2\end{aligned}$$

When $k = 1$,

$$\begin{aligned}\sqrt[5]{32} &= 2\left(\cos\frac{360°}{5}+j\sin\frac{360°}{5}\right)\\ &= 2(\cos 72° + j\sin 72°) = 2(0.30902 + j0.95106)\\ &= 0.61804 + j1.90212\end{aligned}$$

When $k = 2$,

$$\begin{aligned}\sqrt[5]{32} &= 2\left(\cos\frac{720°}{5}+j\sin\frac{720°}{5}\right)\\ &= 2(-0.80902 + j0.58779) = -1.61804 + j1.17558\end{aligned}$$

When $k = 3$,

$$\begin{aligned}\sqrt[5]{32} &= 2\left(\cos\frac{1{,}080°}{5}+j\sin\frac{1{,}080°}{5}\right)\\ &= 2(\cos 216° + j\sin 216°)\\ &= 2(-0.80902 - j0.58779) = -1.61804 - j1.17558\end{aligned}$$

When $k = 4$,

$$\sqrt[5]{32} = 2\left(\cos\frac{1{,}440°}{5} + j\sin\frac{1{,}440°}{5}\right)$$
$$= 2(\cos 288° + j\sin 288°)$$
$$= 2(+0.30902 - j0.95106) = +0.61804 - j1.90212$$

EXERCISE 11

1. Find the three cube roots of 27.
2. Find the four fourth roots of -81.
3. Find the six sixth roots of 729.
4. Show that the square of one of the complex sixth roots of 729 is a cube root of 729.

24·11 Euler's Equation. We have used the binomial form

$$\cos\phi + j\sin\phi$$

to represent a vector of unit magnitude. We shall in this section develop an equivalent monomial expression V. We shall let V equal the above binomial in the special case where $\phi = 1$ radian; *i.e.*, we shall let

$$V = \cos 1 + j\sin 1$$

Raising both sides of the above equation to the α power, we obtain

$$V^{\alpha} = (\cos 1 + j\sin 1)^{\alpha}$$

Following Demoivre's theorem,

$$(\cos 1 + j\sin 1)^{\alpha} = \cos\alpha + j\sin\alpha$$

Therefore

$$V^{\alpha} = \cos\alpha + j\sin\alpha \tag{36}$$

Again from Demoivre's theorem,

$$V^{-\alpha} = (\cos 1 + j\sin 1)^{-\alpha} = \cos(-\alpha) + j\sin(-\alpha) \tag{37}$$

Note that the expression $(\cos\alpha + j\sin\alpha)$ is adequate to represent any vector of unit magnitude in the complex plane.

By inspection of Fig. 17·23,

$$\cos(-\alpha) = \cos\alpha$$

and

$$\sin(-\alpha) = -\sin\alpha$$

Therefore

$$V^{-\alpha} = \cos\alpha - j\sin\alpha \tag{38}$$

Now, solving Eqs. (36) and (38) simultaneously,

$$\begin{aligned} \cos\alpha + j\sin\alpha &= V^{\alpha} \\ \cos\alpha - j\sin\alpha &= V^{-\alpha} \\ \hline 2\cos\alpha &= V^{\alpha} + V^{-\alpha} \\ 2j\sin\alpha &= V^{\alpha} - V^{-\alpha} \end{aligned}$$

Therefore

$$\cos\alpha = \tfrac{1}{2}(V^{\alpha} + V^{-\alpha}) \tag{39}$$

$$\sin\alpha = \frac{1}{2j}(V^{\alpha} - V^{-\alpha}) \tag{40}$$

We shall now define the nature of $\log V$ such that

$$V^{\alpha} = e^{\alpha \log V} \qquad \text{and} \qquad V^{-\alpha} = e^{-\alpha \log V} \tag{41}$$

Then we shall see what significance we must as a consequence attach to $\log V$.

Following Eq. (2), page 328,

$$V^{\alpha} = e^{\alpha \log V} = 1 + \alpha \log V + \frac{\alpha^2(\log V)^2}{2!} + \frac{\alpha^3(\log V)^3}{3!} + \frac{\alpha^4(\log V)^4}{4!} + \frac{\alpha^5(\log V)^5}{5!} \cdots \tag{42}$$

and

$$V^{-\alpha} = e^{-\alpha \log V} = 1 - \alpha \log V + \frac{\alpha^2(\log V)^2}{2!} - \frac{\alpha^3(\log V)^3}{3!} + \frac{\alpha^4(\log V)^4}{4!} - \frac{\alpha^5(\log V)^5}{5!} \cdots \tag{43}$$

Adding Eqs. (42) and (43) gives

$$V^{\alpha} + V^{-\alpha} = e^{\alpha \log V} + e^{-\alpha \log V} = 2 + \frac{2\alpha^2(\log V)^2}{2!} + \frac{2\alpha^4(\log V)^4}{4!} + \frac{2\alpha^6(\log V)^6}{6!} \cdots \tag{44}$$

Substituting Eq. (44) in Eq. (39)

$$\cos\alpha = 1 + \frac{\alpha^2(\log V)^2}{2!} + \frac{\alpha^4(\log V)^4}{4!} + \frac{\alpha^6(\log V)^6}{6!} + \frac{\alpha^8(\log V)^8}{8!} \cdots \tag{45}$$

If we subtract Eq. (43) from Eq. (42) and substitute in Eq. (40) we obtain

$$\sin \alpha = \frac{1}{j}\left[\alpha \log V + \frac{\alpha^3(\log V)^3}{3!} + \frac{\alpha^5(\log V)^5}{5!} + \frac{\alpha^7(\log V)^7}{7!} \cdots\right] \tag{46}$$

If we divide both sides of Eq. (46) by α,

$$\frac{\sin \alpha}{\alpha} = \frac{1}{j}\left[\log V + \frac{\alpha^2(\log V)^3}{3!} + \frac{\alpha^4(\log V)^5}{5!} + \frac{\alpha^6(\log V)^7}{7!} \cdots\right] \tag{47}$$

Equation (47) is equally valid whether α is large or small. If we let α approach zero, by inspection the right-hand side of Eq. (47) approaches $\frac{\log V}{j}$. By Sec. 19·10 the left-hand side approaches 1. Therefore, equating these limits,

$$1 = \frac{\log V}{j} \tag{48}$$

whence

$$\log V = j \tag{49}$$

Consequently

$$V = e^j \tag{50}$$

Substituting for V its equivalent,

$$e^j = \cos 1 + j \sin 1 \tag{51}$$

Now, raising both sides of Eq. (51) to the α power and following Demoivre's theorem,

$$(e^j)^\alpha = (\cos 1 + j \sin 1)^\alpha = \cos \alpha + j \sin \alpha = e^{j\alpha} \tag{52}$$

Equation (52) is known as Euler's equation.

We could multiply both sides of Eq. (52) by some vector magnitude, say $|OP|$; then

$$|OP|e^{j\alpha} = |OP|(\cos \alpha + j \sin \alpha)$$

or

$$\overrightarrow{OP} = |OP|e^{j\alpha} \tag{53}$$

24·12 The Sine and Cosine Series. If we substitute Eq. (50) in Eqs. (39) and (40), we obtain

$$\cos \alpha = \tfrac{1}{2}(e^{j\alpha} + e^{-j\alpha}) \tag{54}$$

and

$$\sin \alpha = \frac{1}{2j}(e^{j\alpha} - e^{-j\alpha}) \tag{55}$$

If we substitute $j\alpha$ and $-j\alpha$ for x in the binomial expansion of e^x (see Sec. 16·3) and substitute the equivalent expansion for $e^{j\alpha}$ and $e^{-j\alpha}$ in Eqs. (54) and (55), we obtain

$$\cos \alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \frac{\alpha^8}{8!} - \frac{\alpha^{10}}{10!} \cdots \tag{56}$$

and

$$\sin \alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \frac{\alpha^9}{9!} - \frac{\alpha^{11}}{11!} \cdots \tag{57}$$

Equations (56) and (57) may be used to calculate numerical values for the sine and cosine of the angle α where α is in radians. The other functions could be calculated by appropriate conversion formulas.

EXERCISE 12

Using Eq. (56) or (57), find the value of the following functions where the angle is in radians:

1. cos 0.5 **2.** sin 2 **3.** sin 0 **4.** cos 1 **5.** sin $-\frac{1}{4}$ **6.** sin 0.1

24·13 Rotating Vectors. In Sec. 23.5 we discussed rotating vectors and showed how a rotating vector could be described in terms of one of its projections in the form

$$y = E \sin (\omega t + \psi) \tag{58}$$

In Sec. 24·7 we developed another nomenclature which can be used to describe a vector; for example,

$$\overrightarrow{OA} = E(\cos \alpha + j \sin \alpha) \tag{59}$$

The angle α may be a variable and a function of time. Therefore we may let

$$\alpha = \omega t \tag{60}$$

To make the situation general, we shall introduce the phase constant ψ and

$$\overrightarrow{OA} = E[\cos (\omega t + \psi) + j \sin (\omega t + \psi)] \tag{61}$$

In Sec. 24·11 we developed an equivalent expression such that we may describe the same rotating vector as

$$\overrightarrow{OA} = Ee^{j\alpha} \tag{62}$$

If α is a function of time, we may let $\alpha = \omega t$ and write

$$\vec{OA} = Ee^{j(\omega t + \psi)} \tag{63}$$

where ψ is a phase constant.

In Eqs. (58), (61), and (63), E is the amplitude of the function, ω is the angular velocity of the rotating vector, t is time, and ψ is a constant phase angle.

Any of these three forms may be used to describe a rotating vector. However, it may very well be that in some specific application one of these three forms is more convenient to use than the others. All three are in common use.

Equation (58) is more convenient to use when we are plotting a projection of the rotating vector. Equation (61) is more convenient when we wish to combine rotating vectors by addition, subtraction, multiplication, or division.

Equation (63) is more convenient when transient terms are present.

EXERCISE 13

Find the amplitude, frequency, period, and phase constant for the following functions where ϵ is the base of natural logarithms:

1. $ve = 10 \sin (120\pi t + 60°)$
2. $ve = 10[\cos (120\pi t + 60°) + j \sin (120\pi t + 60°)]$
3. $ve = 10e^{j(120\pi t + 60°)}$
4. $i = 15\left[\cos\left(628.32t + \frac{\pi}{2}\right) + j \sin\left(628.32t + \frac{\pi}{2}\right)\right]$
5. $i = 25\epsilon^{j(2513.27t)}$
6. $e = 25 \sin (2513.27t)$

24·14 Some Interesting Side Lights on Euler's Equation. In Eq. (52) let $\alpha = \pi$ radians; then

$$\begin{aligned} e^{j\pi} &= \cos \pi + j \sin \pi \\ e^{j\pi} &= -1 + 0 \\ e^{j\pi} &= -1 \end{aligned}$$

If we extract the square root of both sides,

$$e^{\frac{j\pi}{2}} = \sqrt{-1} = j$$

Here we have in one equation three of the most interesting and important numbers in all mathematics, the base of natural logarithms, the ratio of the circumference of a circle to its diameter, and the imaginary unit j.

If we take logarithms of both sides,

$$j\pi = \log_e (-1)$$

Thus negative numbers do have logarithms even though these logarithms are not real numbers.

24·15 The Hyperbolic Sine and Cosine. By definition the quantity $\frac{e^u - e^{-u}}{2}$ is called the *hyperbolic sine of u*. It is abbreviated

$$\sinh u = \frac{e^u - e^{-u}}{2} \tag{64}$$

Also by definition, the quantity $\frac{e^u + e^{-u}}{2}$ is called the hyperbolic cosine of u. It is abbreviated

$$\cosh u = \frac{e^u + e^{-u}}{2} \tag{65}$$

Equations (54) and (55) may now be written

$$\cosh j\alpha = \cos \alpha \tag{66}$$

and

$$\sinh j\alpha = j \sin \alpha \tag{67}$$

UNIT EIGHT

AN INTRODUCTION TO CALCULUS

An introduction to the techniques of calculus with particular attention to their graphical interpretation

Chapter 25 GRAPHICAL METHODS

Thus far we have been emphasizing fixed situations. Now we shall deal for the most part with variable quantities.

25·1 Functions. A *function* can be defined as follows: "When a *variable quantity varies, in some definite way*, with another variable quantity, the first variable is said to be a *function* of the second." The key words are *variable quantity—varies—in some definite way.*

At this point we are introducing a somewhat different interpretation of equations. Take, for example, the equation

$$C = \pi D \tag{1}$$

This is a linear equation, and if plotted, the slope is π (see Fig. 25·1). From a more elementary point of view, Eq. (1) is a recipe for finding the circumference of a circle when the diameter is known. From a more sophisticated point of view, it describes a *functional relationship.*

For example, consider the ripples of water formed on a placid lake when a stone is dropped in. The ripples extend outward in a circular pattern with ever-extending diameter. They move continually; at no two consecutive moments are the diameters identical; consequently at no two consecutive moments are the circumferences identical. Rather, the circumference and the diameter are both *variable quantities.* It is true that looking at Eq. (1) from a strictly elementary point of view, we can see that if the diameter is doubled, the circumference will be doubled. But here we are dealing with a numerical ratio between two fixed quantities, even though they may be fixed only momentarily.

In Eq. (1), then, both C and D are considered to be variable quantities. C depends upon D in a definite way, described by Eq. (1) itself or Fig. 25·1. Therefore C is a *function* of D.

25·2 Dependence. Ordinarily we consider one variable to be dependent on the other. For example, consider a coiled spring being

stretched under the action of a force. The greater the force, the greater the elongation of the spring. Here it is obvious that the force causes the elongation, and not the other way around. In this case the force is the *independent variable*, and the elongation is the *dependent variable*.

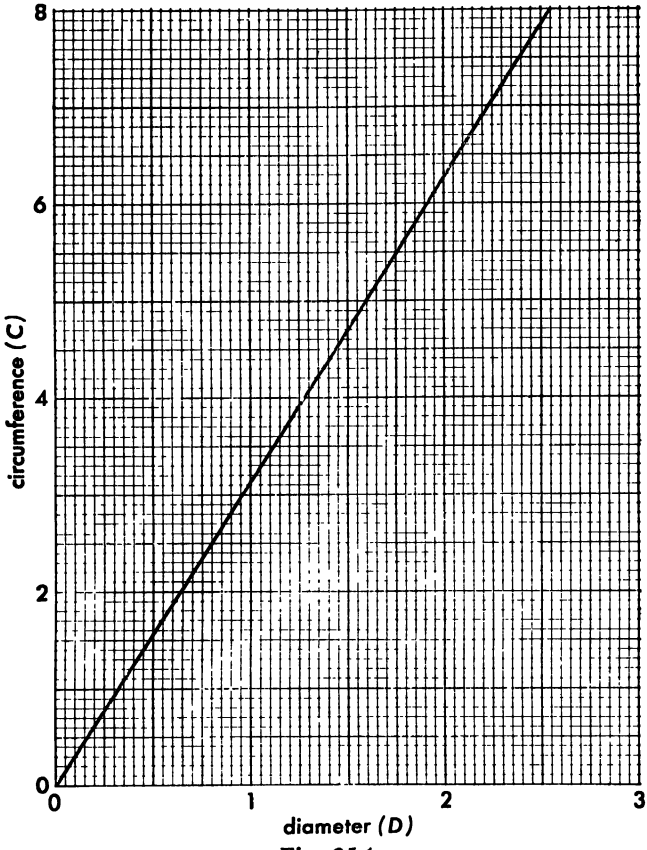


Fig. 25·1

In cases where the distinction is less obvious or nonexistent, the decision is correspondingly more arbitrary. Ordinarily, we consider the variable appearing alone on one side of an equation to be the *dependent* variable. Also, it is usual to consider the variable plotted vertically to be the *dependent* variable.

25·3 Rates of Change. Equation (3), page 117, defines the slope of a straight line as

$$m = \frac{y - y_1}{x - x_1} \tag{2}$$

where m = slope
x_1 and y_1 = coordinates of any point on a straight line
x and y = coordinates of any other point on the straight line

Another way to express the slope of a straight line is as follows:

$$\text{Slope} = \frac{\text{change in dependent variable}}{\text{change in independent variable}} \tag{3}$$

This is usually abbreviated to

$$\text{Slope} = \frac{\Delta y}{\Delta x} \quad \text{(applied to a straight line)} \tag{4}$$

where Δx = change in x (x in this case being independent variable) (5)

Δy = corresponding change in y (y in this case being dependent variable) (6)

Also

$$\Delta x = x - x_1 \tag{7}$$
$$\Delta y = y - y_1 \tag{8}$$

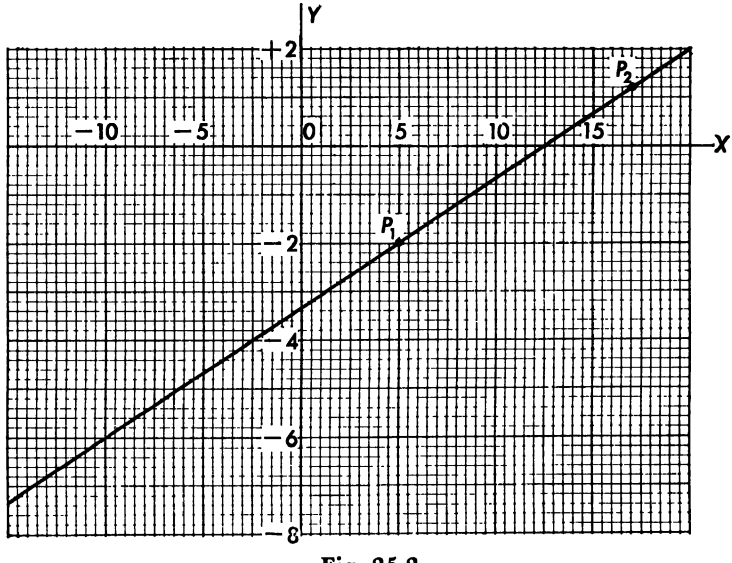


Fig. 25·2

For example, in Fig. 25·2, as x changes from $+5$ to $+17$, y changes from -2 to $+1.2$, or while x increases 12 units, y increases 3.2 units. In other words,

$$\Delta y = 1.2 - (-2.0) = 3.2 \tag{9}$$
$$\Delta x = 17 - 5 = 12 \tag{10}$$

and

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{3.2}{12} = \frac{4}{15} \tag{11}$$

Stated in words, "the slope of a straight line is the amount of change in the ordinate divided by the corresponding change in the abscissa." An equally valid definition of slope is "the slope of a straight line is the change in the dependent variable divided by the corresponding change in the independent variable."

We should bear in mind that these changes may be either positive or negative.

In the above example we could, with equal logic, have reasoned that as x *decreases* from $+17$ to $+5$, then y *decreases* from $+1.2$ to -2.0.

In this case:

$$\Delta x = -12 \tag{12}$$

$$\Delta y = -3.2 \tag{13}$$

$$\text{Slope} = \frac{-3.2}{-12} = +\frac{4}{15} \tag{14}$$

As a matter of practical expediency it is usually more convenient consistently to deal with *increasing* abscissas.

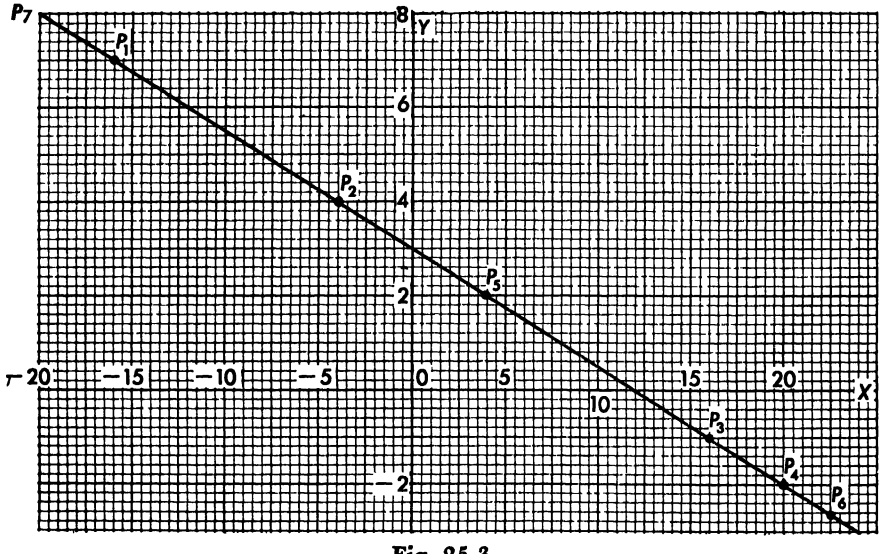


Fig. 25·3

Comparing Fig. 25·2 with Fig. 25·3, we may observe that as x *increases* from $+5$ to $+17$ in Fig. 25·2, y *increases* from -2 to $+1.2$.

In Fig. 25·3, as x *increases* from -20 to $+20$, y *decreases* from $+8$ to -2.

That is, in both cases we are considering x to increase, and consequently Δx is positive. However, in the first case, y increases and Δy is positive, while in the second case y decreases and Δy is negative.

Thus, as x increases, if y also increases the slope is positive, but if y decreases the slope is negative.

Notice that in Fig. 25·2, as x increases one unit, y increases $\frac{4}{15}$ units. In Fig. 25·3, as x increases 1 unit, then y decreases $\frac{1}{4}$ unit.

This suggests another concept of slope. "The slope of a straight line is the amount of change (including the sign) in the ordinate per unit *increase* in the abscissa."

EXAMPLE 1. If a straight line passes through the points $(-5, -16)$ and $(+10, +20)$, how much does y change when x changes 0.005?

$$\text{Slope} = m = \frac{20 - (-16)}{10 - (-5)} = \frac{20 + 16}{10 + 5} = \frac{36}{15} = \frac{12}{5} = +2.4$$

In other words,

$$\frac{\Delta y}{\Delta x} = +\frac{12}{5} = +2.4$$

In this example Δx is given to be 0.005 and we are required to find Δy. Therefore

$$\frac{\Delta y}{0.005} = 2.4$$

$$\Delta y = 2.4 \times 0.005 = 0.012 \quad \text{ANS.}$$

EXERCISE 1

1. a. Find the slope of the line in Fig. 25·3, using points P_1 and P_2.
b. Find Δy as x increases from -12 to $+2.5$.
c. How much does y change when x increases from 2.5 to 2.501?

2. a. Find the change in the ordinate of line in Fig. 25·2 when the abscissa changes from -10 to $+20$.
b. Find $\Delta y/\Delta x$ in part *a*.
c. When x decreases from $+12\frac{1}{2}$ to $+5$, what is Δy? (Remember to include the sign.)

3. a. In moving from P_3 to P_2 in Fig. 25·3, find Δx and Δy.
b. Find $\Delta y/\Delta x$ in part *a*.
c. In moving from P_2 to P_3 find Δx and Δy.
d. Find $\Delta y/\Delta x$ in part *c*.

4. a. Find y in Fig. 25·3 when $x = -15$.
b. What is $\Delta y/\Delta x$ at this point?
c. Between what values of x is y positive in Fig. 25·3 (consider only the region between P_1 and P_4)?
d. What is the slope between the two points in part *c* above?

25·4 Area under a Straight-line Graph. The phrase "area under a graph" refers to the area between the graph and the x axis (see Figs. 25·4 and 25·5).

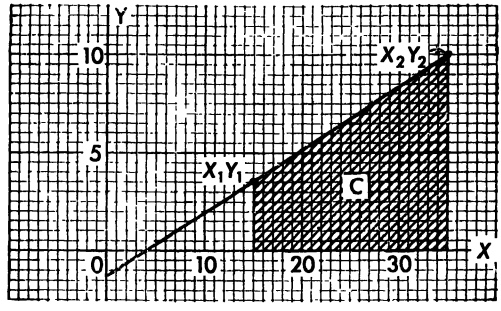


Fig. 25·4

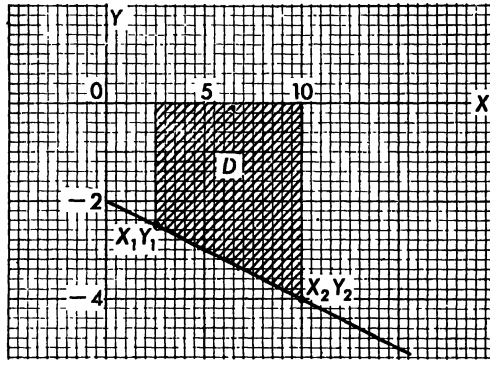


Fig. 25·5

The region C in Fig. 25·4 is the area under the graph between $x = +15$ and $x = +35$. The region D in Fig. 25·5 is the area "under" the graph between $x = +2.5$ and $x = +10$.

In the case of a straight line (and in this case only), the area under the graph is either a triangle or a trapezoid.

In Fig. 25·4

$$A_C = \frac{y_2 + y_1}{2}(x_2 - x_1) \tag{15}$$

In Fig. 25·5 the area under the graph is also

$$A_D = \frac{y_2 + y_1}{2}(x_2 - x_1) \tag{16}$$

By reading the graph,

$$A_C = \frac{10.0 + 3.5}{2}(35 - 15) = 135.0 \tag{17}$$

It is important to observe that we use the scale of the graph in calculating the area:

$$A_D = \frac{-4 + (-2.5)}{2}(10.0 - 2.5) = -24.375 \tag{18}$$

In this case the signs of the ordinates lead to a negative area. In the case of a practical problem the negative sign attached to the area may or may not be significant. We shall retain such negative signs, at least for the present.

The quantity $(y_2 + y_1)/2$ is the arithmetic average of y_1 and y_2. The symbol $\bar{y}$ is adopted to mean "average y." It is read "bar y" or "average y." In general

$$\bar{y} = \frac{y_2 + y_1}{2} \tag{19}$$

and

$$A = \bar{y}(x_2 - x_1) \tag{20}$$

where A = area under graph between ordinates erected at x_2 and x_1
$\bar{y}$ = average ordinate between x_2 and x_1

If we define the quantity $x_2 - x_1$ as the "base" of the area, we can define average ordinate as "that ordinate which, multiplied by the base, equals the area under the graph."

When we are dealing with a relatively long base, we ordinarily (but by no means essentially) denote the "base" of the area under the graph by the symbol $x_2 - x_1$. If we are considering a relatively short base, we often use the symbol Δx.

In preparation for work a little later on, it will be worthwhile to arrive at the average ordinate of a graph by a little different method.

We shall *first* find the area under the graph and *then* find the average ordinate (see Fig. 25·6).

The area under this graph has been divided into relatively *narrow* strips. Accordingly, we shall designate their width by Δx. We shall reserve the symbol $x_2 - x_1$ to designate the base of the entire area.

The area of each strip is the average ordinate of that strip multiplied by the width of the strip.

The total area under the graph is the sum of the areas of these individual strips.

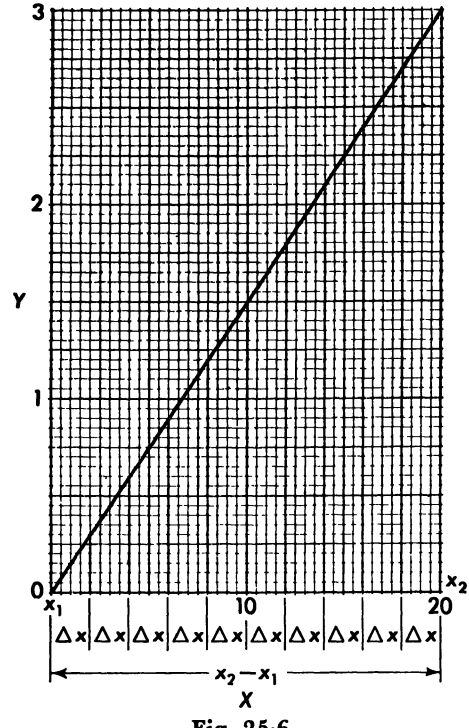


Fig. 25·6

In general, as we scan the graph from left to right, we see that every time x increases, the area under the graph increases; that is, for each increase in x there is a corresponding increase in area. If we let

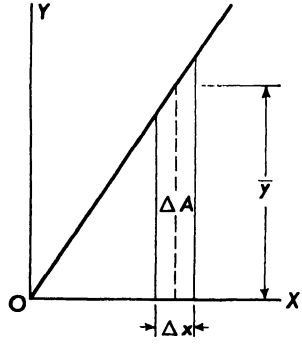


Fig. 25·7

Δx = an increase in x (see Fig. 25·7)

and ΔA = a corresponding increase in area

then

$$\Delta A = \bar{y}\,\Delta x \tag{21}$$

where $\bar{y}$ = average ordinate of a strip Δx units wide

Now returning to Fig. 25·6, the total area is the sum of the areas of all the individual strips.

The Greek letter sigma (Σ) is used to indicate "sum." Therefore we can write

$$A = \Sigma\,(\bar{y}\,\Delta x) \tag{22}$$

where A = total area between ordinates erected at x_1 and x_2

This means that the total area is the sum of all the products of the average heights times the width of the corresponding strip.

$$A = \Sigma \bar{y}\, \Delta x = \bar{Y}(x_2 - x_1) \qquad \star (23)$$

where A = total area between ordinates erected at x_1 and x_2
$\bar{y}$ = average ordinate of each strip Δx units wide
$\bar{Y}$ = average ordinate of entire area under graph

EXAMPLE 2. Find the average height of the graph in Fig. 25·6 between $x = 0$ and $x = 20$. The $\bar{y}$ for each strip is read directly from the graph and tabulated in Table 25·1. The width of all strips in this problem is 2 units, and this value appears in the Δx column in the table below. The area of each strip is

$$\Delta A = \bar{y}\, \Delta x$$

and appears in the ΔA column.

The sum of all the entries in the ΔA column is $\Sigma(\bar{y}\, \Delta x)$ and is 30 square units.

Table 25·1

$\bar{y}$	Δx	ΔA
0 15	2	0.30
0 45	2	0.90
0 75	2	1.50
1 05	2	2 10
1 35	2	2 70
1 65	2	3 30
1 95	2	3.90
2 25	2	4 50
2 55	2	5 10
2 85	2	5.70
		30 00 = $\Sigma \bar{y}\, \Delta x$

Following Eq. (23) the average ordinate $\bar{Y}$ between $x = 0$ and $x = 20$ is

$$\bar{Y} = \frac{A}{x_2 - x_1} = \frac{\Sigma \bar{y}\, \Delta x}{x_2 - x_1} = \frac{30}{20} = 1.5$$

Thus far we have not considered any physical units which may be attached to the scale of the axes. We shall do so now.

The area under the graph is

$$A = \bar{Y}(x_2 - x_1) \qquad (24)$$

Dimensionally, the area under the graph is the product of the dimension of the average ordinate and the dimension of the abscissa.

If the dimension of the average ordinate is *force* and the dimension of the abscissa is *distance*, then the dimension of the area under the graph is

$$\text{Force} \times \text{distance} = \text{work}$$

If the dimension of the average ordinate is *speed* and the dimension of the abscissa is *time*, then the dimension of the area under the graph is

$$\text{Speed} \times \text{time} = \text{distance}$$

If the dimension of the average ordinate is *area* and the dimension of the abscissa is *length*, then the dimension of the area under the graph is

$$\text{Area} \times \text{length} = \text{volume}$$

Now let us consider an idealized but somewhat more applied type of problem. Let a coiled spring be elongated by a variable force F.

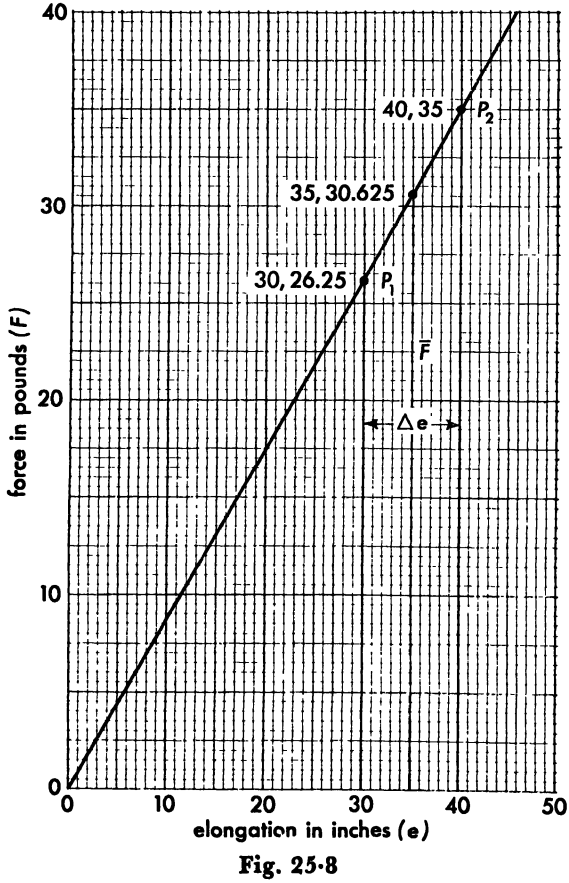


Fig. 25·8

Also, let the elongation caused by the force be e. We shall let F be measured in pounds and e in inches. A graph of the elongation related to F is shown in Fig. 25·8. Actually F is the independent variable and e is the dependent variable. Nevertheless, it will be more convenient to plot F vertically and e horizontally.

The slope of this line is

$$m = \frac{35 - 0}{40 - 0} = 0.875 \text{ lb per in.}$$

This means that a change in elongation of an inch corresponds to a change in force of 0.875 lb, or

$$\frac{\Delta F}{\Delta e} = 0.875$$

The equation for ΔF in terms of Δe is

$$\Delta F = 0.875\ \Delta e$$

EXAMPLE 3. What increase in force is required to elongate the spring from 20 to 20.002 in.? Of course, it is impossible to read this minute increase directly off the graph, but we do know that

$$\frac{\Delta F}{\Delta e} = 0.875$$

throughout the entire range of this graph.

In this particular problem

$$\Delta e = 20.002 - 20.0 = 0.002$$

and

$$\frac{\Delta F}{0.002} = 0.875$$

$$\Delta F = 0.00175 \text{ lb}$$

25·5 Calculation of Work from a Force-Distance Graph. Now let us consider the work done in elongating the spring described in Sec. 25·4. Work in general is a force times a distance in the direction of the force. If we lift a 10-lb weight straight up against the force of gravity for 5 ft, then 50 ft-lb of work is done. Here the 10-lb force is constant throughout the range of the problem.

In the case of the spring under tension, the situation is somewhat different. From the graph in Fig. 25·8 we can read that the elongation is 30 in. when a 26.25-lb force is exerted. But to reach that 30-in. elongation the entire 26.25 lb is not exerted throughout the entire travel. Obviously, then, less than 30×26.25 in.-lb of work was done in elongating the spring to 30 in.

Therefore, to find the actual work done by this varying force, we use an average force which, following Eq. (19), is

$$\text{Average force} = \bar{F} = \frac{F_2 + F_1}{2} \tag{25}$$

where F_1 = force at beginning of interval
F_2 = force at end of interval
$\bar{F}$ = average force
and where F varies linearly with its independent variable

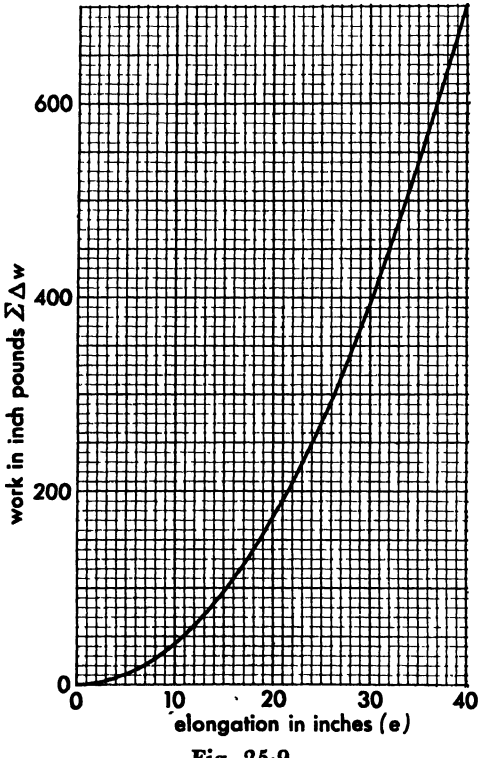


Fig. 25·9

The average force in this case is

$$\bar{F} = \frac{26.25 + 0.00}{2} = 13.125 \text{ lb}$$

Work done to elongate the spring 30 in. is

$$\text{Work} = 13.125 \times 30 = 393.75 \text{ in.-lb}$$

EXAMPLE 4. Find the work done in elongating the spring from 30 to 40 in. (see Fig. 25·8).

The force exerted for a 30-in. elongation is read from the graph to be 26.25 lb. The force exerted for a 40-in. elongation is read from the graph to be 35 lb. The *average* force is

$$\bar{f} = \frac{35 + 26.25}{2} = \frac{61.25}{2} = 30.625 \text{ lb}$$

$$\text{Work} = 30.625 \times 10 = 306.25 \text{ in.-lb}$$

This is precisely the area under the quadrilateral P_2P_1 (30, 0) (40, 0) in terms of the scale of the graph (see Fig. 25.8).

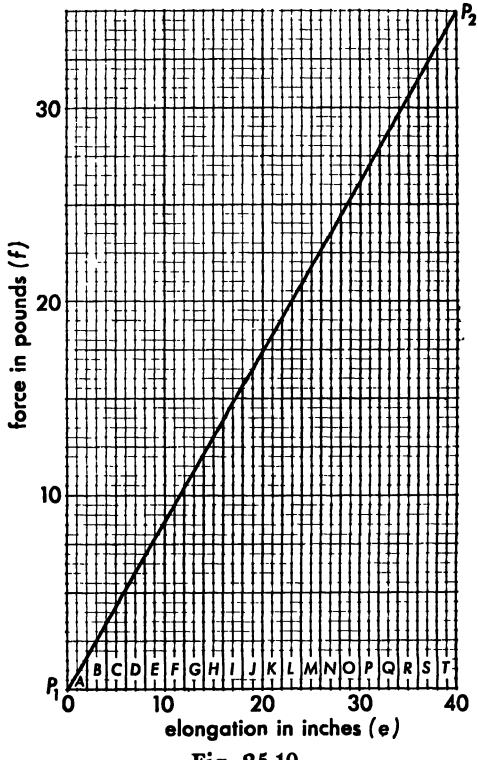


Fig. 25·10

Dimensionally the area under any graph is the product of the dimensions of $\bar{y}$ and x. In this case $\bar{y}$ has the dimension of a force and x has the dimension of a distance. Consequently A has the dimensions of force $\times$ distance, which are the dimensions of work.

Suppose we are required to plot a curve of the work done to reach any elongation between 0 and 40 in. (see Fig. 25·9). We can proceed as follows (see Fig. 25·10). This graph is similar to Fig. 25·8, but the area under this graph is divided into vertical strips. The width of these strips is arbitrarily chosen to be two units, although other widths would do as well in this case.

The average height of each strip is indicated by the dotted lines. The student can verify, precisely by calculation or approximately by direct reading of the graph, the average ordinates in the $\bar{f}$ column in Table 25·2. The strips are lettered alphabetically from left to right. The Δe column indicates 2 units throughout. The width of all strips is identical.

The ΔW column (Table 25·2) indicates the work done in the corresponding interval. While the elongation is identical for each interval, the average force required to produce this elongation varies progressively as the elongation increases. Therefore the work expended to produce each increment of elongation also progressively increases.

Table 25·2

Strip	e	$\bar{f}$	Δe	ΔW	$\Sigma \Delta W$
a	2	0.875	2	1.75	1.75
b	4	2.625	2	5.25	7.00
c	6	4 375	2	8.75	15.75
d	8	6.125	2	12.25	28 00
e	10	7.875	2	15.75	43.75
f	12	9.625	2	19.25	63.00
g	14	11.375	2	22.75	85.75
h	16	13.125	2	26.25	112.00
i	18	14.875	2	29 75	141.75
j	20	16 625	2	33 25	175.00
k	22	18 375	2	36 75	211.75
l	24	20 125	2	40 25	252.00
m	26	21 875	2	43 75	295.75
n	28	23.625	2	47.25	343.00
o	30	25 375	2	50.75	393.75
p	32	27.125	2	54.25	448 00
q	34	28 875	2	57.75	505.75
r	36	30 625	2	61.25	567.00
s	38	32.375	2	64.75	631.75
t	40	34.125	2	68.25	700.00

The column headed ΔW indicates the work required for each corresponding increase of elongation.

The symbol Σ (greek letter sigma) means *sum*. Therefore, $\Sigma\ \Delta W$ means "sum of work increments," and

$$W = \Sigma \bar{f}\ \Delta e \tag{26}$$

where W = total work accomplished
$\bar{f}$ = average force for each strip
Δe = elongation represented by width of each strip in Fig. 25·10

The column headed $\Sigma\ \Delta W$ indicates the total amount of work expended in elongating the spring from 0 in. to the elongation at the end of the interval in question. For example, 63 in.-lb of work were expended in elongating the spring from 0 to 12 in., the end of the f interval. Figure 25·9 is a plot of $\Sigma\ \Delta W$ with respect to elongation. Note that we plot $\Sigma\ \Delta W$ against the elongation at the *end* of the interval in question.

EXERCISE 2

1. The relation between the force and the corresponding elongation of a spring is illustrated by Fig. 25·11.

a. Find the average force over the interval between $e = 1$ and $e = 3$.

b. Find the work done in elongating the spring over this interval.

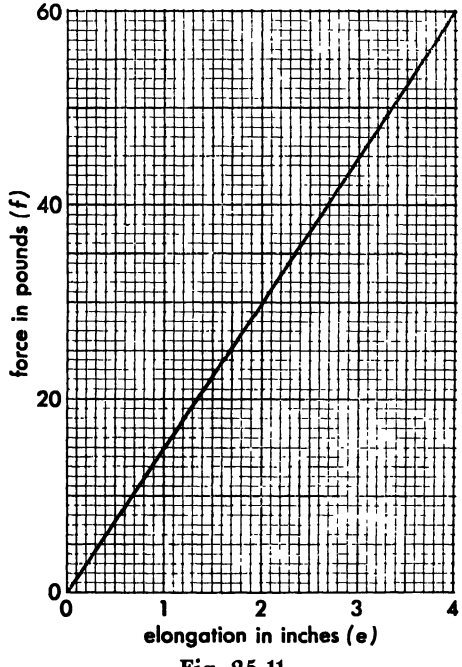


Fig. 25·11

2. Plot a curve of work done vs. elongation between $e = 0$ in. and $e = 4$ in.

25·6 Calculation of Distance from a Speed-Time Graph. If we have given a graph of the speed vs. time for a moving object, we can calculate the distance covered in a way exactly analogous to the calculation of work from a force-distance plot (see Sec. 25·5).

Figure 25·12 shows the graph of speed for a freely falling body with an initial downward velocity of 100 fps.

The area between the graph and the X axis is proportional to distance covered.

Following Eq. (22)

$$D = \Sigma \bar{v}\, \Delta t \tag{27}$$

where D = distance

$\bar{v}$ = average velocity over interval of time Δt

Δt = interval of time

In this problem Δt is chosen to be 0.2 sec.

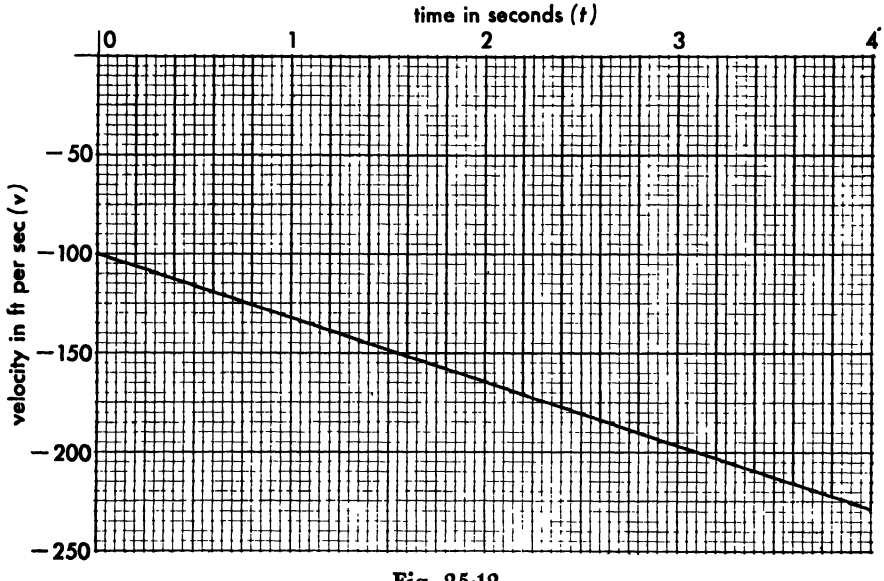


Fig. 25·12

It will be noted that this area carries a negative sign. The reason is that velocities in a downward direction are conventionally negative.

The area under this graph is the product of a negative velocity and a positive time interval. This product is negative (see Fig. 25·5 and Sec. 25·4).

EXERCISE 3

1. Find the distance the object falls for each strip in Fig. 25·12.
2. Plot the distance the object falls vs. time.
3. Plot a curve of altitude vs. time. The object was thrown downward from a height of 800 ft.
4. When does the object hit the ground?
5. Confirm by an algebraic solution the answer in Prob. 4.

25·7 Slope of Nonlinear Functions. Thus far we have been dealing with those functions whose plot is a straight line. This approach

has been an introduction to the more central theme of this work. Hereafter we shall deal with slopes, average ordinates, etc., of functions which are not linear (for example, see Fig. 25·13). This is a plot of an equation in the form

$$y = kx^n \tag{28}$$

Intuitively, we can see that as x increases, the "slope" of the graph increases; that is, it rises "faster" as we scan the curve from left to right. Of course, the graph also rises higher as we progress to the right, but that is beside the point at the moment. The essential point is that the slope of a curved line is *not constant*, as is the case of the straight line.

The slope of a straight line is rigorously defined in Eq. (2). How, then, shall we define the slope of a curved line? We can see intuitively that the line AB in Fig. 25·13 has about the slope we should expect the curved line to have at point P_1, or, what amounts to the same thing, the line AB is "tangent" to the curve at point P_1.

Therefore, to find the slope of a nonlinear function at a given point, we plot the tangent to the curve through the given point and calculate the slope of this tangent by methods described in Sec. 25·3. The slope of the tangent is equal to the slope of the curve at the given point. Before the above method of finding the slope of a curve can be useful, we shall have to define "tangent."

25·8 Tangents Defined. In Fig. 25·13 the line DE is called a secant. A secant is a line of unlimited length cutting a curve. The line segment P_1P_2 is a part of the secant DE and is called a chord. One extremity of this chord will be fixed at P_1.

Now if the secant DE is rotated clockwise while always passing through P_1, the other end of the chord will be at P_2, P_3, P_4, P_5, etc., as well as at intermediate points. The chord will then progressively decrease in length.

Finally, at one particular position (when the secant coincides with the line P_1B), the chord will disappear altogether. If the secant were rotated beyond this position, a chord would appear to the left of P_1. There is then a critical position of the secant DE where the chord is zero.

The tangent to the curve at a given point is therefore defined as "the limiting position of the secant passing through that point as the length of the chord approaches zero."

To draw a tangent to a curve at a given point, therefore, we keep the edge of a straight edge through the point and adjust the direction of the straight edge until the chord appears to vanish. The slope of this straight edge is now equal to the slope of the curve at the point under consideration.

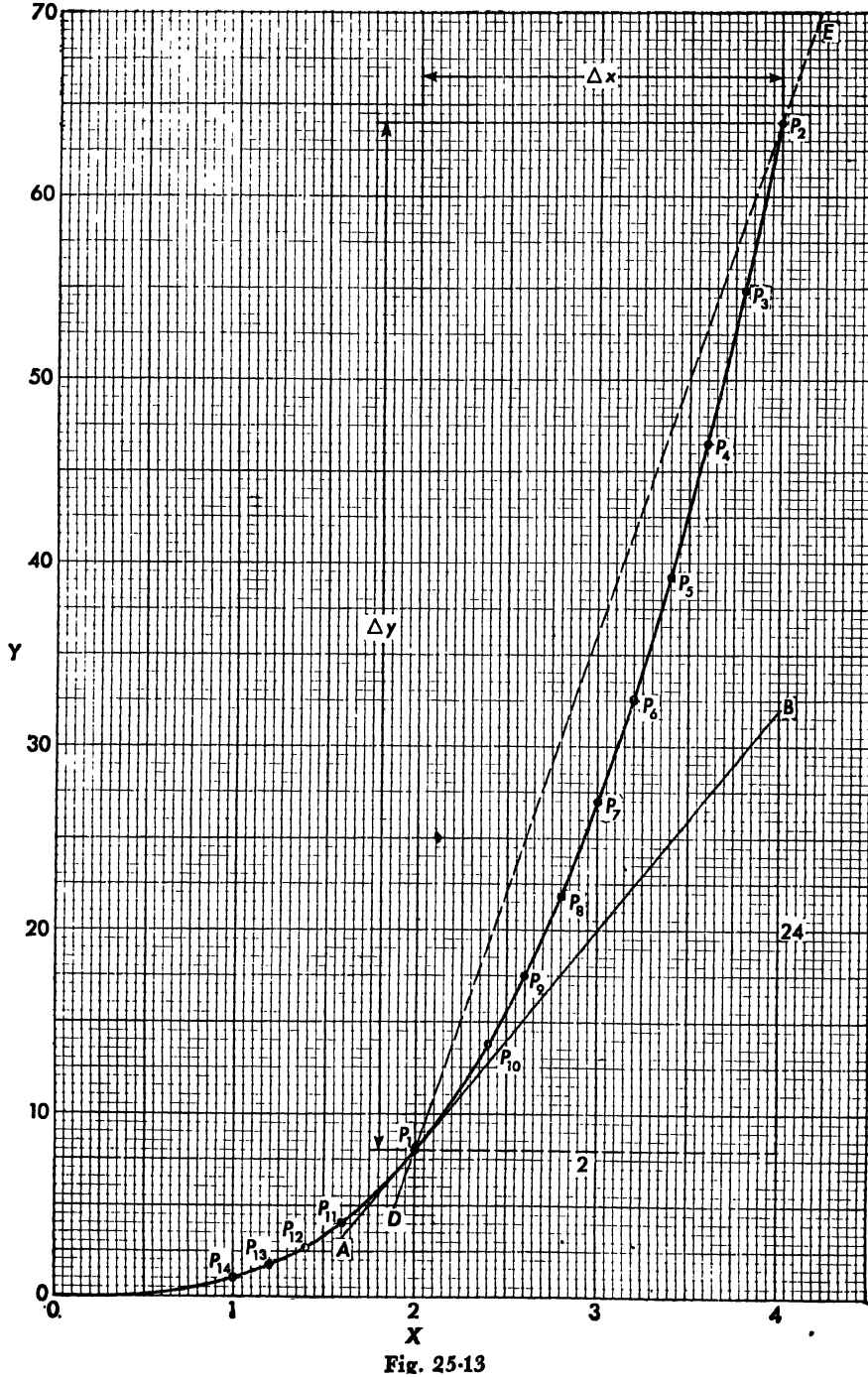

70
60
50
40
30
20
10
0
Y
0
1
2
3
4
X
Δx
Δy
E
B
A
D
P1
P2
P3
P4
P5
P6
P7
P8
P9
P10
P11
P12
P13
P14
24
2

Fig. 25·13

EXERCISE 4

1. In Fig. 25·14, find the slope of line AB.
2. In Fig. 25·14, find the slope of line CD.
3. Finish the sentence: The slope of the curve at point P is greater than ——— and less than ———.

25·9 Average and Instantaneous Slopes and Rates. In the case of the straight line, there is no distinction between average slope and

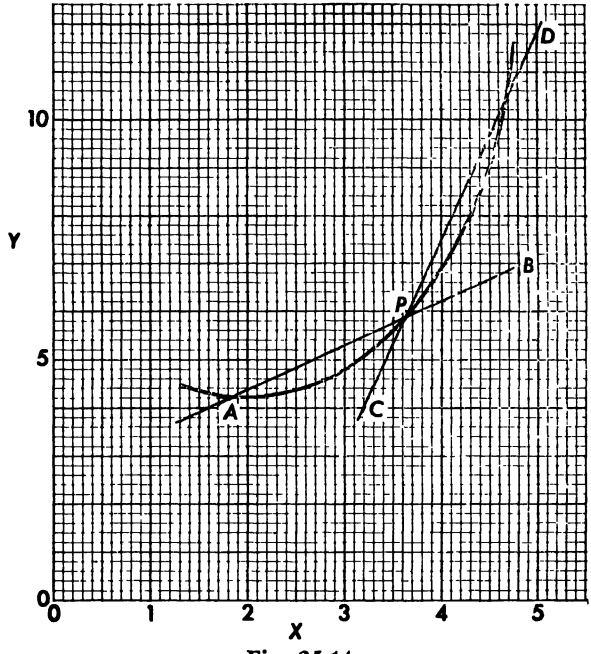


Fig. 25·14

instantaneous slope or between average rate and instantaneous rate. In the case of a curve, there is such a distinction. In Fig. 25·13 the *average* slope of the curve *between* points P_1 and P_2 is

$$\frac{\Delta y}{\Delta x} = \frac{64 - 8}{4 - 2} = \frac{56}{2} = 28$$

The *instantaneous* slope *at* P_1 is

$$\frac{24}{2} = 12$$

It is very important that the student observe that we speak of the *average* slope *between two points* and the *instantaneous* slope *at one single point.* Let us investigate the matter still further. We shall calculate the slopes of

Table 25·3

Points	Δy	Δx	$\Delta y/\Delta x$
Between P_1 and P_2	56.000	2.0	28.00
Between P_1 and P_3	46.872	1.8	26.04
Between P_1 and P_4	38.656	1.6	24.16
Between P_1 and P_5	31.304	1.4	22.36
Between P_1 and P_6	24.768	1.2	20.64
Between P_1 and P_7	19.000	1.0	19.00
Between P_1 and P_8	13.952	0 8	17.44
Between P_1 and P_9	9.576	0 6	15.96
Between P_1 and P_{10}	5.824	0.4	14.56
Between P_1 and P_{11}	−3.904	−0 4	+9.76
Between P_1 and P_{12}	−5.256	−0.6	+8.76
Between P_1 and P_{13}	−6.272	−0.8	+7.84
Between P_1 and P_{14}	−7 000	−1.0	+7.00

lines connecting P_1 and P_2, P_3, P_4, P_5, etc., in Fig. 25·13 and tabulate them in Table 25·3. The values of Δy were read from a much larger graph than the one published here. However, the student can verify them to about three significant figures.

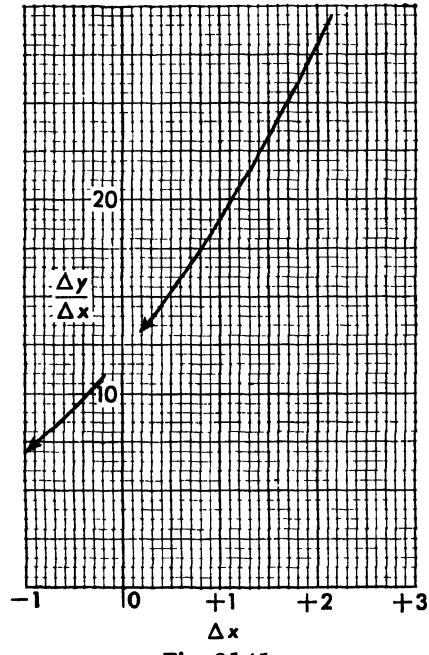


Fig. 25·15

The values of Δx and $\Delta y/\Delta x$ are plotted in Fig. 25·15. As we scan this graph from right to left we note that the trend of the graph is directly toward 12 on the vertical axis. As we pass the vertical axis and continue on toward the left we note that it would appear that the graph had passed through 12 on the vertical axis.

However, it is important to observe that both the graph and the vertical axis are broken at this point. There are two reasons why this was done. In the first place, we are not interested in what $\Delta y/\Delta x$ *is* when $\Delta x = 0$. We are interested in what $\Delta y/\Delta x$ *approaches* as Δx *approaches* zero. In the second place, we would be in all sorts of intellectual difficulties if we tried to figure out what happens if Δx is

precisely zero. In that case Δy would be zero also and $\Delta y/\Delta x$ would be 0/0, which is indeterminate. Certainly it is not 12. As long as Δx and Δy are numbers other than zero, $\Delta y/\Delta x$ has a quotient, no matter how microscopic the value of Δx may be. We can *approach* the vertical axis in Fig. 25·15 as close as we please, but we *must not touch.*

Let us be satisfied to see what $\Delta y/\Delta x$ *approaches* as Δx *approaches* zero. We have observed from this graphical work that $\Delta y/\Delta x$ (the average slope) approaches 12 (the instantaneous slope) as Δx approaches zero. The symbol $\Delta y/\Delta x$ indicates the *actual* change in y divided by the *corresponding* actual change in x. This is an *average* slope.

There is an analogous symbol for *instantaneous* slope. It is

$$\frac{dy}{dx} = \text{instantaneous slope} \qquad \star\ (29)$$

In the case of the straight line they are identical and

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} \qquad (30)$$

In the case of a curve they are not identical and

$$\frac{\Delta y}{\Delta x} \neq \frac{dy}{dx} \qquad (31)$$

The symbol dy/dx is called the "derivative of y with respect to x" and will be discussed in the following section.

25·10 The Derivative. In Fig. 25·13, Table 25·3, and Fig. 25·15, the quantity $\Delta y/\Delta x$ is a variable. In Fig. 25·15 it varies from 28 down to 7. As Δx approaches zero, $\Delta y/\Delta x$ approaches 12 as a limit.

The derivative is defined as "the limit which $\Delta y/\Delta x$ approaches as Δx approaches zero." This symbol is written

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \qquad \star\ (32)$$

In the function represented in Fig. 25·13, therefore, the derivative of y with respect to x is 12 at the point $x = 2$, or

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 12 \qquad (33)$$

Thus

$$\frac{dy}{dx} = 12 \qquad (34)$$

EXAMPLE 5. What is the instantaneous rate of change in y with respect to x when $x = 0.2$ in Fig. 25·16?

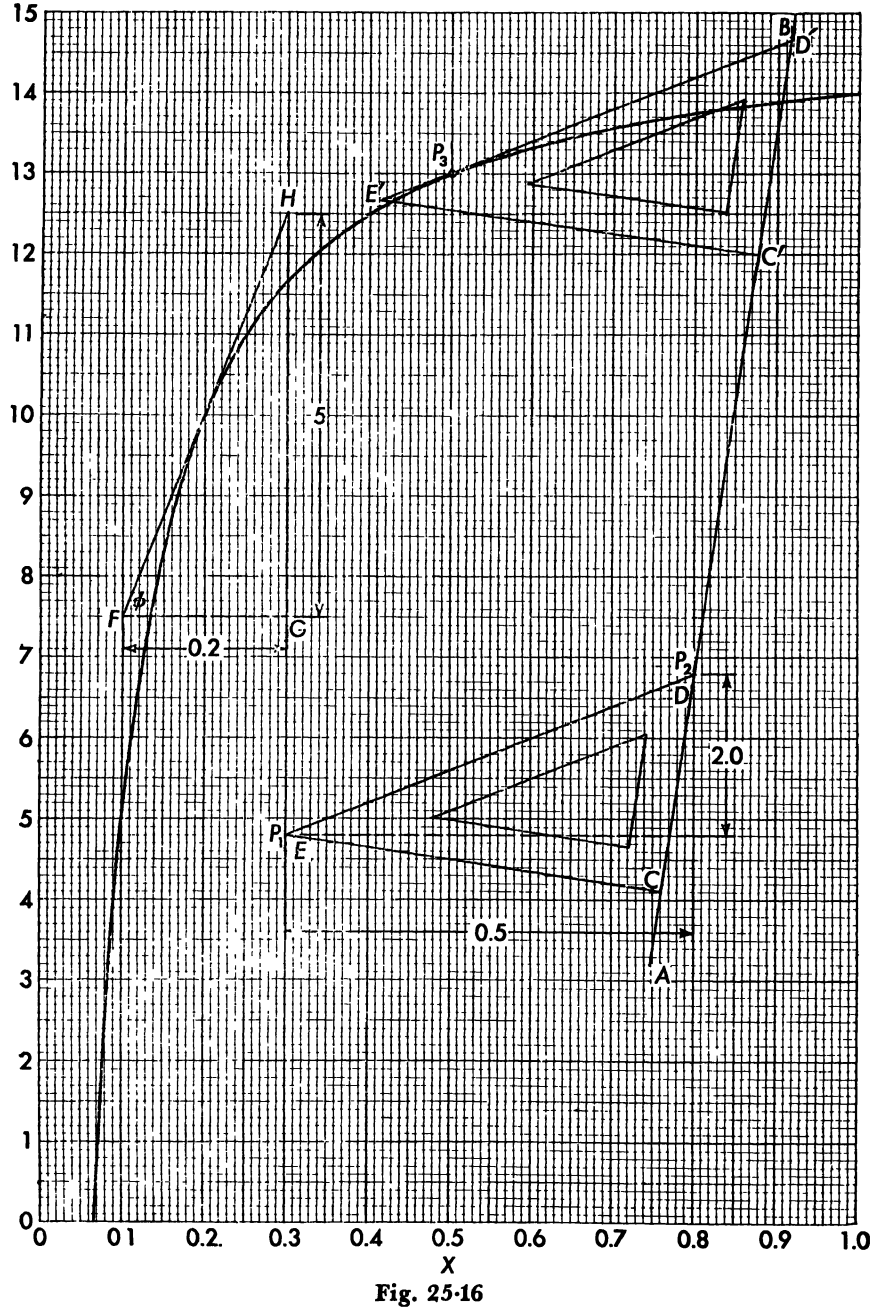

15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
X
B
D'
P3
H
E'
C'
5
F
G
0.2
P3
D
2.0
P1
E
C
0.5
A

Fig. 25·16

*

The instantaneous rate of change in y with respect to x is numerically equal to the slope of the curve.

To find the slope of the curve at $x = 0.2$, a straight edge was placed on the point where the curve passes through the ordinate erected at $x = 0.2$. This

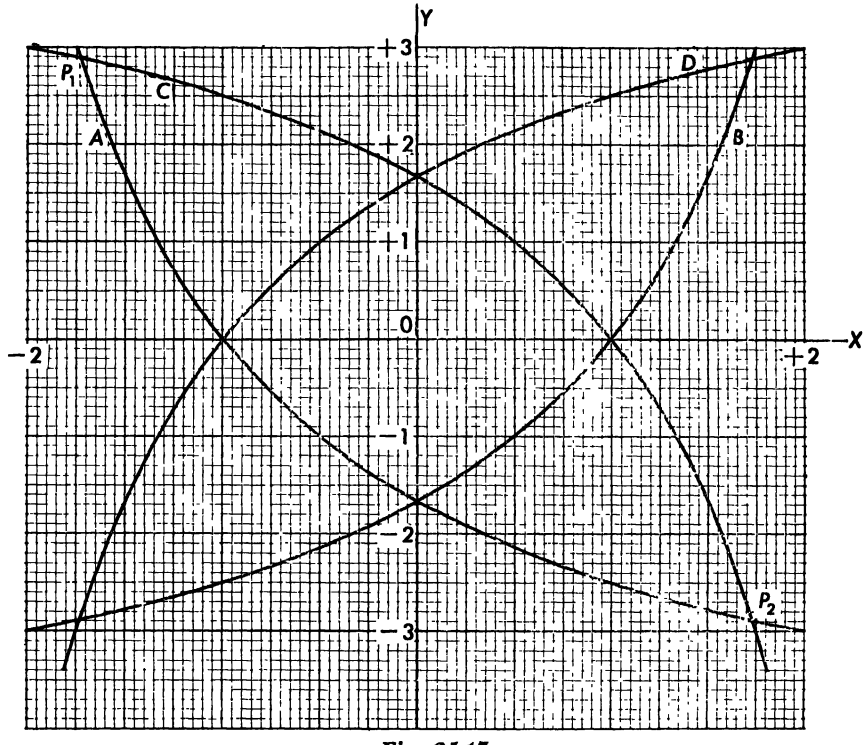


Fig. 25·17

straight edge was adjusted until it appeared to be "tangent" to the curve. The slope of the tangent is

$$\frac{5}{0.2} = 25$$

EXAMPLE 6. In Fig. 25·17 (curve A) find the average rate of change in y with respect to x between $x = -1.73$ and $+1.73$.

$$\frac{\Delta y}{\Delta x} = \frac{2.89 - (-2.89)}{-1.73 - (+1.73)} = \frac{+5.78}{-3.46} = -1.67$$

Thus the average rate of change in y with respect to x between $x = -1.73$ and $x = +1.73$ is -1.67. Observe also that these two points P_1 and P_2 are at the intersections of curves A and C. Therefore the *average* rate of change in y with respect to x is the same for both curves between these two points.

Thus, over a relatively large Δx the *average* rate of change is not necessarily very descriptive of the actual path of the curve throughout that interval.

EXAMPLE 7. In Fig. 25·16 find the coordinates of the point at which the instantaneous slope is $+4$. A pair of convenient points such as P_1 and P_2 are located on the same coordinate system as the graph, such that the slope of the straight line between them is $+4$. Then one edge of a triangle, ED, is placed so that the edge of the triangle passes through P_1 and P_2.

A straight edge, AB, is pressed against another side of the triangle, CD.

The triangle is then moved along the straight edge until the side ED becomes tangent to the curve as shown by the position $(E'D')$.

The point of tangency P_3 has the coordinates (0.5, 13.00).

Thus the coordinates of the point at which the instantaneous slope is $+4$ are (0.5, 13.00).

EXAMPLE 8. What is the angle ϕ in Fig. 25·16?

This angle is one angle of the triangle FGH. The side opposite this angle is 5 large divisions in length. The side adjacent to this angle is 2 large divisions in length. Therefore the tangent of the angle is

$$\tan \phi = \frac{5}{2} = 2.5$$

and

$$\phi = 68° \text{ (approx)}$$

Observe that in finding the tangent of the angle, the *same* scale is used for the opposite and adjacent sides, not the scales used on the axes. Therefore the tangent of the angle and the slope of the curve are identical *only* when the same scale is used on the vertical and horizontal axes.

EXERCISE 5

1. In the curve of Fig. 25·17B find the average rate of change of y with respect to x between $x = -1.5$ and $x = 0$.
2. Find the instantaneous rate of change of y with respect to x when $x = -1.5$ (Fig. 25·17B).
3. Find the average slope of curve B in Fig. 25·17 between $x = +1$ and $x = +1.5$.
4. Find the instantaneous slope of curve B in Fig. 25·17 when $x = +1$.
5. Find $\Delta y/\Delta x$ between $x = -0.5$ and $x = +0.5$ for Fig. 25·17C.
6. Find dy/dx at $x = 0$ for Fig. 25·17A.
7. In Fig. 25·17D find the coordinates of the point where the instantaneous slope is $+\frac{5}{8}$.
8. Find the angle which the tangent drawn to curve C at $x = 0$ makes with the horizontal.

25·11 Approximate Increments. Suppose we wished to find how much y increased when x increased from 0.2 to 0.2001 in Fig. 25·16. Of course it would be impossible to read this small increment directly from

the graph. But from Example 5 above, we find that when $x = 0.2$ the rate of change of y with respect to x is 25. Now it is true that the rate of change of y with respect to x will be a little different at $x = 0.2001$ from what it was at exactly $x = 0.2000$, but not much different. Over this small interval we shall be approximately correct to assume that the rate is constant.

Therefore, we can say that if Δx is *small*, the *instantaneous* rate of change of y with respect to x is *approximately* the same as the average rate of change of y with respect to x.

In mathematical symbols

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} \qquad \text{(when } \Delta x \text{ is small)} \tag{35}$$

In the present example

$$\frac{dy}{dx} = 25$$

and $\Delta x = 0.0001$. Therefore,

$$\frac{\Delta y}{0.0001} \approx 25$$

$$\Delta y \approx 0.0025$$

and y increases approximately 0.0025 unit when x increases 0.0001 unit from $x = 0.2000$ to $x = 0.2001$.

25·12 Differential Notation. The derivative of a function (written dy/dx) was defined in Eq. (32) as "the limit which $\Delta y/\Delta x$ approaches as Δx approaches zero."

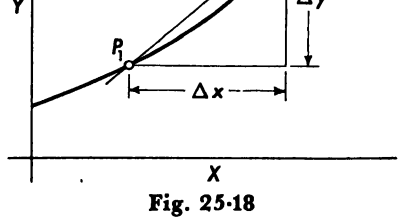


Fig. 25·18

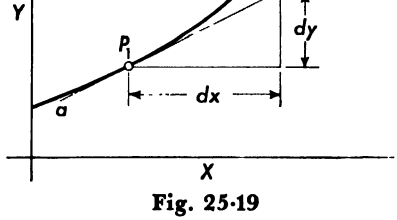


Fig. 25·19

There is another useful interpretation of dy/dx illustrated by a comparison between Figs. 25·18 and 25·19 (refer also to Fig. 25·13). In Fig. 25·18 the slope of the line P_1P_2 is designated by $\Delta y/\Delta x$ where Δy is the actual change in the ordinate of the *graph* between the two points

P_1 and P_2, and Δx is the corresponding actual change in the abscissa of the *graph* between the two points P_1 and P_2.

In Fig. 25·19 the slope of the *tangent line ab* is the slope of the *tangent to the graph at* the point P_1 and is designated by dy/dx where dy is a change in the ordinate of the *tangent to the graph* drawn through the point P_1, and dx is the corresponding change in the abscissa of the *tangent to the graph* drawn through the point P_1.

As applied to a graph, then, the symbol $\Delta y/\Delta x$ denotes a change in the ordinate of the *graph* divided by a corresponding change in the abscissa of the *graph*.

In a similar way, the symbol dy/dx denotes a change in the ordinate of the *tangent* to a graph divided by a corresponding change in the abscissa of the *tangent* to the graph.

Thus Eq. (34) could be transformed to read

$$dy = 12\,dx \tag{36}$$

Remember that Eq. (34) applies to Fig. 25·13 where $x = 2$. Therefore Eq. (36) can be interpreted as meaning that the change dy in the ordinate of the tangent line drawn through the point where $x = 2$ is 12 times the corresponding change dx in the abscissa of the same tangent line.

25·13 Maxima and Minima. In general, a graph has a maximum value when the graph at that point is higher than at any other point

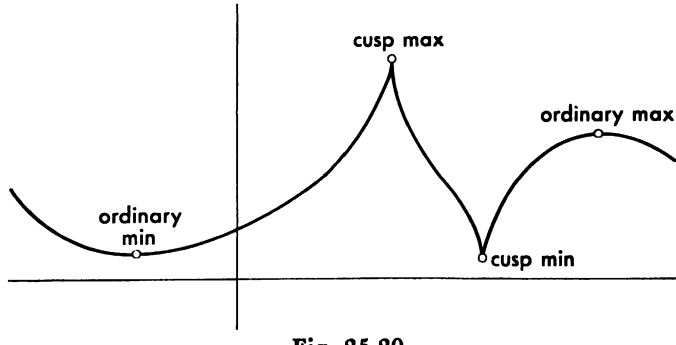


Fig. 25·20

in the immediate vicinity. A graph has a minimum value when it is lower than at any other point in the immediate vicinity.

There are two types of maximum and minimum points: the *cusp* maximum and minimum and the *ordinary* maximum and *ordinary* minimum (see Fig. 25·20).

There are three important characteristics of ordinary maximum and minimum points: (1) the slope at such points is zero; (2) these points

are either higher or lower than any other points in the immediate vicinity; (3) in passing through a maximum or minimum point, the sign of the slope of the curve changes.

In passing through a maximum, if we scan the graph in the direction of *increasing* abscissa (that is, as we scan the graph from left to right), we find a positive slope, a zero slope, and negative slope, *in that order.*

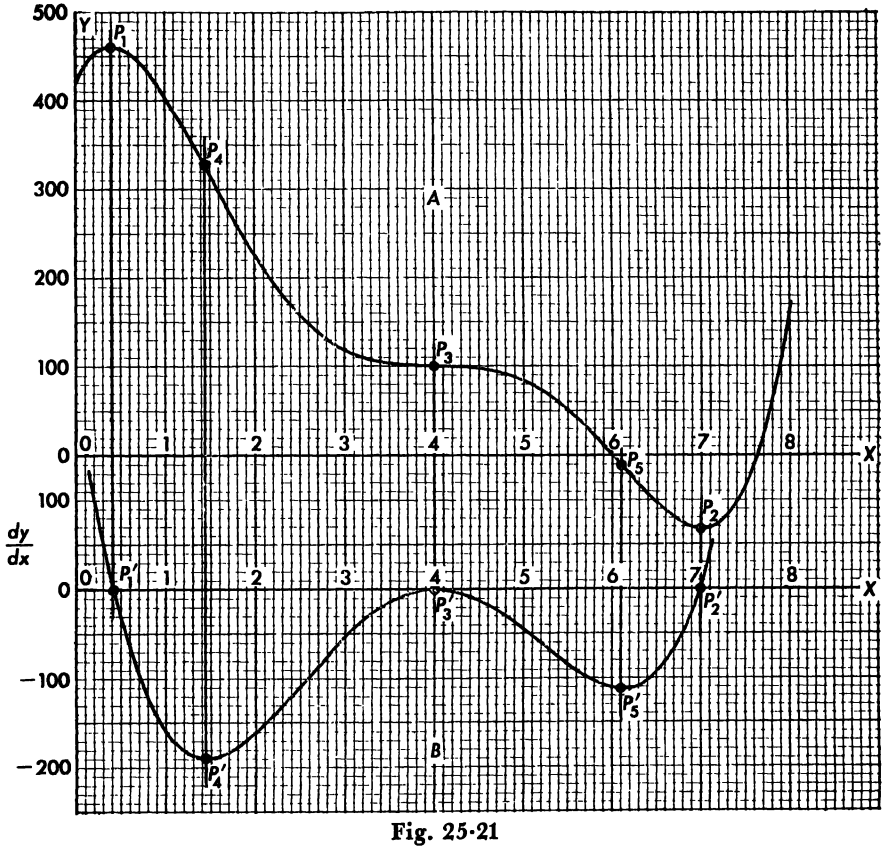


Fig. 25·21

Consider the maximum point P_1 (Fig. 25·21*A*). At the point P_1 the slope is zero; it is higher than any other point in the immediate vicinity. Immediately to the left the slope is positive, and immediately to the right the slope is negative.

A similar situation exists about the minimum point P_2. At the point P_2 the slope is zero. It is lower than any other point in the immediate vicinity, and the slope is negative to the left of P_2 and positive to the right of P_2, being zero at P_2.

As we scan the graph from left to right, if the sign of the slope is in the sequence

$$\text{Positive} \rightarrow \text{zero} \rightarrow \text{negative}$$

the point is a *maximum.*

If the sequence is

$$\text{Negative} \rightarrow \text{zero} \rightarrow \text{positive}$$

the point is at a *minimum.*

If the sequence is

$$\text{Negative} \rightarrow \text{zero} \rightarrow \text{negative}$$

or

$$\text{Positive} \rightarrow \text{zero} \rightarrow \text{positive}$$

the point is neither a maximum nor a minimum.

Thus we cannot assume that just because the slope of a graph is zero at a critical point either a maximum or a minimum must exist at that point.

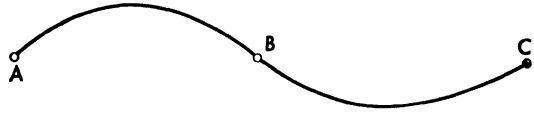


Fig. 25·22

25·14 Points of Inflection. Refer to Fig. 25·22. Consider the reverse curve ABC to be a map of a road. As we drive an automobile along the road from A toward B we are steering to the right. When we reach point B we must steer to the left. At point B the curvature *reverses.*

Any point on a graph where the curvature reverses is called a *point of inflection.*

Applying this notion to Fig. 25·21A we note that points of inflection occur at P_3, P_4, and P_5. True, the slope at P_3 is zero, but this is not a necessary condition for a point of inflection. The slopes at the other two points of inflection, P_4 and P_5, are not zero.

It is a characteristic of a point of inflection having a horizontal tangent (such as P_3) that as we scan the graph in either direction the slope passes through zero but *does not change sign.*

Thus a point of horizontal tangency may be a maximum, a minimum, or simply a point of inflection.

25·15 Derived Curves. Figure 25·21B is a plot of dy/dx versus x, that is, for a given value of x, the numerical value of the slope in Fig. 25·21A is the ordinate of Fig. 25·21B.

For example, we can see that the slope in Fig. 25·21*A* is zero at P_1 because the tangent to the curve at that point is horizontal. The corresponding point P_1' in Fig. 25·21*B* shows that at this point $dy/dx = 0$. Precisely the same information in this respect is obtained from either graph.

Observe points P_2 and P_3 together with their corresponding points in Fig. 25·21*B*, P_2' and P_3'. In each case the slope of Fig. 25·21*A* and its synonym dy/dx are shown to be zero.

If we scan Fig. 25·21*A* from P_1 toward the right, we note that the slope decreases (increases numerically but in a negative direction). The slope continues to decrease until we reach P_4, a point of inflection.

Subsequently, the graph flattens out and the slope becomes zero again at P_3, which is another point of inflection. Considering signs, then, the slope at P_4 is a minimum. Now notice that a minimum dy/dx occurs at P_1' in Fig. 25·21*B*.

Observe that the slope is again a minimum at the point of inflection P_5 and is so indicated at P_5'.

The slope is at a maximum at P_3 and is also so indicated at P_3'. However, the student should be careful to notice that the point P_3 corresponds to a maximum slope in Fig. 25·21*A*. It does not correspond to a maximum ordinate on Fig. 25·21*A*.

EXERCISE 6

1. By laying a straight edge against the curve in Fig. 25·21*A*, find the slope of this curve at the following values of x: 0.4, 1.0, 2.5, 4.0, 4.5, 5.0, 6.0, 6.1, 6.5, 7.0.
2. **a.** Compare the slopes found in Prob. 1 with the corresponding ordinates in Fig. 25·21*B*.
   b. Find where the slope of Fig. 25·21*B* is a maximum. A minimum.
3. Find the maximum speed of the vehicle in Fig. 25·34.
4. Find the maximum speed in Fig. 25·32.
5. When was the speed of the vehicle a maximum in Fig. 25·34?
6. When was the speed a maximum in Fig. 25·32?
7. A rectangle has a perimeter of 40 in. Plot the area of this rectangle against the length of a side. Find the dimensions of the rectangle to give the greatest area.
8. The power lost in resistance R is given by the equation

$$P = \frac{(10 - 0.2i)^2}{R}$$

also

$$i = \frac{10}{R + 0.2}$$

Plot P against R and find the value of R such that the power dissipated in R is a maximum (see Fig. 25·23).

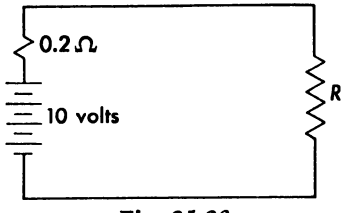


Fig. 25·23

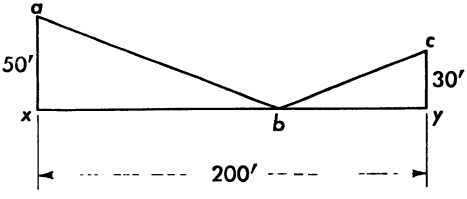


Fig. 25·24

9. The tops of two vertical poles ax and cy are connected by a rope abc. Plot the length of the rope against the distance xb and find the length of xb for a minimum length of the rope (see Fig. 25·24).

10. A sheet-iron trough is to be made by bending up two edges so as to give a rectangular cross-section area. The width of the sheet is 20 in. Therefore 20 in. $= x + 2y$. Plot the cross-section area against y and find y to give a maximum cross-section area (see Fig. 25·25).

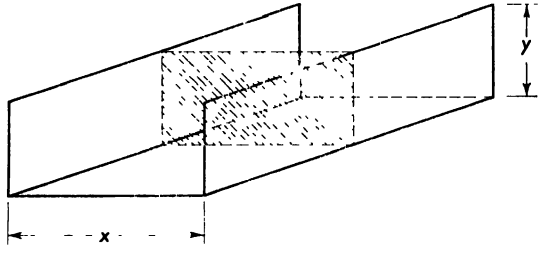


Fig. 25·25

11. Write an equation for the area of the trapezoid in terms of h. Plot the area against h and find h for a maximum area (see Fig. 25·26).

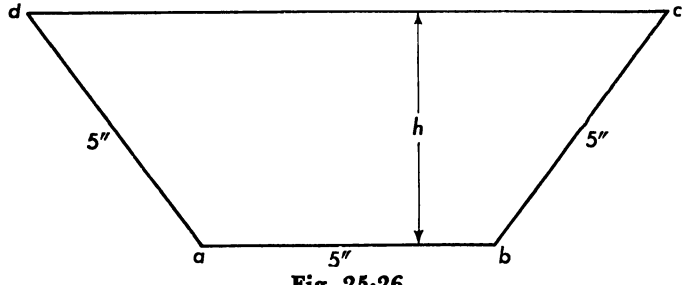


Fig. 25·26

25·16 The Area under a Nonlinear Graph. The average ordinate in Fig. 25·8 was obtained from the equation

$$\bar{F} = \frac{F_1 + F_2}{2} \tag{37}$$

In general,

$$\bar{Y} = \frac{y_2 + y_1}{2} \qquad \text{[See Eq. (19)]} \tag{38}$$

for a straight line only

where $\bar{Y}$ = average ordinate

y_1 = ordinate at beginning of interval

y_2 = ordinate at end of interval

The area under a straight line using the $\bar{Y}$ defined by Eq. (19) is

$$\text{Area} = \bar{Y}(x_2 - x_1) \tag{39}$$

where x_1 = abscissa at beginning of interval

x_2 = abscissa at end of interval

By redefining $\bar{Y}$ we can use Eq. (39) to define the average ordinate between two points on a *curve*. Reduced to words, the average ordinate between two points on a curve is *that ordinate which multiplied by the base equals the area under the curve* between these two points [see Eq. (23)].

To find the average ordinate between two points on a curve, we first find the area under the curve between those two points. Second, we divide the area by the abscissa distance between the points. The quotient is the average ordinate.

In other words, from Eq. (39),

$$\bar{Y} = \frac{\text{area}}{x_2 - x_1} \tag{40}$$ ★

To find the average height of a straight-line graph we can find $\bar{Y}$ directly by Eq. (38).

To find the average height of a curve we must *first* find the *area* under the graph.

Now consider the curve in Fig. 25·27.

Let us find the average ordinate between $x = 0$ and $x = 7$.

The average ordinate is no longer

$$\frac{100 + 51}{2} = \frac{151}{2} = 75.5$$

as would be the case of the straight line connecting the points P_1 and P_2.

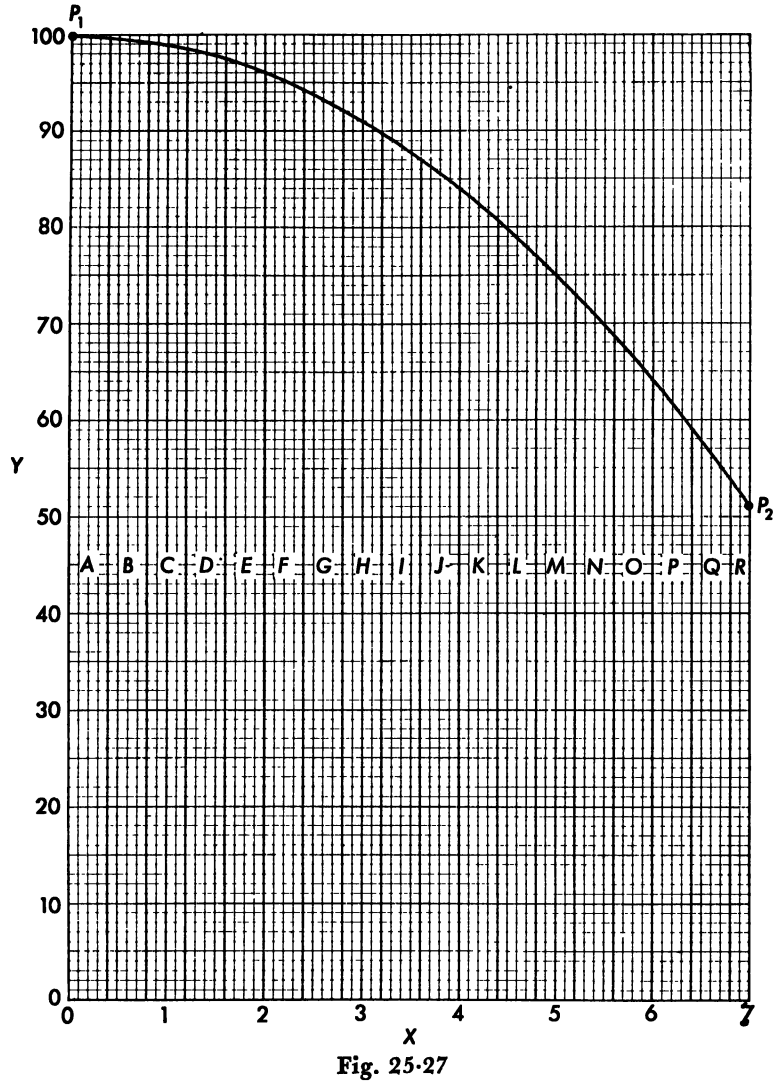


Fig. 25·27

We must first find the area between the curve and the X axis. To accomplish this, we first find the *approximate* area of each strip and then add the *approximate* areas of all strips. We thus obtain an *approximate* area under the curve.

Each strip is made relatively narrow (see Fig. 25·28). This figure shows a single strip similar to any of the strips in Fig. 25·27. We erect $\bar{y}$ midway between *ab* and *cd*. We made the width of this strip narrow enough so that $\bar{y}$ is indistinguishable from either the average ordinate of the *curve bc* or the average ordinate of the *straight line bc*.

To calculate the *approximate* area of each strip we read $\bar{y}$ directly from the graph in Fig. 25·27 and multiply it by the width of the strip Δx, in this case 0.4.

The product is precisely the area of a *rectangle* $\bar{y}$ in altitude and Δx in width, or approximately the area of the *strip*. The area of the strip $\approx \bar{y}(\Delta x)$.

The calculations are tabulated in Table 25·4.

The approximate area of each strip except the last is calculated by multiplying each $\bar{y}$ by 0.4. This product is tabulated under the ΔA column.

The column headed $\Sigma\bar{y}(\Delta x)$ is the cumulative total area. Each entry in the $\Sigma\bar{y}(\Delta x)$ column includes the approximate area of the corresponding strip as well as the approximate area of all strips to the left.

The total area is 585.6 square units. Average ordinate of this entire area between $x = 0$ and $x = 7$ is therefore

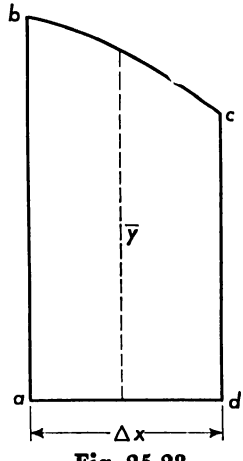


Fig. 25·28

$$\bar{Y} = \frac{585.6}{7} = 83.7$$

Table 25·4

Strip	x	$\bar{y}$	*Width* (Δx)	ΔA	$\Sigma\bar{y}(\Delta x)$
A	0.4	100 0	0.4	40 0	40 0
B	0 8	99.6	0.4	39 8	79.8
C	1.2	99 0	0 4	39.6	119.4
D	1 6	98 0	0.4	39 2	158.6
E	2.0	96.8	0.4	38 7	197.3
F	2.4	95 2	0.4	38.1	235.4
G	2.8	93 2	0.4	37 3	272.7
H	3.2	91.0	0.4	36.4	309.1
I	3 6	88 4	0 4	35.3	344.4
J	4 0	85.6	0.4	34.2	378.6
K	4 4	82.4	0.4	32.9	411.5
L	4.8	78 8	0 4	31.5	443.0
M	5.2	75 0	0 4	30 0	473.0
N	5 6	70 8	0 4	28.3	501.3
O	6.0	66.4	0.4	26 6	527.9
P	6.4	61.6	0 4	24.6	552.5
Q	6.8	56.4	0.4	22.6	575.1
R	7.0	52.4	0.2	10.5	585.6
				585.6	

NOTE that the strip *R* is 0.2 unit wide rather than 0.4 unit wide.

EXAMPLE 9. Find the area under the graph shown in Fig. 25·27 between ordinates erected at $x = 4$ and $x = 6$.

By reference to Table 25·4 we see that the area between $x = 0$ and $x = 6$ is 527.9. The area between $x = 0$ and $x = 4$ is 378.6. Therefore the area between $x = 4$ and $x = 6$ is

$$\underset{4\to 6}{A} = 527.9 - 378.6 = 149.3$$

EXERCISE 7

1. Find the average height of the curve in Fig. 25·13 between $x = 2$ and $x = 3\frac{1}{2}$. In doing this problem, divide the area under the curve into strips such that $\Delta x = 0.3$.
2. Divide the area under the curve in Fig. 25·21*B* into strips 0.2 unit wide. Read $\bar{y}$ for each strip and find the area of each strip, taking into account the signs of $\bar{y}$ and Δx. Record $\Sigma\bar{y}(\Delta x)$ as in Table 25·4. Plot $\Sigma\bar{y}(\Delta x)$ versus x using a scale of 10 small squares to 100 units for the vertical axis and 10 small squares for 1 unit on the horizontal axis. Compare this curve with Fig. 25·21*A*.

EXERCISE 8

(Refer to Fig. 25·29.)

1. Find a when $h = 20$.
2. Find da/dh when $h = 20$. Also find dh/da.
3. Find da/dh when $a = 100$. Also find dh/da.
4. Find the ordinate a when $da/dh = 5$.
5. Find the average rate of change of ordinate with respect to distance between $h = 5$ and $h = 25$.
6. Find the average slope between $h = 15$ and $h = 20$.
7. Find the average slope between $a = 25$ and $a = 100$.
8. Find the instantaneous slope at $h = 22$.
9. Find the instantaneous slope at $a = 100$.
10. How fast is ordinate changing with distance at 12 ft?
11. How fast is ordinate changing with distance at 75 sq ft?
12. Find the angle of inclination of the graph at $h = 10$.
13. How much does area change when distance changes from 10 to 20 ft?
14. How much does distance change when area changes from 25 to 50 sq ft?
15. About how much does ordinate increase when h increases from 10 to 10.001?
16. Find the average ordinate of the curve between $h = 10$ and $h = 20$.

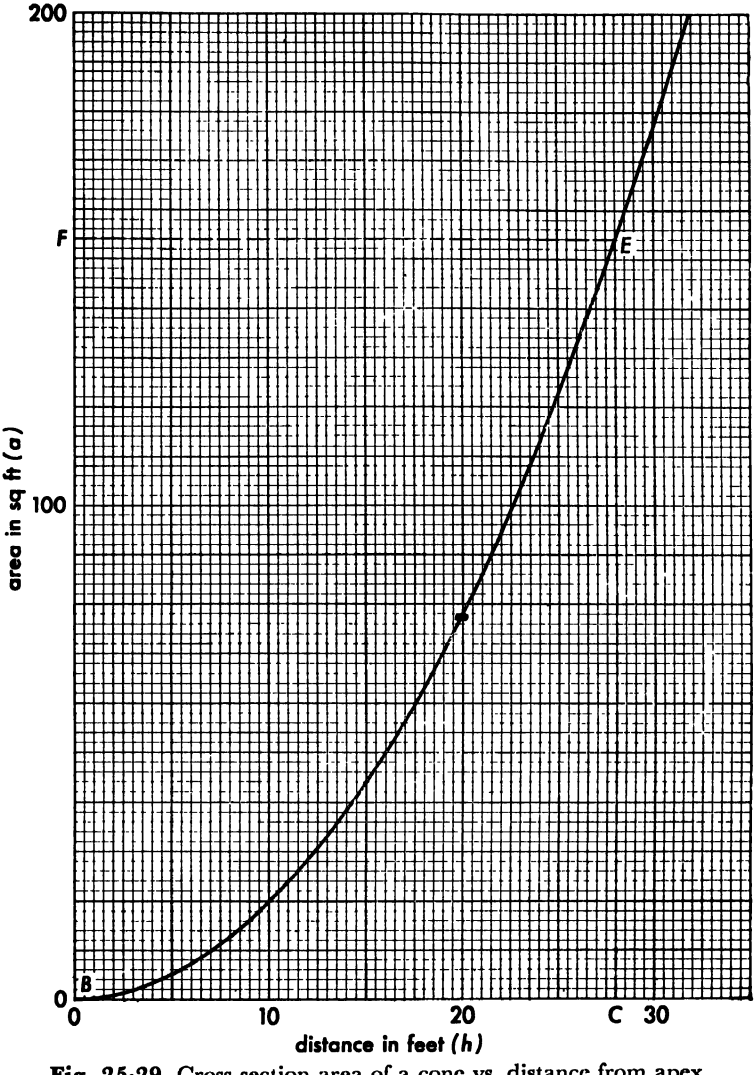


Fig. 25·29. Cross-section area of a cone vs. distance from apex.

17. Find the average ordinate of the curve between $a = 50$ and $a = 100$.

18. What volume is included between the levels $h = 15$ and $h = 25$? Check by the formula for the frustum of a cone [see Eq. (73) of Appendix B].

19. **a.** If the area of *BEC* represents the volume of the cone, what does the area of *BFEC* represent?

b. What does the area of *FEB* represent?

EXERCISE 9

(Refer to Fig. 25·30.)

1. Find dT/dt when $t = 10$.
2. Find dt/dT when $t = 10$.
3. Find dT/dt when $T = 10$.
4. Find dt/dT when $T = 10$.
5. Find t when $dT/dt = -5.0°$ per min.

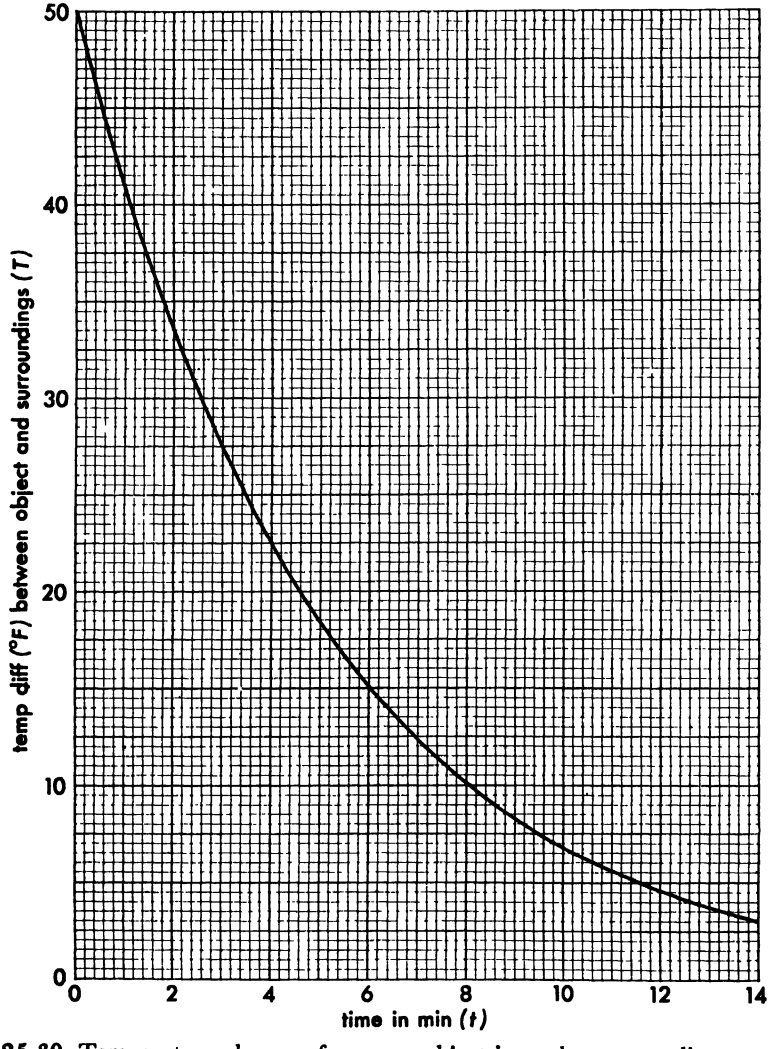


Fig. 25·30. Temperature change of a warm object in cooler surroundings vs. time.

6. Find T when $dT/dt = -2.5°$ per min.
7. Find the average rate of change of T with respect to t between $t = 2$ and $t = 8$.
8. Find the average slope between $t = 6$ and $t = 10$.
9. Find the average slope between $T = 30$ and $T = 10$.
10. Find the instantaneous slope at $t = 6$.
11. Find the instantaneous slope at $T = 30$.
12. How fast is T changing with t at $t = 10$?
13. Find the angle of inclination of the graph at $t = 2$.
14. How much does T change when t changes from 4 to 8?
15. How much does t change when T changes from 40 to 10?
16. About how much does T decrease when t increases from 3.99 to 4.00?
17. Find the area under the curve between $t = 4$ and $t = 8$.
18. Find the average height of the curve between $t = 4$ and $t = 8$.
19. Find the average ordinate of the curve between $t = 4$ and $t = 8$.
20. Find the average T from 3 to 6 min.
21. Find the average rate of change of T between 3 and 6 min.
22. Find the instantaneous rate of temperature change at 4.5 min.
23. When is the rate of fall equal to 4° per min?
24. What is the rate of fall when the temperature difference is 50°, 40°, 30°, 20°, 10°? Plot these rates against corresponding temperature differences. What relationship is revealed by this plot?
25. Approximately how long will it take for the temperature difference to go from 10 to 9.98°?
26. Approximately what will be the change in temperature difference between 2.0 and 2.06 min.?
27. When will the temperature difference be reduced to $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$ of the original value? What relationship is revealed?

EXERCISE 10

(Refer to Fig. 25·31.)

1. How fast does the force decrease with distance when the distance is 4 ft?
2. What is the rate of change in force with respect to distance when the force is 6 lb?
3. Find dF/dD when $D = 5$ ft. Where F = force in pounds, and D = distance in feet.
4. Find dD/dF when $D = 5$ ft.
5. Find $\Delta F/\Delta D$ between $D = 2$ and $D = 3$. Also find $\Delta D/\Delta F$.
6. Find the average force between $D = 2$ and $D = 4$.
7. Plot a curve of work vs. distance between $D = 1$ and $D = 6$.

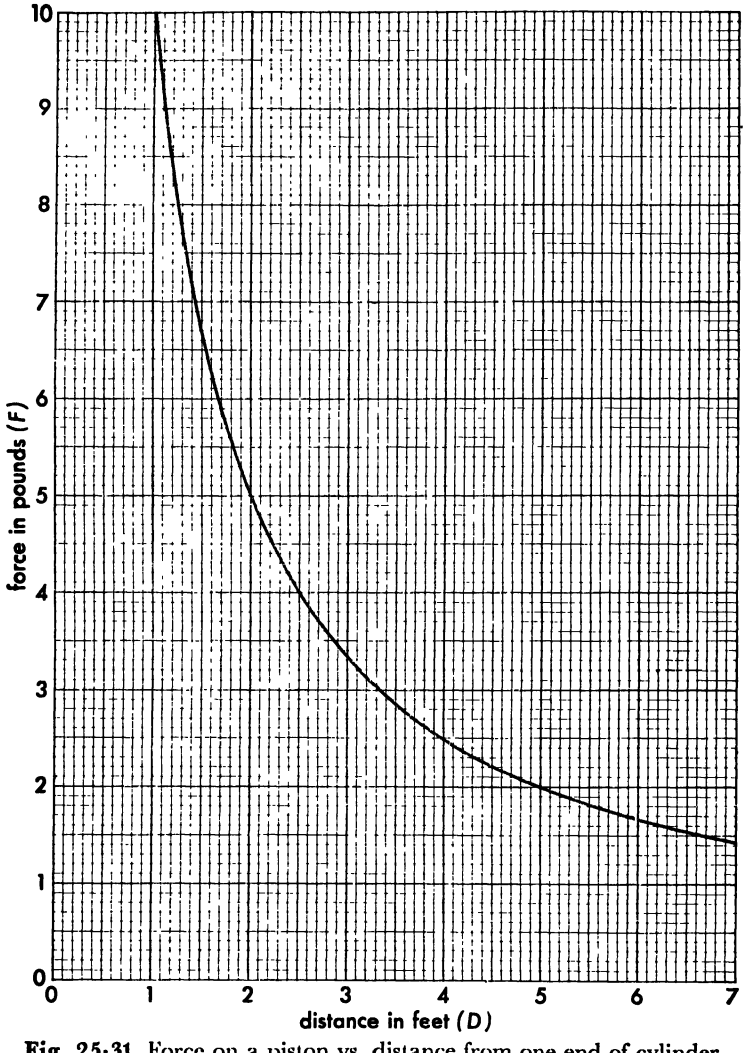


Fig. 25·31. Force on a piston vs. distance from one end of cylinder.

8. For what force is the rate of change of force with respect to distance equal to -2 lb per ft?

9. Find the average rate of change in force with respect to distance between 1 and 4 ft.

10. Find the instantaneous rate of change in force with respect to distance when distance is $3\frac{1}{3}$ ft.

11. Find the approximate change in force when distance changes from $2\frac{1}{2}$ to 2.507 ft.

EXERCISE 11

(Refer to Fig. 25·32.)

1. Find the average acceleration of the point between $t = 0.02$ to $t = 0.03$. (Acceleration is the rate of change in velocity.)
2. Find the instantaneous acceleration at $t = 0.03$.
3. Find the time and value of the maximum speed.

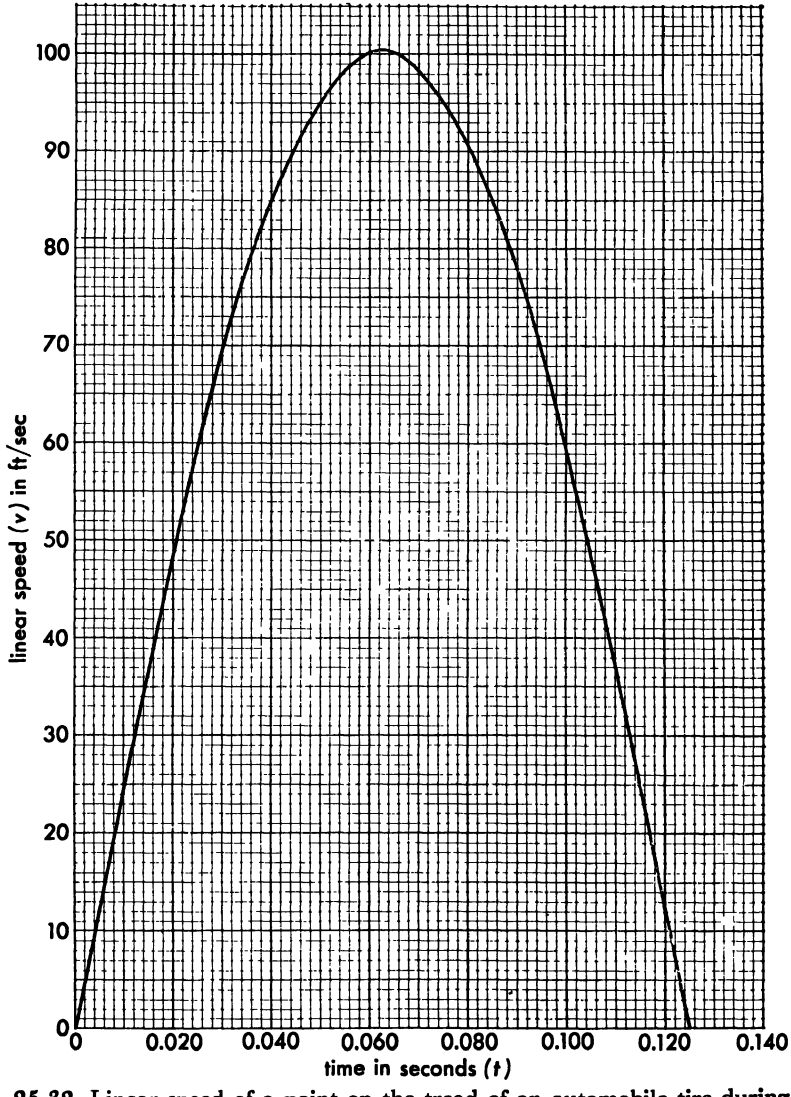


Fig. 25·32. Linear speed of a point on the tread of an automobile tire during one revolution vs. time.

4. Find the speed of the car.
5. Find the outside diameter of the tire.
6. Find the distance the point moves between $t = 0$ sec and $t = \frac{1}{8}$ sec.

EXERCISE 12

(Refer to Fig. 25·33.)

The graph in Fig. 25·33 shows the relation between current and time in a certain series circuit containing inductance and resistance.

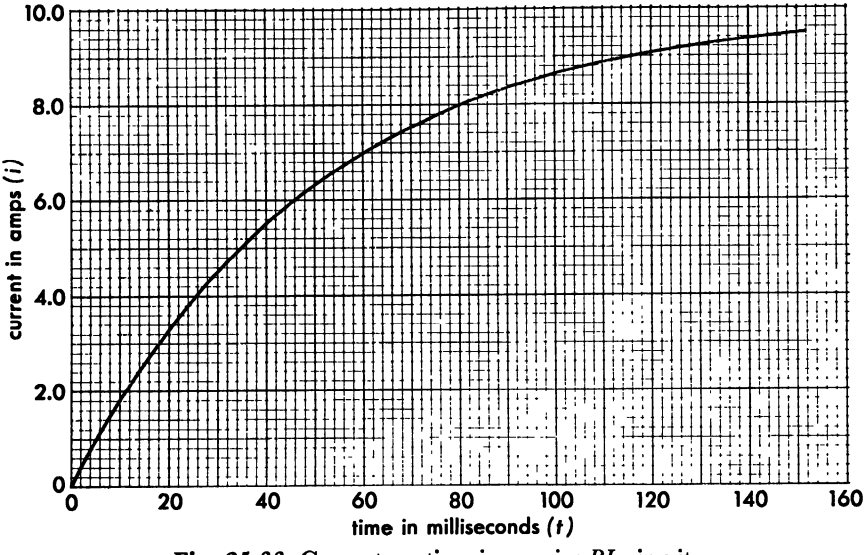


Fig. 25·33. Current vs. time in a series *RL* circuit.

The inductance is 0.75 henry, the resistance is 15 ohms. A battery of 150 volts is connected across this series combination when $t = 0$. Thereafter the current rises as shown in the curve.

1. Find di/dt when $t = 0$ (t is in seconds).
2. Find the "maximum" value of current. When will it be reached?
3. If the current increased at the rate found in Prob. 1 constantly, how long would it take for the current to reach its maximum value?
4. Compare the value found in Prob. 3 with the ratio L/R, where L is inductance and R is resistance. This ratio is called the "time constant" of the circuit.
5. Find the rate of change of current with respect to time at frequent intervals between $t = 0$ and $t = 150$ msec. Plot a graph of di/dt versus t between $t = 0$ and $t = 140$ msec.

EXERCISE 13

(Refer to Fig. 25·34.)

1. Acceleration is the rate of change in velocity with respect to time. Find the average acceleration between 5 and 10 sec.

*2. Find the acceleration at $t = 50$ sec.

3. Find the maximum speed in miles per hour. When did it occur?

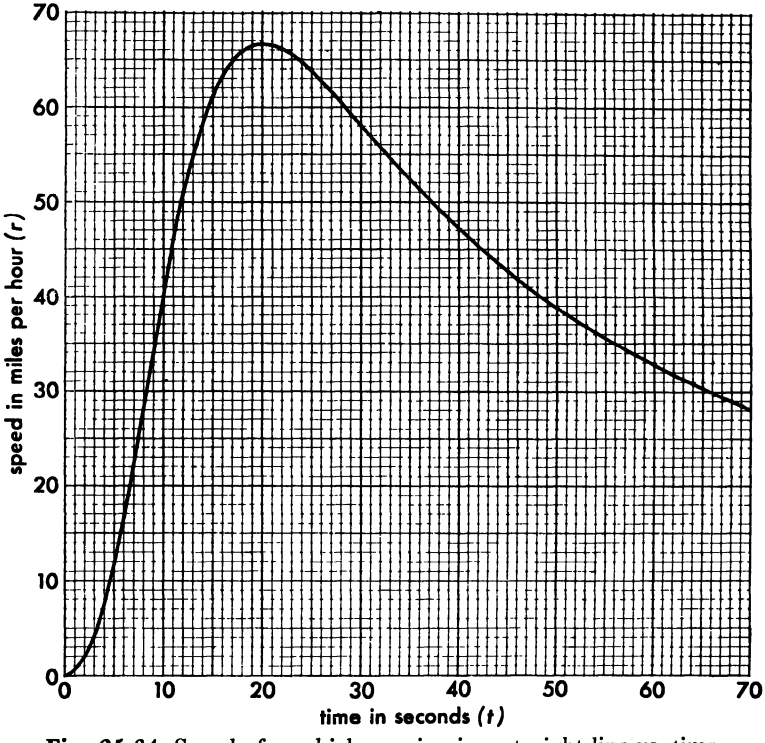


Fig. 25·34. Speed of a vehicle moving in a straight line vs. time.

4. Find and tabulate the distance in feet covered during each 2-sec interval. Plot a cumulative distance curve against time, over the first 70 sec. Find the average speed (in feet per second) for the first 70 sec.

5. When was one-fourth of the total distance covered?

*6. What was the acceleration at $t = 10$? At $t = 15$?

*7. Find the acceleration at $t = 12.5$.

*8. When was the acceleration -1.0 ft per sec^2?

9. What was the gain in speed between $t = 4$ and $t = 8$?

*10. Find the instantaneous acceleration at 2½-sec intervals from $t = 0$ to $t = 70$ sec. Plot acceleration vs. time.

* Find acceleration in feet per second per second.

11. When was the acceleration a maximum?
12. When was the acceleration a minimum?
13. Find the coordinates of any points of inflection.

EXERCISE 14

1. Complete the computations which have been started on Table 25·5. Table 25·5 is an idealized set of trajectory data for a sounding rocket fired in an upward direction when $T = 0$.
2. Plot these data as indicated in the sketch (Fig. 25·35).
3. Explain the physical significance of the shape of the acceleration curve.

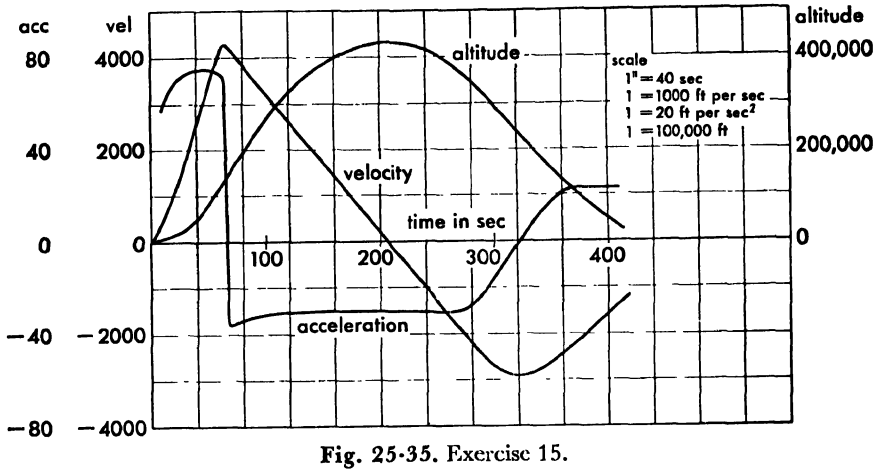


Fig. 25·35. Exercise 15.

4. Find the vertical distance covered between 120 and 140 sec (**a**) from velocity curve, (**b**) from height curve.
5. Find the vertical distance covered between 180 and 230 sec (**a**) from velocity curve, (**b**) from height curve. (**c**) What would be the significance of a zero answer?
6. Find the average velocity from 260 sec to 270 sec (**a**) using velocity curve, (**b**) using height curve.
7. Repeat Prob. 6 for interval 300 to 360 sec.
8. Use height curve to find velocity at 250 sec. Check by reading from velocity curve.
9. What was the probable time of impact?
10. What was the probable velocity of impact?
11. Find the acceleration at 42 sec (**a**) from velocity curve, (**b**) from acceleration curve.
12. Find the average acceleration from 260 to 290 sec (**a**) from acceleration curve, (**b**) from velocity curve.

Table 25·5

Time, sec	Velocity,* fps	Average*† velocity, fps	Average*‡ acceleration, ft/sec²	Change† in altitude Δh, ft	Cumulative altitude, ft
0	0				0
		200	40	2,000	
10	400				2,000
		725	65	7,250	
20	1,050				9,250
		1,410	72	14,100	
30	1,770				23,350
		2,145	75	21,450	
40	2,520				44,800
50	3,270				
60	4,020				
62	4,168				
64	4,308				
66	4,348				
68	4,292				
70	4,220				
80	3,870				
90	3,530				
100	3,200				
120	2,580				
140	1,980				
160	1,380				
180	780				
200	180				
220	−420				
240	−1,040				
260	−1,660				
280	−2,280				
300	−2,760				
320	−2,960				
340	−2,860				
360	−2,520				
380	−2,080				
400	−1,640				
420	−1,200				

* All references to accceleration and velocity are in terms of their vertical components.

† Do not plot. Used only to calculate cumulative altitude.

‡ Plot against mid-point of time interval.

13. Find time of maximum upward velocity, (**a**) using velocity curve, (**b**) using height curve, (**c**) using acceleration curve. (**d**) Find value of maximum upward velocity.
14. Answer the same questions as in Prob. 13 for the maximum downward velocity.
15. Find the time of maximum height (**a**) from height curve, (**b**) from velocity curve.
16. Find the probable value of the maximum height.
17. Calculate each of the following by the best method; also indicate (without calculating) an alternative method of doing the same thing: (**a**) velocity at 50 sec, (**b**) average velocity from 54 to 96 sec, (**c**) acceleration at 290 sec, (**d**) average acceleration from 42 to 58 sec, (**e**) actual distance covered from 160 to 250 sec.

EXERCISE 15

A rubber tire can be "cured" by heating for 2½ hr at 325°F. However, when the tire is placed in the mold it does not heat up to 325° at once, but gradually, as shown in Table 25·6, Part *A*.

During the warm-up period some cure takes place. The problem is to determine how long the tire must remain in the mold after reaching 325°.

Table 25·6

Part *A*		Part *B*	
Time, hr	*Temp., °F*	*Temp., °F*	*Relative curing rate per hour*
0 0	80	60	0.10
0.5	100	120	0.17
0.8	138	140	0.20
1.0	172	160	0.24
1.2	203	187	0.30
1.4	236	211	0.37
1.6	267	233	0.45
1.8	293	245	0.50
2.0	306	270	0.62
2.5	320	280	0.68
1 8	325	290	0.74
3.0	325	304	0.84
		312	0.90
		320	0.96
		325	1.00

This is done by plotting both time and rate of cure over a common temperature base (see Fig. 25·36) and developing Part *C* (time vs. rate of cure). For instance, when the temperature is 155°, the time is 0.90 hr and the rate of cure is 0.23. Therefore, these last two numbers form a pair of values in Part *C*.

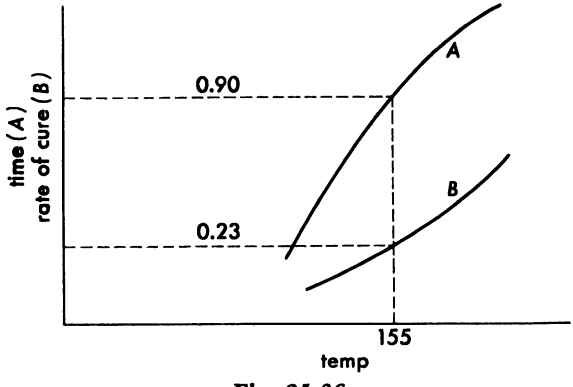


Fig. 25·36

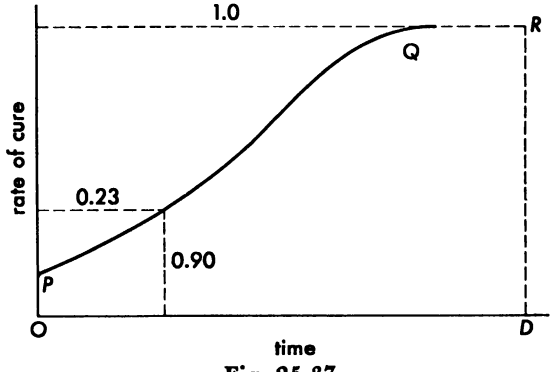


Fig. 25·37

The data in Part *C* are plotted as indicated in Fig. 25·37 and extending straight line *QR* until area *PQRDO* = 2.5.

25·17 Summary. Up to this point we have covered four particularly important topics:

1. The slope of a straight-line graph
2. The area under a straight-line graph
3. The slope of a nonlinear graph
4. The area under a nonlinear graph

All these problems have been approached graphically. They are subject to the precision to which the graph is drawn. With a large graph the precision may be quite high. With a small graph the preci-

sion would be correspondingly less. As a matter of fact, the same thing could be said of arithmetical calculations. Here also the precision of the result is governed by the precision of the data and the number of digits retained.

There are no inherent approximations in the basic *theory* of finding the slope of a straight line, the area under a straight line, or the slope of a curved line. Such approximations as exist arise not from any theoretical approximations but from the relatively inexact nature of the method.

In the case of the area under a nonlinear *curve*, there is a *theoretical* approximation involved. This was brought out in Fig. 25·28 where the area of the strip was calculated as though it were a trapezoid, knowing full well that it is not a trapezoid. If the strip is narrow enough, this approximation is good enough. It is, however, a *theoretical approximation.* The approximate nature of the answer to such problems arises *both* from the approximate nature of the theory and the approximate nature of the method.

In finding the area under a curve we used the approximate formula

$$\Delta A = \bar{y}\,\Delta x$$

In subsequent study we shall learn that if the equation for a function is given (instead of its graph) we may be able to calculate the area under the graph *precisely*. To indicate this we shall use the symbol $\int$.

$\Sigma \bar{y}\,\Delta x$ indicates an approximate area under a graph; the symbol $\int y\,dx$ will indicate the precise area under the graph (see Chap. 27).

Chapter 26 DIFFERENTIATION

In Sec. 25·7 we discussed *graphical* methods of finding slopes and rates when the *graph* of the function is available.

We shall now deal with *algebraic* methods of finding slopes and rates when the *equation* of the function is available.

26·1 Average Slopes and Rates Found by Algebraic Methods. In Sec. 25·9 we found the average slope of the graph in Fig. 25·13 between $x = 2$ and several other values of x. We did this work directly from the graph. The curve shown in Fig. 25·13 was actually plotted from the equation

$$y = x^3 \tag{1}$$

Now we shall work directly from the *equation* (1) rather than from its graph. We shall develop an equation for the average slope between *any* value of x and *any other* value of x.

Equation (1) says in effect that the dependent variable equals the cube of the independent variable. Regardless of what numerical value the independent variable has, there is associated with it some specific value of the dependent variable.

Let us fix our attention on some particular value of x—any value will do. Then we shall let x increase by some amount, large or small, which we shall denote by Δx. As a consequence, y will change by some corresponding amount Δy. Therefore we can write

$$y + \Delta y = (x + \Delta x)^3 \tag{2}$$

The student should bear in mind that we are not adding a constant to both sides of Eq. (1). Rather we are first fixing our attention on Eq. (1) as it stands. Then we are recognizing that if we change the independent variable from the old value x to the new value $x + \Delta x$, the dependent variable will as a consequence change from the old value y to the new value $y + \Delta y$.

Expanding the right-hand side of Eq. (2) we obtain

$$y + \Delta y = x^3 + 3x^2\,\Delta x + 3x\,(\Delta x)^2 + (\Delta x)^3 \tag{3}$$

Subtracting (1) from (3) to find the change in the dependent variable,

$$\Delta y = 3x^2\,\Delta x + 3x\,(\Delta x)^2 + (\Delta x)^3 \tag{4}$$

Dividing both sides of (4) by Δx to find the average rate of change in y with respect to x,

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x\,\Delta x + (\Delta x)^2 \tag{5}$$

Equation (5) is the general equation for the average slope of the graph of $y = x^3$.

EXAMPLE 1. Find the average slope of the graph of $y = x^3$ between $x = 2$ and $x = 2.5$.

In this case $\Delta x = 2.5 - 2.0 = 0.5$ and $x = 2$. Substituting these values in Eq. (5), we obtain

$$\frac{\Delta y}{\Delta x} = 3(2)^2 + 3 \times 2(0.5) + (0.5)^2 \tag{6}$$

$$= 12 + 3 + 0.25 = 15.25 \tag{7}$$

Check this average slope directly from Fig. 25·13.

EXAMPLE 2. Find the equation for the average rate of change in y with respect to x if

$$\begin{aligned} y &= 5x^2 + 3 \\ y + \Delta y &= 5(x + \Delta x)^2 + 3 \\ y + \Delta y &= 5[x^2 + 2x\,\Delta x + (\Delta x)^2] + 3 \\ y + \Delta y &= 5x^2 + 10x\,\Delta x + 5(\Delta x)^2 + 3 \\ y &= 5x^2 \qquad\qquad\qquad\quad + 3 \\ \hline \Delta y &= \qquad 10x\,\Delta x + 5(\Delta x)^2 \\ \frac{\Delta y}{\Delta x} &= 10x + 5\,\Delta x \end{aligned}$$

EXAMPLE 3. Find the equation for the average rate of change in y with respect to x if

$$\begin{aligned} y &= 5x^2 + x^3 + 3 \\ y + \Delta y &= 5(x + \Delta x)^2 + (x + \Delta x)^3 + 3 \\ y + \Delta y &= 5[x^2 + 2x\,\Delta x + (\Delta x)^2] + [x^3 + 3x^2\,\Delta x + 3x(\Delta x)^2 + (\Delta x)^3] + 3 \\ y + \Delta y &= 5x^2 + 10x\,\Delta x + 5(\Delta x)^2 + x^3 + 3x^2\,\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 3 \\ y &= 5x^2 \qquad\qquad\qquad\quad + x^3 \qquad\qquad\qquad\qquad\qquad + 3 \\ \hline \Delta y &= \qquad 10x\,\Delta x + 5(\Delta x)^2 \qquad + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 \\ \frac{\Delta y}{\Delta x} &= 10x + 5\,\Delta x + 3x^2 + 3x\,\Delta x + (\Delta x)^2 \end{aligned}$$

EXAMPLE 4. Find the equation for the average rate of change in y with respect to x if $y = 1/x^2$.

$$y + \Delta y = \frac{1}{(x + \Delta x)^2} = \frac{1}{x^2 + 2x\,\Delta x + (\Delta x)^2}$$

Subtracting

$$\Delta y = \frac{1}{x^2 + 2x\,\Delta x + (\Delta x)^2} - \frac{1}{x^2} = \frac{x^2 - x^2 - 2x\,\Delta x - (\Delta x)^2}{x^2[x^2 + 2x\,\Delta x + (\Delta x)^2]}$$

$$= \frac{-2x\,\Delta x - (\Delta x)^2}{x^2[x^2 + 2x\,\Delta x + (\Delta x)^2]}$$

$$\frac{\Delta y}{\Delta x} = \frac{-2x - \Delta x}{x^2[x^2 + 2x\,\Delta x + (\Delta x)^2]}$$

EXERCISE 1

Find the equation for the average rate of change in the dependent variable with respect to the independent variable in the following problems.

1. $y = 5x^3$
2. $y = 5x^3 + 8$
3. $y = 7x^2 + 3$
4. $y = x^3 - 2x$
5. $g = \dfrac{t}{t + 3}$
6. $w = 1 + 2z - \dfrac{1}{z}$
7. $u = \sqrt{x}$
8. $w = u^2 - 3u - 18$
9. $w = \dfrac{3}{u}$
10. $A = \pi R^2$
11. $V = \frac{4}{3}\pi R^3$

12. Find ΔA in Prob. 10 if R increases from 5 to 5.2 in.
13. Confirm the answers to Probs. 10 and 11 by geometry.

26·2 Limits. It is necessary at this point to interject a discussion of *limits* before we proceed to the following section.

Our discussion of limits will be intuitive for the most part. However, we shall describe a limit and discuss it to some extent, leaving a more complete discussion of the subject for a more advanced work.

When the successive values of a variable P *approach nearer and nearer a fixed number* C *in such a way that the difference between* C *and* P *becomes and remains numerically less than any preassigned positive number however small, the constant* C *is called the limit of* P *and* P *is said to approach the limit* C.

Let us consider these phrases one at a time:

"*When the successive values of a variable*"

Suppose we consider the perimeters of regular polygons inscribed in a circle. The perimeter is a *variable* since as we change the number of sides, the perimeter varies. We are going to consider *successive values* of this variable; that is, we are not going to pick perimeters of random polygons, and we are going to pick them in order: a four-sided polygon, a five-sided polygon, a six-, seven-, eight-sided polygon, etc.

"*approach nearer and nearer a fixed number*"

Now let us compare the perimeters of these polygons with the circumference of the circumscribed circle which is for our purpose a constant.

"*in such a way*"

The approach mentioned above must occur in a certain way to be described presently.

"*that the difference between* C *and* P *becomes and remains numerically less than any preassigned positive number however small*"

The difference between the perimeters of the polygons and the circumference of the circle becomes less as the number of sides of the polygons (and hence their perimeter) increases. In order to make this difference less than any number which you or anyone else may *preassign*, all we have to do is to choose a polygon with enough sides.

As we progress in an orderly way from a polygon of a few sides to a polygon of many sides, the difference between the perimeter and the circumference always decreases, it *never increases;* that is, it *becomes* and *remains* less than any positive number you can think of.

"*the constant* C *is called the limit of* P *and* P *is said to approach the limit* C"

The student must realize that we are not trying to deal with the situation where the number of sides of the inscribed polygon actually becomes infinite. If we did, we would have to deal with an infinite number of sides of zero lengths. The sum of an infinite number of straight lines which do not exist is a rather difficult concept to handle.

Neither are we dealing with a polygon of an enormous number of sides whose perimeter is for all practical purposes an adequate substitute for the circumference of the circle.

We are dealing with a dynamic situation where an inscribed polygon is adding sides to itself, one by one—successively—endlessly. The polygon does not pause in its growth, neither does it ever become a circle.

The student will have to keep his thinking *between* the perimeter of the expanding inscribed polygon and the circumference of the circumscribed circle. If he allows the polygon to become a circle,

then his area of thinking vanishes and with it any conception of the calculus.

EXERCISE 2

1. The cost of publishing a certain pamphlet is given by the following equation:

$$C = \$10.00 + 0.05n$$

where C = total cost, dollars
n = number of pamphlets published

a. Calculate the total cost of publishing 10, 100, 1,000, 10,000, 100,000, 1,000,000, and ∞ pamphlets.

b. Divide the total cost as determined in part *a* by the corresponding number of pamphlets to determine the cost per pamphlet.

c. What limit does the cost per pamphlet approach as the number of pamphlets approaches ∞?

d. Point out the fallacy in the following statement: "The cost per pamphlet when an infinite number are published is \$0.05 per pamphlet."

Let us now return to Eq. (4). Here it is quite obvious that as Δx approaches zero, Δy also approaches zero. As a specific example, let us compute Δy when x changes from 2 to a few other values in the vicinity of 2. When $x = 2$, Eq. (4) can be written

$$\Delta y = 12\,\Delta x + 6(\Delta x)^2 + (\Delta x)^3$$

When $\Delta x = +0.1$

$$\Delta y = 1.2 + 0.06 + 0.001 = 1.261$$

When $\Delta x = +0.01$

$$\Delta y = 0.12 + 0.0006 + 0.000001 = 0.120601$$

When $\Delta x = -0.1$

$$\Delta y = -1.2 + 0.06 - 0.001 = -1.141$$

Column 2 of Table 26·1 was calculated in this way.

The ratio $\Delta y/\Delta x$ can be calculated by dividing each value of Δy above by the corresponding value of Δx. The ratio $\Delta y/\Delta x$ can also be calculated by direct substitution in Eq. (5). Let us adopt the first method.

When $\Delta x = +0.1$

$$\frac{\Delta y}{\Delta x} = \frac{1.261}{0.1} = 12.61$$

When $\Delta x = +0.01$

$$\frac{\Delta y}{\Delta x} = \frac{0.120601}{0.01} = 12.0601$$

When $\Delta x = -0.1$

$$\frac{\Delta y}{\Delta x} = \frac{-1.141}{-0.1} = 11.41$$

Column 3 of Table 26·1 was calculated in this way.

Table 26·1

(1) Δx	(2) Δy	(3) $\Delta y/\Delta x$
+0.1	+1.261	+12.61
+0.01	+0.120601	+12.0601
+0.001	+0.012006001	+12.006001
+0.0001	+0 001200060001	+12.00060001
+0.00001	+0 000120000600001	+12.0000600001
+0.000001	+0.000012000006000001	+12.000006000001
+0.0000001	+0.00000120000060000001	+12.00000060000001
0.00000000	0.0000000000000000000	Indeterminate
−0.0000001	−0.00000119999940000001	+11.99999940000001
−0.000001	−0.000011999994000001	+11.999994000001
−0.00001	−0.000119999400001	+11.9999400001
−0.0001	−0.001199940001	+11.99940001
−0.001	−0 011994001	+11.994001
−0.01	−0.119401	+11.9401
−0.1	−1 1414	+11 41

As we scan Table 26·1 from either the top or bottom, toward the center of the table, we observe that as Δx *approaches* zero, Δy also *approaches* zero. However, their ratio $\Delta y/\Delta x$ approaches the finite constant 12. But if we let $\Delta x = 0$, then Δy also becomes zero and the quotient $\Delta y/\Delta x$ becomes meaningless. Therefore we rest with the statement

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 12 \tag{8}$$

which we read: "The limit which $\Delta y/\Delta x$ approaches as Δx approaches zero is 12."

As was stated in Eq. (32), page 595, this limit is denoted by dy/dx and

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \tag{9}$$

where dy/dx is called "the derivative of y with respect to x."

If y is the ordinate of a curve and x is the abscissa of the same curve, then dy/dx is the instantaneous slope of the curve.

The process of finding the derivative of a function is called *differentiation.*

EXAMPLE 5. Find the derivative of the function $y = x^2$.

$$\begin{aligned} y &= x^2 \\ y + \Delta y &= (x + \Delta x)^2 \\ y + \Delta y &= x^2 + 2x\,\Delta x + (\Delta x)^2 \\ y &= x^2 \\ \Delta y &= 2x\,\Delta x + (\Delta x)^2 \\ \frac{\Delta y}{\Delta x} &= 2x + \Delta x \\ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \frac{dy}{dx} = 2x \end{aligned} \tag{10}$$

(Frequently, it is convenient to write this equation in differential form. See Sec. 25·12.)

Precisely the same answer would have been required if the problem had been stated in any of the following ways:

1. Differentiate $y = x^2$.
2. Find the equation for instantaneous rate of change of y with respect to x if $y = x^2$.
3. Find the equation for the instantaneous slope of the curve of the function $y = x^2$.
4. Find the equation for the slope of the tangent to the curve of $y = x^2$.

EXERCISE 3

Find the derivative of the dependent variable with respect to the independent variable for the problems in Exercise 1.

26·3 General Rule for Differentiating a Power Function. A power function is a function in the form

$$y = kx^n + C \tag{11}$$

where k, n, and C are any constants, and x is the independent variable.

Note that here the independent variable is raised to a constant power. We shall first differentiate Eq. (11) by the Δ method and then point out a short cut to the process which applies to this particular type of function. Let x, in Eq. (11), increase by an amount Δx. Then y will change by a corresponding amount Δy. Accordingly,

$$y + \Delta y = k(x + \Delta x)^n + C \tag{12}$$

Now let us restrict n in Eq. (12) to a positive integer. Then we can expand the right-hand side of Eq. (12) by the binomial theorem. See Eq. (1), page 241.

$$y + \Delta y = k\left[x^n + nx^{n-1}\,\Delta x + \frac{n(n-1)x^{n-2}(\Delta x)^2}{2!} + \frac{n(n-1)(n-2)x^{n-3}\,(\Delta x)^3}{3!} + \cdots (\Delta x)^n\right] + C \quad (13)$$

Multiplying by k in Eq. (13) and subtracting (11) from (13),

$$\Delta y = nkx^{n-1}\,\Delta x + \frac{kn(n-1)x^{n-2}(\Delta x)^2}{2!} + \frac{kn(n-1)(n-2)x^{n-3}(\Delta x)^3}{3!} + \cdots k(\Delta x)^n \quad (14)$$

Dividing Eq. (14) by Δx,

$$\frac{\Delta y}{\Delta x} = nkx^{n-1} + \frac{kn(n-1)x^{n-2}\,\Delta x}{2!} + \frac{kn(n-1)(n-2)x^{n-3}(\Delta x)^2}{3!} + \cdots k(\Delta x)^{n-1} \quad (15)$$

Now observe that all terms on the right-hand side of Eq. (15) except the first contain the factor Δx. Therefore, as Δx approaches zero, all terms on the right-hand side of (15) except the first also approach zero and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = nkx^{n-1} \quad (16)$$

or

$$\frac{dy}{dx} = nkx^{n-1} \quad (17)$$

Therefore, if

$$y = kx^n + C \quad (18)$$

where n is a positive integer, then

$$\frac{dy}{dx} = nkx^{n-1} \quad \star (19)$$

Actually, Eq. (19) is valid whether n is positive or negative, integral or fractional, rational or irrational. We shall prove this in Sec. 26·16. For the present we shall simply assert the fact.

Note that the added constant C subtracts out in the process of subtracting Eq. (11) from Eq. (13). Therefore this added constant does not appear in the derivative.

1. *If a function has an added constant, that constant does not appear in the derivative* (see Sec. 26·4 for supporting reasons).

Also note that the multiplied constant k multiplies the derivative.

2. *If a function has a multiplied constant, that constant multiplies the derivative.*

It is important to note that in Eq. (18) it is *the independent variable itself* which is raised to the nth power. A very different situation exists in the case of a function such as

$$y = (x^2 + 7)^n + C \tag{20}$$

Here it is not the independent variable itself which is raised to the nth power. Differentiating Eq. (20) requires a somewhat different approach, which will be discussed in Sec. 26·5.

EXAMPLE 6. Differentiate $y = 3x^5$.

$$\frac{dy}{dx} = 15x^4$$

EXAMPLE 7. Differentiate $y = 3x^{-5}$.

$$\frac{dy}{dx} = -15x^{-6}$$

EXAMPLE 8. Differentiate $y = -3x^{-5}$.

$$\frac{dy}{dx} = +15x^{-6}$$

EXAMPLE 9. Differentiate $y = 5x^{7/4}$.

$$\frac{dy}{dx} = \frac{35}{4}x^{3/4}$$

EXAMPLE 10. Differentiate $y = 5x^{1/3}$.

$$\frac{dy}{dx} = \frac{5}{3}x^{-2/3}$$

EXAMPLE 11. Differentiate $y = 7x^{-1/2}$.

$$\frac{dy}{dx} = -\frac{7}{2}x^{-3/2}$$

EXAMPLE 12. Differentiate $y = 6x^{\pi}$.

$$\frac{dy}{dx} = 6\pi x^{\pi-1}$$

EXAMPLE 13. Differentiate $a = \frac{\pi}{16}h^2$.

$$\frac{da}{dh} = \frac{\pi}{8}h$$

EXAMPLE 14. Differentiate $y = 5\sqrt{x}$.

Notice that the type of equations from which we are working [Eqs. (18) and (19)] require that the independent variable be expressed as a power. Therefore we must alter the present equation to conform to this. Consequently we write

$$y = 5x^{\frac{1}{2}}$$

and

$$\frac{dy}{dx} = \frac{5}{2}x^{-\frac{1}{2}} = \frac{5}{2\sqrt{x}}$$

EXAMPLE 15. Differentiate $y = 5(3x)^2$.

Referring to Eqs. (18) and (19) we note that it is the *independent variable itself* which is raised to a power. In the present case we have the independent variable times a constant which is raised to a power. Therefore we write

$$y = 5 \times 3^2 \times x^2 = 5 \times 9x^2 = 45x^2$$

and

$$\frac{dy}{dx} = 90x$$

EXAMPLE 16. Differentiate $y = 15/x$.

Again we must change the equation to be differentiated to a power form. Therefore we write

$$y = 15x^{-1}$$

$$\frac{dy}{dx} = -15x^{-2} = -\frac{15}{x^2}$$

EXAMPLE 17. Express the derivative in Example 16 as a function of y. In Example 16

$$\frac{dy}{dx} = -\frac{15}{x^2}$$

But from the original equation

$$x = \frac{15}{y}$$

Therefore

$$\frac{dy}{dx} = -\frac{15}{(15/y)^2} = -15\frac{y^2}{(15)^2} = -\frac{y^2}{15}$$

EXAMPLE 18. In Example 16 find dx/dy.

The equation in Example 16 may be written

$$x = \frac{15}{y} \quad \text{or} \quad x = 15y^{-1}$$

Therefore

$$\frac{dx}{dy} = -15y^{-2} = -\frac{15}{y^2}$$

Compare with Example 17.

EXERCISE 4

Differentiate the following:

1. $y = 3x^2$
2. $y = \frac{4}{3}\pi x^3$
3. $y = 6\sqrt{x}$
4. $y = 12x^{3/4}$
5. $y = \dfrac{14}{x}$
6. $y = \sqrt{6x}$
7. $y = -\dfrac{8}{\sqrt{x}}$
8. $y = 10\sqrt[4]{x}$
9. $y = \dfrac{a}{bx}$
10. $y = 2x^{-3/4}$

EXERCISE 5

1. The graph shown in Fig. 25·29 was plotted from the equation $a = (\pi/16)h^2$ where a is the area of cross section of a right circular cone, and h is the distance between the apex of the cone and the cross-section area a. Using the above equation rather than the graph, solve Probs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, and 15 in Exercise 8, page 608.
2. The graph shown in Fig. 25·31 was plotted from the equation $F = 10/D$ where F is the force on a piston, and D is the distance between the piston and one end of the cylinder. Using the above equation rather than the graph, solve Probs. 1, 2, 3, 4, 5, 8, 10, and 11 in Exercise 10, page 611.

26·4 The Derivative of a Sum or Difference. Let us differentiate the function $y = x^2 + x^3$ by the Δ method.

$$y = x^2 + x^3 \tag{21}$$

$$y + \Delta y = x^2 + 2x\,\Delta x + (\Delta x)^2 + x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 \tag{22}$$

$$\Delta y = 2x\,\Delta x + (\Delta x)^2 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 \tag{23}$$

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x + 3x^2 + 3x\,\Delta x + (\Delta x)^2 \tag{24}$$

$$\frac{dy}{dx} = 2x + 3x^2 \tag{25}$$

Following Eqs. (18) and (19) we note that the derivative of x^2 is $2x$. Also, the derivative of x^3 is $3x^2$. If we compare the above relations with Eqs. (21) and (25), we can see that the following rule applies:

3. *The derivative of a sum (or difference) is the sum (or difference) of the derivatives.*

Following rule 3 we can now support rule 1. Consider the equation

$$y = x^3 + 9 \tag{26}$$

From rule 3 we know that the derivative of y with respect to x is the sum of the derivatives of x^3 and the constant 9. However, we also know that a constant has no rate of change; therefore its derivative is zero, and

$$\frac{dy}{dx} = 3x^2 + 0 \tag{27}$$

or

$$\frac{dy}{dx} = 3x^2 \tag{28}$$

We can express the following rule:

4. *The derivative of an added constant is zero.*

EXAMPLE 19. Differentiate $y = \frac{5}{3x} + x^2$.

$$y = \frac{5}{3}x^{-1} + x^2$$

$$\frac{dy}{dx} = -\frac{5}{3}x^{-2} + 2x = -\frac{5}{3x^2} + 2x$$

EXAMPLE 20. Differentiate $y = 2x^{-6} + 5^e$ (note that 5^e is a constant).

$$\frac{dy}{dx} = -12x^{-7} = -\frac{12}{x^7}$$

EXAMPLE 21. Differentiate $y = a/bx + bx^{-2} - 3\pi x$.

By convention the first letters of the alphabet are usually considered to be constant. Therefore we shall consider a and b to be constants.

$$y = \frac{a}{b}x^{-1} + bx^{-2} - 3\pi x$$

$$\frac{dy}{dx} = -\frac{a}{b}x^{-2} - 2bx^{-3} - 3\pi = -\frac{a}{bx^2} - \frac{2b}{x^3} - 3\pi$$

EXAMPLE 22. Differentiate $y = 3/4x + 2x^3$.

$$y = \frac{3}{4}x^{-1} + 2x^3$$

$$\frac{dy}{dx} = -\frac{3}{4}x^{-2} + 6x^2 = -\frac{3}{4x^2} + 6x^2$$

EXERCISE 6

Differentiate the following:

1. $y = ax^2 + bx + c$

2. $y = \dfrac{4x^3 - 7x + 8}{x}$

3. $y = (2x - 5)(3x + 1)$

4. $y = \dfrac{4}{\sqrt{x}} - 4\sqrt{x}$

5. $y = \dfrac{1}{3x^2} - \dfrac{5}{2x}$

6. $y = \dfrac{ab}{x} - \dfrac{x}{b} + \dfrac{b}{ax}$

7. $y = \sqrt[n]{x^{n-1}}$

8. $y = \dfrac{1}{9x^6}$

9. $y = 3x^{-4} + \pi$

10. $y = -\frac{2}{3}x^{-4} - 10^2$

11. $y = \dfrac{26}{x^4} - 3$

12. $y = \dfrac{-3}{5x^{20}} + \pi^2$

13. $y = kx^{-3} + 15$

14. $y = \dfrac{\pi}{2x^3} - 26$

15. $y = e^2 + 3x^5 + k$

16. $y = 3x^5 - \dfrac{10}{x^2}$

17. $y = \pi x^2 + 30$

18. $y = \dfrac{26}{10^4}$

19. $y = \dfrac{\pi}{x^2} - \dfrac{e}{x^3}$

26·5 The Derivative of $y = u^n$ When u Is a Function of x. Up to this point we have no way of finding dy/dx for a function such as

$$y = (x^2 + 3x + 5)^3 \tag{29}$$

If we let

$$u = x^2 + 3x + 5 \tag{30}$$

then

$$y = u^3 \tag{31}$$

and we can find dy/du (not dy/dx as yet):

$$\frac{dy}{du} = 3u^2 \qquad \text{[see Eq. (19)]} \tag{32}$$

However, the independent variable in (29) is not u, it is x. Therefore Eq. (32) is only a step toward finding dy/dx.

Let us write the identity

$$\frac{dy}{dx} = \frac{dy}{dx} \tag{33}$$

Here both dy and dx are differentials (see Sec. 25·12).

Now let us multiply both numerator and denominator of the right-hand side of (33) by the differential du:

$$\frac{dy}{dx} = \frac{dy}{dx}\frac{du}{du} \tag{34}$$

Rearrange (34) to give

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \star (35)$$

This equation is of great importance and should be carefully memorized. Now from (30) we can write

$$\frac{du}{dx} = 2x + 3 \qquad (36)$$

Substituting (30) in (32),

$$\frac{dy}{du} = 3(x^2 + 3x + 5)^2 \qquad (37)$$

Now substituting (36) and (37) in (35) gives

$$\frac{dy}{dx} = 3(x^2 + 3x + 5)^2(2x + 3) \qquad (38)$$

This is the derivative for which we have been looking, the rate of change in the function y with respect to the *independent variable.*

In general, if

$$y = u^n \qquad (39)$$

where u is a function of x, then

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx} \qquad \star (40)$$

5. *The derivative of the power of a function is the exponent times the next lower power of the function times the derivative of the function.*

Applying rules 2, 4, and 5, if

$$y = ku^n + C \qquad (41)$$

then

$$\frac{dy}{dx} = knu^{n-1}\frac{du}{dx} \qquad \star (42)$$

EXAMPLE 23. Differentiate $y = 7(x^3 + 7x^2 + 2x + 4)^5$

$$\frac{dy}{dx} = 35(x^3 + 7x^2 + 2x + 4)^4(3x^2 + 14x + 2)$$

EXAMPLE 24. If $y = \dfrac{1}{5\sqrt{1 - x^6}}$, find dy/dx.

$$y = \tfrac{1}{5}(1 - x^6)^{-\frac{1}{2}}$$

Then by Eq. (42)

$$\frac{dy}{dx} = \left(\frac{1}{5}\right)\left(-\frac{1}{2}\right)(1 - x^6)^{-\frac{3}{2}}(-6x^5)$$

$$= \frac{3x^5}{5\sqrt{(1 - x^6)^3}}$$

EXERCISE 7

Differentiate the following functions:

1. $y = (3x^2 + 2x + 5)^4$

2. $y = (7x^3 - 5x^2 + 2x)^{-\frac{3}{4}}$

3. $z = \sqrt[3]{(-5x^2 + 3x)}$

4. $w = \sqrt{3t + 7}$

5. $v = \sqrt[4]{6 - 7t^2}$

6. $w = \dfrac{37}{\sqrt{6 - t^2}}$

7. $y = \dfrac{-2}{\sqrt[4]{(1 - t^2)^3}}$

8. $y = (4 - x^2)^3$

9. $y = \sqrt{2t} + 2\sqrt[3]{t}$

10. $R = \dfrac{5}{\sqrt[3]{3t^2 - 6t}}$

11. $S = 2\sqrt[3]{(1 + y^2)^2}$

12. $y = \dfrac{1}{8\sqrt{(3 - 2t)^2}}$

13. $y = \dfrac{4}{1 - x^3}$

14. $y = \dfrac{4}{\sqrt{(1 - x)^3}}$

15. $x = 4\sqrt[5]{5t^3 - t^5}$

16. Sketch the graph of $y = 10/(3 - x) - 5$ between $x = -1.5$ and $x = +1.5$. Compare with curve B of Fig. 25·17. Find the equation for dy/dx from the above equation. Find dy/dx when $x = -1.5$. Compare with Prob. 2 in Exercise 5, page 598. Find dy/dx when $x = +1$. Compare with Prob. 4 in Exercise 5, page 598.

17. Sketch the graph of $y = 10/(x + 3) - 5$ between $x = -1.5$ and $x = +1.5$. Compare with Fig. 25·17A. Find the equation for dy/dx. Find dy/dx when $x = 0$. Compare with Prob. 6 in Exercise 5, page 598.

18. Sketch the graph of $y = 5 - 10/(x + 3)$ between $x = -1.5$ and $x = +1.5$. Compare with Fig. 25·17D. Find the equation for dy/dx. Find x and y when $dy/dx = \frac{5}{8}$. Compare with Prob. 7 in Exercise 5, page 598. Explain the two answers.

26·6 The Derivative of the Function of a Function. In Sec. 26·5 we expressed y—the dependent variable—as a function of u [see Eq. (31)]. We also expressed u as a function of the independent variable

x [see Eq. (30)]. We found that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{43}$$

Equation (43) is an identity and is appropriate when a dependent variable is expressed as a function of an "intermediate" variable, which in turn is expressed as a function of the independent variable. In other words, "if y is a function of u, and u is a function of x," then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{44}$$

This identity will be used in deriving other laws of differentiation and should be memorized.

26·7 Maxima and Minima. Section 25·13 treated the basic theory of problems involving maximum and minimum values. The student should review this section thoroughly. In Sec. 25·13 we discussed so-called "ordinary maximums and minimums." We shall recall that one of the characteristics of a maximum or minimum point on a graph is that at such points the slope of the graph is zero. Consequently, when we are dealing with an algebraic function we can assert that when a maximum or minimum value of this function occurs, the derivative of the function must be zero. We shall call the value of the independent variable which makes the derivative zero a *critical value.* However, there is one further condition which must be met if the function has a true maximum or minimum, that is, the *sign* of the derivative must change as the independent variable passes through such critical values.

If the sign does not change, then the critical value occurs at a point of inflection (see Sec. 25·14).

The following discussion relates to ordinary maximums and minimums. Let us find the critical values of x and identify them in the function

$$y = x^4 - 14x^2 - 24x + 40 \tag{45}$$

We shall identify them as either maximum or minimum points or as points of inflection. It is characteristic of such critical points that they are located where the slope of the curve is zero. Therefore we should find the equation of the slope of the curve and then find what value of x will make the slope zero.

$$\frac{dy}{dx} = 4x^3 - 28x - 24 \tag{46}$$

In order for (45) to have ordinary maximum or minimum values x must have such values as to make (46) equal to zero. Equation (46) is a cubic, and we shall solve it by first finding one root by trial.

Let $x = 0$, then

$$\frac{dy}{dx} = 0 - 0 - 24 = -24 \tag{47}$$

Let $x = +1$, then

$$\frac{dy}{dx} = 4 - 28 - 24 = -48 \tag{48}$$

Let $x = +2$, then

$$\frac{dy}{dx} = 32 - 56 - 24 = -48 \tag{49}$$

Let $x = +3$, then

$$\frac{dy}{dx} = 108 - 84 - 24 = 0 \tag{50}$$

Therefore we have discovered that $x = +3$ is a root of (46). Consequently, Eq. (45) may have a maximum or a minimum value when $x = +3$. If one root of Eq. (46) is $x = +3$, then one factor of (46) is $x - 3$ (see Appendix C).

We may find another factor of (46) by direct division as follows

$$\begin{array}{r|l} & 4x^2 + 12x + 8 \\ x - 3 & \overline{4x^3 - 28x - 24} \\ & \underline{4x^3 - 12x^2} \\ & \quad 12x^2 - 28x \\ & \quad \underline{12x^2 - 36x} \\ & \qquad\quad 8x - 24 \\ & \qquad\quad \underline{8x - 24} \\ & \qquad\quad\quad 0 \end{array}$$

Therefore

$$\frac{dy}{dx} = (x - 3)(4x^2 + 12x + 8) \tag{51}$$

The quantity $4x^2 + 12x + 8$ can be factored by inspection into

$$(x + 2)(x + 1)4 \tag{52}$$

Therefore

$$\frac{dy}{dx} = (x - 3)(x + 2)(x + 1)4 \tag{53}$$

From Eq. (53) we see that y may have a maximum or a minimum value when $x = +3$, -2, or -1. In order to decide which of these points occur at maximum and which at minimum points we shall have to investigate the sign of the derivative in the immediate vicinity of these points. Let us test the sign of the derivative for values of x a little less than and a little more than the critical values $+3$, -2, and -1.

When x is near $+3$

The second and third factors of Eq. (53) are obviously positive if x is near $+3$. However, if x is a little *less* than $+3$, say, $+2.999$, then the first factor is negative and the product of all factors is negative and dy/dx is *negative.*

If x is a little *more* than $+3$, say, $+3.0001$, then the first factor is positive and the product of all factors is positive and dy/dx is *positive.*

As x increases, then, in the vicinity of $x = +3$, the sign of the derivative traverses the following sequence:

$$\text{Negative} \rightarrow \text{zero} \rightarrow \text{positive}$$

and we have a *minimum.*

When x is near -2

The first and third factors of Eq. (53) are obviously negative if x is near -2. However, if x is a little *less* than -2, say, -2.0001, then the second factor is negative and the product of all factors is negative and dy/dx is *negative.* If x is a little *more* than -2, say, -1.9999, then the second factor is positive and the product of all factors is positive and dy/dx is *positive.*

As x increases in the vicinity of $x = -2$, the sign of the derivative traverses the following sequence:

$$\text{Negative} \rightarrow \text{zero} \rightarrow \text{positive}$$

and we have a *minimum.*

When x is near -1

The first factor is obviously negative, and the second factor is obviously positive. However, if x is a little less than -1, say, -1.0001, then the third factor is negative and the product of all the factors is positive and dy/dx is *positive.*

If x is a little *more* than -1, say, -0.9999, then the third factor is positive and the product of all factors is negative and dy/dx is *negative.*

As x increases in the vicinity of $x = -1$, the sign of the derivative traverses the following sequence:

$$\text{Positive} \rightarrow \text{zero} \rightarrow \text{negative}$$

and we have a *maximum.*

EXAMPLE 25. Find the critical values of x, and identify them, if

$$y = 3x^2 - 2x + 7$$

Differentiating the above equation,

$$\frac{dy}{dx} = 6x - 2 = 2(3x - 1)$$

Setting this equation equal to zero

$$0 = 2(3x - 1)$$

and the critical value of x is

$$x = +\tfrac{1}{3}$$

When x is a little less than $\frac{1}{3}$ the derivative is negative; when x is just $\frac{1}{3}$ the derivative is 0; when x is a little more than $\frac{1}{3}$ the derivative is positive. Therefore we can identify the point at which $x = \frac{1}{3}$ the original function as a *minimum* point.

EXAMPLE 26. Find the minimum value of y in the previous example. When $x = \frac{1}{3}$,

$$y = 3(\tfrac{1}{3})^2 - 2(\tfrac{1}{3}) + 7 = \tfrac{1}{3} - \tfrac{2}{3} + 7 = 6\tfrac{2}{3}$$

EXAMPLE 27. A uniform bar weighing $1\frac{1}{2}$ lb per ft is to be used to lift a 100-lb load as shown in Fig. 26·1. What length of bar will require the minimum force, and what is the force?

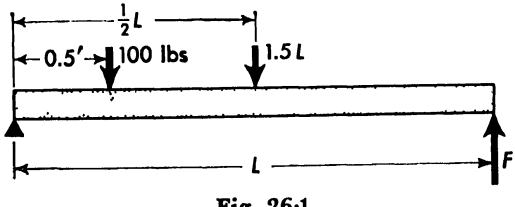


Fig. 26·1

We have two clockwise-force moments 100(0.5) due to the load and $(1.5L)(L/2)$ due to the weight of the bar itself. These are balanced by the moment FL. Accordingly we may write

$$FL = 100(0.5) + 1.5L\left(\frac{L}{2}\right)$$

or

$$F = \frac{50}{L} + 0.75L$$

Differentiating,

$$\frac{dF}{dL} = -\frac{50}{L^2} + 0.75$$

Equating the derivative to zero,

$$0 = -\frac{50}{L^2} + 0.75$$

or $$L = 8.16 \text{ ft}$$

and $$F = \frac{50}{8.16} + 0.75(8.16) = 12.25 \text{ lb}$$

EXAMPLE 28. Find what values of x will make the following function a maximum or minimum.

$$y = x^3 - 3x^2 - 9x + 10$$

Differentiating, we obtain

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

Any maximum or minimum values must occur when $dy/dx = 0$. Accordingly, setting $dy/dx = 0$, we obtain

$$0 = 3x^2 - 6x - 9$$

or $$0 = x^2 - 2x - 3$$

whereby $$x = 3, -1$$

In order to explore the properties of the curve at $x = 3$ and $x = -1$ we shall compute the values of the slope at various points:

x	-2	-1	0	$+3$	$+4$
$\frac{dy}{dx}$	$+15$	0	-9	0	$+15$

In this example let us draw possible tangents whose slopes are in the order positive, zero, negative, zero, and positive. We shall then roughly sketch a reasonable curve to fit these tangents. This is done in Fig. 26·2. Thus we have indicated a maximum at $x = -1$ and a minimum at $x = 3$.

We might confirm this conclusion by substituting values of x in the original equation, noting how y varies in the vicinity of $x = 3$ and $x = -1$. This is done as follows:

x	-3	-2	-1	0	1	2	3	4	5
y	-17	$+8$	$+15$	$+10$	-1	-12	-17	-10	$+15$

At $x = 3$, where $y = -17$, we have a minimum, this point being the lowest point in the immediate vicinity. Likewise at $x = -1$, $y = 15$, we have a maximum point, this point being the highest point in the immediate vicinity.

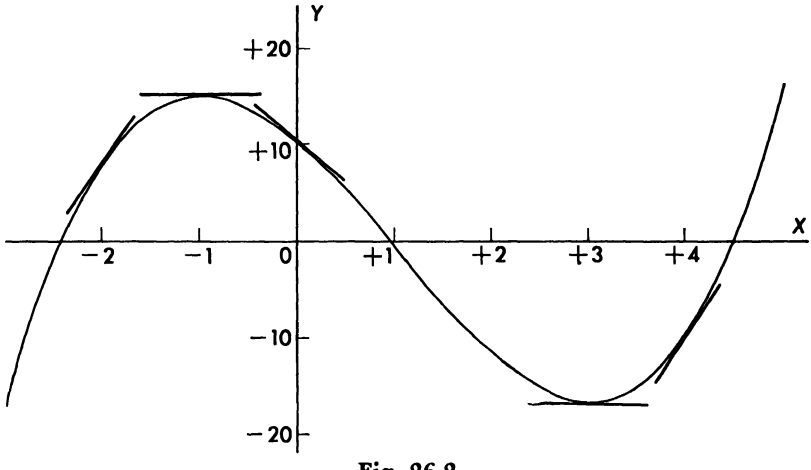


Fig. 26·2

The choice of using the primary function or the derivative in identifying maximum and minimum points depends entirely on convenience. In Sec. 26·21 a third method of identifying maximum and minimum points is suggested. This method involves the use of the "flexion" of the graph.

EXAMPLE 29. Given $y = x^3 - 6x^2 + 12x - 20$.

Let us investigate any horizontal slopes as we did in the preceding example. In this example we obtain

$$\frac{dy}{dx} = 3x^2 - 12x + 12$$

Equating to zero and solving, we obtain

$$x = 2, 2 \qquad \text{(thus the two roots are identical)}$$

If we calculate values of the slope and tabulate, we have

x	+4	+3	+2	+1	0
$\frac{dy}{dx} = 3x^2 - 12x + 12$	+12	+3	0	+3	+12

Graphically, this is shown in Fig. 26·3, and we have neither a maximum nor a minimum but a point of inflection or reversal of curvature (see Sec. 25·14).

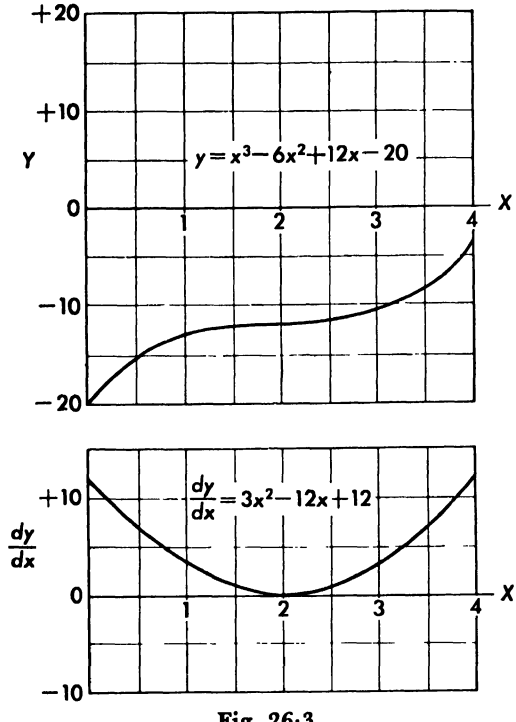


Fig. 26·3

EXAMPLE 30. Locate and identify critical points in the function

$$y = \sqrt{3x^3 - 16x + 20}$$

Change to exponential form, and

$$y = (3x^3 - 16x + 20)^{1/2} \tag{54}$$

Then

$$\frac{dy}{dx} = \tfrac{1}{2}(3x^3 - 16x + 20)^{-1/2}(9x^2 - 16) \tag{55}$$

or

$$\frac{dy}{dx} = \frac{9x^2 - 16}{2\sqrt{3x^3 - 16x + 20}} \tag{56}$$

Eq. (56) can be zero only if the numerator on the right-hand side of the equation is zero. Therefore let us solve the equation:

$$9x^2 - 16 = 0 \tag{57}$$

Solving (57), we obtain

$$x^2 = 16/9$$
$$x = \pm 4/3$$

Therefore when x is either $+4/3$ or $-4/3$ the slope of Eq. (54) is zero and this function may have a maximum or a minimum at these points.

First let us investigate to see if the quantity $3x^3 - 16x + 20$ is real when $x = \pm\frac{4}{3}$.

When $x = +\frac{4}{3}$,

$$\sqrt{3x^3 - 16x + 20} = \sqrt{3 \times 6\tfrac{4}{27} - 16 \times \tfrac{4}{3} + 20} = \sqrt{6\tfrac{4}{9} - 6\tfrac{4}{3} + 20}$$
$$= \sqrt{6\tfrac{4}{9} - 19\tfrac{2}{9} + 18\tfrac{0}{9}} = \sqrt{24\tfrac{4}{9} - 19\tfrac{2}{9}} = \sqrt{5\tfrac{2}{9}}$$

When $x = -\frac{4}{3}$,

$$\sqrt{3x^3 - 16x + 20} = \sqrt{-3(6\tfrac{4}{27}) + 16(\tfrac{4}{3}) + 20}$$
$$= \sqrt{-6\tfrac{4}{9} + 19\tfrac{2}{9} + 18\tfrac{0}{9}} = \sqrt{30\tfrac{8}{9}}$$

Thus the denominator of (56) is real and is positive when $x = +\frac{4}{3}$ or $-\frac{4}{3}$. Consequently the sign of (56) is determined entirely by the sign of the numerator in the vicinity of the critical value of x. The numerator may be written

$$(3x + 4)(3x - 4) \tag{58}$$

When x is in the vicinity of $+\frac{4}{3}$, the left-hand factor of expression (58) is positive and the sign of the numerator of Eq. (56) will be governed by the sign of the quantity $3x - 4$.

When x is a little less than $+\frac{4}{3}$, the expression $3x - 4$ is negative. When x is a little more than $+\frac{4}{3}$, the expression $3x - 4$ is positive.

Thus, as x traverses through this critical point in an increasing direction, the signs of the derivative are in the sequence

$$\text{Negative} \rightarrow \text{zero} \rightarrow \text{positive}$$

and this critical point is a *minimum*.

The student should justify for himself that when $x = -\frac{4}{3}$, the given function is a maximum.

EXERCISE 8

Locate and identify any points where the tangent is horizontal.

1. $y = 2x^3 + 9x^2 - 24x + 12$

2. $y = -x^4 + 24x^2 + 2$

3. $y = x + \dfrac{9}{x}$

***4.** $y = \dfrac{x^2 + 4}{x}$

***5.** $y = \dfrac{x^3 - 16}{x}$

6. $y = \sqrt{3x^3 - 4x + 11}$

7. $y = \dfrac{5}{2x^4 - 9x^2}$

8. Solve Prob. 7 in Exercise 6, page 603, by algebraic methods.

9. Solve Prob. 9 in Exercise 6, page 604, by algebraic methods.

*Divide out before differentiating.

10. Solve Prob. 10 in Exercise 6, page 604, by algebraic methods.

11. A rectangular open-top box is to be made from a sheet of copper 8 by 15 in. by cutting a square from each corner and turning up the sides. What size squares must be cut out to make the capacity of the box a maximum?

12. What are the dimensions of a covered gallon cylindrical container which will make its total surface a minimum?

13. Three towns are to be served by a high-tension line which takes the form of a letter Y 16 miles long and 12 miles across the top. Find the length of the "stem" of the Y which will make the total length of wire a minimum.

14. Two lights whose intensities are as 1:4 are 100 ft apart. How far from the weaker light is the point of minimum total illumination? (Intensity of illumination at any point varies as i/l^2, where i is the intensity of the light source, and l is the distance from the point to the light source.)

15. A man is able to hire one dump truck at \$4 per hour for truck and driver. The truck takes half an hour to deliver a load of sand and return. It takes 4 man-hours to load the truck. Laborers get \$1 an hour whether they are loading the truck or standing around idle while waiting for the truck to return. How many laborers should be hired to realize the minimum cost per load? What is the minimum cost?

16. It is desired to determine the optimum number of checks on the quality of a finished product. It is assumed that the cost of checking one unit is proportional to the number of checks; that is, $y_1 = kn$. It is further assumed that the cost of errors and defective units becomes progressively less with more frequent checks but can never be eliminated entirely. It has been found that the formula $y_2 = b/(n + a)$ applies, where a and b are constants. To summarize, $E = kn + b/(n + a)$,

where E = total cost of checks and errors—to be made a minimum
k = constant = \$0.13
n = number of checks
b = constant = \$1.36
a = constant = 0.32

Find the number of checks for a minimum total cost E. What is the minimum E?

17. It has been determined that a painter will paint 100 sq ft of surface per hour at the beginning of a day and that his rate decreases by 10 sq ft per hr as the day progresses. Allowing 0.4 hr for work preparations and another 0.4 hr to clean up at the

end of the day, what total time, including preparation and cleanup, will result in a maximum over-all square feet per hour?

18. A utility company wishes to determine the size wire to be used on a power transmission line for maximum economy. Initial cost and investment charges may be expressed by the equation $s_1 = ka + b$ where s_1 is the cost, a is the cross-section area of the wire, and k and b are constants. The energy loss may be written $s_2 = c/a$, where s_2 is the cost of lost power and c is a constant. If $k = 0.18$, $b = 15{,}000$, and $c = 40$, find the optimum cross-section area. (This is total area and may be distributed among n wires.)

19. It has been determined that the cutting rate of a tool will diminish linearly to 98 per cent of its original value on 1 hr of use. Loss of 0.2 hr operating time occurs each time the tool is sharpened, and 100 hr total time is available for the job. How often during this interval should the tool be sharpened for maximum output?

20. A rectangular package has rectangular ends. If the sum of the girth and length may not exceed 80 in., what dimensions will correspond to the maximum volume?

21. The hourly cost of operating a ship, in dollars per hour, is $c = 120 + 0.02v^3$, where v equals speed, in miles per hour. Find the speed corresponding to a minimum cost per mile.

26·8 The Derivative of Sin ϕ and Cos ϕ. Let us consider the function

$$y = \sin \phi \tag{59}$$

Referring to Eq. (57), page 572, we may write

$$y = \sin \phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \cdots \tag{60}$$

Differentiating both sides of Eq. (60), we obtain

$$\frac{dy}{d\phi} = 1 - \frac{3\phi^2}{1 \times 2 \times 3} + \frac{5\phi^4}{1 \times 2 \times 3 \times 4 \times 5} - \frac{7\phi^6}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} + \cdots \tag{61}$$

$$= 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \cdots \tag{62}$$

By reference to Eq. (56), page 572, we recognize Eq. (62) as being equal to cos ϕ. Therefore we may conclude that if

$$y = \sin \phi \tag{63}$$

then

$$\frac{dy}{d\phi} = \cos \phi \qquad * (64)$$

Following rules 2 and 4 and Eqs. (63) and (64), if

$$y = k \sin \phi + C \qquad (65)$$

then

$$\frac{dy}{d\phi} = k \cos \phi \qquad * (66)$$

By differentiating both sides of Eq. (56), page 572, we discover that if

$$y = \cos \phi \qquad (67)$$

then

$$\frac{dy}{d\phi} = -\sin \phi \qquad * (68)$$

Following rules 2 and 4 and Eqs. (67) and (68), if

$$y = k \cos \phi + C \qquad (69)$$

then

$$\frac{dy}{d\phi} = -k \sin \phi \qquad * (70)$$

26·9 The Derivative of Sin u Where u Is a Function of θ. Let

$$y = \sin (\theta^2 + 3\theta + 5) \qquad (71)$$

and let

$$u = \theta^2 + 3\theta + 5 \qquad (72)$$

Then

$$y = \sin u \qquad (73)$$

and we can find dy/du (not $dy/d\theta$ as yet):

$$\frac{dy}{du} = \cos u \qquad \text{[see Eq. (64)]} \qquad (74)$$

Also from Eqs. (18), (19), and (72),

$$\frac{du}{d\theta} = 2\theta + 3 \qquad (75)$$

Substituting Eqs. (74) and (75) in (44),

$$\frac{dy}{d\theta} = (\cos u)(2\theta + 3) \qquad (76)$$

Or substituting Eq. (72) in Eq. (76) gives

$$\frac{dy}{d\theta} = [\cos(\theta^2 + 3\theta + 5)](2\theta + 3) \tag{77}$$

In general, if

$$y = \sin u \tag{78}$$

and if u is a function of θ, then

$$\frac{dy}{d\theta} = \cos u \frac{du}{d\theta} \quad * \tag{79}$$

6. *The derivative of the sine of a function is the cosine of the function times the derivative of the function.*

Following rules 2, 4, and 6, if

$$y = k \sin u + C \tag{80}$$

and if u is a function of θ, then

$$\frac{dy}{d\theta} = k \cos u \frac{du}{d\theta} \quad * \tag{81}$$

EXAMPLE 31. If $y = \sin(4\pi\theta - \pi/2)$, find $dy/d\theta$.
Following Eqs. (78) and (79),

$$\frac{dy}{d\theta} = \left[\cos\left(4\pi\theta - \frac{\pi}{2}\right)\right] 4\pi$$

EXAMPLE 32. If $y = \sin \theta^{(t-1)/t}$, find $dy/d\theta$ (t is a constant).
The problem can be simplified somewhat by restating as

$$y = \sin \theta^{1-1/k}$$

Following Eqs. (78), (79), (18), and (19),

$$\frac{dy}{d\theta} = (\cos \theta^{1-1/k}) \left(1 - \frac{1}{t}\right) \theta^{-1/k}$$

EXAMPLE 33. If $y = \sin^2 4\theta$, find $dy/d\theta$.

$$y = (\sin 4\theta)^2$$

Following Eqs. (18), (19), (78), and (79),

$$\frac{dy}{d\theta} = 2[(\sin 4\theta)(\cos 4\theta)]\, 4$$

Following Eq. (9), page 505, we may write

$$\frac{dy}{d\theta} = 8 \sin 4\theta \cos 4\theta = 4 \sin 8\theta$$

EXAMPLE 34. If $y = \sin (4\theta)^2$, find $dy/d\theta$.
Restate the problem to make the expression for the angle match Eq. (18).

$$y = \sin 16\theta^2$$
$$\frac{dy}{d\theta} = (\cos 16\theta^2)32\theta$$

EXAMPLE 35. If $y = -\sin (4\pi\theta^2 - \pi/2)$, find $dy/d\theta$.
Following Eqs. (78) and (79),

$$\frac{dy}{d\theta} = -\cos\left[\left(4\pi\theta^2 - \frac{\pi}{2}\right)\right] 8\pi\theta = -8\pi\theta \sin 4\pi\theta^2$$

26·10 The Derivative of Cos u Where u Is a Function of θ. Let

$$y = \cos (\theta^2 + 3\theta + 5) \tag{82}$$

and let

$$u = \theta^2 + 3\theta + 5 \tag{83}$$

Then

$$y = \cos u \tag{84}$$

and we can find dy/du (not $dy/d\theta$ as yet):

$$\frac{dy}{du} = -\sin u \qquad \text{[see Eqs. (67) and (68)]} \tag{85}$$

Also from Eq. (83) and (19)

$$\frac{du}{d\theta} = 2\theta + 3 \tag{86}$$

Substituting Eqs. (85) and (86) in Eq. (44),

$$\frac{dy}{d\theta} = -(\sin u)(2\theta + 3) \tag{87}$$

Or substituting Eq. (83) in Eq. (87) gives

$$\frac{dy}{d\theta} = [-\sin (\theta^2 + 3\theta + 5)](2\theta + 3) \tag{88}$$

In general, if

$$y = \cos u \tag{89}$$

and if u is a function of θ, then

$$\frac{dy}{d\theta} = -\sin u \frac{du}{d\theta} \qquad * (90)$$

7. *The derivative of the cosine of a function is minus the sine of the function times the derivative of the function.*

Following rules 2, 4, and 7, if

$$y = k \cos u + C \qquad (91)$$

and if u is a function of θ,

$$\frac{dy}{d\theta} = -k \sin u \frac{du}{d\theta} \qquad * (92)$$

EXAMPLE 36. If $y = \cos(4\pi\theta - \pi/2)$ find $dy/d\theta$.
Following Eqs. (89) and (90),

$$\frac{dy}{d\theta} = \left[-\sin\left(4\pi\theta - \frac{\pi}{2}\right)\right] 4\pi$$

EXAMPLE 37. If $y = \cos \theta^{(t-1)/t}$ find $dy/d\theta$.
The problem can be simplified somewhat by restating, as

$$y = \cos \theta^{1-1/t}$$

Following Eqs. (89), (90), (18), and (19),

$$\frac{dy}{d\theta} = (-\sin \theta^{1-1/t})\left(1 - \frac{1}{t}\right)\theta^{-1/t}$$

EXAMPLE 38. If $y = \cos^2 4\theta$, find $dy/d\theta$.

$$y = (\cos 4\theta)^2$$

Following Eqs. (18), (19), (89), and (90),

$$\frac{dy}{d\theta} = (2 \cos 4\theta)(-\sin 4\theta)4 = -8 \cos 4\theta \sin 4\theta = -4 \sin 8\theta$$

[Eq. (9), page 505]

EXAMPLE 39. If $y = \cos (4\theta)^2$, find $dy/d\theta$.
To make the expression for the angle match Eq. (18), we can restate the problem as

$$y = \cos 16\theta^2$$

Then

$$\frac{dy}{d\theta} = (-\sin 16\theta^2)32\theta$$

EXERCISE 9

Differentiate the following:

1. $y = 3 \cos \theta^4$

2. $y = 8 \cos (2x + 3)$

3. $y = 5 \cos \dfrac{2}{x}$

4. $y = 2 \cos \sqrt{x}$

5. $w = -9 \cos \theta^{3/4}$

6. $y = 5 \cos \theta^{1-t}$ (t is a constant)

7. $y = \pi \cos \sqrt{1 - t}$

8. $y = 10 \cos \left(\omega t + \dfrac{\pi}{2}\right)$ where ω = a constant

9. $z = \cos \left(x - \dfrac{1}{x}\right)$

10. $y = \cos \sqrt{1 - x}$

EXERCISE 10

Differentiate the following:

1. $y = \sin x^4$

2. $z = \sin (2x + 3)$

3. $y = \sin \left(\dfrac{2}{x}\right)$

4. $y = \sin \sqrt{x}$

5. $w = -6 \sin x^{3/4}$

6. $y = \sin \theta^{1-t}$ where t = a constant

7. Given $y = 6 \sin (\pi t/2)$, where the angle is in radians, find the smallest positive value of t for which $dy/dt = 0$.

8. The linear speed of a point on the tread of an automobile tire is given by the equation $v = 32\pi \sin (8\pi t)$, where v is in feet per second and t is in seconds (see Fig. 25·32 and Exercise 11, page 613).

a. What significance if any can be attached to the above equation when $\frac{1}{4} > t > \frac{1}{8}$?

b. What time is required for one revolution?

9. From the equation in Prob. 8 find the following:

a. The average acceleration from $t = 0.02$ to $t = 0.03$. (Acceleration is the rate of change in velocity.)

b. The instantaneous acceleration at $t = 0.03$.

c. The maximum speed of the point.

d. The speed of the car.

e. The outside diameter of the tire.

f. When the acceleration is zero.

26·11 Differentiating the Function $y = e^x$.

$$y = e^x \tag{93}$$

From Eq. (2), page 328,

$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \tag{94}$$

Differentiating (94) in accordance with Eqs. (18) and (19) gives

$$\frac{dy}{dx} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \tag{95}$$

Observe that Eqs. (94) and (95) are identical. Therefore, if

$$y = e^x \tag{96}$$

then

$$\frac{dy}{dx} = e^x \qquad \star \tag{97}$$

Observe that in this case, the function and its derivative are identical.

Following rules 2 and 4 and Eqs. (96) and (97), if

$$y = ke^x + C \tag{98}$$

then

$$\frac{dy}{dx} = ke^x \qquad \star \tag{99}$$

26·12 The Derivative of e^u Where u Is a Function of x. Let

$$y = e^{x^2+3x+5} \tag{100}$$

and let

$$u = x^2 + 3x + 5 \tag{101}$$

Then

$$y = e^u \tag{102}$$

and we can find dy/du (not dy/dx as yet):

$$\frac{dy}{du} = e^u \qquad \text{[see Eq. (97)]} \qquad \star \tag{103}$$

Also from Eqs. (101) and (19)

$$\frac{du}{dx} = 2x + 3 \tag{104}$$

Therefore, substituting Eqs. (103) and (104) in Eq. (44)

$$\frac{dy}{dx} = e^u(2x + 3) \tag{105}$$

Or substituting Eq. (101) in Eq. (105),

$$\frac{dy}{dx} = e^{x^2+3x+5}(2x + 3) \tag{106}$$

In general, if

$$y = e^u \tag{107}$$

and if u is a function of x, then

$$\frac{dy}{dx} = e^u \frac{du}{dx} \quad \star \tag{108}$$

8. *The derivative of* e *to a power is* e *to that same power times the derivative of the exponent.*

Following rules 2, 4, and 8, if

$$y = ke^u + C \tag{109}$$

and if u is a function of x, then

$$\frac{dy}{dx} = ke^u \frac{du}{dx} \quad \star \tag{110}$$

Observe that if $C = 0$, we may substitute the value of y from Eq. (109) in Eq. (110), obtaining

$$\frac{dy}{dx} = y \frac{du}{dx} \tag{111}$$

EXAMPLE 40. If $y = 3e^{1/x}$, find dy/dx.
According to Eqs. (109) and (110),

$$\frac{dy}{dx} = 3e^{1/x}\left(-\frac{1}{x^2}\right) = -\frac{3e^{1/x}}{x^2}$$

EXAMPLE 41. If $y = (3e^{\sqrt{x}})^2$, find dy/dx.
Squaring, as indicated, we obtain

$$y = 9e^{2\sqrt{x}}$$

Hence

$$\frac{dy}{dx} = 9e^{2\sqrt{x}}(2)\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{x}}\right) = \frac{9e^{2\sqrt{x}}}{\sqrt{x}}$$

EXAMPLE 42. Find the derivative of $y = A_1e^{k_1x} + A_2e^{k_2x}$
Following Eqs. (109) and (110),

$$\frac{dy}{dx} = A_1k_1e^{k_1x} + A_2k_2e^{k_2x}$$

EXAMPLE 43. Find the derivative of $z = A_1k_1e^{k_1x} + A_2k_2e^{k_2x}$
Following Eqs. (109) and (110),

$$\frac{dz}{dx} = A_1k_1^2e^{k_1x} + A_2k_2^2e^{k_2x}$$

EXERCISE 11

Differentiate:

1. $y = e^{5x^2}$
2. $y = e^{\sqrt{x}}$
3. $y = e^{x+8}$
4. $y = e^{(2-x)/x}$
5. $y = e^{\sqrt{1-x}}$
6. $y = \sin e^t$

7. The graph shown in Fig. 25·30 was plotted from the equation $T = 50e^{-0.2t}$ where T is the temperature difference between a warm body and its surroundings and t is time in minutes. Using the above equation rather than the graph, solve Probs. 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 21, 23, 24 in Exercise 9, page 610.
8. From Prob. 7 write the equation for dT/dt in terms of T. If dT/dt were plotted against T, what is the shape of the plot?
9. Sketch the graph of $i = \dfrac{E}{R}(1 - e^{-Rt/L})$

   where $E = 150$ volts
   $R = 15$ ohms
   $L = 0.75$ henry
   $i =$ amperes
   $t =$ seconds

   This is an equation giving the relation between current and time in a certain series circuit containing inductance and resistance. The inductance is 0.75 henry, the resistance is 15 ohms. A battery of 150 volts is connected across this series combination when $t = 0$. Thereafter the current rises in accordance with the above equation. Compare your sketch with Fig. 25·33.
   a. Find di/dt when $t = 0$.
   b. Find the "maximum" value of current. When will it be reached?
   c. If the current increased at the rate it was increasing when $t = 0$, how long would it take for the current to reach its "maximum" value?
   d. Compare the value found in part *c* with the ratio L/R where L is inductance and R is resistance. This ratio is called the "time constant of the circuit."
   e. Find the rate of change of current with respect to time at frequent points, between $t = 0$ and $t = 150$ msec. Plot a graph of di/dt versus i between $t = 0$ and $t = 150$ msec.

26·13 The Derivative of ln x. Let

$$y = \ln x \tag{112}$$

By definition of logarithm (see Sec. 14·1, page 261)

$$x = e^y \tag{113}$$

Following rule 8,

$$\frac{dx}{dy} = e^y \tag{114}$$

or

$$\frac{dy}{dx} = \frac{1}{e^y} \tag{115}$$

But from Eq. (113)

$$e^y = x \tag{116}$$

therefore

$$\frac{dy}{dx} = \frac{1}{x} \tag{117}$$

In general, if

$$y = \ln x \tag{118}$$

then

$$\frac{dy}{dx} = \frac{1}{x} \tag{119}$$

Following rules 2, 4, and Eq. (119), if

$$y = k \ln x + C \tag{120}$$

then

$$\frac{dy}{dx} = \frac{k}{x} \tag{121}$$ *

If we wish to find dy/dx when

$$y = k \log x \tag{122}$$

where $\log x$ means log to the base 10 of x, we may change to natural logarithms by using Eq. (7), page 333, and

$$y = k \log x = 0.4343k \ln x \tag{123}$$

Therefore, if

$$y = k \log x \tag{124}$$

then

$$\frac{dy}{dx} = \frac{0.4343k}{x} \tag{125}$$

EXAMPLE 44. Differentiate the function $y = \ln x^6$.

Following Eq. (4), page 265,

$$y = 6 \ln x$$

and

$$\frac{dy}{dx} = \frac{6}{x}$$

EXAMPLE 45. Differentiate the function $y = \log x^6$.
Following Eq. (123)

$$y = 0.4343 \ln x^6$$

Following Eqs. (4), page 265,

$$y = 0.4343(6) \ln x = 2.6058 \ln x$$

therefore $$\frac{dy}{dx} = \frac{2.6058}{x}$$

EXERCISE 12

Differentiate the following:

1. $y = \ln x$ **2.** $y = 3 \log x$
3. $y = \ln x^5$ **4.** $y = \ln \theta^6 - 3 \sin \theta + e^\theta$

5. $y = \ln 6x - \ln (5/x^2)$
6. $y = 2 \ln x - 3 \ln x + \ln x - \frac{1}{5} \sin x + 10e^x$
7. $y = 15 \log x^{10}$

26·14 The Derivative of ln u Where u is a Function of x. Let

$$y = \ln (x^2 + 3x + 5) \tag{126}$$

We wish to find dy/dx. Let

$$u = x^2 + 3x + 5 \tag{127}$$

Then $$y = \ln u \tag{128}$$

and we can find dy/du (not dy/dx as yet):

$$\frac{dy}{du} = \frac{1}{u} \qquad \text{[see Eq. (119)]} \tag{129}$$

Also from Eq. (19)

$$\frac{du}{dx} = 2x + 3 \tag{130}$$

Therefore, substituting Eqs. (129) and (130) in Eq. (44)

$$\frac{dy}{dx} = \frac{1}{u}(2x + 3) \tag{131}$$

Or substituting Eq. (127) in Eq. (131),

$$\frac{dy}{dx} = \frac{1}{x^2 + 3x + 5}(2x + 3) \tag{132}$$

In general, if

$$y = \ln u \tag{133}$$

and if u is a function of x, then

$$\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} \tag{134}$$

9. *The derivative of the natural logarithm of a function is one over that function times the derivative of the function.*

Following rules 2, 4, and 8, if

$$y = k \ln u + C \tag{135}$$

and if u is a function of x, then

$$\frac{dy}{dx} = \frac{k}{u}\frac{du}{dx} \tag{136}$$

EXAMPLE 46. If $y = 5 \ln (x^2 - 7)$, following Eqs. (136) and (19)

$$\frac{dy}{dx} = \frac{5}{x^2 - 7} 2x = \frac{10x}{x^2 - 7}$$

EXAMPLE 47. If $y = \ln [1/(1 - x^2)]$ find dy/dx.
First we may rearrange the expression $\ln [1/(1 - x^2)]$, obtaining

$$\ln 1 - \ln (1 - x^2) = 0 - \ln (1 - x^2) = - \ln (1 - x^2)$$

Then, if $y = - \ln (1 - x^2)$,

$$\frac{dy}{dx} = - \frac{1}{1 - x^2} (-2x) = \frac{2x}{1 - x^2}$$

EXAMPLE 48. If $y = \log_{10} [(4 - x^2)/(6 - 2x)]$, find dy/dx.
We may convert to the base e by using Eq. (7), page 333.

$$y = 0.4343[\ln (4 - x^2) - \ln (6 - 2x)]$$

Then

$$\frac{dy}{dx} = 0.4343 \left[\frac{1}{4 - x^2} (-2x) - \frac{1}{6 - 2x} (-2)\right] = 0.4343 \left(\frac{2x}{x^2 - 4} + \frac{2}{6 - 2x}\right)$$

EXAMPLE 49. If $y = \ln^2 (25 - x^2)$, find dy/dx.

$$y = [\ln (25 - x^2)]^2$$

Following Eqs. (18), (19), (133), and (134),

$$\frac{dy}{dx} = 2[\ln (25 - x^2)] \frac{1}{25 - x^2} (-2x) = \frac{4x \ln (25 - x^2)}{x^2 - 25}$$

EXERCISE 13

Differentiate:

1. $y = \ln (x^2 + 2)$

2. $y = \ln \sqrt{x^2 - 1}$

3. $y = \ln (x^2 + 3)^4$

4. $y = \ln \left(\frac{1}{x}\right)$

5. $z = \ln \sqrt[3]{4 - 3x}$

6. $z = \ln \frac{1}{4 - y}$

7. $y = \log (4 - x^2)$

8. $y = \log (4 - x)^2$

9. $y = \log \frac{1}{\sqrt{9 - x^2}}$

10. $z = \log \frac{2x + 5}{x - 3}$

11. $y = \log \frac{4x^2 - 9}{2x + 3}$

12. $y = \ln \left(x - \frac{1}{x}\right)$

13. $z = \ln^2 (x^3 - 8)$

14. $y = \ln^4 \sqrt{9 - x^2}$

15. $y = \log \sin (\theta + \pi)$

16. $y = \log \cos \left(2\theta - \frac{\pi}{2}\right)$

26·15 The Derivative of ka^u Where a Is Any Positive Constant and u Is a Function of x. Let

$$y = 7a^{x^2+3x+5} \tag{137}$$

If a is a positive constant, it is some power of e. Let

$$a = e^b \tag{138}$$

Then
$$y = 7(e^b)^{(x^2+3x+5)} \tag{139}$$

or
$$y = 7e^{bx^2+3bx+5b} \tag{140}$$

We wish to find dy/dx. As in Sec. 26·12, let

$$u = bx^2 + 3bx + 5b \tag{141}$$

Then
$$y = 7e^u \tag{142}$$

and we can find dy/du (not dy/dx as yet):

$$\frac{dy}{du} = 7e^u \qquad \text{[see Eq. (99)]} \tag{143}$$

Also from Eq. (141)
$$\frac{du}{dx} = b(2x + 3) \tag{144}$$

Therefore, substituting Eqs. (143) and (144) in Eq. (44),

$$\frac{dy}{dx} = 7e^u b(2x + 3) \tag{145}$$

Or, substituting Eq. (141) in Eq. (145),

$$\frac{dy}{dx} = 7be^{b(x^2+3x+5)}(2x + 3) \tag{146}$$

Substituting Eq. (138) in Eq. (146),

$$\frac{dy}{dx} = 7ba^{x^2+3x+5}(2x + 3) \tag{147}$$

Applying the basic definition of a logarithm to Eq. (138) yields

$$b = \ln a \qquad \text{[see Eq. (1), page 261]} \tag{148}$$

Therefore, Eq. (147) reduces to

$$\frac{dy}{dx} = 7a^{x^2+3x+5}(2x + 3) \ln a \tag{149}$$

In general, if

$$y = ka^u \tag{150}$$

where a is a positive constant and u is a function of x, then

$$\frac{dy}{dx} = ka^u \frac{du}{dx} \ln a \qquad \star \tag{151}$$

10. *The derivative of any constant* a *to a variable power is* a *to the same power times the derivative of the exponent times the natural logarithm of* a.

EXAMPLE 50. If $y = (3^{x^2+2x})/4$, find dy/dx.
According to Eq. (151),

$$\frac{dy}{dx} = \frac{1}{4} 3^{x^2+2x}(2x + 2) \ln 3$$

$$= \frac{x+1}{2} (3)^{x^2+2x} \ln 3$$

EXAMPLE 51. If $y = 3 \ln^x 4$ this equation can be rewritten as

$$y = 3 (\ln 4)^x$$

Now following Eq. (151),

$$\frac{dy}{dx} = (3 \ln^x 4)(1) [\ln (\ln 4)]$$

EXERCISE 14

Differentiate the following:

1. $y = a^x$ where a is constant

2. $y = a^{1/2}$ where a is constant

3. $z = 2\sqrt{z}$

4. $y = 5^{e(x+1)}$ where e is constant

26·16 General Proof for $dy/dx = knx^{n-1}$. In Sec. 2·4 we developed a formula for the derivative of a power function in the form

$$y = kx^n + C \tag{152}$$

where n is a positive integer.

In Sec. (26·3) we mentioned that in the present section we would prove the validity of Eq. (19) where n is either positive or negative, integral or fractional, rational or irrational. This we shall now proceed to do.

Since the derivative of the added constant is zero and since a multiplied constant multiplies the derivative (see rules 2 and 4), we shall concede these facts and develop an equation for the derivative of

$$y = x^n \tag{153}$$

From Eq. (4), page 265,

$$\ln y = n \ln x \tag{154}$$

regardless of whether n is positive or negative, fractional or integral, rational or irrational. Following rule 9, Sec. (26·14), we shall differentiate both sides of Eq. (154), obtaining

$$\frac{1}{y}\frac{dy}{dx} = \frac{n}{x} \tag{155}$$

or

$$\frac{dy}{dx} = \frac{n}{x}y \tag{156}$$

But substituting Eq. (153) in Eq. (156) gives

$$\frac{dy}{dx} = \frac{n}{x}x^n \tag{157}$$

or

$$\frac{dy}{dx} = nx^{n-1} \tag{158}$$

Following rules 2 and 4 and Eqs. (157) and (158), if

$$y = kx^n + C \tag{159}$$

then

$$\frac{dy}{dx} = nkx^{n-1} \tag{160}$$

26·17 The Derivative of a Product and a Quotient. Suppose we wish to find the derivative of the function

$$y = (x^2 + 2x + 3)(5e^{3x}) \tag{161}$$

Here y is the product of two different functions of x.

Let

$$u = x^2 + 2x + 3 \tag{162}$$

and

$$v = 5e^{3x} \tag{163}$$

Then

$$y = uv \tag{164}$$

Now if x increases by some amount Δx then u will change by some amount Δu, and v will change by some amount Δv. Also y will change by another amount Δy; then

$$y + \Delta y = (u + \Delta u)(v + \Delta v) \tag{165}$$

or, multiplying the factors on the right together,

$$y + \Delta y = uv + v\,\Delta u + u\,\Delta v + \Delta u\,\Delta v \tag{166}$$

Subtracting (164) from (166),

$$\Delta y = v\,\Delta u + u\,\Delta v + \Delta u\,\Delta v \tag{167}$$

Dividing both sides by Δx and reversing the positions of the first two terms on the right,

$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}\Delta v \tag{168}$$

As Δx approaches zero, the quantity $\Delta u/\Delta x$ may very well approach some finite value. But as Δx approaches zero, Δv also approaches zero; therefore the product of $\Delta u/\Delta x$ and Δv will approach zero. Consequently, when we approach the limits

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \tag{169}$$ ★

11. *The derivative of a product is the first factor multiplied by the derivative of the second factor plus the second factor multiplied by the derivative of the first factor.*

In the present example

$$u = x^2 + 2x + 3 \tag{170}$$

$$\frac{du}{dx} = 2x + 2 \tag{171}$$

$$v = 5e^{3x} \tag{172}$$

$$\frac{dv}{dx} = 15e^{3x} \tag{173}$$

Substituting (170), (171), (172), and (173) in (169) gives

$$\frac{dy}{dx} = (x^2 + 2x + 3)(15e^{3x}) + 5e^{3x}(2x + 2) \tag{174}$$
$$= 5e^{3x}(3x^2 + 8x + 11)$$

Now let us consider the case of a quotient. Let

$$y = \frac{x^2 + 2x + 3}{5e^{3x}} \tag{175}$$

and we wish to find dy/dx.

Let

$$u = x^2 + 2x + 3 \tag{176}$$

and

$$v = 5e^{3x} \tag{177}$$

Then

$$y = \frac{u}{v} \tag{178}$$

Now let x increase by the amount Δx, and as a consequence u will change by the amount Δu, v will change by the amount Δv, and y will change by the amount Δy.

Accordingly

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v} \tag{179}$$

Subtracting (178) from (179) gives

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} \tag{180}$$

or, by combining the right-hand side of (180),

$$\Delta y = \frac{uv + v\,\Delta u - uv - u\,\Delta v}{v(v + \Delta v)} \tag{181}$$

or

$$\Delta y = \frac{v\,\Delta u - u\,\Delta v}{v(v + \Delta v)} \tag{182}$$

Dividing both sides of (182) by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{v(\Delta u/\Delta x) - u(\Delta v/\Delta x)}{v(v + \Delta v)} \tag{183}$$

Now as Δx approaches zero, the denominator of (183) approaches v^2, and in the limit

$$\frac{dy}{dx} = \frac{v(du/dx) - u(dv/dx)}{v^2} \tag{184}$$ ⋆

12. *The derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator divided by the denominator squared.*

In Eq. (175), then,

$$\frac{dy}{dx} = \frac{5e^{3x}(2x + 2) - (x^2 + 2x + 3)15e^{3x}}{25e^{6x}} = -\frac{3x^2 + 4x + 7}{5e^{3x}}$$

EXAMPLE 52. If $y = (4x^2 - 5x) \ln 3x^2$, find dy/dx.

Referring to Eq. (169),

$$\frac{dy}{dx} = (4x^2 - 5x)\frac{1}{3x^2}6x + (8x - 5) \ln 3x^2 = 8x - 10 + (8x - 5) \ln 3x^2$$

EXAMPLE 53. If $y = (e^{2x} - e^{-2x})/(e^{2x} + e^{-2x})$, find dy/dx.

If, according to Eq. (184), we represent $e^{2x} - e^{-2x}$ by u and $e^{2x} + e^{-2x}$ by v, we obtain

$$\frac{dy}{dx} = \frac{(e^{2x} + e^{-2x})(2e^{2x} + 2e^{-2x}) - (e^{2x} - e^{-2x})(2e^{2x} - 2e^{-2x})}{(e^{2x} + e^{-2x})^2}$$
$$= \frac{(2e^{4x} + 4 + 2e^{-4x}) - (2e^{4x} - 4 + 2e^{-4x})}{(e^{2x} + e^{-2x})^2} = \frac{8}{(e^{2x} + e^{-2x})^2}$$

EXAMPLE 54. If $y = \sqrt{(3t^2 + 2)}/(t^2 - t)$, find dy/dt.

If we first write

$$y = \frac{(3t^2 + 2)^{1/2}}{(t^2 - t)^{1/2}}$$

and apply Eq. (184), we obtain

$$\frac{dy}{dt} = \frac{(t^2 - t)^{1/2}(\tfrac{1}{2})(3t^2 + 2)^{-1/2}(6t) - (3t^2 + 2)^{1/2}(\tfrac{1}{2})(t^2 - t)^{-1/2}(2t - 1)}{t^2 - t}$$
$$= \frac{3t}{(t^2 - t)^{1/2}(3t^2 + 2)^{1/2}} - \frac{(2t - 1)(3t^2 + 2)^{1/2}}{2(t^2 - t)^{3/2}} = \frac{-3t^2 - 4t + 2}{2(t^2 - t)^{3/2}(3t^2 + 2)^{1/2}}$$

EXAMPLE 55. Find the dimensions of the largest rectangle that can be inscribed in a semicircle of diameter 24 ft.

Referring to Fig. 26·4, the area of the rectangle is given by the equation

$$A = 2h\sqrt{144 - h^2} = 2h(144 - h^2)^{1/2}$$

Differentiating,

$$\frac{dA}{dh} = 2h\left(\frac{1}{2}\right)(144 - h^2)^{-1/2}(-2h) + (144 - h^2)^{1/2}(2)$$

Equating to zero and simplifying, we obtain

$$0 = -2h^2(144 - h^2)^{-\frac{1}{2}} + 2(144 - h^2)^{\frac{1}{2}}$$

or
$$0 = -2h^2 + 2(144 - h^2)$$
$$4h^2 = 288$$
$$h = 8.49$$
$$\text{Width} = 2\sqrt{144 - h^2}$$
$$= 16.98$$

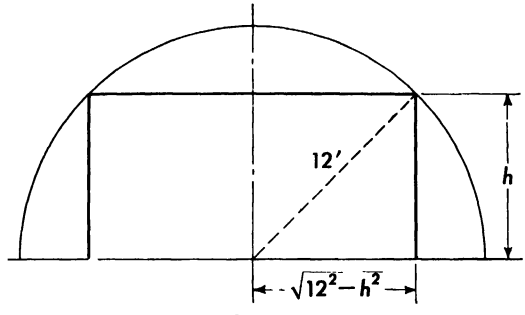


Fig. 26·4

26·18 The Derivative of Tan u. Since

$$\tan u = \frac{\sin u}{\cos u}$$

we can differentiate the tangent by using Eq. (184) and

$$\frac{d(\tan u)}{dx} = \frac{\cos u \cos u(du/dx) + \sin u \sin u(du/dx)}{\cos^2 u}$$

or
$$\frac{d(\tan u)}{dx} = \frac{\cos^2 u(du/dx) + \sin^2 u(du/dx)}{\cos^2 u}$$
$$\frac{d(\tan u)}{dx} = \frac{1}{\cos^2 u}\frac{du}{dx}$$

or
$$\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx} \qquad \star\ (185)$$

By a similar derivation we can discover that

$$\frac{d(\cot u)}{dx} = -\operatorname{cosec}^2 u \frac{du}{dx}$$

EXAMPLE 56. If $y = \cos 2\theta \tan \theta$, find $dy/d\theta$.

$$\frac{dy}{d\theta} = \cos 2\theta \sec^2 \theta - 2 \sin 2\theta \tan \theta$$

$$= (2\cos^2 \theta - 1)\left(\frac{1}{\cos^2 \theta}\right) - 4 \sin \theta \cos \theta \frac{\sin \theta}{\cos \theta} = 2 - \frac{1}{\cos^2 \theta} - 4 \sin^2 \theta$$

EXAMPLE 57. If $y = 4 \tan [(\theta/2) - (\pi/3)]$, find $dy/d\theta$.

By Eq. (185)

$$\frac{dy}{d\theta} = \left[4 \sec^2 \left(\frac{\theta}{2} - \frac{\pi}{3}\right)\right]\left(\frac{1}{2}\right) = 2 \sec^2 \left(\frac{\theta}{2} - \frac{\pi}{3}\right)$$

EXERCISE 15

1. Sketch the graph of the function $r = 2{,}000t^2/(t^3 + 4{,}000)$ between $t = 0$ and $t = 70$. r is the speed in mph of a moving vehicle at time t seconds. Compare with Fig. 25·34.
2. Acceleration is the rate of change in velocity with respect to time. Find the average acceleration between 5 and 10 sec.
3. Find the acceleration* at $t = 50.0$ sec.
4. What was the acceleration* at $t = 10$? At $t = 15$?
5. Find the acceleration* at $t = 12.5$.
6. Find the instantaneous acceleration at $t = 0$, 2.5, 7.5, 10, 12.5, 15, 20, 30, 35, 40, 45, 50, 55, 60, 70, 75, and 80 sec. Plot acceleration vs. time.
7. What was the gain in speed between $t = 4$ and $t = 8$?
8. When was the acceleration -1.0 ft per sec^2?
9. When was acceleration at a maximum?
10. When was acceleration at a minimum?

EXERCISE 16

Find the value of the independent variable at any maximum and minimum points in the following problems. Where the independent variable is an angle, consider only those positive angles less than 360°.

1. $y = x + \dfrac{4}{x}$
2. $y = \dfrac{24x}{x^2 + 1}$
3. $y = \dfrac{x^2}{x + 1}$
4. $y = \dfrac{9(3 - x)}{x^2 + 9}$
5. $y = \sin t + \cos 2t$
6. $y = \frac{1}{4} \cos^2 \theta + \sin^2 \theta$
7. $y = e^x \cos 2x$
8. $y = 3e^{-t} \sin t$
9. $y = \dfrac{2}{x^2} + \ln x$
10. In Exercise 6, page 604, solve by algebraic methods Prob. 9.
11. In Exercise 6, page 604, solve by algebraic methods Prob. 11.
12. Given a sphere of radius 10 in., find:
    a. The altitude of an inscribed right cylinder of maximum volume
    b. The altitude of an inscribed right cylinder of maximum total surface
    c. The altitude of an inscribed right cone of maximum volume

* Find acceleration in feet per second per second.

13. The turning effect of a ship's rudder may be shown theoretically to be $k \sin^2 \theta \cos \theta$, where θ is the angle the rudder makes with the keel and k is a constant. At what value of θ is the rudder most effective?

14. A light is to be placed directly over the center of a 6-ft circular table. How high above the table should the light be placed to obtain the best lighting at the edge of the table? (Intensity of illumination varies directly as the cosine of the angle of incidence and inversely as the square of the distance from the light source.)

15. A minor sector is removed from a disk of brass. The remaining major sector is formed into a cone. Find the central angle of the minor sector required to make the capacity of the cone a maximum.

EXERCISE 17

Differentiate and simplify the following:

1. $y = \ln \sin x$

2. $y = \ln \cos x$

3. $q = \sin^3 5x$

4. $y = \sqrt{\sin 4x^2}$

5. $y = -\frac{1}{ax} + \frac{b}{a^2} \ln\left(\frac{a + bx}{x}\right)$

6. $y = \ln (x + \sqrt{x^2 - a^2})$

7. $y = \sqrt{x^2 + a^2} + \frac{a^2}{\sqrt{x^2 + a^2}}$

8. $y = \sin 3\theta \cos 3\theta$

9. $y = \frac{x}{x^3 - 4}$

10. $y = (x^2 + 3x) \ln x^2$

11. $y = \frac{\sin x}{e^{2x}}$

12. $y = 5 \ln (e^{2x}) - e^2$

13. $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

14. $y = \log (x^2 - 4x + 4)$

15. $y = \frac{5x}{(x - 1)^2} - \frac{5}{x - 1}$

16. $y = \frac{\sqrt{1 - x^2}}{1 - x}$

17. $y = \frac{\sqrt{x^2 - 1}}{x}$

18. $y = x \ln x - x$

19. $y = \frac{1}{2}[x \sqrt{x^2 + a^2} + a^2 \ln (x + \sqrt{x^2 + a^2})]$

20. $y = -\frac{x}{9\sqrt{x^2 - 9}}$

21. $y = \frac{x}{16\sqrt{16 - x^2}}$

22. $y = \frac{\sqrt{x^2 - a^2}}{a^2 x}$

23. $y = -\frac{\sqrt{8x - 2x^2}}{4x}$

24. $y = -\frac{1}{5}(x^2 + 6)(9 - x^2)^{3/2}$

25. $y = \frac{1}{2}x + \frac{1}{4} \sin 2x$

26. $y = \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$ **27.** $y = \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8}$

28. $y = \tan \frac{x}{2}$ **29.** $y = \tan x - x$

30. $y = \frac{1}{2} \tan^2 \theta + \ln \cos \theta$ **31.** $y = \cos x + x \sin x$

26·19 Flexion. The rate of change of slope is called *flexion*. Thus if the ordinate y of a graph is given by the equation

$$y = 3x^3 + 7x^2 + 5x + 2 \tag{186}$$

the equation for slope is

$$\text{Slope} = \frac{dy}{dx} = 9x^2 + 14x + 5 \tag{187}$$

The equation for flexion is the derivative of the equation for slope. Thus for the function shown in Eq. (186)

$$\text{Flexion} = \frac{d^2y}{dx^2} = 18x + 14 \tag{188}$$

26·20 Repeated Differentiation. The nomenclature generally used to indicate successive differentiations (as in Sec. 26·19) is

$$y = 3x^3 + 7x^2 + 5x + 2 \tag{189}$$

$$\frac{dy}{dx} = 9x^2 + 14x + 5 \tag{190}$$

$$\frac{d^2y}{dx^2} = 18x + 14 \tag{191}$$

$$\frac{d^3y}{dx^3} = 18 \tag{192}$$

Equation (190) is called the *first* derivative of y.
Equation (191) is called the *second* derivative of y.
Equation (192) is called the *third* derivative of y.
If we wished, we could differentiate Eq. (192), obtaining the *fourth* derivative of y, and

$$\frac{d^4y}{dx^4} = 0 \tag{193}$$

26·21 Derived Curves. In Fig. 26·5 the graph A is called the *primary* curve. The others are called *derived* curves.

The ordinate of B is numerically equal to the slope of A. The ordinate of C is numerically equal to the slope of B. The ordinate of D is numerically equal to the slope of C. The ordinate of E is numerically equal to the slope of D.

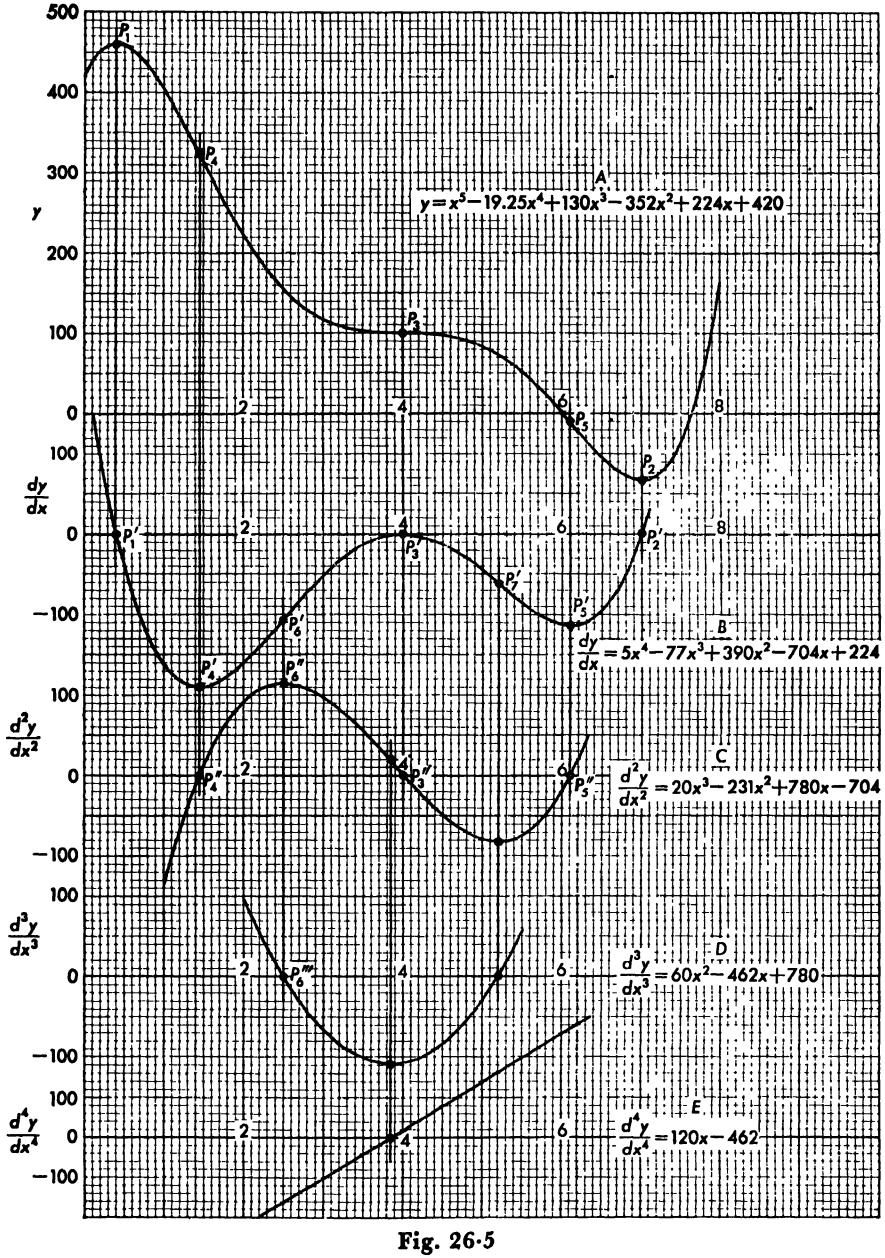

A
$y = x^5 - 19.25x^4 + 130x^3 - 352x^2 + 224x + 420$
B
$\frac{dy}{dx} = 5x^4 - 77x^3 + 390x^2 - 704x + 224$
C
$\frac{d^2y}{dx^2} = 20x^3 - 231x^2 + 780x - 704$
D
$\frac{d^3y}{dx^3} = 60x^2 - 462x + 780$
E
$\frac{d^4y}{dx^4} = 120x - 462$

Fig. 26·5

If the ordinate of A is y and the abscissa of A is x, then the ordinates of graphs B, C, D, and E are, respectively,

$$\frac{dy}{dx} \quad \frac{d^2y}{dx^2} \quad \frac{d^3y}{dx^3} \quad \frac{d^4y}{dx^4}$$

Graphs A and B are shown in Fig. 25·21. The student should review Secs. 25·13, 25·14, and 25·15.

Graph C is the graph of *flexion* referred to the primary curve; that is, the ordinate of C is d^2y/dx^2 (see Sec. 26·19). Observe that if graph C were continued to the left, it is evident that the flexion would be negative at the same value of x which makes the primary graph a maximum. In other words, as the primary curve passes through a maximum, its slope decreases (see Prob. 2 in Exercise 18).

If the graph of flexion were extended to the right it is evident that the flexion is positive at the same value of x which makes the primary curve a minimum. In other words, as the primary curve passes through a minimum its slope is increasing (see Prob. 1 in Exercise 18). This is a general principle. If for any value of the independent variable the function is a maximum, its second derivative will be negative.

If for any value of the independent variable the function is a minimum, the second derivative will be positive.

EXERCISE 18

Solve the following problems by direct reading of Fig. 26·5 rather than by calculations. Observe the vertical "tie lines" which connect corresponding points on successive curves.

1. On curve A find graphically the approximate slope at point P_2, a little to the left of point P_2, and a little to the right of point P_2. Show that the algebraic value of the ordinate at point P_2 is a minimum.
2. On curve A find graphically the slope at point P_1, a little to the left of point P_1, and a little to the right of point P_1. Show that the algebraic value of the ordinate at point P_1 is a maximum.
3. Read the value of x on curve B at which the curve A is a maximum.
4. Read the value of x on curve B at which the curve A is a minimum.
5. Read all the values of x on curve D at which the slope of curve B is a maximum. A minimum.
6. Read the values of x on curve C at which the curve A has points of inflection.
7. Read the value of x on curve D at which the flexion in curve A is a maximum. A minimum.
8. At how many points does graph A have a minimum slope?

9. At how many points does graph A have a maximum slope?
10. At how many points does graph A have a minimum ordinate?
11. At how many points does graph B have a maximum slope?
12. At how many points does graph D have a minimum slope?
13. For what value of x is the rate of change of flexion in curve A a maximum? A minimum?
14. For what value of x does the flexion in curve A increase most rapidly?
15. For what value of x does the flexion in curve A decrease most rapidly?

EXAMPLE 58. If the equation for the graph in Fig. 26·5A is

$$y = x^5 - 19.25x^4 + 130x^3 - 352x^2 + 224x + 420 \tag{194}$$

write the equation for the first, second, third, fourth, and fifth derivatives.

$$\frac{dy}{dx} = 5x^4 - 77x^3 + 390x^2 - 704x + 224 \qquad \text{(see Fig. 26·5, curve } B\text{)} \tag{195}$$

$$\frac{d^2y}{dx^2} = 20x^3 - 231x^2 + 780x - 704 \qquad \text{(see Fig. 26·5, curve } C\text{)} \tag{196}$$

$$\frac{d^3y}{dx^3} = 60x^2 - 462x + 780 \qquad \text{(see Fig. 26·5, curve } D\text{)} \tag{197}$$

$$\frac{d^4y}{dx^4} = 120x - 462 \qquad \text{(see Fig. 26·5, curve } E\text{)} \tag{198}$$

$$\frac{d^5y}{dx^5} = 120$$

EXAMPLE 59. Find the value of x at P_1, P_3, and P_2 (Fig. 26·5A). At these points the slope of the primary curve is zero. The equation for the first derivative is (see Example 58)

$$\frac{dy}{dx} = 5x^4 - 77x^3 + 390x^2 - 704x + 224 \tag{199}$$

Therefore we have to find the value of x which will make

$$0 = 5x^4 - 77x^3 + 390x^2 - 704x + 224 \tag{200}$$

Having the graph available we can see that the *approximate* roots of this equation are 0.4, 4, and 7.

Let us see if these roots are precise. If, for example, $x = 4$ is a root of Eq. (200), then $x - 4$ is a factor of the right side of Eq. (200).

We shall check by division (see Sec. 5·16), obtaining

$$(x - 4)(5x^3 - 57x^2 + 162x - 56) \tag{201}$$

Therefore we know that $x - 4$ is a factor of Eq. (200), and $x = 4$ is a root of Eq. (200).

From the graph it appears that $x = 7$ might be a root of Eq. (200). Therefore $x - 7$ may be a factor of the expression

$$5x^3 - 57x^2 + 162x - 56 \tag{202}$$

By division we can verify the fact that

$$(x - 7)(5x^2 - 22x + 8) = 5x^3 - 57x^2 + 162x - 56 \tag{203}$$

Therefore we know that $x - 7$ is a factor of (202).

The expression

$$5x^2 - 22x + 8$$

may be factored by inspection or by use of the quadratic equation, giving

$$5(x - 4)(x - 0.4) \tag{204}$$

Consequently the factors of Eq. (200) are

$$(x - 4)(x - 4)(x - 7)(x - 0.4)5 \tag{205}$$

and the roots of Eq. (200) are precisely 4, 4, 0.4, and 7.

EXAMPLE 60. Find the value of x for which the slope of the plot of Eq. (194) is a maximum or minimum (see Fig. 26·5*B*).

Maximum or minimum values of a function occur when the derivative of that function is zero (see Sec. 26·7).

The equation for the slope of Eq. (194) is [see Eq. (195)]

$$\frac{dy}{dx} = 5x^4 - 77x^3 + 390x^2 - 704x + 224 \tag{206}$$

The derivative of Eq. (206) is

$$\frac{d^2y}{dx^2} = 20x^3 - 231x^2 + 780x - 704 \tag{207}$$

The slope of the plot of (194) will be at a maximum or minimum when Eq. (207) is zero. Therefore we set Eq. (207) equal to zero and solve for x.

$$0 = 20x^3 - 231x^2 + 780x - 704 \tag{208}$$

There are several ways by which we could solve Eq. (208). By observing Fig. 26·5*C*, which is a plot of Eq. (207), we can see that the slope is zero in the vicinity of $x = 1.4$, $x = 4$, and $x = 6.1$.

Since 4 is a convenient root, let us see if $x - 4$ is a factor of Eq. (208).

$$\begin{array}{r|l} & 20x^2 - 151x + 176 \\ \hline x - 4 & 20x^3 - 231x^2 + 780x - 704 \\ & \underline{20x^3 - 80x^2} \\ & \quad - 151x^2 + 780x - 704 \\ & \quad \underline{- 151x^2 + 604x} \\ & \qquad\qquad + 176x - 704 \\ & \qquad\qquad \underline{+ 176x - 704} \end{array}$$

We now know that $x - 4$ is actually a factor of Eq. (208). The other factor is $20x^2 - 151x + 176$, and

$$\frac{d^2y}{dx^2} = (x - 4)(20x^2 - 151x + 176) \tag{209}$$

We can factor the expression $20x^2 - 151x + 176$ by using the quadratic equation

$$x = \frac{151 \pm \sqrt{(151)^2 - 14{,}080}}{40} = \frac{151 \pm \sqrt{22{,}801 - 14{,}080}}{40} \tag{210}$$

$$= \frac{151 \pm \sqrt{8{,}721}}{40} = \frac{151 \pm 3\sqrt{969}}{40} = \frac{151 \pm 93.38628}{40} \tag{211}$$

$$x = 1.440343,\ 6.109657 \tag{212}$$

Therefore $$\frac{d^2y}{dx^2} = (x - 4)(x - 1.440343)(x - 6.109657) \tag{213}$$

Following the methods illustrated in Sec. 26·7 we can identify the point when $x = 4$ as a maximum. When $x = 1.440343$ we have a minimum. We also have a minimum when $x = 6.109657$.

If we had not been fortunate enough to discover an integral root of Eq. (208), we should have had to resort to the methods discussed in Appendix C.

EXAMPLE 61. Find the value of x when the flexion of the primary graph in Fig. 26·5 is maximum or minimum. State which.

Flexion is the second derivative of the primary graph. In this case (see Example 59)

$$\frac{d^2y}{dx^2} = 20x^3 - 231x^2 + 780x - 704 \qquad \text{(see graph } C \text{ in Fig. 26·5)}$$

The above equation will be a maximum or minimum when $d^3y/dx^3 = 0$.

$$\frac{d^3y}{dx^3} = 60x^2 - 462x + 780$$

Set the above equation to zero and solve for x.

$$0 = 60x^2 - 462x + 780$$

or $$0 = x^2 - 7.7x + 13$$

and $$x = \frac{7.7 \pm \sqrt{(7.7)^2 - 4 \times 13}}{2}$$

$$= \frac{7.7 \pm \sqrt{59.29 - 52}}{2}$$

$$= \frac{7.7 \pm \sqrt{7.29}}{2} = \frac{7.7 \pm 2.7}{2} = \frac{5}{2},\ \frac{10.4}{2}$$

$$x = 2.5 \text{ (max)},\ 5.2 \text{ (min)}$$

EXAMPLE 62. Find the value of x for which the rate of change of flexion is a maximum or minimum. State which.

The equation for the rate of change of flexion is

$$\frac{d^3y}{dx^3} = 60x^2 - 462x + 780$$

The above equation is a maximum or minimum when its derivative is zero. Its derivative is

$$\frac{d^4y}{dx^4} = 120x - 462$$

Therefore we set the above equation to zero and solve for x.

$$0 = 120x - 462$$
$$x = {}^{462}\!/_{120} = 3.85 \qquad \text{(min)}$$

EXAMPLE 63. How rapidly is the rate of change of flexion increasing when $x = 7$.

The rate of change of flexion is given by the equation

$$\frac{d^3y}{dx^3} = 60x^2 - 462x + 780$$

The rate of change of the above equation is

$$\frac{d^4y}{dx^4} = 120x - 462$$

The question is, what is d^4y/dx^4 when $x = 7$?

When $x = 7$,

$$\frac{d^4y}{dx^4} = (120)7 - 462 = 378$$

EXERCISE 19

Find the first, second, and third derivatives of each of the following:

1. $y = 3x^2 - 5x + 7 - \dfrac{4}{x}$ **2.** $z = 7x^2 - 8\sqrt{x}$

3. $v = {}^4\!/_3\pi R^3$ **4.** $y = \cos 2x$

5. $y = 3e^{2t}$ **6.** $y = \ln x^4$

7. $y = \ln \dfrac{2}{x}$ **8.** $y = x \ln x - x$

9. $y = \sin \theta^2$ **10.** $y = \sin^2 \theta$

11. If the height of a ball at any time t is given by the equation $h = 72t - 16t^2$, find the velocity and acceleration at any time.

12. Repeat Prob. 11, if $h = 120 + 72t^2 + t^3$. How fast is the acceleration changing at $t = 2$?

In each of the following, find the slope at any points of inflection:

13. $y = x^3 - 6x^2 + 8$

14. $y = x^2 - \dfrac{1}{x^2}$

15. $y = x^4 - 4x^2 - 21$

16. $y = x^3 - x + 6$

17. $y = \dfrac{4x}{x^2 + 1}$

18. The distance traveled by a moving object is $s = 20t^4 - t^5$. Find when the acceleration is changing most rapidly.

19. The distance traveled by a moving object is given by the equation $s = 10t^3 - t^4$. Find when the speed is a maximum. Also find when the acceleration is a maximum.

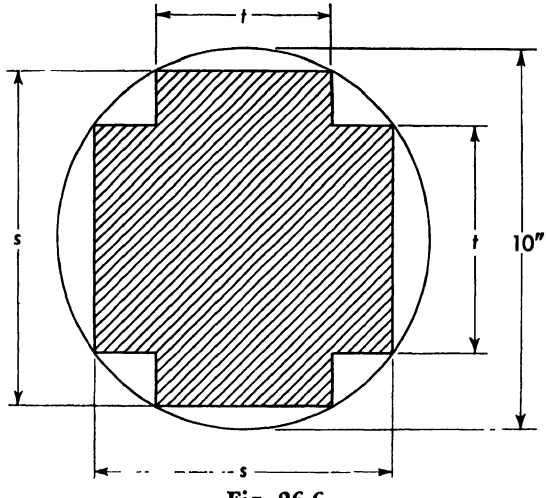


Fig. 26·6

20. A solenoid has a fixed internal diameter D and has a laminated core of the form shown in Fig. 26·6. Determine the dimension s and t of the core so that the cross-section area of the core will be a maximum.

*

Chapter 27 INTEGRATION

In Chap. 26 this general problem was considered. "Given an equation for a *function*, find the equation for the *derivative* of this function." This process is called *differentiation.*

Now we shall be concerned with the associated problem: "Given the equation for the *derivative of a function*, find the equation for the *function itself.*" This process is called *integration.*

Differentiation is an orderly algebraic process based on the delta method (see Sec. 26·2).

27·1 Introduction to Integration. Integration is largely a matter of trial and memory. For example, the student with a good memory will recall from Eqs. (18) and (19), page 628, that if

$$y = 5x^2 + \text{a constant} \tag{1}$$

then

$$\frac{dy}{dx} = 10x \tag{2}$$

or

$$dy = 10x\,dx \tag{3}$$

Consequently, if the student were given the equation

$$dy = 10x\,dx \tag{4}$$

and he were asked to find the corresponding equation for y, he would be expected to *recall* the association between Eq. (4) above and Eq. (19), page 628, and without further reason immediately conclude that if

$$dy = 10x\,dx \tag{5}$$

then y must, as a consequence, be given by the equation

$$y = 5x^2 + \text{some constant} \tag{6}$$

The expression $10x(dx)$ has now been *integrated.* The symbol which demands the process of integration is an elongated s written $\int$. Therefore, the problem could have been proposed as, "Find

$\int 10x(dx)$," and would have been read, "Find the integral of $10x(dx)$." Therefore we may write

$$\int 10x\,dx = 5x^2 + C \tag{7}$$

where C is called the *constant of integration.*

27·2 The Indefinite Integral. There is a certain "indefiniteness" in the above problem. If all we know about a function is its derivative, then we do not know what the added constant is. Consequently, an integral in the form of Eq. (7) is called an *indefinite* integral.

27·3 The Rule for $\int kx^n\,dx$. The *derivative* of a function in the form $kx^n + C$ is "the exponent times the coefficient times the independent variable with the exponent *diminished* by 1" [see Eqs. (18) and (19), page 628]. Conversely, the *integral* of a function in the form $kx^n\,dx$ is "the coefficient times the independent variable with the exponent *increased* by 1, divided by the exponent increased by 1, *plus an added constant.*" Thus (provided $n \neq -1$)

$$\int kx^n\,dx = \frac{kx^{n+1}}{n+1} + C \qquad \star\ (8)$$

Equation (8) can be verified by direct differentiation as follows:

$$\frac{d}{dx}\left(\frac{kx^{n+1}}{n+1} + C\right) = \frac{(n+1)kx^{(n+1)-1}}{(n+1)} \tag{9}$$

$$= kx^n \tag{10}$$

27·4 The Multiplied Constant in Integration. Suppose we wish to find

$$\int 5x^2\,dx$$

According to Eq. (8)

$$\int 5x^2\,dx = \frac{5x^3}{3} + C$$

Now suppose we find $5\int x^2\,dx$. Again following Eq. (8)

$$5\int x^2\,dx = 5\left(\frac{x^3}{3}\right) + C = \frac{5x^3}{3} + C$$

Thus a *multiplied* constant can be transferred across the integral sign, provided the multiplied constant multiplies all the expression to the right of the integral sign. Therefore

$$\int 7(x^2 + 3x + 2)\,dx = 7\int (x^2 + 3x + 2)\,dx$$

$$\int 5\ln(2x^5)\,dx = 5\int \ln(2x^5)\,dx$$

$$\int 7\sin(3\phi^2 + 2\phi + 8)\,d\phi = 7\int \sin(3\phi^2 + 2\phi + 8)\,d\phi$$

EXERCISE 1

Verify the following integrals:

1. $\int 3x^5\,dx = \frac{3x^6}{6} + C = \frac{1}{2}x^6 + C$
2. $\int 2x^{5/4}\,dx = \frac{3}{4}x^{9/4} + C$
3. $\int 7x^{-5}\,dx = -\frac{7}{4}x^{-4} + C$
4. $\int 3x^{-1/2}\,dx = 6x^{1/2} + C$
5. $\int 8x^{-9/5}\,dx = -\frac{40}{3x^{9/5}} + C$
6. $\int -\frac{7}{8}x^{3/4}\,dx = -\frac{7x^{5/2}}{20} + C$
7. $\int 4\sqrt[3]{x^2}\,dx = \frac{12x^{5/3}}{5} + C$
8. $\int -32\,dx = -32x + C$

9. Given the equation for the derivative dy/dx in the following problems, find the equation for y indicating the constant of integration. *Check by differentiation.*

   a. $dy/dx = 2x$ **b.** $dy/dx = 3x^2$ **c.** $dy/dx = x$
   d. $dy/dx = 10$ **e.** $dy/dx = -2$ **f.** $dy/dx = x^{5/2}$
   g. $dy/dx = x^{1/3}$ **h.** $dy/dx = x^{-1/3}$ **i.** $dy/dx = 3x^{-1/2}$

10. Given the following equations for the differential dy, find the equation for y indicating the constant of integration. *Check by differentiation.*

    a. $dy = \frac{5}{6}\sqrt{x}\,dx$ **b.** $dy = -\frac{3}{4}\sqrt[3]{x}\,dx$
    c. $dy = (3x^2 + \sqrt{2})\,dx$ **d.** $dy = x^{-10}\,dx$
    e. $dy = \frac{1}{x^5}\,dx$ **f.** $dy = \frac{k}{x^5}\,dx$
    g. $dy = (\pi + 3x^{5/4})\,dx$ **h.** $dy = (3e + 5\sqrt{x^7})\,dx$

11. Evaluate the following indefinite integrals:

    a. $\int (x^3 - 6x^2 + 500x + 7)\,dx$
    b. $\int (x^{3/4} - 2x^{-5/6} + 2\sqrt[3]{x^2} + \pi)\,dx$

27·5 Evaluating the Constant of Integration. As we mentioned in Sec. 27·2, the integral expressed in the form of Eq. (7) has a certain "indefiniteness" about it. Equation (7) is indefinite to the extent that the constant is not known. As a matter of fact, it could be anything, including zero. If the constant C is to be evaluated, the problem must contain more information than merely to give the derivative of the

function. For example, if we have given that

$$dy = 10x\,dx \tag{11}$$

and we wish to find the value of y, we would naturally write

$$y = \int 10x\,dx \tag{12}$$

and according to Eq. (8)

$$y = 5x^2 + C \tag{13}$$

Now let us suppose that it is also given in the problem that when x is 2, y is 35; then we may write

$$35 = 5(2^2) + C \tag{14}$$

As a consequence,

$$35 = 20 + C \tag{15}$$

and

$$C = 15 \tag{16}$$

Therefore, with this added bit of information, we can evaluate this particular constant of integration, and the entire integral is

$$y = 5x^2 + 15 \tag{17}$$

The question frequently arises, "Is integration unique, that is, is there any other function which can be differentiated and give $dy/dx = 10x$ except $5x^2$ + a constant?" Let us assume for a moment that there is such a function, say z, such that

$$\frac{dy}{dx} = \frac{dz}{dx} \tag{18}$$

Then

$$\frac{dy}{dx} - \frac{dz}{dx} = 0 \tag{19}$$

Now let us integrate both sides of this equation, obtaining

$$y - z = \text{a constant} \tag{20}$$

Therefore, these functions can differ only by the amount of some constant. Consequently, two functions which will differentiate and give the same derivative can only differ by a constant.

Figure 27·1 illustrates the graphs of four equations. The only difference in these equations is in the added constant. The effect of the added constant is simply to raise or lower the graph. It has no effect whatever on the *shape* of the graph. It is the *shape* of the graph that determines the derivative. Therefore the added constant has no effect on the derivative.

From another point of view, we may observe that the derivative of a constant is zero, which means that a constant does not change. If it did it would not be a constant.

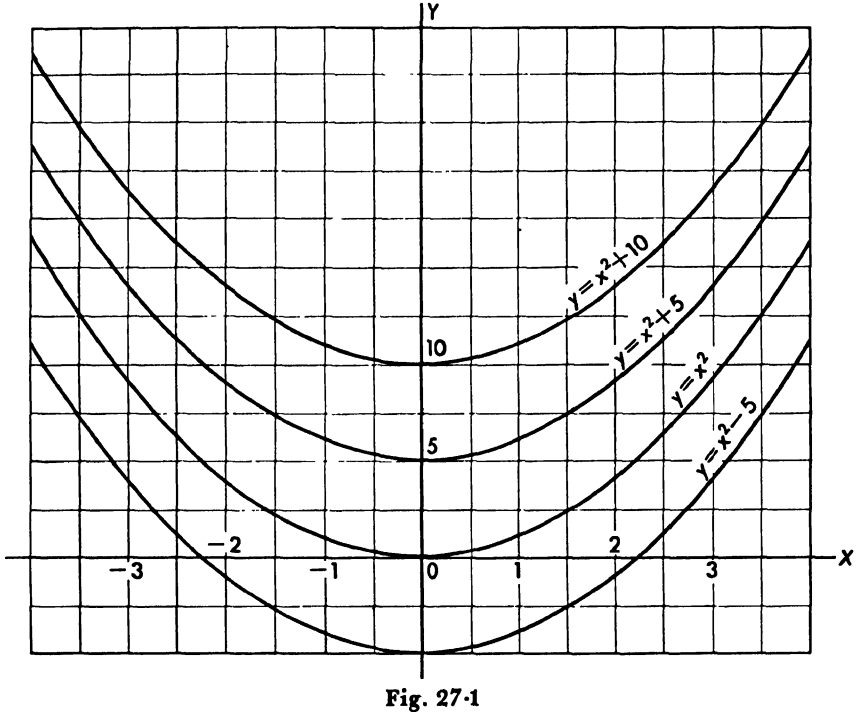


Fig. 27·1

27·6 The Importance of the Differential *du.* Let us differentiate the expression

$$(5x^3 + 7x^2 + 9)^2 \tag{21}$$

Refer to Eqs. (39) and (40) and rule 5, page 634. Following these directions, if

$$y = (5x^3 + 7x^2 + 9)^2 \tag{22}$$

then

$$\frac{dy}{dx} = 2(5x^3 + 7x^2 + 9)(15x^2 + 14x) \tag{23}$$

Now suppose we wish to *integrate* the expression

$$2(5x^3 + 7x^2 + 9)(15x^2 + 14x)\,dx \tag{24}$$

We know without further investigation that

$$\int 2(5x^3 + 7x^2 + 9)(15x^2 + 14x)\,dx = (5x^3 + 7x^2 + 9)^2 + C \tag{25}$$

Observe that the factor $(15x^2 + 14x)\,dx$ is the derivative of $5x^3 + 7x^2 + 9$. If this derivative were not present in Eq. (25), then the integral would not be $(5x^3 + 7x^2 + 9)^2$.

EXAMPLE 1. Find the integral of

$$5(4x^2 + 3x + 8)^2(8x + 3)\,dx \tag{26}$$

At first glance we might say that since $8x + 3$ is the derivative of $4x^2 + 3x + 8$ with respect to x, the integral will be $\frac{1}{3}(4x^2 + 3x + 8)^3 + C$. However, this assumption disregards the factor 5. If we differentiate $\frac{1}{3}(4x^2 + 3x + 8)^3$, we obtain $(4x^2 + 3x + 8)^2(8x + 3)$, which is not the original expression, differing from it by the factor 5.

$$\int 5(4x^2 + 3x + 8)^2(8x + 3)\,dx \tag{27}$$

can be written (see Sec. 27·4)

$$5\int (4x^2 + 3x + 8)^2(8x + 3)\,dx \tag{28}$$

This integral can now be evaluated as

$$5[\tfrac{1}{3}(4x^2 + 3x + 8)^3] + C \tag{29}$$

The latter expression differentiates and gives the original expression.

EXAMPLE 2. Find the integral of

$$(3x^2 + 5)^2x\,dx \tag{30}$$

Here the function $3x^2 + 5$ is not multiplied by its derivative. Its derivative is $6x$. The multiplying factor in the example is not $6x$, it is x. However, we can multiply and divide by 6, obtaining

$$\int \tfrac{6}{6}(3x^2 + 5)^2x\,dx \tag{31}$$

Now we can move the multiplying factor $\frac{1}{6}$ across the integration sign obtaining

$$\tfrac{1}{6}\int 6x(3x^2 + 5)^2\,dx \tag{32}$$

The integral now becomes

$$\frac{1}{6}\left[\frac{(3x^2 + 5)^3}{3} + C\right] \tag{33}$$

Equation (33) will differentiate as follows:

$$\frac{1}{6}\left[\frac{3(3x^2 + 5)^2}{3}6x\right] = \frac{1}{6}(3x^2 + 5)^2 6x = (3x^2 + 5)^2x \tag{34}$$

EXERCISE 2

1. Compute the constant of integration for Prob. 9 in Exercise 1, parts *a* to *i*, if $y = 20$ when $x = 4$.

Find the equation for y (indicating the constant of integration) in each of the following:

2. $\frac{dy}{dt} = (3 - t)^4$

3. $\frac{dy}{dx} = (3 - 5x)^{-6}$

4. $\frac{dy}{dm} = \sqrt{4 - m}$

5. $\frac{dy}{d\phi} = \sqrt{4\phi - 1}$

6. $\frac{dy}{dt} = t^3(3 - t^4)^5$

7. $\frac{dy}{d\phi} = \frac{\phi^3}{(2 - \phi^4)^6}$

8. $\frac{dy}{dt} = \frac{t + 1}{\sqrt{t^2 + 2t}}$

9. $\frac{dy}{dr} = (r - 3)\sqrt{r^2 - 6r - 7}$

10. $\frac{dy}{dx} = 15x(4x^2 - 9)^{\frac{3}{2}}$

27·7 Uniformly Accelerated Motion. If any object is acted on by a *balanced* system of forces, it will either remain at rest or, if already moving, will continue to move in a straight line with a *constant velocity*.

If, on the other hand, it is acted on by a constant but *unbalanced* system of forces, then a *constant acceleration* will be imparted to it.

One of the most common examples of uniformly (or constantly) accelerated motion is a freely falling body. For example, consider a sounding rocket fired straight up. For a time it is propelled by its own motor. During this time it is not a freely falling body. Eventually, the fuel will burn out, and the rocket will then become a "freely falling body"; that is, the only force acting on it is that of gravity, which is constant.

At "burnout" the rocket has attained a certain velocity. If gravity did not exist, the rocket would continue indefinitely with the same velocity it had at "burnout." However, the force of gravity does exist and is a constant force applied to the rocket. Furthermore, the force of gravity is not entirely balanced by any other force on the rocket. Consequently, the rocket undergoes a constant acceleration. The acceleration of gravity is about -32 ft per sec^2. Notice the negative sign attached to the acceleration of gravity. This sign indicates that the velocity of a freely falling body is always decreasing; that is, the velocity is either positive and decreasing in magnitude or is negative and increasing in magnitude. Both situations amount to an algebraic decrease in velocity.

Now let us consider the case of a stone dropped from the roof of a building. At the instant the stone is dropped, the velocity is zero. Thereafter, the total velocity will decrease at the rate of 32 ft per sec^2. (The magnitude of the velocity increases but in a negative direction.)

At the end of the first second it will be moving at the rate of -32 fps; at the end of the second second it will be moving at the rate of -64 fps. At the end of the third second it will be moving downward at the rate of 96 fps, and so on, until it hits the ground. The equation for the acceleration of gravity is

$$a = -32 \text{ ft/sec} \tag{35}$$

where a = acceleration in ft per sec^2

We know that acceleration is the rate of change of velocity. Therefore we know that velocity is the integral of acceleration, and we can write

$$v = \int -32\,dt \tag{36}$$

The differential dt written into Eq. (36) indicates that we are considering rates of change with respect to time and that t is the independent variable. Therefore, in accordance with Eq. (8), we may write

$$v = -32t + C_1 \tag{37}$$

(also see Prob. 8 in Exercise 1). In Eq. (37) we have the usual constant of integration which we have to evaluate by some means other than simply a knowledge of the derivative as shown in Eq. (35). We do have such information implicit in the statement of the problem, namely, that in this particular problem the velocity is zero when t is zero, since we are beginning to count time at the instant the stone is dropped. Consequently, we may write

$$0 = -32(0) + C_1 \tag{38}$$

Therefore the constant of integration in this case is zero, and the complete equation for velocity is

$$v = -32t \text{ fps} \tag{39}$$

We also know that velocity is the rate of change of distance. Therefore distance is the integral of velocity, and we may write

$$\begin{aligned} d &= \int -32t\,dt \\ d &= -16t^2 + C_2 \end{aligned} \tag{40}$$

Equation (40), then, is the equation for distance fallen with respect to time. Here again we know that when t is zero, the distance fallen is also zero. Therefore C_2 is zero, and we may write

$$d = -16t^2 \tag{41}$$

Equation (41) gives a complete statement for the distance which the stone falls, measured from the roof, if we begin to count time when the

stone is dropped. The negative sign is appropriate since distance is measured in a downward direction. Now let us consider a somewhat different case.

A missile is projected directly upward from ground level with an initial velocity of 96 fps. Let us develop the equation for *altitude* in terms of time. Since the missile becomes a freely falling body we may write

$$a = -32 \tag{42}$$

We know that velocity is the integral of acceleration, and therefore we may write

$$v = \int -32\,dt \tag{43}$$

and velocity is given by the equation

$$v = -32t + C_1 \tag{44}$$

From the statement of the problem we know that the initial velocity is 96 fps. That occurs when t is zero; therefore we can write

$$96 = -32(0) + C_1 \tag{45}$$

Solving Eq. (45) for C_1, we obtain

$$C_1 = 96$$

Substituting the constant 96 as found above in Eq. (45) gives

$$v = -32t + 96 \tag{46}$$

The initial velocity is +96 fps. If the force of gravity did not exist, the missile would continue to move indefinitely with that velocity. However, the force of gravity does exist and consequently will influence the velocity of the missile in accordance with the term $-32t$ in Eq. (46). A short table of velocities vs. time is calculated from Eq. (46) and is shown in Table 27·1. Note in Table 27·1 that as time increases, velocity decreases throughout.

Table 27·1

t	0	1	2	3	4	5	6
v	+96	+64	+32	0	−32	−64	−96

We know that, by definition, velocity is the derivative of distance. Therefore distance is the integral of velocity. In this case we are using ground level as the zero reference from which to express distance. Accordingly, we shall use h (height) as the symbol for distance. Then

$$h = \int(-32t + 96)\,dt \tag{47}$$

or

$$h = -16t^2 + 96t + C_2 \tag{48}$$

From the conditions of the problems we know that when $t = 0$, the altitude is zero, and

$$0 = -16(0)^2 + 96(0) + C_2$$

consequently, $C_2 = 0$, and the complete equation for altitude is

$$h = -16t^2 + 96t \tag{49}$$

The initial velocity of this missile is +96 fps. If gravity did not act, then the equation for altitude would be

$$h = 96t$$

However, the force of gravity does exist, and consequently it will influence the altitude reached by the missile. The term $-16t^2$ indicates the effect of gravity on the altitude of the missile. [Compare Eqs. (49) and (41).]

Now suppose this missile were thrown upward from the roof of a building 256 ft high with an initial velocity of 96 fps. The equation for altitude in this case would be identical with Eq. (48), and the constant of integration would be 256 ft, since when $t = 0$, $h = 256$ ft. Therefore, under these conditions the complete equation for altitude is

$$h = -16t^2 + 96t + 256 \tag{50}$$

Equation (50) is the equation for altitude in terms of time. From this equation we can calculate other things with regard to the trajectory of this missile. For example, if we wished to know when the missile hit the ground, we could calculate that from Eq. (50). We know that when the missile hit the ground, the height was zero; consequently, we can write

$$0 = -16t^2 + 96t + 256 \tag{51}$$

We shall now solve Eq. (51) for t. First, factoring out 16 to reduce the problem to simpler numbers:

$$0 = -t^2 + 6t + 16 \tag{52}$$

Equation (52) can be solved by factoring, obtaining

$$0 = (t + 2)(-t + 8) \tag{53}$$

Therefore, t is -2 or $+8$. For the moment we shall consider the negative value of time -2 to be a trivial solution, although a significance can be attached to it. The answer to this particular problem is that it would require 8 sec for this missile to reach the ground if it were thrown upward from a roof 256 ft high with an initial velocity of 96 fps. Now let us find at what time the missile reached its zenith.

We know that when the missile did reach its zenith, its velocity was zero. Therefore the first derivative of altitude with respect to time must be zero. Differentiating Eq. (50) and setting the derivative to zero gives

$$0 = -32t + 96 \tag{54}$$

Solving Eq. (54) for t gives

$$t = {}^{96}\!/_{32} = 3 \tag{55}$$

Thus it takes 3 sec for the missile to reach the highest point in its trajectory. We are now in a position to generalize Eq. (50) to make it suitable for any freely falling body problem. The student should observe that Eq. (50) is a quadratic equation. The coefficient of t^2 is always -16, if we are dealing with a freely falling body. The coefficient of t is the initial velocity; the constant term is the initial height. Therefore, if we let v_0 be the initial velocity and h_0 be the initial height, the height at any time (h) is given by the equation

$$h = -16t^2 + v_0t + h_0 \tag{56}$$

EXAMPLE 3. A stone is dropped from a height of 288 ft. Write the equation for acceleration, velocity, distance fallen, and altitude. This is a freely falling body; therefore acceleration is

$$a = -32 \text{ ft per sec}^2$$

Following Eqs. (36) and (37)

$$v = \int -32\,dt = -32t + C_1$$

Since the stone was dropped and not thrown, $C_1 = 0$, and

$$v = -32t$$

Following Eq. (41),

$$d = -16t^2$$

where d is the distance fallen, not the altitude.

Following Eq. (47)

$$h = \int -32t\,dt = -16t^2 + C_2$$

where h = altitude

In this case, when $t = 0$, $h = 288$, and

$$288 = -16(0)^2 + C_2$$

Consequently

$$C_2 = 288$$

and

$$h = -16t^2 + 288$$

EXAMPLE 4. In Example 3, find the velocity with which the stone hit the ground.

The stone hit the ground when $h = 0$. Therefore

$$0 = -16t^2 + 288$$

or
$$16t^2 = 288$$

and
$$t = \pm\sqrt{\frac{288}{16}} = \frac{1}{4}\sqrt{288} = \pm\frac{1}{4} \times 16.97 = \pm 4.24 \text{ sec}$$

We shall reject the negative root as an extraneous solution to this particular problem.

The stone hit the ground when $t = 4.24$ sec. The equation for velocity is

$$v = -32t$$

Therefore, when the stone hit the ground,

$$v = -32 \times 4.24 = -135.7 \text{ fps}$$

EXAMPLE 5. A stone is thrown straight up from a roof 288 ft high with an initial speed of 48 fps. Find the equations for acceleration, velocity, and altitude. As is always the case with a freely falling body,

$$a = -32$$

and
$$v = \int -32\,dt = -32t + C_1$$

In this problem $v = 48$ when $t = 0$; therefore

$$48 = -32(0) + C_1$$

or
$$C_1 = 48$$

and
$$v = -32t + 48$$

Furthermore,

$$h = \int(-32t + 48)\,dt = -16t^2 + 48t + C_2$$

In this problem $h = 288$ when $t = 0$; therefore

$$288 = -16(0)^2 + 48(0) + C_2$$

or
$$C_2 = 288$$

and
$$h = -16t^2 + 48t + 288$$

EXAMPLE 6. Find the altitude in Example 5 when $t = 1$ and when $t = 2$.

When $t = 1$,

$$h = -16(1)^2 + 48(1) + 288 = 320 \text{ ft}$$

When $t = 2$,

$$h = -16(2)^2 + 48(2) + 288 = 320 \text{ ft}$$

In other words, the stone passes the 320-ft line, going up when $t = 1$. It passes the same level going down when $t = 2$.

EXAMPLE 7. Find the velocity of the stone in Example 5 when $t = 1$ and when $t = 2$.

When $t = 1$,

$$v = -32(1) + 48 = +16 \text{ fps}$$

When $t = 2$,

$$v = -32(2) + 48 = -16 \text{ fps}$$

Since the sign of the velocity when $t = 1$ is positive, we know that the direction of motion at that time is upward. Since the sign of the velocity is negative when $t = 2$, we know that the direction of motion at that time is downward.

EXAMPLE 8. Find the maximum altitude attained by the stone in Example 5.

The maximum altitude occurs when the velocity is zero, that is, when

$$0 = -32t + 48$$

or

$$t = {}^{48}\!/_{32} = 1.5$$

When $t = 1.5$, the altitude is

$$h = -16(1.5)^2 + 48t + 288 = -16(2.25) + 48(1.5) + 288 = 324 \text{ ft (max)}$$

EXAMPLE 9. With what velocity did the stone hit the ground in Example 5?

The stone hit the ground when $h = 0$. Therefore we shall find what value of t makes the altitude zero.

$$0 = -16t^2 + 48t + 288$$
$$0 = -t^2 + 3t + 18$$
$$0 = (-t + 6)(t + 3)$$

or

$$t = 6, -3$$

Discarding $t = -3$ as an extraneous solution for this particular problem, we substitute $t = 6$ in the equation for velocity, obtaining

$$v = -32(6) + 48 = -192 + 48 = -144 \text{ fps}$$

EXAMPLE 10. If a stone is thrown upward from ground level with an initial speed of 144 fps, when will it reach a velocity of +48 fps?

$$48 = -32t + 144$$
$$32t = 96$$
$$t = 3 \text{ sec}$$

(Compare with extraneous solution of Example 9.)

EXAMPLE 11. What maximum altitude will the stone in Example 10 reach?

Maximum altitude occurs when $v = 0$.

$$0 = -32t + 144$$
$$t = 4{}^{1}\!/_{2} \text{ sec}$$
$$h_{max} = -16(4.5)^2 + 144(4.5) = -16 \times 20.25 + 144 \times 4.5$$
$$= -324 + 648 = +324$$

(Compare with Example 8.)

EXAMPLE 12. If the stone had been thrown straight down with a speed of 48 fps from a height of 288 ft, find the equation for acceleration, velocity, and altitude.

$$a = -32$$
$$v = \int -32\,dt = -32t + C_1$$

when $t = 0$, $v = -48$,

$$-48 = -32(0) + C_1$$
$$C_1 = -48$$
$$v = -32t - 48$$
$$h = \int(-32t - 48)\,dt = -16t^2 - 48t + C_2$$

when $t = 0$, $h = 288$,

$$288 = -16(0)^2 - 48(0) + C_2$$
$$C_2 = 288$$
$$h = -16t^2 - 48t + 288$$

EXAMPLE 13. With what velocity will the stone hit the ground in Example 12?

$$0 = -16t^2 - 48t + 288$$
$$0 = -t^2 - 3t + 18$$
$$0 = (-t + 3)(t + 6)$$
$$t = 3, -6$$

Regarding the value -6 as extraneous to this problem, we substitute $t = 3$ in the equation for velocity, obtaining

$$v = -32(3) - 48 = -144$$

Compare with the velocity found in Example 9. Explain.

Figure 27·2 is a plot of the equation

$$h = -16t^2 + 48t + 288$$

Observe that the region of the graph between $t = 0$ and $t = 6$ applies to Examples 5 and 6.

Note that from the symmetry of the graph, the slope at $t = 0$ and $t = 3$ must be equal in magnitude and opposite in sign (see Example 7).

The axis of symmetry of the graph is at $t = 1.5$, and at this time the altitude is 324 ft, which is the maximum altitude (see Example 8).

Considering only the mathematical relationship in the above equation, we find that $h = 0$ when $t = -3$ and when $t = +6$. Within the physical limitations imposed by the wording of Example 5, the root of $t = -3$ is meaningless (see Example 9).

However, if a stone had been thrown directly upward, 3 sec before our arbitrarily chosen zero reference time, with sufficient velocity to pass the 288-ft level with a velocity of +48 fps, the subsequent trajectory would be identical with the one prescribed by the statement of Example 5 (see Examples 10 and 11).

Observe that at the 288-ft level there are two velocities, equal in magnitude and opposite in sign, one occurring at $t = 0$ and the other occurring at $t = 3$. Considering, for example, the negative velocity when $t = +3$, it makes no difference to the subsequent trajectory

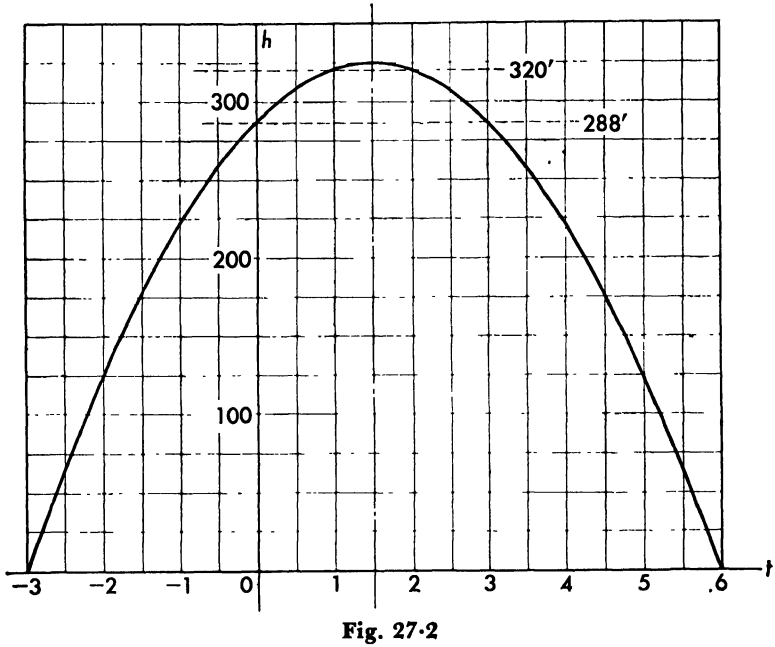


Fig. 27·2

whether this velocity is caused by a momentary thrust as is indicated in Example 12 or whether it is the cumulative effect of the force of gravity (see Examples 12 and 13).

EXERCISE 3

1. A stone is dropped from a certain height h_0. Write the equation for its velocity and altitude in terms of time.
2. A stone is thrown downward with a velocity of 100 fps from a height of 192 ft. Write the equation for its velocity and altitude in terms of time.
3. A stone is thrown upward with a velocity of 100 fps from a height of 192 ft. Write the equation for its velocity and altitude as a function of time.

4. How long did it take the stone to reach the ground in Prob. 2? In Prob. 3?
5. With what velocity did the stone hit the ground in Prob. 2? In Prob. 3?
6. A ball is thrown upward from a roof 96 ft high with an initial speed of 80 fps. When was the maximum altitude reached? What was the maximum altitude? When did it hit the ground? With what speed? When was the ball 150 ft high (two answers)? Find the arithmetical average of the last two answers (see Sec. 10·16).
7. How long will it take for a stone to reach the ground if it is dropped from a height of 1,024 ft?

27·8 Areas under Graphs. One of the most important uses for the integration process is in finding the area under curvilinear graphs.

An approximate graphical method of doing this when the graph of the function is available has been discussed in Sec. 25·16.

We shall now develop a precise algebraic method of accomplishing the same end when the equation of the function is available.

By reference to Fig. 25·27 we illustrated the graphical method of Sec. 25·16.

We shall illustrate the more precise algebraic method by reference to Fig. 27·3, which is a graph of the same function.

The equation for the graph in both Figs. 25·27 and 27·3 is

$$y = 100 - x^2 \qquad (57)$$

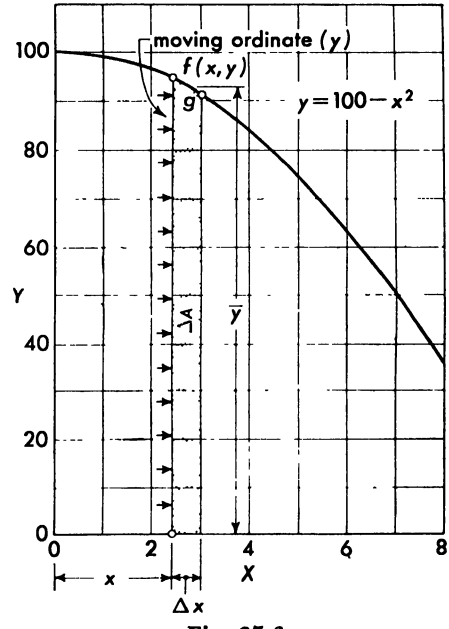


Fig. 27·3

In Sec. 25·16 we found the approximate area under the graph by adding the approximate areas of the narrow strips.

In this section we shall adopt a somewhat different point of view. We shall think of an ordinate as originating near the left-hand margin of Fig. 27·3 and proceeding toward the right. In the process it wipes over the entire area under the graph. One extremity of the moving ordinate is always at the graph itself. The other extremity is always on the X axis.

We shall deal only with functions that are continuous over the region under consideration. Furthermore, for the present we shall require that the moving ordinate does not become negative.

At some arbitrary point we shall arrest the travel of the moving ordinate, as, for example, at the point f in Fig. 27·3. The coordinates of f are x and the corresponding y.

We shall now let the moving ordinate proceed a little farther to the right, say, to the point g. In doing so it wipes over the incremental area ΔA.

By observing Fig. 27·3 we can see that

$$\Delta A = \bar{y}\,\Delta x$$

or

$$\frac{\Delta A}{\Delta x} = \bar{y} \tag{58}$$

where $\bar{y}$ = average ordinate of area ΔA

Δx = base of area ΔA

Now if we let Δx approach zero, the ratio $\Delta A/\Delta x$ by definition approaches dA/dx. Furthermore, $\bar{y}$ approaches y, and from Eq. (58) it follows that

$$\frac{dA}{dx} = y$$

and

$$dA = y\,dx \tag{59}$$

Thus, it is a basic principle in elementary calculus that the ordinate of a graph is equal to the rate of change of the area.

Therefore we have not as yet found an equation for the area under the graph, but we have found an equation for the *rate of change of the area.*

By definition of the integration process we can assert that if Eq. (59) is true, then

$$A = \int y\,dx \tag{60}$$

Equation (60) is an extremely important equation. It states that if y is a function of x, then the area under the graph of y versus x is the integral of y with respect to x.

Figure 27·3 is a plot of the equation

$$y = 100 - x^2$$

Let us find the area under this graph between the ordinate erected at $x = 4$ and the ordinate erected at $x = 6$.

Since $y = 100 - x^2$ and $A = \int y\,dx$ we may write

$$A = \int (100 - x^2)\,dx$$

or

$$A = 100x - \frac{x^3}{3} + C \qquad (61)$$

We must now evaluate the constant of integration C. In order to evaluate C in Eq. (61), we must know *some* value of x (any value will do) and the *corresponding* value of area.

As the moving ordinate proceeds from left to right we have no particular interest in it until it reaches the left boundary of the area we are calculating. Neither do we have any interest in it after it passes the right-hand boundary of the area we are calculating.

At the left-hand boundary (where $x = 4$) it has reached the area we are calculating, but as yet it has passed over none of it. In other words, when $x = 4$, the area $A = 0$, or

$$0 = 100(4) - \frac{4^3}{3} + C = 400 - \frac{64}{3} + C = 400 - 21.33 + C$$

$$0 = 378.67 + C$$

$$C = -378.67$$

When the moving ordinate reaches the right-hand boundary of the area we are calculating, $x = 6$. The area wiped over as the moving ordinate proceeds from $x = 4$ to $x = 6$ is

$$A = 100(6) - \frac{6^3}{3} - 378.67 = 600 - 72 - 378.67 = 149.33$$

(Compare with Example 9, page 608.)

We could have approached the problem by first finding the area under the graph between $x = 0$ and $x = 6$. Then we could have found the area between $x = 0$ and $x = 4$. The difference between these two areas is the area under the graph between $x = 4$ and $x = 6$ (see Example 9, page 608).

Returning to Eq. (61),

$$A = 100x - \frac{x^3}{3} + C$$

in this case, when $x = 0$ the area wiped over is zero; therefore

$$C = 0$$

The complete equation for the area under the graph between the ordinate erected at $x = 0$ and any other ordinate erected at some

other value of x is

$$\underset{0\to x}{A} = 100x - \frac{x^3}{3} \tag{62}$$

When $x = 6$,

$$\underset{0\to 6}{A} = 600 - 216\tfrac{2}{3} = 600 - 72 = 528 \tag{63}$$

When $x = 4$,

$$\underset{0\to 4}{A} = 400 - 64\tfrac{2}{3} = 400 - 21\tfrac{1}{3} = 378\tfrac{2}{3} \tag{64}$$

Therefore

$$\underset{4\to 6}{A} = 528 - 378\tfrac{2}{3} = 149\tfrac{1}{3} \tag{65}$$

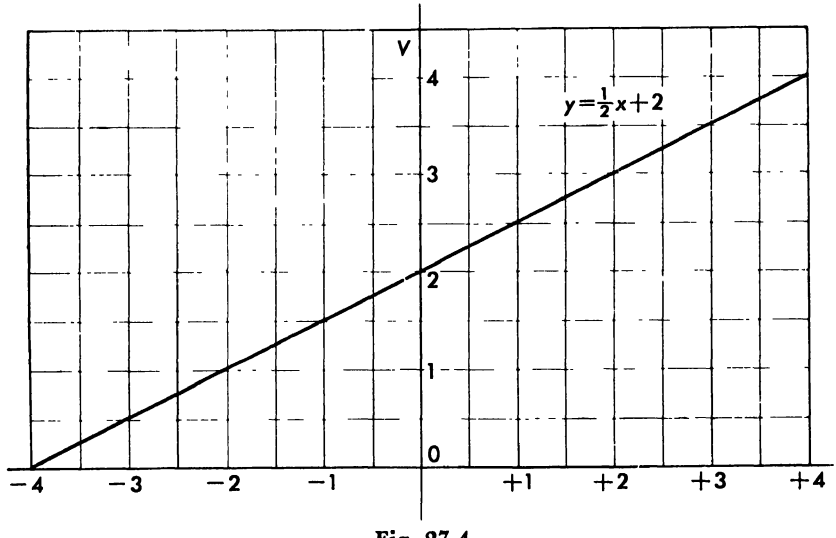


Fig. 27·4

EXAMPLE 14. Find the area under the graph of $y = \frac{1}{2}x + 2$ between $x = -4$ and $x = 0$ (see Fig. 27·4).

$$A = \int y\,dx = \int \left(\frac{1}{2}x + 2\right) dx = \frac{x^2}{4} + 2x + C$$

Let the moving ordinate proceed from left to right. Then when $x = -4$ it has reached the left (or lower) boundary of the area to be calculated, but it has not as yet wiped over any of it. Therefore

$$0 = \frac{(-4)^2}{4} + 2(-4) + C = \frac{16}{4} - 8 + C$$

$$C = 4$$

The complete equation for the area under the graph to the right of $x = -4$ is

$$\underset{-4\to x}{A} = \frac{x^2}{4} + 2x + 4$$

The area between $x = -4$ and $x = 0$ is

$$\underset{-4\to 0}{A} = \frac{0^2}{4} + 2(0) + 4 = 4$$

(Check by elementary geometry.)

We mentioned above that we could have considered the moving ordinate to proceed from right to left. Had we done this we should have reasoned as follows:

$$A = \frac{x^2}{4} + 2x + C$$

Let the moving ordinate proceed from right to left. Then, when $x = 0$, it has reached but not entered the area to be calculated. Consequently, when $x = 0$, the area $A = 0$ and

$$0 = \frac{0^2}{4} + 2(0) + C$$
$$C = 0$$

The complete equation for area under the graph to the left of $x = 0$ is

$$\underset{x\to 0}{A} = \frac{x^2}{4} + 2x$$

The area between $x = 0$ and $x = -4$ is

$$\underset{0\to -4}{A} = \frac{(-4)^2}{4} + 2(-4) = \frac{16}{4} - 8 = 4 - 8 = -4$$

The magnitude of the area so found is the same as before but with the opposite sign. This occurs because, in effect, we are considering dx to be negative.

EXAMPLE 15. Find the area under the graph of $y = \frac{1}{2}x + 2$ between $x = -4$ and $x = +4$.

As before

$$A = \frac{x^2}{4} + 2x + C$$

when $x = -4$, $A = 0$, and

$$0 = \frac{(-4)^2}{4} + 2(-4) + C = 4 - 8 + C$$
$$C = 4$$

Therefore, the area between $x = -4$ and any other ordinate is

$$\underset{-4\to x}{A} = \frac{x^2}{4} + 2x + 4$$

When $x = +4$

$$\underset{-4\to+4}{A} = 16\tfrac{4}{4} + 8 + 4 = 4 + 8 + 4 = 16 \tag{66}$$

(check by geometry).

We could also find the area between $x = 0$ and $x = +4$ and from it subtract the area between $x = 0$ and $x = -4$.

The equation for area under the graph between $x = 0$ and any other ordinate is

$$A = \frac{x^2}{4} + 2x$$

When $x = -4$,

$$\underset{0\to-4}{A} = 16\tfrac{4}{4} - 8 = 4 - 8 = -4$$

When $x = +4$,

$$\underset{0\to+4}{A} = 16\tfrac{4}{4} + 8 = 12$$

The area between $x = -4$ and $x = +4$ is

$$\underset{-4\to+4}{A} = 12 - (-4) = 16$$

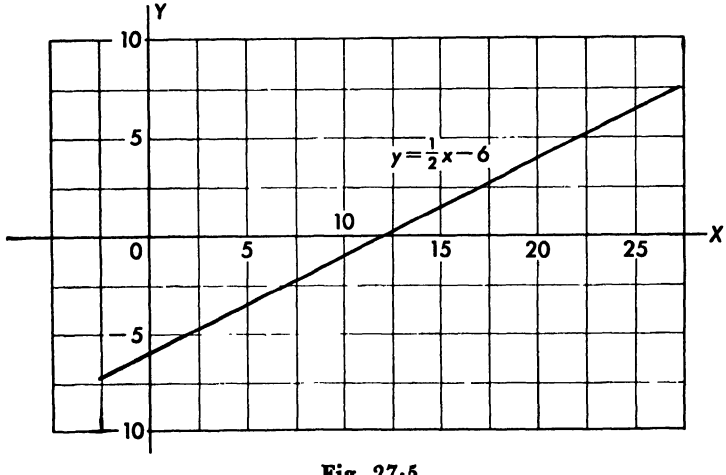


Fig. 27·5

EXAMPLE 16. Find the area under the graphs of $y = \frac{1}{2}x - 6$ between $x = 0$ and $x = 12$ (see Fig. 27·5).

$$A = \int y\,dx = \int \left(\frac{1}{2}x - 6\right) dx = \frac{x^2}{4} - 6x + C$$

When $x = 0$, $A = 0$, and $C = 0$; therefore

$$\underset{0\to x}{A} = \frac{x^2}{4} - 6x \tag{67}$$

When $x = 12$,

$$\underset{0\to 12}{A} = 144\tfrac{4}{4} - 72 = 36 - 72 = -36$$

(see Fig. 27·5). Note that over this region the moving ordinate is negative, and we therefore obtain a negative area.

EXAMPLE 17. Find the area under the graph $y = \frac{1}{2}x - 6$ between $x = 12$ and $x = 24$.

$$A = \frac{x^2}{4} - 6x + C$$

When $x = 12$, $A = 0$, and

$$0 = \frac{12^2}{4} - 6(12) + C = 36 - 72 + C$$

$$C = +36$$

$$\underset{12\to 24}{A} = 576\tfrac{4}{4} - 144 + 36 = 144 - 144 + 36 = 36$$

EXAMPLE 18. Find the area under the graph of $y = \frac{1}{2}x - 6$ between $x = 0$ and $x = 24$.

$$\underset{0\to 24}{A} = \frac{x^2}{4} - 6x \qquad \text{[see Eq. (67)]}$$

$$= 576\tfrac{4}{4} - 144 = 144 - 144 = 0$$

From an observation of Fig. 27·5 we can see the reason for the zero answer. The area between $x = 0$ and $x = 12$ is -36. The area between $x = 12$ and $x = 24$ is $+36$. The sum of these two areas is (algebraically) zero. Thus if we want the absolute value of the area, we disregard the sign and accept the total area as 72 square units.

Therefore, in doing a specific problem of this nature we had better assure ourselves that the graph does not cross the x axis between the boundaries of the area being calculated.

If the graph does cross the x axis between these boundaries and if we want the *absolute* area, we have to do the problem in two parts. This will be illustrated in the example below.

EXAMPLE 19. Find the absolute value of the area under the graph of $y = 100 - x^2$ between $x = 5$ and $x = 15$.

Let us find what value of x makes y equal to zero.

$$0 = 100 - x^2$$

$$x^2 = 100$$

$$x = \pm 10$$

The root $x = +10$ occurs between the two boundary ordinates of the problem. (The boundary ordinates occur at $x = +5$ and $+15$.)

As a first step we shall find the area under the graph between $x = 5$ and $x = 10$. Then we shall find the area between $x = 10$ and $x = 15$.

$$A = \int y\,dx = \int (100 - x^2)\,dx = 100x - \frac{x^3}{3} + C$$

When $x = 5$, $A = 0$; therefore

$$0 = 500 - \frac{125}{3} + C = \frac{1{,}500 - 125}{3} + C$$

$$C = -\frac{1{,}375}{3}$$

$$\underset{5\to 10}{A} = 1{,}000 - \frac{1{,}000}{3} - \frac{1{,}375}{3} = \frac{625}{3} \tag{68}$$

Now let us find the area under the graph between $x = 10$ and $x = 15$. When $x = 10$, $A = 0$; therefore

$$0 = 1{,}000 - \frac{1{,}000}{3} + C$$

$$C = -\frac{2{,}000}{3}$$

$$\underset{10\to x}{A} = 100x - \frac{x^3}{3} - \frac{2{,}000}{3}$$

$$\underset{10\to 15}{A} = \frac{4{,}500 - 3{,}375 - 2{,}000}{3} = -\frac{875}{3}$$

The *absolute* value of area is then

$$A_{\text{abs}} = \frac{625}{3} + \frac{875}{3} = \frac{1{,}500}{3} = 500$$

Had we integrated directly over the entire region between $x = 5$ and $x = 15$, we would have substituted $x = 15$ in Eq. (68), obtaining

$$\underset{5\to 15}{A} = 1{,}500 - \frac{3{,}375}{3} - \frac{1{,}375}{3} = \frac{4{,}500 - 3{,}375 - 1{,}375}{3} = -\frac{250}{3} \tag{69}$$

Observe that the *algebraic* sum of the area between $x = 5$ and $x = 10$ added to the area between $x = 10$ and $x = 15$ is

$$625/3 - 875/3 = -250/3$$

which agrees with Eq. (69).

EXAMPLE 20. Find the area under the graph of $y = 3\sqrt{x}$ between $x = +9$ and $x = +36$.

$$A = \int y\,dx = \int 3\sqrt{x}\,dx = \int 3x^{1/2}\,dx = 2x^{3/2} + C$$

When $x = +9$, $A = 0$, and

$$0 = 2(9)^{3/2} + C$$
$$C = -54$$

Then

$$\underset{9\to x}{A} = 2x^{3/2} - 54$$
$$\underset{9\to 36}{A} = 2(36)^{3/2} - 54 = 432 - 54 = 378$$

EXAMPLE 21. Find the area under the graph of $y = 6/x^2$ between $x = 2$ and $x = \infty$.

$$A = \int y\,dx = \int \frac{6}{x^2}\,dx = \int 6x^{-2}\,dx = -6x^{-1} + C = -\frac{6}{x} + C$$

When $x = 2$, $A = 0$, and

$$0 = -\tfrac{6}{2} + C$$
$$C = +3$$
$$\underset{2\to x}{A} = -\frac{6}{x} + 3$$
$$\underset{2\to \infty}{A} = -\frac{6}{\infty} + 3 = 0 + 3 = 3$$

EXERCISE 4

Find the areas under the following curves between the specified limits:

1. $y = x^2/4$ between $x = 4$ and $x = 10$
2. $y = 6\sqrt{x}$ between $x = 4$ and $x = 9$
3. $y = 9 - x^2$ between $x = -3$ and $x = +3$
4. $y = x^2 - 4$ between $x = -2$ and $x = +2$
5. $y = 16/x^3$ between $x = 1$ and $x = 4$
6. $y = 8/x^2$ between $x = 1$ and $x = \infty$
7. $y = \sqrt{x + 9}$ between $x = 0$ and $x = 16$

27·9 The Definite Integral. If we wish specifically to indicate that we are looking for the area under the graph of $y = 100 - x^2$ between $x = 4$ and $x = 6$, we can use the following notation:

$$A = \int_4^6 (100 - x^2)\,dx \tag{70}$$

The numbers written above and below the integral sign are called the limits of integration. The lower limit, that is, the one defining the left-hand ordinate, is written below the integral sign. The number defining the right-hand ordinate is called the upper limit and is

written above the integral sign. After having written the integral in the form of Eq. (70), we evaluate it thus:

$$A = \left[100x - \frac{x^3}{3}\right]_4^6 \tag{71}$$

You will observe that the quantity enclosed in brackets above is the same quantity as appears in Eq. (62).

In evaluating an expression in the form of (71), we invariably subtract the brackets with the lower limit substituted in from the brackets with the upper limit substituted in. Thus [see Eq. (71)]

$$A = \left[100(6) - \frac{6^3}{3}\right] - \left[100(4) - \frac{4^3}{3}\right] \tag{72}$$

$$= 528 - 378\tfrac{2}{3} = 149\tfrac{1}{3} \tag{73}$$

Compare with Eq. (65).

The type of equation illustrated in Eq. (70) is called a definite integral as distinguished from the type of equation illustrated in Eq. (61), which is called an indefinite integral.

In practice, the differences in these methods are not so much differences in underlying theory as they are differences in nomenclature and calculating procedure. Both are in common use in industrial applications.

EXAMPLE 22. Find the area under the graph of $y = \frac{1}{2}x + 2$ between $x = -4$ and $x = 0$ (see Fig. 27·4 and Example 14, page 692).

$$A = \int_{-4}^{0} \left(\frac{1}{2}x + 2\right) dx = \left[\frac{x^2}{4} + 2x\right]_{-4}^{0} = [0] - \left[\frac{16}{4} - 8\right]$$

$$= -[4 - 8] = -[-4] = +4 \tag{74}$$

EXAMPLE 23. Find the area under the graph of $y = \frac{1}{2}x + 2$ between $x = -4$ and $x = +4$.

$$A = \int_{-4}^{+4} \left(\frac{1}{2}x + 2\right) dx \tag{75}$$

$$= \left[\frac{x^2}{4} + 2x\right]_{-4}^{+4} = \left[\frac{16}{4} + 8\right] - \left[\frac{16}{4} - 8\right] = [4 + 8] - [4 - 8]$$

$$= [12] - [-4] = 12 + 4 = 16 \tag{76}$$

Compare with Example 15, particularly Eq. (66).

EXAMPLE 24. Find the area under the graph of $y = 3\sqrt{x}$ between $x = +9$ and $x = +36$.

$$A = \int_9^{36} 3x^{1/2}\,dx \tag{77}$$

$$= [2x^{3/2}]_9^{36} = [2 \times 216] - [2 \times 27] \tag{78}$$

$$= [432] - [54] = 378 \tag{79}$$

Compare with Example 20, page 696.

EXAMPLE 25. Find the area under the graph of $y = 6/x^2$ between $x = 2$ and $x = \infty$.

$$A = \int_2^{\infty} 6x^{-2}\,dx \tag{80}$$

$$= [-6x^{-1}]_2^{\infty} = \left[-\frac{6}{x}\right]_2^{\infty} = [0] - [-3] = +3 \tag{81}$$

Compare with Example 21, page 697.

EXAMPLE 26. Find the average ordinate of the graph of the equation

$$y = 100 - x^2$$

between $x = 4$ and $x = 6$.

$$A = \int_4^6 (100 - x^2)\,dx = \left[100x - \frac{x^3}{3}\right]_4^6 = [600 - 72] - [400 - 21.33]$$

$$= 528 - 378.67 = 149.33$$

$$\bar{y} = \frac{149.33}{6 - 4} = \frac{149.33}{2} = 74.67 \tag{82}$$

EXAMPLE 27. Find the average ordinate of the graph in Example 24. The area under this graph within the prescribed limits is 378. The base interval is $36 - 9 = 27$. Therefore

$$\bar{y} = {}^{378}\!/_{27} = 14.33$$

27·10 The Definite Integral as the Limit of a Sum. Let us consider Fig. 27·6. Here the ordinate of the curve represents a continuous function of x.

We shall divide the area under the curve into strips of *finite* width. The area of each strip is *approximately* the area of a rectangle y units in altitude and Δx units in width. The area under the curve is *approximately* the sum of the areas of these rectangles. The more rectangles we have, the closer is the approximation.

Each rectangular strip is called an *elemental* rectangle.

The area of one of these elemental rectangles is, in general,

$$y\,\Delta x \tag{83}$$

The sum of all these elemental rectangles between $x = a$ and $x = b$ can be written

$$\underset{a \to b}{A} = \sum_a^b y_1\,\Delta x + y_2\,\Delta x + y_3\,\Delta x + \cdots + y_n\,\Delta x \tag{84}$$

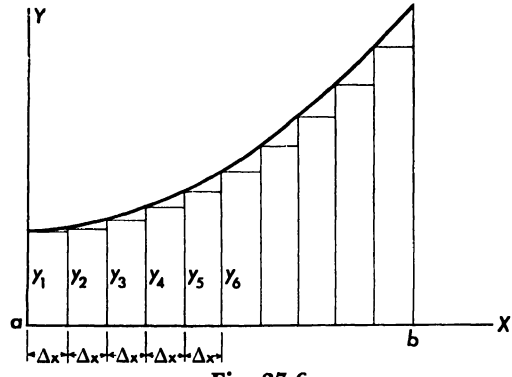


Fig. 27·6

As $n \to \infty$ the sum of all the elemental rectangles approaches the area under the curve, which we know from Eq. (60), page 690, to be

$$\underset{a \to b}{A} = \int_a^b y\,dx \tag{85}$$

Therefore we can write

$$\int_a^b y\,dx = \lim_{n \to \infty} \sum_a^b y_1\,\Delta x + y_2\,\Delta x + y_3\,\Delta x + \cdots + y_n\,\Delta x \tag{86}$$

This equation is a statement of a quite general theorem: "Any quantity which can be expressed as the limit of a sum, such as Eq. (84), is equal to a definite integral."

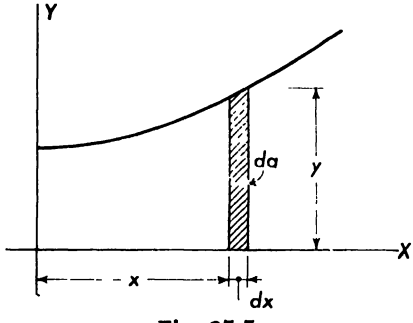


Fig. 27·7

When we are dealing with strips of a finite width it is probably more correct to write their area as $y_1\,\Delta x$, $y_2\,\Delta x$, $y_3\,\Delta x$, etc. However, it is common practice to speak of the area of an "elementary rectangle" as being $y\,dx$. When we do use this symbolism, we mean that the elementary rectangle does actually have *a very small width* and *we are going to use the limit which the sum of the areas of these rectangles approaches as the number of rectangles approaches infinity*.

To avoid the use of a relatively confused figure containing a large number of rectangles such as Fig. 27·6, we usually abbreviate the figure to something like Fig. 27·7. Here we indicate only one elemental

area, but the implication is that the entire area under the curve is filled with similar elementary rectangles.

EXERCISE 5

Do the problems in Exercise 4, page 697, using the definite integral.

27·11 Average Ordinates. The student should recall from Sec. 25·16 that the average ordinate of a curve is that ordinate which, when multiplied by the base interval, equals the area under the curve. We now have a method of precisely finding the average ordinate under a curve, provided we know the equation for the curve. The procedure is first to find the area under the curve by integration, and second, to divide that area by the base interval.

EXERCISE 6

Find the average ordinate between the specified limits. Use the definite integral symbolism (see Exercise 4).

1. $y = x^2/4$ between $x = 4$ and $x = 10$
2. $y = 6\sqrt{x}$ between $x = 4$ and $x = 9$
3. $y = 9 - x^2$ between $x = -3$ and $x = +3$
4. $y = x^2 - 4$ between $x = -2$ and $x = +2$
5. $y = 16/x^3$ between $x = 1$ and $x = 4$
6. $y = \sqrt{x + 9}$ between $x = 0$ and $x = 16$

27·12 The Integral of $ku^n\,du$ When $n = -1$. In Sec. 27·3 it was mentioned that the equation for integrating a power function given in that section fails when $n = -1$.

The student with a good memory will recall rule 9, page 656, and rule 2, page 629. He will remember that if

$$y = k \ln u \tag{87}$$

then

$$\frac{dy}{du} = k\frac{1}{u}$$

or

$$dy = k\frac{1}{u}\,du = k\frac{du}{u} = ku^{-1}\,du \tag{88}$$

Therefore, if he is given the fact that the differential of y with respect to u is

$$ku^{-1}\,du \tag{89}$$

he will, without further investigation, conclude that

$$y = k \ln u + C$$

In other words,

$$\int k\frac{du}{u} = k \ln u + C \qquad * (90)$$

The equation for the graph shown in Fig. 25·31 is

$$F = \frac{10}{D} \qquad (91)$$

The area under this graph is proportional to work w done by the force F in forcing the piston through the distance D. Therefore

$$w = \int \frac{10}{D}\, dD \qquad (92)$$

Work done is

$$w = 10 \ln D + C \qquad (93)$$

Now we wish to find the work accomplished when the piston moves from $D = 1$ to $D = 6$. When $D = 1$ the work in which we are interested is 0; therefore

$$0 = 10 \ln 1 + C \qquad (94)$$

but $$\ln 1 = 0 \qquad (95)$$

Therefore

$$C = 0 \qquad (96)$$

and $$w = 10 \ln D \qquad (97)$$

When $D = 6$,

$$w = 10 \ln 6 \qquad (98)$$

$$= 10 \times 1.792 = 17.92 \qquad (99)$$

EXAMPLE 28. Find the integral of $(9x^2\,dx)/(1 - 2x^3)$.

Refer to Eq. (134), page 656. In this example

$$u = 1 - 2x^3$$

and $$du = -6x^2\,dx$$

Therefore, before we can integrate, we must alter the numerator of the expression

$$\frac{9x^2\,dx}{1 - 2x^3}$$

to contain du in the numerator; thus

$$\frac{-6x^2\,dx}{1 - 2x^3}$$

This can be done by multiplying and dividing by $-\frac{2}{3}$, giving

$$-\frac{3}{2}\int \frac{-\frac{2}{3} \times 9x^2\,dx}{1-2x^3}$$

or

$$-\frac{3}{2}\int \frac{-6x^2\,dx}{1-2x^3} = -\frac{3}{2}\ln(1-2x^3) + C$$

EXERCISE 7

Integrate the following expressions:

1. $\int \frac{dx}{x+3}$
2. $\int \frac{s\,ds}{s^2-4}$
3. $\int \frac{8h^2\,dh}{1-4h^3}$
4. $\int \frac{(2t^2-4)\,dt}{6t-t^3}$
5. $\int \frac{(\theta-1)^2\,d\theta}{\theta^3-3\theta^2+3\theta}$
6. $\int \frac{(x-1)\,dx}{x^2-2x}$
7. $\int \frac{(t+1)(t-1)\,dt}{t^3-3t}$
8. $\int -\frac{h\,dh}{3(9-h^2)}$
9. The equation for the graph shown in Fig. 25·34 is

$$r = \frac{2{,}000t^2}{4{,}000+t^3}$$

where r = speed, miles per hr
t = time, sec

Find the distance (in feet) covered in the first 70 sec. Find the average speed in feet per second over this interval and compare your answer as obtained by integration with the one you obtained for Prob. 4 in Exercise 13, page 615, using graphical methods.

Find when $\frac{1}{4}$ of the total distance is covered. Compare with your answer to Prob. 5 in Exercise 13, page 615.

27·13 The Integral of Sin u (du). The student is referred to rule 6, page 647, and Rule 2, page 629. The student with a good memory will recall that if

$$y = k \sin u \tag{100}$$

then the derivative of y with respect to u is

$$\frac{dy}{du} = k \cos u \tag{101}$$

or

$$dy = k \cos u(du) \tag{102}$$

Therefore, if we are given the fact that the differential of y with respect to u is

$$k \cos u(du) \tag{103}$$

we should without further reason immediately conclude that

$$y = k \sin u + C \tag{104}$$

If we were given the fact that the rate of change of y with respect to u was

$$\frac{dy}{du} = -k \sin u \qquad * \tag{105}$$

we should immediately conclude (based on a memory of rule 7, page 649) that

$$y = k \cos u + C \tag{106}$$

In other words,

$$\int k \cos u \, du = k \sin u + C \qquad * \tag{107}$$

and

$$\int -k \sin u \, du = k \cos u + C \tag{108}$$

One important application of the integral of a sine function occurs in the problem of finding the average ordinate of the sine wave over a half cycle. It will be recalled that we find the average ordinate of a curve by first finding the area under the curve and, second, by dividing that area by the abscissa distance between the limits chosen (see Sec. 27·11). We shall now proceed to find the average ordinate of the curve defined in the equation

$$y = \sin u \tag{109}$$

The area under this curve over one-half the cycle is

$$\text{Area} = \int_0^{\pi} \sin u \, du \tag{110}$$

or

$$\text{Area} = [-\cos u]_0^{\pi} \tag{111}$$

Evaluating (111) we write

$$\text{Area} = [-\cos \pi] - [-\cos 0] \tag{112}$$

or

$$\text{Area} = [-(-1)] - [-(+1)] \tag{113}$$

Therefore

$$\text{Area} = [+1] + 1 \tag{114}$$

and

$$\text{Area} = 2 \tag{115}$$

Thus the area under a sine curve over the interval between 0 radians and π radians is 2. The average height will be, therefore,

$$\text{Average height} = \frac{2}{\pi} = 0.6366 \tag{116}$$

The student should be careful to note that we are here finding the average ordinate of a sine curve, not the so-called effective ordinate of a sine curve. It would also be instructive to find the average ordinate of a sine curve over one entire cycle, that is, between the limits of 0 and 2π radians. Accordingly, let us alter Eq. (110) to find the area under a sine curve between the limits of 0 and 2π.

$$\text{Area} = \int_0^{2\pi} \sin u \, du \tag{117}$$

$$= [-\cos u]_0^{2\pi} \tag{118}$$

$$= [-\cos 2\pi] - [-\cos 0] \tag{119}$$

$$= [-1] - [-1] = -1 + 1 = 0 \tag{120}$$

Thus, the total area under a sine curve between the limits of 0 and 2π is 0. The reason for this is that a sine curve is symmetrical about the x axis, having the same area above the x axis as it has below the x axis. Consequently, it has the same positive area as it has negative area, and the average ordinate of the entire cycle is therefore zero.

Now let us find the *effective* value of a function in the form

$$y = \sin u \tag{121}$$

By definition the effective value of such a function is the square root of the average value of

$$y = \sin^2 u \tag{122}$$

between the limits of $u = 0$ and $u = \pi$. To find this average value we must first find the area under the curve of y versus u and divide this area by π. The area under the above curve would be the integral of $\sin^2 u$. Thus the first problem is to evaluate

$$\int \sin^2 u \, du \tag{123}$$

Thus far we have no method of integrating this function, but we can transform this function into a form that we can integrate [see Chap. 22, Eq. (15)].

$$\sin^2 u = \tfrac{1}{2} - \tfrac{1}{2} \cos 2u \tag{124}$$

Therefore Eq. (123) can be written

$$\int (\tfrac{1}{2} - \tfrac{1}{2} \cos 2u) \, du \tag{125}$$

Now refer to Sec. 26·9. Here we observe that "the derivative of the sine of a function is the cosine of that same function *times the derivative* of that function" (see also Sec. 27·6). We must seek some function which will differentiate and give

$$\tfrac{1}{2} - \tfrac{1}{2} \cos 2u \tag{126}$$

Such a function might appear to be

$$\tfrac{1}{2}u - \tfrac{1}{2}\sin 2u \tag{127}$$

However, Eq. (127) will actually differentiate and give

$$\tfrac{1}{2} - \tfrac{1}{2}\cos 2u \times 2 \tag{128}$$

or

$$\tfrac{1}{2} - \cos 2u \tag{129}$$

which is, by comparison with (126), obviously incorrect. The term $-\cos 2u$ in (129) is just twice what it should be. Therefore let us alter (127) and try

$$\tfrac{1}{2}u - \tfrac{1}{4}\sin 2u$$

This will differentiate and give

$$\tfrac{1}{2} - \tfrac{1}{4}\cos 2u \times 2 \tag{130}$$

or

$$\tfrac{1}{2} - \tfrac{1}{2}\cos 2u \tag{131}$$

Equations (131) and (126) are identical. Therefore we may assert that

$$\int(\tfrac{1}{2} - \tfrac{1}{2}\cos 2u)\,du = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u + C \tag{132}$$

or, substituting (124) in (132),

$$\int \sin^2 u\,du = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u + C \tag{133}$$

We are interested here in finding the area under the graph of $\sin^2 u$ between the limits of 0 and π (that is, over one-half cycle). Therefore we shall apply the definite integral symbolism to Eq. (133), obtaining

$$\int_0^{\pi} \sin^2 u\,du = [\tfrac{1}{2}u - \tfrac{1}{4}\sin 2u]_0^{\pi} \tag{134}$$

Evaluating the integral expressed in Eq. (134), we obtain

$$\left[\frac{\pi}{2} - \frac{1}{4}\sin 2\pi\right] - \left[\frac{0}{2} - \frac{1}{4}\sin 0\right] \tag{135}$$

$$\left[\frac{\pi}{2} - 0\right] - [0 - 0] = \frac{\pi}{2} \tag{136}$$

The average ordinate is, then,

$$\frac{\pi/2}{\pi} = \frac{1}{2} \tag{137}$$

However, the *effective* ordinate of a curve of sin u is, by definition, "the

square root of the average ordinates of the curve of $\sin^2 u$"; that is, if $y = \sin u$

$$y_{\text{eff}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} = 0.707 \tag{138}$$

The effective value of the function

$$y = k \sin u \tag{139}$$

is

$$y_{\text{eff}} = k \times 0.707 \tag{140}$$

EXAMPLE 29. Find y if $dy = 6 \sin (20 + \pi/2)\, d\theta$.

$$y = \int 6 \sin\left(2\theta + \frac{\pi}{2}\right) d\theta \tag{141}$$

Referred to Sec. 27·6,

$$u = 2\theta + \frac{\pi}{2}$$

and

$$du = 2\, d\theta$$

Therefore, before we can integrate, we must alter Eq. (141) to

$$y = -3 \int -\sin\left(2\theta + \frac{\pi}{2}\right) 2\, d\theta$$

Now

$$y = -3 \cos\left(2\theta + \frac{\pi}{2}\right) + C$$

EXERCISE 8

Integrate the following expressions, using a table of integrals where necessary.

1. $\int 4 \cos 2\theta\, d\theta$

2. $\int -\frac{\sin (\phi/2)}{8}\, d\phi$

3. $\int x \cos x^2\, dx$

4. $\int 10 \sin 2\theta\, d\theta$

5. $\int \cos (4 - x)\, dx$

6. $\int \sin\left(\pi - \frac{\phi}{4}\right) d\phi$

7. $\int \frac{\sin \phi}{\cos^2 \phi}\, d\phi$

8. $\int \frac{\cos x}{5 \sin^3 x}\, dx$

EXAMPLE 30. Find, by integration, the area of a circle 10 in. in radius.

We shall first find the area of the quadrant shown in Fig. 27·8. Following Eq. (60), page 690,

$$A = \int y\, dx$$

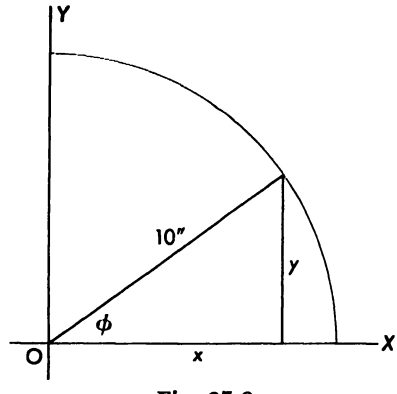


Fig. 27·8

The limits between which we shall integrate are $x = 0$ and $x = 10$ in. Also, we see that

$$y = \sqrt{100 - x^2}$$

Therefore

$$A = \int_0^{10} \sqrt{100 - x^2}\, dx$$

Thus far we have not discussed any method of integrating the above equation. A table of integrals, however, gives the following relation:

$$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right)$$

Applying this relation to the present problem,

$$A = \left[\frac{1}{2}\left(x\sqrt{100 - x^2} + 100 \sin^{-1}\frac{x}{10}\right)\right]_0^{10}$$

$$= \tfrac{1}{2}\left[10\sqrt{100} - \overline{100} + 100 \sin^{-1} 10/10\right]$$

$$- \tfrac{1}{2}[0\sqrt{100} - 0^2 + 100 \sin^{-1} 0/10]$$

Note that

$$\sin^{-1} 10/10 = \sin^{-1} 1 = \frac{\pi}{2} \text{ radians} \qquad \text{[see Eq. (117), page 365]}$$

$$A = \frac{1}{2}\left(100\,\frac{\pi}{2}\right) = 25\pi$$

The area of the entire circle is $4 \times 25\pi$, or 100π.

EXERCISE 9

1. Find the area under the graph of each equation below between the specified limits. Also find the average height between these limits (see Exercise 8).

a. $y = 4 \cos 2\theta$ between $\frac{\pi}{6}$ and $\frac{\pi}{4}$

b. $y = -\frac{1}{8} \sin \frac{\phi}{2}$ between $\frac{\pi}{2}$ and $\frac{2\pi}{3}$

c. $y = x \cos x^2$ between 0.2 and 0.5

d. $y = 10 \sin 2\phi$ between $\frac{\pi}{2}$ and $\frac{3\pi}{4}$

e. $y = \cos (4 - x)$ between 4 and 5

f. $y = \sin\left(\pi - \frac{\theta}{4}\right)$ between 4π and 5π

g. $y = \frac{\sin \theta}{\cos^2 \theta}$ between 0 and $\frac{\pi}{3}$

h. $y = \frac{\cos x}{5 \sin^3 x}$ between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$

2. Find by integration the area of a segment 3 in. high cut from a circle 20 in. in diameter (compare with Example 30, page 707).
3. A circle is 15 ft in diameter. Find the area cut off from this circle by a line $2\frac{1}{2}$ ft from the center.
4. The equation for the linear speed of a point on the tread of an automobile tire traveling at a constant speed is

$$v = 32\pi \sin 8\pi t$$

where t is in seconds, and v is in feet per second. This is the equation from which Fig. 25·32 was plotted.

a. Find the acceleration of the point when $t = 0.03$ (compare with Prob. 2 in Exercise 11, page 613).

b. Find the time and value of the maximum speed (compare with Prob. 3 in Exercise 11, page 613).

c. Find the speed of the car (compare with Prob. 4 in Exercise 11, page 614).

d. Find the outside diameter of the tire (compare with Prob. 5 in Exercise 11, page 614).

e. Find the distance the point moves between $t = 0$ and $t = \frac{1}{8}$ sec (compare with Prob. 6 in Exercise 11, page 614).

27·14 The Integral of $ke^u\,du$. The student should recall rule 8, page 652, and rule 2, page 629, consistent with these rules, if

$$y = ke^u + C \tag{142}$$

Then

$$\frac{dy}{du} = ke^u \tag{143}$$

or

$$dy = ke^u\,du \tag{144}$$

Therefore, if it is given that the rate of change of y with respect to u is some constant times e to the u power, the student will know that

$$y = ke^u + C \tag{145}$$

or

$$\int ke^u\,du = ke^u + C \tag{146}$$ *

EXAMPLE 31. Evaluate $\int 10xe^{x^2}\,dx$

Following Eq. (146) (also refer to Sec. 27·6), let

$$u = x^2$$

and

$$du = 2x\,dx$$

Therefore, before we can integrate, we must alter the original expression to

$$5\int e^{x^2}2x\,dx$$

Then

$$5\int e^{x^2}2x\,dx = 5e^{x^2} + C$$

EXERCISE 10

Integrate:

1. $\int e^{1-4x}\,dx$
2. $\int \frac{dx}{e^x}$
3. $\int e^{\pi x}\,dx$
4. $\int 6x^2e^{x^3}\,dx$
5. $\int \frac{e^{1/x}}{5x^2}\,dx$
6. $\int (e^{2x} + e^{-2x})^2\,dx$
7. $\int e^{3x+1}\,dx$
8. $\int xe^{x^2}\,dx$
9. $\int (e^x - e^{-2x})^2\,dx$
10. $\int (e^{2x} - e^{-2x})^2\,dx$
11. $\int \left(e^{3x} - \frac{1}{e}\right)^2 dx$
12. $\int \frac{e^x + 1}{e^{2x}}\,dx$
13. $\int (e^{2x} - e^{x/2})^2\,dx$
14. $\int \frac{x - 1}{e^{x^2-2x}}\,dx$

27·15 Volume by Integration. Let us consider Fig. 27·9. Here the cross-section area of a pyramid is a continuous function of x. The variable x is the perpendicular distance between a cross-section area and the apex.

We shall divide the volume of the pyramid into slabs of *finite* thickness. The volume of each slab is *approximately* the volume of a rectangu-

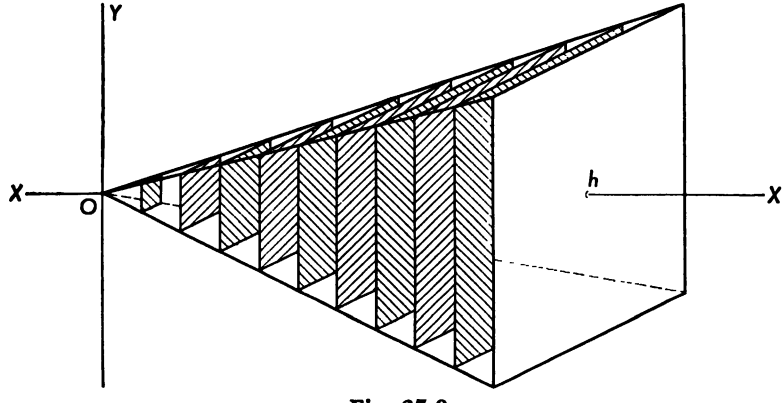


Fig. 27·9

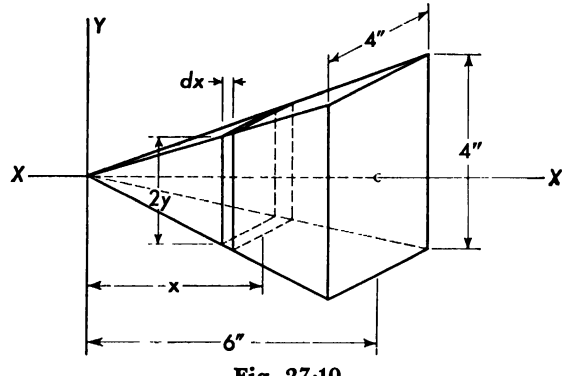


Fig. 27·10

lar solid of area A and thickness Δx. The volume of each slab is then, approximately,

$$\Delta V \approx A(\Delta x) \tag{147}$$

The sum of all such elemental volumes is

$$\underset{0\to h}{V} = \sum_{0}^{h} A_1\,\Delta x + A_2\,\Delta x + A_3\,\Delta x + \cdots + A_n\,\Delta x \tag{148}$$

As $n \to \infty$, the sum of the volumes of all these slabs approaches the volume of the pyramid, and from Sec. 27·10,

$$\int_0^h A\,dx = \lim_{n\to\infty} \Sigma\, A_1\,\Delta x + A_2\,\Delta x + A_3\,\Delta x + \cdots + A_n\,\Delta x \tag{149}$$

The general formula for calculating the volume of a solid is

$$V = \int A_s \, dx \qquad \star (150)$$

where A_s = cross-section area of the solid perpendicular to an axis
x = distance from apex measured along the axis

In setting up problems for finding the volume of solids by integration, we usually abbreviate drawings like Fig. 27·9 to something like Fig. 27·10. Here we show only one of the elemental volumes. Often we designate the thickness of the elemental volume by the symbol dx rather than Δx. This is similar to the analogous case dealing with areas mentioned in Sec. 27·10.

EXAMPLE 32. Find the volume of the square pyramid shown in Fig. 27·10. Following Eq. (150),

$$V = \int_0^6 A_s \, dx \qquad (151)$$

In this case

$$A_s = (2y)^2$$

but by similarity

$$\frac{2y}{x} = \frac{4}{6}$$

and

$$2y = \tfrac{4}{6}x = \tfrac{2}{3}x$$

Therefore

$$A_s = \tfrac{4}{9}x^2$$

and

$$V = \int_0^6 \frac{4}{9} x^2 \, dx = \left[\frac{4x^3}{27}\right]_0^6 = \frac{(4)(6)^3}{27} = \frac{(4)(216)}{27} = 32 \qquad (152)$$

By elementary geometry

$$V = \tfrac{1}{3} \times \text{area of the base} \times \text{height} = \tfrac{1}{3}(4 \times 4) \times 6 = 32 \qquad \text{(check)}$$

In the following examples we shall find the volume of a hemisphere 10 in. in radius.

There are actually several different ways that this can be accomplished. All these ways involve dividing the hemisphere into certain elementary volumes and finding the limit which the total of these volumes approaches as the number of the volumes approaches infinity.

EXAMPLE 33. Find the volume of the hemisphere shown in Fig. 27·11. We shall divide the hemisphere into circular slabs with a finite thickness. One of these circular slabs is shown in the figure.

The volume of each circular slab is *approximately* the volume of a cylinder Δh thick and with a base area of πr^2. The volume of each slab is then, approximately,

$$\Delta V \approx \pi r^2 \, \Delta h \qquad (153)$$

The sum of all such elemental volumes is approximately the volume of the hemisphere; that is,

$$\underset{0\to h}{V} \approx \sum_{0}^{10} \pi r_1^2\,\Delta h + \pi r_2^2\,\Delta h + \pi r_3^2\,\Delta h + \cdots + \pi r_n^2\,\Delta h \qquad (154)$$

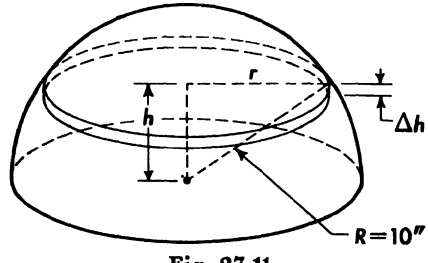


Fig. 27·11

As $n \to \infty$, the sum of the volumes of all these elemental disks approaches the volume of the hemisphere, and from Sec. 27·10

$$V = \lim_{n\to\infty} \sum_{0}^{10} \pi r_1^2\,\Delta h + \pi r_2^2\,\Delta h + \pi r_3^2\,\Delta h + \cdots + \pi r_n^2\,\Delta h \qquad (155)$$

$$= \int_0^{10} \pi r^2\,dh \qquad (156)$$

But in this figure

$$r^2 = 10^2 - h^2 \qquad (157)$$

Therefore, from Eq. (156),

$$V = \int_0^{10} \pi(10^2 - h^2)\,dh = \int_0^{10} (100\pi - \pi h^2)\,dh \qquad (158)$$

$$= \left[100\pi h - \frac{\pi h^3}{3}\right]_0^{10} = 1{,}000\pi - \frac{1{,}000\pi}{3} = \frac{2{,}000\pi}{3}$$

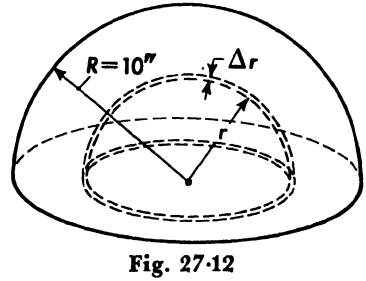


Fig. 27·12

EXAMPLE 34. Find the volume of the hemisphere shown in Fig. 27·12. We shall divide the hemisphere into hollow hemispherical shells. One of these shells is shown in the figure.

The volume of each shell is *approximately* the surface of the shell times the thickness of the shell. The volume of each shell is, then, approximately

$$2\pi r^2\,\Delta r \tag{159}$$

The sum of all such elemental volumes is approximately the volume of the hemisphere; that is,

$$V \approx \sum_{0}^{10} 2\pi r_1^2\,\Delta r + 2\pi r_2^2\,\Delta r + 2\pi r_3^2\,\Delta r + \cdots + 2\pi r_n^2\,\Delta r \tag{160}$$

As $n \to \infty$, the sum of the volumes of all these shells approaches the volume of the hemisphere, and from Sec. 27·10,

$$V = \lim_{n\to\infty} \sum_{0}^{10} 2\pi r_1^2\,\Delta r + 2\pi r_2^2\,\Delta r + 2\pi r_3^2\,\Delta r + \cdots + 2\pi r_n^2\,\Delta r \tag{161}$$

$$= \int_0^{10} 2\pi r^2\,dr \tag{162}$$

$$= \int_0^{10} 2\pi r^2\,dr = \left[\frac{2\pi r^3}{3}\right]_0^{10} = \frac{2\pi 1{,}000}{3} = \frac{2{,}000\pi}{3} \tag{163}$$

EXAMPLE 35. Find the volume of the hemisphere shown in Fig. 27·13. We shall divide the hemisphere into hollow tubes. One of these hollow tubes is shown in the figure.

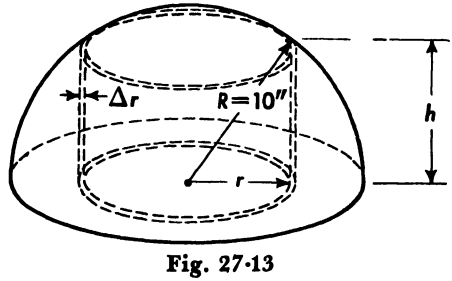


Fig. 27·13

The volume of each hollow tube is *approximately* the volume of a hollow tube with a height h and a wall thickness Δr. The volume of each hollow tube is then approximately

$$2\pi r h\,\Delta r \tag{164}$$

The sum of all such elemental volumes is approximately the volume of the hemisphere; that is,

$$\underset{0\to r}{V} \approx \sum_{0}^{10} 2\pi r_1 h_1\,\Delta r + 2\pi r_2 h_2\,\Delta r + 2\pi r_3 h_3\,\Delta r + \cdots + 2\pi r_n h_n\,\Delta r \tag{165}$$

As $n \to \infty$, the sum of the volumes of all these hollow cylinders approaches the volume of the hemisphere, and from Sec. 27·10,

$$V = \lim_{n \to \infty} \sum_{0}^{10} 2\pi r_1 h_1 \Delta r + 2\pi r_2 h_2 \Delta r + 2\pi r_3 h_3 \Delta r + \cdots + 2\pi r_n h_n \Delta r$$

$$= \int_0^{10} 2\pi r h \, dr \tag{166}$$

But

$$h = \sqrt{100 - r^2} \tag{167}$$

Therefore

$$V = \int_0^{10} 2\pi r \sqrt{100 - r^2} \, dr = \int_0^{10} 2\pi r (100 - r^2)^{1/2} \, dr \tag{168}$$

$$= 2\pi \int_0^{10} r(100 - r^2)^{1/2} \, dr = 2\pi [-\tfrac{1}{3}(100 - r^2)^{3/2}]_0^{10} \tag{169}$$

$$= -\frac{2\pi}{3}[-(100)^{3/2}] = \frac{2{,}000\pi}{3} \tag{170}$$

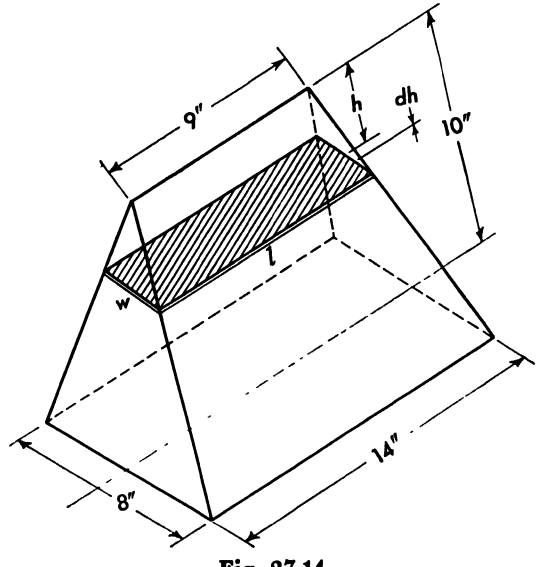


Fig. 27·14

EXAMPLE 36. Find the volume of the wedge illustrated in Fig. 27·14. We shall consider the wedge to be made up of a stack of horizontal sections of width w, length l, and thickness dh. Accordingly,

$$V = \int_0^{10} wl \, dh$$

Since w is at all times proportional to h and is 8 when $h = 10$, we may replace w by $0.8h$. l is a linear function of h, and since l is 9 when $h = 0$ and l is 14 when $h = 10$, it can be readily shown that $l = 9 + 0.5h$.

Therefore we may write

$$V = \int_0^{10} 0.8h(9 + 0.5h)\,dh$$

or
$$V = \int_0^{10} (7.2h + 0.4h^2)\,dh$$

and
$$V = [3.6h^2 + \tfrac{2}{15}h^3]_0^{10} = 360 + 133.3 = 493.3$$

Checking by the prismoidal formula, we obtain

$$V = \tfrac{1}{6}(10)[0 + 4(4 \times 11.5) + (8 \times 14)] = \tfrac{10}{6}(0 + 184 + 112) = 493.3$$

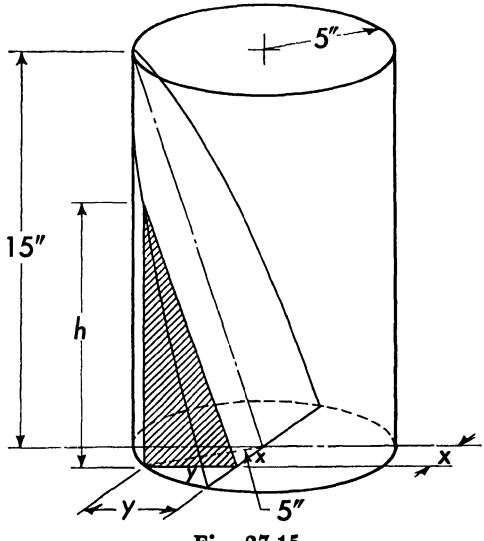


Fig. 27·15

EXAMPLE 37. Find the volume of the wedge cut from the cylinder in Fig. 27·15. If we divide the volume into many vertical right-triangular slabs of thickness dx, we obtain

$$V = 2\int_0^5 \tfrac{1}{2}hy\,dx$$

(Note that we could just as logically have written $V = \int_{-5}^{5} \tfrac{1}{2}hy\,dx$, but the first method is shorter.)

In each triangular element $h = 3y$, or the area of the elemental section is

$$\tfrac{1}{2}hy = \tfrac{3}{2}y^2$$

But

$$y^2 = 25 - x^2$$

Therefore the area of the elemental section is

$$\text{Area} = \tfrac{3}{2}(25 - x^2)$$

Hence we may write

$$V = 2\int_0^5 3\pi(25 - x^2)\,dx = 3\int_0^5 (25 - x^2)\,dx = 3\left[25x - \frac{x^3}{3}\right]_0^5$$

Therefore

$$V = 3[125 - 125/3] = 250$$

EXAMPLE 38. Find the volume common to two cylinders (see Fig. 27·16), each of radius 3 in., whose axes intersect at right angles.

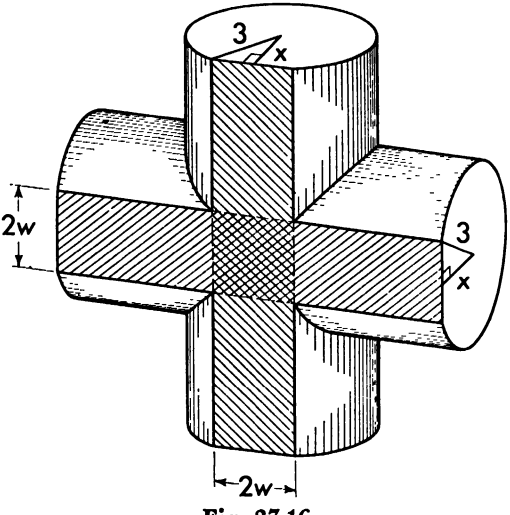


Fig. 27·16

It will be seen that the volume of one of the elemental slabs making up the volume of intersection is

$$\Delta V = (2w)^2\,dx$$

But

$$w = \sqrt{9 - x^2}$$

or

$$4w^2 = 36 - 4x^2$$

Hence

$$V = 2\int_0^3 (36 - 4x^2)\,dx = 2[36x - \tfrac{4}{3}x^3]_0^3 = 2\{[(36)(3) - \tfrac{4}{3}(3)^3] - [0]\}$$
$$= 2(108 - 36) = 144 \text{ cu in.}$$

EXERCISE 11

1. Find, by integration, the volume of a sphere 20 in. in radius.
2. A plane is passed through the sphere in Prob. 1, 10 in. from the center. Find the smaller and the larger volumes cut off.
3. Water is poured from a cylindrical cup 5 in. in diameter and 5 in. high. Find the volume of water remaining when the surface of the water coincides with a diameter of the bottom of the cup.

4. A cylindrical tank is mounted with its axis horizontal. The tank is 15 ft in diameter and 40 ft long. How many gallons of liquid are there in the tank if the surface of the liquid is 5 ft from the bottom? (See Prob. 3, page 709.)
5. A cylindrical block of wood 10 in. in radius has a 20-in. diameter hole bored in it such that the axis of the hole and the axis of the cylinder intersect at right angles. Find the volume of the material bored out.

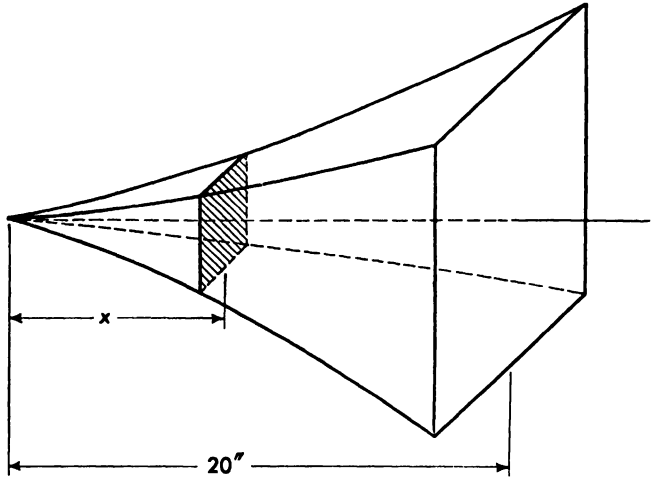


Fig. 27·17

6. Every cross section of the solid shown in Fig. 27·17 is a square whose sides are $0.2x^2$. Find the volume of the solid.
7. A metal sphere 10 in. in radius has a 12-in.-diameter hole bored in it. The axis of the hole passes through the center of the sphere. Find the volume of material removed.

Appendix A[1] COMPUTATION AIDS AND APPROXIMATIONS

A·1 Aids to Computation. In this section we shall attempt to outline some of the more commonly useful short cuts in computation. With the exception of abbreviated multiplication and abbreviated division, these methods are in no sense approximations. They are intended to serve as a means of obtaining the correct answer at a saving of time and work.

Since most short cuts and approximations are soon forgotten unless constantly used, their treatment in this section is not intended to be complete but should indicate what can be done as the need arises. The particular need of the student will depend upon his individual situation.

Division may be checked by multiplying the divisor by the quotient and to the resulting product adding the remainder to get the original dividend.

A·2 Aids to Addition and Subtraction. Addition may be facilitated by adding digits two or more at a time, since the order of addition has no effect on the final answer.

EXAMPLE 1. Add $\underbrace{8+2}_{10}+\underbrace{5+7}_{12}+\underbrace{6+3+2+9}_{20}=42.$

Addition by complements may be illustrated by the addition of 237 and 98. It should be arranged as follows: $237 + 100 - 2 = 335$.

EXAMPLE 2. The method of addition by complements lends itself to the addition of the following typical charges on a department-store bill:

$$\begin{aligned}
\$\ 2.97 &= \$\ 3 \text{ less } \$0.03\\
6.95 &= \quad 7 \text{ less } \ 0.05\\
4.98 &= \quad 5 \text{ less } \ 0.02\\
11.97 &= \ 12 \text{ less } \ 0.03\\
1.99 &= \quad 2 \text{ less } \ 0.01\\
&\overline{\$29 \text{ less } \$0.14} = \$28.86
\end{aligned}$$

[1] Appendixes A, B, and C contain material which is useful for reference, but they are not intended to be used for routine class assignment.

Addition may often be most conveniently checked by reversing the order of addition. Subtraction by complements is often found useful.

For example, $456 - 197 = 456 - (200 - 3) = 456 - 200 + 3 = 259$.

The Austrian, or "making-change," method is commonly used as a means of subtraction. Here we are looking for the number which must be added to the subtrahend to equal the minuend.

For example, subtract 8,379 from 15,584 (see Example 7 below).

15584	$9 + \mathbf{5} = 14$, carry 1
8379	$8 + \mathbf{0} = 8$
7205	$3 + \mathbf{2} = 5$
	$8 + \mathbf{7} = 15$

The digits in boldface type are written down as the remainder.

A·3 Short Cuts in Multiplication. Multiplication tables might be extended to advantage at least to the square of 16 and preferably to the square of 20.

Some short cuts in multiplication will at once be apparent to the student. A few of these are given below.

1. *To multiply by* 25, *divide by* 4 *and point off two places to the right, for* $n \times 25 = n \times 100/4$.

2. *To multiply by* $33\frac{1}{3}$, *divide by* 3 *and point off two places to the right.*

3. *To multiply by* 50, *divide by* 2 *and point off two places to the right.*

4. *To multiply by* 75, *take* $\frac{3}{4}$ *of the number and point off two places to the right.*

For example, multiply 16.8×75 (see Example 7 below).

$$\begin{array}{r} 4/16.8 \\ \underline{\quad 4.2} \\ 12.60 \end{array} = 1{,}260$$

5. *To multiply by* 125, *take* $\frac{1}{8}$ *of the number and point off three places to the right.*

6. *To multiply by* 250, *take* $\frac{1}{4}$ *of the number and point off three places to the right.*

7. *To multiply by* 500, *take* $\frac{1}{2}$ *of the number and point off three places to the right.*

8. *Multiplication by numbers near* 10, 100, *etc., may be effected by the method of complements.*

For example,

$$42 \times 98 = 42(100 - 2) = 4{,}200 - 84 = 4{,}116$$
$$9 \times 64 = 64(10 - 1) = 640 - 64 = 576$$

Since the order of multiplication is immaterial, it may sometimes be reversed to advantage.

EXAMPLE 3. The multiplication 476×588 is best performed using 588 as the multiplier.

$$\begin{array}{r} 476 \\ 588 \\ \hline 3808 \\ 3808 \\ 2380 \\ \hline 279888 \end{array}$$

(Second line may be copied from the first)

EXAMPLE 4. Multiply $2{,}024 \times 3{,}542$, using 3,542 as the multiplier.

$$\begin{array}{r} 2024 \\ 3542 \\ \hline 4048 \\ 8096 \\ 10120 \\ 6072 \\ \hline 7169008 \end{array}$$

The zero in 2,024 allows the multiples of 2 and 24 to be written down at sight. For example, $4 \times 2{,}024 = 8{,}096$, $3 \times 2{,}024 = 6{,}072$, etc.

EXAMPLE 5. $25 \times 5 \times 24 = 25 \times 120 = 3{,}000$

EXAMPLE 6. $75 \times 17 \times 12 = 17 \times 75 \times 12 = 17 \times 900 = 15{,}300$

*Simple compact arrangements for taking certain fractions of numbers are given.

EXAMPLE 7. Multiply $\frac{2}{3} \times 147$.

$$\begin{array}{r} 3\overline{)147} \\ 49 \\ \hline 98 \end{array} \quad \text{(subtract)}$$

EXAMPLE 8. Multiply $\frac{7}{8} \times 344$.

$$\begin{array}{r} 8\overline{)344} \\ 43 \\ \hline 301 \end{array} \quad \text{(subtract)}$$

*L. Meyers, "High Speed Mathematics," D. Van Nostrand Co., Inc., 1947.

EXAMPLE 9. Multiply $1\frac{1}{4} \times 112$.

$$\begin{array}{r} 4\overline{)112} \\ 28 \\ \hline 140 \end{array} \qquad \text{(add)}$$

There are many special arrangements that may be devised for calculating certain products. Just a few are illustrated below.

EXAMPLE 10. $0.22\frac{1}{2} \times 5.6 = 4\overline{)5.6}$

$$\text{(subtract)} \quad \begin{cases} 1.4 \\ 0.14 \\ \hline 1.26 \end{cases} \qquad 5.6[\tfrac{1}{4} - \tfrac{1}{10}(\tfrac{1}{4})]$$

EXAMPLE 11. Take $17\frac{1}{2}\%$ of 440.

$$\text{(add)} \quad \begin{cases} 2\overline{)44.0} \\ 2\overline{)22.} \\ 11. \\ \hline 77.0 \end{cases} \qquad 440\{\tfrac{1}{10} + \tfrac{1}{2}[\tfrac{1}{10} + \tfrac{1}{2}(\tfrac{1}{10})]\}$$

Many other combinations will occur to the student to meet his particular need.

Some multiplications are facilitated by a preliminary multiplication and division by the same quantity.

EXAMPLE 12. Multiply $21\frac{1}{2}$ by 16. If we first double $21\frac{1}{2}$ and halve 16, we get $43 \times 8 = 344$, which is easily done as an oral exercise.

EXAMPLE 13.

$$6\tfrac{2}{3} \times 84 = (6\tfrac{2}{3})(3) \times (84)(\tfrac{1}{3}) = 20 \times 28 = 560$$

EXAMPLE 14.

$$175 \times 360 = (175)(4) \times (360)(\tfrac{1}{4}) = 700 \times 90 = 63{,}000$$

Mixed numbers are often best multiplied by arranging as the product of one binomial by another (see Sec. 5·13).

EXAMPLE 15. Multiply $8\frac{1}{4} \times 6\frac{5}{16}$.

$$(6)(8) + (6)(\tfrac{1}{4}) + (\tfrac{5}{16})(8) + (\tfrac{5}{16})(\tfrac{1}{4})$$
$$= 48 + 1\tfrac{1}{2} + 2\tfrac{1}{2} + \tfrac{5}{64} = 52\tfrac{5}{64}$$

This method is especially useful where large mixed numbers are involved.

A·4 Abbreviated Multiplication. This is a rapid method of obtaining a product if it is to contain no more significant figures than the multiplier or multiplicand.

Round off both multiplier and multiplicand to one more significant figure than desired in the product. Multiply the multiplicand by the left-hand digit in the multiplier. Round off the right-hand digit of the multiplicand, and multiply by the second digit from the left in the multiplier. Continue using successive digits of the multiplier and rounding off digits in the multiplicand. Keep the right-hand digits of the product in line, and determine the position of the decimal point by inspection.

EXAMPLE 16. Multiply 385.216 by 0.17278. The complete product, obtained in the usual way, is 66.55762048. Assume four significant figures are wanted in the product.

38522		
17278		
38522	(1 × 38522)	
26964	(7 × 3852)	
770	(2 × 385)	
273	(7 × 39)	
32	(8 × 4)	
66.561	or 66.56	(correct to four significant figures)

Somewhat greater accuracy can be obtained by attempting to compensate for discarded digits. For purposes of comparison we shall use the same product to illustrate this point.

38522		
17278		
38522	(1 × 38522)	
26965	(7 × 3852) + 1	(adding 1 to compensate for the discarded digit 2)
770	(2 × 385)	
270	(7 × 38) + 4	(adding 4 to compensate for the discarded digits 522)
31	(8 + 3) + 7	(adding 7 to compensate for the discarded digits 8,522)
66.558		

Multiplication may be checked by interchanging the multiplier and the multiplicand.

A·5 Short Division. Time and effort can be saved by extending the process of short division to two-digit divisors. For instance, the

division of 2,148 by 16 can be thought out about as fast as it can be written down.

$$16/\overline{2148}$$
$$\underline{\quad}$$
$$134\tfrac{1}{4}$$

In dividing one mixed number by another, the process is simplified by multiplying the numerator group and the denominator group by the least common multiple of the secondary denominators.

EXAMPLE 17

$$\frac{8\frac{3}{4}}{3\frac{1}{2}} = \frac{35}{14} \qquad \text{(multiplying top and bottom by 4)}$$
$$= 2\tfrac{1}{2}$$

EXAMPLE 18

$$\frac{3\frac{1}{3}}{7\frac{1}{2}} \times \frac{6}{6} = \frac{20}{45} = \frac{4}{9}$$

Where the denominator is an integer, the numerator may be divided into two parts, one of which is the largest possible integral multiple of the denominator.

EXAMPLE 19

$$\frac{13\frac{1}{2}}{4} = \frac{12 + 1\frac{1}{2}}{4} = 3 + \tfrac{3}{8} = 3\tfrac{3}{8}$$

EXAMPLE 20

$$\frac{17\frac{1}{4}}{3} = \frac{15 + 2\frac{1}{4}}{3} = 5 + \tfrac{3}{4} = 5\tfrac{3}{4}$$

A·6 Abbreviated Division. This is a relatively fast method of obtaining a quotient containing a certain number of significant figures.

1. *The dividend and divisor are each rounded off to one more significant figure than is to be contained in the quotient. It may be necessary to retain still another digit in the dividend in order to make it larger than the divisor. Decimal points are disregarded during the operation, the position of the decimal point in the quotient being determined by inspection.*

2. *The first division is made using the entire divisor. Thereafter each successive division is made after rounding off successive digits from the right-hand side of the divisor.*

EXAMPLE 21. Divide 32,586.70143 by 481.60732, carrying out the division to five significant figures in the quotient. Division carried out in the usual manner gives an answer of 67.662388 to eight figures. In the abbreviated division the dividend and divisor are rounded off to seven and six figures, respectively.

$$
\begin{array}{r}
6 \\
481607\overline{)3258670} \\
2889642 \\
48161\overline{)369028}/7 \\
337127 \\
4816\overline{)31901}/6 \\
28896 \\
482\overline{)3005}/6 \\
2892 \\
48\overline{)113}/2 \\
96
\end{array}
$$

ANS.: 67.662

A·7 Least Common Multiple

The least common multiple (LCM) of several numbers may be determined by arranging the numbers in a line and dividing by 2, or the lowest prime number which is contained in at least two of the original numbers. When division by the first prime is exhausted, we pass to the next higher prime number which is contained in two or more of the line of quotients. In any case where division is not even, the number is repeated. The process is continued until there are no longer two or more quotients divisible by a common prime. The least common multiple is the product of all the divisors on the left and quotients remaining in the bottom line.

To illustrate, find the LCM of 75, 50, 12, 42, and 105.

$$
\begin{array}{r}
2/75;\ 50;\ 12;\ 42;\ 105 \\
3/75;\ 25;\ \ 6;\ 21;\ 105 \\
5/25;\ 25;\ \ 2;\ \ 7;\ \ 35 \\
5/\ 5;\ \ 5;\ \ 2;\ \ 7;\ \ \ 7 \\
7/\ 1;\ \ 1;\ \ 2;\ \ 7;\ \ \ 7 \\
1;\ \ 1;\ \ 2;\ \ 1;\ \ \ 1
\end{array}
$$

The LCM $= 2 \times 3 \times 5 \times 5 \times 7 \times 2 = 2{,}100$.

A·8 The Greatest Common Divisor

The greatest common divisor (GCD) of several numbers is obtained by using the process for LCM, except that the entire row must be divisible. The operation ends when the entire row is no longer divisible by a given prime. The product of the primes used as divisors is the GCD.

EXAMPLE 22. Find the GCD of 540, 756, and 72.

$$
\begin{array}{r}
2/540;\ 756;\ 72 \\
2/270;\ 378;\ 36 \\
3/135;\ 189;\ 18 \\
3/\ 45;\ \ 63;\ \ 6 \\
15;\ \ 21;\ \ 2
\end{array}
$$

GCD $= 2 \times 2 \times 3 \times 3 = 36$.

A·9 Fractional Equivalents. Table A·1 lists the percentage equivalents of common fractions.

Table A·1

	Percentage
1/16	6¼
1/12	8⅓
1/8	12½
1/6	16⅔
3/16	18¾
1/4	25
5/16	31¼
1/3	33⅓
3/8	37½
5/12	41⅔
7/16	43¾
1/2	50
9/16	56¼
7/12	58⅓
5/8	62½
2/3	66⅔
11/16	68¾
3/4	75
13/16	81¼
5/6	83⅓
7/8	87½
11/12	91⅔
15/16	93¾

Knowledge of these equivalents is frequently useful, as the following common types of problems illustrate.

EXAMPLE 23. Find the amount of pay for 60 hours' work at \$1.62½.

$$60 \times 1\tfrac{5}{8} = 60 \times 1\tfrac{3}{8} = \$97\tfrac{1}{2}$$

EXAMPLE 24. Find the discount at 18¾ per cent on \$24.

$$24 \times \tfrac{3}{16} = \tfrac{9}{2} = \$4\tfrac{1}{2}$$

EXAMPLE 25. Find the increase in cost of 150 saws if the unit price rose from \$5.50 to \$6.33⅓.

$$150 \times (6.33\tfrac{1}{3} - 5.50) = 150 \times 0.83\tfrac{1}{3} = 150 \times \tfrac{5}{6} = \$125$$

A·10 Short Methods of Finding Squares and Square Roots. Integers and mixed numbers may be squared easily by a process based on the algebraic identity

$$a^2 - b^2 = (a + b)(a - b)$$

Transposing,

$$a^2 = (a + b)(a - b) + b^2$$

EXAMPLE 26. Find $(96)^2$. Here $a = 96$, and a value of b is chosen such that either $a + b$ or $a - b$ becomes a round number facilitating the computation. In this case we choose $b = 4$. Then $(96)^2 = (96 + 4)(96 - 4) + 4^2 = (100)(92) + 16 = 9{,}216$

EXAMPLE 27. Find $(53)^2$.

$$a = 53; \text{ take } b = 3$$

Then

$$(53)^2 = (53 + 3)(53 - 3) + 3^2 = (56)(50) + 9 = 2{,}809$$

EXAMPLE 28. Find $(13\frac{1}{2})^2$.

$$a = 13\frac{1}{2}; \text{ take } b = \frac{1}{2}$$

Then

$$(13\frac{1}{2})^2 = (13\frac{1}{2} + \frac{1}{2})(13\frac{1}{2} - \frac{1}{2}) + (\frac{1}{2})^2 = (14)(13) + \frac{1}{4} = 182\frac{1}{4}$$

This may be generalized into the following rule:

To square a mixed number ending in $\frac{1}{2}$, multiply the next higher whole number by the next lower whole number and add $\frac{1}{4}$.

EXAMPLE 29. Find $(135)^2$.

$$a = 135; \text{ take } b = 5$$

Then

$$\begin{aligned}(135)^2 &= (135 + 5)(135 - 5) + (5)^2 \\ &= (140)(130) + 25 = 18{,}225\end{aligned}$$

The procedure for the above calculation may be stated thus:

To square a number ending in 5, *multiply the number of tens appearing by the same number of tens plus one and annex* 25 *to the result.*

EXAMPLE 30. Find $(7\frac{1}{4})^2$.

$$a = 7\frac{1}{4}; \text{ take } b = \frac{3}{4}$$

Then

$$\begin{aligned}(7\frac{1}{4})^2 &= (7\frac{1}{4} + \frac{3}{4})(7\frac{1}{4} - \frac{3}{4}) + (\frac{3}{4})^2 \\ &= (8)(6\frac{1}{2}) + \frac{9}{16} = 52\frac{9}{16}\end{aligned}$$

EXAMPLE 31. Find $(8\frac{1}{3})^2$.

$$a = 8\frac{1}{3}; \text{ take } b = \frac{2}{3}$$

Then

$$\begin{aligned}(8\frac{1}{3})^2 &= (8\frac{1}{3} + \frac{2}{3})(8\frac{1}{3} - \frac{2}{3}) + (\frac{2}{3})^2 \\ &= (9)(7\frac{2}{3}) + \frac{4}{9} = 69\frac{4}{9}\end{aligned}$$

A·11 Short Cut in the Use of the Pythagorean Theorem. When the sides of a right triangle are given in the common binary system of linear measurement (for example, ½ in., ¼ in., ⅛ in., etc.), the work involved in the application of the Pythagorean theorem may be reduced by taking as a unit of length $1/n$, where n is the least common denominator.

EXAMPLE 32. Find the hypotenuse of a right triangle whose sides are $1\frac{5}{8}$ and $2\frac{3}{16}$ in. Since 16 is the LCD, we shall take $\frac{1}{16}$ in. as the unit of length. Then

$$1\frac{5}{8} = 26 \text{ units}$$

and

$$2\frac{3}{16} = 35 \text{ units}$$

and

$$\sqrt{26^2 + 35^2} = 43.60 \text{ units}$$

$$\frac{43.60}{16} = 2.725 \text{ in.}$$

If the nearest $\frac{1}{64}$ in. is desired,

$$\frac{43.60}{16} = \frac{x}{64}$$

$$x = 174.4 \qquad \text{(call 174)}$$

$$\frac{174}{64} = \frac{87}{32} = 2\frac{23}{32} \text{ in.}$$

The process of calculating square root may be shortened by obtaining the first half of the significant digits in the root by the usual means and the remaining digits in the root by abbreviated division.

EXAMPLE 33. Find $\sqrt{8813476.5625}$.

The answer, obtained in the usual way, is 2,968.75. By the short method,

$$\begin{array}{rl}
 & 2\ 9\ 6\ . \\
 & \sqrt{8813476.5625} \\
 & \underline{4} \\
 & 481 \\
49 \times 9 \rightarrow & \underline{441} \\
 & 4034 \\
586 \times 6 \rightarrow & \underline{3516} \\
2 \times 296 \rightarrow & 592\,/\,5188\,/\,8 \\
 & \underline{4736} \\
 & 59\,/\,452\,/\,7 \\
 & \underline{413} \\
 & 6\,/\,39\,/\,7
\end{array}$$

Approx. root = 2,968.77.

A·12 Suggestions for Cancellation. Cancellation involving very large or very small numbers is more easily accomplished by making use of scientific notation.

The average student recognizes numbers which contain such factors as 2, 4, 5, and 10. Further, if the digits of a number add up to a sum divisible by 3, the number itself is divisible by 3. Similarly, if the digits add up to a sum divisible by 9, the number itself is divisible by 9.

Occasionally it will be advisable not to cancel completely. For example, 1,872/400 should be reduced only to 468/100 instead of completely to 117/25, as the answer 4.68 is at once apparent in the former expression.

Large factors may be revealed by factoring out smaller quantities, even though it can be seen that the smaller factors will not cancel out. A typical example is

$$\frac{141}{235} = \frac{3 \times 47}{5 \times 47} = \frac{3}{5}$$

In making computations involving both multiplication and division, cancel as much as feasible. Then carry out the necessary multiplication, followed by division.

A·13 Averaging by Averaging Departures. If a series of numbers shows but little fluctuation, the average value may be readily obtained by averaging the departures from an arbitrarily chosen reference base.

EXAMPLE 34. Average the following set of numbers, assuming a reference base of 18.60.

	Departure
18.62	+2
18.65	+5
18.59	−1
18.60	0
18.64	+4
18.57	−3
18.56	−4
18.62	+2
	+5

$$\frac{0.05}{8} = +0.006$$

The average = 18.60 + 0.006, or 18.61.

A·14 Absolute and Relative Errors. The *absolute error* is the approximate value minus the true value of a number. It is positive or

negative according to whether the approximate value is larger or smaller than the true value.

The *relative error* is the ratio of the absolute error to the exact value. Since the relative error is a ratio between two like quantities, it is an abstract number and is often expressed as a percentage.

EXAMPLE 35. The actual length of a metal bar is 11.52 in. The length as measured with a scale with a worn end is 11.56 in. Find the actual and relative errors.

1. The actual error $= 11.56 - 11.52 = 0.04$ in.
2. The relative error $= \dfrac{11.56 - 11.52}{11.52} = 0.00347 = 0.347$ per cent.

A·15 Absolute Error of a Sum. *The absolute error of a sum is the sum of the absolute errors of the numbers.* By rounding off the numbers in such a way as to make the positive and negative absolute errors tend to compensate for each other, the absolute error of the sum will be reduced in size.

EXAMPLE 36. Add 6.7295, 0.5436, 7.81752, 0.0058, 681.132, and 23.6836. The sum is to be correct to two decimal places.

$$\begin{array}{r} 6.730\text{(raised)} \\ 0.544\text{(raised)} \\ 7.817\text{(lowered)} \\ 0.006\text{(raised)} \\ 681.132 \\ 23.683\text{(lowered)} \\ \hline 719.912 \end{array}$$

The sum correct to two decimal places is 719.91.

If we add several numbers, each of equal precision, the precision of the sum is decreased according to the number of quantities added. For example, if we were to add ten numbers, each correct to 0.01, the maximum possible error in each number is ± 0.005. The maximum possible error in the sum is ten times as large, or ± 0.05. Therefore the sum should be rounded to the nearest 0.1.

A·16 Absolute Error of a Difference.

The absolute error of the difference of two rounded numbers is equal to the algebraic difference of their absolute errors.

Again, rounding off should be done in such a way as to introduce compensating errors.

EXAMPLE 37. Subtract 13.85653 from 21.27238. The difference is to be correct to two decimal places.

$$\begin{array}{l} 21.272\text{(lowered)} \\ 13.856\text{(lowered)} \\ \hline 7.416 \qquad \text{(call 7.42)} \end{array}$$

It will be noted in performing a subtraction that the absolute error of the difference is usually greater, and the relative error of the difference often much greater, than that of the two original numbers.

In general, computation with rounded numbers should be performed to an accuracy or precision of at least one place better than that desired in the result.

A·17 Relative Error of a Product.

The relative, or percentage, error of the product of two rounded numbers is approximately the algebraic sum of the relative errors of the multiplier and the multiplicand.

If one of these numbers is exact and the other approximate, the relative error in the product will be equal to the relative error in the approximate number.

EXAMPLE 38. Multiply the numbers 30.62584 and 8.23947, rounding off so that the product will not be in error by more than 0.01.

The product will evidently be about 250. $\frac{0.01}{250} \approx 0.004$ per cent relative error. (The symbol $\approx$ represents "approximately equals.") If we round off 30.62584 to 30.625, the relative error is $\frac{-84}{3{,}062{,}584}$, or about -0.0028 per cent. If we round off 8.23947 to 8.240, the relative error is $\frac{53}{823{,}947}$, or about $+0.0064$ per cent. The relative error in the approximate product should, therefore, be about -0.0028 per cent $+$ 0.0064 per cent, or slightly less than $+0.004$ per cent. This is verified by comparing the product of the original numbers, which is 252.3406899048, and the product of the rounded numbers, which is 252.35. This is within the allowable error. Note that slide-rule computation of relative errors is entirely adequate.

A·18 Relative Error of a Quotient

The relative error of the quotient of two rounded numbers is approximately the algebraic difference obtained by subtracting the relative error in the divisor from the relative error in the dividend.

EXAMPLE 39. Divide 574.62713 by 8.957241, obtaining a quotient correct to within 0.01 per cent.

Rounding off 574.62713 to 574.63, the relative error is $\frac{287}{57{,}462{,}713} \approx +0.0005$ per cent. Rounding off 8.957241 to 8.957, the relative error is $\frac{-241}{8{,}957{,}241} \approx -0.0027$ per cent. The relative error in the approximate quotient should be $+0.0005$ per cent $-$ (-0.0027 per cent), or slightly over $+0.003$ per cent. This is confirmed by comparing the division of the original numbers and that of the rounded numbers, the quotients, to six significant figures, being 64.1522 and 64.1543, respectively.

A·19 Relative Error of a Power

The relative error of a power of a rounded number is approximately the relative error of the number multiplied by the degree of the power.

EXAMPLE 40. Compute the value of $(5.761165)^3$, the result to be accurate within 0.01 per cent.

The allowable error in the original number should then not exceed $\frac{0.01}{3}$, or 0.003 per cent. Round off 5.761165 to 5.761; the result is a relative error of $\frac{-165}{5761165}$, or -0.003 per cent, which is within the limit allowed. Checking, $(5.761165)^3 = 191.2189$, whereas $(5.761)^3 = 191.2025$, each being calculated to seven significant figures. The relative error is $\frac{191.2025 - 191.2189}{191.2189}$ $= -0.009$ per cent.

A·20 Relative Error of a Root

The relative error of a root of a rounded number is approximately the relative error of the rounded number divided by the index of the root.

EXAMPLE 41. Find the percentage error involved in taking $\sqrt{335.82}$ as equal to $\sqrt{336}$.

The relative error in the rounded number is $\frac{0.18}{336}$, or 0.054 per cent. The relative error in the square root should approximate $\frac{0.054}{2}$, or 0.027 per cent. Checking, $\sqrt{335.82} = 18.325$, whereas $\sqrt{336} = 18.330$, or a relative error of 0.027 per cent.

A·21 Approximations. Some of the more common approximations are given in Table A·2 below. The third column contains the

term or terms which may be added to the first approximation to eliminate or reduce the error.

Table A·2

*Expression**	*First approximation*	*For closer approximation add*	*Notes*
I $\sqrt{mn}$	$\frac{m+n}{2}$	$-\frac{(\sqrt{m}-\sqrt{n})^2}{2}$†	The less the relative difference between m and n, the closer the approximation.
II $(1 \pm x)(1 + y)$	$1 \pm x + y$	$\pm xy$†	x, y, and z must range between 1 and -1. The nearer they approach zero, the closer the approximation.
$(1 \pm x)(1 - y)$	$1 \pm x - y$	$\mp xy$†	
$(1 + x)(1 + y)(1 + z)$	$1 + x + y + z$	$xy + xz + yz$	
$\frac{1+x}{1+y}$	$1 + x - y$	$y^2 - xy$	
$(1 \pm x)^m$	$1 \pm mx$	$\frac{m(m-1)x^2}{2}$	
$(1 \pm x)^2$	$1 \pm 2x$	x^2†	
$\sqrt[3]{1 \pm x}$	$1 \pm \frac{x}{3}$	$-\frac{x^2}{9}$	
$\frac{1}{1 \pm x}$	$1 \mp x$	x^2	
III $\sqrt{a^2 \pm b}$	$a \pm \frac{b}{2a}$	$-\frac{b^2}{8a^3}$	The smaller b is, relative to a, the closer the approximation.
$\sqrt{a^2 \pm b^2}$ $(b^2 < 2a)$	$a \pm \frac{b^2}{2a}$	$-\frac{b^4}{8a^3}$	
$\sqrt[3]{a^2 \pm b}$	$a \pm \frac{b}{3a^2}$	$-\frac{b^2}{9a^5}$	

* Each of these expressions may be substantiated by using the fundamental operations of algebra and the binomial theorem.

† Including this expression gives the exact value.

If any of the expressions in Group II be multiplied by a constant k, the approximation will also be multiplied by k.

EXAMPLE 42. $(4.014)^3 = 4^3(1.0035)^3 \approx 64[1 + 3(0.0035)] \approx 64.672$. True value to five significant figures $= 64.675$.

A·22 Use of Ordinary Graph Paper as a Means of Calculation. There are a number of commonly occurring computations which are readily solved graphically by means of ordinary cross-section paper. In general, the degree of accuracy is less than that of the 10-in. slide rule. The chief advantage lies in the saving of time and effort. The proof of the validity of each of these graphical solutions will be left as an exercise for the student.

Sum and Difference of Reciprocals

Applications immediately apparent are work problems, parallel resistances, and focal length of lenses. To solve the equation $1/a + 1/b = 1/c$ for c where a and b are given, draw a 45° line from the origin (see Fig. A·1). Draw another line connecting A on one scale with B on the other. The location of the point of intersection of these two lines, which is C, may be referred to either axis.[1]

If three or more reciprocals are to be added, as in the equation $1/a + 1/b + 1/d = 1/e$, the value of c as found above is connected with that of d on the opposite axis and the value e obtained. This

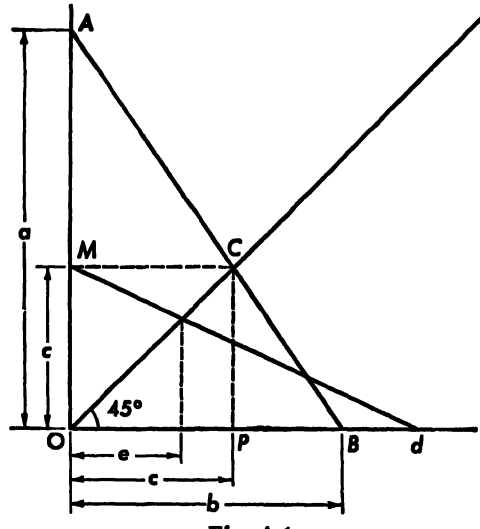


Fig. A·1

process can be continued for the addition of any number of reciprocals.

The problem $1/a = 1/c - 1/b$ can of course be solved by reversing the process. [Connect the points (B) and (C) to find (A).]

If the student finds much occasion to use this method, he would do well to have the 45° line already drawn and to locate point c by laying a straightedge over points a and b rather than drawing an actual line on the paper.

[1] Referring to Fig. A·1, it can be shown that right triangles AMC and CPB are similar and that therefore we may write the equation

$$\frac{a-c}{c} = \frac{c}{b-c}$$

The remainder of the proof is left as an exercise for the student.

Two-component Mixtures

The solution of a typical problem will perhaps best explain the procedure.

Find how many ounces of alloy *A* containing 42 per cent silver and how many ounces of alloy *B* containing 90 per cent silver must be mixed to make 40 oz of an alloy containing 60 per cent silver.

Draw two vertical axes 40 units apart (Fig. A·2) and locate the percentages 42 and 90, one on each axis, and connect with a straight line. Connect the point 60 per cent located on each axis with a horizontal line. The altitude

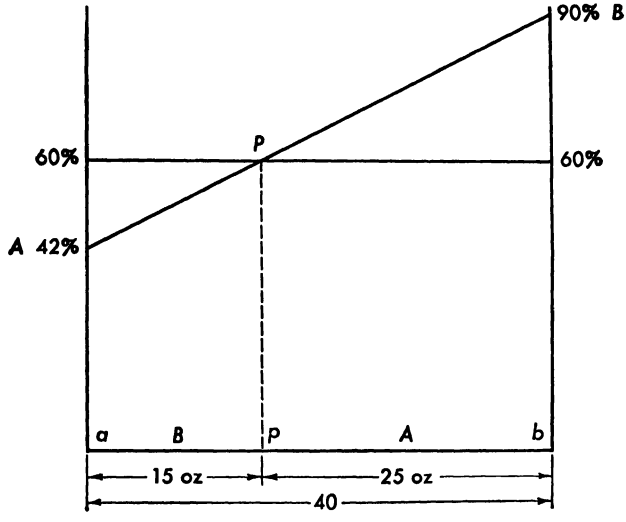


Fig. A·2

Pp drawn from the intersection of these lines divides the base into sections *bp* and *pa*, indicating the amounts of *A* and *B* to be taken. Note that the percentage located on the left will correspond to the constituent on the right, and vice versa. In this example, 25 oz of 42 per cent alloy and 15 oz of 90 per cent alloy must be taken.

A consideration of the properties of similar triangles will suggest a simple arithmetical solution for such a problem; that is, weight of 42 per cent alloy $=$

$$\left(\frac{90 - 60}{90 - 42}\right) 40 = 25 \text{ oz.}$$

It is suggested that the student confirm the validity of this method by setting up a relationship between the similar right triangles in the figure.

The student will see that this same type of diagram could be used to determine the price per pound (60 cents) of an alloy formed by mixing 15 lb of 90-cent alloy and 25 lb of 42-cent alloy.

The method of solution for a variation of this type of problem will now be illustrated.

Determine how much pure (100 per cent) sulfuric acid must be added to 3 qt of 20 per cent acid to raise the strength to 75 per cent. Lay out the left-hand vertical axis and the base line (Fig. A·3). Locate the points 20, 75, and 100 per cent on the vertical axis, and the value 3 qt on the base line. Draw a vertical line from 3 on the base and a horizontal line from 75 per cent on the vertical axis. They will intersect at M. Connect the 100 per cent point with M, and extend the line to the right until it intersects the 20 per cent

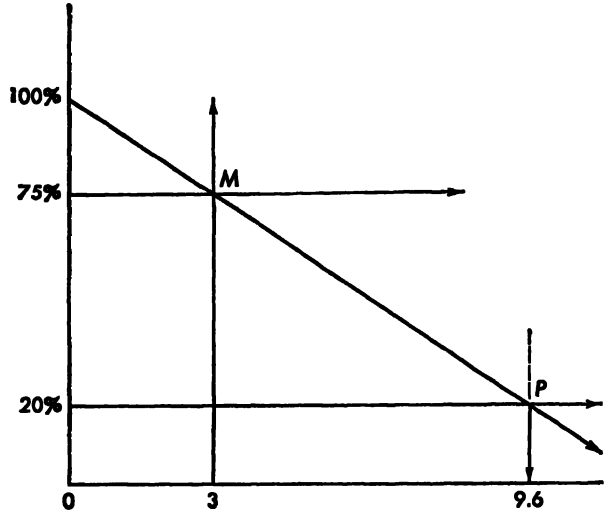


Fig. A·3

horizontal line at P. This will be found to be directly above the value 9.6. Accordingly, 9.6 − 3, or 6.6, qt. of 100 per cent acid must be mixed with the 3 qt of 20 per cent acid to make a solution of 75 per cent strength (9.6 qt).

A·23 Use of Semilog Paper. By drawing a straight line on this type of paper, any required power or root of a given number may be determined.

Some applications are found in compound amount and depreciation calculations, powers of e, natural logarithms, etc.

In order to find any power x of a given quantity a, we plot the equation $y = a^x$ (Fig. A·4) ($a > 1$), with y values on the logarithmic scale and x values on the uniform scale. This is done by drawing a straight line through the point $x = 0$, $y = 1$ and the point $x = 1$, $y = a$ and extending as far as necessary. An extension to the left of the "origin" gives us values of negative powers of a. If $a < 1$, the same principles apply and the chart will have the appearance of Fig. A·5.

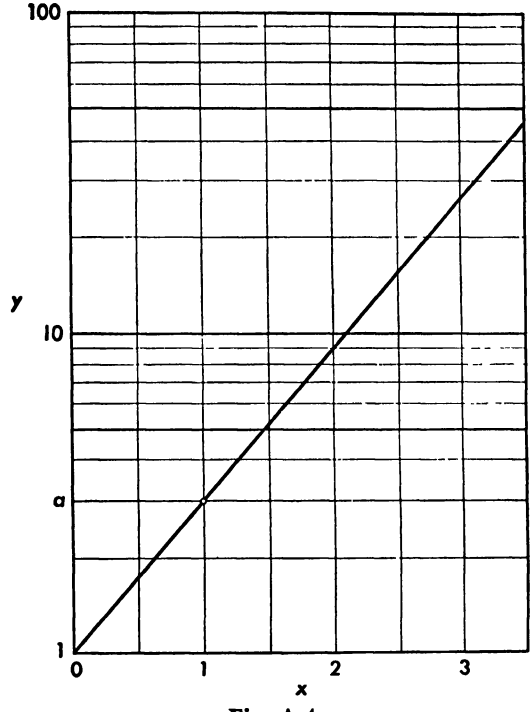


Fig. A·4

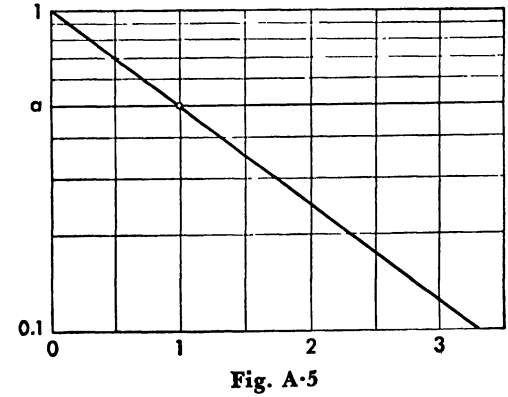


Fig. A·5

If Figs. A·4 and A·5 are to be used for roots, the x scale should be opened up so that the range 0 to 1 occupies most or all of the horizontal space. Although some roots must be rounded values (for example, cube root = 0.333 power), no difficulty should be experienced in locating them.

A·24 Use of Log-Log Paper. A straight line drawn on this type of paper can be used to indicate a given power (or root) of any number.

Among possible applications are calculation of reciprocals and, with some slight modification, of circumferences and areas of circles, volumes of spheres, etc.

In order to find a given power a of any quantity x, we plot the equation $y = x^a$ (Fig. A·6). This is most conveniently done by drawing

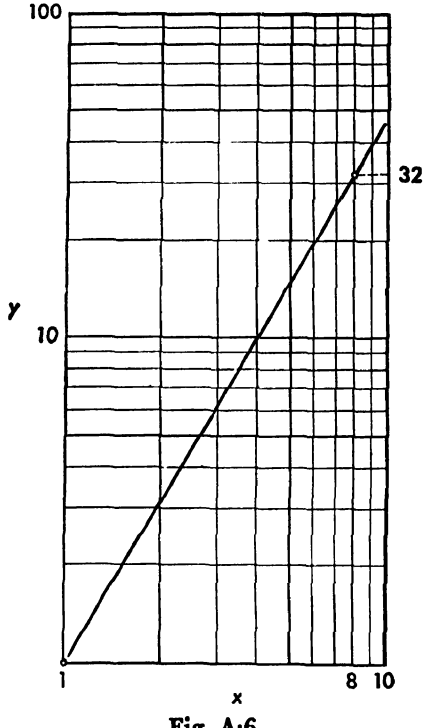


Fig. A·6

a straight line through two easily calculated points. One such point is $x = 1, y = 1$, which we might call the origin. The other point will be dictated by circumstances. If we are interested in reading off $\frac{5}{3}$ powers of various numbers, a good choice would be $x = 8, y = 32$. As before, this line may be extended as far as desired. Areas to the left of the origin represent values of $x < 1$. If we are interested in negative powers of x, we shall need a downward-sloping line starting at the origin, as in Fig. A·5.

Appendix B FORMULAS OF GEOMETRY AND MENSURATION

B·1 Facts in Geometry

1. If one straight line intersects another straight line, the opposite or vertical angles are equal.

2. Two triangles are congruent if two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other.

3. Two triangles are congruent if two angles and the included side of one are equal, respectively, to two angles and the included side of the other.

4. Two triangles are congruent if the three sides of one are equal, respectively, to the three sides of the other.

5. Two right triangles are congruent if the hypotenuse and an acute angle of one are equal to the hypotenuse and an acute angle of the other.

6. Two right triangles are congruent if the hypotenuse and a leg of one are equal to the hypotenuse and a leg of the other.

7. In an isosceles triangle the angles opposite the equal sides are equal.

8. The perpendicular bisector of a line is the locus of all points equidistant from the extremities of the line.

9. The bisector of an angle is the locus of all points equidistant from the sides of the angle.

10. If two parallel straight lines are cut by a transversal, the alternate interior angles are equal.

11. If two parallel lines are cut by a transversal, the exterior-interior angles on the same side of the transversal are equal.

12. Two angles whose sides are parallel, each to each, are either equal or supplementary.

13. Two angles whose sides are perpendicular, each to each, are either equal or supplementary.

14. The sum of the angles of a triangle is equal to 180°.

15. An exterior angle of a triangle is equal to the sum of the two opposite interior angles.

16. In a right triangle the sum of the two acute angles equals 90°.

17. If two angles of one triangle equal two angles of another triangle, their third angles are equal and the triangles are similar.

18. If two right triangles have an acute angle of one equal to an acute angle of the other, they are similar.

19. Two triangles are congruent if two angles and a side of one are equal to two angles and the corresponding side of the second.

20. Two right triangles are congruent if a leg and an acute angle of one are equal to the corresponding leg and acute angle of the other.

21. The opposite sides of a parallelogram are equal, and the opposite angles are also equal.

22. The diagonals of a parallelogram bisect each other.

23. The sum of the interior angles of a polygon of n sides is equal to $(n - 2)180°$.

24. Each interior angle of a regular polygon of n sides is equal to $\left(\frac{n-2}{n}\right)180°$.

25. The sum of the exterior angles of a polygon formed by producing its sides in succession at one extremity equals 360°.

26. Each exterior angle of a regular polygon of n sides is equal to $360°/n$.

27. If three or more parallels intercept equal parts on one transversal, they intercept equal parts on every transversal.

28. The line which joins the mid-points of two sides of a triangle is parallel to the third side and is equal to one-half of the third side.

29. The line which joins the mid-points of the legs of a trapezoid is parallel to the bases and equal to their arithmetic mean.

30. The intersection of the bisectors of the angles of a triangle determines the center of the inscribed circle.

31. The intersection of the perpendicular bisectors of the sides of a triangle determines the center of the circumscribed circle.

32. The medians of a triangle intersect at a point which cuts off two-thirds of each median from its vertex.

33. In the same circle, or in equal circles, equal central angles subtend equal arcs and equal chords.

34. A diameter perpendicular to a chord bisects the chord and the arc subtended by the chord.

35. A radius drawn to the point of contact is perpendicular to a tangent to a circle.

36. Through three points not in the same straight line, one circle, and only one, can be drawn.

37. The two tangents drawn to a circle from an outside point are equal and make equal angles with a line drawn from the point to the center.

38. If two circles intersect, their line of centers is perpendicular to their common chord at its mid-point.

39. If two circles are tangent to each other, the line of centers passes through the point of contact.

40. The number of degrees in a central angle equals the number of degrees in its intercepted arc.

41. An inscribed angle is measured by one-half of its intercepted arc.

42. An angle inscribed in a semicircle is a right angle.

43. An angle formed by two chords intersecting within a circle is measured by one-half of the sum of the intercepted arcs.

44. An angle formed by a tangent and a chord drawn from the point of contact is measured by one-half of the intercepted arc.

45. An angle formed by two secants, or by two tangents, or by a secant and a tangent meeting outside the circle is measured by one-half of the difference of the intercepted arcs.

46. A line parallel to one side of a triangle and meeting the other two sides divides these sides proportionally.

47. If two lines are cut by a number of parallels, the corresponding segments are proportional.

48. If two triangles are similar, their corresponding sides are in proportion.

49. In any triangle the bisector of an angle divides the opposite side into segments which are proportional to the other two sides.

50. If three or more lines pass through the same point and intersect two parallel lines, they intercept proportional segments on the parallel lines.

51. In two similar polygons any two corresponding dimensions are to each other as any other two corresponding dimensions: also the perimeters are to each other as any two corresponding dimensions.

52. The altitude to the hypotenuse of a right triangle forms two right triangles which are similar to each other and to the whole right triangle and is a mean proportional between the segments of the hypotenuse.

53. In a right triangle the square of the hypotenuse is equal to the sum of the squares of the legs.

54. If two chords in a circle intersect, the product of the segments of one chord is equal to the product of the segments of the other chord.

55. The area of a triangle is equal to one-half of the product of its base by its altitude.

56. The area of a triangle equals $\sqrt{s(s-a)(s-b)(s-c)}$ where a, b, and c are the sides and $s = \frac{1}{2}(a+b+c)$.

57. The area of a trapezoid is equal to the altitude multiplied by the average of the bases.

58. The areas of two similar polygons are to each other as the squares of any two corresponding sides.

59. A circle may be circumscribed about, and a circle may be inscribed in, any regular polygon.

60. The area of a regular polygon is equal to one-half of the product of its perimeter by its apothem.

61. Of isoperimetric polygons having the same number of sides, the regular polygon has the greatest area.

62. Of all polygons equivalent in area and having the same number of sides, the regular polygon has the smallest perimeter.

63. The volumes of two similar solid figures are to each other as the cubes of corresponding dimensions.

64. The surfaces of two similar solid figures are to each other as the squares of corresponding dimensions.

65. Pyramidal or conic sections parallel to the base are figures similar to the base. Corresponding dimensions in two sections are proportional to the distances of the sections from the vertex, and corresponding areas are proportional to the squares of these distances.

66. The intersection of a plane with the surface of a sphere is a circle. If the plane passes through the center of the sphere, the intersection is called a great circle.

B·2 Analysis of a Formula. Whenever a formula is developed or applied for the first time, it is instructive to subject it to a few screening tests. The most common of these are the test for dimensional soundness and the test for applicability to special conditions.

The application of these tests can be illustrated to advantage by referring to a formula such as that given on page 749 for the volume of the frustum of a cone. The formula will be seen to be

$$V = \tfrac{1}{3}\pi h(r_1^2 + r_1 r_2 + r_2^2)$$

If the indicated multiplication is carried out, the terms will be hr_1^2, hr_1r_2, and hr_2^2—all third-degree terms in a length unit. Such terms are, therefore, expressions of volume (Sec. 6·6), and the formula is dimensionally sound.

If we assume $r_1 = r_2$, as would be the case in a cylinder, we obtain

$$V = \tfrac{1}{3}\pi h(r_1^2 + r_1 r_1 + r_1^2)$$

or

$$V = \pi r_1^2 h$$

the formula for the volume of a cylinder.

If we assume $r_1 = 0$, as in a cone, we obtain

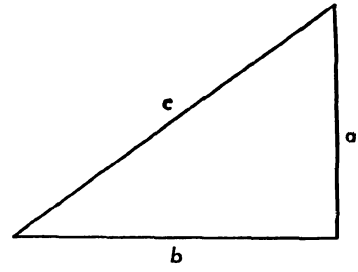
$$V = \tfrac{1}{3}\pi h r_2^2$$

the formula for the volume of a cone.

If we take $h = 0$, the entire expression assumes the value zero, as is to be expected.

B·3 Mensuration. The following listing of formulas for lengths, areas, and volumes is not intended to be complete but rather to include only those formulas most commonly used. Lengths will be represented by the symbols a, b, c, d, l, r, R, and s; areas by A; volumes by V; and angles by θ.

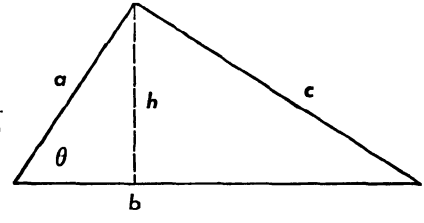
Right Triangle[1]



(1) $A = \tfrac{1}{2}ab$

(2) $c = \sqrt{a^2 + b^2}$ $\quad a = \sqrt{c^2 - b^2}$ $\quad b = \sqrt{c^2 - a^2}$

Oblique Triangle[2]



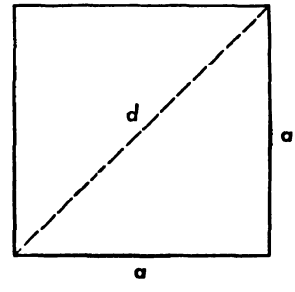
(3) $A = \tfrac{1}{2}bh$

(4) $A = \tfrac{1}{2}ab \sin \theta$

(5) $A = \sqrt{s(s-a)(s-b)(s-c)}$
where $s = \tfrac{1}{2}(a + b + c)$

Equilateral Triangle[3]

Square



(6) $d = a\sqrt{2}$ $\quad a = \dfrac{d\sqrt{2}}{2}$

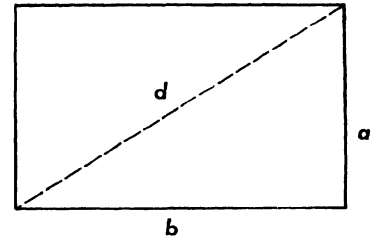
(7) $A = a^2 = \dfrac{d^2}{2}$

[1] See also Eq. (83), page 752, and Sec. 19·9.

[2] For other formulas see Sec. 19·9.

[3] See *Regular Polygons*, page 746.

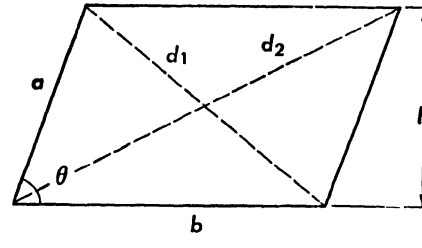
Rectangle



(8) $A = ab$

(9) $d = \sqrt{a^2 + b^2}$

Parallelogram



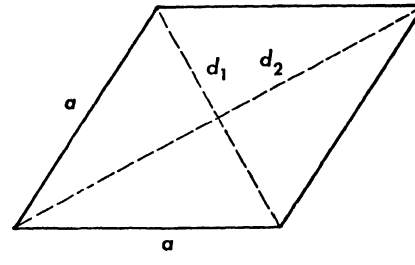
(10) $A = bh$

(11) $A = ab \sin \theta$

(12) $d_1 = \sqrt{a^2 + b^2 - 2ab \cos \theta}$

(13) $d_2 = \sqrt{a^2 + b^2 + 2ab \cos \theta}$

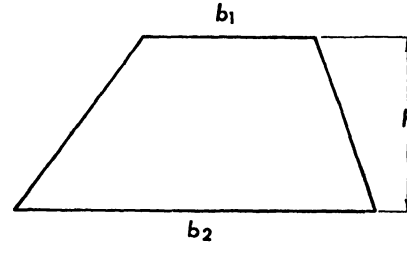
Rhombus



(14) $A = \frac{1}{2} d_1 d_2$

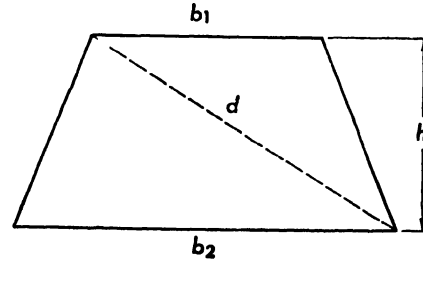
(15) $a = \frac{1}{2} \sqrt{d_1^2 + d_2^2}$

Trapezoid



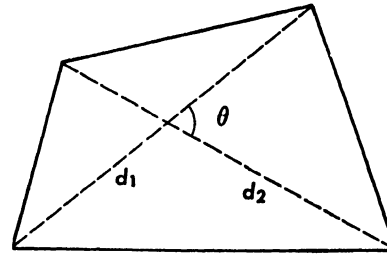
(16) $A = h\left(\frac{b_1 + b_2}{2}\right)$

Isosceles Trapezoid



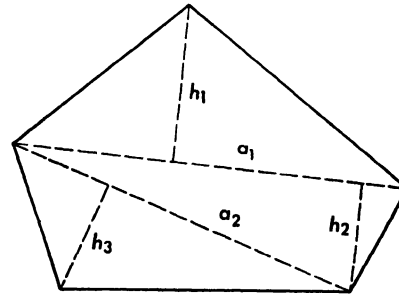
(17) $d = \sqrt{h^2 + \left(\frac{b_1 + b_2}{2}\right)^2}$

Trapezium



(18) $A = \frac{1}{2}d_1 d_2 \sin\theta$

Any Polygon of n *sides*



(19) $A = \frac{1}{2}(a_1h_1 + a_1h_2 + a_2h_3 \cdots)$
= sum of areas of $n - 2$ triangles

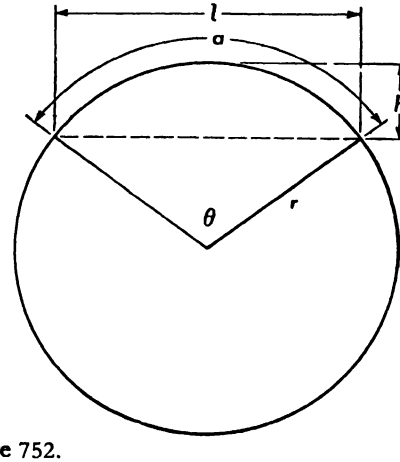
Circle

d = diameter r = radius A = area
c = circumference θ = central angle, degrees

(20) $c = \pi d = 2\pi r$

(21) $d = 2\sqrt{\frac{A}{\pi}} = 1.128\sqrt{A}$

(22) $A = \pi r^2 = \frac{\pi}{4}d^2 = \frac{c^2}{4\pi}$



(23)* $a = \frac{\pi r\theta}{180}$

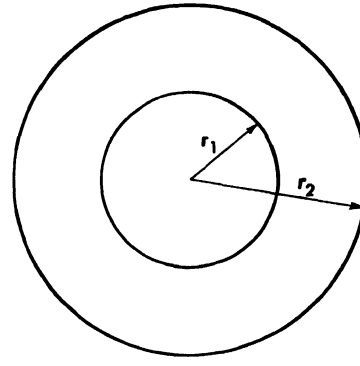
(24) $l = 2\sqrt{2rh - h^2}$

(25) $A_{\text{sector}} = \frac{\pi r^2\theta}{360}$

(26)† $A_{\text{segment}} = \frac{\pi r^2\theta}{360} - \frac{r^2 \sin\theta}{2}$

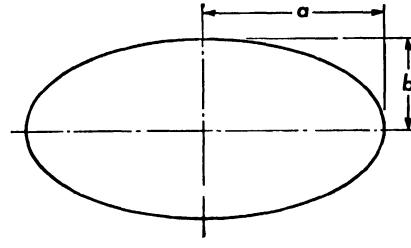
* For an approximation, see Eq. (84), page 752.
† For approximations, see Eqs. (86) to (88), page 753.

(27)* $A_{\text{annulus}} = \pi(r_2 + r_1)(r_2 - r_1)$



Ellipse

(28) $A = \pi ab$



Perimeter†

Regular Polygons

Equilateral Triangle

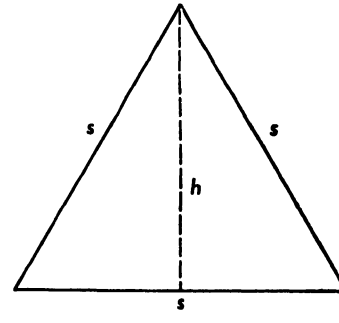
(29) $A = \dfrac{\sqrt{3}}{4} s^2 = 0.433s^2$

(30) $A = \dfrac{\sqrt{3}}{3} h^2 = 0.577h^2$

(31) $h = \dfrac{\sqrt{3}}{2} s = 0.866s$

(32) $s = \dfrac{2\sqrt{3}}{3} h = 1.155h$

(33) $s = 2\sqrt{\dfrac{A}{\sqrt{3}}} = 1.520\sqrt{A}$



* Circles not necessarily concentric.

† For an approximation, see Eq. (85), page 753.

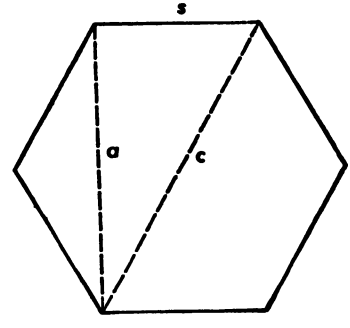
Square[1]

Hexagon

(34) $A = \frac{3\sqrt{3}}{2} s^2 = 2.598s^2$

(35) $A = \frac{\sqrt{3}}{2} a^2 = 0.866a^2$

(36) $A = \frac{3\sqrt{3}}{8} c^2 = 0.650c^2$



(37) $a = \sqrt{3}\, s = 1.732s$

(38) $a = \frac{\sqrt{3}}{2} c = 0.866c$

(39) $a = \frac{1}{3}\sqrt{6A\sqrt{3}} = 1.075\sqrt{A}$

(40) $c = 2s$

(41) $c = \frac{2\sqrt{3}}{3} a = 1.155a$

(42) $s = \frac{\sqrt{3}}{3} a = 0.577a$

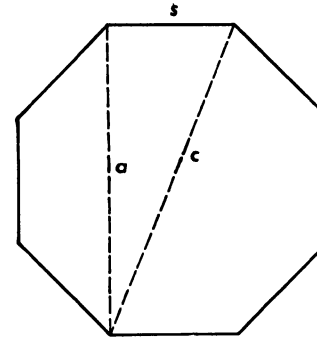
(43) $s = \frac{1}{2}c$

(44) $s = \frac{1}{3}\sqrt{2A\sqrt{3}} = 0.621\sqrt{A}$

Octagon

(45) $A = 2(1 + \sqrt{2})s^2 = 4.828s^2$

(46) $A = \frac{\sqrt{2}\, c^2}{2} = 0.707c^2$



(47) $A = 2(\sqrt{2} - 1)a^2 = 0.828a^2$

(48) $a = (\sqrt{2} + 1)s = 2.414s$

(49) $a = \frac{c\sqrt{2 + \sqrt{2}}}{2} = 0.924c$

(50) $a = \frac{\sqrt{2(\sqrt{2} + 1)A}}{2} = 1.10\sqrt{A}$

(51) $c = s\sqrt{4 + 2\sqrt{2}} = 2.613s$

(52) $c = a\sqrt{4 - 2\sqrt{2}} = 1.083a$

(53) $s = (\sqrt{2} - 1)a = 0.414a$

[1] See Eqs. (6) and (7), page 743.

(54) $$s = \frac{c\sqrt{2 - \sqrt{2}}}{2} = 0.383c$$

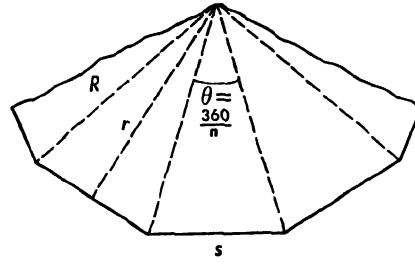
(55) $$s = \frac{\sqrt{2(\sqrt{2} - 1)A}}{2} = 0.455\sqrt{A}$$

Regular Polygon of n *sides*[1]

(56) $$A = \frac{ns^2}{4} \cot\left(\frac{180}{n}\right)$$

(57) $$A = nr^2 \tan\left(\frac{180}{n}\right)$$

(58) $$A = \frac{nR^2}{2} \sin\left(\frac{360}{n}\right)$$



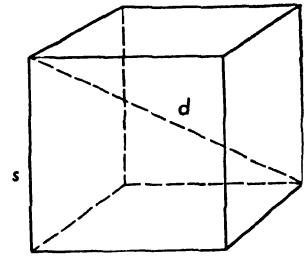
(59) $$r = \frac{s}{2} \cot\left(\frac{180}{n}\right)$$

(60) $$R = \frac{s}{2} \operatorname{cosec}\left(\frac{180}{n}\right)$$

(61) $$s = 2r \tan\left(\frac{180}{n}\right)$$

(62) $$s = 2R \sin\left(\frac{180}{n}\right)$$

Cube

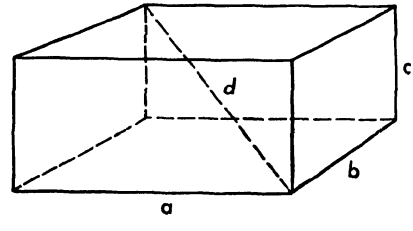


(63) $$V = s^3$$

(64) $$A_{\text{total surface}} = 6s^2$$

(65) $$d = s\sqrt{3}$$

Rectangular Parallelepiped



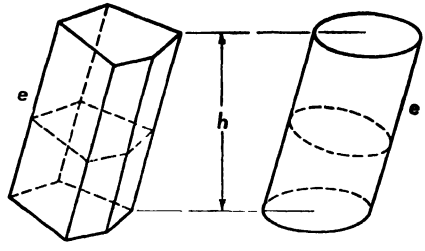
(66) $$V = abc$$

(67) $$A_{\text{total surface}} = 2(ab + ac + bc)$$

(68) $$d = \sqrt{a^2 + b^2 + c^2}$$

[1] See also Sec. 18·15.

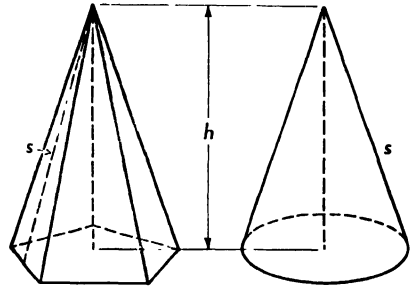
Prism or Cylinder



(69) V = area of base × height

(70) Lateral surface = perimeter of right section × lateral edge

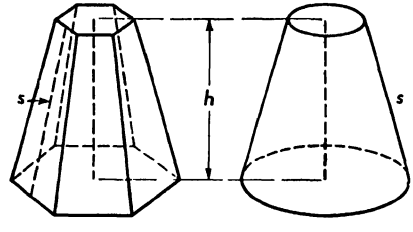
Pyramid or Cone



(71) $V = \frac{1}{3}$(area of base) × (height)

(72) Lateral surface of regular figure = $\frac{1}{2}$(perimeter of base) × (slant height)

Frustum of Pyramid or Cone



(73) $V = \frac{1}{3}h(A_1 + \sqrt{A_1A_2} + A_2)$

where A and A_2 are areas of the bases.[1]

(74) $V = \frac{1}{3}kh(d_1^2 + d_1d_2 + d_2^2)$

where d_1 and d_2 are some corresponding dimension of the two bases and k is a suitable constant. (All sections parallel to the base are similar to the base.)

$$V = 0.2887h(a_1^2 + a_1a_2 + a_2^2)\,*$$

for the frustum of a hexagonal pyramid where the distances across flats a_1 and a_2 are given.

$$V = \frac{1}{3}(\pi)h(r_1^2 + r_1r_2 + r_2^2) = 1.0472h(r_1^2 + r_1r_2 + r_2^2)$$

for the frustum of a cone where radii of the bases r_1 and r_2 are given.

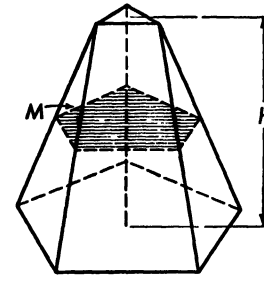
Lateral surface of a regular frustum
= $\frac{1}{2}$(sum of perimeters of bases) × (slant height)
= (perimeter of midsection) × (slant height)

[1] See also Eq. (75), page 750.

* $\frac{1}{3}k = \frac{1}{3}(0.866) = 0.2887$.

*Prismatoid**

(75)† $V = \frac{1}{6}h(A_1 + 4M + A_2)$



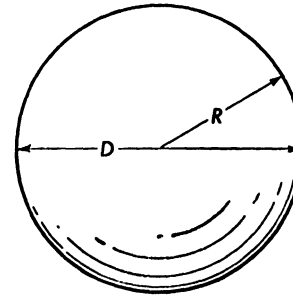
where A_1 and A_2 are the base areas and M is the area of the mid-section. (Applicable also to cones, pyramids, and their frustums.)

Lateral surface = sum of the areas of the individual faces.

Sphere

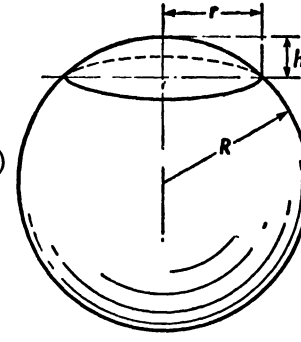
(76) $V = \frac{4}{3}\pi R^3 = \frac{\pi}{6}D^3$

(77) $A = 4\pi R^2 = \pi D^2$



Spherical Segment of One Base

(78) $V = \frac{1}{6}\pi h(3r^2 + h^2) = \frac{1}{3}\pi h^2(3R - h)$

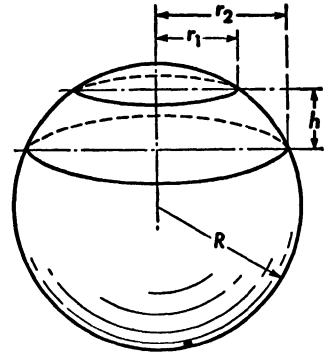


* Bases are in parallel planes, lateral faces are triangles or trapezoids, and edges are straight lines.

† Prismoidal formula.

Spherical Segment of Two Bases

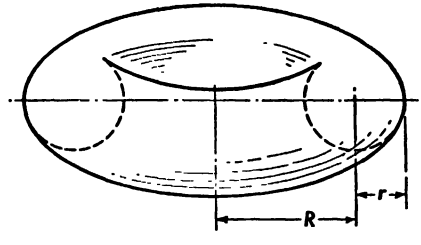
(79) $V = \frac{1}{6}\pi h(3r_1^2 + 3r_2^2 + h^2)$



Zone

(80) $A = 2\pi Rh = \pi Dh$

Ring



(81) $V = 2\pi^2 r^2 R$

(82) $A = 4\pi^2 rR$

B·4 Approximations. A number of very useful approximations exist, some of which have not had the attention they deserve. Possibly the word *approximation* itself suggests a rough answer. In too many cases the degree of "roughness" is left to the student's imagination. Actually, if we have a quantitative idea of the error involved, the correction can be computed by slide rule and added to the approximate answer to give a result correct to four and often five significant figures.

The approximations for the smallest angle of a right triangle and the sine of an angle lend themselves as a stopgap measure in the absence of a table of trigonometric functions.

The approximations for the length of an arc and the area of a segment, when the central angle is not given, are considerably shorter than the usual trigonometric solution.

The derivations of most of the approximations given below involve calculus and are therefore beyond the scope of this text.

EXAMPLE 1. Find the area of a segment 3 in. high cut from a circle 20 in. in diameter. (This calculation arises in estimating the contents of a partially filled horizontal cylindrical tank.)

Using the formula

$$A \approx \frac{4}{3}h^2\sqrt{\frac{2r}{h} - 0.608}$$ [Eq. (87), page 754]

we find

$$A \approx \frac{4}{3}(3)^2\sqrt{\frac{2(10)}{3}} - 0.608, \text{ or } 29.537$$

From the accompanying correction chart we see that for

$$\frac{h}{r} = \frac{3}{10} = 0.3$$

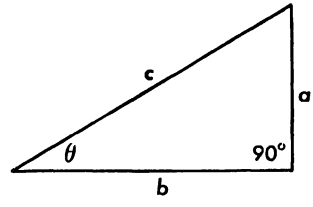
the correction is +0.043 per cent. By slide rule,

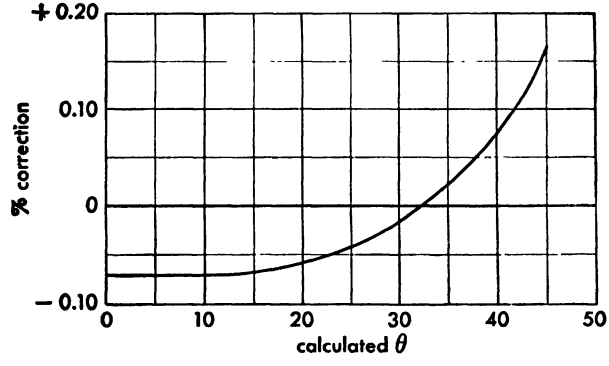
$$(0.00043)(29.5) = 0.0127$$
$$29.537 + 0.013 = 29.550$$

The value accurate to five figures obtained by the usual means is 29.550.

Right Triangle

(83)* $\theta \approx \dfrac{172a}{b + 2c}$ (θ = smallest angle in degrees)



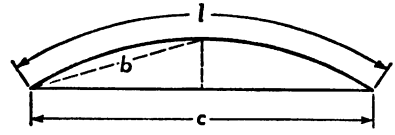


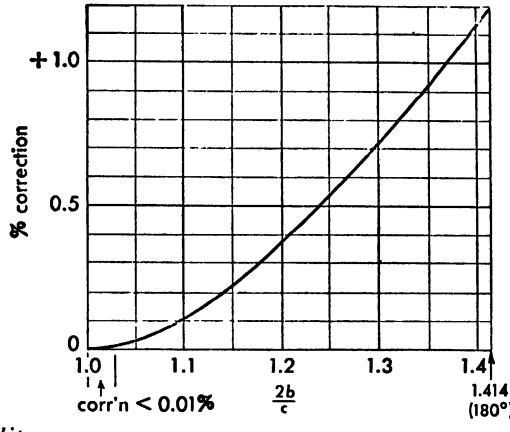
Length of Arc

(84)† $l \approx \dfrac{8b - c}{3}$

* J. S. Frame, Solving a Right Triangle without Tables, *American Mathematical Monthly*, vol. 50, p. 622, 1943. R. A. Johnson, *American Mathematical Monthly*, vol. 27 p. 368, 1920. (Traced to Nicolaus Cusanus, 1514.)

† A. Lodge, "Mensuration for Senior Students," Longmans, Green & Co., Inc. 1945, p. 118.



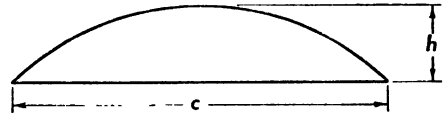


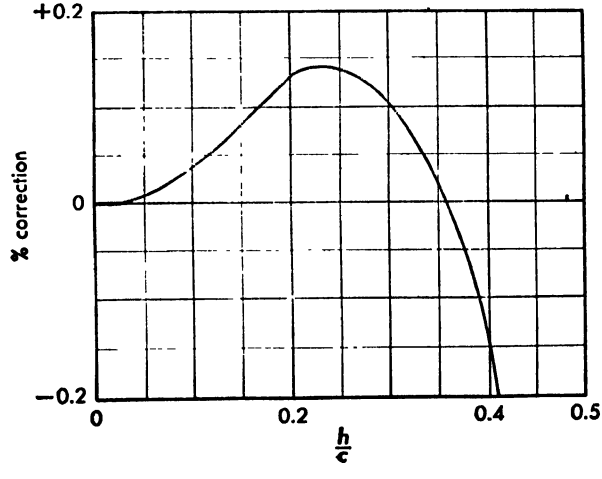
Perimeter of Ellipse

(85) $P \approx \pi \sqrt{2(a^2 + b^2)}$ [For figure see Eq. (28), page 746]

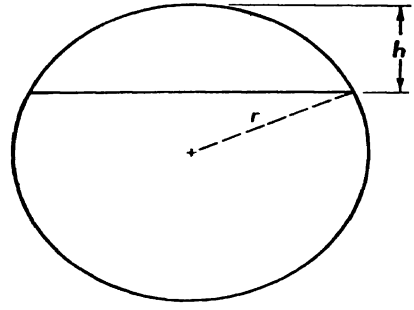
Area of Segment

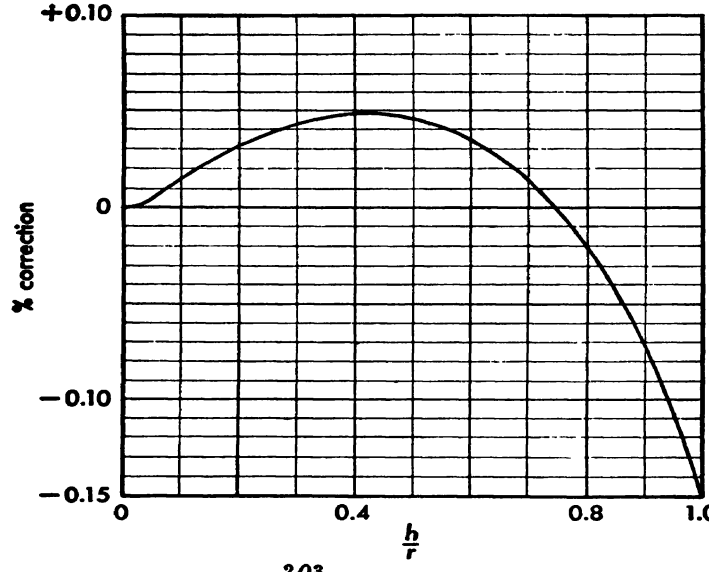
(86) $$A \approx \frac{2}{3}hc + \frac{h^3}{2c}$$



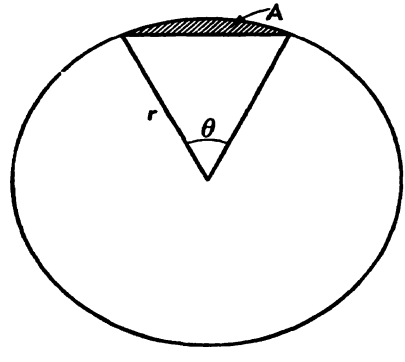


(87) $$A \approx \frac{4}{3} h^2 \sqrt{\frac{2r}{h} - 0.608}$$





(88) $$A \approx \frac{r^2\theta^3}{12} \qquad (\theta \text{ radians})$$



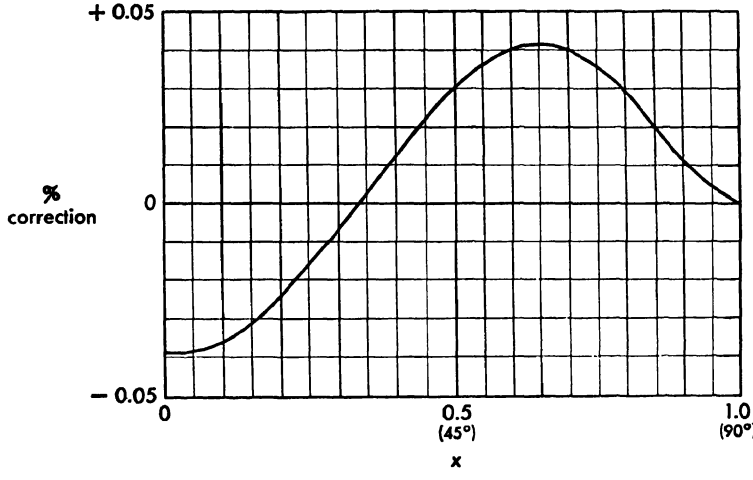
This is a good approximation when θ is small. Correction is roughly proportional to θ^2 and is about $+\frac{1}{2}$ per cent at $\theta = 0.3$ radians or 17°.

For a small angle the error inherent in calculating the very small difference between the area of a sector and the area of a triangle is avoided.

Sine of Angle

$$\sin(90x) \approx x\left(\frac{11 - 3x^2}{7 + x^2}\right) \tag{89}*$$

where x is in degrees.



B·5 Graphical Approximations. A polygon of n sides may be divided into $n - 2$ triangles whose bases and altitudes are scaled off. The area of the polygon is the sum of the areas of the triangles [Eq. (19), page 745.]

The figure whose area is to be approximated may be outlined on cross-section paper and the total number of squares within the figure counted. Evaluation of partial squares can be estimated by eye with sufficient accuracy if the grid work is not too coarse.

B·6 Average Ordinate. The area of an irregular figure may be divided into a number of vertical strips of uniform width. Representative heights of the strips can be gauged visually. The average height of all the strips times the over-all length of the figure approximates the area of the figure. Note that the same result is obtained by multiplying the total of all the heights by the width of any one strip.

* J. S. Frame, A Trigonometric Approximation, *American Mathematical Monthly*, vol. 53, p. 454, 1946.

B·7 Simpson's Rule. The operation of Simpson's rule requires that the figure be divided into any even number of vertical strips of uniform width (Fig. B·1). The heights of the dividing lines are measured and the following rule applied:

$$(90) \quad A \approx \tfrac{1}{3}h(y_0 + 4y_1 + 2y_2 + 4y_3 \cdots 2y_{n-2} + 4y_{n-1} + y_n)$$

where n is an even number.

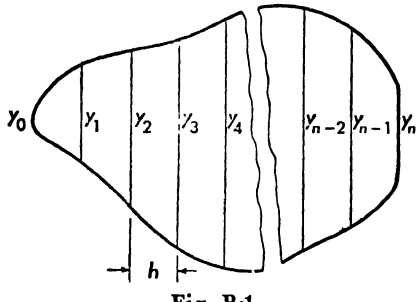


Fig. B·1

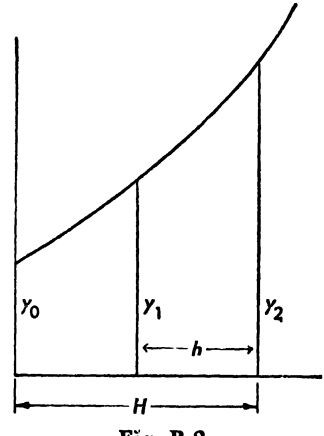


Fig. B·2

The most common version of this rule, which is the abridged form based on two strips, is

$$(91) \quad A \approx \tfrac{1}{6}H(y_0 + 4y_1 + y_2) = \tfrac{1}{3}h(Y_0 + 4y_1 + Y_2)$$

where $H = 2h$. This relationship is exactly true if y is a linear, quadratic, or cubic function of h (Fig. B·2).

If the cross-sectional areas A of a solid be plotted against the distance h from one end (Fig. B·3), the area under the plotted curve

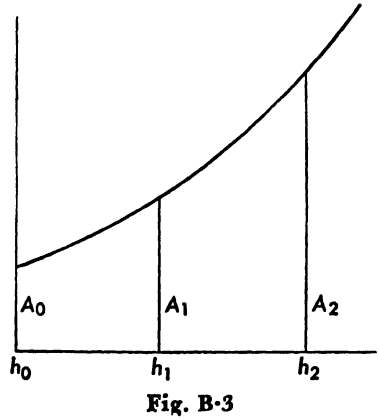


Fig. B·3

between the limits h_0 and h_2 is a measure of the volume of the solid.

If we follow this procedure in the case of a cone or pyramid (or their frustums), we have a situation in which the cross-sectional area is a quadratic function of the distance from one end, and it follows that the formula $V = \frac{1}{6}H(A_0 + 4A_1 + A_2)$ is an exact measure of the volume of such a figure. It can be shown that the same comments apply to a prismatoid [see Eq. (75), page 750].

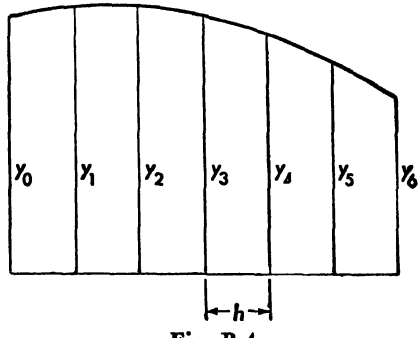


Fig. B·4

B·8 Weddle's Rule. According to this rule the area of a figure such as that illustrated in Fig. B·4 is given by

$$A \approx \tfrac{3}{10}h(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6) \tag{92}$$

The number of strips into which the figure is divided may be 6 or any multiple of 6. If 12 strips, for example,

$$A \approx \tfrac{3}{10}h(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12})$$

As a general rule, of the three approximation formulas just described the average ordinate method is the least accurate and Weddle's rule is the most accurate. Simpson's rule is perhaps the most commonly used of the three formulas, since it is more convenient to use than Weddle's rule and sufficiently accurate for most applications.

Appendix C SOLUTION OF HIGHER-DEGREE EQUATIONS

Occasionally it may be necessary to solve an equation of the type $Ax^n + Bx^{n-1} + \cdots Px + Q = 0$, where n is 3 or some higher integer. This may be done by extending the principles set forth on pages 181, and 193.

In the above-mentioned sections, we had occasion to state in effect that if $(x - a)(x - b) = 0$, then $x - a = 0$, $x - b = 0$, and the roots are $x = a$ and $x = b$. It can be shown that a polynomial expression of the type $Ax^n + Bx^{n-1} + \cdots Px + Q = 0$ may be written as follows:

$$(x - a)(x - b)(x - c) \cdots = 0$$

When the expression is written in the factored form, it can also be shown that there are as many factors involving x (and therefore as many roots) as the degree of the equation indicates.

The greater portion of our solution depends upon plotting the function. Plotting requires a tabulation of coordinate values—a process which can be greatly simplified by *synthetic substitution.*

To illustrate synthetic substitution, let us refer to the equation in Example 3 below. If we substitute any value a for x in the polynomial expression, we obtain

$$2a^3 - 17a^2 + 47a - 24$$

Rearranging,

$$a[a(2a - 17) + 47] - 24$$

It is evident that in this process we have multiplied 2 by a three times, -17 by a twice, and 47 by a once; -24 has not been affected.

EXAMPLE 1. Evaluate $2x^3 - 17x^2 + 47x - 24$ when $x = 3$. The sequence of steps would be

$$(2)(3) = 6$$
$$6 - 17 = -11$$
$$(-11)(3) = -33$$
$$-33 + 47 = 14$$
$$(14)(3) = 42$$
$$42 - 24 = 18$$

The student should satisfy himself that direct substitution of 3 for x in the usual way will lead to the same result.

When a is a less simple number, such as 2.087, synthetic substitution is easily performed on a calculating machine by setting the machine for a constant multiplier of 2.087 and performing the additions and subtractions mentally or on scrap paper.

We shall now illustrate how to evaluate to any desired degree of accuracy the real roots of a higher-degree equation in a single variable.

EXAMPLE 2. Find all the real roots of the equation

$$x^4 - 2x^3 - 14x^2 + 15x + 36 = 0$$

Since this is a fourth-degree equation it may be expected to have four roots, of which some or all may be complex numbers. Using synthetic substitution, we obtain Table C·1:

Table C·1

x	6	5	4	3	2	1	0	−1	−2	−3	−4	−5
y	486	136	0	−18	10	36	36	10	−18	0	136	486

Since, at each end of the table, y is receding from zero at an increasing rate, it is apparent that no points of interest will be outside of this range.

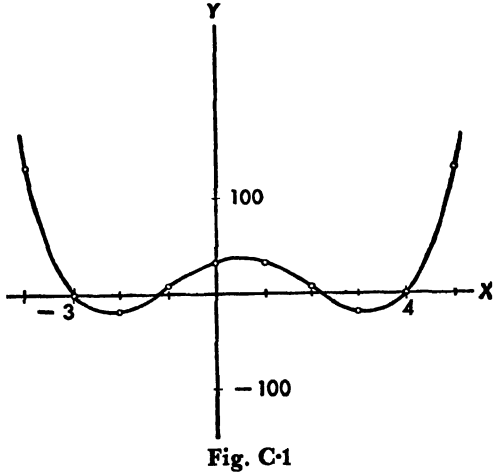


Fig. C·1

These data are plotted as Fig. C·1, which confirms the table in indicating that two roots are $x_1 = 4$ and $x_2 = -3$. It follows that $x - (4)$ and $x - (-3)$, or $x - 4$ and $x + 3$, must be factors of the original expression.

If we divide the original equation through by $x - 4$ and also by $x + 3$ (or by $x^2 - x - 12$), we shall obtain an equation of lower degree from which we can calculate the remaining factors. Accordingly dividing the equation

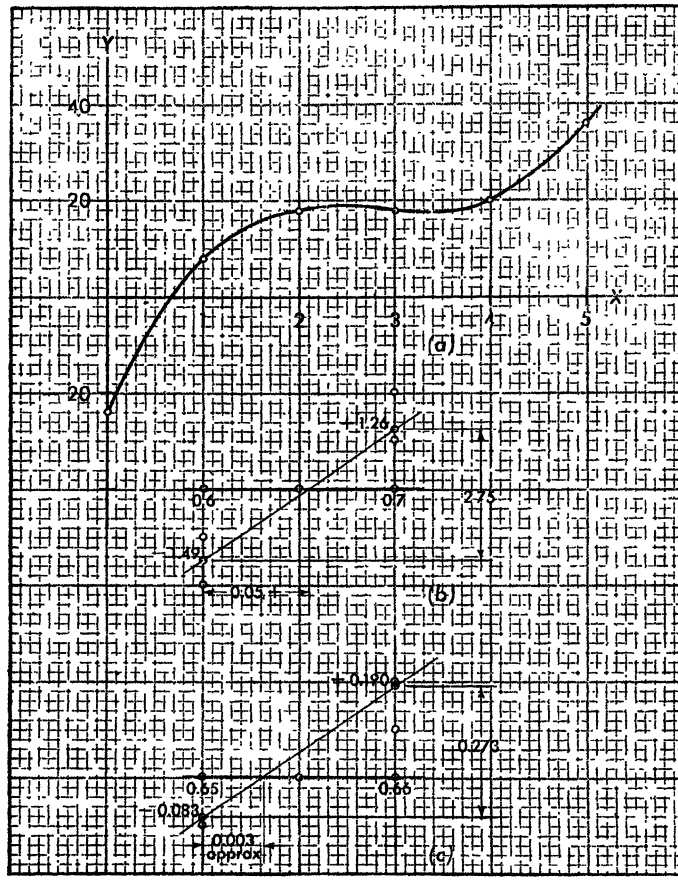


Fig. C·2

$x^4 - 2x^3 - 14x^2 + 15x + 36 = 0$ through by $x^2 - x - 12$, we obtain the equation

$$x^2 - x - 3 = 0$$

Solving by the quadratic formula,

$$x_3 = \frac{1 + \sqrt{13}}{2} = 2.303 \text{ (to three decimal places)}$$

$$x_4 = \frac{1 - \sqrt{13}}{2} = -1.303 \text{ (to three decimal places)}$$

Therefore all four roots are real and are $+4$, -3, $+2.303$, and -1.303.

EXAMPLE 3. Compute to three decimal places the value of any real roots of the equation $2x^3 - 17x^2 + 47x - 24 = 0$. Since this is a third-degree equation, we may expect to find three roots—not all of which will necessarily be real.

Using synthetic substitution, we obtain Table C·2.

Table C·2

x	-1	0	1	2	3	4	5
y	-99	-24	8	18	18	20	36

Plotting these data as Fig. C·2*a*, we note that there is only one real root. It apparently lies between 0.6 and 0.7.

If we compute the value of y when $x = 0.7$, we find $y = +1.26$. When $x = 0.6$, $y = -1.49$. Plotting these two points in Fig. C·2*b* and connecting by a straight line gives us an approximate root between 0.65 and 0.66. Repeating this process for $x = 0.65$ and $x = 0.66$, we obtain corresponding respective values of y of -0.083 and $+0.190$. These points and the connecting straight line are shown in Fig. C·2*c*. The indicated approximate root is 0.653, correct to the nearest thousandth.

The last two approximations might have been calculated by interpolating without graph paper. Figure C·2*b* corresponds to the computation

$$0.1\left(\frac{1.49}{1.49 + 1.26}\right) = 0.05$$

and Figure C·2*c* corresponds to the computation

$$0.01\left(\frac{0.083}{0.083 + 0.190}\right) = 0.003$$

Since 0.653 is the only x intercept, we conclude that the other two roots are complex numbers. These might have been found by dividing the equation $2x^3 - 17x^2 + 47x - 24 = 0$ by $x - 0.653$ and solving the resulting equation by the quadratic formula.

Index

Abbreviated division, 724
Abbreviated multiplication, 723
Abscissa, 105, 348
Absolute error, in addition, 730
  definition, 729
  in subtraction, 730
Absolute numerical value, 52
Accelerated motion, 680
Acceleration, 613, 615, 616, 664, 680
Accumulation factor, 311
Accuracy, degree of, 1
  in interpolation, 26, 385
  in logarithmic computation, 286
  in slide-rule computation, 7
  in solution of triangles, 383, 400
Acute angle, trigonometric functions of, 353, 377
Addition, absolute errors in, 730
  aids to, 719
  algebraic, 52
  of angles, 346
  of complex numbers, 175
  formulas (trigonometric), 501
  of ordinates, 214
  precision of, 2
  of radicals, 166
  rule of signs for, 52
  of sine functions, 529
  and subtraction for solution of simultaneous linear equations, 129
  of trigonometric functions, 501
  of vectors, 473
Algebraic equations (*see* Equations, algebraic)
Algebraic expression, 55
Algebraic methods of finding slopes and rates, 621
Alternation, proportion by, 228
Ambiguous case of sine law, 399
Amortization, 319
Amount, of annuity, 311
  compound, 304
Amount, at simple interest, 303
Amplitude, of periodic functions, 532
  of sine functions, 532
Analytic geometry, 103
Angles, arithmetical operations with, 370
  definition of, 341, 369
  functions having same, 363
  greater than 360°, 345
  initial side of, 342, 369
  interior, of traverse, 447
  measurement of, 343, 369
  negative, 342
  obtuse, 416, 422
  phase (phase constant), 518
  positive, 342
  side of, 342, 369
  small, approximations for, 409
  solid, 467
  standard position of, 342
  terminal side of, 342, 369
  trigonometric functions of, 353, 375
  units of measure, 343, 369
Angular velocity, 526
Annuity, amount of, 311
  ordinary, 311
  present value of, 315
  term of, 311
  whose present value is one, 317
Annulus, 746
Antilogarithms, 271
Applied problems, suggestions for solving, 90
Approximate areas, 606
Approximate increments, 598
Approximation formulas, 733, 751
Approximations, rough, using logarithms, 272
Arc, length of, 745, 752
Arc functions, 365
Area, approximate computation of, 755
  by integration, 689
  rate of change of, 690
  of a traverse, 449

Area, of triangles, 125, 392, 439
under a nonlinear curve, 605
under a straight-line graph, 580
Arithmetic means, 246
Arithmetic progression, 244
arithmetic means of, 246
elements of, 245
common difference, 244
first term, 245
last term, 245
number of terms, 245
sum of terms, 245
extremes, 246
Asymptote, 213
Average ordinate, 581, 605, 701
of sine curve, 520, 704
of $\sin^2$ curve, 520
Averaging by departures, 729
Axes, coordinate, 105, 341
Axis of parabola, 183
Azimuth angle, 447

Base, 55
change of, 268
for exponents, 55
of logarithms, 261
Bearing of line, 445
Binomial, 55
expansion of powers of, 240
general term of, 242
square of, 67
Binomial theorem, in compound interest calculations, 311
in computation of powers and roots, 243

Cancellation, suggestions for, 729
Capitalized cost, 322
Cartesian coordinates, 105
Characteristics, of logarithms, 270
methods of determining, 271
of trigonometric functions, 394
Chord, 745
Circle, area of, 745
circumference of, 745
diameter of, 745
equation for, 211
radius of, 745
Circular measure, 343
Coefficient, 55
Cologarithm, 281
Common difference of arithmetic progression, 244
Common logarithm, 269
Common ratio of geometric progression, 251
Comparison method of solving simultaneous linear equations, 131
Completing square, 184
Complex conjugate defined, 177
Complex fractions, 77
Complex numbers, addition of, 175
angle of, 544
conjugate of, 177
definition, 175
Demoivre's theorem for, 565
division of, 176
exponential form of, 571
graphical representation of, 544
multiplication of, 176
subtraction of, 176
Components of vector, 351
Compound amount, 304
Compound angles, 467
Compound interest, 304
conversion of, 304
Compound-interest law, 326
Computation, using logarithms, 286
using slide rule, 18, 400
Conditional equation, 82
Cone, frustum of, 749
volume of, 749
Conjugate, of complex number, 177
of denominator, 168
Constant, definition of, 103
of integration, 675
evaluation of, 676, 691
of proportionality, 230
units of, 233
Continued proportion, 228
Convergent series, 256
Conversion period, 304
Coordinate system, 105
Coordinates, polar, 350, 478
rectangular, 105, 341
Cosecant, definition of, 353, 375
graph, 515
Cosine, definition of, 353, 375
of difference of two angles, 504
of double an angle, 505
graph of, 515, 517
law, 422
of sum of two angles, 502
variation of, 364
Cotangent, definition of, 353, 375
of difference of two angles, 504
of double an angle, 505
graph of, 515
of sum of two angles, 502
variation of, 364
Course, 445
Cramer's rule, 145
Critical values, 636

Cube root of number, 566
Cusp, 600
Cycle defined, 525
Cylinder, lateral surface of, 749
  volume of, 749

Data, measured, 1
Decibel, 291
Decimal, position in slide-rule calculation, 10
  repeating, 257
Definite integral, 697
Degree, angular measurement, 343, 369
  of equation, 83
  of term, 83
Delta method, 621
Delta notation, 26, 118
Demoivre's theorem, 565
Denominator, least common, 74
  primary, 78
  rationalizing, 164
  secondary, 78
Departure, 445
Dependence, 575
Dependent equations, 128
Dependent variable, 104
Depreciation, 323
  constant-percentage, 323
  straight-line, 323
Derivative, constant, 632
  defined, 595
  e^u, 650
  exponential function, 651
  function of a function, 635
  logarithm, 653
  logarithmic function, 655
  power, of a constant, 657
    of a function, 633
  power function, 627
  product, 659
  quotient, 659
  sine and cosine, 645
  sum or difference, 631
  tangent, 663
Derived curves, 602, 666
Determinant, 143
  elements of, 144
    minor of, 148
  expansion of, 148
  order of, second, 144
    third, 147
  solution of linear equations by, 144, 151
  of system, 145, 151
Difference, absolute error in, 730
  tabular (*see* Interpolation)
  of two squares, 67
Differential, 678
Differential notation, 599
Differentiation, 627
  repeated, 666
Dimensional units, operations with, 89, 742
Dimensions of area under graph, 584
Direct variation, 230
Direction of vector, 339
Discriminant, 194
Distance, between two points, 124
  calculated from speed-time graph, 589
Division, 176
  abbreviated, 724
  accuracy in, 2
  of complex numbers, 76
  of fractions, 76
  of imaginary numbers, 174
  by irrational polynomial, 168
  by logarithms, 266
  of monomials, 64
  of polynomials, 65
  of powers, 57, 155
  of radicals, 167
  relative error in, 731
  rule of signs for, 54
  short, 723
  by slide rule, 9
  of vectors, 494
  by zero, 54

e (base of natural logarithms), 327
Effective rate, 310
Effective value of sine curve, 522, 705
Element, of determinant, 144
  of volume, 711
Ellipse, area of, 746
  equation for, 210
  perimeter of, 753
Equations, algebraic, conditional, 82
    definition, 81
    degree of, 83
    dependent, 128
    equivalent, 82
    fractional, 84
    higher-degree, 758
    identical, 83
    inconsistent, 128
    intercept form, 120
    linear, definition, 83
      one unknown, 81
      in reciprocals, 133
      three or more unknowns, 135, 151
      two unknowns, 145
    literal, 85, 190
    members of, 81

Equations, algebraic, operations on, 83
quadratic (*see* Quadratic equation)
in quadratic form, 196
radical, 205
roots of, 83
sides of, 81
simultaneous, 145, 151
linear, 127
quadratic, 210
slope-intercept form, 117
slope-point form, 117
solutions, 83
exponential, 287
graphical solution, 137
members of, 81
of periodic functions, 514
reduction of order of, 760
roots of, 83, 191, 194, 216
standard form of, 145
trigonometric, 510
Equivalent equations, 82
Error, absolute, 729, 730
relative, 729
Euler's equation, 713
Expansion of determinant, 148
Exponential equations, 287
Exponential function, 326
Exponents, fractional, 55, 155, 158
laws of, 57, 155, 159
negative integral, 160
positive integral, 155
zero, 159
Expression, algebraic, 55
Extraneous root, 192
Extremes of a proportion, 228

Factor theorem, 193
Factorial notation, 241
Factors, common, 72
determination by quadratic formula, 193
of difference of two cubes, 68
of difference of two squares, 67
monomial, 69
of sum of two cubes, 68
of trinomial, 69
Falling bodies, 680
Figures, significant, 1
Flexion, 666
Force-distance graphs, 585
Forces as vectors, 489
Formula, analysis of, 742
mensuration, 743
quadratic, 187
transformation, 87
Fractional equations, 84
Fractional equivalents, 726
Fractional exponents, 158
Fractions, addition of, 74
complex, 77
division of, 76
multiplication of, 76
properties of, 71
reduction of, 72
subtraction of, 74
Frequency of sine functions, 525
Frustum, of cone, 749
of pyramid, 749
Functional notation, 105
Functional relationship, 575
Functions, definition, 103, 575
exponential, 326
graphical representation, 107
linear, 107
not defined by equations, 109
power, 326
trigonometric, of acute angles, 353
of complementary angles, 355
of difference between angles, 503
of double angles, 505
of general angles, 363
graphs of, 515
of half angles, 506
inverse, 365
as lines, 376
logarithms of, 394
of negative angles, 364
of obtuse angles, 363, 416, 422
signs of, 359
of small angles, 409
of special angles, 356–358, 360–362
of sum of two angles, 501
Fundamental operations in algebra, 51

Geometric means, 253
Geometric progressions, 251
elements of, 251
common ratio, 251
sum of terms, 251
terms of, 251
extremes of, 253
infinite, 255
means of, 253
Geometry, facts in, 739
Graph paper, use of, in calculating, 733
Graphical representation, of quadratic functions, 215
of trigonometric functions, 515
Graphical solutions, of linear equations, 127, 137
of polynomial equations, 758
of quadratic equations, 183, 216

Graphs, of functions, 107
  suggestions for plotting, 110
Greatest common divisor, 725
Grouping, symbols of, 58

Half angle formulas (trigonometry), 506
Harmonic motion, simple, 535
Hexagon, regular, 393
Hyperbola, equations for, 213
Hypotenuse, 373

i, imaginary unit, 172
Identical equations, 83
Identities, algebraic, 83
  trigonometric, 366, 507
Imaginary numbers, 170
  addition and subtraction of, 173
  division of, 174
  multiplication of, 173
  operations with, 172
Imaginary roots, 171
Imaginary unit, 172
Inconsistent equations, 128
Increments, 588
  approximate, 598
Indefinite integral, 675
Independent variable, 104
Index of radical, 158
Infinite geometric progression, 255
Infinite series, 256
Infinity, 359
Inflection, points of, 602
Initial side of angle, 369
Instantaneous rate, 593
Integral, of $\cos u\, du$, 704
  definite, 697
  indefinite, 675
  of $ke^u\, du$, 709
  of $\sin u\, du$, 703
  of $u^{-1}\, du$, 701
Integration, 674
  constant of, 675, 691
  limits of, 697
Intercept form of linear equation, 120
Intercepts of graph, 118
Interest, 303
  compound, 304
  conversion, 304
  simple, 303
Interpolation, in logarithms, 275
  in numerical tables, 26
  in trigonometric tables, 385
Intersecting straight lines, 212
Inverse trigonometric functions, 365
Inverse variation, 230
Inversion, proportion by, 228
Irrational number, 158

j operator, 172
  powers of, 172
Joint variation, 230

Kirchhoff's law, 153

Latitude, 445
Law, of cosines, 422
  of sines, 415
  of tangents, 428
Least common denominator, 74
Least common multiple, 73
Length of arc, 745, 752
Limit, 256, 327, 595, 623, 699
Limits of integration, 697
Line, bearing of, 445
  slope of, 114
Line segment, mid-point of, 125
Linear equations, literal, 132
  in one unknown, 81
  in three or more unknowns, 151
  in two unknowns, 127
Linear functions, 107
Lines, parallel, 123, 147
  parallel to axis, 122
  perpendicular, 123
  through origin, 122
Lissajous figures, 539
Locus, 107, 212, 216
Log-log graph paper, 737
Logarithmic computation, accuracy in, 286
  aids to, 289
  on slide rule, 300
Logarithmic notation vs. exponential notation, 263
Logarithms, 261
  base b, 261
  base e, 331
  base 10, 269
  change of base, 268
  characteristic of, 270
  common, 269
  in computation, 279
  definition, 261
  graphical representation, 261
  mantissa of, 270
  Napierian, 331
  natural, 331
  of power, 266, 282

Logarithms, of product, 265, 279
properties of, 265
of quotient, 266, 279
of reciprocals, 267
of roots, 266, 284
in scientific notation, 271
slide rule, relation to, 299
table of, formation of, 262
use of, 273, 331
of trigonometric functions, 394
of unity, 267
use in rough approximation, 272
Lowest terms, reduction to, 72

Machine and tool design, applications to, 459
Magnitude of vector, 340
Mantissa of logarithms, 270
table of, 273
Maximum and minimum values, 109, 195, 207, 600
identification, 636
Mean ordinate, 581, 605, 701, 704
Mean proportional, 228
Means, arithmetic, 246
geometric, 254
logarithmic, 294
of proportion, 228
Measured data, 1
Members of equation, 81
Mid-point of line segment, 125
Minors of elements of determinant, 148
Mixture problems, graphical solution of, 735
Monomial, 55
division by, 64
multiplication by, 60
Multinomial, 55
Multiple, least common, 73
Multiplication, abbreviated, 723
accuracy in, 2
aids and short cuts, 720
of complex numbers, 176
of fractions, 76
of imaginary numbers, 173
with logarithms, 279
of monomials, 60
of polynomials, 61
of powers, 57, 155
of radicals, 167
relative error in, 731
rule of signs, 53
special products, 67
of vectors, 487, 493
zero in, 54

*n*th power of vector, 560
*n*th root, principal, 157
of vector, 561
Natural logarithms, 331
relation to common logarithms, 333
use of tables of, 331
Negative angles, 342
Negative characteristics, 271
Negative of expression, 72
Negative exponent, 160
Nominal rate, 310
Notation, scientific, 2, 271
Number, complex, 175–178
explicit, 51
imaginary, 170–174
irrational, 51, 118
literal, 51
negative, 52
positive, 52
pure imaginary, 175
rational, 51, 158
real, 51, 170
rounding off, 2
signed vs. unsigned, 52
Numerator, primary, 78
Numerical value, absolute, 52

Oblique triangles, solution of, 415
Octagon, regular, 393
Operation, signs of, 54
Ordinates, 105
addition of, 214
average, 581, 605, 701, 704
Origin, of coordinates, 105
lines passing through, 122

Parabola, 210
equations for, 211
Parallel lines, 123
Parallelogram, mensuration formulas for, 744
Parallelopiped, volume of, 748
Parametric equations, 538
Parentheses, insertion of, 58
removal of, 58, 62
Pascal's triangle, 241
Percentage, 227
Period, angular, 516
time, 525
Periodic function, 514
Periodic payment, 312
Perpendicular lines, 123
Perpetuities, 321
Phase constant, 518
Phase difference, 518
Plane surveying, 444

Points, distance between two, 124
Polar coordinates, 350, 478
Polar diagram, 482
Polar notation of vectors, 478
Polygon, area from coordinates of vertices, 126
  mensuration formulas, 745, 748
  regular, circumscribed, 393
    inscribed, 393
Polynomial, definition, 55
  division, by monomial, 64
    by polynomial, 65
  multiplication, by monomial, 60
    by polynomial, 61
Positive angle, 342
Positive integral exponents, 155
Positive number, 52
Power, 55
  logarithm of, 266
  relative error in, 732
Power function, 326
Powers, computation, by binomial theorem, 243
    using logarithms, 282
  of j, 172
  product of, 57, 155
  quotient of, 57, 155
Precision, 1
Present value, 307
  of annuity, 315
Pressure-volume equations, 292
Principal, 303
Principal nth root, 157
Prism, mensuration formulas for, 749
Prismatoid, 750
Prismoidal formula, 750
Product, logarithm of, 265
  relative error in, 731
  of sum and difference, 67
  of two fractions, 76
  of two polynomials, 61
  of two radicals, 167
Progression (*see* Arithmetic progression; Geometric progressions)
Projections, 451
Proportion, 228
  by alternation, 228
  by composition, 228
  continued, 228
  extremes of, 228
  by inversion, 228
  means of, 228
  terms of, 228
Proportional, mean, 228
Proportional parts table, 276
Proportionality, constant of, 230
  dimensionality of, 233
Pyramid, mensuration formulas for, 749
Pythagorean theorem, short method, 728
  slide-rule computation, 19

Quadrants, 106
Quadratic curves, 210
Quadratic equation, 180
  affected, 180
  axis of symmetry, 195
  discriminant of, 194
  graph of, 183
  incomplete, 180
  linear in x^2 and y^2, 219
  literal, 190
  maximum and minimum values, 195
  nature of roots of, 194
  pure, 180
  simultaneous, 216
  solution, by completing square, 184
    by elimination of constants, 220
    by factoring, 184
    by formula, 187
    by graphical means, 182, 188, 216
    by reduction to simpler systems, 222
    by substitution, 218
Quadratic formula, 187
  use of, in factoring, 193
Quotient, logarithm of, 266
  of powers, 57, 155
  relative error in, 731

Radian, 343
Radical equations, 205
Radicals, 157
  addition and subtraction of, 166
  changing order of, 167
  division of, 168
  index of, 158
  laws of, 162
  multiplication of, 167
  rationalizing denominator of, 164
  reduction of order of, 165
  same, 166
  sign, 158
  simplest form, 165
Radicand, 158
Radius vector, 349
Rate of interest, 303
  effective, 310
  nominal, 310
Rates, average, 597
    by algebraic methods, 621
  of change, 576
  instantaneous, 597
Ratio, 227

Rational number, 51, 158
Rationalization, of complex denominators, 168, 176, 555
of denominator in radical, 164
Real number, 51, 170
Reciprocal relationships in trigonometry, 366
Reciprocals, equations linear in, 133
graphical addition of, 734
Rectangle, mensuration formulas for, 744
Rectangular coordinates, 105, 341
Rectangular notation of complex numbers in vectors, 350
Rectangular parallelopiped, mensuration formulas for, 748
Reduction, of fraction to lowest terms, 72
of order of radical, 165
Reference vectors, addition of, 543
Regular polygons, formulas, 748
Relative errors, 729
Reliability, 1
Removing signs of grouping, 58
Repeating decimals, 257
Resolution, of forces, 490
of vectors, 491
Resultant, of forces, 489
ρ (rho), 350
Rhombus, mensuration formulas for, 744
Right triangles, solution, graphical, 372
logarithmic, 394
natural functions, 380
by slide rule, 400
Ring, mensuration formulas for, 751
Root, index of, 158
principal, 157
Roots, computation of, by binomial theorem, 243
by logarithms, 266, 284
by slide rule, 12, 14
of equation, extraneous, 192
graphical estimation of, 127, 152, 216
identical, 217
irrational, 194
rational, 194
third degree or higher, 758
vanishing, 190
of numbers, imaginary, 171
real, 157, 568
of unity, 566
Rounding off numbers, 2

Scalar quantities, 399
Scientific notation, 2
relation to logarithms, 271
Secant of angle, 353, 375
Sector, mensuration formula for, 745
Segment, mensuration formual for, 745, 753
Semi-log graph paper, 737
Sense of vector, 339
Sequence, 240
Series, 240
convergent, 256
for e, 327
exponential, 328
infinite geometric, 256
for sine and cosine, 572
Sides of equation, 81
Significant digits, 1
Significant figures, 1
Signs, of functions, trigonometric, 359
laws of, 52
of operation, 54
Simple harmonic motion, 535
Simple interest, 303
Simpson's rule, 756
Simultaneous equations, linear, 127
quadratic, 210
Sine, approximation for, 443, 452
defined, 353, 375
of difference between two angles, 503
of double an angle, 505
graph of, 515, 516
ratio of, to angle, 441
of sum of two angles, 501
variation of, 364
Sine and cosine series, 572
Sine functions, adding harmonically related, 534
Sine law, 415
Sine waves, adding, 529
average ordinate of, 520
$\sin^2 \phi$, 520
plotting by geometric methods, 523
root mean square of, 522
Sinking fund, 319
Slide rule, accuracy of, 7
circles, circumference and areas of, 17
cubes and cube roots, 14
decimal point, location of, 10
description of, 5
division, 9
gearing problems, 21
location of numbers on, 7
logarithmic computation on, 300
multiplication, 7
proportion, 11
Pythagorean theorem, 19
relation to logarithms, 299
scope of, 5
sines, 400
solution, of oblique triangles, 406
of right triangles, 400

Slide rule, squares and square roots, 12
suggestions for operation of, 18
tangents, 403
theory of, 299
Slope, average, 593
by algebraic method, 621
instantaneous, 593
of nonlinear functions, 590
of straight line, 577
characteristics of, 116
defined, 114
Slope-intercept formula, 117
Slope-point formula, 117
Solid figures (compound angles), 467
Special products, 67
Sphere, mensuration formulas for, 750
Square of binomial, 67
Square root, short method, 726
Squares, short method, 726
Standard form of linear equations, 145
Straight line, determination of, 117
Straight lines, intersecting, 212
Substitutions, method of solving equations, quadratic, 218
simultaneous linear, 130
synthetic, 758
Subtraction, absolute error in, 730
aids to, 719
algebraic, 52
of angles, 347
of complex numbers, 176
of fractions, 74
of radicals, 166
of vectors, 484
Successive differentiation, 666
Sum, absolute error of, 730
of fractions, 74
of radicals, 166
of two angles, functions of, 501
Symbols of grouping, 58
Symmetry, axis of, 183
as related to falling body problems, 687
Synthetic substitution, 758

Tables, of logarithms, 262, 273, 331
numerical, areas of circles, 24
circumferences, 24
cube roots, 25
cubes, 24
extending range of, 24
interpolation, rules for, 26
reciprocals, 30
square roots, 25
squares, 24
of proportional parts, 276
Tabular difference, 275
Tangent (to a curve), defined, 591
(trigonometry), defined, 353, 375
of difference between two angles, 504
of double an angle, 505
graph of, 515
law, 428
of sum of two angles, 502
variation of, 364
Terminal side of angle, 369
Terms, 55
of annuity, 311
of arithmetic progression, 245
collection of, 59
degree of, 83
dissimilar, 59
of geometric progression, 251
of interest, 304
lowest, 72
of proportion, 228
similar, 59
sum of, in arithmetic progression, 245
in geometric progression, 251
transposition of, 82
Time payments, 315
Topographical diagrams, 482
Transformation of formulas, 87
Transposition, 82
Trapezium, mensuration formulas for, 745
Trapezoid, mensuration formulas for, 744
Traverse, 448, 471
Triangle, area of, from coordinates of vertices, 125
equilateral, 602
mensuration formulas for, 743
oblique, trigonometric solution, 415
right, trigonometric solution, 369
Trigonometric equations, 510
Trinomial, 55
factors of, 67, 193
Two-point formula for straight line, 119

Unity, roots of, 566

Value, absolute numerical, 52
Variable, definition, 103
dependent, 104
independent, 104
Variation, 230
direct, 230
of functions, trigonometric, 516
inverse, 230
joint, 230
Vector, components of, 351
defined, 339

Vector, direction of, 339
exponential form of, 573
as force, 489
multiplication, of two vectors, 493
of vector and scalar, 487
*n*th power of, 560
*n*th root of, 561
projection of, 351
quantity, 339
and rectangular coordinates, 341
resolution of, 491
rotating, 524
sense of, 339
as velocity, 488
Velocity, angular, 526
of falling bodies, 682
linear, 52
as vector, 488
Versed sine, 453
Vertex, of angle, 342
of parabola, 195
Vertical lines, 122
Vinculum, 58
Volume, by integration, 710

Weddle's rule, 757
Word problems, suggestions for solving, 90
Work, 585

X axis, 105
X coordinate, 105
x intercept, 118

Y axis, 105
Y coordinate, 105
y intercept, 118

Zero, division by, 54
exponent, 159
in multiplication, 54
Zone, surface of, 751

Answers

CHAPTER 1

Problem: **1.** 5,180,000; 0.000376; 0.000000075 **3.** 3.75×10^9 **5.** 0.2 **7.** 19,500

CHAPTER 2

Exercise 1: **1a.** 1,270 **1c.** 1,872 **1e.** 341 **1g.** 792 **1h.** 207 **1j.** 153 **1l.** 432 **1n.** 875 **1o.** 138 **1q.** 243 **1s.** 476 **1u.** 96

Exercise 2: **1.** 68.0 **3.** 29.7 **5.** 110,000 **7.** 887,000 **9.** 907,000 **11.** 416,000 **13.** 2,240 **15.** 1.16 **17.** 0.252 **19.** 2,100,000 **21.** 7,500

Exercise 3: **1.** 0.360 **3.** 136 **5.** 14.6 **7.** 0.0440 **9.** 0.0267 **11.** 0.000497 **13.** 2.40

Exercise 4: **1.** 160 **3.** 0.0154 **5.** 121.6 **7.** 0.0612 **9.** 0.0031 **11.** 0.0000656

Exercise 5: **1.** 9.41 **3.** 5.43 **5.** 8.34 **7.** 11.9

Exercise 6: **1.** 324 **3.** 36.6 **5.** 28,200 **7.** 0.01111 **9.** 268,000 **11.** 720,000,000 **13.** 0.136 **15.** 6.27 **17.** 1.677 **19.** 4.66 **21.** 0.0838 **23.** 0.915 **25.** 0.233 **27.** 214 **29.** 2,660 **31.** 343 **33.** 1,685 **35.** 0.00387 **37.** 12,470 **39.** 109,200,000 **41.** 480,000,000 **43.** 8.48 **45.** 0.191 **47.** 2.38 **49.** 42.4 **51.** 211 **53.** 7.79 **55.** 34.55

Exercise 7: **1.** $d = 12.6$ miles; $C = 39.6$ miles; $A = 124.7$ sq. miles **3.** $d = 156$ ft; $C = 490$ ft; $A = 19{,}100$ sq ft **5.** $r = 5.6$ yd; $C = 35.2$ yd; $A = 98.5$ sq yd **7.** $r = \frac{7}{16}$ in.; $C = 2.75$ in.; $A = 0.602$ sq in. **9.** $r = 1.343$ in.; $d = 2.685$ in.; $A = 5.66$ sq in. **11.** $r = 1.785$ in.; $d = 3.57$ in.; $C = 11.22$ in.

Exercise 8: **1.** 60.6 **3.** 663 **5.** 27.4 **7.** 16.65 **9.** 4,150 **11.** 2.53

Exercise 9: **1.** 52.0865 **3.** 15.0047 **5.** 1.75183

Exercise 10. **1.** 26, 19; 89, 65; 63, 46 **3.** 85, 41 **5.** 37, 42; 59, 67; 52, 59 **7.** 29, 38; 45, 59; 74, 97

Exercise 11. **1.** 23.4 sq ft **3.** 7.48×10^{-5} cm **5.** 11,570 miles **7.** 217 gal **9.** 79 hp **11.** 2.65×10^9 **13.** 0.0117

CHAPTER 4

Problem: **1.** 57°46′14″ **3.** $a = b = 132°32'$; $c = d = 47°28'$ **5.** $x = 1.1$; $y = 4.675$ **7.** 34° **9.** 75° **11.** 4.75 **13.** 135° **15.** $2\frac{9}{32}$ in. **17.** 5.59 in.

19. $1\frac{3}{16}$ in. **21.** 450 sq in. **23.** 45° **25.** 17 in. **27.** 20.34 in. **29.** 9:16 **31.** 4.8 **33.** 2:1 **35.** 373.8 yd **37.** 84.8 lb **39.** 67.95 sq in. **41.** 1.27 in. **43.** 12π sq in. **45.** $57\frac{1}{2}$ hr **47.** Vol = 4,432 cu in.; surf = 1,760 sq in. **49.** 169π sq in. **51.** Surf = 935 sq in.; vol = 1,780 cu in. **53.** 73°15′ **55.** $17\frac{1}{2}$ in. **57.** 345.6 in. **59.** 0.996 acre = 43,400 sq ft **61.** 0.395 in. **63.** 121.7 gal **65.** Length = 43.75 in.; width = 3.03 in. **67.** 20 miles **69.** 0.510 in. **71.** 183.2 acres **73.** 10,150 lb **75.** 2.888 in. **77.** 114°7′ **79.** 76.5+ in. **81.** 1.03 in. **83.** 41°49′ **85.** 4.33 in., 1.444 in. **87.** 13°39′ **89.** 3.39 in. **91.** 32.9 gal **93.** 1.277 in. **95.** 36.25 sq ft **97.** 2 ft 10 in. **99.** 10,050 lb **101.** 9.08 sq in. **103.** 761 rpm **105.** 23,560 ft **107.** 13 ft 8 in. **109.** 100.4 in. **111.** 300 rpm; 1,125 to 180 rpm **113.** 400 ft **115.** 39.8 ft per sec **117.** Radius = 10.77 in.; angle = 133.6° **119.** 48.9 cu in. **121.** 37.7 in. **123.** $11.196s^2$ **125.** 0.65% high. No. **127.** 44.22 sq in. **129.** 227 ft 7 in. **131.** 0.0016 in. **133.** 0.42 in. **135.** 50 gal, 9.0 in.; 100 gal, 14.8 in.; 150 gal, 20.0 in.; 200 gal, 25.2 in.; 250 gal, 31.0 in. **137.** 80.83 cu in. **139.** 3.41 gal **141.** 165.2 ft **143.** 90°

CHAPTER 5

Exercise 1: **1a.** 13 **1c.** 14 **1e.** −14 **1g.** 4 **2a.** 3 **2c.** −6 **2e.** 8 **2g.** −22 **3a.** 4 **3c.** 5 **4a.** 40 **4c.** 48 **4e.** −24 **4g.** −36 **5a.** 4 **5c.** −5 **5e.** $\frac{1}{8}$ **5g.** $-\frac{5}{8}$

Exercise 2: **1.** $3a$ **3.** y^4 **5.** $4xy^2z^3$ **7.** w^4x^2 **9.** 7 **11.** 77 **13.** 20 **15.** 15 **17.** 49 **19.** 25 **21.** 9 **23.** 5 **25.** −8 **27.** 0 **29.** 900 **31.** $11x$ **33.** $13mn$ **35.** $6x + 9y$ **37.** $12m + 8p$ **39.** $4k^2 - 2km + 5m^2$ **41.** $9x - 2y - 3z$ **43.** $4x - y$ **45.** $12p$ **47.** $2k^2 - 2km - 3m^2$ **49.** $-x - 2y + 3z$ **51.** $-a^2 + a + 1$ **53.** $-6x^2 - 3xy + y^2$

Exercise 3: **1.** x^6 **3.** y^7 **5.** a^{10} **7.** m^{15} **9.** $25x^6$ **11.** $64b^{24}$ **13.** x^3 **15.** y^5 **17.** $-b$

Exercise 4: **1.** $5a + 3b$ **3.** $2x$ **5.** $4m - n$ **7.** $-6b - c$ **9.** $16ab - a^2b - 2ab^2$ **11a.** $5a + 2b - (4c - m + x)$ **11b.** $5a + 2b + (-4c + m - x)$ **13a.** $7mn + 3m^2 - 4n^2 - (8m - 5n - 2mn)$ **13b.** $7mn + 3m^2 - 4n^2 + (-8m + 5n + 2mn)$ **15a.** $9w^2 + 5y^3 - (2w^2y + 3wy^2 - w^3)$ **15b.** $9w^2 + 5y^3 + (-2w^2y - 3wy^2 + w^3)$

Exercise 5: **1.** $32m^2nx$ **3.** $-36ab^2c^2d$ **5.** $a^2m^3x^3$ **7.** $-72a^5b^{11}c^9m^{11}$ **9.** $-18my + 15ty$ **11.** $30a^4b - 42a^3b^2 - 54a^2b^3$ **13.** $24a^3h^4k^4 - 66bh^2k^5$

Exercise 6: **1.** $ac + bc + ad + bd$ **3.** $h^2 - k^2$ **5.** $6m^2 + mw - 35w^2$ **7.** $24x^2 - 78x - 39y - 6y^2$ **9.** $b^2 - x^2 - 2xy - y^2$ **11.** $n^3 - 8$ **13.** $x^5 + 3x^3 + 3x^2 + 1$

Exercise 7: **1.** $17 - 120x$ **3.** $120x$

Exercise 8: **1.** $2x + 2y$ **3.** $2c + 2y + 6x$ **5.** $xy - ab$ **7.** $cy - 12x^2$ **9.** $20x^2$ **11.** Surf = $2a^2 + ab - 2b^2$; vol = $ab(a - b)$ **13.** $\dfrac{D - d}{2}$

Exercise 9: **1.** $3c$ **3.** $-5c^2$ **5.** $4x^9$ **7.** $\dfrac{8xy^3}{wz^2}$ **9.** $3b - 4c$ **11.** $-2m + 3k$ **13.** $r + 2$ **15.** $\dfrac{5kr}{3h} + \dfrac{5}{2r} - \dfrac{7h}{4r}$

Exercise 10: **1.** $a + 8$ **3.** $4m - 5w$ **5.** $2k^4 + 4k^2 + 8$ **7.** $h^4 - h^3 - h + 1$

9. $32x^5 + 16x^4 + 8x^3 + 4x^2 + 2x + 1$ **11.** $\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4} + \cdots$

Exercise 11: **1.** $20axy - 28bxy$ **3.** $6a^2b^2cx - 8ab^3cy + 14ab^2c^2z$ **5.** $x^2 - 9$
7. $64k^2 - 9m^2$ **9.** $81p^2q^2 - 49r^2$ **11.** $64 + 16n + n^2$ **13.** $16b^2 + 24b + 9$
15. $144a^2m^2 + 168amy + 49y^2$ **17.** $36 - 12m + m^2$ **19.** $25x^2 - 10x + 1$
21. $49t^2 - 126t + 81$ **23.** $25h^2 - 120hn + 144n^2$ **25.** $m^2 + 4m - 21$
27. $a^2 - 15a + 44$ **29.** $z^2 - 16z + 48$ **31.** $8c^2 + 18c - 5$ **33.** $30x^2 - 17x + 2$

Exercise 12: **1.** $3(5x - 6y)$ **3.** $6(5ab - 7km)$ **5.** $a(a - c)$ **7.** $c(ab + bm + mx)$
9. $6x^2y(7a - 4bxy + 3cx^2)$ **11.** $5(2a^2x^2 - 3abxy + 4b^2y^2)$

Exercise 13: **1.** $(x + 2)(x - 2)$ **3.** $(6 + y)(6 - y)$ **5.** $(7m + 1)(7m - 1)$
7. $(5d + 8m)(5d - 8m)$ **9.** $(10yz + 7cd)(10yz - 7cd)$ **11.** $(4n^4 + 1)(2n^2 + 1)$
$(2n^2 - 1)$ **13.** 49

Exercise 14: **1.** $10x$ **3.** $14z$ **5.** $4a$ **7.** $30n$ **9.** $80bh$ **11.** 25 **13.** $49q^2$
15. $25x^2$

Exercise 15: **1.** $(x - 2)^2$ **3.** $(6 - m)^2$ **5.** $(3 + 7z)^2$ **7.** $(8a - 5n)^2$
9. $(9m^2 - 4p)^2$

Exercise 16: **1.** $(y + 3)(y + 4)$ **3.** $(m - 5)(m - 7)$ **5.** $(x + 3)(x + 16)$
7. $(h - 5)(h + 6)$ **9.** $(k^2 + 9)(k^2 - 8)$ **11.** $(x + 3y)(x + 18y)$
13. $(a + 1)(2a + 1)$ **15.** $(4m - 3)(m - 2)$ **17.** $(4b + 3)(2b + 3)$
19. $(9h - 10)(6h + 5)$

Exercise 17: **1.** $3c(3a + 5b)$ **3.** $4(a + 5)(a - 5)$ **5.** $6(b - 2)(b - 1)$
7. $4(4k^2 + 9)$ **9.** $(9h - 25)(h - 1)$ **11.** $(9x^2 + 4)(3x + 2)(3x - 2)$
13. $2(2y - 3)(y - 6)$ **15.** $4(3t + 5)^2$ **17.** No factors

Exercise 18: **1.** $-\frac{y}{5}$ **3.** $-\frac{x}{2}$ **5.** $\frac{3 - c}{4}$ **7.** $\frac{1 - a}{a + 1}$ **9.** $\frac{b + c}{a - c}$

11. $\frac{x - y}{(z - x)(y + z)}$ or $\frac{y - x}{(x - z)(y + z)}$ **13.** $\frac{(d - c)^3}{p + r}$

Exercise 19: **1.** $\frac{y}{w + z}$ **3.** $\frac{m + n}{b - c}$ **5.** $\frac{1}{a + b}$ **7.** $\frac{2}{a - 2b}$ **9.** $\frac{3}{2a - 3b}$

11. $-\frac{a + 3}{3a}$ **13.** Not reducible **15.** $\frac{2(b - a)}{b + a}$ **17.** Not reducible

19. $-\frac{(b - 2c)^2}{2b + c}$

Exercise 20: **1.** $\frac{8x}{15}$ **3.** $\frac{x^2 + y^2 + z^2}{xyz}$ **5.** $\frac{109a - 39b}{72}$

7. $\frac{24bc - 6c^2 - 4a^2 - ab - 3b^2}{24abc}$ **9.** $\frac{m}{30}$ **11.** $\frac{a^2 - c^2}{a}$ **13.** $\frac{b^2 + 2b + 1}{b}$

15. $\frac{3a}{a^2 - b^2}$ **17.** $\frac{x - 3}{x - 6}$ **19.** $\frac{11}{c^2 - 9}$ **21.** $\frac{3c^2 - 2cd - 6d^2}{(c - d)^3}$

Exercise 21. **1.** $\frac{3(a + b)}{4}$ **3.** $\frac{d(x^2 - x + 12)}{a(x^2 + x - 6)}$ **5.** 1 **7.** $\frac{3m}{2(n - m)}$ **9.** $\frac{2a}{3(a - 2)}$

11. $-\frac{8y}{9a}$ **13.** $\frac{21yz}{5x(2x - 3)}$ **15.** $\frac{3(a + 1)(5 - a)}{2}$

Exercise 22. 1. $\frac{y-x}{y+x}$ 3. $\frac{c+d}{cd}$ 5. -1 7. $q-1$ 9. $\frac{1}{r-1}$ 11. $2m^2-1$ 13. x

CHAPTER 6

Exercise 1: 1. 2 3. -2 5. $\frac{3}{5}$ 7. 4 9. 3 11. 4 13. $\frac{3}{5}$ 15. 1 17. 6 19. 8 21. -3 23. 0.7 25. 7 27. 4 29. 6 31. 7 33. 1 35. 13 37. -1 39. $\frac{t-r}{s}$ 41. $\frac{a^2}{4}$ 43. $\frac{m}{k}$ 45. 3 47. $\frac{m+n}{2}$ 49. $\frac{6}{a-b}$ 51. $ac+bc$ 53. $\frac{a}{b+c}$ 55. $x=-\frac{a^2}{2b}$ 57. $y=\frac{b+c}{2}$ 59. $y=\frac{2r^2}{p-q-r}$ 61. $x=-c-m$ 63. $w=\frac{cd}{c+d}$ 65. $z=\frac{n^2-p^2}{m}$

Exercise 2: 1. $\frac{WL}{Q}$ 3. $E-IR$ 5. $\frac{eR}{E-e}$ 7. $\frac{Cb}{Kb+C}$ 9. $\frac{\rho(d^2-L^2)}{2L}$ 11. $T=\frac{2{,}097Q}{EI}$ 13. $\frac{9}{5}C+32$ 15. $\frac{HL}{0.4\pi N}$ 17. $\frac{f(n+1)}{n}$ 19. $\frac{1.299}{T-S}$ 21. $\frac{r_1r_2}{r_1+r_2}$ 23. $\frac{T_1h}{T_1h-(T_1-T)(h_0)}$ 25. $\frac{6V-Bh-bh}{4h}$ 27. $\frac{274(V_1-V_0)}{V_0}$

Exercise 4: 1. 2,400 lb 3. 100 gal 5. 12½ oz, 20% alloy; 27½ oz, 12% alloy 7. 115 ft 9. \$3,000 11. 88.1°F 13. 114 15. \$1,115 17. 1550°F approx 19. 8.7 hr 21. $A = 11.92$ ft; $B = 16.40$ ft 23. 15 men 25. 11.25 27. 23⅓% 29. 83⅓ lb 31. \$3.30 33. $x = 5\frac{1}{8}$ in.; $y = 8\frac{1}{8}$ in. 35. 2.4 gal 37. 4.10 in. 39. $R=\frac{30}{3+\pi}$ 41. 65 @ \$50; 35 @ \$18 43. 75 lb 45. 0.852 fps 47. 12.9 miles 49. 40 miles from B 51. 15 min 53. 42 min 55. \$250 57. 18 days 59. $D = 0.586S$ 61. 59.3% gold 63. 25.5 in. 67. 22¢; Nov. 13 69. 68 in. 71. $r=c-3w$ 73. $V=\frac{wh(l+2L)}{6}$ 75. $V=\frac{\pi h(d^2+2D^2)}{12}$

CHAPTER 7

Exercise 1: 1. -5 3. -5 5. 24 7. $-2\frac{6}{7}$ 9. $\frac{4(3a-4)}{a(8-3a)}$

Exercise 3: 1. Square 3. Triangle 5. Right triangle

Exercise 5: 17. 2.57 in. deep

Exercise 6: 1. ⅛ 3. $-\frac{3}{4}$ 5. ½ 7a. $\frac{\sqrt{3}}{3}$ 7b. 1 7c. $\sqrt{3}$

Exercise 7: 1. $x-2y=-8$ 3. $2x-y=11$ 5. $x-y=-4$ 7. $x\sqrt{3}-y=5+2\sqrt{3}$ 9. $y=7$ 11. $y=-x+8$ 13. $y=\frac{1}{3}x-4$ 15. $y=\frac{1}{4}x+\frac{5}{2}$ 17. $y=\frac{7}{5}x+1\frac{4}{5}$ 19. $y=\frac{3}{2}$ 21. $x-3y=-3$ 23. $3x+4y=-10$ 25. $x+4y=3$ 27. $y=5$ 29. $x-2y=-4$ 31. $2x+3y=-18$ 33. $3x-4y=-24$ 35. $2x+3y=-30$

Exercise 8: **1a.** $C = 0$ **1b.** $B = 0$ **1c.** $A = 0$ **1d.** $A = 24$ **1e.** $B = 2$ **3.** 6 **5.** 9 **7.** 5 **9.** $\sqrt{85} = 9.23$

Exercise 9: **1.** 48 **3.** 56 **5.** 55½ **9.** 56 **11.** 66½

CHAPTER 8

Exercise 1: **1.** $x = 5; y = 3$ **3.** $x = -\frac{1}{2}; y = -4$ **5.** $x = -1; y = 2$ **7.** $x = \frac{7}{2}; y = 1$ **9.** Dependent **11.** $x = \frac{7}{3}; y = -\frac{1}{3}$ **13.** $x = \frac{1}{4}; y = -\frac{3}{8}$

Exercise 2: **1.** $x = 4; y = 4$ **3.** $x = 6; y = 6$ **5.** $x = 5; y = 4$ **7.** $x = 6; y = -2$ **9.** $x = -3; y = 11$ **11.** $x = 9; y = 4$ **13.** $y = 5; z = \frac{7}{2}$ **15.** $x = 4; w = 3$ **17.** $x = 5; y = -2$ **19.** $x = -2; y = 4$ **21.** $x = 8; y = 3$ **23.** $x = \frac{1}{5}; w = \frac{2}{5}$ **25.** $x = 1\frac{6}{13}; y = 2\frac{9}{13}$ **27.** $w = 5; z = 1$

Exercise 3: **1.** $x = c + d; y = c - 3d$ **3.** $x = 2a - b; y = a - 2b$ **5.** $x = \dfrac{3}{a}; y = \dfrac{1}{2b}$ **7.** $x = \dfrac{3c}{a}; y = \dfrac{4c}{b}$ **9.** $x = c; y = b$ **11.** $x = \dfrac{m+n}{2}; y = \dfrac{m-n}{2}$ **13.** $x = \frac{1}{5}; y = 1$ **15.** $x = 2; y = 3$ **17.** $x = \frac{1}{12}; y = \frac{1}{18}$

Exercise 4: **1.** $x = 3; y = 4; z = 5$ **3.** $x = 7; y = 3; z = -2$ **5.** $x = -3; y = 5; z = 8$ **7.** $x = 6; y = 5; z = 4$ **9.** $x = \frac{2}{3}; y = \frac{3}{4}; z = -1$

Exercise 5: **1.** Plumber: \$2.40 per hr; helper: \$1.20 per hr **3.** Man: \$12 per day; boy: \$7.20 per day **5.** Sound: 375 yd per sec; bullet: 428.6 yd per sec **7.** $A = 0.986$ in.; $B = 1.252$ in.; $C = 1.632$ in. **9.** 10 word minimum, 35¢; each additional word, 2½¢ **11.** ⅕, ⅙ **13.** 280 lb A; 80 lb B; 240 lb C **15.** Distance = 144 miles; time = 3 hr **17.** $W_3 = -\dfrac{pd^2}{14}$; $W_2 = W_4 = -\dfrac{3pd^2}{28}$ **19.** 16 days; A: \$12 per day; B: \$10.50 per day **21.** \$12 fixed charge; 200 1st order; 280 2d order; \$0.15 unit cost **23.** 12.5 lb A; 28.6 lb B; 8.9 lb C **25.** 110 mph; 90 mph **27.** $a = 2; b = 3; c = 4$

Exercise 6: **1.** -1 **3.** 32 **5.** $9a^2 - 20b^2$

Exercise 7: **1.** Inconsistent **3.** Dependent **5.** $x = a/2; y = a/3$ **7.** $x = a - b; y = b - a$ **9.** $x = 4; y = -5$

Exercise 8: **1.** -47 **3.** -48 **5.** -9 **7.** $15a$ **9.** $8a + 36b + 6c$ **11.** -812 **13.** -445

Exercise 9: **1.** $x = 7; y = 5; z = 3$ **3.** $x = -4; y = \frac{3}{2}; z = 5$ **5.** $x = 3a + 2b + c; y = a + 2b + 3c; z = a + 3b + c$ **7.** $x = 5; y = 4; z = 3; w = 2$

Exercise 10: **1a.** $I_1 = -0.638; I_2 = 6.68$ **1c.** $I_1 = -13.50; I_2 = 14.97$ **2a.** $I_1 = \dfrac{E_1 - I_2R_3}{R_1 + R_3}$; for negative I_1, $I_2R_3 > E_1$ **3.** $R_2 = 2$

Exercise 11: **1a.** $I_1 = 1.474; I_2 = 1.068; I_3 = 0.0642$ **1c.** $I_1 = 0.165; I_2 = 0.101; I_3 = 0.0306$ **1e.** $I_1 = 2.252; I_2 = 1.492; I_3 = -0.0446$

CHAPTER 9

Exercise 1: **1.** x^6 **3.** b^{x+3} **5.** $3^5 = 243$ **7.** a^{5y} **9.** -16 **11.** $-8x^3$ **13.** $8x^3$ **15.** y^{12} **17.** a^{2n} **19.** a^{n^2} **21.** $2\frac{5}{9}$ **23.** $\dfrac{32x^5}{243}$ **25.** $0.008x^3$ **27.** $a^{3n}b^6$

29. $4^n x^{2n} y^{n^2}$ **31.** x^5y^8 **33.** a^2 **35.** $\frac{1}{y^4}$ **37.** $\frac{1}{a^4}$ **39.** $\frac{x}{y}$ **41.** $9/16$ **43.** $\frac{y^{2n-2}}{x^n}$

45. $\frac{2}{3a^2}$ **47.** $\frac{c^n}{a^n}$

Exercise 2: **1.** $\sqrt[3]{-8} = -2$ **3.** $\sqrt[4]{81} = 3$ **5.** $\sqrt{1/25} = 1/5$ **7.** $\sqrt{49} = 7$
9. 5 **11.** 8 **13.** 3 **15.** 12 **17.** 2 **19.** -1 **21.** 0.2 **23.** 40 **25.** $-3/5$

27. $2ab^2c^3$ **29.** $\frac{ax^2}{c^3}$ **31.** $0.13x^8$

Exercise 3: **1.** 2 **3.** $1/25$ **5.** $-1/27$ **7.** $2\frac{7}{8}$ **9.** -2 **11.** 2.5 **13.** 81
15. $1/81$ **17.** 0.001 **19.** $1/2$ **21.** -32 **23.** $-1/2$ **25.** $x^{7/6}$ **27.** $4x^8$ **29.** $25x^2y^4$

31. $243a^2b^3$ **33.** $\frac{18y^2z^2}{x^3}$ **35.** $\frac{w^4}{x^2y}$ **37.** $x - 2 + \frac{1}{x}$ **39.** $a - \frac{9}{a}$ **41.** ab^{-3}

43. $4^{-2}z^2$ **45.** $x^{1/4}y^{-1/2}$ **47.** $8a^{-9}b^3x^3y^{-6}$ **49.** $\sqrt{x}$ **51.** $\frac{1}{\sqrt{a}}$ **53.** $\sqrt[4]{x^3}\sqrt[4]{y}$

55. $\frac{1}{\sqrt[3]{5x^2}}$ **57.** $a^{3/2}$ **59.** $(x+y)^{1/4}$ **61.** $(a^2b^4c^6)^{1/4}$ **63.** $\left(\frac{a^2}{8x^6}\right)^{1/3}$

Exercise 4: **1.** $2\sqrt{2}$ **3.** $2\sqrt{10}$ **5.** $15\sqrt{2}$ **7.** $12\sqrt{2}$ **9.** $2\sqrt[3]{3}$
11. $15\sqrt[3]{2}$ **13.** $-5\sqrt[3]{2}$ **15.** $10\sqrt[4]{6}$ **17.** $x^3\sqrt{x}$ **19.** $xy\sqrt{xy}$ **21.** $2xy^2\sqrt{3xy}$
23. $x\sqrt{a+b}$ **25.** $2\sqrt{m^2 - 4n^2}$ **27.** $a\sqrt[n]{a}$ **29.** $a^{n+1}\sqrt[n]{a^2}$ **31.** $a^{2n}\sqrt[n]{a}$
33. $x^2y^2\sqrt[3]{y^a}$ **35.** $x^2y^2\sqrt[3]{x^2}$ **37.** $10\sqrt{53} = 72.80$ **39.** $5\sqrt{55} = 37.08$
41. $2\sqrt[3]{754} = 18.20$

Exercise 5: **1.** $\frac{1}{2}\sqrt{2} = 0.7071$ **3.** $\frac{3}{2}\sqrt{2} = 2.121$ **5.** $\frac{2}{5}\sqrt{15} = 1.549$
7. $\frac{1}{2}\sqrt[3]{4} = 0.7937$ **9.** $2\sqrt[3]{4} = 3.175$ **11.** $2\sqrt[4]{3}$ **13.** $\frac{1}{3}\sqrt{6} = 0.8165$
15. $\frac{3}{5}\sqrt{15} = 2.324$ **17.** $\frac{1}{15}\sqrt[3]{180} = 0.3764$ **19.** $\frac{1}{x^2}\sqrt{x}$ **21.** $\frac{1}{a}\sqrt[3]{a}$

23. $\frac{1}{x}\sqrt[3]{x}$ **25.** $\frac{2a}{9c^2}\sqrt{6abc}$ **27.** $\frac{\sqrt{a^2 - b^2}}{a+b}$ **29.** $\frac{\sqrt[n]{x^{n-1}}}{x}$ **31.** $\frac{1}{a^{n+1}}\sqrt{a}$

Exercise 6: **1.** $\sqrt{5}$ **3.** $\sqrt[3]{4}$ **5.** $\sqrt{xy}$ **7.** $yz\sqrt[3]{5x^2z}$ **9.** $\sqrt{x}$ **11.** $\frac{1}{2}\sqrt[3]{4}$
13. $\frac{1}{c}\sqrt[3]{c^2}$

Exercise 7: **1.** $7\sqrt{2}$ **3.** $13\sqrt{3}$ **5.** $17\frac{1}{2}\sqrt{2}$ **7.** $7\frac{1}{7}\sqrt{7}$ **9.** $\sqrt{6}$
11. $-6\sqrt{7}$ **13.** $(1 + x + x^2)\sqrt{x}$ **15.** $\frac{2y}{x^2 - y^2}\sqrt{x^2 - y^2}$

17. $(a + b - 1)\sqrt{a+b}$ **19.** $1\frac{1}{4}\sqrt{2} + \frac{5}{3}\sqrt{3}$

21. $\left(\frac{\sqrt{b}}{b} + \frac{2\sqrt{a}}{a} - 3b\right)\sqrt{a - 3b}$

Exercise 8: **1.** $\sqrt{6} = 2.449$ **3.** $7\sqrt{2} = 9.899$ **5.** 66 **7.** $2a\sqrt{3b}$ **9.** $bc\sqrt{ad}$
11. $3\sqrt[3]{10} = 6.463$ **13.** $2\sqrt[6]{54}$ **15.** $2\sqrt[4]{2}$ **17.** 7 **19.** 7
21. $9 - 6\sqrt{2} = 0.515$ **23.** $5 + \sqrt{6} = 7.449$ **25.** $6\sqrt{10} + 20\sqrt{6} - 6$
$- 4\sqrt{15} = 46.47$ **27.** $\frac{1}{3}\sqrt{21} = 1.528$ **29.** $\frac{9}{5}\sqrt{5} = 2.683$ **31.** $\frac{1}{2z}\sqrt{6xz}$

33. $\sqrt[6]{24}$ **35.** $\sqrt[4]{3}$ **37.** $2(\sqrt{7}+2) = 9.292$ **39.** $\dfrac{2(5+\sqrt{7})}{3} = 5.097$

41. $\dfrac{3(2\sqrt{5}+\sqrt{6})}{2} = 10.38$ **43.** $\dfrac{15\sqrt{2}-4\sqrt{5}}{37} = 0.3316$

45. $\dfrac{78+17\sqrt{15}}{33} = 4.359$ **47.** $\dfrac{x\sqrt{z}+z\sqrt{x}}{xz}$

Exercise 9: **1.** -1 **3.** $-j$ **5.** $-j$

Exercise 10: **1.** $2(-1+j5)\sqrt{3}$ **3.** $(-4+j9)\sqrt{2}$ **5.** $j7\sqrt{5}$
7. $j6+(4+j4)\sqrt{5}$ **9.** $-j2\sqrt{2}$ **11.** $-15xy^2$ **13.** $45ab$ **15.** $\sqrt{105}$ **17.** -40
19. -6 **21.** $-5-2\sqrt{6}$ **23.** $-j5\sqrt{6}$ **25.** $-\sqrt{30}+\sqrt{10}-3\sqrt{2}+\sqrt{6}$
27. $j(6\sqrt{19}+\sqrt{627}+6\sqrt{95}+\sqrt{3135}-6\sqrt{10}-\sqrt{330}-30\sqrt{2}$
$-5\sqrt{66})$ **29.** $\frac{7}{8}$ **31.** $j2\sqrt{5}$ **33.** $\dfrac{\sqrt{14}}{2}$ **35.** $\dfrac{\sqrt{2}}{2}$ **37.** $-\dfrac{j4\sqrt{5}}{5}$

Exercise 11: **1.** $-1+j31$ **3.** $-51+j484$ **5.** -53 **7.** $4{,}282-j1{,}475$
9. $11.0754-j3.465$ **11.** 1 **13.** $\dfrac{-9+j12}{20}$ **15.** $\dfrac{1-j}{2}$ **17.** $\dfrac{2-j3\sqrt{5}}{7}$
19. $-\dfrac{1+j\sqrt{2}}{3}$ **21.** $-\dfrac{j\sqrt{10}}{2}$ **23.** $\dfrac{-3+j5}{34}$ **25.** $\dfrac{504+j332}{157}$
27. $\dfrac{1.21+j0.47}{1.3}$ **29.** $\dfrac{317.3+j114.94}{65.96}$

CHAPTER 10

Exercise 1: **1.** $y = \pm 3$ **3.** $x = \pm 7$ **5.** $y = \pm 11$ **7.** $z = \pm 2\sqrt{2}$
9. $x = \pm\sqrt{30}$ **11.** $y = \pm 3a$ **13.** $w = a, 0$ **15.** $z = \pm\dfrac{1}{c}$ **17.** $x = \pm\frac{1}{4}$
19. $w = \pm\dfrac{\sqrt{3}}{6}$ **21.** $w = \pm 1\frac{1}{2}$ **23.** $x = \pm 12$ **25.** $y = \pm\frac{2}{7}$ **27.** $x = \pm\dfrac{m}{5}$
29. $z = \pm 4$ **31.** $x = \pm 6$ **33.** $w = \pm 6$

Exercise 2: **1.** $x = 3, -1$ **3.** $x = 7, -8$ **5.** $x = 3, -8$ **7.** $x = 1, -\frac{2}{3}$
9. $x = 5, 3$

Exercise 3: **1.** $x = 2, 2$ **3.** $x = 8, -9$ **5.** $x = -3, -4$ **7.** $x = 8, -6$
9. $x = 5, -8$ **11.** $x = -\frac{1}{2}, -1$ **13.** $x = \frac{3}{4}, -2$ **15.** $x = 8, 4$

Exercise 4: **1.** $x = 1, -3$ **3.** $x = 1, -7$ **5.** $x = 2, -12$ **7.** $x = 13, -3$
9. $x = -b \pm a$ **11.** $x = 3b - 2a, -3b$ **13.** $x = 1, -\frac{5}{3}$ **15.** $y = 10 \pm j\sqrt{5}$
17. $w = 2 \pm 2\sqrt{3}$ **19.** $y = 7 \pm 2\sqrt{15}$ **21.** $x = -4 \pm 4\sqrt{3}$
23. $x = 7 \pm j\sqrt{10}$

Exercise 5: **1.** $x = 12, 5$ **3.** $x = 15, -11$ **5.** $y = 1, -\frac{1}{6}$ **7.** $w = \dfrac{3 \pm j\sqrt{6}}{3}$
9. $x = \frac{3}{4}, -\frac{4}{9}$ **11.** $w = \dfrac{-5 \pm j\sqrt{3}}{2}$ **13.** $p = \dfrac{3 \pm j\sqrt{11}}{10}$

Exercise 6: **1.** $x = \dfrac{2m}{5}, -\dfrac{m}{3}$ **3.** $w = c, -a - b$ **5.** $y = \dfrac{a-b}{b-c}, 1$

7. $z = \dfrac{h \pm \sqrt{h^2 + 60}}{6}$ 9. $x = h + k, 0$

Exercise 7: 1. $y = 2, -\frac{1}{2}$ 3. $z = 24, -1$ 5. $y = m, n$ 7. $d = \dfrac{-7 \pm \sqrt{5}}{22}$

9. $h = 6, {}^{25}\!/_{6}$ 11. $c = 4 \pm j4\sqrt{3}$ 13. $y = \dfrac{a}{3}, -\dfrac{a}{5}$ 15. $x = -n \pm \sqrt{n^2 + \dfrac{A}{\pi}}$

17. $m = 12$ 19. $x = \dfrac{b - c}{c - a}, 1$ 21. $x = \pm 4a$ 23. $x = \dfrac{h \pm jh\sqrt{35}}{18}$

Exercise 8: 1. 64, real, rational, unequal 3. 0, real, rational, equal 5. 49, real, rational, unequal 7. 49, real, rational, unequal 9. 0, real, rational, equal 11. 201, real, irrational, unequal 13. 289, real, rational, unequal

Exercise 9: 1. Min; 1, −16 3. Min; $-\frac{7}{2}, -{}^{25}\!/_{4}$ 5. Max; 3, 1
7. Max; $\frac{7}{12}, \frac{1}{24}$ 9. Max; $-\frac{1}{2}, 16$

Exercise 10: 1. $x = \pm 3, \pm 1$ 3. $x = 1, 3, \dfrac{-3 \pm j3\sqrt{3}}{2}, \dfrac{-1 \pm j\sqrt{3}}{2}$

5. $x = 27, -1$ 7. $x = -\frac{1}{8}, +\frac{1}{2}$

Exercise 11: 1. 14, 9 3. 13, 6 5. 8 7. 14, 16 9. 8, −9 11. 36 ft
13. 0.60 in. 15. 1 sec., 6 sec. 17. 4.23 in. 19. 160 yd 21. 22 23. 3.87 in.
25. 3.52 in. 31. $1\frac{1}{2} \times 5$ in.; $2\frac{1}{2} \times 3$ in. 33. 5 in. 35. 29.3 in. 37. 14 mph
39. 0.230 in. 41. 2.54 in. 43. 1.08 in. 45. Bus 25 mph; Train 40 mph
47. 82.0 ft 49. 493 ft 51. 15 ft, 20 ft, 25 ft 53. 40.4°F 55. 1.14 in.

57. 540°C 59. 8.94 in. 61. $d = \sqrt{\dfrac{3h}{2}}$

Exercise 12: 1. 3 3. 18 5. No real roots 7. 15 9. 7 11. 9 13. $2, -1\frac{1}{8}$
15. 2.69 in. 17. 1.87 in. 19. 11.38 in.

Exercise 13: 1. $8\frac{1}{2}, 8\frac{1}{2}$ 3. $V = 13\frac{1}{3}$ cu ft; $W = 560$ ft-lb 5. 536 ft
7. 12 in. each 9. 50¢ adm.; $1,250 receipts 11. 150 fps, 12.5 kw

13. 6,500; $42.25 15. $F = \dfrac{24 + 0.0032A - 0.00043A^2}{20}$ 17. 2.81 in. diam circle; 2.79 in. side square

CHAPTER 11

Exercise 2: 1. $x = 3, -{}^{2}\!/_{3}; y = 2, -9$ 3. $x = 1, 1; y = -4, -4$ 5. $x = -5, -5; y = 4, 4$ 7. $x = 1.9, -2.9; y = 3.1, -1.0$ 9. $x = 2.6, 1.3, -1.6, -2.3; y = 2.6, -2.3, -1.6, 1.3$ 11. Imaginary roots 13. $x = 3, -2.8; y = -1, 0.46$

Exercise 3: 1. $x = 4, -4; y = 12, -12$ 3. $x = 10, -10; y = 8, -8$
5. $x = 3, -4; y = -4, 3$ 7. $x = 7, -{}^{47}\!/_{3}; y = 3, {}^{43}\!/_{3}$
9. $x = 2, -{}^{22}\!/_{19}; y = 4, {}^{71}\!/_{19}$

Exercise 4: 1. $x = 1, -1, 1, -1; y = 3, 3, -3, -3$ 3. $x = 2, 2, -2, -2; y = 0, 0, 0, 0$ 5. $x = \sqrt{{}^{155}\!/_{44}}, -\sqrt{{}^{155}\!/_{44}}, \sqrt{{}^{155}\!/_{44}}, -\sqrt{{}^{155}\!/_{44}}; y = \sqrt{\frac{5}{22}}, \sqrt{\frac{5}{22}}, -\sqrt{\frac{5}{22}}, -\sqrt{\frac{5}{22}}$ 7. $x = \frac{1}{11}\sqrt{21}, -\frac{1}{11}\sqrt{21}, \frac{4}{11}\sqrt{21}, -\frac{4}{11}\sqrt{21}; y = \frac{2}{11}\sqrt{91}, \frac{2}{11}\sqrt{91}, -\frac{2}{11}\sqrt{91}, -\frac{2}{11}\sqrt{91}$

Exercise 5: **1.** $x = \frac{3}{13}\sqrt{13}, -\frac{3}{13}\sqrt{13}; y = 1\frac{9}{13}\sqrt{13}, -1\frac{9}{13}\sqrt{13}$ **3.** $x = 7, -7, 6, -6; y = 2, -2, 1, -1$ **5.** $x = 7, -7, \frac{j6}{5}\sqrt{3}, -\frac{j6}{5}\sqrt{3}$; $y = 2, -2, \frac{j14}{15}\sqrt{3}, -\frac{j14}{15}\sqrt{3}$ **7.** $x = 2, -2, \frac{4}{19}\sqrt{19}, -\frac{4}{19}\sqrt{19}$; $y = -5, 5, \frac{5}{19}\sqrt{19}, -\frac{5}{19}\sqrt{19}$

Exercise 6: **1.** $x = 6, \frac{3}{2}; y = 1, -\frac{7}{2}$ **3.** $m = 7, -7, 1, -1; n = 1, -1, 7, -7$ **5.** $x = \frac{1}{2}, \frac{1}{2}; y = 2, 2$ **7.** $x = 2, -1\frac{7}{5}, \frac{-3 - j\sqrt{671}}{10}, \frac{-3 + j\sqrt{671}}{10}$; $y = 5, -1\frac{7}{2}, \frac{3 + j\sqrt{671}}{4}, \frac{3 - j\sqrt{671}}{4}$ **9.** $x = 0, \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}$; $y = 0, 1, 1$ **11.** $m = 1\frac{9}{3}, -1\frac{9}{3}; n = \frac{8}{3}, -\frac{8}{3}$ **13.** $x = 1, -2, \frac{-3 + j5\sqrt{3}}{6}, \frac{-3 - j5\sqrt{3}}{6}; y = -2, 1, \frac{-3 - j5\sqrt{3}}{6}, \frac{-3 + j5\sqrt{3}}{6}$ **15.** $m = 3, -4, 3, -4$; $n = 3, -4, 3, -4$ **17.** $u = 3, 1\frac{1}{7}; z = 5, 2\frac{9}{7}$ **19.** $x = 8, 0, 1 + \sqrt{7}, 1 - \sqrt{7}$; $y = 8, 0, 1 - \sqrt{7}, 1 + \sqrt{7}$

Exercise 7: **1.** 12, 4 **3.** $\frac{1}{2}, \frac{1}{3}$ **5.** 3.53 amp **7.** Stream: 4 mph; tributary: 3 mph

CHAPTER 12

Exercise 1: **1.** 6:1 **3.** 1:9 **5.** 10:3 **7.** 10:3 **9.** 3:8 **11.** $3x:4a$ **13.** 4:3

Exercise 2: **1.** 8 **3.** $2\frac{2}{3}$ **5.** $52\frac{1}{2}$ **7.** 6 **9.** 21 **11.** $\frac{ac}{b}$ **13.** $\frac{bc}{a + b}$ **15.** $\frac{ad - bc}{a - b + c - d}$ **17.** 12 **19.** $\frac{5}{4}$ **21.** $4abx$ **23.** $2(x + y)$ **25.** $6ab$

Exercise 3: **1.** $W = kxy$ **3.** $V = \frac{kx^3}{d}$ **5.** $R = \frac{kw\sqrt{x}}{h^3}$ **7.** $N = \frac{7}{y}$ **9.** $V = \frac{4.8m}{t^2}$

Exercise 4: **1.** $k = 2{,}520$ in.-lb; $V = 40$ cu. in. **3.** $k = 1.8$ ft per sec^2; $V = 108$ fps **5.** $k = 43{,}560$ sq ft per acre; $N = 1440$ plants **7.** $k = 1.63$ (no units); $V = 7.3$ fps **9.** 56 **11.** $V = 819.2$

Exercise 5: **1.** 54.5 lb **3.** 12 **5.** 895 **7.** $17\frac{1}{2}$ cu yd cement; 35 cu yd sand; $52\frac{1}{2}$ cu yd gravel **9.** $174\frac{1}{2}$ miles **11.** 6.30 in. **13.** 5.13 lb **15.** 2465 sq ft **17a.** $v = 2\frac{9}{3}\sqrt{h}$ **17b.** 25.8 gal per min **17c.** 3.24 ft **19.** 199 lb **21.** 23.9 miles **23.** 36¢ per doz **25.** 1 qt: 8.32 in.; 2 qt: 10.5 in. **27.** Doubled **29.** 13.8 hp **31.** $83\frac{1}{3}$ min; 145 miles **33.** 8.1% zinc oxide; 32.4% titanium dioxide; 13.5% lithopone **35.** 17.1 lb **37.** 29.8 ft per sec^2 **39.** 10.42 lb **41.** Double m, x mult. by 2; double f, x mult. by 1.414; double w, x div. by 2; double a, x decreased; double b, x decreased; double k, x increased **43.** 28.1 mph

CHAPTER 13

Exercise 1: **1.** $x^4 + 12x^3y + 54x^2y^2 + 108xy^3 + 81y^4$ **3.** $w^{12} + 3w^{10}x + \frac{15w^8x^2}{4} + \frac{5w^6x^3}{2} + \frac{15w^4x^4}{16} + \frac{3w^2x^5}{16} + \frac{x^6}{64}$ **5.** $1{,}024x^5 + 640x^3 + 160x$

$+\frac{20}{x}+\frac{5}{4x^3}+\frac{1}{32x^5}$ **7.** $\frac{x^6}{y^6}+\frac{6x^5}{y^4z}+\frac{15x^4}{y^2z^2}+\frac{20x^3}{z^3}+\frac{15x^2y^2}{z^4}+\frac{6xy^4}{z^5}+\frac{y^6}{z^6}$
9. $(100)^3-3(100)^2(2)+3(100)(2)^2-(2)^3=941{,}192$

Exercise 2: **1.** $84x^3y^6$ **3.** $1{,}512w^5z^3$ **5.** $-\frac{5x^7y^3}{18}$ **7.** $-34{,}560x^3y^3$ **9.** 90,720

Exercise 3: **1.** $1-y+y^2-y^3+\cdots$ **3.** $1+\frac{x}{4}+\frac{5x^2}{32}+\frac{15x^3}{128}+\cdots$

5. $4-x-\frac{x^2}{16}-\frac{x^3}{96}-\cdots$ **7.** 7.0711 **9.** 1.2247 **11.** 3.9149 **13.** 2.0362
15. 8.5499

Exercise 4: **1.** 26 **3.** 9 **5.** $7\frac{2}{3}$ **7.** $-9x+5y$ **9.** -112 **11.** $11a-44b$
13. $5a+15b$ **15.** $S=372$; $a=-2$ **17.** $S=385$; $d=-5$ **19.** $n=12$; $l=-33$ **21.** $a=-4$; $l=24$ **23.** $l=15$; $d=2$ **25.** 18 **27.** $-2\frac{1}{2}$
29. $17\frac{3}{10}$ **31.** $x+y$ **33.** $\frac{y^2+1}{2y}$ **35.** 17, 20, 23, 26, 29, 32
37. $-10, -5\frac{1}{2}, -1, 3\frac{1}{2}, 8, 12\frac{1}{2}, 17$

Exercise 5: **1.** \$1,115 **3.** 4,920 **5.** 12,400 **7.** \$1,524 **9a.** \$4,040;
9b. \$37,400 **11.** 2, 8, 14 **13.** $S=4035$; $l=236$; $n=30$

Exercise 6: **1.** 192 **3.** $\frac{1}{64}$ **5.** 104,976 **7.** $36\sqrt{2}$ **9.** 18; $r=\frac{3}{2}$
11. -19.2; $r=-\frac{4}{5}$ **13.** 5; $r=\frac{3}{2}$ **15.** $-1, \frac{1}{2}$ **17.** $\frac{2}{3}\sqrt{3}, \frac{1}{2}\sqrt{5}$
19. $47\frac{5}{8}$ **21.** $665\frac{1}{288}$ **23.** 3,066 **25.** $S=936$; $n=4$ **27.** $S=122$; $l=162$
29. $n=4$; $a=13\frac{1}{2}$ **31.** $n=6$; $r=-\frac{2}{3}$ **33.** 6 **35.** 20 **37.** 4 **39.** $\frac{3}{2}$
41. x^n **43.** $-12, 36$ **45.** $\pm 135, -45, \pm 15$ **47.** $\pm 10x^6, 20x^8, \pm 40x^{10}$, $80x^{12}, \pm 160x^{14}$

Exercise 7: **1.** 24 **3.** $62\frac{1}{2}$ **5.** $\frac{9}{20}$ **7.** $\frac{1}{1-x}$ **9.** $\frac{2}{3}$ **11.** $\frac{3}{11}$ **13.** $\frac{5}{37}$
15. $\frac{9}{22}$

Exercise 8: **1.** 18, 32 **3.** \$409.60 **5.** 1.05% **7.** 27.2°C **9.** 23.298
11. 68.6% **13.** 0.003% **15.** 36.4% **17.** 29.3%

CHAPTER 14

Exercise 1: **1a.** $\log_2 8=3$ **1b.** $\log_2 \frac{1}{64}=-6$ **1c.** $\log_7 \frac{1}{49}=-2$
3a. $\log_5 625=4$ **3b.** $\log_3 \frac{1}{27}=-3$ **3c.** $\log_{\frac{2}{3}} \frac{8}{27}=3$ **5a.** $\log_3 \frac{1}{81}=-4$
5b. $\log_6 36=2$ **5c.** $\log_{\frac{1}{5}} 25=-2$ **7a.** $\log_{16} 8=\frac{3}{4}$ **7b.** $\log_{25} 5=\frac{1}{2}$
7c. $\log_{27} 9=\frac{2}{3}$ **9a.** $\log_{27} \frac{1}{3}=-\frac{1}{3}$ **9b.** $\log_{16} \frac{1}{32}=-\frac{5}{4}$
9c. $\log_{36} \frac{1}{216}=-\frac{3}{2}$

Exercise 2: **1a.** 2 **1b.** 3 **1c.** 4 **3a.** $\frac{3}{2}$ **3b.** $\frac{4}{3}$ **3c.** $\frac{2}{3}$ **5a.** -3
5b. -3 **5c.** $-\frac{5}{2}$

Exercise 3: **1a.** 8 **1b.** 25 **1c.** 81 **3a.** 100 **3b.** $\frac{1}{9}$ **3c.** 9 **5a.** 3 **5b.** 2
5c. 8 **7a.** 15 **7b.** any constant other than zero **7c.** 2 **9a.** $\frac{1}{27}$ **9b.** 0.1 **9c.** 8

Exercise 4: **1.** $b^c=a-x$ **3.** $b^{a+c}=(3x-2)^2$ **5.** 1.40 **7.** $-\frac{1}{4}$ **9.** 4
11a. $\frac{1}{3}\log x$ **11b.** $-3\log x$ **11c.** $-\frac{1}{2}\log x-\log 2$ **13a.** $\log x^6$ **13b.** $\log \frac{x^2}{3}$
13c. $\log_b b\sqrt{a}$

Exercise 5: **1d.** 8.61×10^3; 3.935 **1f.** 86,100; 8.61×10^4 **1h.** 0.861; 0.935 − 1 **1j.** 8.61×10^6; 6.935 **1l.** 0.00861; 8.61×10^{-3} **3a.** 0.549 − 2 **3b.** 0.549 − 1 **3c.** 3.902 **5a.** 354 **5b.** 79.8 **5c.** 0.0798 **7a.** 0.000354 **7b.** 7,980 **7c.** 0.00354

Exercise 6: **1.** 5th to left of decimal point **3.** 2d to right of decimal point **5.** 10th to left of decimal point **7.** 6th to left of decimal point **9.** 14th to right of decimal point **11.** 9th to left of decimal point

Exercise 7: **1.** 3.49178 **3.** 2.49136 **5.** 4.49360 **7.** 0.49859 **9.** 6.49136 **11.** 1.49136

Exercise 8: **1.** 0.49443 − 3 **3.** 0.49748 − 1 **5.** 0.50065 − 2 **7.** 0.50120 − 1 **9.** 0.50379 − 6 **11.** 0.49206 − 2 **13.** 31.03 **15.** 3167 **17.** 0.0032 **19.** 32,040 **21.** 311,100 **23.** 31.63

Exercise 9: **1a.** 3.09132 **1c.** 0.09132 − 2 **1e.** 2.60032 **3a.** 2.25455 **3c.** 0.03743 − 2 **3e.** 0.84911 − 4 **5a.** 0.008605 **5c.** 400,000 **5e.** 57,550

Exercise 10: **1a.** 2.42843 **1c.** 1.32950 **1e.** 0.61802 **3a.** 4.07766 **3c.** 0.89962 − 4 **3e.** 8.07266 **5a.** 0.0010071 **5c.** 0.00079367 **5e.** 139,700

Exercise 11: **1.** 228,380 **3.** 4.4216 **5.** 9.5530 **7.** 6,999,200,000 **9.** 8,157,900,000

Exercise 12: **1.** 8.0424 **3.** 0.077289 **5.** 0.081484 **7.** 0.27963 **9.** 24,476 **11.** 0.018578

Exercise 13: **1.** 1.3430×10^9 **3.** 4,096.4

Exercise 14: **1a.** 308.35 **1b.** 612.61 **1c.** 1359.3 **3a.** 0.00084606 **3b.** 0.021703 **3c.** 0.0043712 **5a.** 23.789 **5b.** 140.84 **5c.** 6,809.3 **7a.** 0.37876 **7b.** 69.553 **7c.** 16.150 **9a.** 2.3428 **9b.** 29.829 **9c.** 0.31216 **11a.** 1.19405 **11b.** 1.11028 **11c.** 1.31209

Exercise 15: **1a.** 9.9058 **1b.** 4.5979 **1c.** 2.1341 **3a.** 8.5711 **3b.** 5.5195 **3c.** 7.4326 **5.** 23.967 **7.** 0.031250 **9.** 634.98 **11.** 21.598 **13.** 195.72 **15.** 0.40880 **17.** 0.74308 **19.** −3.2836 **21.** −2.8923 **23.** −0.041612

Exercise 16: **1.** 6.6257 **3.** 6.9078 **5.** 1.2987

Exercise 18: **1.** 6.2288 **3.** 1.2027 **5.** 20.939 **7.** 7/4 **9.** 7.3365 **11.** 3/2

Exercise 19: **1.** 3.6%; 34,612 **3.** 77.8° **5.** 177.2 **7a.** 0.39 **7b.** 91.7 lb; **7c.** 3.56 turns **9.** 0.315 ohm per 1,000 ft **11.** 0.162 in. **13.** 3.18:1 **15.** 2.399 in. **17.** 1.55 **19.** 0.21 to 1.97 **21.** 24 *db* **23.** −10.8 **25.** 19.64 **27.** 39.9 **35.** 195°F **37.** 1.26 **39.** 1.434 in. **41.** log mean diam = 4.57 in.; arithmetic mean diam = $4\frac{15}{16}$ in. **43.** $28\frac{3}{4}$ in. **45.** $H = 211{,}650 \log \frac{T_2}{T_1}$

47a. 135.5 **47b.** $t = 78.2M^{0.398} - 393$ **51.** 573 hr **53.** 92.1% **55.** 0.623 lb per lb dry stock **57.** 4500°F **59.** 834,100 lb

CHAPTER 15

(Throughout this chapter, a star is used to indicate amounts obtained from interest tables.)

Exercise 1

1.

	Compound amount		*Amount at simple interest*
1%	$1,051.01	*$1,051.01	$1,050
2%	1,104.10	*1,104.08	1,100
3%	1,159.30	*1,159.27	1,150
4%	1,216.70	*1,216.65	1,200
5%	1,276.30	*1,276.28	1,250
6%	1,338.20	*1,338.23	1,300

3. 6.34% **5.** $6,880.70 (*$6,880.67) **7a.** $657.96 (*$657.96) comp annually **7b.** 659.73 (*$659.74) comp semiannually **7c.** 660.65 (*$660.64) comp quarterly **9.** $4,481.80 (*$4,481.84) **11.** $2,886.10 (*$2,886.10) **13.** $1,338.20 (*$1,338.23) comp annually; 1,343.90 (*$1,343.92) comp semiannually; 1,346.80 (*$1,346.86) comp quarterly; 1,348.80 (*$1,348.85) comp monthly **15.** $1,867.00 (*$1,866.99) **17.** 23 yr **19.** 3.06%

Exercise 2: **1.** 8.24% **3.** 5.61% **5.** 6.11%

Exercise 3: **1.** 1.31948 **3.** 1.10490 **5.** 1.16169 **7.** 0.82075 **9.** 0.59201

Exercise 4: **1.** $1,829.80 (*$1,829.98) **3.** 15½ yr **5.** $3,049.75 **7.** $23.52

Exercise 5: **1.** $679.33 (*$679.34) **3.** $53,146 (*$53,148.34) **5.** $8,638.90 (*$8,638.79) **7.** $357.32 (*$357.31) **9.** $1,297.24 **11.** *$2,409.46 **13.** 8.36%

Exercise 6: **1.** $3,679.10 (*$3,679.38) **3.** $643.44

Exercise 7: **1.** $36,000 **3.** *$17,962.33 **5.** *$1,119.40

Exercise 8: **1.** $782.70 **3a.** $850.00 **3b.** $612.56 **5.** 8 yr approx

CHAPTER 16

Exercise 1: **1.** 4.9530 **3.** 0.27253 **5a.** 16.069 **5b.** 6,482 **7.** 15.83 g **9.** 49.2%

Exercise 2: **1a.** 1.81482 **1c.** 0.67803 **1e.** 6.03310 **1g.** −4.44818 **2a.** 3.90661 **2c.** 8.18423 **3a.** 38.6 **3c.** 0.0496 **3e.** 79,500 **4a.** 59.6 **4c.** 0.001205 **4e.** 1973 **5a.** 44.70 **5c.** 0.2466 **5e.** 6.205 **5g.** 9.875 **6a.** 0.05734 **6c.** 77.95 **6e.** −0.07458 **7.** 32.19° **9.** 6.58 min **11.** 6,242 ft

Exercise 3: **1a.** $\log x = 1.32133$ **1c.** $\log x = 0.10041 - 1$ **1e.** $\log x = 0.25710 - 1$ **2a.** $\ln x = 5.0045$ **2c.** $\ln x = -2.8861$ **2e.** $\ln x = -2.9370$ **3a.** 20.95 **3c.** 0.1260 **3e.** 0.1808 **4a.** 1.7618 **4c.** 9.2103 **4e.** −5.2035

Exercise 4: **2a.** $y = e^x$ **2c.** $y^a = e^x$ **3a.** $\ln z = w$ **3c.** $\ln\left(\frac{y}{3}\right) = -\frac{1}{2}$ (or $\ln y = 0.5986$) **4a.** 12.18 **4c.** 2.890 **5a.** $y = 5e^{0.47x}$ **6a.** $y = 0.63\,(14.88)^x$ **7.** 0.4343 **9.** 2.197 **11.** −4.828 **13.** −2.079 **15.** 3.303 **17.** 2.230 **19.** 6.253 **21.** −0.693

Exercise 5: **1.** $i = \frac{E}{R}e^{-\frac{t}{RC}}$ **3.** 0.398 amp **5.** $i \approx 0$ **7a.** 0.114 amp **9.** 13.29 ft **11b.** 22.8 hr **13.** $i = 0.054$ **15.** $i \approx \frac{E}{R}$ **17.** 63.2%

CHAPTER 17

Exercise 3: **1.** 0.47124_r **3.** 4.38077_r **5.** 77.9222° **7.** 210.2755°

Exercise 5: **1.** $r = 8.06; \phi = 29.7°$ **3.** $r = 10.66; \phi = 110.3°$ **5.** $r = 7.92;$

$\phi = 349.8°$ **7.** $r = 6.00; \phi = 120°$ **9.** $r = 8.02; \phi = 80°$ **11.** $r = 8.54;$ $\phi = 290.6°$ **13.** $r = 9.05; \phi = 20°$ **15.** $r = 12.93; \phi = 140.7°$ **17.** $x = \pm 3.84;$ $\phi_1 = 320.2°; \phi_2 = 219.8°$ **19.** $y = \pm 4.31; \phi_1 = 151.4°; \phi_2 = 208.6°$ **21.** $x = \pm 10.32; \phi_1 = 200.2°; \phi_2 = 339.8°$ **23.** $x = \pm 9.44; \phi_1 = 19.5°;$ $\phi_2 = 160.5°$

Exercise 6: **1.** $x = +4.70; y = +1.710$ **3.** $x = -3.86; y = -4.60$ **5.** $x = +5.00; y = +8.66$ **7.** $x = +2.25; y = -3.90$ **9.** $x = +10.00; y = 0.00$ **11.** $\rho = 5.39; \phi = 68.2°$ **13.** $\rho = 6.40; \phi = 231.3°$ **15.** $\rho = 7.62; \rho = 66.8°$

Exercise 7

	sin	*cos*	*tan*	*cos*	*sec*	*cosec*
1a.	$\frac{80}{89}$	$\frac{39}{89}$	$\frac{80}{39}$	$\frac{39}{80}$	$\frac{89}{39}$	$\frac{89}{80}$
1c.	$\frac{12}{13}$	$\frac{5}{13}$	$\frac{12}{5}$	$\frac{5}{12}$	$\frac{13}{5}$	$\frac{13}{12}$
3a.	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
3c.	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{1}$	$\frac{1}{\sqrt{3}}$	$\frac{2}{1}$	$\frac{2}{\sqrt{3}}$

Exercise 8: **1.** cos 70° **3.** sin 30° **5.** tan 73° **7.** sin 51.5° **9.** cosec **65°**

Exercise 9:

1.

	30°	45°	60°
sin	0.500	0.707	0.866
cos	0.866	0.707	0.500
tan	0.577	1.000	1.732
cot	1.732	1.000	0.577
sec	1.155	1.414	2.000
cosec	2.000	1.414	1.155

Exercise 11: **1.** 30 and 210° **3.** 30 and 150° **5.** 120 and 240° **7.** 45 and 225° **9.** 150°

CHAPTER 18

Exercise 1: **1.** 139°37′31″ **3.** 89°26′49″ **5.** 87°48′40″ **7.** 64°23′50″ **9.** 47°13′43″ **11.** 53°31′36″ **13.** 18°01′12″ **15.** 88°46′32″ **17.** 1°27′38″ **19.** 5°14′37″ **21a.** 52°45′47″ **21b.** 00°31′37″ **21c.** 146°53′42″

Exercise 2: **1.** Possible **3.** Impossible **5.** Impossible **7.** Two possible triangles **9.** Possible **11.** Inconsistent

Exercise 7

	sin	*cos*	*tan*	*cot*	*sec*	*cosec*
1.	0.35021	0.93667	0.37388	2.6746	1.0676	2.8555
3.	0.70690	0.70731	0.99942	1.0006	1.4138	1.4146
5.	0.43445	0.90070	0.48234	2.0732	1.1102	2.3018
7.	0.98619	0.16562	5.9545	0.16794	6.0379	1.0140
9.	0.00291	1.0000	0.00291	343.77	1.0000	343.78
11.	0.23203	0.97271	0.23854	4.1922	1.0281	4.3098
13.	0.82822	0.56040	1.4779	0.67663	1.7844	1.2074
15.	0.03054	0.99953	0.03055	32.730	1.0005	32.746

17. 10°24′ **19.** 48°57′ **21.** 44°03′ **23.** 21°16′ **25.** 3°6′ **27.** 50°29′ **29.** 55°35′ **31.** 48°03′ **33.** 25°05′ **35.** 11°10′ **37.** 67°25′ **39.** 50°08′ **41.** 41°06′ **43.** 29°15′ **45.** 5°10′

Exercise 8: **1.** $a = 55.744$; $b = 76.072$; $B = 53°46'$ **3.** $a = 60.200$; $b = 117.47$; $A = 27°08'$ **5.** $a = 0.31011$; $b = 2.1698$; $B = 81°52'$ **7.** $a = 232.39$; $b = 92.168$; $B = 21°38'$ **9.** $a = 50.810$; $b = 39.389$; $A = 52°13'$ **11.** $b = 5{,}516.9$; $c = 40{,}998$; $B = 7°44'$ **13.** $b = 134.96$; $c = 143.04$; $B = 70°39'$ **15.** $b = 48.090$; $c = 50.103$; $A = 73°42'$ **17.** $b = 342.03$; $c = 463.29$; $A = 42°25'$ **19.** $a = 9{,}237.3$; $c = 12{,}302$; $B = 41°20'$ **21.** $A = 64°18'$; $B = 25°42'$; $c = 385.09$ **23.** $A = 69°25'$; $B = 20°35'$; $c = 717.07$ **25.** $A = 34°47'$; $B = 55°13'$; $c = 247.17$ **27.** $A = 18°10'$; $B = 71°50'$; $b = 2{,}556.9$ **29.** $A = 24°18'$; $B = 65°42'$; $b = 39{,}782$ **31.** $A = 21°20'$; $B = 68°40'$; $a = 4{,}082.8$ **33.** $A = 36°36'$; $B = 53°24'$; $a = 1{,}394.0$ **35.** $A = 16°49'$; $B = 73°11'$; $a = 297.70$

Exercise 9

	sin	*cos*	*tan*	*cot*	*sec*	*cosec*
1.	0.04311	0.99907	0.04315	23.175	1.0009	23.196
3.	0.24678	0.96907	0.25466	3.9269	1.0319	4.0522
5.	0.64564	0.76365	0.84546	1.1827	1.3095	1.5489
7.	0.79646	0.60470	1.3172	0.75923	1.6538	1.2555
9.	0.98767	0.15653	6.3098	0.15848	6.3886	1.0125
11.	0.45795	0.88898	0.51514	1.9412	1.1249	2.1836
13.	0.95282	0.30354	3.1391	0.31856	3.2945	1.0495
15.	0.29674	0.95496	0.31073	3.2182	1.0472	3.3700

Exercise 10: **1.** 6°04′06″ **3.** 35°04′20″ **5.** 46°35′24″ **7.** 12°01′10″ **9.** 88°37′48″ **11.** 40°59′27″ **13.** 24°54′20″ **15.** 46°16′30″ **17.** 86°23′12″ **19.** 32°52′07″ **21.** 87°30′45″ **23.** 51°24′30″ **25.** 64°44′48″ **27.** 6°08′51″ **29.** 46°17′30″

Exercise 11: **1.** $a = 0.54882$; $b = 0.68148$; $B = 51°09'15''$ **3.** $a = 12.437$; $b = 9.1720$; $B = 36°24.5'$ **5.** $a = 2{,}517.9$; $b = 1{,}044.7$; $A = 67°27.9'$ **7.** $a = 6.3335$; $b = 8.5310$; $A = 36°35'26''$ **9.** $a = 6.8772$; $b = 12.194$; $A = 29°25'17''$ **11.** $b = 12{,}217$; $c = 12.281$; $B = 84°09'29''$ **13.** $b = 977.88$; $c = 1{,}389.3$; $B = 44°44.2'$ **15.** $b = 2{,}222.7$; $c = 2{,}334.5$; $B = 72°11.7'$ **17.** $a = 50.357$; $c = 757.90$; $A = 3°48.6'$ **19.** $a = 20.466$; $c = 55.313$; $B = 68°17'02''$ **21.** $A = 27°45'34''$; $B = 62°14'26''$; $c = 42.941$ **23.** $A = 14°30'38''$; $B = 75°29'22''$; $c = 18.517$ **25.** $A = 25°13'12''$; $B = 64°46'48''$; $c = 196.42$ **27.** $A = 18°49'13''$; $B = 71°10'47''$; $c = 1{,}357.5$ **29.** $A = 34°58'13''$; $B = 55°01'47''$; $c = 54.523$ **31.** $A = 20°33'36''$; $B = 69°26'24''$; $b = 391.92$ **33.** $A = 67°37'33''$; $B = 22°22'27''$; $b = 16.358$ **35.** $A = 55°45'08''$; $B = 34°14'52''$; $b = 639.64$ **37.** $A = 65°21'30''$; $B = 24°38'30''$; $a = 272.50$ **39.** $A = 42°21'28''$; $B = 47°38'32''$; $a = 12.577$

Exercise 12: **1.** 10.719; 2.3694 **3.** Area when 6 in. wide = 20.622 sq in.; area when 6 in. deep = 27.000 sq in.; greater when 6 in. deep **5.** Side = 7.8728; angles = 62°22′43″ and 55°14′34″; area = 25.461 sq in. **7.** 16°51′47″ **9.** 3.9314 and 3.1458

Exercise 13: **3.** 82.273 **5.** 3.0000 sq in. **7.** π sq in.

Exercise 14: **1.** $a = 89.488$; $b = 115.76$ **3.** $a = 47.868$; $b = 150.21$

5. $a = .65841$; $b = 1.2073$ **7.** $b = 0.42347$; $c = 15.364$ **9.** $a = 1{,}369{,}100$; $c = 1{,}369{,}200$ **11.** $A = 5°28'05''$; $b = 142.33$ **13.** $A = 30°07'41''$; $b = 9.3669$ **15.** $A = 38°39'08''$; $b = 0.45661$ **17.** $A = 84°47'01''$; $c = 96.028$ **19.** $A = 26°35'27''$; $c = 8.1702$

Exercise 15: **1.** $A = 22°37'11''$; $B = 108°55'28''$; $C = 48°27'21''$ **3.** No solution **5.** $a = 17$; $c = 25$; $B = 25°03'27''$ **7.** $c = 73.756$; $A = 3°43'53''$; $B = 139°23'55''$ **9.** $c = 75$; $B = 8°47'48''$; $C = 118°04'23''$ **11.** $a = 28.441$; $B = 59°14'24''$; $A = 16°30'36''$

CHAPTER 19

Exercise 1: **1.** $\sin 140° = \sin 40° = 0.64279$ **3.** $\sin 95°30' = \sin (180° - 95°30') = \sin 84°30' = 0.99540$ **5.** $\sin 113°29'50'' = \sin 66°30'10'' = 0.91708$ **7.** $\sin 175°09'55'' = \sin 4°50'05'' = 0.08428$ **9.** $\sin 179°40'30'' = \sin 00°19'30'' = 0.00568$ **11.** $\sin^{-1} 0.33244 = 19°25'$ and $160°35'$ **13.** $\sin^{-1} 0.72837 = 46°45'$ and $133°15'$ **15.** $\sin^{-1} 0.99824 = 86°36'$ and $93°24'$

Exercise 2: **3.** $\dfrac{a}{\sin A} = b$ **5.** $A = 28°05'$; $C = 106°15'$; $B = 45°40'$

7. $a = 7.3205$; $b = 5.1764$ **9.** $a = 1{,}000$; $b = 1{,}732.05$; $h = 866.025$ **11a.** $b = 66.913$ **11b.** $b = 17.365$ **13.** $a = 1.2007$; $c = 1.9355$; $B = 53°31'$ **15.** $a = 2.8438$; $c = 1.7307$; $B = 130°13'$ **17.** $a = 1.3670$; $c = 6.0563$; $B = 122°46'$ **19.** $a = 1.1165$; $b = 11.795$; $C = 90°46'$ **21.** $b = 4.9509$; $c = 2.9099$; $A = 104°30'$ **23.** $a = 11.080$; $b = 32.048$; $C = 128°12'$ **25.** $a = 1.2045$; $B = 86°55'$; $A = 44°44'$; $A' = 38°34'$; $B' = 93°05'$; $a' = 1.0669$ **27.** $c = 6.3591$; $B = 67°46'$; $C = 54°37'$; $c' = 1.3745$; $B' = 112°14'$; $C' = 10°09'$ **29.** $c = 3.418$; $B = 71°58'$; $C = 46°40'$; $c' = 0.8644$; $B' = 108°02'$; $C' = 10°36'$ **31.** $b = 2.0257$; $B = 32°50'$; $C = 84°39'$; $b' = 1.4076$; $B' = 22°08'$; $C' = 95°21'$ **33.** $b = 2.7276$; $c = 4.3829$; $A = 59°12'$ **35.** $b = 20.678$; $c = 12.131$; $B = 87°44'31''$ **37.** $a = 56.356$; $b = 188.79$; $C = 10°13'15''$ **39.** $c = 30.238$; $A = 82°54'$; $C = 11°56'05''$ **41.** $a = 23.305$; $A = 42°20'07''$; $C = 28°26'37''$ **43.** $b = 10.002$; $A = 60°38'51''$; $B = 30°25'06''$ **45.** $b = 6.1150$; $c = 10.132$; $C = 65°51'09''$ **47.** $b = 43.227$; $c = 24.339$; $B = 130°19'$ **49.** $a = 13.398$; $c = 29.091$; $B = 111°36'58''$ **51.** $b = 7{,}363.2$; $c = 6{,}152.9$; $A = 56°54'48''$ **53.** $a = 61.048$; $c = 54.981$; $A = 107°26.7'$ **55.** $b = 1{,}964.2$; $c = 2{,}272.1$; $C = 122°45.7'$ **57.** $b = 413.14$; $c = 615.36$; $B = 30°45'51''$ **59.** $b = 96.050$; $c = 74.784$; $A = 113°50.8'$

Exercise 3: **9.** 7.1414 **11.** 16.894 **13.** 10.890 **15.** 33.734 **17.** 13.416 **19.** $A = 46°23'49''$; $B = 104°15'00''$; $C = 29°21'09''$ **21.** $A = 61°55'42''$; $B = 81°12'10''$; $C = 36°52'11''$ **23.** $A = 34°02'53''$; $B = 44°24'54''$; $C = 101°32'13''$ **25.** $A = 44°24'54''$; $B = 57°07'18''$; $C = 78°27'47''$ **27.** $A = 42°32'13''$; $B = 56°18'45''$; $C = 81°09'02''$ **29.** $A = 17°36'47''$; $B = 28°57'17''$; $C = 133°25'57''$

Exercise 4: **1.** $A = 41°50'$; $C = 27°26'$; $b = 36.828$ **3.** $B = 12°51'47''$; $C = 18°28'13''$; $a = 3.1647$ **5.** $B = 38°02'07''$; $C = 132°21'53''$; $a = 29.775$ **7.** $B = 78°25'56''$; $C = 96°10'04''$; $a = 0.33528$ **9.** $B = 137°31'58''$; $C = 33°10'02''$; $a = 1.8279$ **11.** $B = 115°26'56''$; $C = 53°07'48''$; $a = 24.999$ **13.** $B = 15°37'10''$; $C = 151°42'00''$; $a = 132.37$ **15.** $B = 126°52'10''$; $C = 30°30'38''$; $a = 43.633$ **17.** $B = 143°07'50''$; $C = 8°47'50''$; $a = 6.9813$ **19.** $B = 100°22'06''$; $C = 46°12'58''$; $a = 61.024$

Exercise 5: **1.** $A = 56°15'02''$; $B = 71°41'42''$; $C = 52°03'18''$ **3.** $A = 18°37'$; $B = 13°50'58''$; $C = 147°32'$ **5.** $A = 37°11'06''$; $B = 58°29'24''$; $C = 84°19'36''$ **7.** $A = 12°59'36''$; $B = 51°13'12''$; $C = 115°47'6''$ **9.** $A = 11°02'18''$; $B = 14°18'38''$; $C = 154°39'10''$ **11.** $A = 32°20'36''$; $B = 119°57'10''$; $C = 27°42'18''$ **13.** $A = 114°22'26''$; $B = 51°43'12''$; $C = 13°54'22''$ **15.** $A = 14°20'56''$; $B = 28°19'24''$; $C = 137°19'44''$ **17.** $A = 31°20'02''$; $B = 130°13'00''$; $C = 18°27'00''$ **19.** $A = 24°10.2'$; $B = 36°53.6'$; $C = 118°56.2'$

Exercise 6: **1.** 803.00 **3.** 1,200.1 **5.** 4,853.6 **7.** 206,930 **9.** 72,477 **11.** 75.013 **13.** 4,064,500 **15.** 3,912,400

CHAPTER 20

Exercise 1: **1.** Error in departure = 0.06 ft; error in latitude = 0.18 ft; linear error = 0.19 ft **3.** S 29°44'37" E; 909.90 ft **5.** Error in departure = 0.63 ft; error in latitude = 0.19 ft; linear error = 0.66 ft **7.** $CJ = 711.85$ ft; $BK = 864.53$ ft (some variation permissible because of choice of method)

Exercise 2: **1.** 198,218 sq ft **3.** 30,641,000 sq ft

Exercise 3: **3.** 597.75 **5.** 133⅓

Exercise 4: **9.** $AB = 636.10$ **11.** $ac = 4{,}542.9$ yd; $bc = 3{,}673.4$ yd; $ad = 5{,}038.2$ yd

Exercise 5: **1.** 8°3'58" **3.** 0.98059 **5.** 24°49'47" **7.** 43°55'10" **9.** 1.1516 **11.** 02°59'32" **13.** 1.0774 **15.** 25°22'11" **17.** 4.997 **19.** 122°10'21" **21.** 5.8743 **23.** 1.6043 **25.** $x = 0.3211$; $y = 1.4756$

Exercise 6: **5.** 33°51'16" **7.** 28°02'32" **9.** $\phi = 17°29'43''$ **11.** $\phi = 55°28'54''$

CHAPTER 21

Exercise 1: **1a.** 100' W; 400' S **1b.** 200' E; 200' N **1c.** 200' E; 600' S **3a.** $x = +6.3$; $y = -2.8$ **3c.** $x = -3.2$; $y = +15.8$ **3e.** $x = +6.0$; $y = -3.0$

Exercise 2: **1a.** $x = +2.67$; $y = +1.362$ **1c.** $x = -8.66$; $y = +5.00$ **1e.** $x = 0.00$; $y = +7.00$ **1g.** $x = +13.12$; $y = +7.27$ **1i.** $x = -14.14$; $y = -14.14$ **1k.** $x = -1.532$; $y = -1.286$ **1m.** $x = 0.00$; $y = -3.00$ **1o.** $x = +4.33$; $y = +2.50$

Exercise 3: **1a.** $x = -22.9$; $y = -6.43$; $\rho = 23.8$; $\phi = 195.7°$ **1c.** $x = 1.589$; $y = -6.07$; $\rho = 6.28$; $\phi = 284.7°$ **1e.** $x = 0$; $y = 40$; $\rho = 40$; $\phi = 90°$ **3d.** $x = -111.2$; $y = -73.2$; $\rho = 133.2$; $\phi = 213.4°$; **3f.** $x = -18.91$; $y = -7.93$; $\rho = 20.5$; $\phi = 202.7°$ **3h.** $x = -55.8$; $y = 45.9$; $\rho = 72.2$; $\phi = 140.6°$ **3j.** $x = -30.4$; $y = 19.54$; $\rho = 36.2$; $\phi = 147.3°$

Exercise 4: **1.** $R = 821.90$; $\phi = 5°35'33''$ **3.** 40°32'09" and 32°21'34" **5.** $F_1 = 44.425$; $F_2 = 422.67$ **7.** F to submerge 19.966; F horizontal 15.045 **9.** Length $BC = 31.855'$; force in $BC = 740.81$ lb; force in $AB = 976.73$ lb

Exercise 5: **1.** $\rho = 6$; $\phi = 350°$ **3.** $\rho = 20$; $\phi = 70°$ **5.** $\rho = 24$; $\phi = 139°$ **7.** 26.2/139.6° **9.** 28.2/96.1° **11.** $\rho = 3.00$; $\phi = 60°$ **13.** $\rho = 2.50$; $\phi = 320°$ **15.** $\rho = -5.00$; $\phi = 350°$; $\rho = +5.00$; $\phi = 170°$ **17.** 2.86/347.9° **19.** 1.414/202.4° **21.** 4.24/126.3° **23.** 6.87/29.9° **25.** 8.78/32.7° **27.** 17.19/219.9°

790

Exercise 6: **1.** 2.18/36.2° **3.** 10.12/42.3° **5.** 33.9/61.3°

CHAPTER 22

Exercise 2: **1.** 30°, 150°, 270° **3.** 45°; 225° **5.** no solution **7.** 78°
9. 35°16′, 324°44′; 144°44′, 215°16′

CHAPTER 23

Exercise 1: **1.** 0.868; 3.83; 1.71; −4.70; −4.33

Exercise 2: **1a.** 8.55 **1b.** 25 **1c.** 19.15 **1d.** 0 **1e.** −12.5 **1f.** −25
1g. −21.7 **1h.** 0 **5.** 25.4 **7.** 45.6°; 134.4°; 225.6°; 314.4° **9.** 40° **11.** 26.1°

Exercise 3: **3.** 28.6 **5.** 31.4 **7.** 31.5 **9.** −91.0

Exercise 4: **7a.** $y = 200 \sin\left(60\pi t + \frac{5\pi}{18}\right)$ **7c.** $y = 5 \sin (2\pi \times 10^{12} t + \pi/2)$
7e. $y = 10 \sin (2{,}000\pi t + \pi/4)$ **7g.** $y = 20 \sin 50\pi t$ **7i.** $y = 100 \sin (100\pi t + \pi/2)$

Exercise 5: **2a.** $Y_3 = 4.11$; $B = 9.3°$ **2c.** $Y_3 = 52.9$; $B = -40.9°$

CHAPTER 24

Exercise 1: **1.** 5/53.1° **3.** 7.62/113.2° **5.** 5.00/306.9° **7.** 5.83/211.0°
9. 2.83/135° **11.** 2.83/225°

Exercise 2: **1.** 10/53.1° **3.** 7.21/33.7° **5.** 5.83/121.0° **7.** 11.71/20.0°
9. 10.63/131.2° **11.** 8.06/209.7° **13.** 5.10/258.7° **15.** 16.55/245.0°
17. 15.81/288.4° **19.** 8.02 + *j*13.25 **21.** 10.62 + *j*24.8 **23.** 10.13 + *j*11.84
25. 2.02 + *j*2.37 **27.** 37.1 − *j*17.31 **29.** −10.72 − *j*3.19

Exercise 3: **1.** 1 + *j*8; 8.06/82.9° **3.** −5 + *j*; 5.10/168.7° **5.** 3 + *j*0; 3/0°
7. −4.28 + *j*4.32; 6.08/134.7° **9.** 7.59 − *j*11.75; 13.94/302.9°

Exercise 4: **1.** 26.2/139.6° **3.** 28.2/96.1° **5.** −29 + *j*29 **7.** 11 − *j*27
9. −41 + *j* **11.** 29 − *j*3 **13.** 41 − *j* **15.** −29 + *j*3 **17.** 29 − *j*29
19. −11 + *j*27

Exercise 5: **1a.** 0.1509 − *j*0.472; 0.495/287.7° **1b.** 2.80 − *j*0.6; 2.86/347.9°
1c. 2.08 − *j*0.615; 2.17/343.5° **1d.** −1.308 − *j*0.538; 1.414/202.4°
3. 0 + *j*0.667; 0.667/90° **5.** 1.154 − *j*1.769; 2.11/303.1° **7.** 0.533 + *j*0.4000; 0.667/36.9° **9.** −2.08 − *j*0.385; 2.11/190.5° **11.** −0.533 − *j*0.400; 0.667/216.9°
13. 2.08 + *j*0.385; 2.11/10.5° **15.** 0 − *j*0.667; 0.667/270°
17. −1.154 + *j*1.769; 2.11/123.1°

Exercise 6: **1.** 7.5000 − *j*1.5000 **3.** 4.6180 − *j*1.8371 **5.** 2.9311 + *j*1.8857
7. 33.449 − *j*10.829

Exercise 7: **1.** 5.83(cos 59° + *j* sin 59°) **3.** −96.4 + *j*114.9

Exercise 8: **1.** 125/60° **3.** 16/240° **5.** 625/0°

Exercise 9: **1.** 2/10°; 2/130°; 2/250° **3.** 2/0°; 2/180° **5.** 3/70°; 3/160°; 3/250°; 3/340°

Exercise 10: **1.** 1; −1 **3.** 1; (0.309 + *j*0.951); (−0.809 + *j*0.588);

Exercise 11: **1.** $3; \left(-\frac{3}{2}+j\frac{3\sqrt{3}}{2}\right); \left(-\frac{3}{2}-j\frac{3\sqrt{3}}{2}\right)$ **3.** $\left(\frac{3}{2}+j\frac{3\sqrt{3}}{2}\right); \left(-\frac{3}{2}+j\frac{3\sqrt{3}}{2}\right); (-3+j0); \left(-\frac{3}{2}-j\frac{3\sqrt{3}}{2}\right); \left(\frac{3}{2}-j\frac{3\sqrt{3}}{2}\right); (3+j0)$

Exercise 12: **1.** 0.87758 **3.** 0 **5.** −0.24740

Exercise 13

	Amplitude	*Frequency*	*Period*	*Phase constant*
1.	10	60	0.0167	+60°
3.	10	60	0.0167	+60°
5.	25	400	0.0025	0

CHAPTER 25

Exercise 1: **1a.** $-\frac{1}{4}$ **1b.** −3.6 **1c.** −0.00025 **3a.** $\begin{cases}\Delta x = -20\\ \Delta y = +5\end{cases}$ **3b.** $-\frac{1}{4}$ **3c.** $\begin{cases}\Delta x = +20\\ \Delta y = -5\end{cases}$ **3d.** $-\frac{1}{4}$

Exercise 2: **1a.** 30 lb **1b.** 60 in.-lb

Exercise 4: **1.** 0.909 **3.** The slope of the curve at point P is greater than 0.909 and less than 4.3

Exercise 5: **1.** 0.74 **3.** +3.33 **5.** −1.145 **7.** $x = 1; y = 2.5$

Exercise 6: **1a.** 0 **1b.** −162 **1c.** −106 **1d.** 0 **1e.** −13 **1f.** −46 **1g.** −112 **1h.** −113 **1i.** −95 **1j.** 0 **3.** 66⅔ mph **5.** 20 sec **7.** 10 by 10 in. **9.** 125 ft **11.** $\frac{5\sqrt{3}}{2} = 4.33$ in.

Exercise 7: **1.** 22.3

Exercise 8: **1.** 78.5 **3.** 8.86; 0.113 **5.** 5.87 **7.** 6.64 **9.** 8.86 **11.** 7.68 **13.** 58.9 **15.** 0.00393 **17.** 73.6

Exercise 9: **1.** −1.35 **3.** −2 **5.** 3.5 **7.** −3.9 **9.** −3.7 **11.** −6.0 **13.** −69½° **15.** 6.93 **17.** 61.85 **19.** 15.46 **21.** −4.13 **23.** 4.58 **25.** 0.01 **27.** 3.47; 6.93; 10.4; 13.9

Exercise 10: **1.** $-\frac{5}{8}$ **3.** −0.4 **5.** $-\frac{5}{3}; -\frac{3}{5}$ **9.** −2.5 **11.** −0.0112

Exercise 11: **1.** 2,038 **3.** 0.062 and 100.3 **5.** 2 ft

Exercise 12: **1.** 200 amp/sec **3.** 0.05 sec

Exercise 13: **1.** 8.16 ft per sec² **3.** 66⅔ mph at $t = 20$ sec **5.** 20 sec **7.** 6.26 **9.** 20.5 mph **11.** $t = 8$ **13.** (8, 28) and (30, 58)

Exercise 14: **5.** Approx 2,000′ change in elevation (approx 61,000′ total movement) **7.** 2,820 fps downward **9.** 437.2 sec **11.** 75 ft per sec² **13.** 66 sec 4,348 fps **15.** 206 sec **17a.** 3,270 fps **17b.** 3,880 fps **17c.** −24.5 ft per sec² **17d.** 75 ft per sec² **17e.** 61,400 ft (31,700 ft up and 29,700 ft down) approx values

Exercise 15: 0.842 hr

CHAPTER 26

Exercise 1: 1. $15x^2 + 15x(\Delta x) + 5(\Delta x)^2$ 3. $14x + 7(\Delta x)$

5. $\dfrac{3}{(t + \Delta t + 3)(t + 3)}$ 7. $\dfrac{1}{\sqrt{x + \Delta x} + \sqrt{x}}$ 9. $-\dfrac{3}{u(u + \Delta u)}$

11. $4\pi R^2 + 4\pi R(\Delta R) + \frac{1}{3}\pi(\Delta R)^2$

Exercise 2: 1a. \$10.50, \$15, \$60, \$510, \$5,010, \$50,010
1b. \$1.05, \$0.15, \$0.06, \$0.051, \$0.0501, \$0.05001 1c. \$0.05

Exercise 3: 1. $15x^2$ 3. $14x$ 5. $\dfrac{3}{(t + 3)^2}$ 7. $\dfrac{1}{2\sqrt{x}}$ 9. $-\dfrac{3}{u^2}$ 11. $4\pi R^2$

Exercise 4: 1. $6x$ 3. $\dfrac{3}{\sqrt{x}}$ 5. $-\dfrac{14}{x^2}$ 7. $\dfrac{4}{x^{3/2}}$ 9. $-\dfrac{a}{bx^2}$

Exercise 6: 1. $2ax + b$ 3. $12x - 13$ 5. $\dfrac{15x - 4}{6x^3}$ 7. $\dfrac{n - 1}{n\sqrt[n]{x}}$

9. $-12x^{-5}$ 11. $-\dfrac{104}{x^5}$ 13. $-3kx^{-4}$ 15. $-15x^4$ 17. $2\pi x$ 19. $\dfrac{3e - 2\pi x}{x^4}$

Exercise 7: 1. $8(3x + 1)(3x^2 + 2x + 5)^3$ 3. $\dfrac{3 - 10x}{3(-5x^2 + 3x)^{2/3}}$

5. $-\dfrac{7t}{2(6 - 7t^2)^{3/4}}$ 7. $-3t(1 - t^2)^{-7/4}$ 9. $\dfrac{\sqrt{2}}{2\sqrt{t}} + \dfrac{2}{3t^{2/3}}$ 11. $\dfrac{8y}{3\sqrt[3]{1 + y^2}}$

13. $\dfrac{12x^2}{(1 - x^3)^2}$ 15. $\dfrac{4t^2(3 - t^2)}{(5t^3 - t^5)^{4/5}}$ 17. $\dfrac{dy}{dx} = -\dfrac{10}{(x + 3)^2}; -\dfrac{10}{9}$

Exercise 8: 1. Max at $x = -4$, min at $x = 1$ 3. Max at $x = -3$, min at $x = 3$
5. Min at $x = -2$ 7. Max at $x = \frac{3}{2}$ and $x = -\frac{3}{2}$ 9. 125 ft 11. $1\frac{3}{8}$ in.
13. $12\frac{1}{2}$ miles 15. 5 or 6 men 17. 4.08 hr 19. 21 or 22 times 21. 14.4 mph

Exercise 9: 1. $-12\theta^3 \sin \theta^4$ 3. $\dfrac{10}{x^2} \sin \dfrac{2}{x}$ 5. $27\frac{1}{4}\theta^{-1/4} \sin \theta^{3/4}$

7. $\dfrac{\pi}{2} - \dfrac{1}{\sqrt{1 - t}} \sin \sqrt{1 - t}$ 9. $-\left(1 + \dfrac{1}{x^2}\right) \sin \left(x - \dfrac{1}{x}\right)$

Exercise 10: 1. $4x^3 \cos x^4$ 3. $-\dfrac{2}{x^2} \cos \dfrac{2}{x}$ 5. $-9\frac{1}{2}x^{-1/2} \cos x^{3/4}$ 7. $t = 1$

9. See Exercise 11, page 613 9f. Acceleration = zero at $t = \frac{1}{16}$

Exercise 11: 1. $10xe^{5x^2}$ 3. e^{x+8} 5. $\dfrac{1}{2\sqrt{1 - x}} e^{(1-x)^{1/2}}$

7. See Exercise 9, page 610
9. See Exercise 12, page 614.

Exercise 12: 1. $\dfrac{1}{x}$ 3. $\dfrac{5}{x}$ 5. $\dfrac{3}{x}$ 7. $\dfrac{65.14}{x}$

Exercise 13: 1. $\frac{2x}{x^2+2}$ 3. $\frac{8x}{x^2+3}$ 5. $\frac{1}{3x-4}$ 7. $\frac{0.8686x}{x^2-4}$ 9. $\frac{0.4343x}{9-x^2}$

11. $\frac{0.8686}{2x-3}$ 13. $\frac{6x^2}{x^3-8}\ln(x^3-8)$ 15. $0.4343\cot(\theta+\pi)$

Exercise 14: 1. $a^x \ln a$ 3. $\frac{2\sqrt{x}-1}{\sqrt{x}}\ln 2$

Exercise 15: See Exercise 13, page 615.

Exercise 16: 1. Max at $x=-2$, min at $x=+2$ 3. Min at $x=0$, max at $x=-2$
5. Min $t=90°$, $270°$, Max $t=14°28'39''$, $165°31'21''$
7. Max $13°17'$; min $103°17'$; max $193°17'$; min $283°17'$ 9. 2
11. $h=4.33$ 13. $54°44'$ 15. 66° approx.

Exercise 17: 1. $\cot x$ 3. $15(\sin^2 5x)(\cos 5x)$ 5. $\frac{1}{x^2(a+bx)}$

7. $\frac{x^3+2a^2x}{(x^2+a^2)^{3/2}}$ 9. $-\frac{2(x^3+2)}{(x^3-4)^2}$ 11. $e^{-2x}(\cos x-2\sin x)$

13. $-\frac{4}{(e^x-e^{-x})^2}$ 15. $\frac{10}{(1-x)^3}$ 17. $\frac{1}{x^2\sqrt{x^2-1}}$ 19. $\sqrt{x^2+a^2}$

21. $(16-x^2)^{-3/2}$ 23. $\frac{1}{x\sqrt{8x-2x^2}}$ 25. $\cos^2 x$ 27. $x\sin^2 x$ 29. $\tan^2 x$

31. $x\cos x$

Exercise 18: 3. 0.4 5. Max 2.5; min 5.2 7. Max 2.5; min 5.2 9. One
11. One 13. $x=3.84$ min 15. $x=3.84$

Exercise 19: 1. $6x-5+\frac{4}{x^2}$; $6-\frac{8}{x^3}$; $\frac{24}{x^4}$ 3. $4\pi R^2$; $8\pi R$; 8π 5. $6e^{2t}$; $12e^{2t}$; $24e^{2t}$

7. $-\frac{1}{x}$; $\frac{1}{x^2}$; $-\frac{2}{x^3}$ 9. $2\theta\cos\theta^2$; $2\cos\theta^2$; $-4\theta^2\sin\theta^2$; $-8\theta^3\cos\theta^2$; $-12\theta\sin\theta^2$

11. $V=72-32t$; $a=-32$ 13. -12 at $x=2$ 15. $1\frac{5}{8}\sqrt{\frac{3}{8}}$ at $x=-\sqrt{\frac{3}{8}}$; $-1\frac{5}{8}\sqrt{\frac{3}{8}}$ at $x=\sqrt{\frac{3}{8}}$ 17. $-\frac{1}{2}$ at $x=-\sqrt{3}$; 4 at $x=0$; $1-\frac{1}{2}$ at $x=\sqrt{3}$

19. Max v at $t=5$; max a at $5\frac{1}{2}$

CHAPTER 27

Exercise 1: 9a. x^2+C 9b. x^3+C 9c. $\frac{1}{2}x^2+C$ 9d. $10x+C$
9e. $-2x+C$ 9f. $\frac{3}{7}x^{7/3}+C$ 9g. $\frac{3}{4}x^{4/3}+C$ 9h. $\frac{3}{2}x^{2/3}+C$ 9i. $6x^{1/6}+C$

11a. $\frac{x^4}{4}-2x^3+250x^2+7x+C$ 11b. $\frac{6}{7}x^{7/6}-12x^{1/6}+\frac{6}{5}x^{5/6}+\pi x+C$

Exercise 2: 1a. 4 1b. -44 1c. 12 1d. -20 1e. 28 1f. $-16\frac{4}{7}$
1g. 15.24 1h. 16.22 1i. 8 3. $+\frac{(3-5x)^{-5}}{25}+C$ 5. $\frac{(4\phi-1)^{3/2}}{6}+C$

7. $\frac{(2-\phi^4)^{-5}}{20}+C$ 9. $\frac{1}{3}(r^2-6r-7)^{3/2}+C$

Exercise 3: **1.** $V = -32t$; $h = -16t^2 + h_0$ **3.** $V = -32t + 100$; $h = -16t^2 + 100t + 192$ **5.** -149.28 fps; -149.28 fps **7.** $t = 8$ sec.

Exercise 4: **1.** 78 **3.** 36 **5.** $7\frac{1}{2}$ **7.** $65\frac{1}{3}$

Exercise 5: See Ans. to Exercise 4

Exercise 6: **1.** 13 **3.** 6 **5.** $5\frac{1}{2}$

Exercise 7: **1.** $\ln(x + 3) + C$ **3.** $-\frac{2}{3}\ln(1 - 4h^3) + C$
5. $\frac{1}{3}\ln(\theta^3 - 3\theta^2 + 3\theta) + C$ **7.** $\frac{1}{3}\ln(t^3 - 3t) + C$
9. Dist = 4366′; Arvel = 62.37 ft/sec; $\frac{1}{4}$ dist at 20.47 sec

Exercise 8: **1.** $2\sin 2\theta + C$ **3.** $\frac{1}{2}\sin x^2 + C$ **5.** $-\sin(4 - x) + C$
7. $\sec\phi + C$

Exercise 9: **1a.** 0.26795; 1.0235 **1b.** -0.051777; -0.98887 **1c.** 0.10371; 0.3457
1d. -5; -6.3662 **1e.** 0.84147; 0.84147 **1f.** -1.1716; -0.373 **1g.** 1; 0.955
1h. 0; 0 **3.** 51.56

Exercise 10: **1.** $-\frac{1}{4}e^{1-4x} + C$ **3.** $\frac{1}{\pi}e^{\pi x} + C$ **5.** $-\frac{\epsilon^{1/x}}{5} + C$ **7.** $\frac{1}{3}e^{3x+1} + C$

9. $\frac{1}{2}e^{2x} + 2e^{-x} - \frac{1}{4}^{-4x} + C$ **11.** $\frac{\epsilon^{6x}}{6} - \frac{2\epsilon^{3x-1}}{3} + \frac{x}{\epsilon^2} + C$

13. $\frac{\epsilon^{4x}}{4} - \frac{4\epsilon^{5x/2}}{5} + \epsilon^x + C$

Exercise 11: **1.** $33{,}510.292 = \frac{32{,}000\pi}{3}$ **3.** 20.834 **5.** $5{,}333\frac{1}{3}$ **7.** 2,044 cu in.