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ON STOCHASTIC DIFFERENTIAL EQUATIONS

BY

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ON STOCHASTIC DIFFERENTIAL EQUATIONS

By

KIYOSI ITO

Let x_t be a simple Markoff process with a continuous parameter t, and $F(t,\xi;s,E)$ be the transition probability law of the process:

(1) $F(t,\xi;s,E) = Pr\{x_s \in E/x_t = \xi\}$,

where the right side means the probability of $x_s \in E$ under the condition: $x_t = \xi$. The differential of x_t at $t = s$ is given by the transition probability law of x_t in an infinitesimal neighborhood of $t = s$:

(2) $F(s-\Delta_2, \xi; s+\Delta_1, E)$.

W. Feller[1] has discussed the case in which it has the following form:

(3) $F(s-\Delta_2, \xi; s+\Delta_1, E) = (1-p(s,\xi)\ (\Delta_1+\Delta_2)G(s-\Delta_2, \xi; s+\Delta_1, E)$

$+(\Delta_1+\Delta_2)p(s,\xi)P(s,\xi,E) + o(\Delta_1+\Delta_2)$,

where $G(s-\Delta_2, \xi; s+\Delta_1, E)$ is a probability distribution as a function of E and satisfies

(4) $\frac{1}{\Delta_1+\Delta_2} \int_{|\eta-\xi|>\delta} G(s-\Delta_2, \xi; s+\Delta_1, d\eta) \longrightarrow 0$,

(5) $\frac{1}{\Delta_1+\Delta_2} \int_{|\eta-\xi|\leq\delta} (\eta-\xi)^2 G(s-\Delta_2, \xi; s+\Delta_1, d\eta) \longrightarrow 2a(t,\xi)$,

(6) $\frac{1}{\Delta_1+\Delta_2} \int_{|\eta-\xi|\leq\delta} (\eta-\xi) G(s-\Delta_2, \xi; s+\Delta_1, d\eta) \longrightarrow b(t,\xi)$,

for $\Delta_1+\Delta_2 \longrightarrow 0$ and $p(s,\xi) \geq 0$ and $P(s,\xi,E)$ is a probability distribution in E. The special case of "$p(s,\xi) = 0$" has already been treated by A. Kolmogoroff[2] and S. Bernstein.[3]

We shall introduce a somewhat general definition of the differential of the process x_t (Cf. §5). Let $P_{s,\xi,\Delta_1,\Delta_2}$ denote the conditional probability law:

$Pr\{x_{s+\Delta_1} - x_{s-\Delta_2} \in E/x_{s-\Delta_2} = \xi\}$, $\Delta_1, \Delta_2 \geq 0$.

If the $[1/\Delta_1+\Delta_2]$- times[4] convolution of $P_{s,\xi,\Delta_1,\Delta_2}$ tends to a probability law $L_{s,\xi}$ with regard to Lévy's law-distance as $\Delta_1+\Delta_2 \longrightarrow 0$, then $L_{s,\xi}$ is called the stochastic differential coefficient at s. $L_{s,\xi}$ is clearly an infinitely divisible law. In the above Feller's case the logarithmic characteristic function [5]

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$\psi(z, L_{s,\xi})$ of $L_{s,\xi}$ is given by

(7) $$(z, L_{s,\xi}) = ib(s,\xi)z - a(s,\xi)z^2 + p(s,\xi)\int_{-\infty}^{\infty}(e^{iuz}-1)P(s,\xi,du(+)\xi).$$ [6]

A problem of stochastic differential equations is to construct a Markoff process whose stochastic differential coefficient $L_{t,\xi}$ is given as a function of (t,ξ). W. Feller has deduced the following integro-differential equation from (3), (4), (5) and (6):

(8) $$\frac{\partial}{\partial t}F(t,\xi;s,E) + a(t,\xi)\frac{\partial^2}{\partial \xi^2}F(t,\xi;s,E) + b(t,\xi)\frac{\partial}{\partial \xi}F(t,\xi;s,E) - p(t,\xi)F(t,\xi;s,E) + p(t,\xi)\int_{-\infty}^{\infty}F(t,\eta;s,E)P(t,\xi,d\eta) = 0.$$ He has proved the existence and uniqueness of the solution of this equation under some conditions and has shown that the solution becomes a transition probability law, and satisfies (3),(4),(5) (6). He has termed the case: $p(t,\xi) \equiv 0$ as continuous case and the case: $a(t,\xi) \equiv 0$ and $b(t,\xi) \equiv 0$ as purely discontinuous case.

It is true that we can construct a simple Markoff process from the transition probability law by introducing a probability distribution into the functional space R^R by Kolmogoroff's theorem,[7] but it is impossible to discuss the regularity of the obtained process, for example measurability, continuity, discontinuity of the first kind etc., as was pointed out by J. L. Doob.[8] To discuss the measurability of the process for example, J. L. Doob has introduced a probability distribution on a subspace of R^R and E. Slutsky has introduced a new concept "measurable kernel".[9] We shall investigate the sense of the term "continuous case" and "purely discontinuous case" used by W. Feller from the rigorous view-point of J. L. Doob and E. Slutsky. A recent research of J. L. Doob[10] concerning a simple Markoff process taking values in an enumerable set has been achieved from this view-point. A research of R. Fortet[11] concerning the above continuous case seems also to stand on the same idea but the author is not yet informed of the details.

In his paper "ON STOCHASTIC PROCESSES (I)" [12] the author has deduced Lévy's canonical form of differential processes with no fixed discontinuities by making use of the rigorous scheme of J. L. Doob. Using the results of the above paper, we shall here construct the solution of the above stochastic differential equation in such a way that we may be able to discuss the regularity of the solution. For this purpose we transform the stochastic differential equation into a stochastic integral equation.

The first and most simple form of stochastic integral is Wiener's integral[13] which is an integral of a function $\sigma(t) \in L_2$ based on a brownian motion $g(t)$: $\int \sigma(t)dg(t)$. In this integral $\sigma(t)$ is not a random function. The author has ex-

extended this notion and defined an integral in case $\sigma(t)$ is a random function satissying some conditions.[14] A brownian motion is a temporally homogeneous and differential (i.e. spatially homogeneous) process with no moving discontinuity. The process $x(t) = \int_a^t \sigma(t)dg(t)$ obtained by Wiener's integral is not temporally homogeneous but spatially homogeneous. In order to obtain a simple Markoff process--which is in general neither temporally nor spatially homogeneous--we shall have to solve a stochastic integral equation:

$$x(t) = \int_a^t \sigma(\tau, x(\tau))dg(\tau)$$

or more generally

$$x(t) = c + \int_a^t m(\tau, x(\tau))d\tau + \int_a^t \sigma(\tau, x(\tau))dg(\tau).$$

The author has published a note[15] on this stochastic integral equation, which concerns the continuous case above mentioned.

In order to discuss the general case we shall have to consider a stochastic integral equation where the integral is based not on a brownian motion but on a more general temporally homogeneous differential process, which will be called a fundamental differential process (Cf. § 6) in this paper.

Chapter I is devoted to the explanation of the fundamental concepts. Some of them are well-known but we shall explain them in a rigorous form for the later use. In Chapter II we shall introduce a stochastic integral of a general type. The results of the author's previous paper[16] will be contained here in an improved form. The aim of this paper will be attained in Chapter III, where we shall investigate a stochastic differential equation and a stochastic integral equation.

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I. Fundamental concepts.

§1. Function of random variables. Let X be any set and B_X be a completely additive class of subsets of X. When we consider X together with B_X we call it a Borel field (X, B_X). It is evident that B_X may be arbitrarily taken, but in case X is the real number space R^1, then we usually take the system B^1 of all Borel subsets of R^1 as B_X, and in case X is R^A, B_X is usually the least completely additive class that contains all Borel cyclinder subsets of R^A, which we denote by B^A. If B_X and B_Y are associated respectively with X and with Y, then we usually associate with the product space the least completely additive class that contains all the sets of the form: $E_X \otimes Y$, $X \otimes E_Y$, $E_X \in B_X$, $E_Y \in B_Y$; this class will be denoted by $B_X \otimes B_Y$. The product of many Borel fields can be similarly defined.

Let (X,B_X) and (Y,B_Y) be Borel fields. A mapping $f(x)$ from X into Y is called to be B-measurable if $f^{-1}(E_Y) \in B_X$ for any $E_Y \in B_Y$. If $f(x)$ is a B-measurable mapping from (X,B_X) into (Y,B_Y) and if $g(x)$ is a B-measurable mapping from (Y,B_Y) into (Z,B_Z), then $g(f(x))$ will be a B-measurable mapping from (X,B_X) into (Z,B_Z).

Let (Ω,B_Ω,P) be a probability field, where Ω is a set, B_Ω is a completely additive class of subsets of Ω, and P is a probability distribution (p.d.) on (Ω,B_Ω). An (X,B_X)-valued function $x(\omega)$ defined on Ω is called an (X,B_X)-valued random variable, if it is B-measurable i.e. $x^{-1}(E_X) \in B_\Omega$ for any $E_X \in B_X$. If we put $P_x(E_X) = P(x^{-1}(E_X))$ for $E_X \in B_X$. P_x is a p.d. on a Borel field (X,B_X) which is called the probability law (P.ℓ.) of x; we also say that x is governed by P_x.

Let $x(\omega)$ be an (X,B_X)-valued random variable and $f(.)$ be a B-measurable mapping from (X,B_X) into (Y,B_Y). Put $y(\omega) = f(x(\omega))$. Then $y(\omega)$ will be a (Y,B_Y)-valued random variable. $y(\omega)$ is called a B-measurable function of $x(\omega)$.

Theorem 1. Let $y_n(\omega)$, $n=1,2,\ldots$, be real-valued B-measurable functions of an (X,B_X)-valued random variable. If $y_n(\omega)$ be convergent in probability, then the limit variable $y(\omega)$ is also coincident with a B-measurable function of $x(\omega)$ up to P-measure 0.

Proof. By taking a subsequence if necessary, we may assume that $y_n(\omega)$ be convergent with P-measure 1. Put $y_n(\omega) = f_n(x(\omega))$. Then

$$P_x\Big(\bigcap_p \bigcup_q \bigcap_{m,n>q} \{\xi; |f_m(\xi) - f_n(\xi)| < 1/p\}\Big) = P\Big(\bigcap_p \bigcup_q \bigcap_{m,n>q} \{\omega; |f_m(x(\omega)) - f_n(x(\omega))| < 1/p\}\Big) = 1.$$

Put $f(\xi) = \lim f_n(\xi)$ in the above ξ-set and $= 0$ elsewhere. Then $f(\xi)$ is a B-measurable function of $\xi \in (X,B\ )$, since the above ξ-set belongs to B_X. We have clearly, with probability 1,

$$f(x(\omega)) = \lim_n f_n(x(\omega)) = \lim_n y_n(\omega) = y(\omega),$$

which completes the proof.

§2. Conditional probability law. Let $x(\omega)$ and $y(\omega)$ be random variables taking values in (X,B_X) and (Y,B_Y) respectively. A function $P_y(E_Y/\xi)$ of $E_Y \in B_Y$ and $\xi \in X$ will be called the (conditional) probability law of $y(\omega)$ under the condition that $x(\omega) = \xi$ and will be denoted by $P_y(E_Y/x(\omega) = \xi)$ or $\Pr\{y \in E_Y/x = \xi\}$, if and only if

(2.1) $P_y(E_Y/\xi)$ is a p.d. on (Y,B_Y) for any ξ,

(2.2) $P_y(E_Y/\xi)$ is a B-measurable function of $\xi \in (X,B)$ for any $E_Y \in B_Y$, and

(2.3) $\int_{E_X} P_y(E_Y/\xi) P_x(d\xi) = \Pr\{x \in E_X \,\&\, y \in E_Y\} = P(x^{-1}(E_X) \cap y^{-1}(E_Y))$.

The existence and uniqueness (up to P-measure 0) of $P_y(E_Y/\xi)$ was proved by J. L. Doob[17] in the case that (Y,B_Y) is the n-dimensional space (R^n,B^n).

$P_y(E_Y/x(\omega))$ i.e. the function of ω obtained by replacing ξ with $x(\omega)$ in $P_y(E/\xi)$ will be called the conditional p.ℓ. of $y(\omega)$ under the condition that $x(\omega)$ is determined and it will be also denoted by $\Pr\{y \in E_Y/x(\omega)\}$: this is clearly a real-valued random variable for any assigned E_Y. By (2.3) we have

(2.4) $\mathcal{E}\, P_y(E_Y/x(\omega)) = \Pr\{y \in E_Y\}$ ($\mathcal{E}$ = expectation.).

If the p.ℓ. of the combined random variable $(x(\omega),y(\omega))$, which clearly takes values in $(X \otimes Y, P_X \otimes B_Y)$, is coincident with the direct product measure of P_x and P_y: $P_x \otimes P_y$ on $(X \otimes Y, B_X \otimes B_Y)$ then $x(\omega)$ and $y(\omega)$ are called to be independent. The independence of many random variables can be similarly defined. Clearly we have

Theorem 2.1. $x(\omega)$ and $y(\omega)$ be independent. Then

$$P_y(E_Y/x(\omega) = \xi) = P_y(E_Y) \quad \text{for almost all } (P_x)\ \xi,$$

i.e.

$$P_y(E_Y/x(\omega)) = P_y(E_Y) \quad \text{for almost all } (P)\ \omega.$$

Theorem 2.2. $x(\omega)$ and $y(\omega)$ be independent. $G(\xi,\eta)$ be a B-measurable mapping from $(X \otimes Y, B_X \otimes B_Y)$ into (R^1,B^1). Put $z(\omega) = G(x(\omega),y(\omega))$. Then we have

$$P_z(E/x(\omega) = \xi) = \Pr\{G(\xi,y(\omega)) \in E\}$$

for almost all (P_x) ξ.

Proof. Since $x(\omega)$ and $y(\omega)$ are independent, we can make use of Fubini's theorem.

$$\Pr\{z \in E, x \in E_X\} = (P_x \otimes P_y)(\{(\xi,\eta); f(\xi,\eta) \in E, \xi \in E_X\})$$

$$= \int_{E_X} P_y(\{\eta; f(\xi,\eta) \in E\}) P_x(d\xi) = \int_{E_X} \Pr\{f(\xi, y(\omega)) \in E\} P_x(d\xi),$$

which completes the proof.

Theorem 2.3. $x(\omega)$ and $y(\omega)$ be independent. $G(\xi,\eta)$ be any real-valued B-measurable function in (ξ,η). If $G(x(\omega), y(\omega)) = 0$ with P-measure 1, then $G(\xi, y(\omega)) = 0$ with P-measure 1 for almost all (P_x) ξ.

Proof. By Theorem 2.2 we have $\int_X \Pr\{G(\xi, y(\omega)) = 0\} P_x(d\xi)$ $\Pr\{G(x(\omega), y(\omega)) = 0\} = 1$ and so $\Pr\{G(\xi, y(\omega)) = 0\} = 1$ for almost all (P_x) ξ.

§3. Transition probability law. $x(\tau,\omega)$ be a real random variable for any τ, $a \leq \tau \leq b$. The system $x(\tau,\omega)$, $a \leq \tau \leq b$, is called a stochastic process, which is also considered as an (R^I, B^I)-valued random variable, I being the interval $[a,b]$[18]. The p.l. of $x(s,\omega)$ under the condition that $(x(\tau,\omega), a \leq \tau \leq t)$[19] is determined:

(3.1) $\Pr\{x(s,\omega) \in E / x(\tau,\omega), a \leq \tau \leq t\}$ $(t < s)$

is called the transition probability law of this process. If this is equal to

(3.2) $\Pr\{x(s,\omega) \in E / x(t,\omega)\}$

for almost all (P) ω, the process is called a simple Markoff process. In such a process we put

(3.3) $F(t,\xi; s,E) = \Pr\{x(s,\omega) \in E / x(t,\omega) = \xi\}$.

Then we can easily prove, for almost all $(P_{x(t,\omega)})$ ξ,

(3.4) $$F(t,\xi; s,E) = \int_{-\infty}^{\infty} F(t,\xi; u, d\eta) F(u,\eta; s,E), \qquad (t < u < s),$$

which is well-known as Chapman's equation.

If $x(s_\nu,\omega) - x(t_\nu,\omega)$, $\nu = 1,2,\ldots,n$, are independent random variables for any system of non-overlapping intervals (t_ν, s_ν), $\nu = 1,2,\ldots,n$, then we call $x(\tau,\omega)$, $a \leq \tau \leq b$, a differential process. This is evidently a simple Markoff process whose transition p.l. is given by

(3.5) $F(t,\xi; s,E) = F_{t,s}(E(-)\xi)$,

where $F_{t,s}$ is the p.l. of $x(s,\omega) - x(t,\omega)$ and $E(-)\xi$ is the set $\{\eta - \xi; \eta \in E\}$; (3.5) will be obtained at once if we substitute $(x(\tau,\omega), a \leq \tau \leq t)$, $x(s,\omega) - x(t,\omega)$ and $x(s,\omega)$ respectively for $x(\omega)$, $y(\omega)$ and $z(\omega)$ in Theorem 2.2.

§4. THREE ELEMENTS OF AN INFINITELY DIVISIBLE LAW OF PROBABILITY.

The logarithmic characteristic function (l.c.f.) of an infinitely divisible law of probability (i.d.l.) can be expressed in the form:

$$(4.1)\quad imz - \frac{\sigma^2}{2} z^2 + \int_{|u|>1} (e^{if(u)z} - 1)\frac{du}{u^2} + \int_{|u|\le 1} (e^{if(u)z} - 1 - if(u)z)\frac{du}{u^2}$$

in one and only one way, where m is real, $\sigma \geq 0$, and $f(u)$ is monotone non-decreasing and right-continuous and

$$\int_{|u|\le 1} f(u)^2 \frac{du}{u^2} < \infty;$$

this formula is deduced at once from Levy's formula.[20] These m, σ, $f(u)$ will be called the <u>three elements</u> of this i.d.l. . The i.d.l. whose l.c.f. is

$$(4.2)\quad \Psi_0(z) = iz - \frac{z^2}{2} + \int_{|u|>1} (e^{iuz} - 1)\frac{du}{u^2} + \int_{|u|\le 1} (e^{iuz} - 1 - izu)\frac{du}{u^2},$$

i.e. $m = \sigma = 1$, $f(u) \equiv u$,

will be called the fundamental i.d.l. in this note.

Theorem 4.1. Let $m(L), \sigma(L)$ and $f(u,L)$ be the three elements of an i.d.l. L. Then $m(L), \sigma(L)$ and $f(u,L)$ (for any fixed u) are all B-measurable in $L=(L(E);E\in B^1)\in(R^{B^1},B^{B^1})$

Remark. By the expression " $m(L)$ is B-measurable in $L\equiv(L(E),E\in B^1)\in(R^{B^1},B^{B^1})$ we mean that there exists at least one B-measurable function $M(L)$ defined on the whole space (R^{B^1},B^{B^1}) such that we have $M(L) = m(L)$ for any $L \equiv (L(E), E\in B^1)$ that is an i.d.l. as a function of E.

Proof. Let $\phi(z,L)$ be the characteristic function of any i.d.l. L. For any z, $\phi(z,L)$ is B-measurable in $L\in(R^{B^1},B^{B^1})$, because, if we define $\Phi_z(L)$ by

$$\Phi_z(L) = \lim_{n\to\infty} \sum_{k=-n^2}^{n^2} \exp(ikz/n)L((k-1/n,k/n)) \text{ (if this limit exists)}$$

$$= 0 \text{ (if otherwise)},$$

then $\Phi_z(L)$ (for each z) is B-measurable function defined on the whole space (R^{B^1},B^{B^1}) and $\Phi_z(L) = \phi(z,L)$ for any i.d.l. L.

Let $\psi(z,L)$ be the logarithmic characteristic function of any i.d.l. Since $\psi(z,L)$ is the branch of $\log \phi(z,L)$ which is obtained from $\log \phi(0,L)=1$ by the analytic prolongation along the curve:

$$\phi(\lambda,L),\ o \leq \lambda \leq z \ (\text{or } o \geq \lambda \geq z)$$

and so it is expressible as

$$\psi(z,L) = \lim_{n\to\infty} \sum_{k=1}^{n} \int_{o}^{1} \frac{\phi(\frac{k}{n}z,L) - \phi(\frac{k-1}{n}z,L)}{(\phi(\frac{k}{n}z,L) - \phi(\frac{k-1}{n}z,L))t+1}\,dt$$

we see that $\psi(z,L)$ is also B-measurable in L for any z. By virtue of the Levy's formula $\psi(z,L)$ is written in the form

$$\psi(z,L) = i\bar{m}(L)z - \frac{\sigma^2(L)}{2}z^2 + \int_{-\infty}^{\infty}\left(e^{izu}-1-\frac{izu}{1+u^2}\right)n(du,L),$$

where the measure n is determined by the following procedure (Cf. A. Khintchine: Déduction nouvelle d'une formule de P. Lévy, Bull. d. l'univ. d'etat a Moscou, Serie International, Sect. A, Vol. 1, Fasc. 1, 1937),

$$\Delta(t,L) = \int_{t-1}^{t+1}\psi(z,L)dz - 2\psi(t,L),$$

$$K(u,L) = \frac{1}{2\pi}\lim_{c\to\infty}\int_{-c}^{c}\frac{1-e^{itu}}{it}\Delta(t,L)dt,$$

$$G(v,L) = -\int_{-\infty}^{v}\frac{1}{2(1-\frac{\sin u}{u})}dK(u,L),$$

$$n((a,\infty),L) = \int_{a-o}^{\infty}\frac{1+v^2}{v^2}dG(v,L) \qquad (a > 0),$$

$$n((-\infty,a),L) = \int_{-\infty}^{a+o}\frac{1+v^2}{v^2}dG(v,L) \quad (a < 0).$$

Therefore we can prove recursively the B-measurability of the above functions of L. Thus we obtain, for each $a > 0$, a B-measurable functions $N_a(L)$ defined on the whole space (R^{B^1},B^{B^1}) such that $N_a(L) = n((a,\infty),L)$ for any i.d.l. L. We may assume that $N_a(L)$ is monotone-decreasing and left-continuous in a for each L, by taking the supremum of

$N_r(L)$, r running over all rational numbers $r < a$, instead of $N_a(L)$, if necessary.

Now we shall prove, for each $u > 0$, that $f(u,L)$ is B-measurable in L. $f(u,L)$ is written in the following form by the definition.

$$f(u,L) = \inf\{a; n((a,\infty),L) < \frac{1}{u}\} \quad (u > 0).$$

Therefore, if we put

$$F_u(L) = \inf\{a; N_a(L) < \frac{1}{u}\},$$

$F_u(L)$ (for each $u > 0$) is a function defined on the whole space RB^1 and $F_u(L)=f(u,L)$ for any i.d.l. L. The B-measurability of $F_u(L)$ is clear on account of the fact that "$F_u(L) < a$" is equivalent to "$N_a(L) < \frac{1}{u}$", which follows from the definition of $F_u(L)$ and the monotone-property (in a) of $N_a(L)$. Thus we see that $f(u,L)$ (for each $u > 0$) is B-measurable in L. Similarly we can show that $f(u,L)$ $(u < 0)$ is B-measurable in L. It is clear that $f(0,L)$ $(\equiv 0)$ is B-measurable in L.

Now we put

$$\Phi(z,L) \equiv im(L)z - \frac{\sigma^2(L)}{2}z^2 \equiv \psi(z,L) - \int_{|u|>1} (\exp(izf(u,L))-1)\frac{du}{u^2}$$

$$- \int_{|u|\leq 1} (\exp(izf(u))-1-izf(u,L))\frac{du}{u^2}.$$

Then $\Phi(z,L)$ (for each z) is B-measurable in L, since $\psi(z,L)$ and $f(u,L)$ are B-measurable. But we have

$$m(L) = \frac{1}{2i}(4\Phi(1,L) - \Phi(2,L))$$

and

$$\sigma^2(L) = 2\Phi(1,L) - 4\Phi(2,L),$$

from which follows the B-measurability of $m(L)$ and $\sigma(L)$.

Theorem 4.2. Let L_α, $\alpha \in A$, be any system of i.d.l. depending on $\alpha \in A$ and m_α, σ_α and $f_\alpha(u)$ be the three elements of L_α. In order that L_α, $\alpha \in A$, be totally bounded in the sense of Levy's law-distance,[21] it is necessary and sufficient that each of $|m_\alpha|$, σ_α and $||f_\alpha||_n$, $n=1,2,\ldots$, is bounded, where

$$||f_\alpha||_n^2 = \int_{|u|\leq n} f_\alpha(u)^2 \frac{du}{u^2}.$$

Proof. L_α is decomposed as

$$L_\alpha = L_\alpha^{(1)} * L_\alpha^{(2)} * L_{\alpha\, n}^{(3)} * L_{\alpha\, n}^{(4)} * L_{\alpha\, n}^{(5)},$$

where the l.c.f. of the factors are respectively

$$im_\alpha z,\ -\frac{\sigma_\alpha^2}{2} z^2,\ iz\int_{1<|u|<n} f_\alpha(u)\frac{du}{u^2},\ \int_{o<|u|<n}(e^{if_\alpha(u)z}-1-if_\alpha(u)z)\frac{du}{u^2}$$

and $\displaystyle\int_{|u|\geq n}(e^{if_\alpha(u)z}-1)\frac{du}{u^2}$.

Sufficiency. If the condition is satisfied, $\{L_\alpha^{(1)}\}$, $\{L_\alpha^{(2)}\}$ are clearly totally bounded and $\{L_{\alpha\, n}^{(3)}\}$ is also totally bounded for any fixed n, since we have, by Schwarz' inequality,

$$\left|\int_{1<|u|<n} f_\alpha(u)\frac{du}{u^2}\right|^2 \leq 2\int_{1<|u|<n} f_\alpha(u)^2\frac{du}{u^2}.$$

$L_{\alpha\, n}^{(4)}$ has the expectation 0 and the standard deviation $\|f_\alpha\|$ and so $\{L_{\alpha\, n}^{(4)}, \alpha\in A\}$ is totally bounded. Therefore

$$\{L_{\alpha\, n}^* \equiv L_\alpha^{(1)} * L_\alpha^{(2)} * L_{\alpha\, n}^{(3)} * L_{\alpha\, n}^{(4)}, \alpha\in A\}$$

is totally bounded, and so we have

$$\lim_{c\to\infty} \inf_\alpha L_{\alpha\, n}^*((-c,c)) = 1.$$

But

$$L_\alpha((-c,c)) \geq L_{\alpha\, n}^*((-c,c))\, L_\alpha^{(5)}(\{0\}) = L_{\alpha\, n}^*((-c,c))\exp(-2/n).$$

Consequently we have

$$\lim_{c\to\infty} \inf_\alpha L_\alpha((-c,c)) \geq \exp(-2/n) \text{ and so } \lim_{c\to\infty} \inf_\alpha L_\alpha((-c,c))=1,$$

which completes the proof.

Necessity. Let Q(L,c) be Levy's concentration function[22] of the p.d.L. Suppose that $\{L_\alpha\}$ is totally bounded. Then

$$\inf_\alpha Q(L_\alpha^{(2)}, c) \longrightarrow 1 \qquad \text{as } c\longrightarrow\infty.$$

But we have $Q(L_\alpha^{(2)},c) \geq Q(L_\alpha,c)$ by Levy's theorem concerning the non-decreasing of concentration function. Therefore

$$\inf_\alpha Q(L_\alpha^{(2)}, c) \longrightarrow 1 \quad \text{as} \quad c \longrightarrow \infty,$$

and so σ_α will be bounded since $L_\alpha^{(2)}$ is a Gaussian distribution with the mean 0 and the standard deviation σ_α .

$L_{\alpha\,n}^{(5)}$ is decomposed as $L_{\alpha\,n}^{(5)} = L_{\alpha\,n+}^{(5)} * L_{\alpha\,n-}^{(5)}$, where the factors has the l.c.f.

$$\int_n^\infty (\exp(if_\alpha(u)z) - 1)du/u^2 \qquad \text{and} \qquad \int_{-\infty}^{-n} (\exp(if_\alpha(u)z) - 1)du/u^2$$

respectively. By the above-cited Lévy's theorem we see

$$(4.3) \quad \inf_\alpha Q(L_{\alpha\,n+}^{(5)}, c) \geq \inf_\alpha Q(L_\alpha, c) \longrightarrow 1 \qquad \text{as } c \longrightarrow \infty.$$

But $c < f_\alpha(n)$ implies $Q(L_{\alpha\,n+}^{(5)}, c) = \exp(-1/n)$, i.e.

$$(4.4) \quad Q(L_{\alpha\,n}^{(5)}, c) > \exp(-1/n) \text{ implies } \quad c \geq f_\alpha(u).$$

By (4.3) there exists c such that $Q(L_{\alpha\,n+}^{(5)}, c) > \exp(-1/n)$ and so that $c \geq f_\alpha(n)$ for $\alpha \in A$. Thus we see that $f_\alpha(n)$ is bounded for any assigned n. This is also the case for $f_\alpha(-n)$. Consequently we see that $f_\alpha(u)$ is bounded whenever $\alpha \in A$ and $|u| \leq n$, for any fixed n.

If $L_{\alpha(p)}, p=1,2,\ldots,$ be chosen from $\{L_\alpha\}$ such that $||f_{\alpha(p)}||_n$ increases indefinitely with p, $L_{\alpha(p)n}^{(4)}$ is approximately a Gaussian distribution with the mean 0 and the standard deviation $||f_{\alpha(p)}||$ as $p \longrightarrow \infty$ by the central limit theorem. Thus we have

$$Q(L_{\alpha(p)n}^{(4)}, ||f_{\alpha(p)}||_n) \longrightarrow \int_{-1}^{1} 1/\sqrt{2\pi} \exp(-t^2/2)dt < 1$$

as $p \longrightarrow \infty$, which contradicts with the fact that

$$\inf_\alpha Q(L_{\alpha\,n}^{(4)}, c) \geq \inf_\alpha Q(L_\alpha, c) \longrightarrow 1 \text{ (as } p \longrightarrow \infty).$$

Thus $||f_\alpha||_n$ proves to be bounded for any fixed n. Therefore

$\{L_{\alpha\,n}^{(3)} * L_{\alpha\,n}^{(4)} * L_{\alpha\,n}^{(5)}\}$ is totally bounded. Therefore $L_\alpha^{(1)}$ must be totally bounded and so $\{m_\alpha\}$ will be bounded.

§5. STOCHASTIC DIFFERENTIATION. $x(\tau,\omega)$, $a \leq \tau \leq b$, be a stochastic process on (Ω,B,P) and $F(E,\omega;\Delta_1,\Delta_2)$ be the conditional p.l. of $x(t+\Delta_1,\omega)-x(t-\Delta_2,\omega)(\Delta_1,\Delta_2 \geq 0)$ under the condition that $x(\tau,\omega)$, $a \leq \tau \leq t-\Delta_2$, be determined. If the $[1/\Delta_1+\Delta_2]$-times convolution of $F(E,\omega;\Delta_1,\Delta_2)$ ρ-converges to a distribution $L(E,\omega)$ in probability as $\Delta_1+\Delta_2 \longrightarrow 0$, i.e. for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that $0 < \Delta_1+\Delta_2 < \delta$ implies

$$\Pr\{\rho(H(E,\omega;\Delta_1,\Delta_2), L(E,\omega)) > \varepsilon\} < \varepsilon,$$

ρ being Lévy's law-distance, then we say that $x(\tau,\omega)$ is differentiable at t and we call $L(E,\omega)$ the differential coefficient of $x(\tau,\omega)$ at t, and we denote it with $D_t x(\tau,\omega)$ or briefly with $D(t,\omega)$. This is considered as an (R^{B^1},B^{B^1})-valued random variable. By taking a convenient sequence $\Delta_1'+\Delta_2' > \Delta_1''+\Delta_2'' > \dots \longrightarrow 0$, we see that $L(E,\omega)$ is the ρ-limit of the $[1/\Delta_1^{(n)}+\Delta_2^{(n)}]$-times convolution of $F(E,\omega;\Delta_1^{(n)},\Delta_2^{(n)})$ with P-measure 1, and so we obtain

Theorem 5.1. $DX(t,\omega)$ is an i.d.l. with P-measure 1.

From the definition we obtain, by making use of Theorem 1,

Theorem 5.2. $DX(t,\omega)$ is a B-measurable function of $(x(\tau,\omega), a \leq \tau \leq t)$. If $x(\tau,\omega)$, $a \leq \tau \leq b$, is a simple Markoff process, then $DX(t,\omega)$, if it exists, is a B-measurable function of $x(t,\omega)$; the form of the function clearly depends on t. If $x(\tau,\omega)$, $a \leq \tau \leq b$, is a differential process, then $DX(t,\omega)$, if it exists, does not depend on ω but on t. If $x(\tau,\omega)$, $a \leq \tau \leq b$, is a temporally homogeneous differential process, then $DX(t,\omega)$ exists and depends neither on ω nor on t; the l.c.f. of $DX(t,\omega)$ is equal to the l.c.f. of the p.l. of $x(b,\omega) - x(a,\omega)$ divided by b-a.

We can easily see that, if F_n^{*n} ρ-converges to a probability law, then F_n ρ-converges to the unit distribution. and so we have

Theorem 5.3. If $x(\tau,\omega)$ is differentiable at t, then it is continuous at t in probability i.e. t is not a fixed discontinuity of this process.

II. Stochastic Integral.

The integral of the form:

$$\int \sigma(\tau)dg(\tau,\omega),$$

where $\sigma(\tau) \in L_2$ and $g(\tau, \omega)$ is a brownian motion, is well-known as Wiener's integral.[23] The author has extended this integral to the case in which σ depends not only on τ but also on ω and called it a stochastic integral.[24] In this Chapter we treat a more general stochastic integral for the later use.

§6. FUNDAMENTAL DIFFERENTIAL PROCESS. Let $l(t,\omega)$, $a \leq t \leq b$, be a temporally homogeneous differential process such that both $l(t+0,\omega)$ and $l(t-0,\omega)$ exist and $l(t+0,\omega) = l(t,\omega)$, i.e. $l(t,\omega)$ is continuous in t except possibly for discontinuities of the first kind (hereafter we term this property with "belong to d_j-class"). Further we require that the p.l. of $l(s,\omega) - l(t,\omega)$ has the l.c.f. $(s-t)\psi_0(z)$, where $\psi_0(z)$ is the l.c.f. of the fundamental i.d.l. . Then $l(t,\omega)$, $a \leq t \leq b$, is defined to be a fundamental differential process. Such a process can be realized on a conveniently defined probability field (Ω, B_Ω, P), where the p.l. of $l(a,\omega)$ can be arbitrarily assigned.

Any jump of $l(t,\omega)$ is expressed by a point $(t,u) \in [a,b] \otimes R^1$, t being its position and u being its height: $l(t,\omega) - l(t-0,\omega)$. The number $p(E,\omega)$ of the jumps in E, E being a Borel subset of $[a,b] \otimes R^1$, can be considered a real random variable, which proves to be governed by the Poisson distribution with the mean:

$$\pi(E) = \int_E d\tau du/u^2.$$

$p(E,\omega)$ is evidently a function of $l(t,\omega)$, $a \leq t \leq b$. The system $\{p(E,\omega)\}$ is called the discontinuous part of $l(t,\omega)$, $a \leq t \leq b$, $l(t,\omega)$ can be expressed as

$$l(t,\omega) = l(a,\omega) + t + g(t,\omega) + \int_a^t \int_{|u|>1} up(d\tau du,\omega) + \int_a^t \int_{|u|<1} uq(d\tau du,\omega)$$

for any t, $a \leq t \leq b$, for almost all (P) ω, where $q(E,\omega) = p(E,\omega) - \pi(E)$ and $g(t,\omega)$ is a brownian motion which is also a function of $l(\tau,\omega)$, $a \leq \tau \leq t$, and is called the continuous part of $l(\tau,\omega)$.

For any disjoint system $E_1, E_2, \ldots, E_n$, $p(E_1,\omega), p(E_2,\omega), \ldots, p(E_n,\omega)$ and $(g(\tau,\omega), a \leq \tau \leq b)$ are independent.

All these properties can be immediately deduced from the results in the above-cited paper.[25]

§7. STOCHASTIC INTEGRAL BASED ON g. We shall define here an integral of the form:

(7.1) $\int_E \sigma(\tau,\omega)dg(\tau,\omega)$, E being a Borel subset of (a,b], in such a way that it may be a natural extension of Wiener's integral.

First we shall consider the case in which E is an interval: $I_1=(\alpha,\beta]$. By $S(I_1)$ we denote the class of all functions $\sigma(\tau,\omega)$, $\alpha \leq \tau \leq \beta$, $\omega \in \Omega$, satisfying the following three conditions:

(S.1) $\sigma(t,\omega)$ is measurable in (t,ω),

(S.2) $\int_\alpha^\beta \sigma(\tau,\omega)^2\, d\tau < \infty$ for almost all ω, and

(S.3) for any t, $\alpha \leq t \leq \beta$, the system $(\sigma(\tau,\omega), \alpha \leq \tau \leq t;$ $g(\tau,\omega) - g(\alpha,\omega), \alpha \leq \tau \leq t)$ is independent of $(g(\tau,\omega) - g(t,\omega),\ t \leq \tau \leq \beta)$. As is easily verified, $S(I_1)$ is conditionally complete; if $\sigma_n \in S(I_1)$ tends to σ_∞ for almost all (t,ω) and if $|\sigma_n| \leq \sigma_o \in S(I_1)$, then $\sigma_\infty \in S(I_1)$.

Theorem 7.1. We can determine, for $\sigma \in S(I_1)$,

(7.1') $\int_\alpha^\beta \sigma(\tau,\omega)dg(\tau,\omega)$ or $\int_{I_1} (\tau,\omega)dg(\tau,\omega)$ or briefly $\int(\sigma,\omega)$ in one and only one way so that it may satisfy (G.1) and (G.2). Furthermore it satisfies (G.3), (G.4), (G.5) and (G.6).

(G.1) When $\sigma(t,\omega)$ is a uniformly stepwise function, i.e., when there exist $\alpha = t_o < t_1 < \dots < t_k = \beta$ independent of ω such that $\sigma(t,\omega) = \sigma(t_{\nu-1},\omega), t_{\nu-1} \leq t < t_\nu$, we have

$$\int(\sigma,\omega) = \sum_{\nu=1}^{k} \sigma(t_{\nu-1},\omega)(g(t_\nu,\omega) - g(t_{\nu-1},\omega)).$$

(G.2) If $\sigma_n \in S(I_1)$ tends to σ_∞ for almost all (τ,ω), and if $|\sigma_n| \leq \sigma_o \in S(I_1)$ and further if every B-measurable function $\sigma(t,\omega)$ of $(\sigma_1, \sigma_2, \dots)$

satisfies (S.3), then $\int(\sigma_n, \omega)$ converges to $\int(\sigma_\infty, \omega)$ in probability.

(G.3) $$\int(c_1\sigma_1+c_2\sigma_2,\omega) = c_1 \int(\sigma_1,\omega) + c_2 \int(\sigma_2,\omega)$$

if σ_1, σ_2, $c_1\sigma_1 + c_2\sigma_2 \in S(I_1)$.

(G.4) $$\mathcal{E}((\int_{I_1}(\sigma,\omega))^2) = \int_{I_1}\mathcal{E}(\sigma^2(\tau,\omega))d\tau$$

if the right side is finite.

(G.5) If $\sigma_1(\tau,\omega) = \sigma_2(\tau,\omega)$ for $\tau \in I_1$, $\omega \in \Omega_1$, Ω_1 being a P-measurable set, then $\int(\sigma_1,\omega) = \int(\sigma_2,\omega)$ for almost all (P) $\omega \in \Omega_1$.

(G.6) If $\int_{I_1}\mathcal{E}(\sigma^2(\tau,\omega))d\tau < \infty$, then $\mathcal{E}(\int(\sigma,\omega)) = 0$.

Proof of the existence. In case σ is a uniformly stepwise function we define by (G.1). It is evident that this definition satisfies (G.3), (G.4), (G.5) and (G.6). The condition (S.3) will be used in the proof of (G.4) and (G.5) and (G.6).

In order to define $\int(\sigma,\omega)$ for $\sigma \in S(I_1)$ such that

(7.2) $$\|\sigma\|^2 = \int_{I_1}\mathcal{E}(\sigma^2(\tau,\omega))d\tau < \infty,$$

we shall establish

Lemma 7.1. For any $\sigma \in S(I)$ satisfying (7.2) we can find a sequence of uniformly stepwise functions $\sigma_n \in S(I)$ such that

(7.3) $$\|\sigma_n - \sigma\|^2 = \int_{I_1}\mathcal{E}((\sigma_n(\tau,\omega) - \sigma(\tau,\omega))^2)d\tau$$

may tend to 0.

The proof can be achieved by the method[26)] J. L. Doob has used in his research of measurable stochastic processes. By defining $\sigma(\tau,\omega) = 0$ for $\tau \leq \alpha$ or $\tau > \beta$, we may assume that $\sigma(\tau,\omega) \in L_2(R^1 \times \Omega)$, and so, for almost all ω, $\sigma(\tau,\omega) \in L_2(R^1)$.

Now put

$$\phi_n(t) = (k-1)/n \quad \text{for } (k-1)/n \leq t < k/n, \quad k=0, \pm 1, \pm 2, \cdots, n=1,2,\cdots.$$

Then $\phi_n(t) \longrightarrow t$ as $n \longrightarrow \infty$, and so we have for all t and for almost all (P) ω,

$$\int_{-\infty}^{\infty} (\sigma(\phi_n(t) + s, \omega) - \sigma(t+s, \omega))^2 ds \longrightarrow 0 \ (n \longrightarrow \infty),$$

since $\sigma(t+s, \omega)$ belongs to $L_2(R^1)$ as a function of s for almost all (P) ω. The left side is always less than $4\int_{-\infty}^{\infty} \sigma(s, \omega)^2 ds$, since $(a-b)^2 \leq 2a^2 + 2b^2$, and so

$$\int_{\Omega} \int_{t=-1}^{\beta} \int_{s=-\infty}^{\infty} (\sigma(\phi_n(t)+s, \omega) - \sigma(t+s, \omega))^2 ds dt P(d\omega) \longrightarrow 0.$$

Therefore there exists a sequence $a_1 < a_2 < \cdots$ for almost all s such that

$$\int_{\Omega} \int_{\alpha}^{\beta} |\sigma(\phi_{a_n}(t)+s, \omega) - \sigma(t+s, \omega)|^2 dt P(d\omega)$$

$$= \int_{\Omega} \int_{\alpha-s}^{\beta-s} |\sigma(\phi_{a_n}(t)+s, \omega) - \sigma(t+s, \omega)|^2 dt \, P(d\omega)$$

$$\leq \int_{\Omega} \int_{\alpha-1}^{\beta} |\sigma(\phi_{a_n}(t)+s, \omega) - \sigma(t+s, \omega)|^2 dt \, P(d\omega) \longrightarrow 0.$$

Put

$$\sigma_n(t, \omega) = \sigma(\phi_{a_n}(t-s)+s, \omega), \quad n=1,2,\cdots.$$

Then $\{\sigma_n\}$ is the required sequence. Thus the lemma is proved.

Since σ_n, $n=1,2,\cdots$, are all uniformly stepwise, we have already defined $\int(\sigma_n, \omega)$ and by (G.4)

$$\mathcal{E}(|\int(\sigma_n,\omega) - \int(\sigma_m,\omega)|^2) = \mathcal{E}(|\int(\sigma_n-\sigma_m,\omega)|^2)$$

$$= \int_{I_1}\mathcal{E}((\sigma_n(\tau,\omega) - \sigma_m(\tau,\omega))^2)d\tau \longrightarrow 0,$$

as $m,n \longrightarrow \infty$. Thus we have $f(\omega)$ such that $\mathcal{E}((\int(\sigma_n,\omega) - f(\omega))^2) \longrightarrow 0$. This $f(\omega)$ does not depend on the special choice of the sequence $\{\sigma_n\}$, but it is determined by σ only. We denote it by $\int(\sigma,\omega)$. For this extended definition we can easily verify the properties: (G.3), (G.4) and (G.5). The proof of (G.6) is following. Let $\{\sigma_n\}$ be the sequence obtained above from σ. Then we have

$$|\mathcal{E}\int(\sigma,\omega)| = |\mathcal{E}\int(\sigma,\omega) - \mathcal{E}\int(\sigma_n,\omega)| = |\mathcal{E}\int(\sigma-\sigma_n,\omega)|$$

$$\leq\sqrt{\mathcal{E}((\int(\sigma-\sigma_n,\omega))^2)} = \sqrt{\int\mathcal{E}((\sigma-\sigma_n)^2)\,dt} \longrightarrow 0.$$

For any $\sigma \in S(I_1)$ we define σ_n by

$$\sigma_n(t,\omega) = \phi_n(\int_\alpha^t \sigma^2(\tau,\omega)d\tau)\sigma(t,\omega),$$

where $\phi_n(\lambda) = 1$ for $|\lambda| \leq n$, and $= 0$ for $|\lambda| > n$.

We shall prove that $\sigma_n(t,\omega)$ satisfies (S.1) and (S.3). It is sufficient to show that $\int_\alpha^t \sigma^2(\tau,\omega)d\tau$ is B-measurable in $(\sigma(\tau,\omega),\ a \leq \tau \leq t)$. Since

$$\int_\alpha^t \sigma^2(\tau,\omega)d\tau = \lim_n \int_\alpha^t \phi_n(\sigma(\tau,\omega))\sigma^2(\tau,\omega)d\tau$$

for any ω, it is sufficient to show the B-measurability of

$$\int_\alpha^t \phi_n(\sigma(\tau,\omega))\sigma^2(\tau,\omega)d\tau,\ n=1,2,\cdots$$

by Theorem 1, which is evident since we have, by Lemma 7.1,

$$\int_\alpha^t \phi_n(\sigma(\tau,\omega))\sigma^2(\tau,\omega)d\tau = \lim_{p\longrightarrow\infty}\int_\alpha^t \phi_n(\sigma(\phi_{a_p}(\tau-s)+s,\omega))\sigma^2(\phi_{a_p}(t-s)+s,\omega)d\tau$$

$$= \lim_{p \to \infty} \sum_{a_p(t-s) < \nu < 1 + a_p(t-s)} \phi_n\left(\sigma\left(\frac{\nu - 1}{a_p} + s, \omega\right)\right) \sigma^2\left(\frac{\nu - 1}{a_p} + s, \omega\right) \frac{1}{a_p}$$

for some s and some sequence $a_1 < a_2 < \dots$.

Let Ω_n be the set of all ω such that $\int_{I_1} \sigma^2(\tau, \omega) d\tau \leq n$. Then we have $\Omega_1 \subseteq \Omega_2 \subseteq \dots$ and $\bigcup \Omega_n = \Omega - N$, $P(N) = 0$ by (S.2). Furthermore $\sigma_n(t, \omega) = \sigma(t, \omega)$, $t \in I_1$, $\omega \in \Omega_n$. Since $\|\sigma_n\| < \infty$, we have already defined $\int(\sigma_n, \omega)$. We define $\int(\sigma, \omega)$ as $\int(\sigma_n, \omega)$ on $\Omega_n - \Omega_{n-1}$, and 0 on N.

This extended definition satisfies (G.3), (G.4), (G.5) and (G.6), as is easily verified. We shall prove (G.2). If $\|\sigma_o\| < \infty$, then $\|\sigma_n - \sigma_o\| \longrightarrow 0$. Therefore we have $\mathcal{E}((\int(\sigma_n, \omega) - \int(\sigma_\infty, \omega))^2) \longrightarrow 0$ by (G.4) and so $\int(\sigma_n, \omega)$ will tend to $\int(\sigma_\infty, \omega)$ in probability. In the general case we consider

$$\sigma_n^{(m)}(t, \omega) = \phi_m\left(\int_\alpha^t \sigma_o^2(\tau, \omega) d\tau\right) \sigma_n(t, \omega), \quad n = 1, 2, \dots, \infty.$$

By virtue of (S.2) we have

$$P(\{\omega ; \sigma_n^{(m)}(t, \omega) = \sigma_n(t, \omega), \ n = 1, 2, \dots, \infty\})$$

$$\geq P(\{\omega ; \int_\alpha^\beta \sigma_o^2(\tau, \omega) d\tau < m\}) > 1 - \frac{\varepsilon}{2}$$

for a sufficiently large m. Since $\sigma_n^{(m)}(t, \omega) \longrightarrow \sigma_\infty^{(m)}(t, \omega)$, $|\sigma_n^{(m)}| \leq \sigma_o^{(m)}$ and $\|\sigma_o^{(m)}\| < \infty$, we have, for a sufficiently large n,

$$P(\{\omega ; \ |\int(\sigma_n^{(m)}, \omega) - \int(\sigma_\infty^{(m)}, \omega)| > \varepsilon\}) < \frac{\varepsilon}{2} .$$

We obtain from the above two conditions

$$P(\{\omega ; \ |\int(\sigma_n, \omega) - \int(\sigma_\infty, \omega)| > \varepsilon\}) < \varepsilon .$$

Proof of the uniqueness. Let $\int_1(\sigma,\omega)$ and $\int_2(\sigma,\omega)$ satisfy (G.1) and (G.2). We see by (G.1) that $\int_1(\sigma,\omega) = \int_2(\sigma,\omega)$ for any uniformly stepwise function $\sigma \in S(I_1)$. If $\sigma \in S(I_1)$ is bounded ($|\sigma| \leq m$), we have $\|\sigma\| \leq m(\beta-\alpha)$. The sequence $\{\sigma_n\}$ obtained by Lemma 7.1 from σ is uniformly bounded ($|\sigma_n| \leq m$). By taking a subsequence we may assume that $\sigma_n \longrightarrow \sigma$ for almost all (τ,ω). σ_n being uniformly stepwise, we have $\int_1(\sigma_n,\omega) = \int_2(\sigma_n,\omega)$, $n = 1,2,\ldots$, and so $\int_1(\sigma,\omega) = \int_2(\sigma,\omega)$ by (G.2).

By (G.2) and (G.3) we have

Theorem 7.2. Let $\sigma_n \in S(I_1)$, $n=1,2,\ldots$. If every measurable function of the joint variable $(\sigma_1, \sigma_2, \ldots)$ satisfies (S.3) and if $\sum|\sigma_n| \in S(I_1)$, then we have

$$\int_{I_1} \sum_{n=1}^{\infty} \sigma_n(\tau,\omega)dg(\tau,\omega) = \sum_{n=1}^{\infty} \int_{I_1} \sigma_n(\tau,\omega)dg(\tau,\omega)$$

in the sense of 'limit in probability'.

Let E be any Borel subset of $(\alpha,\beta]$. For $\sigma \in S(I_1)$ we define as follows:

$$\int_E \sigma(\tau,\omega)dg(\tau,\omega) = \int_I \sigma(\tau,\omega)C_E(\tau)dg(\tau,\omega),$$

where $C_E(\tau)$ is the characteristic function of the set E. This definition is clearly independent of the choice of $(\alpha,\beta]$ containing E, and so it is an extension of (7.1').

Theorem 7.3. Let $\sigma \in S(I)$. Then we can define $\int_E \sigma\, dg$ for $E \subseteq I$. If $\{E_n\}$ be a disjoint sequence of Borel subsets of I. Then we have

$$\int_{\sum_n E_n} \sigma \, dg = \sum_n \int_{E_n} \sigma \, dg$$

in the sense of 'limit in probability'.

This is clear by the previous theorem.

§8. THE CONTINUOUS KERNEL OF THE INTEGRAL. Let $\sigma \in S(I)$, $I = (\alpha,\beta]$. Then $\sigma \in S(I_t)$, $I_t = (\alpha,t]$ for any t, $\alpha \leq t \leq \beta$, and so we can define

$$(8.1) \int_{\alpha}^{t} \sigma(\tau,\omega)dg(\tau,\omega),$$

which is uniquely (up to P-measure 0) determined for any assigned t.

Theorem 8. We can determine, for $\sigma \in S(I)$, a stochastic process:

$$\left(*\int_{\alpha}^{t} \sigma(\tau,\omega)dg(\tau,\omega),\ \alpha \leq t \leq \beta \right)$$

in one and only one way (up to P-measure 0) so that the process may be continuous in t with P-measure 1 and that

$$(G') \qquad *\int_{\alpha}^{t} \sigma(\tau,\omega)dg(\tau,\omega) = \int_{\alpha}^{t} \sigma(\tau,\omega)dg(\tau,\omega)$$

with P-measure 1 for any assigned t.

For this integral we have

$$(G'.4) \qquad c^2 P(\{\omega; \sup_{\alpha \leq t \leq \beta} |*\int_{\alpha}^{t} \sigma(\tau,\omega)dg(\tau,\omega)| \geq c\})$$

$$\leq \mathcal{E}((\int_{\alpha}^{\beta} \sigma(\tau,\omega)dg(\tau,\omega))^2) = \int_{\alpha}^{\beta} \mathcal{E}\sigma^2(\tau,\omega)d\tau ,$$

if the right side is finite.

Proof of the existence. If $\sigma(\tau,\omega)$ is uniformly stepwise in $\alpha \leq \tau \leq \beta$, then it is so in $\alpha \leq \tau \leq t$. In this case we shall define

$$*\int_{\alpha}^{t} \sigma(\tau,\omega)dg(\tau,\omega)$$

by (G') i.e. by (G.1), which is clearly continuous on account of the continuity of $g(\tau,\omega)$. In order to show (G'.4) we need

Lemma 8. Let $y_\nu(\omega), x_\nu(\omega)$, $\nu = 1,2,\ldots,m$, be random variables satisfying the following conditions:

(8.2) $\mathcal{E}(x_\nu) = 0$, $\mathcal{E}(x_\nu^2)$, $\mathcal{E}(y_\nu^2) < \infty$, $\nu = 1,2,\ldots,m$,

(8.3) $(x_k(\omega), x_{k+1}(\omega), \ldots, x_m(\omega))$ is independent of $(y_1(\omega), x_1(\omega), y_2(\omega), x_2(\omega), \ldots, y_{k-1}(\omega), x_{k-1}(\omega), y_k(\omega))$ for $k = 1,2,\ldots,m$.

Then we have

$$(8.4) \qquad c^2 P(\{\omega; \max_{k=1}^{n} |\sum_{\nu=1}^{k} y_\nu(\omega)x_\nu(\omega)| > c\})$$

$$< \mathcal{E}((\sum_{\nu=1}^{m} y_\nu(\omega)x_\nu(\omega))^2) = \sum_{\nu=1}^{m} \mathcal{E}(y_\nu(\omega)^2)\mathcal{E}(x_\nu(\omega)^2),$$

In case $y_\nu(\omega) \equiv 1, \nu=1,2,3,\dots,m$, this lemma is nothing but the so-called Kolmogoroff's inequality, whose proof will be available for our lemma, if we give it a slight modification.

Now let $\{s_n\}$ be a sequence dense in $(\alpha,\beta]$. By the continuity of $*\int_\alpha^t \sigma\, dg$, we have

$$\sup_{t \in I} |*\int_\alpha^t \sigma\, dg| = \sup_{\nu=1}^{\infty} |*\int_\alpha^{s_\nu} \sigma\, dg|.$$

Since $\{\omega ;\ \max_{\nu=1}^m |\int_\alpha^{s_\nu} \sigma\, dg| > c\}$, m=1,2,...,is a monotone increasing sequence tending to $\{\omega ;\ \sup_{\nu=1}^{\infty} |\int_\alpha^{s_\nu} \sigma\, dg| > c\}$, we need only prove

$$c^2 P\{\omega ;\ \max_{\nu=1}^m |\int_\alpha^{s_\nu} \sigma\, dg| > c\} \leq \mathcal{E}((\int_\alpha^\beta \sigma\, dg)^2) = \int_\alpha^\beta \mathcal{E}(\sigma^2) d\tau$$

for the proof of (G'.4). Let $\alpha = t_0 < t_1 < \dots < t_n = \beta$ be chosen so that $\sigma(\tau,\omega)$ may be constant in $(t_{\nu-1}, t_\nu)$ for each ω, $\nu=1,2,\dots$. If we rearrange $s_1, s_2, \dots, s_m$, $t_0, t_1, \dots, t_n$ in the order of magnitude and denote it with $u_0, u_1, \dots, u_{m+n}$, it is sufficient to prove

$$c^2 P(\{\omega ;\ \max_{\nu=0}^{m+n} |*\int_\alpha^{s_\nu} \sigma\, dg| > c\}) \leq \mathcal{E}((\int_\alpha^\beta \sigma\, dg)^2) = \int_\alpha^\beta \mathcal{E}(\sigma^2) d\tau$$

, which we obtain at once by putting $y_\nu = \sigma(u_{\nu-1},\omega), x_\nu = g(u_\nu, \omega) - g(u_{\nu-1},\omega)$ in the above Lemma 8.

In order to define $*\int_\alpha^t \sigma\, dg$ for $\sigma \in S(I)$ such that

$$(8.5) \quad \|\sigma(\tau,\omega)\|^2 \equiv \int_\alpha^\beta \mathcal{E}\sigma^2(\tau,\omega) d\tau < \infty,$$

we define σ_n, n=1,2,..., from σ by Lemma 7.1. By choosing a convenient subsequence, we may assume

$$(8.6) \quad \|\sigma_{n+1} - \sigma_n\|^2 \leq 1/8^n, \quad n=1,2,\dots .$$

$\sigma_{n+1} - \sigma_n$, n=1,2,..., being uniformly stepwise, we can apply (G'.4) to it and we have

$$P(\{\omega ;\ \sup_{\alpha \leq t \leq \beta} |*\int_\alpha^t \sigma_{n+1} - *\int_\alpha^t \sigma_n| \geq 1/2^n\}) \leq 1/2^n.$$

Therefore $*\int_\alpha^t \sigma_n dg$ is uniformly (in t) convergent with P-measure 1 by Borel-Cantelli's theorem; the limit depends only on $\sigma(t,\omega)$ and it is independent of $\{\sigma_n\}$, if we choose them so that $\sigma_n(t,\omega)$ may be a function of $\sigma(\tau,\omega)$, $a < \tau \leq t$, for any t. We define $*\int_\alpha^t \sigma\, dg$ as this limit. But $*\int_\alpha^t \sigma_n dg = \int_\alpha^t \sigma_n dg$ and so $*\int_\alpha^t \sigma\, dg = \int_\alpha^t \sigma\, dg$ with P-measure 1 for any assigned t.

By the uniformity of the convergence, $*\int_\alpha^t \sigma\, dg$ is continuous in t and it satisfies (G'.4). For any function σ for which (8.5) does not hold, we can define $\int_\alpha^t \sigma\, dg$ in the same way as the previous § 7.

Proof of the uniqueness. Let $*\int_1$ and $*\int_2$ satisfy the assigned conditions. By (G') we have

$$*\int_{1,\alpha}^{\gamma} \sigma\, dg = *\int_{2,\alpha}^{\gamma} \sigma\, dg = \int_\alpha^{\gamma} \sigma\, dg$$

for any rational number γ with P-measure 1, and by the continuity of $*\int_1$ and $*\int_2$ we obtain

$$*\int_{1,\alpha}^{t} \sigma\, dg = *\int_{2,\alpha}^{t} \sigma\, dg$$

for any t with P-measure 1.

Definition 8. The above $*\int_\alpha^t \sigma\, dg$ is called the continuous kernel of $\int_\alpha^t \sigma\, dg$.

§9. STOCHASTIC INTEGRALS BASED ON p AND q AND THEIR REGULAR KERNEL.

We consider p and q appearing in the resolution of a fundamental differential process (§6). Let I be a half-open 2-dimensional interval $(\alpha, \beta] \times (\gamma, \delta]$, $0 < \gamma, \delta < \infty$, $a \leq \alpha \leq \beta \leq b$. By F(I) we denote the class of all functions $f(t,u,\omega)$, $\alpha \leq t \leq \beta$, $x \leq u \leq \delta$, such that

(F.1) $f(t,u,\omega)$ is measurable in (t,u,ω),

(F.2) $\int_I |f(t,u,\omega)| d\tau\, du/u^2 < \infty$ with P-measure 1, and

(F.3) $(f(t,u,\omega), \alpha \leq \tau < t; p(E,\omega), E \subseteq R^2(t))$ is independent of

$(p(E,\omega),\ E \subseteq I \frown R^2_+(t))$, $R^2_+(t), R^2_-(t)$ being respectively the half-plane $\{(\tau,u); \tau \geq t\}$,

In the same way as in §7 we obtain

Theorem 9.1. We can determine, for $f \in F(I)$,

$$(9.1) \qquad \int_\alpha^\beta\int_\gamma^\delta f(\tau,u,\omega)dp(\tau,u,\omega) \text{ or } \int_I f(\tau,u,\omega)dp(\tau,u,\omega)$$

or briefly $\int(f,\omega)$

in one and only one (up to P-measure 1) way so that it may satisfy (P.1) and (P.2). Furthermore it satisfies (P.3), (P.4), (P.5) and (P.6).

(P.1) When $f(\tau,u,\omega)$ is a uniformly stepwise function, i.e. when there exists $\alpha = t_0 < t_1 < \dots < t_n = \beta$, $\gamma = u_0 < u_1 < \dots < u_n = \delta$ independent of ω such that $f(\tau,u,\omega) = f(t_{\mu-1},u_{\nu-1},\omega)$, $t_{\mu-1} \leq t < t_\mu$, $u_{\nu-1} \leq u < u_\nu$, $\mu = 1,2,\dots,m$, $\nu = 1,2,\dots,n$, we have

$$\int(f,\omega) = \sum_{\mu=1}^{m}\sum_{\nu=1}^{n} f(t_{\mu-1},u_{\nu-1},\omega)p((t_{\mu-1},t_\mu) \times (u_{\nu-1},u_\nu),\omega).$$

(P.2) If $f_n(\tau,u,\omega) \in F(I)$ tends to $f_\infty(\tau,u,\omega)$ for almost all (τ,u,ω), and if $|f_n| \leq f_0 \in F(I)$ and furthermore if every B-measurable function $f(\tau,u,\omega)$ of $(f_1,f_2,\dots)$ satisfies (F.3), then $\int(f_n,\omega)$ converges to $\int(f_\infty,\omega)$ in probability.

(P.3) $\int(c_1f_1+c_2f_2,\omega) = c_1\int(f_1,\omega) + c_2\int(f_2,\omega)$ if $f_1,f_2,c_1f_1+c_2f_2 \in F(I)$.

(P.4) $\mathcal{E}(\int(f,\omega)) = \int_I \mathcal{E} f(\tau,u,\omega)d\tau\, du/u^2$ if $\int_I \mathcal{E}|f(\tau,u,\omega)|d\tau\, du/u^2 < \infty$.

(P.5) If $f_1 = f_2$ for $(\tau,u) \in I$, $\omega \in \Omega_1$, Ω_1 being a P-measurable set, then $\int(f_1,\omega) = \int(f_2,\omega)$ for almost all ω on Ω_1.

Let $f(\tau,u,\omega)$, $\alpha \leq t \leq \beta$, $\gamma \leq u$, $\omega \in \Omega (\gamma > 0)$ belong to $F((\alpha,\beta] \times (\gamma,n])$ for any integer $n > \gamma$. Then we can define

$$\int_\alpha^\beta\int_\gamma^n f(\tau,u,\omega)dp(t,u,\omega), \quad n > \delta .$$

Let Ω_n denote the set $\{\omega; p((\alpha,\beta]\times(n,\infty),\omega) = 0$. Then we have, for $\omega \in \Omega_n$ and $m > n$,

$$\int_\alpha^\beta \int_\gamma^m fdp = \int_\alpha^\beta \int_\gamma^n fdp.$$

But $\Omega_n \subseteq \Omega_{n+1}$, and $P(\Omega_n) = \exp\{ -\int_{\tau=\alpha}^{\beta}\int_{u=n}^{\infty} d\tau\, du/u^2\} \longrightarrow 1$ as $n \longrightarrow \infty$.

Therefore $\lim\limits_{n\to\infty} \int_\alpha^\beta \int_\gamma^n fdp$ exists with P-measure 1 and it has the above properties: (F.1), (F.2), etc., and so we denote it with

$$\int_\alpha^\beta \int_\gamma^\infty f(\tau,u,\omega)dp(\tau,u,\omega).$$

Similarly we can define, for $\delta < 0$,

$$\int_\alpha^\beta \int_{-\infty}^\delta f(\tau,u,\omega)dp(\tau,u,\omega).$$

Let E be any Borel subsets of $I = (\alpha,\beta] \times (\gamma,\delta]$, $\gamma, \delta > 0$, $a < \alpha \leq \beta \leq b$. Then we define, for $f \in F(I)$, as follows,

$$\int_E f(t,u,\omega)dp(t,u,\omega) = \int_I f(t,u,\omega)C_E(t,u)dp(t,u,\omega),$$

where $C_E(t,u)$ is the characteristic function of E. This definition is clearly independent of the choice of the interval $I \supsetneq E$.

From (P.2) and (P.3) we obtain

Theorem 9.2. If E_n, n=1,2,..., are disjoint Borel subsets of I, and if $f \in F(I)$, then we have

$$\int_{\sum E_n} fdp = \sum_{\nu=1}^{n} \int_{E_n} fdp$$

in the sense of limit in probability.

From (P.4) we obtain

Theorem 9.3. If $f \in F(I)$, and if $\int_E |f(\tau,u,\omega)| d\tau du/u^2 < \infty$, then

$$\mathcal{E}(\int_E fdp) = \int_E \mathcal{E}(f(\tau,u,\omega))dt\, du/u^2.$$

Now we define the regular kernel of $\int_E f dp$, $f \in F(I)$, which will be denoted by $*\int f dp$. Firstly we consider the case in which $f \geq 0$ and

(9.2) $$\int_I \mathcal{E}(f(t,u,\omega))dt du/u^2 < \infty.$$

Let $\{\alpha_\nu\}$, $\{\gamma_\nu\}$ be dense in $(\alpha, \beta]$ and in $(\gamma, \delta]$ respectively. We assume that $\{\alpha_\nu\} \ni \alpha, \beta$ and $\{\gamma_\nu\} \ni \gamma, \delta$. We shall here call any finite sum of intervals of the form $(\alpha_\mu, \alpha_\nu] \times (\gamma_\lambda, \gamma_\kappa]$ elementary set. Then the system of all elementary sets is enumerable and forms a finitely additive class. Let $E_1, E_2, \ldots, E_n$ disjoint elementary sets and E their sum. Then we have

$$\int_E f dp = \sum_{\nu=1}^{n} \int_{E_\nu} f dp$$

with P-measure 1. Since this system of all these equalities is enumerable, we see that they hold simultaneously with P-measure 1. Since $f \geq 0$, we obtain $\int_E f dp \geq 0$ for any elementary set E with P-measure 1. Thus $\int_E f dp$ is a finitely additive measure for almost all ω.

Let G be any set open in I and B any Borel subset of I. We define

$$*\int_G f dp = \sup\{\int_E f dp;\ E \text{ is any elementary set whose closure } \leq G\}$$

$$*\int_B f dp = \inf\{\int_G f dp;\ G \text{ is any open (in I) set that contains } B\}.$$

As is easily proved, we have

$$*\int_B f dp = \sum_{n=1}^{\infty} *\int_{B_n} f dp, \qquad B = \sum_{n=1}^{\infty} B_n,$$

for any disjoint system of Borel sets $\{B_n\}$.

Now we shall prove that

(9.3) $$*\int_B f dp = \int_B f dp$$

with P-measure 1 for any Borel set B. For any open set G we can choose a monotone-increasing sequence of elementary sets $\{E_n\}$ so that the closure of E_n is contained in G, n=1,2,... and that $G = \bigcup E_n$. By Theorem 9.2 we see that

$$\int_G fdp = \text{l.i.P.} \int_{E_n} fdp \quad (\text{l.i.P.} = \text{limit in probability})$$

and so that $\int_G fdp \leq {}^*\int_G fdp$ with P-measure 1. Besides $\int_G fdp \geq {}^*\int_G fdp$, since $\int_E fdp$ is monotone in E on account of $f \geq 0$. Thus we see that (9.3) holds for any open set. By (9.2) and (P.4), $\int_I fdp$ is finite for almost all. Therefore by Theorem 9.3 we have

$$\int_B fdp = \text{l.i.P.} \int_{B_n} fdp,$$

if $B_1 \supseteq B_2 \supseteq \dots \longrightarrow B$. Consequently (9.3) holds also for any closed (in I) set B.

Let B be any Borel set. By (9.2) we can find a sequence of open sets G and a sequence of closed sets $\{F_n\}$ such that

$$F_1 \subseteq F_2 \subseteq \dots \subseteq B \subseteq \dots \subseteq G_2 \subseteq G_1$$

and that

$$\int_{G_n - F_n} \mathcal{E}(f(t,u,\omega))dtdu/u^2 \leq 1/n, \quad n=1,2,\dots .$$

It is clear that

$$\int_{F_1} fdp \leq \int_{F_2} fdp \leq \dots \leq \int_B fdp \leq \dots \leq \int_{G_2} fdp \leq \int_{G_1} fdp$$

with P-measure 1. Furthermore we have

$$\mathcal{E}\left(\int_{G_n} fdp - \int_{F_n} fdp\right) = \int_{G_n - F_n} \mathcal{E}(f)dtdu/u^2 \leq 1/n \longrightarrow 0.$$

Therefore

$$(9.4) \quad \int_B fdp = \text{l.i.P.} \int_{G_n} fdp = \text{l.i.P.} \int_{F_n} fdp.$$

But we have

$$^*\int_{F_1} fdp \leq {}^*\int_{F_2} fdp \leq \dots \leq {}^*\int_B fdp \leq \dots \leq {}^*\int_{G_2} fdp \leq {}^*\int_{G_1} fdp$$

by the definition, and so

$$(9.5) \qquad \lim_n {}^*\int_{F_n} fdp \leq {}^*\int_B fdp \leq \lim_n {}^*\int_{G_n} fdp.$$

As has been already proved,

$$^*\int_{F_n} fdp = \int_{F_n} fdp, \quad {}^*\int_{G_n} fdp = \int_{G_n} fdp, \; n=1,2,\dots,$$

with P-measure 1 and so we obtain (9.3) at once from (9.4) and (9.5).

For $f \in F(I)$ for which $f \geq 0$ but (9.2) does not hold, we define $^*\int_B fdp$ by

$$^*\int_B fdp = \lim_{N \to \infty} {}^*\int_B \phi_N\left(\int_{I \cap R^2_-(t)} f(\tau, \lambda, \omega) d\tau d\lambda/\lambda^2\right) f(t,u,\omega) dp(t,u,\omega),$$

where $\phi_N(\lambda)$ is the characteristic function of the interval $[-N,N]$.

For any general f we shall define

$$^*\int_B fdp = {}^*\int_B \frac{|f|+f}{2} dp - {}^*\int_B \frac{|f|-f}{2} dp.$$

Now, we shall define $\int_I fdq$. For $f \in F(I)$, $I = (\alpha, \beta] \times (\gamma, \delta]$, $\gamma, \delta > 0$, we put

$$(9.6) \qquad \int_I fdq = \int_I fdp - \int_I f(\tau, u, \omega) d\tau du/u^2.$$

Then we can easily prove that

$$(9.7) \qquad \mathcal{E}((\int_I fdq)^2) = \int_I \mathcal{E}(f^2) d\tau du/u^2, \quad \mathcal{E}(\int_I fdq) = 0,$$

for f such that

$$\int_I \mathcal{E}(f^2) d\tau du/u^2 < \infty.$$

For $I = (\alpha, \beta] \times (0, \delta]$, $a \leq \alpha \leq \beta \leq b$, $0 < \delta < \infty$, we define $F_1(I)$ as the class of all functions $f(t,u,\omega)$ satisfying (F.1), (F.3) and

$$(F'.2) \qquad \int_I (f(\tau,u,\omega))^2 d\tau du/u^2 < \infty.$$

We shall define $\int_I fdq$ for $f \in F_1(I)$. Firstly we consider the case:

$$(F''.2) \quad \int_I \mathcal{E}(f(\tau,u,\omega)^2)\, d\tau du|u^2 < \infty$$

Let I_n denote $(\alpha,\beta] \times (1/n, \delta]$. Then

$$\mathcal{E}((\int_{I_n} |f(\tau,u,\omega)| d\tau du/u^2)^2) \leq \int_{I_n} d\tau du/u^2 \int_{I_n} \mathcal{E}(f(\tau,u,\omega)^2) d\tau du/u^2$$

with P-measure 1. Therefore $f \in F(I_n)$ and so we can define

$$\int_{I_n} fdq$$

by (9.6). By (9.7) and (F'' .2) we have

$$\mathcal{E}((\int_{I_n} fdq - \int_{I_m} fdq)^2) = \mathcal{E}((\int_{I_n - I_m} fdq)^2)$$

$$= \int_{I_n - I_m} \mathcal{E}(f^2) d\tau du/u^2 = \int_\alpha^\beta \int_{\frac{1}{n}}^{\frac{1}{m}} \mathcal{E}(f^2) d\tau du/u^2 \longrightarrow 0.$$

We shall define

$$\int_I fdq = \text{l.i.m.} \int_{I_n} fdq \text{ (l.i.m. = limit in mean).}$$

This extended definition satisfies (9.7) evidently.

In order to define the integral in the general case we put

$$f_n(t,u,\omega) = \phi_n(\int_{\tau=\alpha}^t \int_{u=0}^\delta (f(\tau,u,\omega)^2) d\tau du/u^2) f(t,u,\omega),$$

where ϕ_n is the characteristic function of the interval $(-n,n)$. By (F'.2) we have

$$P(\Omega_n) \uparrow 1, \text{ where } \Omega_n = \{\omega; f_n(\tau,u,\omega) = f(\tau,u,\omega) \text{ for } (\tau,u) \in I\}.$$

We shall define

$$\int_I fdq \text{ as } \int_I f_n dq \text{ on } \Omega_n.$$

Similarly we can define the integral in case $I = (\alpha,\beta]x(\gamma, 0]$. In case $I = (\alpha,\beta]x(\gamma,0]$. In case $I = (\alpha,\beta]\, x(0,\delta]$, we define

$$\int_I fdq = \int_{I_1} fdq + \int_{I_2} fdq,\ I_1 = (\alpha,\beta]\, x\, (\gamma,0],\ I_2 = (\alpha,\beta]\, x(0,\delta].$$

As in the previous integrals we can define $\int_E fdq$ as $\int fC_E dq$, $C_E(t,u)$ being the characteristic function of E.

For the regular kernel of this integral we establish

Theorem 9.4. Let $I = (\alpha,\beta]x(\gamma,\delta]$ and $E \subseteq (\gamma,\delta]$. For $f \in F_1(I)$ we can determine $*\int_\alpha^t \int_E fdq$, $\alpha \le t \le \beta$, which belongs to d_1-class as a function of t with P-measure 1 and satisfies

$$*\int_\alpha^t \int_E fdq = \int_{[\alpha,t]xE} fdq$$

with P-measure 1. We have, for this regular kernel,

$$c^2 \Pr\{\sup_{\alpha \le t \le \beta} |*\int_\alpha^t \int_E fdq\,| > c\} \le \int_\alpha^\beta \int_E \mathcal{E}((f(t,u,\omega))^2)dtdu/u^2.$$

The proof can be achieved in the same way as that of Theorem 8.

III. STOCHASTIC DIFFERENTIAL EQUATION AND STOCHASTIC INTEGRAL EQUATION.

§10. Stochastic differential equation. We shall solve a stochastic differential equation:

(10.1) $Dx(t,\omega) = L(t,x(t,\omega)),\ a \le t \le b,$

under the condition:

(10.2) $P_{x(a,\omega)} = L,$

where L is a given distribution on R^1.

Theorem 10. We can construct a simple Markoff process $x(t,\omega)$ on a convenient probability field (Ω, B_Ω, P) so that $x(t,\omega)$ may satisfy (10.1) and (10.2) and that $x(t,\omega)$ may belong to d_1-class with P-measure 1, if the three elements $m(\tau,\xi)$, $\sigma(\tau,\xi)$ and $f(\tau,u,\xi)$ of $L(\tau,\xi)$ satisfies the following conditions (A) and (B):

(A) $|m(\tau,\xi) - m(\tau,\eta)| \le M|\xi - \eta|,$

$$|\sigma(\tau,\xi)-\sigma(\tau,\eta)| \leq S|\xi-\eta|,$$

$$\| f(\tau,u,\xi) - f(\tau,u,\eta)\|_n \equiv \left(\int_{|u|\leq n}(f(\tau,u,\xi)-f(\tau,u,\eta))^2 du/u^2\right)^{\frac{1}{2}}$$

$$\leq F_n|\xi-\eta|,$$

where $M, S, F_n, n=1,2,\ldots,$ are independent of (τ,ξ,η),

(B) $L(\tau,\xi)$ is continuous in τ with regard to Lévy's distance for any fixed ξ.

Proof. We construct a fundamental differential process $\ell(\tau,\omega)$, $a \leq \tau \leq b$, on a convenient probability field so that $\ell(a,\omega)$ may be governed by L, and we consider a stochastic integral equation:

$$(10.3) \quad x(t,\omega) = \ell(a,\omega) + \int_a^t m(\tau,x(\tau,\omega))d\tau + \int_a^t \sigma(\tau,x(\tau,\omega))dg$$

$$+\int_a^t \int_{|u|>1} f(\tau,u,x(\tau,\omega))dp + \int_a^t \int_{|u|<1} f(\tau,u,x(\tau,\omega))dq,$$

whose solution is the required stochastic process by virtue of the following §11 (Hereafter the stochastic integral means the continuous or regular kernel, even if the notation ' * ' be omitted.

§11. Stochastic integral equation.

Theorem 11. Let $L(\tau,\xi)$ satisfy the condition (A) and (B) in Theorem 10. Let $c(\omega)$ be independent of $\ell(\tau,\omega) - \ell(a,\omega)$, $a \leq \tau \leq b$. Then there exists one and only one (up to P-measure 0) stochastic process satisfying a stochastic integral equation:

$$(11.1) \quad x(t,\omega) = c(\omega) + \int_a^t m(\tau,x(\tau,\omega))d\tau + \int_a^t \sigma(\tau,x(\tau,\omega))dg$$

$$+\int_a^t \int_{|u|>1} f(\tau,u,x(\tau,\omega))dp + \int_a^t \int_{|u|<1} f(\tau,u,x(\tau,\omega))dq$$

for $a \leq t \leq b$ with P-measure 1 and fulfilling the following property:

(11.2) $(x(\tau,\omega), \ell(\tau,\omega) - \ell(a,\omega); a < \tau \leq t)$ is independent of $(\ell(\tau,\omega) - \ell(t,\omega), t \leq \tau \leq b)$ for $a \leq t \leq b$. This solution is a simple Markoff process, which belongs to d_1-class with P-measure 1 and satisfies a stochastic differential equation:

(11.3) $Dx(t,\omega) = L(t,x(t,\omega))$.

Proof: We shall firstly remark that the condition (B) implies that each of $|m(t,\xi)|$, $|\sigma(t,\xi)|$, $\| f(\tau,u,\xi)\|$, n=1,2,..., are bounded in $a \leq t \leq b$ for any assigned ξ, which is deduced from Theorem 4.2, since $\{L(\tau,\xi), a \leq \tau \leq b\}$ is compact and so totally bounded as a continuous image of a compact set $[a,b]$ for any assigned ξ.

Next we shall remark that (A) and (B) imply that $m(\tau,\xi)$, $\sigma(\tau,\xi)$ and $f(\tau,u,\xi)$ are all B-measurable. Since $L(t,\xi)$ is continuous in t for any ξ by (B) and $m(t,\xi)$ is B-measurable in $L(t,\xi)$ by virtue of Theorem 4.1, $m(t,\xi)$ is also B-measurable in t for any ξ. Besides $m(t,\xi)$ is continuous in ξ for any fixed t by (A). Thus $m(t,\xi)$ is B-measurable in (t,ξ). Similarly $\sigma(t,\xi)$ and $f(t,u,\xi)$(for any fixed u) are B-measurable in (t,ξ). Therefore $f(t,u,\xi)$ is B-measurable in (t,u,ξ) since $f(t,u,\xi)$ is right-continuous in u.

For brevity we introduce the following notations. We put

$$c_1K_1+c_2K_2 = (c_1m_1+c_2m_2,\ c_1\sigma_1+c_2\sigma_2,\ c_1f_1+c_2f_2)$$

for $K_1=(m_1,\sigma_1,f_1)$ and $K_2=(m_2,\sigma_2,f_2)$. When $K(\tau,\omega) = (m(\tau,\omega),\sigma(\tau,\omega),f(\tau,u,\omega))$, we define

$$\int_\alpha^\beta K(\tau,\omega)d\mathcal{L} = \int_\alpha^\beta m(\tau,\omega)d\tau + \int_\alpha^\beta \sigma(\tau,\omega)dg + \int_\alpha^\beta\int_{|u|>1} f(\tau,u,\omega)dp$$
$$+ \int_\alpha^\beta\int_{|u|<1} f(\tau,u,\omega)dq.$$

The triple of three elements of an i.d.l. L is also denoted by the same notion L. We put

(11.4) $c_N(\omega) = \phi_N(c(\omega))c(\omega)$,

(11.5) $f_N(\tau,u,\xi) = \phi_N(u)f(\tau,u,\xi)$,

(11.6) $L_N(\tau,\xi)$ = the i.d.l. whose elements are $m(\tau,\xi)$, $\sigma(\tau,\xi)$ and $f_N(\tau,u,\xi)$,

where ϕ_N is the characteristic function of the interval $(-N,N)$.

Firstly we shall prove the existence and uniqueness of the solution of the stochastic integral equation:

$$(11.7) \qquad x(t,\omega) = c_N(\omega) + \int_a^t L_N(\tau, x(\tau,\omega))d\ell$$

such that

(11.8) $(x(t,\omega), \ell(\tau,\omega) - \ell(a,\omega), a \le \tau \le t)$ is independent of $(\ell(\tau,\omega) - \ell(t,\omega), t \le \tau \le b)$.

In order to find a solution we make use of the method of successive approximations; we define $x_n(t,\omega)$, n=1,2,..., recursively by

(11.9.1) $x_o(t,\omega)$ = any measurable process such that $x_o(t,\omega)$ is a function of $(c_N(\omega); \ell(\tau,\omega) - \ell(a,\omega), a \le \tau \le t)$ for $a \le t \le b$ and that $\mathcal{E}(x_o(t,\omega)^2)$ is bounded,

$$(11.9.2) \qquad x_n(t,\omega) = c_N(\omega) + \int_a^t L_N(\tau, x_{n-1}(\tau,\omega))d\ell, \quad a \le t \le b, \ n=1,2,\ldots .$$

From (11.9.2), n=1, we have, for any fixed ξ_o,

$$x_1(t,\omega) = c_N(\omega) + \int_a^t L_N(\tau, \xi_o)d + \int_a^t (L_N(\tau, x_o(\tau,\omega)) - L_N(\tau, \xi_o))d\ell ,$$

$$\int_a^t L_N(\tau, \xi_o)d\ell = \int_a^t m(\tau, \xi_o)d\tau + \int_a^t \sigma(\tau, \xi_o)dg + \int_a^t \int_{1<|u|<N} f(\tau, u, \xi_o)du/u^2dt$$

$$+ \int_a^t \int_{|u|<N} f(\tau, u, \xi_o)dq = I_1 + I_2 + I_3 + I_4,$$

$$\mathcal{E}((\int_a^t L_N(\tau, \xi_o)d\ell)^2) = 4I_1^2 + 4\mathcal{E}(I_2^2) + 4I_3^2 + 4\mathcal{E}(I_4^2),$$

$$I_1^2 \le (t-a)\int_a^t m(\tau, \xi_o)^2 d\tau \le (b-a)\int_a^b m(\tau, \xi_o)^2 d\tau ,$$

$$\mathcal{E}(I_2^2) = \int_a^t \sigma(\tau, \xi_o)^2 d\tau \le \int_a^b \sigma(\tau, \xi_o)^2 d\tau ,$$

$$I_3^2 \le \int_a^t \int_{1<|u|<N} d\tau\, du/u^2 \int_a^t \int_{1<|u|<N} f(\tau,u,\xi_0)^2\, d\tau\, du/u^2 \le 2\int_a^b \|f(\tau,u,\xi_0)\|_N^2\, d\tau$$

$$\mathcal{E}(I_4^2) = \int_a^t \int_{|u|<N} f(\tau,u,\xi_0)^2\, d\tau\, du/u^2 \le \int_a^b \|f(\tau,u,\xi_0)\|_N^2\, d\tau\ .$$

Thus $\mathcal{E}((\int_a^t L_N(\tau,\xi_0)d\ell)^2)$, $a \le t \le b$, is bounded by Theorem 4.2 and the condition (B). Similarly we can prove the boundedness of

$$\mathcal{E}((\int_a^t (L_N(\tau,x_0(\tau,\omega)) - L_N(\tau,\xi_0))d\ell\ )^2),\ a \le t \le b,$$

by making use of the condition (A). Consequently $\mathcal{E}(x_1(t,\omega)^2)$ is bounded.

Furthermore $x_1(t,\omega)$ belongs to d_1-class with P-measure 1 and so measurable in (t,ω). Besides $(x(\tau,\omega), \ell(\tau,\omega) - \ell(a,\omega);\ a \le \tau \le t)$ is independent of $(\ell(\tau,\omega) - \ell(t,\omega);\ t \le \tau \le b)$ for any t, as is easily verified.

Thus we can define $x_2(t,\omega)$ by (11.9.2) and so recursively $x_n(t,\omega)$, $n=3,4,\dots$, and we have

$$x_{n+1}(t,\omega) - x_n(t,\omega) = \int_a^t (m(\tau,x_n(\tau,\omega)) - m(\tau,x_{n-1}(\tau,\omega)))d\tau$$

$$+ \int_a^t (\sigma(\tau,x_n(\tau,\omega)) - \sigma(\tau,x_{n-1}(\tau,\omega)))dg + \int_a^t \int_{1<|u|<N} (f(\tau,u,x_n(\tau,\omega))$$

$$- f(\tau,u,x_{n-1}(\tau,\omega)))\ \frac{du}{u^2}$$

$$+ \int_a^t \int_{1<|u|<N} (f(\tau,u,x_n(t,\omega)) - f(\tau,u,x_{n-1}(\tau,\omega)))du|u^2 = I_1 + I_2 + I_3 + I_4,$$

$$\mathcal{E}(I_1^2) \leq \mathcal{E}((\int_a^t |m(\tau, x_n(\tau,\omega)) - m(\tau, x_{n-1}(\tau,\omega))| d\tau)^2)$$

$$\leq M^2(b-a)\int_a^t \mathcal{E}((x_n(t,\omega) - x_{n-1}(t,\omega))^2) d\tau \ ,$$

$$\mathcal{E}(I_2^2) \leq S^2 \int_a^t \mathcal{E}((x_n(\tau,\omega) - x_{n-1}(\tau,\omega))^2) d\tau \ ,$$

$$\mathcal{E}(I_3^2) \leq \mathcal{E}((\int_a^t \int_{1<|u|<N} |f(\tau,u,x_n(\tau,\omega)) - f(\tau,u,x_{n-1}(\tau,\omega))| d\tau du/u^2)^2)$$

$$\leq 2(b-a)F_N^2 \int_a^t \mathcal{E}((x_n(\tau,\omega) - x_{n-1}(\tau,\omega))^2) d\tau \ ,$$

$$\mathcal{E}(I_4^2) \leq F_N^2 \int_a^t \mathcal{E}((x_n(\tau,\omega) - x_{n-1}(\tau,\omega))^2) d\tau \ ,$$

$$\mathcal{E}((x_{n+1}(t,\omega) - x_n(t,\omega))^2) \leq 4(M^2(b-a) + S^2 + 2(b-a)F_N^2 + F_N^2)$$

$$\times \int_a^t \mathcal{E}((x_n(\tau,\omega) - x_{n-1}(\tau,\omega))^2) d\tau \ .$$

But $\mathcal{E}((x_1(t,\omega) - x_0(t,\omega))^2)$ has a finite upper bound (G), as is above proved. We obtain recursively ($\alpha = 4(M^2(b-a)+S^2+2(b-a)F_N^2+F_N^2)$)

$$(11.10) \qquad \mathcal{E}((x_n(t,\omega) - x_{n-1}(t,\omega))^2) \leq \alpha^{n-1} G(t-a)^{n-1} / \underline{|n-1},$$

$$(11.11) \qquad \mathcal{E}((\int_a^t |m(\tau,x_n(\tau,\omega)) - m(\tau,x_{n-1}(\tau,\omega)) d\tau)^2)$$

$$\leq \alpha^{n-1} GM(b-a)(t-a)^n / \underline{|n},$$

$$(11.12)\qquad \mathcal{E}\left(\left(\int_a^t (\sigma(\tau, x_n(\tau,\omega)) - \sigma(\tau, x_{n-1}(\tau,\omega))dg\right)^2\right)$$

$$\leq \alpha^{n-1} GS^2 (t-a)^n / \lfloor n,$$

$$(11.13)\qquad \mathcal{E}\left(\left(\int_a^t \int_{1<|u|<N} |f(\tau, x_n(\tau,\omega)) - f(\tau, x_{n-1}(\tau,\omega)))dt\, du/u^2\right)\right)$$

$$\leq 2\alpha^{n-1} GS\, (F_N^2 (b-a)(t-a)^n / \lfloor n,$$

$$(11.14)\qquad \mathcal{E}\left(\left(\int_a^t \int_{|u|>N} (f(\tau, x_n(\tau,\omega)) - f(\tau, x_{n-1}(\tau,\omega)))\, dq\right)^2\right) \leq \alpha^{n-1} GF_N^2 (t-a)/\lfloor n.$$

Now, putting t=b in (11.11) and using Bienaymé-Tschebycheff's inequality, we obtain

$$P(\{\omega;\ \sup_{a\leq t\leq b} \left|\int_a^t (m(\tau, x_n(\tau,\omega)) - m(\tau, x_{n-1}(\tau,\omega)))d\tau\right| \geq \lambda_n^{\frac{1}{4}}\})$$

$$\leq P(\{\omega;\ \int_a^b |m(\tau, x_n(\tau,\omega)) - m(\tau, x_{n-1}(\tau,\omega))|d\tau \geq \lambda_n^{\frac{1}{4}}\}) \leq \lambda_n^{\frac{1}{2}},$$

where $\lambda_n = \alpha^{n-1} GM^2 (b-a)/\lfloor n-1$. Since $\sum \lambda_n^{\frac{1}{4}}$, $\sum \lambda_n^{\frac{1}{2}} < \infty$,

$$\sum_n \int_a^t (m(\tau, x_n(\tau,\omega)) - m(\tau, x_{n-1}(\tau,\omega)))d\tau, \qquad a \leq t \leq b,$$

is uniformly convergent in $a \leq t \leq b$ with P-measure 1 by Borel-Cantelli's theorem. If we make use of (G'.4) (§8) and (11.12), we can prove in the same way that

$$\sum_n \int_a^t (\sigma(\tau, x_n(\tau,\omega)) - \sigma(\tau, x_{n-1}(\tau,\omega)))dg$$

is uniformly convergent with P-measure 1. Similarly

$$\sum_n \int_a^t \int_{N>|u|>1} (f(\tau,u,x_n(\tau,\omega)) - f(\tau,u,x_{n-1}(\tau,\omega)))d\tau du/u^2 \text{ and}$$

$$\sum_n \int_a^t \int_{N>|u|} (f(\tau,u,x_n(\tau,\omega)) - f(\tau,u,x_{n-1}(\tau,\omega)))dq$$

are uniformly convergent in $a \leq t \leq b$ with P-measure 1. Consequently $x_n(t,\omega)$ is also uniformly convergent in $a \leq t \leq b$ with P-measure 1. We denote this limit with $x(t,\omega)$. Then $x(t,\omega)$ belongs to d_1-class with P-measure 1, and so measurable in (t,ω), since it is so the case with $x_n(t,\omega)$, as is recursively proved. By letting $n \longrightarrow \infty$ in (11.9.2), we can easily see that $x(t,\omega)$ satisfies (11.7).

Let $y(t,\omega)$ and $z(t,\omega)$ be any two solutions of (11.7) such that $\mathcal{E}(y(t,\omega)^2)$ and $\mathcal{E}(z(t,\omega)^2)$ are bounded ($\leq G_1$). Then in the same way as above we see

$$\mathcal{E}((y(t,\omega) - z(t,\omega))^2) \leq 4(M^2(b-a) + S^2 + 2F_N^2(b-a) + F_N^2)$$

$$\times \int_a^t \mathcal{E}((y(\tau,\omega) - z(\tau,\omega))^2)dt \leq \frac{\alpha^n}{\lfloor n} (4G_1)(t-a)^n \longrightarrow 0.$$

Therefore $y(t,\omega) = z(t,\omega)$ with P-measure 1 for t. Since $y(t,\omega)$ and $z(t,\omega)$ belong to d_1-class as the solution of (11.7), we have $y(t,\omega) = z(t,\omega)$, $a \leq t \leq b$, with P-measure 1.

For the solution $x(t,\omega)$ obtained above by the successive approximations, $\mathcal{E}(x(t,\omega)^2)$ is bounded and so it does not depend on $x_0(t,\omega)$. Let $y(t,\omega)$ be any solution of (11.7). Starting from $x_0(t,\omega) = \phi_M(y(t,\omega))y(t,\omega)$, ϕ_M being the characteristic function of $(-M,M)$, we define $x_n(t,\omega)$, $n=1,2,\ldots$, by (11.9.2), and obtain the solution $x(t,\omega)$. Now we put

$$\Omega_M = \{\omega; |y(t,\omega)| \leq M,\ a \leq t \leq b\}.$$

Then Ω_M increases with M and $P(\Omega_M) \longrightarrow 1$ since $y(t,\omega)$ belongs to d_1-class with P-measure 1. $y(t,\omega)$ satisfying (11.7), we have, by (G.5) and (P.5). $y(t,\omega) = x_0(t,\omega)$

$=x_1(t,\omega)$ ... and so

$$x(t,\omega) = \lim_n x_n(t,\omega) = y(t,\omega)$$

for almost all ω on Ω_M and so with P-measure $1 (M \longrightarrow \infty)$. Thus we have proved the existence and uniqueness of the solution of (11.7), say $x(t,\omega;N)$.

We put

$$E_M = \{\omega; |c(\omega)| \leq M, \int_a^b \int_{|u|\geq M} dp(\tau,u,\omega) = 0\}.$$

Then E_M increases with M and $P(E_M) \longrightarrow 1$ as $M \longrightarrow \infty$. Starting from $x_o(t,\omega)$ $=x(t,\omega;M)$, we define $x_n(t,\omega)$ by (11.9.2) and so we obtain the solution of (11.7) $x(t,\omega;N)$ as the limit. For $\omega \in E_M (M > N)$ we have $x(t,\omega;M) = x_o(t,\omega) = x_1(t,\omega)$ $=\ldots=x(t,\omega;N)$. Therefore $x(t,\omega;M)$, $a \leq t \leq b$, does not depend on M for a sufficiently large M with P-measure 1 and so $\lim_{M\longrightarrow\infty} x(t,\omega;M)$ exists and satisfies (11.1).

Let $y(t,\omega)$ be any solution of (11.1). We put

$$F_N = E_N \frown \{\omega; |y(t,\omega)| \leq N, a \leq t \leq b\}.$$

Then $y(t,\omega)$ satisfies (11.7) in F_N. Starting from

$$x_o(t,\omega) = \phi_N(y(t,\omega))y(t,\omega)$$

we obtain $x_n(t,\omega)$, n=1,2,...,by (11.9.2). For $\omega \in F_N$ we have, by (G.5) and (P.5),

$y(t,\omega) = x_o(t,\omega) = x_1(t,\omega) = \ldots \longrightarrow x(t,\omega;N)$. Since $y(t,\omega)$ belongs to d_1-class with P-measure 1, we have $P(F_N) \longrightarrow 1$. Therefore $y(t,\omega)$ coincides with the above obtained solution. Thus we have proved the existence and uniqueness of the solution.

Let $x(t,\omega)$ be the solution. Then we have

$$(11.15) \qquad x(t,\omega) = x(s,\omega) + \int_s^t L(\tau, x(\tau,\omega))d\ell, \quad s \leq t \leq b.$$

This is also considered as a stochastic integral equation of the above type concerning $x(\tau,\omega)$, $s \leq \tau \leq b$. By the uniqueness of the solution, $x(t,\omega)$ is obtained by the above procedure and so $x(t,\omega)$ is a B-measurable function of $x(s,\omega)$ and $(\ell(\tau,\omega) - \ell(s,\omega),\ s \leq \tau \leq b)$, as is easily verified if we use Theorem 1. We put

$$x(t,\omega) = f_t(x(s,\omega);\ \ell(\tau,\omega) - \ell(s,\omega),\ s \leq \tau \leq b)$$

and

$$x(t,\omega;s,\xi) = f_t(\xi;\ \ell(\tau,\omega) - \ell(s,\omega),\ s \leq \tau \leq b).$$

From (11.15) we obtain

$$(11.16) \qquad x(t,\omega;s,\xi) = \xi + \int_s^t L(\tau, x(\tau,\omega;s,\xi))d\ell$$

with P-measure 1 for almost all $(P_{x(s,\omega)})\ \xi$, by replacing, in Theorem 2.3, $x(\omega), y(\omega)$ and $G(x(\omega), y(\omega))$ respectively with $x(s,\omega), (\ell(\tau,\omega) - \ell(s,\omega), s \leq \tau \leq b)$ and $x(t,\omega) - x(s,\omega) - \int_s^t L(\tau, x(\tau,\omega))d\ell$; the measurability condition can be easily verified by the definition of the stochastic integral, if we use Theorem 1.

By Theorem 2.3 we have, for almost all (with regard to the probability law of $(x(\tau,\omega),\ a \leq \tau \leq s)$) $(\xi_\tau,\ a \leq \tau \leq s)$,

$$Pr\{x(t,\omega) \in E / x(\tau,\omega) = \xi_t, a \leq \tau \leq s\} = Pr\{x(t,\omega;s,\xi_s) \in E\},$$

$$Pr\{x(t,\omega) \in E / x(s,\omega) = \xi_s\} = Pr\{x(t,\omega;s,\xi_s) \in E\},$$

and so

$$Pr\{x(t,\omega) \in E / x(\tau,\omega) = \xi_\tau,\ a \leq \tau \leq s\} = Pr\{x(t,\omega) \in E / x(s,\omega) = \xi_s\},$$

which implies that $x(t,\omega)$ is a simple Markoff process.

It remains only to prove (11.3). By the above discussion we need only prove that, as $\Delta_1 + \Delta_2 \longrightarrow 0$,

(11.17) the $[1/\Delta_1+\Delta_2]$ - times convolution of the p.l. of $x(s+\Delta_1,\omega;s-\Delta_2,\xi)$ tends to $L(s,\xi)$.

Let $x_N(t,\omega;s-\Delta_2,\xi)$ be the solution of the stochastic integral equation:

$$x_N(t,\omega;s-\Delta_2,\xi) = \xi + \int_{s-\Delta_2}^t L_N(\tau, x_N(\tau,\omega;s-\Delta_2,\xi))d\ell .$$

Then we have

$$(11.18) \qquad x_N(t,\omega;s-\Delta_2,\xi) = x(t,\omega;s-\Delta_2,\xi) \text{ for } s-\Delta_2 \leq t \leq s+\Delta_1 \text{ and for}$$

ω such that

$$(11.19) \qquad \int_{s-\Delta_2}^{s-\Delta_1} \int_{|u|>N} dp = 0.$$

Now we have

$$(11.20) \qquad x(s+\Delta_1, \omega; s-\Delta_2, \xi) = x(\omega) + y(\omega) + z(\omega),$$

where

$$x(\omega) = \int_{s-\Delta_2}^{s+\Delta_1} L(\tau, \xi) d\ell$$

$$y(\omega) = \int_{s-\Delta_2}^{s+\Delta_1} (L_N(\tau, x_N(\tau, \omega; s-\Delta_2, \xi)) - L_N(\tau, \xi)) d\ell \quad ,$$

$$z(\omega) = \int_{s-\Delta_2}^{s+\Delta_1} (L(\tau, x(\tau, \omega; s-\Delta_2, \xi)) - L_N(\tau, x_N(\tau, \omega; s-\Delta_2, \xi))) d\ell$$

$$+ \int_{s-\Delta_2}^{s+\Delta_1} (L_N(\tau, \xi) - L(\tau, \xi)) d\ell \quad).$$

$L(\tau, \xi)$ being continuous in τ for any ξ by (B), the logarithmic characteristic function $\phi(z; \tau, \xi)$ of $L(\tau, \xi)$ will be uniformly continuous in t, whenever z runs over any assigned bounded region, and so the l.c.f. of the $[1/\Delta_1+\Delta_2]$-times convolution of P_x:

$$[1/\Delta_1+\Delta_2] \int_{s-\Delta_2}^{s+\Delta_1} \phi(z; \tau, \omega) d\tau$$

will tend to $\phi(z; \tau, \omega)$ and so

$$P_x^{*[1/\Delta_1+\Delta_2]} \quad \text{(* means convolution)}$$

is arbitrarily near $L(s, \xi)$. By the property of stochastic integrals we have

$$\mathcal{E}(y(\omega)) = o(\Delta_1+\Delta_2),\quad \mathcal{E}(y(\omega)^2) = o(\Delta_1+\Delta_2),$$

$$\Pr\{z(\omega) \neq o\} \;(= (\Delta_1+\Delta_2)(1-e^{-\frac{2}{N}})) = o(\Delta_1+\Delta_2).$$

Thus it is sufficient to prove the following lemma.

Lemma 11. $x(\omega)$, $y(\omega)$ and $z(\omega)$ be real random variables on (Ω,B,P) such that

$$(11.21)\qquad |\mathcal{E}y(\omega)| < \frac{\alpha}{n},\quad \mathcal{E}(y(\omega)^2) < \frac{\alpha}{n},\quad \Pr\{z(\omega) \neq 0\} < \frac{\alpha}{n}\ (\alpha < 1).$$

Then we have

$$\rho\,(P_x^{*^n},\ P_\phi^{*^n}) < 4\sqrt{2}\,\alpha^{\frac{1}{3}},$$

where $\phi = x + y + z$ and ρ is the Levy's distance.

Proof. Let (Ω^*,B^*,P^*) be the product probability field $(\Omega,B,P)^n$. For $\omega^* = (\omega_1,\omega_2,\dots,\omega_n)$ we define $x^*_\nu(\omega^*) = x(\omega_\nu)$, $y^*_\nu(\omega^*) = y(\omega_\nu)$, $z^*_\nu(\omega^*) = z(\omega_\nu)$, $\nu = 1,2,\dots,n$. Then $(x^*(\omega^*),y^*(\omega^*),z^*(\omega^*))$, $\nu = 1,2,\dots,n$, are independent random vectors.

P_x^{*n} and P_y^{*n} are respectively the p.l. of $X^*(\omega^*) = \sum x^*_\nu(\omega^*)$ and $\Phi^*(\omega^*) = \sum x^*_\nu(\omega^*) + \sum y^*_\nu(\omega^*) + \sum z^*_\nu(\omega^*)$. But we have

$$\mathcal{E}((\sum y^*_\nu(\omega^*))) \leq (\sum_\nu \mathcal{E}(y^*_\nu(\omega^*)))^2 + \sum_\nu \mathcal{E}(y^*_\nu(\omega^*)) \leq (n\frac{\alpha}{n})^2$$
$$+\, n\frac{\alpha}{n} = \alpha^2+\alpha < 2\alpha\ .$$

Therefore we have

$$\Pr\{|\sum_\nu y^*_\nu(\omega^*)| > \alpha^{\frac{1}{3}}\} \leq 2\alpha^{\frac{1}{3}}.$$

But

$$\Pr\{\sum_\nu z^*_\nu(\omega^*) \neq 0\} \leq \sum_\nu \Pr\{z^*_\nu(\omega^*) \neq 0\} \leq n\frac{\alpha}{n} = \alpha\ .$$

Thus we have

$$\Pr\{|\Phi^*(\omega^*) - X^*(\omega^*)| > \alpha^{\frac{1}{3}}\} < 2\alpha^{\frac{1}{3}} + \alpha < 3\alpha^{\frac{1}{3}},$$

from which we obtain

$$\rho\ (\text{p.l. of } X^*(\omega^*),\ \text{p.l. of } \Phi^*(\omega^*)) < 4\sqrt{2\alpha}^{\frac{1}{3}},\ \text{q.e.d.}$$

Remark. In case $L(t,\xi)$ is a Gaussian distribution, the above obtained process is continuous with P-measure 1(continuous case). In the case when the l.c.f. of $L(t,\xi)$ is of the form:

$$\psi(z;t,\xi) = \int_{-\infty}^{\infty} (e^{i\lambda u} - 1)n_{t,\xi}(du),$$

the above process increases only with jumps with P-measure 1 (purely discontinuous case). The most simple case of $\sigma = 0$, $f \equiv 0$ is reduced to that of an ordinary differential equation:

$$\frac{dx}{dt} = m(t,x).$$

IV. Appendix I.

We shall show an interesting property of the stochastic integral which does not appear in the ordinary integral.

Theorem.[27] Let $\lambda(\xi)$ be any function of ξ with the continuous second derivative. Then we have

$$(1) \quad \int_a^b \lambda'(g(t,\omega))dg(t,\omega) = \lambda(g(b,\omega)) - \lambda(g(a,\omega)) - \frac{1}{2}\int_a^b \lambda''(g(t,\omega))dt.$$

Proof. For the brevity of the notation we may assume $a = 0, b=1$. We have clearly

$$(2) \quad \lambda(g(1,\omega)) - \lambda(g(o,\omega)) = I_1 + \frac{1}{2} I_2 + I_3 + I_4,$$

where

$$I_1 = \sum_{k=1}^{n} \lambda'(g(\tfrac{k-1}{n},\omega))(g(\tfrac{k}{n},\omega) - g(\tfrac{k-1}{n},\omega))$$

$$I_2 = \sum_{k=1}^{n} \lambda''(g(\tfrac{k-1}{n},\omega))\frac{1}{n}$$

$$I_3 = \sum_{k=1}^{n} \lambda''(g(\tfrac{k-1}{n},\omega))((g(\tfrac{k}{n},\omega) - g(\tfrac{k-1}{n},\omega))^2 - \tfrac{1}{n})$$

$$I_4 = \sum_{k=1}^{n} \delta(n,k,\omega)(g(\tfrac{k}{n},\omega) - g(\tfrac{k-1}{n},\omega))^2;$$

$\delta(n,k,\omega)$ tends to 0 uniformly in k as $n \longrightarrow \infty$ for almost all ω on account of the continuity of $\lambda''(\xi)$ and $g(t,\omega)$.

We can choose a sufficiently large N for any $\delta > 0$ such that $n > N$ implies

(3) $\Pr\{|\delta(n,k,\omega)| < \delta,\ k=1,2,\ldots,n\} > 1-\delta$.

If we define

$\delta^*(n,k,\omega) = \delta(n,k,\omega)$ for $|\delta(n,k,\omega)| < \delta$ and $= 0$ elsewhere, and if we put

$$I_4^* = \sum_{k=1}^{n} \delta^*(k,n,\omega)(g(\tfrac{k}{n},\omega) - g(\tfrac{k-1}{n},\omega)),$$

then we have

(4) $\Pr\{I_4 = I_4^*\} > 1-\delta$.

(5) $\mathcal{E}((I^*)^2) \leq \sum_k \delta^2 \mathcal{E}((g(\tfrac{k}{n},\omega) - g(\tfrac{k-1}{n},\omega))^2) = n\,\delta^2 \tfrac{1}{n} = \delta^2.$

By (4) and (5) we have, as $n \longrightarrow \infty$, $I_4 \longrightarrow 0$ in probability.

By the continuity (in t), of $\lambda''(g(t,\omega))$ we can choose a sufficiently large M for any $\delta > 0$ such that

(6) $\Pr\{|\lambda''(g(\tfrac{k-1}{n},\omega))| < M, k=1,2,3,\ldots,n\} > 1-\delta$.

If we define

$\mu_M(\xi) = \lambda''(\xi)$ for $|\lambda''(\xi)| \leq M$ and $= 0$ elsewhere, and if we put

$$I_{3,M} = \sum_{k=1}^{n} \mu_M(g(\tfrac{k-1}{n},\omega))((g(\tfrac{k}{n},\omega) - g(\tfrac{k-1}{n},\omega))^2 - \tfrac{1}{n}),$$

then we have, by (6),

(7) $\Pr\{I_3 = I_{3,M}\} > 1-\delta$,

and by making use of the fact that $g(\tau,\omega)$ is a differential process,

$$(8)\ \mathcal{E}(I_{3,M}^2) \leq M^2 \sum_{k=1}^{n} \mathcal{E}(((g(\tfrac{k}{n},\omega) - g(\tfrac{k-1}{n},\omega))^2 - \tfrac{1}{n})^2)$$

$$= M^2 n \int_{-\infty}^{\infty} \sqrt{\frac{n}{2\pi}}\, e^{-n\xi^2/2} (\xi^2 - \tfrac{1}{n})^2 d\xi = M^2/n \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\xi^2}{2}} (\xi^2-1)^2 d\xi \longrightarrow 0.$$

Therefore we have, as $n \longrightarrow \infty$, $I_3 \longrightarrow 0$ in probability.

Similarly we can prove that

$$I_1 \longrightarrow \int_0^1 \lambda'(g(t,\omega))dg(t,\omega)$$

in probability. By the continuity (in t) of $\lambda''(g(t,\omega))$ we have

$$I_2 = \sum \lambda''(g(\frac{k-1}{n}, \omega)) \frac{1}{n} \longrightarrow \int_0^1 \lambda''(g(t,\omega))dt.$$

Thus our theorem is completely proved.

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FOOTNOTES

1) W. Feller: Zur Theorie der stochastischen Prozesse (Existenz-und Eindeutigkeitssätze), Math. Ann. 113.

2) A. Kolmogoroff: Über die analytische Methoden in der Wahrscheinlichkeitsrechnung, Math. Ann. 104.

3) S. Bernstein: Equations differentielles stochastiques, Act. Sci. et Ind., 738,1938.

4) $[\alpha]$ is the greatest integer n such that $n \leq \alpha$.

5) The characteristic function of an infinitely divisible law is expressible in the form exp $\psi(z)$ by Levy's theorem. This $\psi(z)$ will be called hereafter the logarithmic characteristic function of the infinitely divisible law. Cf. P. Lévy: Théorie de l'addition des variable aléatoires (1937), Chap. VII, and also my previous paper: On stochastic processes (I), Japanese Jour. of Math. Vol. 18, 1942.

6) $E(+)$ is the set $\{\eta + \xi ; \eta \in E\}$.

7) A. Kolmogoroff: Grundbegriffe der Wahrscheinlichkeitsrechnung, Ergeb. der Math. Vol. 2, No. 6.

8) J. L. Doob: Stochastic processes depending on a continuous parameter Trans. Amer. Math. Soc. Vol. 42, 1937.

9) E. Slutsky: Sur les fonctions aléatoires presque périodiques et sur la décomposition des fonctions aléatoires stationaires en composantes, Act. Sci. et Ind. 738, 1938.

10) J. L. Doob: Markoff chains -- denumerable case. Trans. Amer. Math. Soc. 58, 1945.

11) R. Fortet: Les fonctions aléatoires du type de Markoff associées à certaines équations linéaires aux derivees partielles du type parabolique.

12) K. Ito: loc. cit. 5).

13) R. E. A. Paley and N. Wiener: Fourier transforms in the complex domain, Amer. Math. Soc. Coll. Publ. 1934, Chap. IX.

14) K. Ito: Stochastic integral, Proc. Imp. Acad. Tokyo, Vol. 20, No. 8.

15) K. Ito: On a stochastic integral equation, forthcoming in Proc. Imp. Acad. Tokyo, Vol. 22.

16) loc. cit. 14)

17) J. L. Doob: Stochastic processes with an integral-valued parameter, Trans. Amer. Math. Soc. Vol. 44, 1938. The proof of the theorem concerning the conditional probability law seems to be somewhat incomplete, but it is available for our special case.

18) By $[a,b]$ we understand the closed interval: $a \leq x \leq b$, and by $(a,b]$ the half-open interval: $a < x \leq b$. $[a,b)$ and (a,b) are to be understood similarly.

19) By $(x(\tau,\omega),\ a \leq \tau \leq t)$ we understand the joint variable of $x(\tau,\omega)$ for τ such that $a \leq \tau \leq t$.

20) loc. cit. 5).

21) P. Lévy: loc. cit. 5) p. 55.

22) P. Lévy: loc. cit. 5) p. 90.

23) loc. cit. 13).

24) loc. cit. 14).

25) loc. cit. 5).

26) J. L. Doob: loc. cit. 8) Lemma 2.1.

27) This theorem has been reported by the author without the proof, loc. cit. 14).

Appendix 2. A generalized Fokker-Plank equation.

When I sent this paper to Professor J. L. Doob, he suggested to me to show that the process of Theorem 11 satisfies the Fokker-Plank equation. Though I cannot yet prove it in its complete generality, I have been able to solve it to a certain extent. It seems to be of some use to add it as an appendix.

Let $x(t,\omega)$ be the solution of the stochastic integral equation in Theorem 11, and $x(t,\omega;s,\xi)$ be the solution of the stochastic integral equation:

$$(1)\quad x(t,\omega;s,\xi) = \xi + \int_s^t L(\tau, x(\tau,\omega;s,\xi))d\ell\ , \quad s \leq t \leq b.$$

By putting $a=s$ and $c(\omega) \equiv \xi$ in Theorem 11, we see that $x(t,\omega;s,\xi)$ is uniquely determined for each ξ and obtained by the procedure stated in Theorem 11, so that $x(t,\omega;s,\xi)$ is B-measurable in ξ. Denote the probability law of $x(t,\omega;s,\xi)$ with $F(s,\xi;t,E)$. By the argument in the proof of Theorem 11 we have

$$(2)\quad F(s,\xi;t,E) = \Pr\{x(t,\omega)\in E/x(s,\omega) = \xi\}$$

for almost all $(P_{x(t,\omega)})$ ξ.

Theorem. $F(s,\xi;t,E)$ has the following properties:

I. Chapman's equation.

$$(3)\quad F(s,\xi;t,E) = \int_{-\infty}^{\infty} F(s,\xi;u,d\eta)F(u,\eta;t,E) \quad (a \le s \le u \le t \le b).$$

II. Let $\Phi(\xi)$ any bounded function with the second derivative $\Phi''(\xi)$. Then we have

$$(4)\quad \lim_{\substack{\Delta_1+\Delta_2 \downarrow 0 \\ \Delta_1,\Delta_2 \ge 0}} \frac{1}{\Delta_1+\Delta_2}\left[\int \Phi(\eta)F(s-\Delta_1,\xi;s+\Delta_2,d\eta) - \Phi(\xi)\right]$$

$$= m(s,\xi)\frac{d\Phi}{d\xi}(\xi) + \frac{\sigma^2(s,\xi)}{2}\frac{d^2\Phi}{d\xi^2}(\xi) + \int\Big(\Phi(\xi+u) - \Phi(\xi)$$

$$- \frac{u}{1+u^2}\Phi'(\xi)\Big)n(du,s,\xi),$$

where $m(s,\xi)$, (s,ξ) and $n(E,s,\xi)$ are determined by

(5) the logarithmic characteristic function of $L(s,\xi)$

$$= im(s,\xi)z - \frac{\sigma^2(s,\xi)}{2}z^2 + \int_{-\infty}^{\infty}\left(e^{izu} - 1 - \frac{izu}{1+u^2}\right)n(du,s,\xi).$$

III. Generalized Fokker-Plank equation.

If $\frac{\partial^2}{\partial\xi^2}F(s,\xi;t,E)$ exists, then

$$(6)\quad D_s^- F(s,\xi;t,E) = \lim_{\Delta\downarrow 0}\frac{1}{\Delta}(F(s,\xi;t,E) - F(s-\Delta,\xi;t,E))$$

exists and we have, in using m, σ and n in II

$$(7)\quad D_s^- F(s,\xi;t,E) + m(s,\xi)\frac{\partial}{\partial\xi}F(s,\xi;t,E) + \frac{\sigma^2(s,\xi)}{2}\frac{\partial^2}{\partial\xi^2}F(s,\xi;t,E)$$

$$+\int_{-\infty}^{\infty}\left(F(s,\xi+u;t,E) - F(s,\xi;t,E) - \frac{u}{1+u^2}\frac{\partial}{\partial\xi}F(s,\xi;t,E)\right)n(du,s,\xi) = 0.$$

Remark 1. By specializing m, σ, n, we obtain the Fokker-Plank equations that have already been known.

Case 1. $m(s,\xi) = m(\xi), \sigma(s,\xi) = \sigma(\xi), n(E,s,\xi) \equiv 0$... Fokker-Plank's original case.

Case 2. $n(E,s,\xi) \equiv 0$ ... Kolmogoroff's continuous case.

Case 3. $p(s,\xi) \equiv n((-\infty, \infty), s, \xi) < \infty$ . . . Feller's mixing case.

In this case, by putting

$$n(E,s,\xi) = p(s,\xi)P(s,\xi,E(+)\xi)$$

$$m(s,\xi) - \int \frac{u}{1+u^e} n(du, s, \xi) = b(s,\xi),$$

and

$$\frac{\sigma^2(s,\xi)}{2} = a(s,\xi),$$

we obtain Feller's equation:

$$\frac{\partial}{\partial s} F(s,\xi;t,E) + a(s,\xi)\frac{\partial^2}{\partial \xi^2} F(s,\xi;t,E) + b(s,\xi)\frac{\partial}{\partial \xi}F(s,\xi;t,E)$$
$$+ p(s,\xi)\int (F(s,\eta;t,E) - F(s,\xi;t,E)) P(s,\xi,d\eta) = 0.$$

Remark 2. It is desirable to prove the existence of $\frac{\partial}{\partial \xi^2} F(s,\xi;t,E)$ under some adequate restrictions concerning m, σ and n, but it is an open problem for the author.

Proof 1. By the proof of Theorem 11 we see that $x(t,\omega;s,\xi)$ is a B-measurable function of ξ and $\ell(\tau,\omega) - \ell(s,\omega)$, $s \leq \tau \leq t$. Put

$$x(t,\omega,s,\xi) = f_{st}(\xi, \ell(\tau,\omega) - \ell(s,\omega), s \leq \tau \leq t).$$

Since

$$x(t,\omega;s,\xi) = \xi + \int_s^t L(\tau, x(\tau,\omega;s,\xi))\, d\ell$$

$$= \xi + \int_s^u + \int_u^t$$

$$= x(u,\omega;s,\xi) + \int_u^t L(\tau, x(\tau,\omega;s,\xi))d\ell, \ (s < u \leq t),$$

we have, by the uniqueness of the solution in Theorem 11,

$$x(t,\omega;s,\xi) = f_{ut}(x(u,\omega;s,\xi),\ell(\tau,\omega) - \ell(s,\omega),\ u \le \tau \le t)$$

for almost all ω. $x(u,\omega;s,\xi)$ being a function of $\ell(\tau,\omega) - \ell(s,\omega)$, $s \le \tau \le u$, and so independent of $\ell(t,\omega) - (s,\omega)$, $u \le \tau \le t$, we have, by Theorem 2.2,

$$\Pr\{x(t,\omega;s,\xi)\in E\} = \Pr\{f_{ut}(\eta,\ell(\tau,\omega) - \ell(s,\omega), u \le \tau \le t)\in E\} P_x(u,\omega;s,\xi)(d\eta)$$

$$= \int \Pr\{x(t,\omega;u,\eta)\in E\}\ P_x(u,\omega;s,\xi)(d\eta),$$

which proves Chapman's equation.

Next we shall prove III, assuming II. Since we have, by I,

$F(s-\Delta,\xi;t,E) = \int F(s-\Delta,\xi;s,d\eta)\ F(s,\eta;t,E)$, we obtain III at once, by putting $\phi(\xi) = F(s,\xi;t,E)$, $\Delta_1 = \Delta$ and $\Delta_2 = o$ in II.

We have only to prove II. In the proof of Theorem 11 we have shown that

$$(8)\quad F(s-\Delta_1,\xi;s+\Delta_2,E(+)\xi)^*\left[\frac{1}{\Delta_1+\Delta_2}\right] \longrightarrow L(s,\xi).$$

Therefore it is sufficient to show that (8) implies (4). For this we state some preliminary lemmas.

Lemma 1. Let the logarithmic characteristic function (l.c.f.) of an infinitely divisible law (i.d.l.) P be given by

$$(9)\quad \psi(z) = imz + \int_{-\infty}^{\infty} \left(e^{izu} - 1 - \frac{izu}{1+u^2}\right) \frac{1+u^2}{u^2}\, d\,G(u),\quad G(-\infty) = 0$$

and $\{P_k\}$ be a sequence of probability laws such that

$$P_k^{*k} \longrightarrow P.$$

Then

$$(10)\qquad G_k(u) \equiv k\int_{-\infty}^{u} \frac{v^2}{1+v^2}\ P_k(dv)$$

is bounded for $k=1,2,\ldots$ and $-\infty < u < \infty$, and

$$(11.a)\qquad G_k(u) \longrightarrow G(u)$$

for any continuity point u of $G(u)$, and

(11.b) $$\int \frac{d\,G_k(u)}{u} \longrightarrow m.$$

This lemma can be proved in the same way as A. Khintchine's proof of Lévy's formule (A. Khintchine: Déduction nouvelle d'une formule de P. Lévy, Bull. d. l'univ. d'état à Moscou, Sér. inter. Sect. A. Math. et Meca. Vol. 1, Fasc. 1, 1937).

Lemma 2. Let P be an i. d. l. with the l. c. f. $\psi(z)$ in (9). Let $\{P_h\}$ be a system of probability laws, h running over (o,c), (c=positive constant) such that

(12) $$P_h^{*[\frac{1}{h}]} \longrightarrow P.$$

Then

(13) $$G_h(u) = \frac{1}{h}\int_{-\infty}^{u} \frac{v^2}{1+v^2} P_h(dv)$$

is bounded for $o < h < c$ and $-\infty < u < \infty$, and

(14.a) $$G_h(u) \longrightarrow G(u)$$

for any continuity point u of G(u), and

(14.b) $$\int \frac{dG_h(u)}{u} \longrightarrow m.$$

This lemma follows immediately from Lemma 1.

Fundamental Lemma. Let P be an i.d.l. with the l.c.f.:

(15) $$\psi(z) \equiv imz - \frac{\sigma^2}{2} z^2 + \int_{-\infty}^{\infty} (e^{izu} - 1 - \frac{izu}{1+u^2})\, n(du),$$

and $\{P_h\}$ be a system of probability laws, h running over (o,c). Then (12) implies that we have, for any bounded function $\Phi(\xi)$ $(-\infty < \xi < \infty)$ with the second derivatives $\Phi''(\xi)$,

(16) $$\lim_{h \downarrow o} \frac{1}{h}\left[\int \Phi(\xi+u)\, P_h(du) - \Phi(\xi)\right] = m\,\Phi'(\xi) + \frac{\sigma^2}{2}\Phi''(\xi)$$

$$+ \int_{-\infty}^{\infty} (\Phi(\xi+u) - \Phi(\xi) - \frac{u}{1+u^2}\Phi'(\xi))\, n\,(du).$$

Proof. (Concerning the following proof the author has obtained many suggestions in his discussions with Professor K. Yosida, whose research on a generalization of Fokker-Plank equation will soon be published in some journal.)

(15) is written in the form (9) if we put

$$G(u) = \sigma^2 + \int_{-\infty}^{u} \frac{v^2}{1+v^2} n(dv) \quad (u > o) \tag{17}$$

$$G(u) = \int_{-\infty}^{u} \frac{v^2}{1+v^2} n(dv) \quad (u < o).$$

Defining $G_h(u)$ by

$$G_h(u) = \frac{1}{h} \int_{-\infty}^{u} \frac{v^2}{1+v^2} P_h(dv),$$

we obtain

$$\frac{1}{h}\left[\int \Phi(\xi + u)P_h(du) - \Phi(\xi)\right]$$

$$= \frac{1}{h}\left[\int (\Phi(\xi+u) - \Phi(\xi))\, P_h(du)\right.$$

$$= \int (\Phi(\xi+u) - \Phi(\xi)) \frac{1+u^2}{u^2}\, d\, G_h(u)$$

$$= \Phi'(\xi) \int \frac{d\, G_h(u)}{u} + \int \left(\Phi(\xi+u) - \Phi(\xi) - \frac{u}{1+u^2}\Phi'(\xi)\right) \frac{1+u^2}{u^2}\, d\, G_h(u),$$

$$= \Phi'(\xi) \int \frac{d\, G_h(u)}{u} + \int \psi(u,\xi)\, d\, G_h(u),$$

where $\psi(u,\xi)$ is defined by

$$\psi(u,\xi) = \left(\Phi(\xi+u) - \Phi(\xi) - \frac{u}{1+u^2}\Phi'(\xi)\right) \frac{1+u^2}{u^2} \quad (u \neq o) \tag{18}$$

$$\psi(o,\xi) = \Phi''(\xi).$$

Then $\psi(u,\xi)$ is bounded and continuous outsides of any neighborhood of u=o for any fixed ξ by the boundedness of Φ, while $\psi(u,\xi)$ is also bounded and continuous within any neighborhood of u=o for any fixed ξ, since

$$\Phi(\xi+u)-\Phi(\xi)-\frac{u}{1+u^2}\Phi'(u)=u\Phi'(\xi)+\frac{u^2}{2}\Phi''(\xi)-u\Phi(\xi)+o(u^2),$$

$$=\frac{u^2}{2(1+u^2)}\Phi''(\xi)+o(u^2).$$

Therefore we have, by Lemma 2, (17) and (18),

$$\lim_{h\downarrow o}\frac{1}{h}\left[\int\Phi(\xi+u)P_h(du)-\Phi(\xi)\right]=m\Phi'(\xi)+\int_{-\infty}^{\infty}\psi(u,\xi)d\,G(u)$$

$$=m\Phi'(\xi)+\frac{\sigma^2}{2}\Phi''(\xi)+\left(\int_{-\infty}^{-o}+\int_{o}^{\infty}\right)\psi(u,\xi)\,d\,G(u)$$

$$=m\Phi'(\xi)+\frac{\sigma^2}{2}\Phi''(\xi)+\int_{-\infty}^{\infty}\left(\Phi(\xi+u)-\Phi(\xi)-\frac{u}{1+u^2}\Phi'(\xi)\right)n(du),$$

which proves the fundamental lemma.

Proof of II of the above theorem. We can modify the above fundamental lemma without any essential change in the proof as follows:

" Let $\{P_{\Delta_1\Delta_2}\}$ be a system of probability laws, where $o\leq\Delta_1,\Delta_2<c$ and $\Delta_1+\Delta_2>o$, and P be an i.d.l. stated in the lemma. Then

(12') $$P_{\Delta_1\Delta_2}^{*[\frac{1}{\Delta_1+\Delta_2}]}\longrightarrow P$$

implies

(16') $$\lim_{\Delta_1+\Delta_2\downarrow o}\frac{1}{\Delta_1+\Delta_2}\left[\int\Phi(\xi+u)P_{\Delta_1\Delta_2}(du)-\Phi(\xi)\right]$$

$$=m\Phi'(\xi)+\frac{\sigma^2}{2}\Phi''(\xi)+\int_{-\infty}^{\infty}\left(\Phi(\xi+u)-\Phi(\xi)-\frac{u}{1+u^2}\Phi'(\xi)\right)n(du)".$$

Now put

$$\Phi(\xi) = F(s, \xi; t, E),$$
$$P_{\Delta_1 \Delta_2}(E) = F(s-\Delta_1, \xi; s, E),$$

and

$$P = L(s, \xi).$$

Then (8) i.e. (12') implies (16') i.e. (4). Thus the theorem is completely proved.

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