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THE TEACHING OF MATHEMATICS

THE TEACHING OF MATHEMATICS

ISSUED BY THE
INCORPORATED ASSOCIATION OF
ASSISTANT MASTERS IN
SECONDARY SCHOOLS

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CONTENTS

FOREWORD *page* viii

MEMBERS OF THE COMMITTEE ix

I MATHEMATICS, TEACHERS AND PUPILS

1. The Origin and Purpose of this Report 1
2. Changes since 1919 which have affected the Teaching of Mathematics 2
3. The Changing Attitude to Mathematics 3
4. What is Mathematics? 4
5. The Influence of Psychology 6
6. Nunn's Principles 8
7. Mathematics as Mental Training 9
8. Immediate Ends 11
9. Ultimate Ends 15
10. Syllabuses—Traditional and Alternative 16

II PRE-SECONDARY SCHOOL MATHEMATICS

1. The Two Stages of Development 17
2. The Nursery and Infants' Schools 17
3. The Junior School 18
4. Addition and Subtraction 19
5. Multiplication and Division 21
6. Money, Weights and Measures 24
7. Fractions 27
8. Decimals 27
9. Areas and Volumes 28
10. Unitary Method 28
11. Problems in the Junior School 28
12. Attainment on Entry to Secondary Grammar Schools 30

III AIMS AND SYLLABUSES FOR THE FIRST FIVE YEARS IN A SECONDARY GRAMMAR SCHOOL

1. Why Secondary Grammar School only? *page* 36
2. Fundamental Aims of the First Two Years 36
3. Some General Comments on Method 37
4. A Teaching Syllabus for the First Two Years 40
5. Mathematics as One Subject—an Integrated Syllabus 44
6. A Teaching Syllabus for the Third Year 47
7. Analysis of the Third-Year Syllabus 48
8. A Teaching Syllabus for the Fourth Year 50
9. A Teaching Syllabus for the Fifth Year 51
10. Analysis of the Fourth- and Fifth-Year Syllabuses 52

IV MATHEMATICS IN THE SCHOOL AND CLASSROOM

1. Inculcating a Love for Mathematics 54
2. The Teacher's Attitude to his Class 59
3. Methods in Class Work 60
4. The Organization of the Work of a School 74

V SOME GENERAL TEACHING TOPICS

1. Degrees of Accuracy 81
2. The Teaching of Logarithms 85
3. Tables, Slide Rules, and Graph Paper 89
4. Written and Deductive Work in Geometry 91
5. The Teaching of Area 104
6. Three-Dimensional Geometry 109
7. Calculus in the Fifth Year 115
8. Surveying, Meteorology, Astronomy and Navigation 123
9. Correlating Mathematics with other Subjects 128
10. Visual Aids 131
11. The Environmental Approach 133
12. The History of Mathematics 136
13. The Philosophy of Mathematics 140

VI SIXTH FORM WORK

1. General Discussion *page* 146
2. A Syllabus of Pure Mathematics 156
3. Mechanics in relation to Physics 158
4. A Syllabus of Mechanics 160
5. The Scholarship Level 161
6. The Scholarship Level Syllabus 163
7. Sixth Form Geometry 165

VII THE MATHEMATICAL LIBRARY

1. General Remarks 171
2. Some Suggestions for Books for the Mathematical Library 174

VIII THE MATHEMATICS ROOM

1. Introduction 179
2. The Purpose of the Mathematics Room 180
3. Description of the Mathematics Room 183
4. Apparatus 187
5. The Use of the Mathematics Room in Teaching 189
6. Conclusion 193

IX THE TEACHING OF STATISTICS

1. Statistics up to the Ordinary Level 194
2. Statistics in the Sixth Form 200
3. Conclusion 216

APPENDIX

The Jeffery Report 218

INDEX 227

FOREWORD

The Teaching of Mathematics, issued by the Incorporated Association of Assistant Masters in Secondary Schools, is the seventh book in the Association's post-war series on the teaching of the main subjects in the secondary school curriculum. *The Teaching of Science*, *The Teaching of Modern Languages*, *The Teaching of History*, *The Teaching of English*, *The Teaching of Geography*, and *The Teaching of Classics* have already been warmly welcomed. In the preparation of these publications the Association has had the good fortune to be able to draw upon the devotion, knowledge and experience of many members and friends, to all of whom it would express its most grateful thanks.

The far-reaching reforms in the administration of secondary education foreshadowed in the Education Act of 1944, the development of a wide variety of types of secondary education suited to the abilities and aptitudes of all children in the secondary stage of their education, and the need for further advancement of the 'grammar-school type' of secondary education combine to make it essential that there should be a thorough examination of the whole content of what is taught in the schools.

It is the earnest hope of the Executive Committee that this present publication will bring to that study a contribution of no mean value and that its wide circulation will exert an important influence on the theory and practice of the teaching of mathematics in secondary schools of all types.

October 1956

A. W. S. HUTCHINGS
Secretary, Incorporated Association of Assistant Masters in Secondary Schools

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I

MATHEMATICS, TEACHER AND PUPILS

1. The Origin and Purpose of this Report

This report is one of a series which deals with the teaching of the principal subjects in the secondary school curriculum and whose preparation has been suggested by members of the I.A.A.M. No such general survey of the teaching of Mathematics has been made since 1919 when a committee of the Mathematical Association reported on the Teaching of Mathematics in Public and Secondary Schools. Their report was revised and reissued in 1928 but is now out of print. The only mathematical report which has been issued by the I.A.A.M. is that on the sequence of theorems in geometry which appeared in 1923. Since then mathematical studies in secondary schools have changed in many ways. In view of these changes, and of the transformation of the secondary stage in education, it seemed appropriate to consider afresh the teaching of mathematics at this stage. The present report is the work of a committee representing both the I.A.A.M. and the Mathematical Association. Its chief concern is with the work of the secondary grammar school, but some reference is made to work in other types of secondary school.

The preparation of this report has not been free from difficulties. The teaching of mathematics is still in a state of uneasy transition from the traditional forms which prevailed up to the beginning of the present century. Changes since then have affected aims, methods and syllabuses. The final outcome of these changes is still undecided but it may be possible to suggest some of its main features from collective experience available. The purpose of this report is to summarize this experience after discussion of fundamental principles. It is convenient to begin by indicating some of the changes which have affected the teaching of mathematics in secondary grammar schools since the issue of the 1919 report.

2. Changes since 1919 which have affected the Teaching of Mathematics

The first change is a substantial increase in the secondary grammar school population, with some alteration in its average intellectual quality. The second is a profound change in the influence of the examinations for the General Certificate of Education. Those who devised these examinations in the period from 1918 to 1925 planned a system that would follow the school curriculum rather than control it. For several reasons the result has on the whole been otherwise. Furthermore, to meet the syllabuses of these examinations, a variety of text-books has been produced which have exerted a subtle influence on the teacher's outlook and his choice of material. On the other hand there has been considerable improvement in methods of teaching. New elements have been incorporated into the school syllabus and others have been discarded. A series of helpful reports issued by the Mathematical Association has discussed the teaching of separate subjects. Wider issues not immediately influencing work in the classroom have been considered in a steady stream of official reports from the Hadow Report of 1931 to the Scottish Advisory Council Report on Secondary Education of 1947. In these reports will be found discussions on the nature of the curriculum and the place of mathematics therein, on the aims of school mathematics and on the part to be played by the examination system.

These changes in the content and treatment of school mathematics have occurred against a background of change in the content and treatment of mathematics at the university level. During the past fifty years we can observe:

(*a*) The growth of abstract mathematics following the work of Russell and Whitehead on the foundations of mathematics.

(*b*) The consequent increasing rigour and thoroughness of the presentation of mathematics in our universities and therefore in the higher forms of secondary schools.

(*c*) A tremendous increase in the range of mathematical knowledge through the rapid development of new fields and techniques, as in modern algebra and probability theory, some knowledge of which may be required by scientists as well as by mathematicians.

These changes have led to difficulties in relating school and university mathematics and have stimulated the expansion of the school

curriculum to include calculus, mechanics and statistics. Future teachers in secondary schools may have studied mathematics in special honours courses, designed primarily to produce creative or research workers in their subject; or they may have attended courses far less exacting in which a less abstract presentation of mathematics has been given. The gradual evolution of the professional mathematician and the arrangement of university work to keep pace with the ever increasing advances in mathematics have consequences which can hardly be assessed, beyond the bare statement that future teachers in secondary schools are unlikely to be drawn from those who have attained high places in their university work. The general pattern of mathematical studies at school and university is very different in 1956 from what it was in 1919.

Other changes are less easily defined. There have been fundamental changes in the teacher's attitude to the work of the psychologists. There have been popular accounts of mathematics such as *Mathematics for the Million* (1936) by Hogben, whose success may indicate some inadequacy in the traditional treatment of mathematics in schools. The Second World War has revealed new material suitable for the curriculum in the mathematical courses for the Air Training Corps. These courses have confirmed the work of the psychologists by showing the surprising efforts that pupils will make in learning mathematics if their studies have some clear and immediate purpose. Finally there have been changes in the examination requirements. The most notable is the trend towards the unification of school mathematics into a single subject, which led to the introduction in 1944 of an alternative syllabus prepared by the Jeffery Committee.

In preparing this report we have borne in mind these changes and their possible concequences.

3. The Changing Attitude to Mathematics

A generation ago the teaching of mathematics in grammar schools was in several ways a simpler matter than it can be today or is likely to be in the future. The curriculum—despite the onslaughts of Professor Perry and the Association for the Improvement of Geometrical Teaching, later the Mathematical Association—was still determined by tradition. In this tradition the value of mathematics

in providing mental training was unchallenged: it was universally agreed that mathematics was taught partly for its utility but mainly because it trained the mind. And for this training almost any mathematical material was adequate. The relation of the pupil to this process was hardly considered. The fundamental importance of his co-operation was ignored: the influence of his emotional reactions was unrecognized. Today the position is altogether different. We have to face:

(*a*) Recurrent demands for a searching revision of the mathematical curriculum. Their most recent and scathing expression is found in the Scottish Council Report already mentioned.

(*b*) Increasing doubt of the value of mathematics *as usually taught* in providing training in clear thinking.

(*c*) Widespread recognition of the importance of arousing the pupil's interest and providing a worth-while motive for learning. The emphasis is increasingly on the student rather than on the subject, but in so far as it concerns the subject it is with the substance of what is taught and less with the structure.

In consequence it is necessary to reconsider our aims, immediate and ultimate, when we set out to teach mathematics. We must examine our choice of material and determine, from the point of view of the psychologist, the most effective ways of presenting it. It is first necessary to consider the nature of mathematics itself.

4. What is Mathematics?

Teachers of mathematics who discuss their work with other people discover widespread misunderstanding of the nature of their subject; mathematics means profoundly different things to different people. For effective teaching it is important to recognize and to appreciate these different aspects of mathematics. Let us look at the matter from the point of view of the mathematical graduate fresh from the university about to begin his teaching in a secondary school.

He is likely to regard mathematics as a series of methods arranged in logical order and dealing with abstract conceptions. He accepts that there are certain broad divisions such as algebra and analysis, and further subdivisions within these. The chief purpose of mathematics appears to him to be the increase of knowledge of some subdivision. He agrees that at the university level, mathematics is

largely abstract, and that its problems are suggested by its own subject matter. This self-creating aspect of mathematics is of extreme importance and is frequently misunderstood. It is not always realized that the historic roots of mathematics lie deep in problems arising in the physical world, but that the elements of abstraction and of generalization soon enable mathematical studies to evolve independently of these problems. Further, the trend is constantly towards abstraction and generalization though continual new calls are made on mathematics by problems suggested by the outside world, as in the fields of statistics and atomic physics. In considering these new problems all kinds of new methods and techniques may be evolved. Some of these, as in the theory of operators, may be used before their logical structure is completely understood. In other words, precision of reasoning need not be opposed to the use of imagination and intuition: everything depends on the stage of development of an inquiry. The final stage, however, is always one of ordering logically the methods and results that have been discovered.

With this preliminary discussion our young teacher would probably agree that for purposes of teaching at a lower level such as the average secondary school, the studies classified as mathematics might be described as a collection of methods and results created through solving problems suggested originally by the physical world, but latterly by the results themselves.

Thus geometry developed through problems in surveying, astronomy and navigation. The methods and theorems discovered were found, by the Greeks, to possess in their own right, and independently of the original problems, a logical coherence and ordered structure, so that they could be deduced by reasoning from certain axioms. The truths of geometry and of many other branches of mathematics do in fact reveal themselves in two aspects. With the one they face and have contact with the world of outer reality, with the other they face and have relations with one another. The full exposition of this discovery may best be read in the original words in *The Teaching of Algebra, including Trigonometry* by the late Sir Percy Nunn. Its full appreciation will enable the young teacher to avoid those unfortunate antitheses which commonly divide mathematics into useful and academic, or into practical and theoretical, or suggest that the subject should be studied either for its 'utility'

or for its 'outlook' value. These classifications are unhistorical and unhelpful.

Two further points arise from this double aspect of mathematics. The first is concerned with the logical organization of the subject. During the last fifty years there has been a concentration on foundations, on mathematical logic, on 'axiomatik' to use Klein's term. In teaching at the secondary school level this may lead to premature abstraction, to omission of the free use of imagination and intuition, to emphasis on the deductive-postulation side of mathematics. It may lead teachers to overlook the advantages of presenting mathematical ideas historically and the importance of inductive reasoning in mathematics. It has been said that the greatest gift a young mathematician can have is 'creative intuition'. It is necessary for those teaching the subject to recognize the importance of both the inductive and deductive treatment of mathematics according to the immediate needs of the pupils.

The second point is the resemblance between mathematics and music. Both activities are recognized as essential elements of well-ordered living: both are self-creating, both have direct contact with daily living. It might be suggested that for both subjects formal teaching is best regarded as largely an interpretation, and that the teacher of mathematics may find inspiration in his work if he regards his aim not as that of developing creative mathematicians but as that of interpreting as many aspects of mathematics as he can. Finally, the words of Courant in a notable work, *What is Mathematics?*, may be quoted: 'For scholars and laymen alike it is not philosophy but active experience in mathematics itself that alone can answer the question: What is Mathematics?'

5. The Influence of Psychology

The importance of the twofold aspect of mathematics discussed above is due to Nunn. His also is the suggestion that the motives which give rise to mathematical activities in the human mind are three: wonder, utility and the urge to systematize. The importance of motive in learning mathematics has been stressed by the work of the psychologists in studying more general problems of learning and thinking. It is essential to say something of this as it seems indispensable to the proper understanding of aims in the teaching of mathematics.

The notable changes in the education of young children which have taken place in the past forty years as a result of careful study of how children learn have, as yet, had comparatively little effect on education at the secondary school level. It is however realized, in Sir John Adam's famous phrase, that 'if we are to teach John Latin, we must understand John as well as understand Latin'.

The essential conclusions of the work of the past forty years seem to be:

(*a*) The fundamental importance for the pupil of motive and purpose.

(*b*) The disabling effects produced in the pupil by emotional reactions, such as fear or dislike of the subject taught and loss of confidence through lack of success.

The first point is sufficiently illustrated by the keenness for mathematics shown by Air Training Corps cadets when mathematics was seen by them to be necessary for their work. With regard to the second we may note the common experience of teachers of mathematics in their contact with intelligent adults. Most adults, or at any rate those over the age of thirty or so, seem to have disagreeable memories of their mathematical lessons: few seem to have enjoyed their teaching, though an increasing number are grateful for some aspects of it, usually because it helped them with other work. Many feel wistfully that there must be something worth learning about mathematics, but that it has always eluded them. Both *Mathematics for the Million* and *Mathematician's Delight* (W. W. Sawyer) were written for those who were unable to profit by school mathematical teaching because 'they were put off'. Failure to appreciate school mathematics may lie in the 'abstract' approach to the subject through rules and definitions, instead of in the 'interest' approach through problems within the pupil's experience. It may also be caused by loss of confidence on the part of the pupil, usually through missing some step in an argument, or through failing to master some piece of technique—e.g. in manipulating numbers. If, however, it is admitted that lack of success leads to loss of confidence, then it is equally true of learning mathematics that 'nothing succeeds like success', a fact which implies that material must be carefully selected or modified to suit the pupils' abilities and interest.

Behind this discussion is a deeper educational principle which is considered in various official reports such as the Hadow Report of

1931, the Spens Report of 1938 and the more recent Scottish Reports. It is that the curriculum for young people must be thought of in terms of activity and experience, not as hitherto in terms of passive assimilation by the pupil of material set before him by the teacher. This view of the curriculum affects the whole position of mathematics in schools. In particular it influences the presentation and the selection of material which have also been altered by the challenge of psychologists to the traditional belief that 'mathematics trains the mind'.

6. Nunn's Principles

The preceding discussion suggests that the nature of mathematics is an obstacle to the successful teaching of mathematics to young people. In no other subject of the curriculum do we encounter this difficulty: to overcome it we need to associate *ultimate* ends with *immediate* ends. It is essential for the keen mathematician to realize that the nature of much of his material can become a barrier between his pupils and himself, however noble his aim, unless he relates his material to his pupils' world of ideas. Our ultimate ends cannot now be as simple as they used to be: our immediate ends are far more important than they were.

The most satisfying analysis of this difficult matter is Nunn's. He derives it from two fundamental principles of exposition which may be stated thus:

(*a*) The ultimate end of teaching mathematics is to demonstrate the internal logical coherence of its subject-matter; but this logical coherent structure has to be re-erected for our pupils by our teaching; it is not the datum from which our teaching begins.

(*b*) At each stage of instruction the subject-matter offered to our pupils must be determined by the motive, or the combination of motives (wonder, utility, the urge to systematize) which is most likely to evoke from them the intellectual effort which the assimilation of the subject-matter demands.

In discussing the application of these principles, Nunn showed how the twofold nature of mathematics, that is, its contacts with the external world and its own internal logical coherence, is demonstrated by the historical development of mathematics. He summed up the purpose of teaching mathematics in schools in words which have often been quoted:

Our purposes in teaching mathematics in school should be to enable the pupil to realize, at least in an elementary way, this twofold significance of mathematical progress. A person, to be really 'educated', should have been taught the importance of mathematics as an instrument of material conquests and of social organization, and should be able to appreciate the value and significance of an ordered system of mathematical ideas. There is no need to add that mathematical instruction should also aim at 'disciplining his mind' or giving him 'mental training'. So far as the ideals intended by these phrases are sound they are comprehended in the wider purpose already stated.

Thus Nunn distinguishes the ultimate from the immediate ends of mathematical teaching. The ultimate ends may seem beyond the comprehension of young people with only a limited knowledge either of mathematics or of the world in which they live. The teacher must keep the ultimate ends in view, but he can lead his pupils towards them only by using the immediate ends most appropriate to each stage of teaching, that is, by evoking feelings of wonder in mathematics, or by demonstrating its utility or its systematizing power.

It is perhaps helpful to consider ultimate and immediate ends separately, but the ultimate end, usually expressed in the form that 'mathematics is training for the mind', must be considered first. Our attitude to this professed purpose is important because it has a critical influence on our discussions about what and how to teach in school mathematics.

7. Mathematics as Mental Training

It is widely supposed that mathematics is a valuable study because it disciplines the mind and thereby enables the mathematician, because of his training in mathematics, to deal more successfully with problems *outside* mathematics than he would have been able to do without that training. By solving exercises the pupil learns to concentrate, to draw deductions, to think clearly, to express himself precisely and to achieve, in mathematics, accuracy. All these are very desirable qualities, and no doubt most teachers of mathematics believe, in their heart of hearts, that in themselves they are promoted by some if not all of their work in school. And so indeed they *may* be: but there is less evidence than we suppose to show that these qualities *can* be transferred from work in mathematics to other fields of activity. The possibilities of this transfer of training are much

smaller than has formerly been supposed. A clear and comforting account of the position is printed in Appendix V to the Spens Report of 1938:

> Briefly, the study of mathematics may have beneficial results in developing accuracy, precision and reasoning ability but everything depends on the conscious recognition by the learner of appropriate ideals of neatness or logical thinking, and on conscious transfer from mathematical to other material. Transfer of improvement in some desirable quality occurs only under appropriate conditions and does not occur appreciably simply through use of conventional or traditional material. Geometry, for instance, must be taught in such a way that the logical relations developed from it are not associated only with geometrical material: they must be made free and accessible, available for use in non-geometrical situations. Thus little or no 'transfer' follows unintelligent memory drill, but considerable 'transfer' follows the conscious application of principles and methods of procedure.

Some valuable comments on the consequences of this discovery so far as it affects the teaching of science are found on page 13 of the *Report on the Teaching of General Science* of the Science Masters' Association (1936). After pointing out that the possibilities of transfer of training are smaller than is generally believed the report continues:

> On the other hand it has been shown that the importance of the sentiments which grow up round a subject cannot be overestimated. It is essential for future success that a child's first introduction to Science should not lead to boredom, to a feeling of failure and insufficiency, or to an impression that it is a formalized study. . . .

Later it is pointed out that 'We must not include in our syllabuses any matter which is taught only for the sake of the training it gives', nor can we 'defend a narrow curriculum by urging the claim that it trains boys to think and to appreciate scientific method'.

There are clear implications here for the teacher of mathematics. The intelligent response to this challenge from the experimental psychologists is a resolve to modify the material and the treatment of the mathematical curriculum so that the maximum transfer shall occur. This would imply a closer association of mathematics with everyday life, and a wider field of exercises in which reasoning can be applied. Geometry for instance might be re-inspired by becoming genuinely three-dimensional (including many different ways of representing three-dimensional space on flat paper), by including some discussion of its logical bases and logical implications (e.g.

non-Euclidean geometry), and by showing methods of analysis and argument other than those commonly used in solving riders. Much of the geometrical and spatial material involving comparisons and relationships such as are found in intelligence tests might be drawn upon.

The one consequence that seems inescapable is that far more depends on wise choice of material and on psychologically intelligent presentation of material than has hitherto been supposed. No longer can we justify the teaching of *any* mathematical material on the ground that it trains the mind. Attention must be given to the kind of material, to the conscious co-operation of the pupil, to explicit attention to possible transfer from mathematical to non-mathematical processes. A fundamental change of attitude is implied here. So long as mathematics was felt to impart a mental discipline the content of the syllabus was comparatively unimportant. Nor, at that time, was the pupil's attitude relevant. Indeed it was generally believed that the more repulsive and exacting was the mathematical work, the more valuable was the mental training it developed, exemplifying the ancient heresy that 'it doesn't matter what you teach him so long as he dislikes it'. Few specialist mathematical teachers would admit the validity of this dictum when considering their own school and university careers, and it may well be discarded in favour of more wholesome doctrines.

The first of these is that in teaching any subject in the curriculum the primary aim is to arouse the interest of the pupil—the emphasis must be on the student and not on his subject. The others derive from this. They are:

(*a*) The material for the mathematical curriculum must be carefully selected,

(*b*) It is important to make the relevance and aims of mathematics clear to the student,

(*c*) It is necessary to find psychologically effective means of presentation.

It is clear that immediate ends and means are of the utmost importance.

8. Immediate Ends

The motives which lead young people to specialize in mathematics at the university are not always simple, but presumably one motive

is a powerful interest in the subject: a real love for mathematics. The immediate aim of the teacher must be to arouse in his pupils some of this enthusiasm, interest and love. A great headmaster once remarked that for success in teaching 'a man must know what he wished his pupils to know, he must be what he wished his pupils to be, and to these he must add enthusiasm'. Unfortunately not even knowledge allied with enthusiasm will wholly succeed in encouraging a liking for mathematics unless at every stage we study the needs of the pupil. For example, it is a commonplace of all intelligent teaching that difficulties must be presented one at a time, but in mathematics special attention is needed to determine what *are* the individual pupil's difficulties. Furthermore, the order of presentation of mathematics tends to be logical instead of psychological. In particular, it is tempting to begin with the final form—with the ultimate definitions, complete rules, the perfected theorems—instead of developing a topic in the form through which it evolved historically. There is a perpetual temptation to show the tool and then apply it to the problem, whereas the reverse is the more effective procedure: first show the problem, then the evolution of the tool, lastly the application of the tool in wider fields. In effective teaching of mathematics the question 'Why are we learning this stuff?' should not arise, since the immediate material through its presentation is an obvious and satisfying answer. Clearly the best time to systematize a topic, to show pupils the relations of this topic to others of which they already have some understanding, is at the *end* of the class instruction, or during revision of the topic.

It will usually be found that wise teaching can achieve a liking and a respect for mathematics with nearly all pupils, and even those with some inherent or temperamental aversion from mathematics can be aroused by one or other of its many aspects—its wonder, its utility, its creativeness. Most children wisely taught can acquire sufficient technical skill to pass examinations, but they may also feel, when they leave mathematics for studies more congenial to them, that they have not wasted their time, and that when they were studying mathematics they were in contact with something relevant and great.

In achieving our immediate ends some of the conditions which seem to be necessary will now be outlined. First the order, selection and treatment of topics must be psychological rather than logical.

It is essential to stimulate the pupil's interest, to suggest a relevant purpose, and to secure confidence by successful achievement on the part of the individual. This does not imply that the work need be 'easy' or 'soft': indeed work properly planned calls for greater effort on the part of the teacher, stimulates greater industry on the part of the pupil and gives deeper satisfaction to both. Nor does the need to arouse the pupils' interest imply that the course should consist wholly of interesting snippets. No study can profitably be pursued without periods of routine drill when necessary techniques are perfected. What is essential is that this drill be related to some purpose which is significant to the pupil. It is well known that children will labour most keenly when the incentive comes from within themselves.

Secondly the work in the classroom must constantly be related to other fields of knowledge; to the pupil's immediate experience, and to other activities (e.g. science and geography) in which he is engaged. Obviously the conventional subdivisions of mathematics into arithmetic, etc., may be disregarded, even though an immediate integration of these elements is not readily attained.

It may be remarked that no final arrangement or order of topics in a course of mathematics organized on modern lines is immediately to be recommended. The ideas on which such a course should be organized are not yet widely appreciated. It is quite likely that several arrangements are possible, and more than one may well be equally convenient. A good deal of experiment is desirable. It is already known that many topics in elementary arithmetic are more efficiently learnt at a stage far later than that at which they have been taught in the past.* There are good grounds for teaching simple spatial knowledge through handwork, patterns, plans and models at an earlier age than usual (see the *Report on Arithmetic and Spatial Knowledge in Primary Schools* (Longmans, 1940)). The impact of alternative syllabuses in school certificate mathematics encourages rearrangement of advanced topics in trigonometry and geometry. But only provisional suggestions for the arrangements of modern courses can be made in the light of present knowledge. We deal with this matter more fully later (Ch. III). Meanwhile the reader may adjust his particular course in the light of the suggestions we have made.

* See *Studies in Education*, No. 7, 'The Bearings of Recent Advances in Psychology on Educational Problems' (Evans, 1955).

In general the treatment of a particular topic should follow the course suggested by the natural motives for studying that topic, as shown in its history. A problem is indicated, methods are devised for its solution, and these methods are organized into a logically related body of knowledge. The problem should therefore precede the technique: briefly, *one should teach through problems not for problems.* A lesson on a new topic should begin, as it were, with a challenge.

The logical sequence which is the final form of a branch of mathematics is the form usually used in research papers, and thus in text-books at the university level. Therefore a university graduate new to teaching has to reconsider his presentation in the classroom. It is fundamentally important for him to succeed in this; a formal logical treatment of his material can create a barrier between him and his pupils. The ideal method of presentation is to investigate a new topic as a voyage of discovery. This implies an active and creative presentation: a lesson not a lecture. It should reveal the substance of mathematics before organizing its structure. A lesson should begin with a challenge; it should then devise a method or a tool, and finally exhibit the use of this tool in other fields.

The selection of material for lessons in mathematics must depend on the capacity of the learners. It is, however, not necessary to defer the study of one branch, e.g. geometry, until high achievement has been attained in some unrelated branch, e.g. arithmetic. It seems wise to let the development of any branch follow a concentric plan. Thus some acquaintance with decimal notation through tenths may come after simple common fractions have been considered. Later this may be revised and extended to work with hundredths and thousandths. Probably work with metric units will be used here, though this can be deferred. Similarly, in algebra it is unwise to work exhaustively through every known species of quadratic equation. Such work should be broken up by other studies requiring formulae and graphs. The general plan should be a concentric development implying revision-and-extension.

For success in teaching mathematics to children of less than average ability it is essential to select and to simplify the material ruthlessly. Thus work with compound units can be restricted to two units at most, fractions to those represented on a ruler, trigonometry to one or two ratios. The aim here is to cover as wide a field

of genuine mathematics as possible by eliminating needless difficulties such as arise in cumbrous manipulation or involved techniques. *In general a process should be studied only so far as it is needed to solve a problem or illustrate a principle.* And at all stages the approach should be through media other than the eye—by measuring, modelling, surveying, use of instruments such as slide rules and verniers and every device making for comprehensive perception, for insight.

9. Ultimate Ends

It is easy in writing of ultimate aims in any field of education to utter pious platitudes. Perhaps the preceding discussion will make clear, however, that some of the aims which can inspire the teacher of mathematics are simple, clear and precise. They are intended to show that mathematics is a creative activity, with many aspects —more aspects than are generally recognized. We have already mentioned the external or problem aspect—sometimes inadequately described as the practical side of mathematics. We have indicated the internal aspect—the systematizing aspect of mathematics—again often described imperfectly as the abstract side of the subject. There are others. For instance there is the fascinating self-creating aspect of mathematics, shown in the evolution of kinds of number, of the theory of equations, of abstract algebra and geometry. Good teaching will reveal all these aspects. It will show the immense range of mathematical studies, their deep roots in history and in science, and will stress that mathematics is a dynamic activity and not a static body of accepted dogma.

To help teachers in framing their courses, especially in the earlier stages of mathematical work, it may be suggested that they regard mathematics as the study of problems (using the word in a wide sense) about number, quantity, space, and change. Such a study will reveal important methods of investigation—such as symbolic and graphic representation or the varieties of deductive and inductive reasoning—and will lead on to generalization and increasing abstraction. It may be added that material providing mathematical activity is to be found in many fields—for instance, in the sciences, mechanics, geography, astronomy—and a mathematical course will gain in richness and variety by drawing on such resources as these offer.

Perhaps in no other activity in the curriculum is it so important to reveal ends through means. If our aims in the teaching of mathematics are to be achieved then both materials and methods need re-examination. With present syllabuses much can be done by intelligent teaching to make an acceptable course of instruction, since the new alternative syllabuses were constructed from successful teaching of appropriate parts of the traditional syllabus. But the challenge to the traditional course concentrates on comparative values: on the emphasis still laid on needless techniques and unessential topics. To meet this challenge new syllabuses have been devised. For some years their interpretation will be a matter of experiment. Only one comment may be made: that in experimenting with a new syllabus it is fatal to allow oneself to be wholly circumscribed by that syllabus.

10. Syllabuses—Traditional and Alternative

At a later stage in this report there will be found examples of mathematical syllabuses, but it is necessary to point out at this stage that there are already substantial variations in content and interpretation of syllabuses, and we do not desire to pronounce emphatically in favour of one as against another. The present report attempts to promote fruitful teaching at all stages, and its authors realize that for many years much mathematical teaching will follow the ancient ways. It has seemed more realistic to discuss separately two types of course rather than to belittle the one to the greater glory of the other.

It is perhaps convenient, however, to classify syllabuses as *traditional* (by which we mean the conventional syllabuses of the secondary grammar school which divide mathematics sharply into subject divisions and these into subdivisions such as practical and theoretical geometry) and *alternative* (by which we mean those implying a regrouping of mathematics through topics or projects, or containing unusual material, as well as those based on interpretations of the Jeffery Report). In the following discussions we have considered the two types separately since it is impracticable to group them together. Finally we make constructive suggestions for unifying mathematics at the secondary school level.

II

PRE-SECONDARY SCHOOL MATHEMATICS*

1. The Two Stages of Development

The development of a sense of number and of space in the child before the age of eleven can be considered to fall into two stages.

In the first stage, as the result of his own experience and through his play, the child gains a considerable body of knowledge informally. This is a period in which the progress the child makes is largely the result of his own mental growth. He is entering the province of abstract thought and no amount of teaching can accelerate his progress; it may, in fact, even confuse or retard him.

In the second stage he develops the power to deal with number with greater abstraction and no longer needs the aid of continual concrete illustration; he is ready to treat number as of interest in itself, and the mechanization of the fundamental processes is begun and completed. Further, his continually increasing power of muscular co-ordination and visual imagination allows him a wider exploration of the world of shape, size and position.

Though the age at which a child makes the transition from the first stage to the second varies widely, the change is assumed to be in progress about the age of seven, when the child is transferred from the Infants' to the Junior School. The work of these schools is arranged to correspond with these stages of mental growth.

2. The Nursery and Infants' Schools

Number Work. 'Activity' methods are thekey note throughout the first stage. The child is provided with opportunities and experiences in which number forms a natural and inevitable part, and so he becomes acquainted with the simple number ideas. He will learn to count and to recognize the number symbols; he will begin to develop an idea of place value. With the help of apparatus he

* For an authoritative and comprehensive discussion the recent report of the Mathematical Association, 'The Teaching of Mathematics in Primary Schools', should be consulted.

will perform easy exercises in addition and subtraction and will begin to build up a knowledge of the number combinations. Multiplication and division will occur informally, the first from counting equal groups and the second from sharing and grouping.

Space Knowledge. The apparatus which the child uses in his number work does in itself help in the development of a knowledge of spatial relationships, but many Infants' Schools are now provided with apparatus which is specifically devoted to this object. Children find great satisfaction in the use of this material, as all who have seen young children using such things as a 'posting box' will agree. The provision of such material is of great value in extending a child's idea of shape and size, and the predominance of the three-dimensional aspect in the apparatus is an important feature.

Vocabulary. Number and space knowledge demand a quite extensive vocabulary of their own; an important aim of the work in the Infants' Schools is to begin to provide each child with a vocabulary of necessary words and phrases, and to ensure that he is able to understand and to use them. Such a vocabulary will cover most simple qualitative and quantitative expressions; it will also include the units of measurement of time and space which come within the child's own experience.

3. The Junior School

The main object of the work of the Junior School is to give facility and confidence in the fundamental processes and operations. The arithmetical ideas which have been met informally in the first stage are developed formally; the attempt is made to give a thorough understanding and mastery of the number combinations, the tables, the four rules, and the straightforward use of the units of measurement including those of area and volume. The fundamental processes applied to simple fractions are usually completed by the end of the course and the idea of a decimal fraction is introduced, though formal work on decimals is rarely attempted. Easy problems involving proportion form a normal part of the syllabus, unitary method being commonly employed. The idea of ratio may be introduced through scale drawing, but formal work is deferred. Graphical work, when it occurs, is generally in the form of single temperature charts, etc.; the emphasis is more often on the drawing of graphs rather than on their interpretation.

Space knowledge receives comparatively little attention and is frequently restricted to 'ruler-work', which is often linked with work in handicrafts. At present it is rare to find any attempt to pursue the subject beyond these somewhat narrow limits. In a comparatively small number of schools, however, much good work is being done by way of elementary surveying, the drawing of plans and elevations of a very simple kind, the construction of easy three-dimensional figures such as the pyramid and of models whose origin may be in other lessons such as geography and history.

No description of the work of the Junior School would be complete without some mention of selection examinations at 11+. While it is no part of our function to enter into a critical discussion of this examination, its influence upon the work of the Junior School, particularly in the last year, cannot be neglected. Its demands have often led in the past to undue pressure upon children to cover a wide syllabus, to learn processes for which they were not ready, and to devote an undue amount of time to 'problems' at the expense of a sound knowledge and understanding of fundamental processes. This evil, while less prevalent now than in the past, still exists. With the research and attention which is now devoted to this examination and the closer co-operation between teachers and examiners it is hoped that this unfortunate state of affairs will soon be entirely a thing of the past.

The methods commonly used in the Junior School are of direct interest to the teacher in the Grammar School and it is therefore thought that a summary dealing with features of the teaching of the fundamental processes will prove useful.

4. Addition and Subtraction

At the Infants' School the child builds up for himself a knowledge of the simple number combinations, both addition and subtraction being dealt with, though the former always leads. The results the child discovers are expressed first in concrete form by the use of apparatus such as counters and cards on which 'pips' arranged in definite formation represent each number. These results are subsequently expressed by the child in symbolic form.

In the earliest years at the Junior School the aim is to consolidate the knowledge of the fundamental combinations discovered in the

Infants' School and to extend their use to larger numbers. Concrete representation is gradually dropped though a return to it is made on any occasion when difficulty arises.

Formal Addition. Many teachers at first allow children to add by 'making up to 10', e.g. 8+7 is treated as 8 and 2 are 10; 10 and 5 are 15. It is generally recognized, however, that speed and accuracy in addition cannot be attained until the number combinations are memorized and much practice is devoted to this end.

Formal Subtraction. There are three possible methods: (*a*) equal additions, (*b*) decomposition, (*c*) complementary addition.

Illustrations of these methods which show the reasoning the child employs follow:

(*a*) *Equal additions*

	H	T	U		H	T	U
573	5	7	3 +10		5	7 +10	3 +10
−289 →	2	8 +1	9	→	2 +1	9	9
			4		2	8	4

The verbal statement of the child usually takes the form: '9 from 3, I cannot. Give it a 10 and add 1 to 8 "to be fair". Then 9 from 13, etc.'

The sum is, of course, recorded once only. It is split into the parts shown above in order to illustrate the process.

(*b*) *Decomposition*

	H	T	U		H	T	U
573	5	6	13		4	16	13
−289 →	2	8	9	→	2	8	9
					2	8	4

Verbal statement: '9 from 3, I cannot. Take 10 from the 7 tens. Then 9 from 13, etc.'

(*c*) *Complementary Addition.* Using the same example as above the process consists in finding what addition to 289 is needed to make 573.

The full explanation is as follows:

(i) To produce the 3 of 573, 4 must be added to the 9 of 289; 289 has therefore now become 293.

(ii) To produce the 7 tens of 573, 8 tens must be added to the 9 tens of 293; 289 has therefore now become 373.

(iii) To produce the 5 hundreds of 573, 2 hundreds must be added to the 3 hundreds of 373; 289 has therefore now become 573.

The total amount added in steps (i), (ii) and (iii) is 284, and therefore $573-289 = 284$.

In practice the process is performed as follows:

(i) $9+?=13$. Record 4 and carry 1 to the 8, giving 9.

(ii) $9+?=17$. Record 8 and carry 1 to the 2, giving 3.

(iii) $3+?=5$. Record 2.

The first of these methods, equal additions, is undoubtedly the most popular and research appears to show that in general it is likely to give more accurate results. Decomposition is sometimes taught, but, while easy to demonstrate in concrete form, its use does not appear to give as high a degree of accuracy as equal additions. A further drawback is the difficulty which arises in such cases as $5000-1387$. Complementary addition is rarely taught as a standard method though the idea usually receives some attention when dealing with the shopkeeper's method of giving change; it can also be very useful in dealing with logarithmic subtraction where 'bars' are involved. Whichever method is adopted, its use is carried on to subtraction when it occurs in money, weights and measures.

An interesting discussion of the whole subject with the results of various researches appears in vol. II of *Studies in Arithmetic* published by the Scottish Council for Research on Education (University of London Press). Some local authorities have formed co-ordinating committees of teachers in primary and secondary schools to standardize methods: we recommend the formation of such committees.

5. Multiplication and Division

Informal work on this subject begins in the Infants' School, and children there begin to build up their own tables by a process of continued addition. This work is carried on in the Junior School on a more formal basis though the idea of continued addition is still

employed. The process is preserved in the form in which the tables are now commonly learnt and referred to, e.g. the 'table of sevens' instead of the 'seven times table'.

Table of sevens	Seven-times table
One seven is seven	Seven ones are seven
Two sevens are fourteen	Seven twoes are fourteen
Three sevens are twenty-one, etc.	Seven threes are twenty-one, etc.

Tables are now generally learnt by practice in haphazard order so that results are memorized as entities and can be given without the necessity of repeating the table from the beginning, which was often the result of the traditional method of learning the tables by repetition. It must be said, however, that while the modern method is clearly desirable if as high a standard of accuracy can be attained as by the older method, there appears to have been something of a decline in the general standard of table knowledge. Whether this is due to the change of method or to other circumstances it is difficult to say, but the decline has been noticeable for a considerable period. The tables are sometimes extended beyond 12×12 but this is not desirable. A knowledge of some simple squares, e.g. $15^2 = 225$ is, however, useful and these should be learnt.

Short Multiplication. This is universally taught in traditional form.

Long Multiplication. Most children are now taught to begin this operation using the left-hand figure of the multiplier first. There is a substantial body of opinion which considers that the grounds for preferring this method to the traditional one are somewhat slender and that no great advantage is gained by its use. The main argument of those who support the use of the left-hand figure first is that this is the most important figure of the multiplier and that the initial product gives the main part of the answer. Further, in the new method there is less likelihood of error owing to transfer of eye in the most important product, which will be recorded only two lines below the multiplicand. In the case of the older method this product may have to be recorded several lines below the multiplicand, with a consequent greater chance of mistake. It is sometimes urged that the new method is also an advantage from the point of view of later work on decimals and contracted methods. Whether this is true of decimals will depend upon the method employed. Con-

tracted methods are now so rarely taught that they can hardly be considered to provide a valid argument. Advocates of the traditional method state that many children find difficulty in the correct placing of the first product when the left-hand figure of the multiplier is used; no such difficulty arises when the unit figure is employed. While not attempting to decide which is the better of the two methods, we can certainly say that it would be unwise to attempt to change a child from the use of one method to the other unless there were very good reasons.

Short Division. The fact that this is only an abbreviated form of long division is often recognized in the early teaching stages by setting out the work in long division form. This is found to have the advantage that the child finds not only that the process is easier to understand but also that the way is paved for the later work on long division. If the method of approach indicated above is employed, the quotient is usually placed above the dividend when the abbreviated form of short division is introduced. This practice is, in fact, becoming increasingly common whatever teaching method is employed. Admittedly this may lead to difficulties if factor division is taught, but this process, particularly the calculation of the remainder, is found so difficult by children that it is better omitted.

Long Division. This is one of the most difficult arithmetical processes the children learn in the Junior School. The difficulties are (i) the determination of the quotient figure, (ii) the somewhat complicated form of lay-out, (iii) zeros in the quotient.

Learning how to decide upon the quotient figures is the greatest difficulty. There are two popular methods of teaching for this purpose.

Method A. Preliminary work is performed on the division of numbers by multiples of 10, 100, etc., with the aim of making the children realize that division by a number such as 30 reduces itself to division by 3.

The first work on long division proper will employ divisors which are very near multiples of 10, e.g. 31 (such numbers as 29 are dealt with a little later owing to the greater difficulties of calculation, though the method employed is the same). The result of dividing by 31 will be considered as being near that obtained when dividing by 30. The child therefore estimates the figure of his quotient by using the divisor 3. The early examples are chosen so that the

estimated figure obtained in this way is the correct one. When the child has attained a measure of confidence and success with such divisors, examples are introduced where the estimated figure is not correct, but the method will always give an approximation to the quotient figure and prevent blind and aimless guessing.

Method B. The children first build up the complete table of their divisor and then select the appropriate multiples as the division proceeds. They may then be introduced to Method A, or, in the light of the experience of several sums performed by Method B, may proceed to deal with the process without such help as is afforded by Method A or the table.

The complications of the lay-out of long division and of zeros in the quotient are greatly reduced if the children have already seen short division performed in long division form. Many teachers now preface their work in long division by a few such examples.

The 'Italian' method in which the processes of calculation of the multiple and subtraction are performed in one step is occasionally taught but its use is not recommended as it is difficult and the standard of accuracy attained is usually low.

6. Money, Weights and Measures

Money

Addition of Pence. Two methods are employed:

Method A. Completion of shillings as the addition progresses, e.g.

$$\begin{aligned} 8d. + 9d. &= 12d. + 5d. \\ &= 1s.\ 5d. \end{aligned}$$

Method B. Addition of pence and final conversion to shillings, e.g.

$$\begin{aligned} 8d. + 9d. &= 17d. \\ &= 1s.\ 5d. \end{aligned}$$

Method A is often favoured since it is held that there is less likely to be confusion over carrying than in Method B, where one ten may be carried instead of one 12. For speed and accuracy in complex addition, Method A is certainly the better.

Addition of Shillings. The conversion to £'s is often dealt with by the use of the ten-shilling note as a unit.

£	*s.*	*d.*	
3	15	0	$5s.+7s$ = 1 ten-shilling note + $2s.$
2	17	0	3 ten-shilling notes = £1 + 1 ten-shilling note.
6	12	0	

Subtraction. Whichever of the three methods described in the section on the subtraction of number is adopted, it will be used in subtraction of money.

Short Multiplication. The standard method is universally employed. Conversion from shillings to pounds is often taught by using the ten-shilling note as a unit, as in addition.

Long Multiplication. Before commenting on the details of this process we recommend that less time be given to mechanical computations which are unreal, and that more time be given to arithmetical problems; some comments on problems will be found on p. 28.

The '10–10' method, where multiplication by such a number as 273 is based on a calculation of the results of multiplying by 10 and 100, is still taught but children find it laborious and are often very inaccurate in its use.

The 'top-line' or 'wholesale' method by which the result is obtained by one process of multiplication is now very popular and the excellent results obtained by its use fully justify this popularity.

Time is still being spent on examples with sums of money which involve half-pennies and farthings and large multipliers; work on this process should be restricted to the simpler forms of examples which are likely to occur in real life.

Short Division. This is taught in the usual form, the quotient often being placed above the dividend.

Long Division. The compact form of statement illustrated below is now usually employed:—

£27 16*s.* 1*d.* ÷ 17.

	£	*s.*	*d.*	
	1	12	8	Rem. 9*d.*
17)	27	16	1	
	17	216	145	
	—	17	136	
	10	—	—	
		46	9	
		34		
		—		
		12		

We advise that in the pre-secondary stage there should be no remainders.

The use of this process later in secondary school arithmetic in evaluating such fractions as £37/73 to the nearest penny is not always realized. It is the experience of many teachers that pupils will later perform a calculation quite correctly up to the point at which they arrive at such a fraction but find difficulties in expressing it as a compound quantity. While one method is to decimalize £37/73 and then convert it to shillings and pence, pupils, and particularly the more backward ones, are often found to be quicker and more accurate when the fraction is treated as the long division sum £37 ÷ 73.

Weights and Measures

Work with these, particularly in the early stages, is made as concrete as possible. In the Infants' School children weigh and measure and begin to find out the relations between the various units for themselves. This work is continued in the Junior School. Difficulties arise, of course, in dealing with the larger units, but as far as possible these are related to the child's experiences. Thus the mile will be related to the distance between home and school, the ton to coal, and so on.

The methods for the four rules learnt in dealing with money are applied to similar calculations in weights and measures, though the tendency nowadays is to restrict the units employed to those in common usage and to set only such simple examples as are likely to occur in practice.

In the same way, examples set on the process known as 'reduction' have shown a welcome change. While in the past it has been the custom to spend a long time and much labour on such questions as 'convert 107,501 oz. to tons, cwt., qr., lb., oz.' or 'express 3 m. 7 fur. 5 ch. 16 yd. 3 ft. in feet', the process is now largely restricted, as it should be, to change from a given unit *to the one or two immediately above or below*, since such change is almost all that is ever needed in practice. In fact, a little reflection shows that the use of a wide range of units in everyday life is rare; for instance, the draper works almost entirely in yards (and to girls, therefore, such fractions as $\frac{1}{2}$ yd. and $\frac{1}{4}$ yd. have an important significance); the engineer often works entirely

in inches, and so on. Some units we should omit entirely; others we should include when teaching a particular group of pupils if these pupils meet them in their daily life. We suggest that children need to have a real *comprehension* of everyday units of weight, length and time, and of the pint, quart and gallon.

7. Fractions

The understanding and manipulation of simple fractions are usually completed during the Junior School stage. On entry to the secondary school the child will have covered the following syllabus: the idea of a fraction to represent a part of a whole; equivalence of fractions; the four rules.

There is little to say about this work, the methods employed being the normal ones. There is, perhaps, rather too much emphasis on 'cancelling' at the expense of the opposite process, with the result that in addition and subtraction, when it is necessary to express fractions with a new denominator, difficulties sometimes arise. It may be remarked that a formal method for the determination of L.C.M. should be omitted, the denominators employed being sufficiently simple for the L.C.M. to be determined by inspection. Many secondary school teachers would desire less formal manipulation of addition and subtraction of fractions, and they would exclude multiplication and division of fractions entirely. They believe that junior children cannot understand the nature of these processes and that they would be better employed in analysing problems requiring the simpler processes of arithmetic.

8. Decimals

This topic is often excluded from the Junior School syllabus. When it does appear the work is usually restricted to the idea of decimal representation of such concrete examples as $2\frac{8}{10}$ in., and informal addition and subtraction with simple measurements using tenths of an inch on a ruler. There are good arguments for excluding it from the Junior School syllabus if it is taught as a series of mechanical rules.

9. Areas and Volumes

The area of a rectangle is normally included but many junior schools do no work on volumes, and in the opinion of the Committee this is the correct policy. We deal with the teaching of areas on p. 104.

10. Unitary Method

Unitary method is taught and used to solve questions about proportionately related quantities: the 'middle line' is worked out. A more extensive treatment of proportion should come only in the secondary school.

11. Problems in the Junior School

A problem is any question concerning which the preliminary inquiry is to decide which process or processes are involved. Problems should be taught from the earliest stages. As the reading difficulty is profound and the experience of the child is limited, questions should in all cases be simple; language should be clear and precise, and the situations considered should be worthwhile to the child and within the orbit of his own experience. Artificial problems should be avoided.

A child should not be heard to say, 'I don't like problems'. This implies that the process of doing problems has been detached from the mathematics stream. Work in problems should appear, not as a second or third stage in the development of the subject begun formally at nine or ten, but as the natural way of doing arithmetic. *The word 'problem' should not be mentioned:* problems are the 'beginnings' of arithmetic.

In general, no lesson in arithmetic should be complete without (*a*) some practice in mental arithmetic, (*b*) simple mechanical arithmetic, (*c*) simple problems on the mechanical arithmetic of the lesson. For example, if, after some oral mental arithmetic, a card of twenty addition questions of the type 13+28 is to be done, the last question could well be: 'There are 23 boys in a class and 14 girls, how many children altogether?' This simple inquiry will not be regarded as a 'problem', something apart from the main items of the lesson, but as a perfectly straightforward, clear and easy application of a rule already known. Without consciously thinking of the distinction

between mechanical work and problems, the child can be gradually led to bring reality to the subject, and overcome the difficulty that at present appears to exist.

Pupils should be encouraged to use their imagination in *inventing* problems on questions of mechanical arithmetic. They may be given $4 \times 9 = 36$ and be asked to state a problem leading to this. They may give:

A man earns £9 a week for 4 weeks and thus earns £36.
A baker delivers 4 loaves a day for 9 days.
How many pints of milk in 9 vessels each containing 2 quarts?
When the Goode quads are nine years old, how many candles will there be on their birthday cakes?
Nine motor-cars—How many wheels?

With the experienced teacher there may develop a real enthusiasm for this type of question—and the converse mathematical process will develop naturally. In the later stages, there will be a demand from the children for more complicated processes. The question (4×3) plus $5 = 17$ or (4×3) add $5 = 17$ will lead to innumerable situations. The problems in division, e.g. $100 \div 25 = 4$, will not only be useful in themselves, but will lead to a keener appreciation of the process of division.

It is essential to do mental problems, not only orally, but also with the medium of the written word. The child must be encouraged to read questions. It is quite a simple matter to prepare a list of twenty mental questions, and to ask the child to jot down the answer after reading the questions. Answering mental arithmetic questions should not be an entirely oral process.

Problems should be real. Not only should they come from books or cards, but the teacher should arrange for them to arise naturally from the activities of the child. Local needs may in fact demand a careful editing of any text-book. Every Junior School should be equipped with weights and measures in everyday use and articles of household use such as scales, buckets, bottles, jugs, balls of string, weighing machines, clocks, springs, tools and calendars. There should be available charts dealing with local industries, maps of local railway lines, bus and tram tables, charts of gas and electricity consumption in the school. Diagrams should be available dealing with comparative weights and measures. Children should tot up the week's bank money, make approximations of interest and keep records of

milk consumed. They should forecast future consumption. If a function is to be held in the school to raise funds for the school Christmas party, this can be the theme of a large number of problems. The school sports day will introduce plans of the field, layout, time-tabling of events, comparisons of time and distance. The local park and the school garden are useful sources of material.

Problems set in the Junior School should be those in which method is more important than arithmetical computation. The standard of the arithmetical computation for problems should be lower than that of the mechanical questions. Problems should always be easy, very easy, and the child in the Junior School should develop confidence in dealing with them.

Children should be encouraged, but not compelled, to set down briefly the stages in the solution of a problem. No stereotyped methods should be insisted upon. With the abler child simple intermediate mental steps should be encouraged. Pupils should see partial credit given for method and intermediate steps in their marked questions. They should be encouraged and expected to describe their solutions in words.

A series of 'serial' problems, the final answer to which is dependent on the accuracy of all the preliminary ones, is not appropriate for the Junior School.

The key to the solution of problems is the one-step problem. It is self-evident that proficiency in one-step problems is an essential preliminary to all future work. The transition from the one-step problem is critical; here diagrams are helpful.

11. Attainment on Entry to Secondary Grammar Schools

In considering the teaching and syllabus of mathematics in secondary grammar schools, one of the first questions that arises is 'At what point do we start?' Secondary school entrants have already spent several years learning arithmetic if not mathematics. The content of at least the first-year of the secondary grammar school course depends on the answer to this question. When it was put to members of the Committee in the form 'What is the level of attainment of the pupils who enter your school?' it was found that the level varied over a wide range, depending in part on the type of secondary school,

on the method of selection of entrants, and on the differing requirements of the respective education authorities. For example, some members said that it could not be assumed that the entrants knew the multiplication tables up to 12×12; others said that entrants to their schools were familiar with decimals, with elementary algebra and geometry. It is apparent that there is no standard level of attainment. This lack of uniformity may be desirable, but the fact that there is no generally agreed minimum level of attainment makes it difficult to set down an agreed plan for the early work of the secondary schools. Later in this section there is given in detail a minimum which is desirable from the secondary school point of view. It suggests no change in the content of what is usually attempted in the primary school, but it does suggest that more consideration should, in general, be given to the mathematical background of the pupil.

It is comparatively easy to delimit the content of the mathematical syllabus of the primary school; it is more difficult to ensure that the syllabus is given a wide interpretation in terms of class work and approach to the subject. It is not the purpose of the memorandum to discuss the teaching of mathematics in primary schools, but if it be accepted that one of the aims of primary school teaching is to prepare pupils for secondary school work, then it is not unreasonable for us to state our opinion on whether the primary schools achieve that aim. Moreover, it is hoped that those responsible for teaching, examining and selecting entrants to secondary schools will be interested to read the comments of secondary school teachers on their work. Too frequently, individual pupils who have done well in primary schools fail to maintain their relatively high class position in the secondary school. Allowing for the differences in the rates of development of individuals, and the effects of possible adverse external circumstances, the failure of some of these pupils is undoubtedly due to unsuitable teaching methods. Because this retrogression of the pupil occurs in the secondary school, it is too readily assumed that the cause lies in faults in the secondary school teaching. Sometimes it does, but it is also possible that the roots of the trouble lie deeper and lie back in the days of the primary school. In criticizing the standard of some secondary school entrants, we do appreciate the difficulties under which the primary school teacher often works, with large classes of pupils, wide ranges of intellectual ability, some-

times under the close and unimaginative control of local educational administrations, in an atmosphere of competition by written examinations that often leads to teaching for stereotyped questions, and the judging of the teacher's effectiveness by those examination results.

The requirements of written examinations at the age of eleven, where these still exist, may lead to serious difficulties in the secondary school. The simplest operations of arithmetic are particularly suitable, from the administrator's point of view, for providing a ready means of setting papers and obtaining lists of marks. From the teaching point of view examinations which can be passed by pupils possessing facility in simple arithmetic operations, and nothing else, can easily be prepared for by applying sufficient drill. Little imagination on the part of the teacher is required. While it is not disputed that quickness and accuracy in arithmetical operations is a desirable aim at this stage, it is disputed that it is the only aim. Quickness and accuracy should not be the only mathematical criteria which determine whether a child of 11 will profit by a grammar school education or not. Many authorities already appreciate this point, but there are still many who are ready to sacrifice educational principles to administrative convenience.

Apart from the learning of simple arithmetical processes, the primary school pupil has to be taught to appreciate the significance of the quantities, measures and relations with which he deals. This increasing of the pupils' awareness of numbers, quantity and spatial relations is a 'process of slow growth' which continues past the critical age of 11 into the secondary school age, and, we hope, beyond that. At the school age, however, it is of vital importance to the pupil that his ability to 'do sums' should not run ahead of his appreciation of the significance of the operations he is performing. If this does happen, there usually comes disillusionment when the pupil finds that mathematics is but an empty shell with no meaning for him, and with no relation to the external world of which he is beginning to be aware. Pupils, who in the primary school were successful and contented so long as they could 'get the right answer' to every question by mere mechanical manipulation, may find themselves outclassed in the early years of the secondary school and therefore decide that they 'never could do mathematics'. One of the most important of the 'processes of slow growth' is the ability to abstract the mathematical content from verbal statements of

problems, to realize what operations are required to produce the solution, and, conversely, to be able to give verbal interpretation of numerical relations and results. Mathematics is a dead subject to pupils without this ability.

In attempting to draw up a minimum syllabus, we must consider, therefore, not only the content of the syllabus; the teaching approach to that syllabus is more important than the content. Those responsible for examining secondary school entrants in mathematics must encourage the primary school teacher to give this broader interpretation to his syllabus by devising tests in which its importance is recognized.

For the purposes of this report we assume that children will have covered most of the ground indicated below. The minimum content recommended is given together with notes to indicate the level of attainment required:

(*a*) The four rules applied to whole numbers, including short and long multiplication and division.

This requires that knowledge of the multiplication tables up to 12×12 should be:

(i) direct and inverse, i.e. that $7 \times 11 = 77$ and also that the factors of 77 are 7 and 11;

(ii) applied automatically, i.e. that it should be possible to write or say $6 \times 7 = 42$ without beginning $6 \times 1 = 6$, $6 \times 2 = 12$, etc., or by counting on fingers, etc.

(*b*) The *common* tables of weights and measures:

Money:	Pounds, shillings, pence; coinage.
Length:	Miles, furlongs, chains, yards, feet, inches.
Weight:	Tons, hundredweights, pounds, ounces.
Time:	Years, months, weeks, days, hours, minutes, seconds; the clock; the calendar.
Capacity:	Gallons, quarts, pints.
Area:	Square inch, square foot, square yard, applied to rectangles.

A knowledge of these measures implies not only their relations one to another, but an ability to estimate them. The pupil should be able to show that he appreciates their magnitude in simple practical situations; e.g. an ability to estimate such things as the lengths of lines drawn on paper or on the blackboard, distances in the playground or from the school to the station, the library, etc., the dimensions of the classroom, the capacity of cans, bottles, and buckets,

the weights of articles up to about 28 lb. by handling them, a hundredweight of coal or coke by eye, and the time required to walk or cycle a given distance.

Too frequently the need for appreciation of the magnitudes of these measures is ignored in the classroom; the pupil acquires a knowledge of them only in activities out of school. An early acquaintance with these measures brings interest and concreteness to calculations that otherwise remain abstract and remote from the pupils' experience.

In calculations involving these measures, it is suggested that not more than three consecutive measures should be required at once, e.g. yards, feet, inches may be combined, but questions involving the whole gamut of measures of length from miles to inches should be avoided. They are impractical and meaningless. The calculations would include addition, subtraction, reduction and short multiplication and division.

(*c*) Simple fractions, limited to fractions with denominators of 2, 3, 4, 5, 6, 8, 10, 12 and 20 which can arise out of problems on the tables of measures. The operations should include simple addition, subtraction, multiplication and division, but not combinations of these. Formal methods of determining L.C.M. and H.C.F. should not be required.

The syllabus outlined above is a suggested *minimum*. What is the primary teacher to do if his pupils have attained this minimum and yet have to spend another year, perhaps, in the primary school? He must go forward. The range of problems can be widened and new topics may be introduced so long as they are treated practically and correlated as closely as possible with the pupils' experience. There is, however, one further point to realize in linking the work of the primary to that of the secondary school. When the new entrant arrives at his secondary school to begin a new phase of his life, he does expect to hear of new ideas and see new things. He arrives with an air of expectancy, elated probably at his success over some of his former fellows and in the state of being 'ready for anything'. Here is a teaching opportunity. If, however, he finds in the mathematics class the same old topics, decimals perhaps, that belonged to his primary school days, disillusionment sets in. It may have been found that his and his fellows' ability to handle decimals is very uncertain, but, even so, it is difficult to convince or interest him—the 'gilt is off the gingerbread'. The great opportunity has been missed, and it may

not be entirely the fault of the secondary school teacher. It is therefore suggested that any work additional to the minimum syllabus should be undertaken only when it is certain that this essential groundwork is thoroughly assimilated. Any new topics that are introduced should also be given equally thorough treatment. Above all else, the primary teacher should avoid giving his pupil a merely superficial acquaintance with topics that are better left to the first year of the secondary school course. Such topics include the beginnings of albegra and geometry. Additional work might include: further practice in drawing; the extension of fractions to hundredths as an introduction to the decimal notation; the addition and subtraction of decimals involving not more than two figures after the decimal point.

III

AIMS AND SYLLABUSES FOR THE FIRST FIVE YEARS IN A SECONDARY GRAMMAR SCHOOL

1. Why Secondary Grammar School only?

A discussion of the early years of secondary education is, at the present time, both important and difficult. It is difficult because the whole conception of secondary education is changing under the administrative interpretations of the Education Act of 1944. This conception and its administrative interpretation in different types of secondary education make it undesirable to discuss the work of the earliest years in a secondary *grammar* school without reference to other forms of secondary education. On the other hand this committee has little direct experience of these forms. It would indeed find the collection of information about them laborious and perplexing. It realizes that there are at least three substantial problems requiring attention: those relating to selection, to a common curriculum in mathematics, and to transfers from one type of education to another. If it is unable at present to make a substantial contribution towards the solution of these problems, it can at least make some comment on the early years in the school it knows best, the secondary grammar school, bearing in mind these wider aspects.

2. Fundamental Aims of the First Two Years

Amongst fundamental aims in the work of these two years we would propose:

(*a*) The inculcation of a feeling, almost a love, for mathematics. We elaborate this aim on a later page.

When a teacher meets a new group of children for the first time his primary aim should be to awaken within them some of the affection which presumably he has for the subject he has chosen to teach.

(*b*) The maintenance and increase of confidence, even if the pace be slow.

(*c*) Accuracy over a limited range of operations.

(*d*) The association of mathematics with the experience of the pupil.

To attain these aims we suggest the planning of the work on a psychological rather than a logical basis, the grading of exercises so that one difficulty is encountered and mastered at a time, and a general development which is concentric or spiral rather than a linear sequence of unrelated topics.

3. Some General Comments on Method

All children arrive at a secondary grammar school with many years' experience of one aspect of mathematics, namely arithmetic, and for most children this is the only aspect. Often their arithmetic is just written 'sums' on the four rules with extensions to fractions, decimals, areas, and mental arithmetic associated with everyday transactions. If children are 'good' at this kind of work then they are clearly ready for something better, in the mathematical sense. But, as usual throughout mathematical education, the good children do not present the difficulty. The difficulty centres round those children whose view of mathematics is likely to be determined by earlier inability to achieve success in arithmetic.

We wish to stress as strongly as we can that in dealing with this problem the secondary grammar school teacher must attend to two points.

(*a*) In dealing with any topic of arithmetic he needs to consider the individual weaknesses of his pupils, and to attempt diagnostic and remedial treatment. For instance many errors occur through weaknesses in subtraction, or where a zero digit occurs, in multiplication, or in a particular table. A whole process, e.g. long division, may be so partially comprehended that some careful reteaching is necessary. Weakness in fundamental number processes is a constant handicap in later mathematical work.

(*b*) He must try to discover, by suitable teaching and by comparing his results with the previous records of his pupils, their true intellectual capacity, which may well have been masked by performance in an entrance examination.

It is, of course, the tragedy of much mathematical teaching in junior, or primary, schools that it is still directed towards unnecessary

facility in written calculations of a kind that have no real purpose, e.g. long multiplication and division of money—apart from the assumed training in 'mental discipline', which seems in the light of all modern educational research to be comparatively valueless. The further assumption that secondary grammar schools require this skill in written arithmetic on entry is not justified. It is far more desirable for the pupil to have a really firm grasp of essential number relations, and of tables—for addition and subtraction as well as for multiplication—together with a direct experience of such concepts and units as are met with in daily living, and a curiosity and interest in the world of number and space. We agree with the *Report on Teaching in Primary Schools* of the A.T.C.D.E. (Longmans) that handwork leading to spatial knowledge, i.e. geometry, should replace some of the written arithmetic now studied in the primary school.

It is a general practice in secondary schools to begin the course at 11+in one or other of two ways:

(*a*) By a revision and consolidation of the previous work, i.e. of arithmetic.

(*b*) By one or two lessons per week in each of arithmetic, algebra and geometry.

We think both these practices need reconsideration.

Concerning (*a*), unless the work is handled skilfully it easily degenerates into drill on routine processes. If treated diagnostically and remedially so that children recognize their weaknesses and can achieve improvement by their own guided efforts this work can be of great benefit. Nevertheless it is hardly an inspiring opening to a new stage in a child's life.

The view is sometimes expressed that this beginning is necessary 'to make sure of the groundwork'. This ignores the need to provide immediate incentives for learning and overlooks the gradual process whereby items of knowledge link themselves together: for instance, the most effective way to recall forgotten elements in a multiplication table is not merely to grind away at relearning a table but to provide situations in which a knowledge of the table is needed for successful achievement.

As to (*b*) there is no justification for dividing mathematics at this stage into the subject classifications appropriate for adults, and beginning two new and very different aspects of the subject. It must be remembered that new entrants to a secondary grammar school are

subjected to many demands of specialist teachers—so that they may begin within a few days a new language, a science and two new branches of mathematics. Furthermore, arithmetic, algebra and geometry appear as very different aspects of mathematical thinking to young children. Arithmetic is most natural, geometry (regarded of course as properties of the space in which we move) certainly most familiar, and algebra comparatively abstract and artificial.

There would therefore seem to be good reasons for introducing the pupil to only one aspect of mathematics during his first weeks in a secondary grammar school. It is obviously desirable to associate some new work with a new phase in a child's education. It would appear wise therefore to make geometry the main subject in mathematics, giving some periods to revision, consolidation and extension of arithmetic. Geometry properly handled is stimulating and demands very little previous knowledge, thus enabling children to start level. It also reveals fascinating and permanent aspects of mathematics.

There is one aspect of mathematics which may too easily be neglected at all levels: namely, the solving of problems, using the word in its widest sense, as opposed to a piece of calculation. Problems require the translation of language into symbolism, the selection of operations and the choice of logical as against illogical procedures, and are of supreme value in genuine mathematical education. The material comprises real-life situations, number games and puzzles, 'intelligence test' exercises, newspaper diversions and the immortal works of Henry E. Dudeney and others.

Since the trend at all levels is away from 'formal' exercises on stereotyped lines there should be more time for work of this kind. The kind of thinking required is usually mathematical since it is concerned with relations between elements, but does not require specialized mathematical techniques. And the expression of the solution by a concise argument is an obvious training in purposeful use of language and symbolism. The treatment of problems is not easy. It should therefore be attempted in the earliest stages with the simplest instances, and be extended on suitable occasions throughout the course.

Another possibility which arises when we think of *mathematics* rather than of the separate categories of arithmetic, algebra, and geometry, is the devising of a course based on topics or projects of

elementary mathematics. Some teachers might find it feasible to set aside a regular part of their weekly time to developing the mathematics arising from travel, or lighting and heating, or map-making, or associated with the factory or the farm, the school garden, or the construction of model aircraft or ships. Children are usually very anxious to make a good showing in a new school with new teachers. It is worth devoting to the work of this stage some of the thought and energy that tend to be concentrated on the examination preparation at the later stages of the secondary grammar school course.

The syllabus which follows this introduction is a *teaching* syllabus, not an examination syllabus. The pressure of conditions imposed on the school has led teachers to assume that an examination syllabus must necessarily become the syllabus to be followed in the classroom. Educationally this is short-sighted. We therefore begin by insisting on the distinction between the two. As a consequence it follows that the (teaching) syllabus to be described should be modified, enlarged, curtailed or otherwise adapted first to local conditions —e.g. social or occupational interests outside the school—secondly to the temperament, interest and capacity of the teacher, and thirdly (or should it be firstly?) to the aptitudes of the children.

4. A Teaching Syllabus for the First Two Years

The Subject-Matter of the Syllabus. It is during the first two years that the attitude of the pupil towards the subject is usually determined. It is therefore important that he should be given as broad a picture as possible, the depth of detail varying with the particular class. A list of topics is given below; it is not suggested that B or C streams will complete this list. We take it as obvious that a teaching syllabus is adapted in particular schools to the pupils who are to be taught.

Although it is recognized that the subject is Mathematics and will be taught as such, the list is subdivided under the conventional headings of Arithmetic, Algebra, Geometry, Trigonometry. The exact sequence will be determined by conditions in a particular school. Most teachers now prefer to give a week or more to develop one topic fully, rather than to allocate one or two periods each week regularly to each branch; but difficulties of time-tables and staffing,

particularly in cases where the mathematics time for one form is divided between two teachers, will often make the traditional division unavoidable.

Even in these cases the links between the branches will be forged at every stage. In dealing with the Theorem of Pythagoras, for example, the teacher wll not fail to deal with arithmetical, algebraical, and trigonometrical aspects of the result, even though the particular period is described in the timc-table as Geometry.

Arithmetic

1. Simple properties of number. Tests of divisibility; factors; unitary method and its applications; averages; squares (by compiling a table of squares to 20^2); recognition of common factors and multiples of groups of easy numbers (such as 5, 10, 15, or 14, 21, 28), but not a formal treatment of H.C.F. or L.C.M. Interesting number relations such as that the sum of a series of odd numbers from 1 onwards is always a perfect square.

2. Fractions. Numbers used will be simple and will not call for heavy manipulation (for example, denominators greater than 30 will be rare) or for formal determination of L.C.M. Questions involving difficulties in deciding the order of operations and the use of 'of' as a major bond will be omitted. Problems will, as far as possible, be real-life problems, within the range of experience of the pupil. Easy fractions of £1, a ton, etc., will give rise to simplified calculations of the type known as 'practice', but a formal treatment of 'compound practice' will be avoided.

3. Decimals. Work beyond the third or fourth place will not be necessary; the idea of degree of accuracy will be introduced, but not formal contracted multiplication or division. Simple fractions expressed as decimals and vice versa. Decimalization of parts of £1 (to 1*s.*, 6*d.*, and 3*d.*) from first principles, but not three-place decimalization by rule. The metric system, limited to commonly used units (kilometre, metre, centimetre, and millimetre; kilogram and gram; hectare, square metre and square centimetre; and litre and cubic centimetre); approximate equivalents in everyday use, such as 1 kilometre = $\frac{5}{8}$ mile.

4. Ratio and Proportion. Comparison by ratio; its application to proportion (the 'fractional method'); conversion and ready reckoner graphs; growth by ratio; similar figures (maps and scale drawings).

5. Percentages. Their use as a basis for comparison; percentage increases or decreases; finding a percentage of a given quantity; simple equivalent fractions, decimals, etc., (e.g. $2\frac{1}{2}\ \% = 1/40 = 6d.$ in the £). Simple equivalents within the range of experience of the pupil but little conventional treatment of profit and loss and interest.

6. Area of triangle, parallelogram, trapezium and circle. Simple volumes (cuboid and pyramid, but not, except with good forms, cylinder, cone and sphere). Relevant applications, using real data at all stages, including exercises on the determination of areas by the counting of squares on squared paper.

Algebra

1. Generalized Arithmetic. The 'shorthand of algebra'.

2. The formula. This will provide an introduction to algebraic manipulation, the four rules, brackets (order of operations) and easy fractions.

3. The equation as a result of substitution in a formula; one and two unknowns; fractional forms, limited to those with easy numerical denominators. The use of equations in solving easy problems and puzzles. (Many teachers prefer to reverse this sequence, leading from the problem to the formula, rather than from the formula to the problem.)

4. Directed numbers.

5. Multiplication of single terms and binomials at sight. Factors —monomial, simple trinomials with small coefficients, difference of two squares; the use of algebraic factors to simplify arithmetical calculations, as in finding areas.

Geometry

1. Geometry arising from space measurement and drawing (e.g. maps, plans, etc.). Measurement to the nearest inch, centimetre, etc., provides a basis for future work on degree of accuracy.

2. Shape, symmetry. Simple constructions. Elementary work on solids. Construction of regular polyhedra from their nets.

3. Angles; making and using a protractor; horizontal and vertical.

4. Angles at a point; vertically opposite angles; parallels.

5. Angle-sum properties; their use in deductive arguments.

6. Pattern drawing, including those involving the regular hexagon and octagon, and elementary chord and tangent properties of a circle.

7. Construction of triangles from given data, leading to the tests for congruence; the use of the triangle and quadrilateral in real life. Simple proofs involving congruent triangles; these will provide an introduction to style, which should not, at this stage, be too formal.

8. Similar figures; the shadow stick; enlarging and reducing; this should be linked with arithmetical work on ratio and proportion.

9. Use of theodolite and clinometer; elementary surveying (traverse and triangulation).

10. Elementary idea of loci, not limited to straight lines and circles, or to two dimensions.

11. Rhombus, parallelogram, kite.

12. Freehand sketches of simple solids; representation of three dimensions in two (plan and elevation).

13. Pythagoras' Theorem and its use.

(N.B. The notions of geometrical reasoning are developed throughout the course.)

Trigonometry

The introduction of one trigonometrical ratio by means of a practical problem. The conventional approach is through problems of height or distance to the tangent, but some teachers prefer to begin with the sine—as it offers scope, through the sine formula, for further work before introducing a second ratio. There is something to be said for both approaches.

Graphical Work

Although it is sometimes placed under the heading of algebra, graphical work is so important that it deserves a place in its own right. It should deal with many aspects of the pictorial representation of number—the column, sector, pictograph, isotype—before the chart (i.e. the statistical graph), and should include a treatment of co-ordinates, but not, at this stage, the algebraic graph, e.g. of $y = x(x+2)$.

5. Mathematics as One Subject—an Integrated Syllabus

There is a general agreement that it is the task of the teacher of mathematics to show the unity of all he teaches, as there is a danger, especially in the early stages, that the pupil may fail to see the interrelation of the various branches. Although difficulties of time-table, staffing, and text-books, often necessitate a strict allocation of teaching time under the various sub-headings, no good teacher fails to forge strong links at every opportunity; the fact that the particular period may be officially described as geometry, for example, does not prevent him from diverging to the fields of arithmetic, algebra, or trigonometry as the occasion warrants.

Some teachers feel, however, that this is not enough. The demand has grown recently for an integrated syllabus, in which the paths are signposted so as to traverse the whole field in one continuous route. Such a syllabus is given below. It is not suggested that this is the only such sequence, or even the best. At this stage it is no more than experimental, but may be of assistance to teachers whose thoughts lie in this direction.

An Integrated Syllabus

1. Geometry arising from space measurement, such as drawing plans, measuring maps, etc. In the course of this measurement the pupil will meet with tenths (decimals) and the centimetre and metre. He will also meet with angles and will learn about angles at a point and parallels. Calculations of cost, etc., arising from these measurements will provide the opportunity for consolidating the work of the primary school, for dealing with multiplication and division of compound quantities and the unitary method. The use of symbols in the course of this work will lead to the formula.

2. The formula; the use of symbols; a new language—translation into algebra. This will provide an introduction to algebraic manipulation, the four rules and brackets. The use of the formula will lead to simple equations and problems. At this stage the transformation from $a = b/c$ to $ac = b$ and so to $b = ac$ is of first importance, as it is the key to ratio work and the use of a ratio as a multiplier.

3. The use of index notation in the formula will lead to factors; tests of divisibility; common factors and multiples; L.C.M. of easy groups of numbers at sight (a formal treatment is not necessary). Fractions, using simple algebraic forms as summaries.

4. Decimals; these will arise naturally from fractions, as an alternative notation. Simple fractions expressed as decimals and vice versa. Decimalization of parts of £1 etc., from first principles, but not three-place decimalization by rule. The pupil will already have met decimal fractions in the course of measurement, and the opportunity will arise of introducing first ideas on degree of accuracy.

5. The metric system. The pupil will already have met the centimetre and possibly the gram. He should now meet the other metric units in common use and learn some approximate equivalents. Where possible he should meet these units in the course of actual measurement, as, for example, his milk might be measured out in litres instead of pints, or his weight be measured in kilograms instead of stones and pounds.

6. Problems involving consideration of area may be developed into a study of shape, symmetry and similarity. The triangle, polygon; regular polygons, angle-sum and exterior angle properties of the triangle; isosceles and equilateral triangles; ruler and compass constructions. Construction of triangles, leading to the tests for congruence, and notions of deductive proof. Patterns, especially those which give rise to new curves, such as the parabola, ellipse and hyperbola.

7. These is ample scope in the above section to lead to a treatment of the pictorial representation of number and to the chart (the graph involving the idea of co-ordinates, but not yet of the algebraical function). The pupil should learn how to interpret a graph, as well as how to draw one, sufficient time being allowed for the purpose. Large-scale charts should be available for this purpose, to supplement those drawn by the pupil. First ideas on loci.

8. Solids from their nets. Polyhedra (at least the regular forms). A few simple line and plane properties, from observation of walls, books, etc. Plan and elevation and other methods of representing three dimensions in two. Sections and sketches of simple solids; an examination (but not, at this stage, very deep) of crystal shapes.

9. Scale-drawing arising out of the above will lead to the development of the fundamental idea of ratio. This is a vast topic and will include: equal ratios or proportion; similar figures; scale drawing and bearings; equal intercept properties and the construction of scales; the functional relation $y = kx$; the ratio as multiplier, leading to the trigonometrical ratios; the use of two-figure tables (which can be made up in class, by division of labour, in two or three lessons); percentages. It is not suggested that all this ground should be covered at this stage; some will, in fact, have been dealt with earlier; but the idea of ratio, often treated as little more than an alternative to the unitary method in arithmetic, should be developed as fully as time will allow.

10. Some of the problems arising in the course of the work above will now lead to a fuller treatment of the triangle and the quadrilateral, including the parallelogram, trapezium and kite.

11. Area of triangle, parallelogram, trapezium, quadrilateral, circle (the treatment of the ratio of the radius to the circumference although included under § 9, may be taken at this stage). Volumes of simple solids. The use of the Theorem of Pythagoras; no formal treatment of square root is needed at this stage.

6. A Teaching Syllabus for the Third Year

Arithmetic	Algebra	Geometry	Trigonometry	Graphical Work
Money in everyday life, e.g. Post Office Savings Bank, household income and expenditure, gas and electricity accounts, bills in general, school camp expenditure, ready reckoners, wages, simple interest and extension of percentage work generally	Transformation of a formula Use of suffixes Determination of constants in formulae e.g. in $y = ax + b$ Manipulation of simple fractions involving the use of factors Revision of $(a \pm b)^2$ and possibly extension to cover $(a+b+c)^2$ Geometrical illustrations of simple algebraical identities	Mid-point and intercept theorems, simple scale problems and the diagonal scale Revision and extension of elementary work on similarity of triangles Proof of the Theorem of Pythagoras by e.g. (i) similar triangles, or (ii) trigonometry Converse of Pythagoras' Theorem Angles made by lines and planes with the horizontal The cuboid and its diagonals and the square pyramid, with applications of Pythagoras' Theorem and trigonometry to these figures Plane sections of a sphere, great and small circles, latitude and longitude Plans and elevations of simple solids, e.g. the square pyramid Chord, angle and tangent properties of a circle, including the construction and symmetrical properties of tangents‡ Development of work on loci. Simple envelopes Constructions involving the intersection of loci	Sines, cosines and tangents as multipliers Northings, eastings, etc., and other simple trigonometrical applications Use of 3- or 4-figure tables, results to be given to a reasonable degree of accuracy Contours and gradient (= V.I./H.E.) The area of a triangle ($= \frac{1}{2}bc \sin A$) and of a parallelogram ($= bc \sin A$). (A acute in both cases) Simple relationships between trigonometrical ratios, such as § $\sin^2\theta + \cos^2\theta = 1$ and $\frac{\sin\theta}{\cos\theta} = \tan\theta$	Graphs of x^2 and x^3 and their use to find squares, cubes, and roots Introduction to graphical form through the graphs of various functions, including such forms as $12/x$, $\sin x$, $x(x^2 - 4)$, mx^2 (for different values of m), and $(x+a)^2$ (for different values of a)¶ Uniform speed graphs Speed as a gradient Graphs of the ready reckoner type and conversion graphs
Use of tables of squares and square roots Mensuration of the right prism and cylinder Use of positive integral indices, as in $9{\cdot}3 \times 10^7$ Use of logarithm tables for numbers > 1*				

* With first ideas on tabulation, at discretion.

† Too much time should not be spent on the last two sections and possibly exercises on completing the square should be limited to equations in which the coefficient of x^2 is unity.

‡ Defining a tangent as the perpendicular to a radius through its extremity.

§ To show the interdependence of the ratios and not as material for identities, or for the calculation of one ratio from another. The latter should be dealt with by using right-angled triangles.

¶ This will consist in the main of informal blackboard work.

7. Analysis of the Third-Year Syllabus

In view of the demand for an integrated syllabus, an analysis of the teaching syllabus for the third year was made in order to discover possible alternative lines of development and unifying general principles. The results are given below.

General Principles. During the third year the concepts of area and volume are further developed and the concepts of similarity and symmetry of form are introduced algebraically and geometrically.

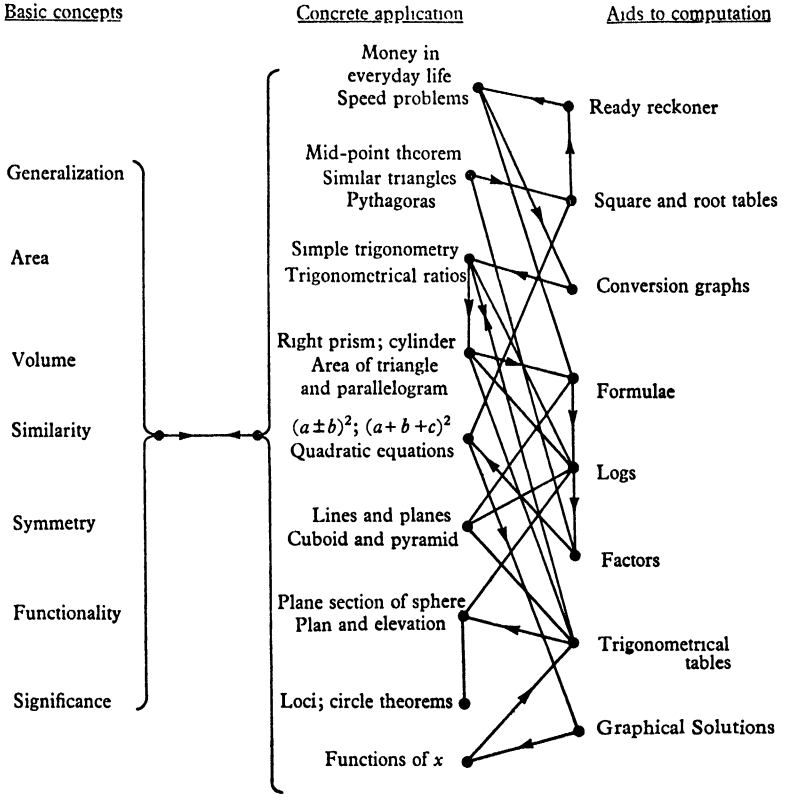


Fig. 1.

In the plotting of non-linear graphs the first steps leading to the idea of functionality are taken. Concurrently, the use of mathematical tables, including logarithms, greatly increases the scope of the pupil's arithmetical operations, allowing him to explore the new concepts through the investigation of concrete numerical examples.

The lines in Fig. 1 join related topics to show those which can be linked together. In order to simplify the chart, separate items are combined in small groups; some possible lines are omitted.

The topics in the centre colume are arranged so as to descend in a possible order of treatment, but to cover the syllabus as an 'integrated' whole each line should be traversed at least once in the year. The column of 'Basic concepts' is not entirely satisfactory, but it does not seem worth while to carry the analysis further.

8. A Teaching Syllabus for the Fourth Year

Arithmetic	Algebra	Geometry	Trigonometry	Graphical Work
Use of negative integral indices, as in $9{\cdot}3 \times 10^{-5}$	Simple treatment of fractional and negative indices	Further properties of the circle, including the alternate segment theorem, its converse, and the converses of the angle theorems	Projections and resolved parts	Graphical approach to direct and inverse variation; $y = mx$ leading to $y = mx+c$
Logarithms for all positive numbers; the slide rule	Simple manipulation of formulae	Circle constructions, to include circles in contact, a segment to contain a given angle, and the circumscribed, inscribed, and escribed circles of a triangle	Composition of two perpendicular vectors	Solution of quadratic equations graphically
Tables of reciprocals	Solutions of quadratic equations by formula	The ratio theorem and its converse	Extensions of definitions to cover obtuse angles; graphs of the functions	Graphs with asymptotes, e.g. $y = \frac{a}{x}$, $y = (ax+b)/(cx+d)$
The '*s*' formula for the area of a triangle	Easy fractional equations, including those reducible to quadratics which can be solved by factors	Ratio property of angle bisector of a triangle and its converse	The area of a triangle $= \frac{1}{2}bc \sin A$ (A obtuse)	Gradients of graphs by drawing, with applications to easy linear kinematics and maxima and minima
The mensuration of the sphere, cone and pyramid; density and specific gravity	Simultaneous equations in two variables, one linear and one of the second degree, with graphical illustrations	The similarity theorems, including the relation between the areas of similar triangles	Sine and cosine rules	
The ratio of the surface areas and volumes of similar figures	The factor theorem	The rectangle properties of a circle and their converses	Logarithms of trigonometrical functions	
Local and national finance, including compound interest, applications of ratios and percentages (numbers should be simpler than those used in current professional practice)		The distance of the visible horizon		
		Further plans and elevations		

9. A Teaching Syllabus for the Fifth Year

The fifth year is to be regarded as a year of consolidation of all previous work, a year in which there should be some real integration of the subject. With these objects in view, the amount of new subject matter suggested is slight.

During this final stage of the main school course, which will complete the mathematical education of most of our pupils, the tendency to spend an undue amount of time on papers set in past examinations should be resisted and every effort made to stimulate interest. To this end every opportunity should be taken of linking revision with practical applications of the subject, e.g. in engineering, physics, statistics, navigation, astronomy, mechanics, surveying, etc., thus ensuring that the pupils appreciate, at least in some measure, that aspect of mathematics which makes contact with the world of reality.

A concentric development of the subject makes it necessary that, sooner or later, threads associated with various topics should be gathered together. Important examples are degree of accuracy, which is discussed in a separate chapter, and locus work.

Before the introduction of alternative syllabuses, considerable time was often spent during the fifth year on intensive revision of geometry. It is now generally agreed that much of this time could be better spent, but time should still be found for analysing the logical structure of the attenuated sequence of theorems of which specific knowledge is now demanded, if only to demonstrate the inward-facing aspect of mathematics, in which mathematical truths are related to one another.

Arithmetic	Algebra	Geometry	Trigonometry	Geographical Work	Calculus†
Degree of accuracy* Miscellaneous problems illustrating the principles developed during the previous four years	Fractional and negative indices; theory of logarithms Simple work on surds, e.g. $\sqrt{48} = 4\sqrt{3}$, and easy rationalization Further work on variation, including joint variation Factors of $a^3 \pm b^3$ Values of $(a+b)^3, (a+b)^4$, etc., as a preliminary to the binomial theorem	Centroid and orthocentre Construction of a square equal in area to a rectangle	Miscellaneous problems. Practical applications of trigonometry, such as surveying and navigation	Graphs of $y = 10^x$ and $y = \log_{10} x$ Interpretation of the area under a (v, t) graph Approximate evaluation of the area under a curve	Interpretation of gradients of graphs and their determination by drawing and calculation Differential coefficient of x^n where n is an integer Application to rates of increase, easy linear kinematics, maxima and minima Applications to the area under a graph, volumes of revolution, and linear kinematics

* This topic is considered in detail on p. 81.

† The development of this branch of the subject at fifth-form level is discussed on p. 115.

10. Analysis of the Fourth- and Fifth-Year Syllabuses

An analysis of the fourth- and fifth-year syllabuses corresponding to that given for the third year follows below.

1. (General Principles—Fourth Year.) During the fourth year the new concepts of variation, limits, and rate of change are introduced

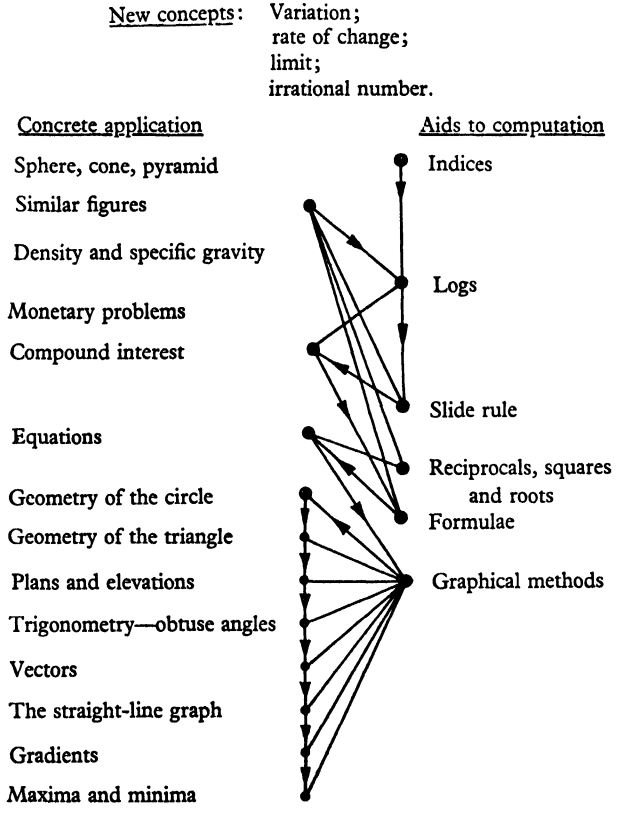


FIG. 2.

through graphs. The concept of number is extended by a first treatment of irrationals and vectors. There is a rapid extension of computational techniques so that by the end of the year the pupil is

equipped with all the tools necessary for elementary computation and should have achieved reasonable facility in handling and selecting appropriate tools for a piece of mathematical work.

2. (General Principles—Fifth Year.) The main aim of the fifth year is to review, consolidate, and round off the work already done. Emphasis is given to the clarification of mathematical concepts already introduced, especially to the concept of logical structure as exemplified by elementary geometry. The few new topics are introduced as extensions of earlier work, the calculus as early as possible so that the strangeness of the new symbols is forgotten by the end of the year (the last year of mathematics for many). A wide range of practical applications is used to demonstrate the power of mathematical methods; computation becomes incidental. To give the subject cohesion wide topics extending over several consecutive periods are dealt with and every possible link with other topics is emphasized.

3. Fig. 2, for the fourth year, is constructed similarly to that for the third year.

There is no Figure for the fifth year.

IV

MATHEMATICS IN THE SCHOOL AND CLASSROOM

1. Inculcating a Love for Mathematics

Enthusiasm, guided and controlled by a sympathetic appreciation of the learner's tastes and capacity, is probably the mainspring of the most successful teaching. It is certainly essential in any teaching likely to inculcate a love for mathematics. That such a feeling for mathematics may be aroused by skilful teaching, at any stage in school or university education, is a commonplace amongst students and successful teachers. The root of the matter seems to lie in a personal reinterpretation of some aspect of mathematical knowledge which strikes the interest or imagination of the pupil. Enthusiasm for mathematics on the part of the teacher, however, is not enough. It must be manifest enthusiasm for the *kind of mathematics* to which the pupil can respond. Only experience can decide what material and what response are likely at any given stage.

In this matter all teachers need to reflect on the immense variety of material available for mathematical education, on their own and on their pupils' emotional response to mathematics, and on the need to draw on the widest experience of successful teachers. In our teaching we all seek keys to our various locks: to every lock the experience of successful teachers suggests there is some reasonably adequate key. But these keys are not always those most fashionable at that moment. It may be wise to reconsider keys once fashionable but now discarded, or keys approved in other countries. Mathematics is a universal subject; and we should attend to experience of its teaching outside those chains of custom and convention in which secondary education is confined in this country. The young teacher should not wait passively for time to bring him experience. If he wishes to inculcate a love for mathematics he must attend to three things:

(*a*) The experience of others—by continual study of books, reports, discussions. He should not confine himself to this country,

nor to schools of his own type. Some of the most stimulating mathematical material is found in French, German, or American literature, and most valuable experience is available from junior, from technical, and from evening schools.

(*b*) The intelligent and sensitive teaching of 'the daily round, the common task'. Unless he can cope with this he is bound to feel uncertain and harassed: but for this work he must continually seek inspiration, material and spiritual, from the wider cultural, historical and recreational aspects of his subject.

(*c*) The emotional response of his pupils. To win a fruitful response from his pupils may call for a real adjustment of attitude on the part of the teacher. He has to persuade a group of children to get the best out of themselves. We say 'persuade' deliberately; though more vigorous methods are not ruled out where they are necessary, it is unquestionably true that many children are easily disturbed by their early experience of mathematics and that loss of confidence, bewilderment, or repeated failure gives rise to a distaste for the subject. The teacher's approach needs therefore to be sympathetic and encouraging. He needs to arouse interest, to see that there is genuine learning and not mere memorization by rote, and to adapt his material and his methods to meet individual difficulties. He must remember that some children who come to secondary schools have already decided that 'they are no good at maths'; he must ensure that, even if his pupils are not skilful at all branches of his subject, they shall enjoy mathematics. Some children may be persuaded' to like mathematics.

There need be no conflict between disciplined effort and mere pleasure. Children gain great emotional satisfaction through genuine learning, through gaining mastery by their own efforts, through setting themselves goals within their capacity, and in watching their progress towards those goals. Much pleasure comes through successful achievement. Winning the mastery of the simplest processes will provide the sense of achievement provided their relevance is clear. Further enjoyment may come through the aesthetic appreciation of algebra and geometry, even if that appreciation is not matched by executive ability. And finally it is important to remember that constructive and sympathetic school reports on his pupils' progress in mathematics may also help to evoke the fruitful emotional response which the teacher must win from them.

The teaching of mathematics may be expressed very simply as a compromise between the logical and the psychological. Since 1900 the emphasis in creative mathematics has been emphatically logical; in general education it has been excessively psychological. The task imposed on the teacher of mathematics, particularly at the secondary school stage, has become correspondingly more difficult. In the present state of knowledge he can only be advised to study how his pupils learn, to find what interests they have at different stages, and to attempt to enlist these interests in their mathematical education. For instance he must consider their social and economic backgrounds and possible future employment. Ideally his aim should be the *self-education* of children through mathematics. This could occur most effectively if mathematics were studied by groups of pupils in a room which was a library at one end and a physical laboratory at the other, with opportunities for making things, measuring and experimenting. In some boarding schools such an ideal mathematical *self-education* may be possible; if teachers in other schools have some clear notion of this ideal they may find its achievement easier than some would suppose.

Some comments should be made about the particular difficulties of the specialist teacher of mathematics. The advance of knowledge has curious results, not always immediately apparent. In the period between the two world wars there was a great influx into secondary schools of university graduates highly qualified in mathematics. The period from 1950 onwards is likely to see the disappearance from secondary education of the most highly qualified mathematicians graduating from our universities, simply because the advance of mathematical knowledge and technique has created alternative occupations for them. Indeed, the advance of knowledge, with the corresponding advance in the standard of university work for those specializing in mathematics, brings with it the paradox that those now qualifying with the highest qualifications in mathematics may thereby be gravely handicapped for teaching mathematics to children —and especially to those children who, lacking mathematical genius, are most in need of sympathetic teaching. Somewhat exceptional personal qualities are needed in the specialist mathematician if he is to adapt his values and aims to the curious world of mathematics of his pupils. He will find himself compelled to reinterpret his mathematical knowledge. Certainly he will need to select, to reorder and

reassess. For instance he will learn that children need to *discover* mathematics for themselves: it is hopeless to lecture them on mathematics. They are often impatient about argument and think more of results. For them, proof and demonstration are not absolute but relative to their stage of development. There are some wise comments on this in the 1923 Report on Geometry of the Mathematical Association, and in an article by Nunn on the 'Training of the Teacher of Mathematics', in *Special Reports*, vol. 27, of the Board of Education (1912). It may be noted that all specialists have to make some adaptation from the work of their university studies to that of the schoolroom, but the studies of the geographer or biologist or physicist seem to have more aspects common to both worlds than do those of the mathematician. To make contact with the world of the child the specialist mathematician will need to find out more about the 'external relations' of his subject, and to view its 'inner relations' in the light of their historical development. As a result he may well find himself using methods and material that previously would have been displeasing to his mathematical conscience. He must test himself by the measure of successful and happy achievement attained by his pupils. We offer some detailed comments and illustrations.

1. All children are deeply content when they have mastered some method of calculation with number and can apply it confidently and successfully. Suitable topics from number theory which are not too difficult and which stimulate interest are: tests for divisibility (with reasons of course) for 3, 9 and 11: the use of 2 as a base of notation —including the expression of any number as the sum of a series of powers of 2 and so-called 'Russian multiplication': the properties of recurring decimals—fully discussed in Westaway's *Craftsmanship*— of magic squares, of factorization. Some teachers have experienced amongst their pupils a passion for calculation revealing itself in summing series or evaluating π, or square roots, or logarithms.

2. Children enjoy making up their own mathematics, by constructing 'sums' for one another, or their own exercises in square root, or in factors of quadratics, or by making their own measurements and experiments in mensuration. They work successfully in pairs when discussing riders or problems, or making surveys.

3. Joint efforts by groups of children are possible for constructing a series of plans, or models (e.g. regular or semi-regular polyhedra), or graphs of functions or of families of curves, or illustrations of

geometrical theorems. The design and construction of a piece of apparatus or of a model showing some application of mathematics is a promising field for experiment—see the *18th Year Book* on 'Multisensory aids' and 'Designing and Making' by Sawyer and Crawley.

4. Mathematics as an aesthetic experience offers an immense field of possibilities for stimulating a love for, at any rate, certain aspects of the subject. Many children, more interested in acquiring experience than in organizing knowledge, are thoroughly happy in constructing patterns, loci, envelopes, and diagrams illustrating mathematical theorems, especially if they can use coloured pencils and ink. There is unlimited material available in geometry, and to reveal it seems to satisfy deep cravings for aesthetic or rhythmic satisfaction.

5. There are innumerable books of popular mathematics, of puzzles, of interpretations, which the teacher should examine for topics which he can incorporate into his usual courses to replace material of lesser value. For instance the paradoxes of mathematics —from those arising from division by zero to those depending on the contradictions of the continuum—are always fascinating, and stimulate an interest in the study of mathematics as an ordered structure. Again, the mathematics of the infinite offers all kinds of opportunities for arousing interest.

6. In the past the separation of mathematics into different branches has sometimes led to a distaste for one branch—e.g. for problems in algebra; this may be overcome by removing barriers between different branches and using algebra to solve some problems in geometry or arithmetic. The more links that can be devised to bring together aspects of mathematics, the greater the likelihood of some transfer of interest from one branch to another. The free use of intuitional methods and notions, especially available in geometry, is a help in revitalizing aspects that may be formal and arid. The reader may best be referred to such books as *Mathematical Snapshots* by Steinhaus, *Intuitional Geometry*, by Hilbert and Cohnvassen, and many popular expositions of mathematics.

7. Clearly the individual enthusiasms of different teachers may be infectious if they deal with mathematics within the child's experience. A master with a passion for riders and constructions in geometry may awaken enthusiasm for mathematics in pupils who have been drilled

in definitions and theorems. So also with puzzles and problems from magazines and newspapers as against the less stimulating exercises from the text-book. It should be clearly understood that such excursions into mathematics should not be restricted to the kind of mathematics assumed suitable for a particular class. For instance there is much scope for stimulating mathematics through problems about sequences of numbers or through problems depending on the solution of an indeterminate equation.

We add some comments from individual teachers:

1. A material aid towards raising the status of the subject in the eyes of pupils and staff is the institution of an annual Mathematics Prize given not necessarily to the best boy in the subject in the highest form but to the boy most deserving for his attitude to mathematics.

2. The occasional use of a library period enables pupils to see, handle and discuss all kinds of books on mathematics.

3. Some schools make use of a mathematical notice board, or run a mathematical club.

4. It is useful to get a senior boy, specializing in mathematics, to talk to junior forms about his interest in the subject: they gain thereby a standard they might not accept from their teacher.

5. A boy may be encouraged if he is assessed separately for effort (say A, B, C) and for attainment (say α, β, γ) with regard to his group or set, and later with regard to the school as a whole.

6. A mathematics classroom, which may be a half-way stage to a mathematical laboratory, should be set aside in every school. In it there should be mathematical material, some library books, portraits and displays, a daily diary of events of mathematical interest such as those in the lives of the great mathematicians. The room should, of course, be used for teaching and as many forms should come in contact with it as possible.

2. The Teacher's Attitude to his Class

Though some of the following general remarks may be equally applicable to the teaching of other subjects, they have been selected as being of special importance in the teaching of mathematics.

In the first place, as we have already emphasized, it is vital that the teacher should have a love of his subject. In the second place,

it is important that the teacher should be able to command his pupils' respect. There are several ways in which he can achieve this:

(*a*) Through his knowledge of his subject. It is by means of this that he can immediately produce an 'example' to meet the occasion; or, at the psychological moment, give a foretaste of work that lies further ahead; or be in a position to give a satisfactory answer to the sometimes awkward questions that may be asked of him. His qualifications should therefore be as high as possible.

(*b*) Through his discipline. It is sufficient to say that his class control should at least be adequate.

(*c*) Through his clearness of exposition. His explanations require much care and patience, for until the teacher is understood the pupil can do virtually nothing.

(*d*) Through his care in the arrangement of his own written work. High quality always commands respect, and acts as a spur to do likewise.

3. Methods in Class Work

It is most convenient to comment on methods of class work by discussing a number of selected topics.

Mental Work. Some teachers make a practice of having 5 to 10 minutes of 'mental work' at the beginning or end of each lesson; it usually consists of single-step operations, such as to calculate $299 \div 13$, or to find the number of pence in 15*s.* 7*d.*, etc. Such work appears to be intended to form a kind of 'mental tonic' and may or may not be related to the work of the remainder of the lesson. Others use the device of 'the Nine Questions' described in *The Teaching of Elementary Mathematics* by Godfrey and Siddons (Cambridge). Yet others prefer organization less rigid than either of these. By them mental work is given at irregular intervals, usually with some assigned objective in view, such as the elimination of a weakness which has suddenly become apparent, or, after the exposition of a new topic, the setting of a few mental examples to give the 'feel' of the new topic before written examples are set.

The second method is much to be preferred. Mental work should come only as and when required, should have a definite and immediate value, and should be confined almost entirely to calculations which one would expect to be done mentally in written work. It is

worth noting that there is a difference in difficulty between answering a question which has been dictated and answering the same question written on the blackboard.

It is freely admitted that there are subjects for which continuous drill over a number of periods is necessary. Examples which come readily to mind are the multiplication tables (in which boys of 11+ are by no means perfect) and the commoner fractions of £1.

Apart from drill of the type just described, it is doubtful whether mental work has any value for the weaker pupils. They have little sense of number; they are more likely to maintain a reasonable standard of accuracy if they are allowed to work out on paper a calculation which an average boy would do 'in his head'. If forced away from this support, they are more likely to lose confidence than to gain anything of value.

With the better pupils the position is different. They can be encouraged to perform parts of calculations mentally so long as accuracy is preserved; for example, they could decimalize £2 12*s*. 6*d*. at sight, whereas with weaker pupils such a procedure would be dangerous. Little deliberate practice is necessary, as the ability develops very quickly when the possibility is once recognized. All that is required is some form of general guidance, and occasional practice will probably surprise some of the pupils, if it is directed to show how much they can achieve in mental calculation.

It is necessary to stress that although this tendency to shorten written work by means of mental calculations should be encouraged among the better pupils, it must be kept under control; it should not be allowed to reach a stage at which accuracy begins to be impaired. It is dangerous to allow younger boys to attempt to simplify mentally an expression of the type: $(2x-3y)^2-(x+4y)^2$.

One possible objective of the type of training outlined (especially in the higher branches of mathematics) may not be generally realized: it is to encourage the habit of exploring mentally several different methods of approach to a problem with a view to selecting the most promising.

Memory Work. One of the most pleasing features of mathematical exposition is its conciseness of method and economy of effort. A wide knowledge of facts and formulae is of great help in securing this; but there must be a limit to the amount of material memorized. Two considerations should help to decide whether or not a given fact

or formula should be memorized; firstly, whether a mathematician would usually bother to memorize it (this test would exclude the equation of the normal at $(a \cos \theta, b \sin \theta)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, since one can write this down immediately from the equation to the tangent), and secondly, whether its use is sufficiently wide to justify the memorization (this test would exclude memorizing the number of inches in a mile).

This point is important in text-books intended for the use of pupils in schools. It is not uncommon to find a 'worked example' solved by extracting one or two 'formulae' from the preceding bookwork, and modifying them to suit the conditions of the worked examples. The 'formulae' quoted do not always satisfy the conditions mentioned above, and as they are unfamiliar to the pupil they may actually make the problem appear more difficult to him.

As a guide, a suggested list of facts and formulae which might be memorized up to the Ordinary Level stage is given at the end of this section. It is well worth while to construct similar lists for subsequent stages.

Another point must now be stressed. As he develops his facility for solving problems, the pupil also builds up for himself a mathematical structure. From many points of view the building of the structure is the more important; for the budding mathematician it is vital. A formula simply summarizes the results of some of the steps in the development of this structure; and until he understands these steps, the pupil should not be provided with the formula to memorize. He is in fact then in a much better position to appreciate the value of the formula (consider for example the formula for the roots of a quadratic equation).

The building of the mathematical structure is very much in evidence in middle-school geometry, where concurrently with the solving of riders the pupil is expected to learn a certain number of theorems so that he can become acquainted with the development of the subject from its beginnings to a stage appropriate to his power of understanding. The learning here is not done by rote, as in the case of the multiplication tables, but by insight or understanding. The pupil thus builds his own structure, but expresses the proofs in his own words. As his structure grows so does his understanding of the vocabulary of geometry. His memory of the early theorems becomes

uncertain partly because they are expressed by him in terms which he is beginning to discard for terms more appropriate. It is therefore necessary to help him strengthen his memory of the early work by reviewing it periodically, thus enabling him to interpret it anew in the light of his wider vocabulary and deeper knowledge.

If in algebra there is a danger of a boy using a formula blindly, without being able to give any suggestion of its origin, there is in geometry an equal danger of an opposite kind. A boy may be able to demonstrate a theorem perfectly, but be entirely incapable of applying it in practice. Each geometrical theorem involves learning two things: first, the proof of the theorem itself, and second, what has been proved. When the second has been learnt, the theorem can be applied as is a formula in algebra.

Revision. To some teachers revision implies merely a repetition of work already done, and it is apt to be confined to the period just preceding an examination, or to the beginning of a new school term or year. In such circumstances the work is likely to be dull and uninteresting.

Revision, in the sense of review of work previously done, should always be purposeful, and for this reason need not be confined to specific periods of the school year. One purpose may be to explore for weak points in work previously done. In this case, the time spent on it may be quite short, prehaps only part of a lesson. Questioning will play a large part in this process; or, as some prefer, a series of short questions may be given which demand written answers.

Another purpose may be to consolidate work that has previously been done. This would be specially appropriate in a case where the original work had proved difficult. In such a case there is much to be said for allowing time for 'digestion'. In the original exposition examples would probably have been confined to straightforward uses of the process taught. In the review an element of freshness can be added by using harder examples and examples showing useful applications of the processes taught, so relating the new work more closely to the rest of the mathematical structure.

A third purpose of revision could be to extend previous work. From this point of view opportunities increase with the pupils' increasing mathematical background. In some instances, later knowledge will simplify methods of approach to work that has already been done, as for instance the application of logarithms to previous

calculations on areas and volumes; in other cases, it may allow the earlier work to be extended to a further degree of development (for instance, after studying factors in algebra, a pupil can simplify more complex fractional expressions and solve more complicated equations). Such opportunities as these should be used fully, and the exact time at which revision is undertaken should be chosen accordingly.

We may also distinguish what may be referred to as an *ad hoc* revision in preparation for new work. This will probably only take a few minutes; but none will deny the advisability of making sure that there will be no previous difficulties which will form an obstacle towards concentration of the whole of the attention upon the new work itself.

Tests and Examinations. Testing, like revision, should be purposeful, and not just a periodic 'infliction'. It might be valuable for discovering weaknesses in previous work and for testing the grasp of a new section of work. Yet, in view of the time taken in devising the test, in performing and marking it, it may seem reasonable to ask whether, in many cases, the required information may not be obtained more quickly and as effectively by judicious questioning of pupils or by the examination of a few suitably chosen exercise books. The conclusion would seem to be that testing has only a limited utility.

On the other hand, so long as external examinations continue to play a large part in our educational system, there is a type of test of some value: namely, the 'training' type of test. It is in such a test, worked under conditions approaching as near as possible to those of the actual examination, that the pupil gradually acquires 'examination technique'. Such an accomplishment rarely emerges under ordinary classroom conditions. Further, the pupil has an opportunity of passing judgment on himself, and, if he so desires, of adjusting his own work accordingly. But the part played by such tests as these should be governed not only by the importance the teacher attaches to successful results in external examinations, but also by the value which the pupil attaches to such success.

Questioning. This can be one of the teacher's most powerful weapons. Used in an exploratory manner, it can quickly bring to light weak points in the pupil's mathematical knowledge. Used as a preliminary to new work it can ascertain whether the pupil's background is adequate to make the new work intelligible and acceptable. In such cases, most questions will of course be directed towards the

weaker pupils. The practice of asking questions of the whole class must be deprecated.

Even when approaching new work, the teacher can obtain much help from a class by judicious questioning. Geometrical theorems provide a perfect illustration. For instance, in proving that the opposite sides of a parallelogram are equal, questions of the type 'In what ways have we previously been able to prove two lines equal?', 'How can we obtain two triangles to contain the required lines?', should enable a class to provide a complete proof with little further help. The process may in fact provide new and unexpected proofs. This is well illustrated by the theorem that the opposite angles of a cyclic quadrilateral are supplementary. If the starting-point is taken to be known angle properties of a circle, there are several ways of completing the proof besides the one commonly given in text-books. In fact, speaking generally, text-book proofs of theorems are not always the best, and there still remains scope for originality in proofs of standard theorems.

Even theorems wth long proofs (e.g. the Theorem of Pythagoras) can be approached by means of questions. Each proof would need to be broken up into easy stages, and a general strategy of the proof would have to be indicated before the separate stages are tackled. More leading questions will of course be required, and some of them may have to be framed almost as hints.

This method of approach is strongly recommended. There is great psychological value in allowing the pupils to feel that they are helping with the new work instead of being mere passive recipients of knowledge. More than this, they are receiving some training in developing procedures and methods. Most answers may of course come from the better pupils, who can see into the heart of a problem more quickly. This is when they need to be encouraged, as we want them to get into the habit of contributing their ideas freely.

Exposition of New Work. The target is of course simple to state: it is that every pupil should eventually understand what has been done. To achieve it may often require much patience. Frequent repetitions, varying the emphasis on the separate details or experimenting with different forms of words, may be necessary until the required objective has been attained. Two things will always help the alert teacher. One is the response of the class; pupils are more alert when they follow the work, and more subdued when they are in difficulties.

The other is to question one or two of the weaker members. Even after several repetitions it may be found that a few do not seem to understand as thoroughly as required. In such cases it is probably better to deal with them as a separate group.

With large topics it is far better to break up the work into units and to spread it over several lessons than to attempt to cover the whole in one lesson. If the latter course were to be followed with young pupils the early stages of the lesson might proceed satisfactorily, but after a time the pupils' receptivity would drop rapidly. At the middle-school stage not more than about half a lesson should be devoted to 'talking'.

With the work divided into separate assimilable doses, progress is in the long run much quicker. The intervals between the lessons give time for the pupils to 'digest' the doses, a process that is of great help to the teacher. In fact, with a particularly difficult piece of work it may be necessary to make deliberate use of this process, and, after beginning the work, to leave it for a time and return to it later.

There still remains the question of how is the new work to be expounded. The previous section supplies the answer. Whenever possible the pupil should himself help in the development, as the training he thereby obtains in analysis is invaluable. The teacher is then teaching, and not lecturing. All attention is concentrated on the objective to be attained, and the effort is co-operative because the pupils are helping. External distractions ought to be at a minimum. For instance it is doubtful whether it is wise to let a class work out the next step of a problem in their books while it is being demonstrated on the blackboard. The varying speeds of working cause a break in the proceedings for most pupils and the tempo of the lesson is slowed down. If steps are familiar, it is much quicker to let a pupil dictate them.

Although lecturing as the normal method of procedure has been condemned, this does not mean that it has no place in schools. Some of it may well be introduced judiciously from time to time in the sixth form as a help in softening the sudden change in conditions on passing from school to university. In such a case it is advisable to examine pupils' summaries of such lectures with a view to trying to inculcate some rational method of procedure. With younger boys leaving school at sixteen or thereabouts, the habit of being able to

take brief notes on lectures may also be regarded as desirable, but the material of mathematics is much less suitable than that of other subjects.

Use of the Blackboard. In teaching mathematics a high standard of blackboard work is of special importance. The teacher's own arrangement should be beyond criticism, and easy both to read and to understand. One step at a time on the blackboard is urged, so that in each line the pupil has only to follow the carrying out of one process. Let him have the option of condensing his own work in accordance with his own ability.

A teacher should not develop habits in his blackboard work which he intends to discourage his class from adopting in their own written work. For instance, if he intends to discourage rough working on scraps of paper (as we shall recommend), he also must avoid doing rough working in odd corners of the blackboard. Yet it is easy to overlook this. On the other hand, we do wish to encourage compactness in written work; and abbreviations or symbols, once understood, and provided they make the meaning clear and preserve good English when translated, should be freely used. Here the teacher sets the standard in his blackboard work, and the class will usually imitate him. A watch will still have to be kept for the use by the class of ambiguous or otherwise unacceptable abbreviations, as the habit once adopted can easily get out of control.

It is suggested that in the working of a problem on the blackboard, the separate lines of working should be well separated (there are no lines on a blackboard to guide the pupil). The corresponding gain in clarity is more than might be expected.

A mathematics room therefore needs plenty of blackboard space, but there is a stronger argument for this than the need for wide spacing. It is very desirable that all the stages of a problem, or of some other piece of work, should be visible simultaneously. It is particularly necessary to have plenty of blackboard space for sixth form work.

It is also of advantage if part of the blackboard is ruled off in squares for graphical demonstrations. While much effective teaching of graphs can be carried out by good freehand sketches, there are times when in order to demonstrate precisely how results can be obtained from a graph something more accurate than a sketch is necessary. Further, pupils can come out from the class and read off

results for themselves with the rest of the class to comment upon the results obtained.

The use of coloured chalks (and of coloured pencils by the pupils) can be very effective, particularly in geometry, in graphical work, and in some branches of mechanics. The following are just a few examples of their use:

(*a*) In proving three points A, B, C collinear, AB and BC can be drawn in different colours. This helps to prevent pupils assuming the result.

(*b*) The essential features of a geometrical figure can be drawn coloured, the framework being left white. Equal angles can be marked by the same coloured arcs.

(*c*) On a graph of some such function as $y = 4-3x-2x^2$, parts where the function is positive can be shown in one colour, and where it is negative in another colour. Lines such as $y = 5$ or $y = -2$ needed for deducing results from the graph can be put in with different colours.

(*d*) In solid geometry, figures such as tetrahedra and parallelepipeds can be outlined in white, and interior lines and planes shown in several different colours.

(*e*) In statics the resultant of a set of forces can be shown coloured, and in dynamics the external forces can be shown in one colour, and the mass-accelerations in another.

There is some division of opinion about the use of blackboard compasses, protractors, etc. Although we are urging plenty of practice in good freehand sketching, at which of course the teacher should be proficient, there still remain occasions when blackboard instruments can be very useful. They are needed in lower forms particularly in order to demonstrate exactly how the common geometrical instruments are used, for this is better than all the explanations that can be given in a book. In middle forms they can often be used to draw an accurate diagram to illustrate practically the truth of some unexpected result that has turned up in a rider or theorem. On the whole it is considered desirable that blackboard instruments should be available for use when they are required.

As a final suggestion, we are not in favour of encouraging pupils to work on the blackboard. The different writing position is a handicap and the quality of writing or drawing is always poor and below the standard of their normal written work. Further it pro-

vides the rest of the class with a standard of work which we certainly do not wish them to aim at. The development of good blackboard technique is of value only to intending teachers, and such pupils do not form a large proportion of the class.

Written Work in Class. As a general principle it should be urged that whenever any new work has been done, there should be some written application of it by the pupils during the same lesson. When understanding has come, it does not always follow that retention has come, and methods of procedure are sometimes forgotten with surprising rapidity. The written application is intended to be a help towards retention.

Such written application may also serve another purpose: it may bring to light difficulties unsuspected by the teacher. But choice of examples is of importance. At first, the examples must necessarily be straightforward illustrations of the new process or technique; as the number of such examples required will differ for different pupils, the teacher must either use a text-book containing plenty of varied examples, or have a store in reserve (either his own or from other text-books) to use in emergencies for extra practice. The setting of examples should not end here, however, for the new procedures are not intended to be kept in isolation, but to be fused with the rest of the pupils' mathematical knowledge. The pupil must therefore be led on eventually, not just to harder examples, but to examples providing applications of the new technique in situations falling within, or extending, his former experience.

One of the qualities we desire to develop in our pupils during their mathematical training is accuracy. In many cases accuracy of working can be checked easily, and checking should be encouraged (e.g. addition of a column of figures by reverse addition, solution of an equation by substitution). In other cases there may not be such a short test, but the pupil can be taught to use tests which will give an estimate of the answer. Instances might be the use of approximate answers in long multiplications, or the testing of an answer to a problem in mechanics by the use of dimensions. It is in fact interesting and instructive to devise simple tests as the need arises.

Above all a pupil must be taught to realize that his answers must pass the test of common sense, and that he has not finished a problem if its answer is obviously improbable. He must search, either alone or aided, for the cause of error before proceeding to the next

problem. It sometimes happens that the answers to text-book questions do not satisfy this test; such questions should be ruthlessly deleted.

The teacher may have to convince his pupils that neatness and methodical arrangement have a practical as well as an aesthetic value. He may have to demonstrate the connection between clearly written figures and accuracy, between a good sketch and the chances of solving a rider, that bad alignment of figures causes errors in addition, etc. Sometimes working is pushed to the edge of the page by long statements or by several = signs in the same row, and becomes illegible in the effort to squeeze it in. Such faults must be constantly watched for. In the very early stages books ruled with large squares are recommended as a help in securing an orderly arrangement.

Most pupils will just imitate the teacher's method of setting out a problem, but a few are often able and willing to experiment on different lines. This should be allowed, the good being encouraged, and the unsatisfactory being discouraged. But we are not in favour of 'rough work' being done on scraps of paper. These scraps invariably become lost, and checking of working subsequently becomes difficult. Some working can be done mentally, but written answers should show all other working. Some may appear in the body of the answer, and some (if it is long) might more conveniently be placed to the side of the main working, but this must not be called 'rough' working, or neatness and accuracy are apt to be lost immediately.

A separate pad for working out the consequences of a tentative line of procedure may be useful to a sixth former, but he must also be advised not to overuse this method so that he works out all problems 'in rough' first, and then transcribes them as 'fair copies'. Much time can be wasted in this way.

As pupils begin to spend more time in solving riders they must learn to draw good freehand sketches. Success with a rider depends not so much on drawing a perfect figure with instruments as on producing a figure that exhibits the relations between its elements as clearly as possible. Pupils definitely need training in this process. At first their lines are far from straight, and their circles have most unexpected shapes; but they improve with practice. Once the facility is acquired, several riders can be quickly tackled in one session with little time spent on the actual drawing.

Concurrently with this training in freehand sketching must also be

stressed the need for drawing a figure which is general. How often a 'triangle' becomes an isosceles triangle, a 'quadrilateral' becomes a a parallelogram, or 'two circles' become two equal circles! The use of instruments furnishes no guarantee that such things as these will not happen.

There is of course a place in geometry for accurate drawing. If instruments are largely reserved for accurate work, the pupils will probably be more likely to be impressed with the need for the extra care it requires.

Lastly, in writing out his answers to questions, the pupil is liable to believe that if he himself understands what he has written, then it automatically follows that anyone else can understand it. It is therefore necessary to train him to realize when his explanations are clear enough to be understood by others. Verbosity is not needed, but frequent criticism must be given to make him appreciate whether his explanations are adequate. Much can be achieved during oral work by demanding necessary explanations, or by asking for verbal introductions to various stages in a worked example. The opportunities which mathematics provides in helping to train a boy to express himself intelligibly should not be missed—for the sake both of his English and of his mathematics.

Homework. Although there are doubts about the value of homework, we do not suggest its abolition. In fact homework serves a purpose—that of giving training in working without help. From this point of view, the practice of parents helping pupils with homework should be discouraged; pupils' difficulties are intended to be discussed subsequently. The pupil should in general find out that he is able to master difficulties by himself; and such a process, carried out over several years, can only be a valuable training.

The following principles determine the kind of homework to be set. The teacher must not be too exacting, and as a general rule he should not introduce points which have not already been discussed in class. The pupil should be able to do most of the work successfully, so that his success will serve as a stimulus for future endeavour.

The varying speeds of pupils must also be kept in mind. The value of an otherwise sensible homework assignment to a boy is lost if he needs over an hour to complete it.

Occasionally, one can, with interesting results, set work which gives scope for originality. In all such cases, the pupils should be

warned not to exceed the allotted time, except perhaps by a few minutes. Variety can be introduced by including practical work as part of the homework. A problem may be centred round measurements of the bathroom cistern, or may involve making a plan of a room, or the construction of a model.

Assessment, Marking and Correction. For the purposes of this section the word 'marking' will be used for the giving of a numerical mark; the word 'assessment' will be used for the giving of a grade. The indicating of individual questions as right or wrong will be called 'scoring'.

The collection and perusal of exercise books by the teacher varies from an occasional event to a regular drudgery. There is no reason why every separate piece of written work done by a pupil should subsequently be closely scrutinized. Much scoring or marking can be done by the pupils themselves. The teacher should however examine books (or a selected number of them) as soon as possible after new work has been begun. The object is to look for unforeseen difficulties, and to find out whether there have been any details that were not completely understood. It is suggested that the books of younger pupils should be collected periodically as their habits are still in the formative stage, and standards are liable to fall if vigilance is not maintained.

At all stages, standards can also be maintained if the teacher moves round among the class while written work is being done, paying particular attention to the known untidy pupils.

Whether each separate piece of work should be marked or assessed is largely a matter of taste. Sometimes this amounts almost to a fetish, and most elaborate 'marking systems' are used. We do not consider that every piece of written work should be rigidly marked or assessed. We feel that there is no point in doing this unless the mark or grade has some special value either to the teacher or the pupil. The teacher might often find a value in a 'term mark', which, studied in comparison with a pupil's examination mark, could provide useful information (in some schools he may have to provide such a mark). This could be easily obtained by marking suitably selected pieces of work during the term.

The pupil also may find a value in a mark if there is a stimulus to improvement attached to it, or if his competitive spirit is aroused. In illustrating these points, one or two marking or assessment

schemes will be mentioned out of many that have been suggested. These are intended to be illustrative only, and no attempt is made to suggest that any one scheme is better than another.

If a stimulus to improvement is desired, a suitable atmosphere must be deliberately created and many schemes can be devised which help. Perhaps the greatest difficulty arises with pupils of the lowest sets. For this group, a popular system is to give one mark or grade for attainment and another for effort; so that even if their attainment is limited, there is still a possibility of encouragement from rewarded effort. There is, of course, no reason why such a system should be confined to the lowest sets. Another system which works simply is to assess a particular set of work on a number of grades, e.g. on a 5-point scale; some teachers arrange the numbers in each grade on some simple ratio. When this is done the grading will, of course, be for that particular set only, and can have no reference to other standards. The stimulus here is for the pupil to work his way into a higher grade, the letters being simpler to remember than a set of numerical marks (and probably more reliable). Another way of creating a stimulus for improvement is by awarding an average mark for the whole class, and charting it over a number of lessons. This can be done in two or three minutes by collecting relevant data from the assembled class. It obviously only has meaning when the material is similar on each occasion. In all of the above, the emphasis is on individual improvement.

The competitive spirit can be aroused and a little variety can be introduced by original systems of marking. For instance, a short test might be marked on the system: 0 marks if everyone gets a question right, 1 mark if one person gets it wrong; and so on.

Mistakes should be dealt with according to their nature. As numerical accuracy is one of the essential requirements of arithmetic, it is good for the pupil to find and correct his own errors. It remains at the teacher's discretion to decide whether the calculation should be written out again, or neatly corrected (pupils seem to get little benefit from this operation). An error in method will almost certainly require a fresh start after the pupil has been shown the correct procedure. Amongst a miscellaneous collection of problems there will probably be many in which there is no point in the pupil's repeating the problem after being shown the best method of approach.

Sometimes there is value in a pupil indicating in some distinctive manner in his own book the mistakes that he has made. This can help to bring home to him some of his weaknesses, and stimulate him to eradicate them.

4. The Organization of the Work of a School

Classification of Pupils. It is advantageous, wherever possible, to divide the school into mathematical 'sets'. In an average form or class, the spread of mathematical ability is usually so great that teaching becomes complicated. The greater homogeneity achieved in a set considerably simplifies the teacher's work. A greater advantage, especially in a large school, is that the syllabus can be taken at varying rates, the top sets proceeding most rapidly because of their greater ability. With the lower sets emphasis can be varied, and the content may be different so as to allow them also to meet with success in their efforts.

It is advisable to keep the lowest set as small as possible in order to allow for more individual attention. It may also help in the earlier years to keep the numbers in the top sets a little higher than the others, so that transfers may be downward rather than upward.

It is inadvisable to introduce sets before the end of the first year. Pupils usually enter a secondary school from different schools, and their mathematical backgrounds on entry may differ widely. The first year helps to bring them to a common level of experience, and makes possible a more accurate subdivision into homogeneous sets.

Allocation of Teachers. The first point to consider is the number of mathematicians required. Some must be specialists, and for satisfactory organization of the work as a whole the number of these must be adequate. It is recommended that the minimum number of specialists should be one more than the number of streams of entry. This is intended to give adequate specialist teaching to each entry stream, and the extra specialist ensures that the sixth form is also adequately covered. If the sixth form is large, or the mathematical teaching in it is extensive, the minimum would have to be raised by one or even by two more specialists. In addition to the specialists, some non-specialist teachers will be required for junior forms. It is now necessary to allocate the teaching staff to the best advantage. Here we deal with the non-specialist first.

It will generally be found that non-specialists can be used to the best advantage in the middle school. They often have practical or other interests which help in the teaching of mathematics. And it is at this stage that full use may be made of them, after the pupils have received a firm grounding, and before the influence of external examinations has begun to make itself felt. There is also the 'form master with general subjects' type of teacher, who, possessing much patience and sympathy, can often be invaluable with the lower sets.

The allocation of specialists need not be so rigid. It is advisable for each teacher to have experience of classes over a wide range of ability; it is this experience, more than anything else, which improves his technique. A guiding principle for allocating teachers in any particular year would be to allocate men of high teaching ability to lower sets and men of high mathematical ability to higher sets.

A procedure which has many supporters is to arrange for an individual teacher to take a particular group of pupils for several years as they travel up the school. It enables the pupils to obtain a more integrated course of teaching, and the closer personal knowledge of the pupils obtained by the teacher makes it easier for their difficulties to be dealt with. Some feel that to give 'A' sets experience with different teachers, each having their own distinctive characteristics, may in the long run be better for the pupils, and provide them with a richer and more varied experience.

In any case, methods of allocation must be governed by questions of practicability. We feel that in general a set should be taught during its last two or three years in the main school by the same teacher, and that it should not be taught by more than three different teachers during its passage through the school, e.g. one teacher in the first year; one in the second and third years; and a third in the fourth and fifth years.

Time devoted to Mathematics. The amount of teaching time devoted to mathematics calls for no special comment. The range appears to be from $3\frac{1}{2}$ to $4\frac{1}{2}$ hrs per week, spread over five or six lessons; and this seems reasonable.

Homework time seems to range from about 1 to $1\frac{1}{2}$ hrs per week; the allocation for older pupils is usually greater than that for younger. This is accepted as reasonable; but one or two comments are necessary. There ought first to be a definite homework time-table throughout the main school, as a mathematical teacher (as much as any

other) realizes that the value of the work he sets is greatly minimized when it has to compete with a large number of other assignments. Again, great care should be taken to see that the weakest pupils do not have to exceed the allotted time.

Another point of importance is that a teacher should have the option not to set homework if he does not consider it necessary. Homework is something that is to have value in the development of the subject, and a master should not have to search for something to fill up the time because there happens to be a 'homework that night'. Some teachers even feel that the value of mathematical homework in the early stages is in fact so slight that there should be none at all during the first year.

In dividing the subject among the available teaching periods, three methods seem to be popular:

(*a*) Teaching the same branch on each day, e.g. Monday—Arithmetic, Tuesday—Algebra, etc.

(*b*) Teaching on a 'weekly' basis, e.g. a week of Arithmetic, followed by a week of Algebra, etc.

(*c*) Teaching on a 'topical' basis, one topic being taught until it is completed, then another started, and so on.

Of these, (*a*) and (*b*) have the disadvantage of tending to split mathematics into watertight compartments, whereas the various traditional branches ought to fuse together and reinforce one another. The subject of areas, for instance, does not belong solely to any *one* of the usual branches.

On any integrated syllabus (*c*) is of course the obvious way of dividing the available time; but even if it is felt desirable to preserve the normal divisions of the subject, some form of (*c*) is still preferable to (*a*) or (*b*), so that a particular item of work in any one branch is completed before the next item is chosen from another branch. Some links between the successive topics here would seem to be desirable in order to preserve a unity during a continuous course of instruction.

The Choice of Text-Books. Plenty of adequate text-books have been published, and their suitability has on the whole consistently improved. Many of the old-fashioned artificial examples are gradually disappearing. On the other hand many text-books are prepared solely to satisfy examination requirements.

The actual books chosen will depend largely on the type of school. The main point to bear in mind is that they should contain a sufficient

number and variety of examples to cater for the pupils adequately when they are working by themselves. A plan in many recent text-books which is worthy of favourable comment is that of providing examples in groups of three, marked *A*, *B*, and *C*. Groups *A* and *B* are of equal difficulty, and group *C* is harder. This plan achieves the double purpose of providing further practice for weaker pupils while at the same time providing work for the better pupils to do. A set of exercises should in general provide plenty of practice in the mechanics of a new operation, and also plenty of examples on its possible applications.

It does not follow that a book suitable for one set will also be suitable for another set in the same year. The teacher in charge of mathematics in a school should be familiar with a fairly wide range of text-books, and take some care in trying to provide each set with a text-book as nearly as possible appropriate to its needs.

A master in charge of a new group of boys should go carefully through the text-books intended to be used. Some obviously unsuitable examples will probably be found, and these can be struck out. On the other hand, he may find some especially good examples, and these can be suitably marked. Some of these may come from 'Miscellaneous Examples' or 'Revision Papers', and the chance of finding these during the pressure of a normal lesson is not great. Other examples may be marked '1', '2', '3', etc., for first, second, third, etc., reading. This scheme would be advisable if he decided to develop a subject partly and to return to it later. Here the examples marked '2' should extend in scope beyond those marked '1', so that the second reading does not become a mere repetition of what was done at first.

Blank pages at the front and back of the teacher's copy can be profitably used in providing a summary of the work to be done, or a classification of examples so that any which are required can be found at once.

A number of single copies of other text-books is also very useful, as further practice examples may frequently be required, and more good examples may be found. Such examples can be clearly marked, with a brief classification written in the front of the book. If a number of supplementary text-books are available a separate index covering various topics might be found useful.

Nevertheless the teacher must be the main 'text-book', and books in use by the pupil are merely subsidiary. The teacher varies his methods of approach and the order of his topics according to his pupils. It is obviously extremely unlikely that any text-book will exactly reflect the course as he himself will have planned it. Further, in the choice of examples to be done by pupils the examples should be chosen in accordance with the teacher's rather than the author's principles.

Thus, for the younger pupils, the text-book becomes largely a set of examples, and those chosen are governed partly by the teacher's scheme, and partly by the pupils' needs. If a teacher finds himself due to take a particular form, say III B, in two successive years, it does not follow automatically that the examples chosen for the first III B can be used as they stand for the second III B. Some elasticity is necessary.

With older pupils more use can be made of the subject-matter of text-books; and, especially with sixth form pupils, training in the intelligent use of them should be undertaken as a preparation for university work. Too often a sixth former is liable to run through pages simply to find a formula appropriate to the problem in hand, and after that to have no further interest in the author's expositions. He should gradually be taught to compare and contrast the work of different authors, and to study their methods of approach and development of their subject critically. He should begin to learn that no one book provides a perfect development of any one topic, and be prepared to select from several different sources in building up his own scheme of knowledge.

From this point of view a well-stocked mathematical section in the school library is of paramount importance, and it is to be hoped that the school time-table is organized so that a sixth former can spend some periods during the week working in it.

Answers in Text-Books. Should text-books issued to pupils include answers or not? This section summarizes opinions for and against the inclusion of answers.

Some say that all pupils should have answers available since they provide checks on wrong methods of working. We agree that it is bad for a pupil to do many examples each of which contains the same fundamental error, and that every effort should be made to detect the error as soon as possible. On the other hand, the

difficulty would be met if the pupil's text-book contained answers, perhaps grouped together in some way, of a selection of the examples. This experiment has not been tried in any text-book, so far as we know. The advantage of this scheme is that the pupil cannot rely on knowing the answer to every problem that confronts him. It is helpful for the pupil to have answers to check his problems while he is learning a new technique, but this purpose is well served by providing answers to a limited number of problems only.

Others say that abler or older pupils can use answers responsibly, while others may not; they assume that a greater respect for the value of answers arises as pupils develop mathematically.

A decision on the advisability of supplying answers depends on the kind of problem to be solved. For example, in the development of mechanical skills, such as the multiplication of decimals, the teacher would need to check accuracy, and so only a limited number of answers should be made available to the pupils. Again, in factorization in algebra, where the working is very short, a limited number of answers would be adequate. On the other hand, a greater proportion of answers could reasonably be given to problems which are more complicated, e.g. problems leading to equations, permutations and combinations, and some harder problems in mechanics. In these cases, knowledge of the correct answer can encourage a pupil to keep trying until he finds the key to the correct solution.

There is, therefore, much to be said for producing, as an experiment, a pupils' text-book with selected answers only. The younger pupils would, on the whole, be given fewer answers, since they are mainly concerned with the acquisition of mechanical skills. The older and abler pupils would be given more answers, since they are expected to solve problems of greater complexity. For mature pupils, who can be relied upon to use answers responsibly, answers to every problem should be provided.

A List of Facts and Formulae Required up to Ordinary Level

Facts

Multiplication tables to 12×12.
Squares of numbers up to 20^2 or 25^2.
$\pi = 3\frac{1}{7}$ or 3·142.

Simpler divisibility tests.

Usual fractions for £'s or shillings.

Equivalence of fractions, decimals, percentage.

Table of money: £1 = 240 pence.

Table of length; 1 mile = 1760 yd. = 5280 ft.

Table of square measure from square inch to square yard:
1 acre = 4840 sq. yd.

Table of cubic measure from cubic inch to cubic yard.

Table of capacity from pint to gallon.

Table of weight: 1 ton = 2240 lb.; 1 cwt. = 112 lb.; 1 stone = 14 lb.

Metric units in common use: 1 litre = $1\frac{3}{4}$ pints; 1 kg. = $2\frac{1}{5}$ lb.
1 km. = $\frac{5}{8}$ mile; 1 metre = 39·37 in.

Basic geometrical facts and properties.

Formulae

Areas of rectangle, parallelogram, triangle, trapezium.

Circumference and area of circle.

Volume of rectangular solid.

Curved surface and volume of cylinder, cone, sphere.

Simple Interest formula $I = \frac{PTR}{100}$.

Compound Interest formula $A = P\left(1+\frac{r}{100}\right)^n$.

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$a^2 - b^2 = (a-b)\,(a+b)$.

Formula for roots of quadratic equation.

The index notation.

Basic formulae in trigonometry.

V

SOME GENERAL TEACHING TOPICS

1. Degrees of Accuracy

An important function of the mathematics teacher and one which he should constantly bear in mind is that of training his pupils towards an intelligent appreciation of the degree of accuracy of the data they use and of the results they derive from the use of these data. This objective cannot be attained merely by teaching pupils how to give an answer correct to a required number of significant figures or by working through a few stereotyped examples on relative and percentage errors—indeed there is a grave doubt whether the latter items are worth retaining in the normal school mathematics course. It is of little use to be able to give an answer to a required degree of accuracy unless there is deeper and wider knowledge of fundamentals than the mere mechanical ability that is required to give that answer. That the understanding of underlying principles is often lacking is frequently revealed in examination scripts when after using $g = 32$ ft. per sec.^{2} or $\pi = 22/7$ candidates will, without a qualm, give an answer correct to four significant figures.

Success in teaching this subject is possible only if the fundamental ideas are brought to the attention of the pupil on every relevant occasion. Like graphical work, degree of accuracy should run like a theme throughout the whole of the mathematical teaching. Every mathematics lesson in which measurements are encountered provides the teacher with chances to introduce the topic. Science lessons, of course, are particularly fruitful of such opportunities and the collaboration of the science staff in this respect is of the greatest value.

The fundamental idea to be grasped is that no measurement, however fine, is exact. Until this has really been understood and become part of the pupil's attitude of mind to all the measurements he reads about or makes for himself, either in school or elsewhere, the remainder of the work can have no meaning. The development of this mental outlook is a slow one and its growth must be fostered over a long period. It cannot be built up in one or two lessons but

is a long-term process, one which can be begun before the child enters the secondary school. For instance, early work on the units of length is commonly made concrete by encouraging the children to measure the common objects in their surroundings. With the simple rulers used at this stage, usually graduated only in inches and half inches, there is no difficulty but great advantage in training the children to measure to the nearest inch. Practical work of this nature with increasing degrees of accuracy constantly recurs throughout school life and is of great value.

But it must always be remembered that the ability to measure or to give an answer to a given degree of accuracy is only the half of the story; it is also most important that the pupil should attain an understanding of the nature of the error involved in his measurement and the limits between which the real measurement must lie. It is not suggested that this is work for the early stages but rather that it should gradually be introduced as occasion serves and that ultimately no measurement should be made or employed without some mention of this aspect.

The principles enunciated above are the indispensable bases for all further work. Until the pupil has begun to appreciate them, at least in part, it is useless to teach him how to give an answer correct to a stated number of significant figures or later to expect him to have an intelligent and critical eye for the reliability of his answers rather than that touching faith in the results of a multiplication or division sum which is so often seen.

Significant Figures. Significant figures can be regarded as having a dual purpose. First they provide a weapon which allows us to give a result to a required degree of accuracy or to assess at a glance the standard of accuracy of given data. Secondly they provide a simple means of deciding upon the reliable figures in a result calculated from data whose standard of accuracy is known.

While significant figures in themselves present no great difficulty, the principles underlying the uses described above are not easy for the pupil to grasp and there is no real advantage to be gained by introducing significant figures until real need for their use appears. It is felt that to associate significant figures with the teaching of decimal manipulation is wrong. It is better to delay their introduction until a later stage when experience of measurement, particularly in the metric system, will have become much wider and the

fundamental idea that no measurement is exact will have become engrained. Every effort should be made to make the work arise naturally from measurements which the pupil has made; surveying and experiments in the mathematical or scientific laboratory provide endless opportunities which can be used when the time is ripe.

Significant figures can most easily be introduced by identifying a measurement in different units, e.g. a length as 5·037 metres to the nearest metre or 503·7 cm. to the nearest centimetre. The degree of accuracy is clearly the same though one form of the measurement employs three decimal places and the other only one. Consideration of examples of this type can be used to lead to some such definition as: 'All the figures in a number are significant except the noughts which are put in to show the position of the decimal point.'

Besides making sure that the pupil can give an answer to a required degree of accuracy it is also most important that he should be brought to realize that when he sees data expressed with a certain number of significant figures he is being given all the necessary information about the degree of accuracy of that measurement.

Noughts at the end of a number are always a difficulty which can best be dealt with by the consideration of simple examples, e.g.

(*a*) The issued capital of a company is £50,000.

(*b*) The displacement of a ship is 50,000 tons.

In (*a*) it is clear that the capital is exactly £50,000 and the noughts are therefore significant, 50,000 in this case being a number of five significant figures.

In (*b*) the last three noughts are almost certainly not significant and 50,000 is here only a number of two significant figures.

Accuracy of Results from Given Data. This is work for a later stage but should not present any great difficulty if the way has been prepared by earlier consideration of the limits of accuracy of a variety of measurements as they have occurred. The main aim is to arrive by consideration of a number of examples such as those shown below at a working rule which will quickly decide upon the number of reliable figures in the result of a calculation based upon data of a known degree of accuracy.

Addition and Subtraction. Consider the sum of the following weights, each being given correct to the number of significant figures shown: 16·8 kg.+70·42 kg.+3·614 kg.

There are two methods which can be employed to determine the reliable figures of the answer:

(*a*) 16·8*xx*
70·42*x*
3·614

90·8*xx* kg. Unknown figures are indicated by *x*.

It is clear that the final 8 is suspect and the most that can be said is that the weight is 91 kg. correct to two significant figures.

(*b*) The maximum error is $\pm(0{\cdot}05+0{\cdot}005+0{\cdot}0005)$ kg., i.e. $\pm 0{\cdot}05555$ kg.

Since $16{\cdot}8+70{\cdot}42+3{\cdot}614 = 90{\cdot}834$, the total weight must lie between $90{\cdot}834 \pm 0{\cdot}0555$ kg., i.e. between 90·7785 kg. and 90·8895 kg.

It follows as in (*b*) that the answer can be given correct to two significant figures only, as 91 kg.

Similar methods and reasoning can be applied to subtraction.

Multiplication and Division

A method similar to that shown at (*a*) above can be employed, but most teachers prefer to use the method shown below, which corresponds to (*b*).

(*a*) *Multiplication.* From a survey an area in square yards is given by the product 107×43, each measurement being correct to the number of significant figures shown.

The true area lies between $107{\cdot}5 \times 43{\cdot}5$ sq. yd. and $106{\cdot}5 \times 42{\cdot}5$ sq. yd., i.e. between 4676·25 sq. yd. and 4526·25 sq. yd.

The product $107 \times 43 = 4601$.

An examination of these results shows that the area can only be given correct to 1 significant figure, as 5000 sq. yd.

(*b*) *Division.* Evaluate as accurately as possible $\frac{47{\cdot}28}{3{\cdot}14}$, each number being correct to the number of significant figures shown.

The true result lies between $\frac{47{\cdot}285}{3{\cdot}135}$ and $\frac{47{\cdot}275}{3{\cdot}145}$, i.e. between 15·08 and 15·03.

The quotient $\frac{47{\cdot}28}{3{\cdot}14}$ is 15·06.

Hence the most accurate result that can be given is 15, which is correct to two significant figures.

In (*a*) the data have 3- and 2-figure accuracy, and the result can be given only to 1 significant figure.

In (*b*) the data have 4- and 3-figure accuracy, and the result can be given only to 2 significant figures.

Consideration of examples like these will show that we can define a working rule as follows: 'In multiplication and division the degree of accuracy of the answer is governed by that of the least accurate of the numbers involved. In general, if that number is correct to n significant figures, then $(n-1)$ figures can be replied upon in the answer.'

It should be emphasized that this is a working rule only, and that sometimes, though not very frequently, n figures will be reliable in the answer; but for all ordinary purposes the rule is a very useful weapon with which every boy should become familiar and which he should be encouraged to use.

Contracted Methods. It is very doubtful indeed whether the time and labour which have to be devoted to teaching and learning these methods are worth while; and having taught them, it is very difficult to persuade a pupil to use them, since he finds it quicker to perform his operation in full and then to reject the figures he does not need or which are unreliable. In the case of the product or quotient of numbers of three or four significant figures such as are normally met, there is probably very little time gained by the use of contracted methods even in the hands of the experienced user. It is accordingly recommended that contracted methods be omitted from the syllabus.

2. The Teaching of Logarithms

Logarithmic tables play such an important part in a pupil's mathematics from the middle school onwards that careful grounding in their use, and the inculcation of good habits of working with them, are essential. Fortunately this is a branch of mathematics in which it is not difficult to secure the co-operation of even the dullest pupils. The fact that logarithmic tables are of considerable practical use,

and can save much tedious calculation, is sufficient to arouse curiosity and interest.

There are two main ways of introducing logarithms. One method is to teach the purely mechanical use of the tables. A preliminary talk is first given on how to read 4-figure tables, pointing out that the left and upper borders give the numbers, considered to be in standard form, and that the logarithms forming the body of the tables are to be read as decimal fractions. By comparing operations of multiplication and division of 3-figure numbers with the results obtained by the addition or subtraction of their logarithms, a *modus operandi* of the tables is acquired.

E.g.

No.	Log	
2·31	0·3636	3·64
×3·64	+0·5611	2·31
8·409	0·9247	728
		1092
		364
		8·4084

The degree of accuracy at this stage need not be stressed. Some practice in such examples should be given, with the extension to 4-figure numbers.

Characteristics of numbers greater than 10 can be obtained as exercises in multiplication.

E.g.

No.	Log
1·712	0·2335
×10·00	+1·00
17·12	1·2335

The use of negative characteristics can be demonstrated by tabulating numbers and logarithms thus:

No.	Log
364·5	2·5617
36·45	1·5617
3·645	0·5617

Each division of a number by ten reduces the logarithm by 1. Hence it is reasonable to assume that, the process being continued, decimal fractions have negative characteristics as follows:

No.	Log
0·3645	−1+0·5617
0·03645	−2+0·5617

As this method of introducing logarithms makes no attempt to justify the operations $\log a + \log b = \log ab$, $\log a - \log b = \log a/b$ its use cannot be advised, except possibly when teaching very young or dull pupils.

Most teachers prefer to develop the subject from the four laws of indices. As an introduction the laws are deduced for positive, integral indices.

x^0 is then defined to be 1, to comply with these laws, and definitions are also obtained for fractional and negative indices. Some practice in simple numerical evaluations of the type 7^0, $16^{\frac{1}{4}}$, $8^{\frac{2}{3}}$, 5^{-2}, $9^{-\frac{1}{2}}$, should follow.

The graph of 10^x is then drawn for value of x between 0 and 1; values of $10^{\frac{1}{2}}$, $10^{\frac{1}{4}}$, $10^{\frac{1}{3}}$, being found from square root tables, and hence $10^{\frac{2}{3}}$, $10^{\frac{3}{4}}$, $10^{\frac{1}{6}}$, $10^{\frac{1}{8}}$.

From this graph numbers, in standard form, are expressed as powers of 10 to two significant figures; and vice versa. The logarithmic tables are then used to obtain these values to four significant figures.

Operations of multiplication and division of numbers are performed as exercises in the laws:

$$10^a \times 10^b = 10^{a+b}; \quad 10^a \div 10^b = 10^{a-b}.$$

The expression of logarithms of numbers with positive and negative characteristics, and the evaluation of powers, or roots of numbers, follow as further operations in the laws of indices.

It is suggested that even where the first method of introducing logarithms has been adopted, the second method should be taken at some later date in a revision course.

A methodical arrangement of working should be insisted upon at all times. Setting out the work in columns probably yields the best results:

$$\frac{\sqrt[3]{62{\cdot}16} \times 0{\cdot}8617}{12{\cdot}34} = 0{\cdot}277 \text{ (to 3 sig. figs)}$$

$$\text{(Check: } \frac{4 \times 1}{12} = 0{\cdot}3\text{)}$$

No.	Log
62·16	1·7935
$\sqrt[3]{62{\cdot}16}$	$\bar{1}{\cdot}5978$
0·8617	$\bar{1}{\cdot}9354$
	$0{\cdot}5332$
12·34	1·0913
0·2767	$\bar{1}{\cdot}4419$

(Later, when logarithms are ancillary to the working out of a problem, the number column may be omitted.)

Teachers who have introduced logarithms through indices will, at first, set out examples as follows:

$$\frac{8{\cdot}682\times 62{\cdot}74}{2{\cdot}317} = \frac{10^{0{\cdot}9386}\times 10^{1{\cdot}7976}}{10^{0{\cdot}3649}} = \frac{10^{2{\cdot}7362}}{10^{0{\cdot}3649}}$$
$$= 10^{2{\cdot}3713}$$
$$= 235{\cdot}2 = 235 \text{ (to 3 sig. figs.).}$$

Some prefer to retain this arrangement in later work, and it has the advantage of showing the principles of the working clearly. We do not, however, advise it because there is a redundancy of 10's, and a greater liability of mistakes being made in adding or subtracting 'across' instead of 'up and down'.

Any tendency to set out working in the form:

$$\log \frac{8{\cdot}682\times 62{\cdot}74}{2{\cdot}317} = \log 8{\cdot}682+\log 62{\cdot}74-\log 2{\cdot}317$$
$$= 0{\cdot}9386+1{\cdot}7976-0{\cdot}3649 = 2{\cdot}3713, \text{ etc.}$$

must be discouraged. Not only is this method fruitful in arithmetical errors, but it also leads to incorrect statements such as:

$$\frac{8{\cdot}682\times 62{\cdot}74}{2{\cdot}317} = \frac{\log 8{\cdot}682\times \log 62{\cdot}74}{\log 2{\cdot}317} = \frac{0{\cdot}9386+1{\cdot}7976}{0{\cdot}3549}.$$

To begin with, anti-logarithms should preferably be obtained from the backward reading of logarithmic tables. It will not be long, however, before some of the class will be eager to know the use of anti-logarithmic tables, if they have not already found out for themselves. There would be no point in disallowing their use at this stage. It might prove a useful exercise to compare results obtained from the anti-logarithmic tables, logarithmic tables and by interpolation. Further practice in the reverse use of tables will come, if it has not already done so, with trigonometrical tables.

The checking of answers is to be encouraged. It is particularly useful when dealing with negative characteristics, as mistakes are very liable to occur in placing the decimal point in the answer. We advise, however, that the rough check should come last, to test an answer already obtained; otherwise there is the danger that the check, having been made at the beginning, will not be compared later with the final result. When considerable working is involved, discretion should be exercised in insisting upon a rough check; a mistake may often be more easily incurred in this than in obtaining the answer itself.

The degree of accuracy of results obtained by using tables should be stressed at all times. Although 4-figure tables may give some answers trustworthy to 4 figures, it is safer if pupils make a practice of taking them to 3 significant figures only. They should also understand that, when the subtraction of numbers is involved in the working, 4-figure tables may not even give this degree of accuracy. It is necessary too, to prevent the abuse of tables in questions, such as those connected with money, where they cannot give reliable results.

To conclude the work on logarithms, some time can be profitably spent on slide rules, including their construction out of logarithmic paper. If possible, pupils should also use real slide rules. Not only can considerable interest be aroused in these lessons, but they serve to drive home the principles of logarithms and to give further practice in approximations, in the correct placing of the decimal point, and in estimating the accuracy of results.

3. Tables, Slide Rules, and Graph Paper

Tables. Before the First World War the early introduction of various tables and calculating aids, and any extensive use of them, was unusual and would have been regarded with considerable suspicion. The arguments against such a step would have been that: (*a*) underlying principles might be obscured by using tables, and (*b*) it is impossible for a pupil to know the limitation of the tables, and when they could, or should not, be used. We were then coming to the end of a business age, when complaints against grammar schools were frequent because boys leaving them were unable to tot up long columns of figures, or to do 'Office Arithmetic' with mechanical accuracy. There was thought to be virtue in finding square root by rule, rather than by tables, and contracted methods of multiplication were more generally taught than now. It was also not uncommon in text-books of that time to find answers given to a degree of accuracy quite out of relation to the practical aspects of problems, e.g. compound interest expressed to one hundredth of a penny.

Since the First World War, mathematics, and particularly arithmetic, has become increasingly technical. In shops, offices and banks, much of the labour of mechanical calculation has been taken over by machines. The grammar-school boy is no longer so likely to enter

a black-coated office job, but is just as likely to become a technician or a scientist. This change has inevitably been reflected in the teaching of mathematics, and during these years there has been a steady revolution in outlook and practice. A modern boy leaving school, well equipped mathematically, is now expected to use a variety of tables confidently. The opinion in most schools now can probably be summarized as follows: Fundamental principles and processes in arithmetic must be thoroughly understood, and then practised by simple numerical examples in which accuracy is insisted upon. The use of tables is introduced early in the curriculum; the first tables could be of square roots and squares, or logarithmic, or trigonometric, according to the sequence of teaching. By the end of the third year many pupils will be familiar with all these tables, and during the next two years will probably add logarithmic sines, cosines, and tangents, and tables of reciprocals. During this later period boys should learn how to obtain differences by interpolation. The necessity will probably arise in using tangent tables for those angles for which differences are not given. The reason for this can be illustrated by graphing the variation in values of the tangent over a range of one degree for large angles, and comparing with graphs obtained similarly for small angles.

This early introduction of tables has, of course, its dangers. The teacher must be on constant guard, for example, against the use of logarithmic tables for multiplication or division that could be done mentally or by other simpler means. It is necessary to check continually that answers do not imply a greater accuracy than the tables allow. At first pupils should be told the number of significant figures to be taken, but as they grow older they should be increasingly aware of the limitations of the particular tables they are using and the circumstances in which the use of their logarithmic tables can be justified.

A comprehensive set of 4-figure tables for general school use should include logarithms, anti-logarithms, natural and logarithmic sines, cosines, tangents, secants, cosecants, and cotangents, and also square roots, squares and reciprocals. In addition, the sixth form require tables of Napierian logarithms, radians to degrees, trigonometrical functions of angles given in radians, and preferably, e^x, e^{-x}, $\sinh x$ and $\cosh x$. The tables should be clearly printed on good paper and securely bound; they will have considerable use and it is false

economy to save on quality. Sine and cosine tables should not be combined for the use of children, and, preferably, the characteristics of log. sines, etc. should not be increased by 10.

Slide Rule. It has already been suggested that boys should be shown how to construct a simple slide rule, and should learn how to use it, at any rate, for simple multiplication and division. We would not advise the general use of the slide rule in the classroom, apart from any question of expense, on the following grounds: (*a*) it cannot give the degree of accuracy usually required and it is almost impossible for many boys to appreciate this, (*b*) with no working on paper faults of calculation cannot be traced, and errors of principle may also be obscured. On the other hand, there is much to be said for the use of the slide rule as a practical aid to laboratory calculations. Its very limitations may deter boys from giving the results of their experiments to a greater degree of accuracy than is justified.

Graph Paper. Graphical work throughout the school has a continuity of development which it is useful for pupils to record. For this purpose suitable specimens from their graphs could be filed for later reference and revision. Alternatively a graph book, possibly interleaved with ordinary ruled paper, could be issued to each boy, early in the school, in which to build up his record.

There is some danger that when graph paper of the same size is always in use, scales, or ranges of values for the variables on the axes, may always be chosen to fit that particular size of paper. It is possible to obtain large sheets of graph paper which could be cut as required, and it is suggested that older boys, at least, should have some practice in using graph paper of different shapes and sizes.

In the sixth form there is a further extension of graphical work, particularly in relation to polar co-ordinates. Here is an opportunity to introduce polar graph paper. Logarithmic and other types of ruled graph paper may also be useful. A selection of various types of graphical and ruled papers may be obtained from various commercial sources.

4. Written and Deductive Work in Geometry

The Drift from Euclid. At the beginning of this century the study of geometry at schools in England meant the study of Euclid's *Elements*. This great masterpiece determined the material, the order and the

style of a geometrical argument, and as it was essentially a sequence of theorems the style of a geometrical argument became fixed in the form of the proof of a theorem. It may however be noted that Euclid's original style was less rigid than the style employed in school books written to replace Euclid, many of which required three or four pages adequately to demonstrate a theorem.

Since then the trend of teaching has been 'away from Euclid' though in no clearly defined direction. There were good reasons for this movement, which had been preceded on the Continent by an earlier movement for reform consequent upon the trend of thought which developed with the French Revolution. This movement produced, in Legendre's *Elements*, an authoritative text-book which assisted the reform and which has since led to courses of geometry, based on space perception and leading to an axiomatic treatment, which are unfamiliar in this country. During the past fifty years, substantial improvements have been attained in the movement 'away from Euclid'. They may best be summarized historically.

In the first stage of the revolt from Euclid, an attempt was made to appeal to hand and eye by using drawing as a substitute for reasoning about diagrams. This soon led to courses in Practical Geometry based on a box of instruments. Though this was a more acceptable introduction to geometry, it was frequently misused, especially in verification of a general theorem by measurement from more or less carefully drawn diagrams. There also grew up a tendency to spend too long over this work, and to follow it by courses of theoretical geometry which were still Euclidean, and sharply divorced from geometry in the form of the study of space. Comparatively little attention was given to solid geometry or even to similarity, since these were not discussed in the earlier books of Euclid's *Elements*. The movement away from Euclid soon modified the number and sequence of the theorems considered necessary. It was found desirable to extend the basis of assumption so that proofs of theorems on congruence and on parallels were not required. The growth of the examination system stimulated this since examination schedules had to indicate precisely which theorems could be assumed and which were to be proved. Attempts to popularize an agreed sequence of theorems were not wholly successful, though the sequence contained in the pamphlet on *The Teaching of Elementary Geometry* published by the I.A.A.M. in 1923 was very largely used

as a basis by a considerable number of writers of successful textbooks of geometry. Furthermore, the need to produce an adequate mark distribution led examiners to give less importance to bookwork and more to riders and calculations. Theorems could be (and of course were) crammed; to test intellectual capacity, other material had to be drawn upon, and it was realized that the value of a geometry course lay not in proofs of theorems but in the solution of riders.

It may be remarked that the Civil Service Commissioners have always taken, in their syllabuses and examination papers, a far more mature and mathematical view of elementary geometry, in which emphasis is laid on using the tools of geometry, with little regard for formal proofs of theorems. One of the strangest features of the mathematical scene in this country has been the contrast between the papers set for these examinations and those set in G.C.E. examinations. Recent changes have diminished the contrast through modifications in the G.C.E. syllabus.

A decisive stage was reached by the publication, in 1923, of the Mathematical Association's *Report on the Teaching of Geometry*. This scholarly and difficult document proposed the division of the course into three stages; it suggested a theoretical line of development which has failed to find support, and made many less ambitious but more fruitful suggestions for teaching different topics of school geometry. The nomenclature of the Report has now passed into general usage. The first stage in the geometry course, Stage A, should be experimental, comprising the investigation of properties of space using material and methods appropriate for children. Both instruments and deductive reasoning would be required but the stress would gradually pass towards deductive reasoning until, at the age of thirteen or so, Stage B, the deductive stage, would be attained. Here the emphasis would be on the substance of geometry, on riders, constructions and calculations, rather than on proofs of theorems. A Stage C, the systematizing stage, was intended to complete the course but it is doubtful, in practice, if more than a vigorous drill on School Certificate theorems is ever attempted.

A Second Report on the Teaching of Geometry, published by the Mathematical Association in 1938, amplified many topics of the first report without producing substantial modifications in practice. The report still emphasized the importance of the solution of riders,

and the value of solid geometry was re-emphasized; in addition, much valuable material was collected, dealing with perspective drawing, non-Euclidean geometry and other aspects of the axiomatic development which are more commonly taught on the Continent.

The Alternative Syllabuses recommended by the Jeffery Report (1944) (see the Appendix) have tried to embody these various trends in a workable scheme of school mathematics in which due attention is paid to what is mathematically, and not only geometrically, significant. Many problems associated with deductive work still remain, however, and we offer the following comments towards their solution.

Deduction in general. The essence of mathematics is abstract reasoning, and this essence is perhaps best revealed for school children through geometry. A course of geometry should begin by inquiries about space and lead to as great a variety of forms of reasoning as are available. *From the earliest stages* the deductive argument should be used wherever it is natural; in particular there should not be a preliminary course in which measurements are used to 'prove' results, e.g. that the sum of the angles of a triangle is two right angles, which can be better demonstrated by deduction from agreed properties (of parallel lines cut by a transversal). The principle of the deductive process is: 'Given A, we can (by processes of reasoning appropriate to the age and experience of those receiving instruction), show that B must follow.' In practice it is wise to begin by making A as simple as possible and arranging that B shall be a rather unexpected result. This emphasizes the value of the deductive process. Some teachers believe that simple riders are better material for this work than proofs of theorems. In any case one important distinction must be made involving the word 'proof'.

When teaching geometry to children it is necessary to distinguish between two senses in which we use the word 'proof'. In the first sense, we use the word in its primary purpose of explanation or verification, as when we say 'Prove the Theorem of Pythagoras.' The statement of this theorem calls out for explanation. The philosopher Hobbes on meeting the statement for the first time expressed himself simply—'By God, this is impossible!' The purpose of proof here is to reveal why the 'impossible' *is* possible. But in sharp contrast with this we use the word proof with a wholly different meaning as when we ask for a proof of propositions which appear 'obvious'

—to prove, for example, that the base angles of an isosceles triangle are equal. The meaning here means proof in the sense of logical relation, showing how this result follows from what we choose to assume as the simple elements from which reasoning must begin. For most pupils proof in the sense of explanation is interesting and significant, but when proof in the second sense is required it is most desirable to make clear what is implied, and, in particular, the logical points involved.

Deduction in the Class Room. Deduction is essentially the statement of an argument, and it seems important for mathematical education to set exercises which would require a logical explanation not necessarily using geometrical material. There are many books of puzzles, inferential problems, cross numbers, intelligence tests and the like: a good deal of the problem material familiar a generation ago might reappear in these more appropriate forms. Such exercises stimulate inventiveness in methods and economy of expression in stating a solution. They would have a valuable effect in stressing the elements of mathematical intuition, so necessary in creative thinking. They would enlarge the reasoning aspect of mathematics from the somewhat restricted deductive work on the equating of angles or lengths or areas, which constitutes much of the rider and theorem work in geometry at present, by including other forms of argument. Even in rider work it is good to set 'discovery' exercises in which pupils are asked to find out properties of a given configuration, or to ask what follows from given data, or to vary the type of question which has to be answered. Thus, sometimes more than one solution is possible, or a solution changes its form through some apparently slight change in the data. All these devices stimulate independent thinking on geometrical material.

General educational principles would suggest that any framework of 'proof' should be evolved by the pupils themselves through oral work and by using marked diagrams and coloured pencils. It is suggested that this evolution of geometrical 'proof' should at first deal with simple riders and not with theorems. The most suitable are those in which the result is not obvious. More attention than seems usual might be given to constructions, not merely to those required in G.C.E. examinations, but to others using data from maps and charts and introducing methods required in geography or navigation. The description, in the simplest and most precise terms,

of a construction, can be an excellent exercise in geometrical argument, and appeals to many pupils because it has an obvious purpose.

Only when pupils can solve simple riders or draw accurate diagrams from given data should they be expected to appreciate what is required in the writing out of a theorem.

Amongst the points to be borne in mind in this work will be the following:

(*a*) At first the term theorem will be restricted to properties of major importance, e.g. those on the angle sum of a triangle, or the angle at the centre of a circle, and during the first year of geometry not more than half-a-dozen such properties may formally arise. They may be discovered by informal methods; for each a general proof will be evolved, and the theorem will be used as a basis for further deductive work. The emphasis will be on the *use* of the theorem, on its value as an element of substance rather than as a portion of the structure of geometry. It is most desirable also that results should be illustrated by diagrams in many different positions so as to free the mind of the pupil from associating a particular result solely with triangles of one shape or with circles in one position.

(*b*) In subsequent years the deductive development will deal with groups of theorems, e.g. the angle-properties of the circle, and here the importance of the key theorem of the group will need emphasis. In writing out proofs of theorems and riders, references to key-theorems or to other standard theorems should be made explicitly, though teachers seem divided in opinion as to the extent to which abbreviations should be encouraged. It must be remembered that for the large majority of pupils, geometry is a means and not an *end*; the aim of teachers should be to give the substance of the subject and not simply to demand written exercises in a formalized style.

(*c*) The logical aspects which must be considered will include:

(i) The gradual evolution of definitions. It must be stressed that a definition is evolved from a description for convenience of classification and deductive reasoning.

(ii) The distinction between a direct theorem and its converse theorems. It may be pointed out when appropriate that a theorem may have several converses according to the separate elements of the data. How far the classical method for converses, *reductio ad absurdum*, should be employed is uncertain; in general it is not preferred by children to a separate and direct proof. Proofs using

this method will be demonstrated at the blackboard and their relevance in a logical framework will be explained but their reproduction by the pupils should be deferred as late as possible.

(iii) The notion of an axiom. Possibly the most effective approach to this is to retrace the steps by which a theorem of some complexity has been reached and to discover some of the implicit assumptions that have been accepted. Work of this kind needs taking at the appropriate stage with pupils of some maturity. Whilst all will agree that reasoning must be based on some agreed premises it is almost impossible to persuade ordinary human beings to abandon long-held notions of geometry as space perception and in particular to abandon assumed properties of the straight line. A course of lessons on the geometry of lines on a sphere is an essential preliminary to any discussion of the axiom of parallels. The axiom of congruence presents fewer psychological difficulties, but an attempt to create an abstract geometry from axioms alone seems unattainable in most schools, possibly because of the unfamiliarity of the work to the teacher, but more likely from lack of appeal of the work to the pupils.

(iv) The sequence of theorems. It is inevitable that any sequence of theorems in school geometry in the future will be much looser than any in the past, and since this should go with a wider knowledge of the structure and substance of geometry and of mathematics generally there is no need to regret the past. It is clear that the level of reasoning acceptable at twelve or thirteen is unlikely to be as high as that at sixteen or seventeen, which implies a gradual increase in the complexity of the sequence studied. But whereas a generation ago it was believed that mental training was provided by advancing link by link along a chain of theorems there is now little justification for this belief even in the teaching of mature pupils. Certainly in schools it is essential to survey the *important* links in the chain and to indicate the range of the whole system before examining the connections between separate links. Most teachers will feel that work of this kind is less fruitful than an extension of the substance of geometry, for instance into three-dimensional work, considered theoretically and practically, as in perspective drawing. Certainly some acquaintance with ideas of projection or transformation would widen the geometrical horizon for most pupils. An extension of this kind may well be possible in the immediate future and is urgently

needed if geometrical studies at school are to bear any relation to those at the university.

Riders. As has already been stated, at the beginning of the present century the study of geometry was the study of Euclid's *Elements*. Most pupils failed to pass beyond the initial stage of memorizing the theorems and reproducing them in the pattern laid down by tradition. Success in applying theorems to riders was attainable only by the few, and for those there was no special difficulty in expressing the demonstration of a rider in the traditional form. Since then the trend has been 'away from Euclid' and less emphasis is now laid on learning and reproducing formal proofs of theorems, and more on the applications of theorems to riders and to situations involving practical mathematics. This change in emphasis is one of the most significant trends of the past thirty years.

In public examinations the tendency is towards requiring formal proofs of only the major or key-theorems. These proofs are expressed in a form which varies from the traditional Euclidean form (now very seldom encountered) to a compact 'Given', 'To Prove' and 'Proof'. Many teachers prefer to keep this arrangement for rider work also, though there is a tendency for solutions to be expressed more informally.

Developing Skill in Rider Work. We have assumed that rider work is more effective as an introduction to the deductive form of argument than is a sequence of theorems. Most teachers would demonstrate a major theorem—such as the exterior angle property of a triangle—and would then show the application of this simple and easily grasped result in a variety of one-step and two-step riders. Another favourite introduction to rider work is based on one or other of the congruence conditions. Such a clear basis of argument shows the deductive process very explicitly. It is more effective than the kind of rider work which depends upon a specific definition, e.g. riders about properties of a parallelogram. Children do not appreciate readily the charm of a mathematical definition in which we may assume only that the opposite sides of a parallelogram are parallel but not that they are equal. This is an unnecessarily sophisticated type of argument in the early stages. Likewise any stress on axioms as a basis for early deductive work is perhaps likely to make more difficulties than are really necessary. For most pupils, school geometry is the study of physical space and the

axiomatic development of geometry should evolve only gradually from this.

Given a set of riders from a text-book the left-hand side of an open exercise book may be used for trial attempts at a series of riders. The methods used will be practical, 'working from both ends', a mixture of analysis and synthesis. The solution of a particular rider will appear in the form of a marked diagram, often with the aid of coloured pencils, with some side notes showing the steps of the pupil's thinking. If necessary, different pupils can expound their own solutions at the blackboard; difficulties may then be resolved and solutions may be improved—usually by cutting out redundant steps. On the right-hand pages of the pupils' books the deductive demonstrations of *a few* of the worked riders can then be set out in a style appropriate to the age and knowledge of the class. Full use should be made at the appropriate stage of any notation that leads to ease of comprehension—abbreviations, single letters for angles (the a, a_1, a_2, notation explained in the Second Report on Geometry of the Mathematical Association), perhaps symbols like $\equiv$ for congruence and some symbol such as $\simeq$ for similarity (this last is commonly used in German text-books).

The number of riders solved in this 'marked diagram' form should greatly exceed the number written out deductively. It must be emphasized that writing is best required only when there is something to be expressed that cannot be expressed by marks on a diagram or by other simple devices. Too much formal written expression at too early an age leads to a distaste for geometry, to a feeling that it is a matter of incomprehensible patterns and theorems that have to be 'done' in obedience to authority, with consequent loss of interest and power. What is needed is to precede this written work with more oral explanation using marked diagrams, more appreciation of 'well-known' theorems and their application to simple one-step and two-step riders, the aim being the creation of a feeling of achievement. The whole procedure corresponds to the process by which the experienced geometrician 'solves' a rider—a combination of analysis and synthesis leading to a deductive proof. Teachers who insist on a formal set-out before the proof has been discovered are sometimes apt to forget that a satisfactory arrangement of the proof is possible only when the solution has been seen through to the end. Only when the child is confident that he *has* the correct solution is it

reasonable to attend to the separate business of expressing the arguments in the clearest and simplest form. It is necessary to stress explicitly that simplicity and directness of expression are fundamental elements in mathematical thinking. The essentials of the deductive process should be revealed—but gradually.

A good deal of attention should be given to the grading of rider work, especially at these early stages. In addition to the possible difficulty of solving the riders, a pupil in the early stages has two further difficulties with which to contend:

(*a*) The interpretation of the given information in the form of a figure.

(*b*) The production of a useful figure after this interpretation has been made.

In view of these difficulties there seems to be a case for providing the figures to all early riders. If this is not done by the text-book it should be done by the teacher. (This assumes that the first emphasis is on the *process of solution*—which seems to be natural). Simultaneously with, or very shortly after, the above, there would be given some systematic guidance in the most effective drawing of the figures. At first the objective would be simply the correct expression of the data in the form of a figure. Then the teacher should try to bring home to the pupil the importance (or value) of the good spacing of the elements of the figure, and of good proportions generally. At this stage he can also bring in the need for drawing *general* figures, e.g. a triangle is not to be an isosceles triangle, a quadrilateral is not to be a parallelogram, two circles are not to be equal circles, a chord of a circle is not to be a diameter, and so on. There is a common tendency for pupils to draw such special cases unwittingly.

The number of steps in a rider should be limited until signs of confidence and success become apparent, and this may not happen for two or three years. In fact, at the beginning, riders might even be confined to the one-step type, with an occasional two-step type included to keep the better pupils usefully occupied. The teacher can then begin gradually to include the more difficult types, though if there are a number of weak pupils in the class there should still be a fair proportion of the easier types, particularly on the earlier theorems.

In all mathematical education it is necessary to *review* work done previously far more frequently than is usual. For instance a set of

riders that were worked a few weeks ago may be worked through again, or worked with altered diagrams, or a selection may be taken —and extended to more difficult riders of the same set. It is, in fact, most important to arrange for systematic consolidation by attempting harder riders on the more familiar theorems as the pupils' knowledge is advancing. The importance of unconscious memory, of association of ideas, is probably greater in rider work than anywhere else in mathematics. This process of revision and extension in which one may deal with alternative proofs, or special diagrams, or applications to particular figures, enables pupils to grasp more quickly the essential items in a geometrical diagram. It helps them to knit together that tenuous fabric of properties, relations and consequences, without which success in the solution of riders cannot be acquired. Above all, this process encourages children to study their own progress in solving riders.

Some teachers already realize that rider work is often more effective if it uses, not the theorems under discussion, but those studied some two or three weeks ago—work on theorems being kept some distance ahead of work on riders. A process of unconscious consolidation is here in operation. Perhaps once a term it is useful briefly to review the major results of the geometry course, and, as has been suggested in the previous paragraph, to explore some early but more difficult exercises.

We may here make a remark about the 'congruent triangle complex'. A great failure of teaching in an earlier generation was revealed by examination candidates whose attempts at proving almost any rider began by 'In the triangles ABC, XYZ, $AB = XY$ (given) . . .' and proceeded no further. Younger pupils, and especially the weaker ones do like a 'rule' to follow; and the stereotyped pattern which a 'congruent triangle' proof follows naturally appeals to them. We cannot blame them too much for wanting to solve as many riders as possible by this stock method. The use of other deductive material in the early stages now tends to prevent wrong use of congruent triangles; in any case, undue emphasis on congruent triangles should not appear during the early stages.

Style in Rider Work. The style in which the proof of a rider should be expressed has been considerably modified, in recent years, from the traditional Euclidean style. Abbreviations are used more freely, shorter references to necessary theorems are encouraged, a useful

suffix notation for angles (already mentioned) has been developed into a definite technique, and the general tendency is towards a more descriptive symbolic form of proof such as is usual in other branches of mathematics. In stating the deductive argument most teachers prefer to use a clear framework such as 'Given', 'To Prove' and 'Proof'. Any such formal expression must be adjusted to the age and maturity of the pupil: the terms themselves have special meanings in geometry, and may well be introduced separately and gradually. The full Euclidean style was originally for proofs of *theorems* and it cannot be too clearly emphasized that it was intended for *adult* mathematicians interested in the structure of geometry. This precision of demonstration is perhaps appropriate in specialist work but changing conditions make it unsupportable in educating young children. The nearest approximation to it tends to occur in middle- and upper-school geometry under the influence of text-books and examination requirements. The formal framework tends to disappear after the first school examination, though in the opinion of many teachers it is still a help to the pupil: the influence of such books as Forder's *Higher Geometry* and Maxwell's *Geometry for Advanced Pupils* is towards clarity and precision without a formal framework. There may, however, be a good case for retaining it in solid geometry in order to show the basis of assumption on which a deductive proof rests.

There are still some teachers who would insist on a formal style for proofs of theorems and riders from the first or second year of the school course; others would prefer to evolve some simple formal style more gradually. Certainly one should not expect the same level of formality in the second year as in the fifth, neither for theorems nor riders, yet this seems commonly to be required if one may judge by text-book standards.

Most children can express themselves with adequate reasoning quality when they have something to express which they consciously realize, or when they feel clearly that there is a problem to be solved or a result to be demonstrated. Their expression will be ambiguous, roundabout and confused, but they are then in a better position to appreciate the mature contribution their fellows or their teacher can make towards clarifying their statements. In practice this means that expression of geometrical arguments must develop as a conscious process, as part of the pupil's self-education.

Although, as we have said, most teachers prefer, at some stage, a formal framework for the written solution of riders, yet there is some difference of opinion about the matter. Modern text-books wisely set many simple riders by presenting a diagram, e.g.

'In Fig. 3, B is the centre of the circle, and ED is equal to the radius. Show that $b = 3d$.'

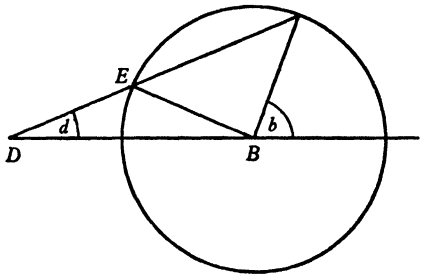


FIG. 3.

The pupil is presumably expected to copy the diagram and complete the argument. There seems little value in translating the data of the diagram and the requirements into words.

When the rider is set in the standard form, e.g. 'In a quadrilateral $ABCD$, $AB = CD$, and $\hat{B} = \hat{C}$, prove that AD is parallel to BC', the matter is not quite so straightforward. Many teachers would simply draw their diagram according to these instructions and would then evolve a proof. Only if copying out the data, they maintain, assists the pupil to evolve a proof does it seem profitable to spend time in this way. They would advise the pupil to try first the short way of attempting the solution from the book with the help of a marked diagram; if this fails he would be advised to write out, in a reflective and meaningful manner, the statement of the data and the requirements, hoping that some chord of association may thereby be aroused. Mechanical copying out is probably worthless.

It is perhaps worth remembering that there was once a fashion for 'copying out' the question in elementary arithmetic in the belief that this time-consuming process helped towards achieving a solution. Few teachers, if any, would tolerate such a routine today. A mechanical process, such as the copying out of the data may easily become, and which does not actively promote the solution of a problem, cannot be justified on other grounds. If the problem has been solved, there may be a case for the full expression, in writing, of the data, but only at a fairly late stage of the pupil's education when he can write out the solution rapidly.

Conclusion. It will be clear that pupils can attain some success in the solution of two-step and three-step riders and in the proper expression of their solutions. The construction of a deductive chain of theorems is a different matter psychologically, since it implies a more mature conception of 'proof'. Like all other abstract and fundamental notions in mathematics it is, for the majority, best approached gradually. The links in the chain are of unequal significance and one must attend to the major links first and so downwards to those which are least obvious and therefore logically more essential. At all stages the major assumptions need explicit mention—reference to a key-theorem or to some assumed condition for similarity or congruence or symmetry. The assumptions made at any stage must be such as are acceptable to the geometrical perception of the pupils; the great danger in mathematical teaching is premature formalization.

To encourage the conscious growth of a feeling for deductive argument in geometry it is wise to use initiative and imagination in class-teaching, and in setting examination questions to use freely such terms as 'deduce from' or 'assuming that'. For instance, 'Assuming the angle property of the isosceles triangle, but not that of the angle at the centre of the circle, deduce the properties of a cyclic quadrilateral'; 'Show that the SSS test for congruence follows from the SAS test': 'Compare three arguments for deducing the angle property of a triangle'; 'Apply the method of *reductio ad absurdum* to deduce the converse of the isosceles triangle theorem'; 'In the triangles ABC, XYZ, the angles A and X are equal; prove that the ratio of their areas is $AB.AC/XY.XZ$. Deduce that the internal bisector of the angle A of a triangle ABC divides the base in the ratio AB/AC'; 'Making use of the known properties of the rhombus, deduce constructions for (*a*) bisecting an angle, (*b*) the perpendicular bisector of a line, (*c*) the perpendicular from a point to a line'.

5. The Teaching of Area

Nearly all children entering a secondary school have been taught something about areas, but their understanding of the concepts and principles involved is usually so slight that it is invariably necessary to consider the topic afresh. This offers an excellent opportunity for enlightened teaching within the secondary school and may illustrate what might be called the important principle of 'It's never too late

to mend', a principle of constant application at all levels of mathematical teaching. (It may be remarked that the common assumption that a great mass of previous knowledge is necessary for further progress in learning mathematics is erroneous. It is in fact remarkable how little essential knowledge is required as a foundation for mathematical education and how easily deficiencies in earlier work may be remedied by skilful teaching.)

The traditional teaching of areas does, in fact, illustrate a cardinal defect of much mathematical teaching—the tendency to hurry on too quickly to 'sums' whilst ignoring concepts and realities. There was sufficient excuse for this in the days when doing sums was regarded as wholesome mental training, and in the primary school there is still unfortunately some excuse if the mathematical paper for secondary school selection still contains written calculations of areas. Though it is impossible to say that the teaching of areas in schools follows an agreed line of development there is every reason to suppose that it follows the lines of development in widely used school-books, and in these the emphasis from the start is on the area of the rectangle. Areas of irregular figures receive little consideration. We suggest that this order of ideas needs revision. Once again we have to assert the value of the psychological approach as against the logical.

In this approach it is essential to lead the pupil to discover for himself the main concepts and methods by reflecting on problems posed by his experience. Area is a kind of surface-size. One approach to the measurement of surface-size arises when we wish to find which of two surfaces is greater. It is more important to compare irregular areas than regular, since this emphasizes the notion of measuring areas by covering them with a suitable network. For example, on a map are two areas (lakes or woods, say). Which is larger? It may be possible to make tracings, and by fitting one tracing on top of another to settle the question without measurement. But if this is not decisive an obvious solution is to make tracings on squared paper and to count squares within the boundary. The notion that we find area-size by counting squares within a boundary is fundamental but tends to be ignored in the hurry ro rush on to the actual rules for calculating rectangular areas. (As a consequence it is not unusual to find secondary school pupils very doubtful whether an irregular figure has an area or not!) From this notion will be developed the

use of an appropriate unit of area and the mental estimation of areas in terms of appropriate units.

There is unlimited scope here for actual measurements, for construction of scale diagrams, for problems with data taken from local maps, with consequent unification of arithmetic and geometry. We wish to stress also the importance of constant mental estimation of areas, e.g. the area of a floor in square feet, the discovery of some area in the classroom representing a square yard. Later some emphasis should be given to estimating larger areas so that an acre has some kind of tangible meaning for the pupil. Such exercises should replace the conventional written calculations on reduction and conversion of units in area 'tables'. Compound measures of area seldom occur outside areas of estates of a limited size. Other areas are given in terms of an appropriate unit with fractions or decimals, e.g. 0·683 acres or 2·7 sq. miles. At a suitable later stage the discussion of the comparative accuracy of the digits in such a statement as 'area equals 0·683 acres' can lead to important ideas of degree of accuracy of results from measured data.

In the line of development suggested above, the rule for the calculation of the area of a rectangle appears at a comparatively late stage. Most pupils entering a secondary school have encountered this rule but if they have not they should be led to discover it and to appreciate its generality. There is usually no great difficulty about this: a far more stimulating exercise in teaching method is the discovery of the rule for the area of a triangle. Fully to realize the generality of this rule is a longer and more intricate matter of teaching than is usually supposed and there is consequently greater inducement to tell pupils things which they should think out for themselves if they are really to understand what they are about.

Amongst other matters which should receive attention at appropriate stages are:

(1) *The Independence of Area and Perimeter.* Pupils do not realize that a fixed area can be enclosed by widely differing perimeters or that a constant length can enclose a very small or a much larger area.

(2) *The Equivalence of Areas.* It is of mathematical importance to show that areas can be *compared* without actual measurement, and some discussion of the properties of parallelograms on the same base and between the same parallels may bring this point home. It

is an aspect of mathematical reasoning familiar to an older generation through Euclid Book I and II and may well be restored to the curriculum. Whilst we have insisted throughout this report on the paramount significance of the psychological as against the logical, for most children in our secondary schools, we would plead equally strongly for an insistence on, and an extension of, logical discussions provided they arise at a psychologically appropriate stage. If the historical evidence, and the evidence of teachers of experience, is of value, the logical treatment of areas, such as appears in Euclid's *Elements* can only be undertaken after pupils have acquired much direct experience of measuring, describing, and comparing areas. The abstract may arise from the concrete. But if they have this full experience and realization of the area concept it is stimulating to discuss the deeper logical points which have been discovered by mathematicians. This does not mean that the teacher expounds a sequence of theorems on area from Euclid or any of his rivals. But we suggest that he shows that areas may be equivalent, and may be studied and compared without calculation, and that calculation itself poses interesting problems. Some reference to incommensurables will be appropriate here as elsewhere in the course. Teachers who are interested in the possibilities of this logical development will no doubt need to refer to such modern authorities as Forder's *Euclidean Geometry*. Provided one begins in the middle, so to speak, and develops the logical theory upwards as well as downward, there are unrealized possibilities in the abstract treatment of school mathematics. For most people logic is best taken in small doses. For children in school a wholly impeccable logical treatment is impossible. A perpetual problem in mathematical education is to determine the appropriate logical dosage. We suggest that the logical aspects of area offer a field for investigation.

(3) *The Subtraction of Areas.* Young children are found to have curious ideas on the areas of rectangles from which inner rectangles have been removed, and it seems important to let children tackle in their own way problems on 'adding or subtracting areas' before solving these problems in more sophisticated ways. It is very easy, in dealing with these problems, to ignore the psychological twists which beset children and to hurry on to a formal calculation which is usually meaningless to them. All teachers are acquainted with the query 'And is that the answer?' at the end of some piece of

demonstration of this kind. It is a danger signal and suggests caution in dealing with these questions.

(4) *Importance for Algebra.* Many teachers, following Nunn, believe in the importance of the formula in the earliest stages of algebra. It is not easy, however, to find suitable rules for translation into formulae and amongst those which are convenient are the rules for calculating areas of rectangular figures. It is very desirable for these rules to be familiar to pupils some time before algebra is systematically considered. Perhaps the most satisfactory treatment is to use algebra informally to express these rules, e.g. to write Area $= l \times b$ or $A = l \times b$, at a convenient stage in the discussion of areas, and later to represent them with the conciseness of the algebraic notation.

Much use is now made of area diagrams for illustrating algebraic formulae and identities and we think this might be extended. For instance in 'completing the square' for solving a quadratic equation it is very helpful to insist on the drawing of some actual squares in a geometrical diagram. It may help, certainly in blackboard demonstration, invariably to shade in areas when a diagram is concerned with equality or comparison of areas. One of the eternal difficulties in geometry is the perception of what is relevant in a given diagram, and shading, and using coloured chalks, can help when areas are to be considered.

(5) *Area and Volume.* It is usual in text-books for exercises on volume to be taken immediately after those on area, but there seems to be no justification for this as the complexity of three dimensions is far greater psychologically than it is logically. We should suggest a more complete separation of the two concepts of area and volume; the approach to volume could follow lines similar to that for areas. In teaching problems about volume (or capacity) it is both necessary and easy to restrict ourselves to problems and units that are in everyday use and to exclude those which survive only in books of tables. Perhaps the ultimate ideal in this work is the solution by the pupil of some problem in which he has had to make all the necessary measurements or to find all necessary data himself, e.g. to find the cost of laying a cement floor or of planting a field or the weight of a metal casting or the capacity of a reservoir.

(6) *Metric Units.* Respect for the metric system of units should not carry us so far that we teach to our children units of area and

volume which are unfamiliar on the Continent. It is most important to have a firm grasp of one or two simple equivalents, e.g. to know that a litre is approximately $1\frac{3}{4}$ pints and a kilogram approximately $2\frac{1}{5}$ lb.

(7) *Calculation of Areas in Secondary Selection Tests.* Many secondary grammar school teachers feel that exercises on the formal conversion of area units or the calculation of areas of rectangular figures are undesirable in secondary selection tests since they often lend themselves to undesirable cramming, while the principles involved are not understood. We support this view though we insist on the value of a careful, unhurried, and psychological approach to the mensuration of surfaces, which may best be secured by teaching at intervals over a period of some years in the real and active atmosphere of a well-ordered primary school. In other words, let the primary school teach whatever can be learnt about this aspect of mathematics, but not set formal examinations on it. There are better means of assessing children's abilities and the formal treatment may best be given later when adequate foundations have been laid.

6. Three-Dimensional Geometry

Here we put forward some proposals for extending and improving the teaching of three-dimensional geometry. Proposals of this kind are not unfamiliar. The arguments justifying them have been set out in reports of the Ministry of Education and of the Mathematical Association: we need not repeat these arguments. There may, however, be some gain in offering practical suggestions, particularly as the influence of the Euclidean tradition (still powerful amongst us) does not encourage the teaching of solid geometry. In this tradition, or at any rate in that form of it usually encountered in schools, a full course of plane geometry and a long sequence of preliminary theorems precede any genuine advance into a third dimension. It is only fair to point out that the finest teaching of Euclid's *Elements* in the nineteenth century did include much more solid geometry than is usually taught today. The late F. W. Westaway in his book *Craftsmanship in the Teaching of Mathematics* (1933) observes, in the course of a chapter on the polyhedra: 'In my day we taught the interrelations of all nine solids.' Exceptional teaching such as this could scarcely determine the general tradition: the substance of

school geometry became two-dimensional and has largely so remained. This judgment may appear pessimistic, but a glance at school-books in geometry and at examination syllabuses supports it. Few school-books attempt to combine solid geometry with plane, and although there are excellent separate volumes on solid geometry they are little used. Examination syllabuses do not show all that is taught but they have a profound influence on teaching. Even today the teacher who has to prepare children for the examination in Ordinary Level mathematics of the G.C.E. will not be encouraged to teach three-dimensional geometry by the questions he finds in papers on geometry, though questions involving solid geometry occur in papers in arithmetic and trigonometry.

Perhaps it is not out of place to suggest a simple immediate remedy for this state of affairs. Useful and interesting results in solid geometry may be reached by commonsense reasoning: very few major theorems are essential. In the *First Report of the Mathematical Association* (1925) on 'The Teaching of Geometry', in which the context makes it clear that 'results' does not mean theorems, there is the recommendation on p. 24 that 'results' in two dimensions 'should be extended into three', and in the second Report (1938) there is a collection of easy riders in solid geometry associated with the corresponding two-dimensional results. Nor should the syllabuses and papers of the Civil Service Commissioners be overlooked. The Commissioners have in the past held an enlightened view of mathematics, including in their syllabuses the elements of descriptive geometry ('the solution of problems by the method of Monge'). Their practical exercises, to be solved by drawing and calculation, are admirable. Finally, a rich store of three-dimensional material is available in most continental text-books. It is clear that a little enterprise on the part of examiners and text-book writers could easily remedy the inadequate treatment of three-dimensional geometry in ordinary mathematical examinations and text-books.

Fortunately, a more encouraging treatment of solid geometry is possible for teachers who arrange their work to cover the alternative syllabuses in mathematics, drawn up in 1944 by the Conference of Examining Bodies and outlined in its Report—the Jeffery Report. These syllabuses require knowledge of solid forms, ability to solve three-dimensional problems by analysis into plane figures, and the construction of plans and elevations. But the improvement of

mathematical education was not the sole reason behind the movement for alternative syllabuses. Another reason, powerful in those desperate days, was the desire to replace the trivial and purposeless material in school mathematics by applications more related to modern conditions. Inevitably this implied the inclusion of mathematics hitherto associated with technical schools and colleges. For deep social and cultural reasons it is not easy for specialist mathematical teachers in grammar schools to interpret this material satisfactorily. Whilst many of them speak enthusiastically of their experience in teaching the alternative syllabus many confess that they do not know 'how to teach plans and elevations'. They feel that it is not enough simply to expound the rules of the conventional engineering drawing text-book. To meet this difficulty it is necessary to consider what seem to be the only two alternatives to the quasi-Euclidean geometry course now followed in schools.

The first alternative was evolved by Mr W. C. Fletcher, from continental sources, and is fully expounded in the *Second Report of the Mathematical Association* (1938). It proposes a complete reversal of order. The study of space should begin with solid forms. Models should be constructed and two-dimensional geometry should arise from plane sections of these. At least one school-book has shown how solid geometry may be taken at the beginning of the school course, but no book, so far as we know, has fully explored the possibilities of this new course. Whilst approving this course in principle we believe that some teachers could approach it, in practice, in a piecemeal way by working out for themselves certain three-dimensional topics. This is the second alternative. Some of these topics arranged in order of development are:

(1) The construction of models, including those representing common objects, with some polyhedra, regular and semi-regular.

(2) The representation of solid bodies in two dimensions, including notions of perspective drawing, of plans and elevations, of shadows and sections of simple solids.

(3) The geometry of the sphere with applications from map projections and astronomy.

Any elaboration of these suggestions is only possible if we emphasize some points of method which apply throughout:

(*a*) The most effective treatment of any of these topics will be through a series of problems. These may be selected, or constructed,

so that whilst the earliest of them can be solved by the pupil using common sense, the succeeding problems introduce, one by one, new difficulties, and their solution will require the joint efforts of pupils and teacher. In solving these problems the necessary techniques or principles will be evolved. This order of procedure is the reverse of that usually assumed in text-books on applications of geometry, most of which enunciate rules and principles, illustrate them abstractly and then apply them to concrete problems. Naturally the construction of a graded course of problem material calls for some reflection on the part of the teacher but this is overwhelmingly repaid by the increase in attention and in effective learning by his pupils. For suitable material one must consult a wide variety of books—on architectural drawing, applied geometry, polyhedra, perspective and so on.

(*b*) Some elementary problems from these topics can be taken at almost any stage in a geometry course, and it is always better to spread out the study of a topic over two or three years than to take it *en bloc* towards the end of the course. The representation of solid bodies by plane drawings, including representation in plan and elevation, is a case in point. Exercises on drawing elevations of simple solids may first be taken—preferably with models, constructed by the pupil, in view. The drawing of plans may follow. Next the combination of drawings of plans and of elevations on the same diagram, using special devices, can be evolved. At each stage a series of exercises can be devised by tilting or turning a model. A corresponding treatment of accuracy is possible. In the earliest diagrams the emphasis will be on interpretation, on appreciating the solid through plane sections. Diagrams may be drawn on squared paper, or using set squares. There is a confusion of aim if the standards of precision of the engineering draughtsman are attempted too soon. The diagram is for the engineer only a means to an end —the construction of the solid—and this end is rarely possible of attainment in a mathematics course. But towards the end of this work it is desirable for pupils to have some experience in the use of drawing boards and T-squares without aiming at a vocational treatment of geometrical and mechanical drawing. Perhaps the drawing of sections or of shadows may be useful alternatives to the technical exercises in some books on workshop drawing.

(*c*) Each of these topics may be extended into fields of mathematics, indefinite in extent, but sadly neglected in most schools. The

construction of models, which may begin as in Westaway's book, and continue through the study of polyhedra in Rouse Ball's *Mathematical Recreations and Essays*, could lead to the immense field of knowledge represented by Coxeter's treatise on *Polytopes*. The representation of solids in two dimensions leads to descriptive geometry and to the projective geometry of the nineteenth century from which the more abstract developments of the twentieth century may be approached. Finally, the geometry of the sphere is a useful preliminary to the study of non-Euclidean systems. We suggest that, by exploring these topics along the lines we have indicated, teachers may be led to a better appreciation of three-dimensional geometry, from which it is but a short step to the adoption of the proposal of the Mathematical Association's Second Report whereby three-dimensional geometry becomes the starting point for the entire subject.

(*d*) The need for studying three-dimensional geometry is greater today than ever it was. All pupils who study mathematics or sciences beyond the Ordinary Level in the G.C.E. need training in visualizing three-dimensional figures and their sections; so do those who become technicians in building or engineering. Most pupils will need to read maps, plans and elevations when in National Service or when arranging their homes and gardens. This need is not adequately met by courses in 'Geometrical and Machine Drawing'. Elementary courses in descriptive geometry would be easier to justify and we feel that these may emerge from the suggestions we have proposed.

At present we are not in a position to offer a scheme of work beyond these suggestions: few teachers of mathematics have at the moment the experience from which to devise a teaching syllabus. Experience is best acquired by discussing suitable problems—for instance, given the plan and elevation of a roof, find the true lengths and inclinations of different members; construct vertical, horizontal and oblique sections—and by consulting teachers of woodwork and handicraft. There is plenty of technical literature available but it will be necessary to select from it and to guard against developing technical skill at the expense of mathematical insight.

The treatment of formal solid geometry deserves a few comments. Fifty years ago the subject was studied as Euclid Book XI and for most pupils this was a succession of minor theorems with a few results of mathematical value. The aim was logical thinking, not

insight into spatial relations. More attention was given to structure than to substance. So long as Euclid was the authority it was possible to ask for proofs in public examinations and to maintain a definite standard of argument. The movement away from Euclid, the inclusion of new subjects in the curriculum and the need to relate mathematics to the interests and future requirements of pupils, have completely altered this position. For some years there has been little teaching of formal solid geometry in the former sense of these words.

Most teachers now show the more important relationships of solid figures using deduction from a wide basis of intuition. They may expound perhaps half-a-dozen results as 'theorems'—for instance those about the properties of tetrahedra, of perpendiculars to planes or to skew lines—and they would encourage good diagrams and the drawing of sections and elevations. Occasionally a proof of some theorem is required in a public examination but the marking of such proofs is difficult since there is no agreement about the theoretical basis and sequence. The valuable elements in solid geometry are now taught earlier and the formal treatment of the subject is crowded out of sixth-form work by other topics. For those who require a formal treatment there are modern books by Forder and Lines. Some teachers with a taste for rigorous thinking will feel time well spent in showing how solid geometry can be built up logically; perhaps there is need to give some attention to work of this kind with mature pupils. Unfortunately there are many claimants for time at this stage, and it is arguable that logical sequence and rigorous thinking are best conveyed by other topics—e.g. modern algebra. Also the multiplicity of subsidiary theorems, and thus the time required, make the teaching of formal solid geometry not attractive to many teachers: they prefer to spend their time on the applications and the interesting developments of the subject, and to take the early stages of the formal development 'as read'.

It is not likely that agreement could be reached over a more formal treatment of a sequence of theorems in solid geometry in public examinations: the necessary theoretical sequence could not easily be drawn up; there would probably be more support for increasing the attention given to riders. Perhaps the increasing interest in the wider aspects of solid geometry will suggest to examiners that a wider range of riders and exercises would be useful additions to present geometry syllabuses of public examinations.

7. Calculus in the Fifth Year

The suggestion that calculus should be introduced to pupils below the sixth form level was first made many years ago. In practice, however, until recently, any such teaching of calculus was usually confined to a few abler pupils destined to specialize in mathematics or science. The publication of the Jeffery Report, and the subsequent introduction of calculus by a number of examination authorities in their Alternative Syllabuses at the Ordinary Level, has had the effect of stimulating the teaching of calculus more generally to fifth form pupils. We anticipate that this practice will grow in time; and this is all to the good. The methods of calculus are so powerful, and of such practical application, that it is regrettable that the majority of pupils now leaving grammar schools should be completely ignorant of this branch of mathematics. If it is well presented, calculus can be made interesting to the majority of pupils in their fifth school year, and the time devoted to its study will be profitably spent.

The reader is urged, if he has not already done so, to study the report on *The Teaching of Calculus in Schools*, issued by the Mathematical Association. It is not the purpose of this article to replace the report in any respect, but rather to indicate certain methods of approach in teaching the early stages of calculus to fifth form pupils of no particular mathematical ability.

Of all branches of mathematics calculus requires the most careful introduction and development. Its ideas are so novel that there must be no attempt to rush the early stages, and it must be emphasized that the initial approach should not be stereotyped, but varied according to the capabilities of the pupils being taught. With a good class not only will the progress be quicker, but the calculus notation can be introduced at an earlier stage than with others. Unfortunately the intrinsic difficulties of calculus are met at the start, and it is impossible to deal with them in any rigorous sense at this level. For example, although some idea of the meaning of a 'limit' is essential, we cannot attempt any study of the theory of limits. On the other hand we must, at all costs, avoid letting pupils perform operations of differentiation and integration mechanically, without any understanding of what they are doing. It should be our aim

to develop the subject so as to give them, at any rate, an intelligent comprehension of the processes they are using.

Preliminary Work. It is unlikely, at this stage, that any preliminary co-ordinate geometry, in the general sense, will have been done. The background of graphical work acquired earlier in the school will, however, be helpful. We should expect pupils to be familiar with the idea of the equation of a curve, e.g. $y = x^2$. They should be able to calculate the y co-ordinate, given the x co-ordinate, in simple cases, and vice versa; and also be able to determine whether a point with known co-ordinates does, or does not, lie on a curve with a given equation. It would be helpful if they knew the forms of the graphs of $y = ax+b$; $y = x^2$; $y = x^3$; $y = 1/x$; and could draw freehand sketches of graphs of simple equations such as $y = x(3-x)$. There will indeed be opportunities throughout the calculus course to practice the drawing of freehand graphs.

Introduction of the Ideas of Calculus. The fact that calculus is so different from the other branches of mathematics makes it all the more necessary to base its introduction upon ideas which are already familiar. For this reason there is much to be said for the kinematical approach to the subject. From their previous work pupils will already be aware that the distance-time graph of a particle moving with constant velocity is a straight line, and that the speed is represented by the gradient of this line. Now let us suppose that a marble rolls down an inclined plane. It was first shown by Galileo (and his experiment can easily be repeated) that its distance (s ft.) from

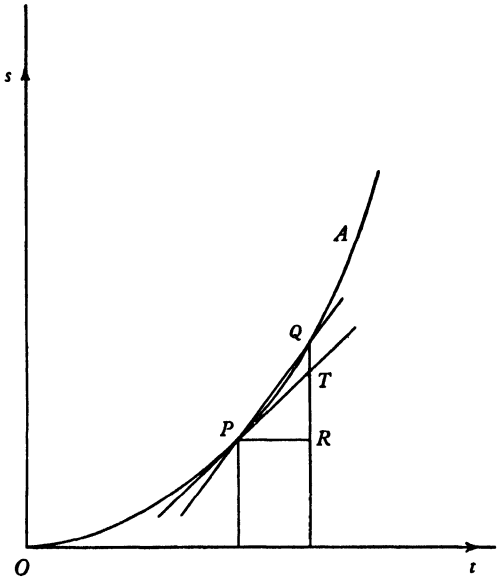


FIG. 4.

the starting point varies as the square of the time (t secs.) taken. We will assume that the plane is so adjusted that $s = t^2$. If s is plotted against t the curve obtained is a part OA of a parabola, as drawn. From the graph the average speed of the ball down the plane during the interval of time represented by PR is given by the ratio QR/PR, i.e. by the gradient of the secant PQ.

We may now discuss what is meant by the speed at a particular instant. How can we, for example, find the instantaneous speed of a motor car? In practice we obtain an approximation to this speed by finding the average speed over the smallest distance for which the time can be measured with a sufficient degree of accuracy.

From the graphical point of view the closer Q is taken to P the more nearly does the average speed, represented by the gradient of PQ, approach the actual speed at P. But as Q approaches P, so does the secant PQ approach the tangent PT at P. This suggests that the gradient of the tangent gives us the speed at the time corresponding to P on the graph.

Now let us take the time at P to be 1 sec. and work out the average speeds for diminishing intervals of time t. (As $s = t^2$ it is more convenient to do this than to take diminishing intervals of distances s).

For $t =$ 1,	1·1,	1·01,	1·001,	1·0001.....
$s =$ 1,	1·21,	1·0201,	1·002001,	1·00020001.....

and the average speeds in the intervals

0·1,	0·01,	0·001,	0·0001 seconds after 1 sec. are
2·1,	2·01,	2·001,	2·0001 ft. per sec.

Similarly the average speeds in the same intervals of time before 1 sec. are

1·9,	1·99,	1·999,	1·9999 ft. per sec.

These results indicate that the smaller the time interval before, or after, 1 sec., the more closely does the average speed approach 2 ft. per sec. and that, although we cannot speak of average speed over no time interval, the sequence of the results does suggest that we may speak of the speed at the time 1 sec. as 2 ft. per sec. In this way the idea of a limit may be introduced in terms with which pupils are not unfamiliar.

Some practice should be given in finding speeds from the gradients of tangents, and by calculation, as above. The method may then be generalized. Let s be the distance of the marble at time t, and let it

travel a distance k after a further time h. The total distance $s+k$ is then connected with the total time $t+h$ by the relationship

$$s+k = (t+h)^2$$

whence (since $s = t^2$) the average speed, which $= k/h = 2t+h$, can be obtained.

Introduction to Calculus Notation. We may now turn our attention to the general problem of finding the gradient at a point on the curve $y = f(x)$. Here there is some diversity of opinion as to the calculus notation to be used. If we use the incremental notation we should proceed thus:

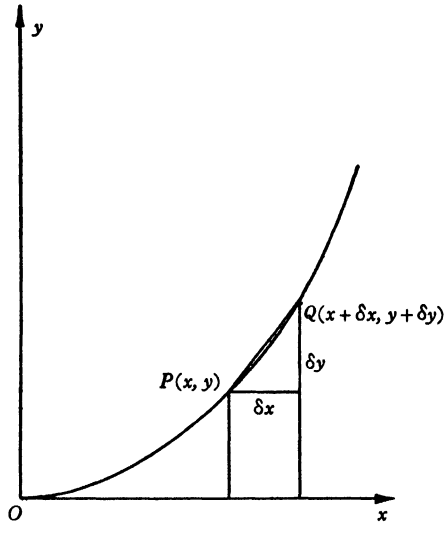


FIG. 5.

Given $y = x^2$, let P be any point (x, y) on the curve of this equation and let Q be a near point $(x+\delta x,\ y+\delta y)$ on the curve, where δx is a small increment in x and δy the corresponding increment in y. Then since Q is a point on the curve

$$y+\delta y = (x+\delta x)^2$$

whence

$$\frac{\delta y}{\delta x} = 2x+\delta x.$$

As $\delta x \to 0$, $\quad 2x+\delta x \to 2x$.

This gives us the gradient of the tangent, and hence of the curve, at P and we will write this result $\dfrac{dy}{dx} = 2x$.

In this notation it must be understood that $\dfrac{dy}{dx}$ is not a ratio in the sense of $\dfrac{\delta y}{\delta x}$, but is a calculus notation for the gradient of the curve

at a point. The result might also be written $\frac{d}{dx}(x^2) = 2x$, where $2x$ is the gradient of the function x.[2]

There is a strong body of opinion that the difficulties of the above notation can be avoided by using no notation except D in the sense of 'derivative of'. Thus $D(x^2) = 2x$, or, if $y = x^2$, $Dy = 2x$. Others again prefer to proceed without introducing any special calculus notation at all. For a fuller discussion of the problem of notation the reader is referred to the calculus report mentioned above. We shall adopt the incremental notation in the remainder of this article, although we recognize the points of view of those who hold other opinions.

The gradients of such simple functions as x^3, x^4, cx, $1/x$ and c (c being a constant) can now be obtained from first principles, leading to the formula $\frac{d(x^n)}{dx} = nx^{n-1}$ for positive and negative integral values of n. Anything in the nature of a general proof of this formula, even for n a positive integer, is probably out of place at this stage. Practice should be given in finding the gradients of simple functions such as $3x^2-2x+5$; $3/x$; $(2x+1)(x^2-2)$; $\frac{1+x^2}{x}$.

The teacher can now, at his discretion, introduce such alternative terms and notations as are appropriate to the development of calculus, e.g. differentiation, differential coefficient, derivative, $f'(x)$. Applications of differentiation to maxima and minima, kinematics, and possibly to finding the equation of a tangent to a curve may follow, or the teacher may prefer to defer these until he has dealt with integration as the reverse of the procedure of differentiating. Merely for convenience of presentation we will follow the second course.

Integration. No new notation is here necessary or advisable. It should be made quite clear, however, that an equation such as $\frac{dy}{dx} = 2x$ represents a family of curves, all having the same gradients for given values of x, and that in this example the family has the general equation $y = x^2+c$ where the value of c depends upon the particular curve of the family which is required. It is not difficult for pupils to understand this diagrammatically, but considerable and

varied practice will be necessary to obtain c from initial sets of conditions, e.g. a point through which the curve passes.

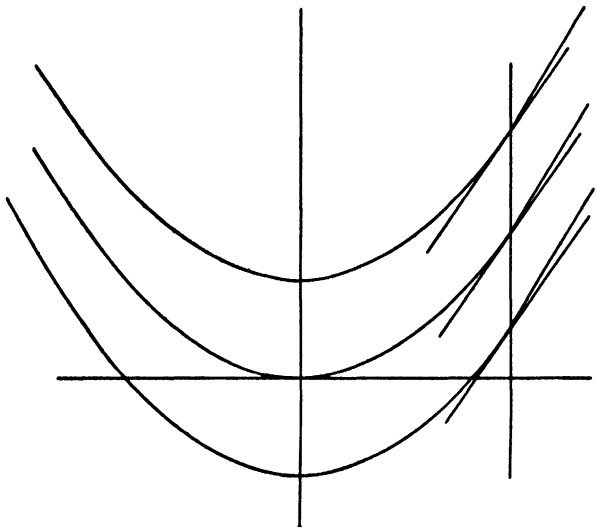


FIG. 6.

Applications

(i) *Kinematics* can again be helpful in giving reality to processes of differentiation and integration. Starting with $s = f(t)$ we can derive velocity, and, by differentiating again, acceleration. (Incidentally there is interest in finding the second differential, and in giving it a meaning, even if it is unnecessary at this stage.) The problem of obtaining s given $v = f(t)$, and initial values for t and s, assists in clarifying the idea of the constant of integration.

(ii) *Maxima and minima.* Finding maximum and minimum values of functions furnishes some of the most interesting applications of calculus and can be taken quite early in the course.

The introduction should be through simple specific problems, e.g. 'A rectangular enclosure is formed against a wall, the other three sides being formed by fencing of total length 240 yd.; find the greatest possible area enclosed.' Graphs of the resulting equations are drawn and solutions to the problems are first obtained from these graphs. A consideration of the gradients of the curves shows that at a maximum or a minimum point the gradient is zero, and

that it changes from positive to negative in going from left to right of a maximum value; and vice versa for a minimum value. At this stage the discrimination between maximum and minimum values of a function $f(x)$ should be made by considering the change in sign of $f'(x)$, and not by the sign of $f''(x)$ at the turning point. When

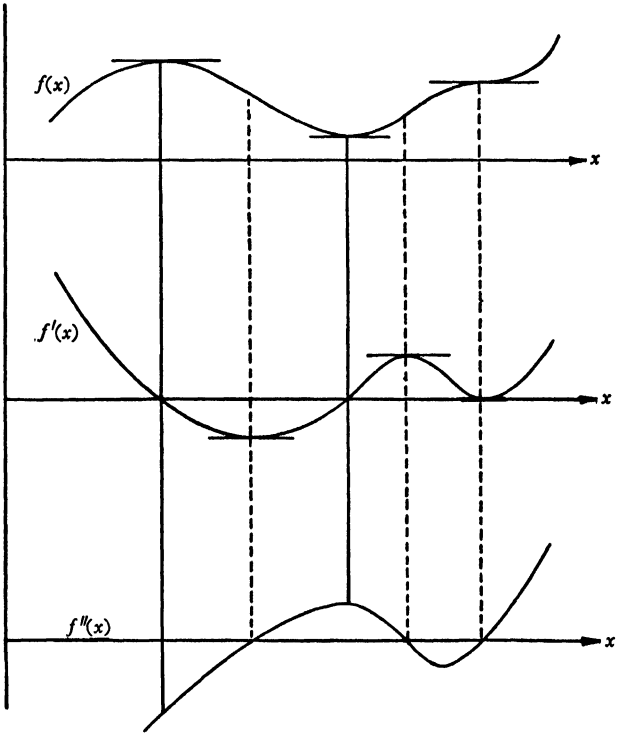


FIG. 7.

working out problems the nature of the result may indicate quite clearly whether it is a maximum or a minimum without need of a further test. Simple problems in which one variable is expressed explicitly in terms of another should cause little difficulty. Confusion often arises, however, when the dependent variable is expressed in terms of two others which are related to each other e.g. in finding

the maximum volume of a rectangular box on a square base having been given the sum of the areas of its four faces and base. Considerable practice is necessary to obtain facility in solving problems of this type.

Although detailed consideration of points of inflexion may be out of place, it is worth while inserting a point of inflexion at which the gradient is zero to emphasize that stationary values of $f(x)$ are not necessarily maxima or minima, and also because such a point of inflexion will already have been seen in the curve $y = x^3$. It is instructive to trace the changes in the values of $f'(x)$ in a curve corresponding to $f(x)$, and possibly in the curve of $f''(x)$ as well.

(iii) *Area under a curve and volumes of revolution.* No new notation is needed in finding the area under a curve. Let LM be part of the curve $y = f(x)$ above the x-axis. We require to find the area between this curve, the x-axis and two ordinates LL', MM'.

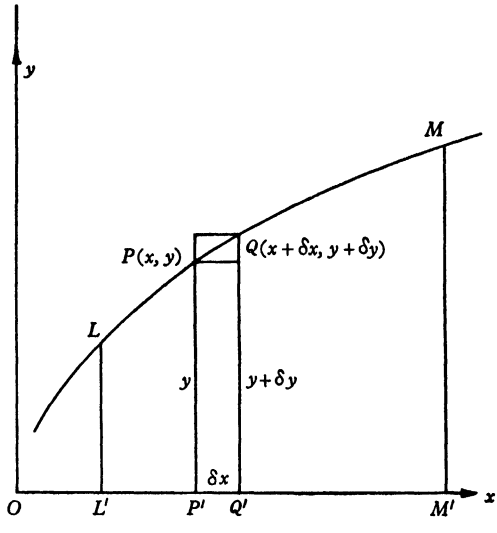


FIG. 8.

Imagine a moving ordinate PP' to start at LL' and move towards MM', (in the direction of x increasing) covering an area A as it does so. For any position (x, y), of P, let $Q(x+\delta x, y+\delta y)$ be a point on the curve close to P. Then the small increment of area δA between PP' and QQ' is such that, in our diagram,

$$y\,\delta x < \delta A < (y+\delta y)\,\delta x$$

or

$$y < \frac{\delta A}{\delta x} < y+\delta y.$$

Then when

$$\delta x \to 0, \quad \frac{dA}{dx} = y.$$

Now let us apply this formula to a simple numerical problem, for example, to find the area between the curve $y = 12x - 3x^2$ the ordinates $x = 1$ and $x = 4$ and the x-axis.

We have $\frac{dA}{dx} = 12x - 3x^2$.

Therefore $A = 6x^2 - x^3 + C$.

When $x = 1$, $A = 0$. $\therefore C = -5$.

$\therefore$ $A = 6x^2 - x^3 - 5$.

When $x = 4$, $A = 96 - 64 - 5 = 27$.

The evaluation of the constant of integration C in this method links up with the work already done.

Volumes of revolution can be treated similarly.

With a good class the teacher may naturally prefer to introduce the integral symbol, and the idea of a definite integral, before proceeding to find areas and volumes of revolution. It is doubtful, however, if it is advisable in this introductory stage to consider a definite integral as the limit of a summation. How far the teacher can proceed in considering the sign of an area and in calculating areas enclosed between a line and a curve, or between two curves, will depend upon the abilities of his class and the time available.

8. Surveying, Meteorology, Astronomy and Navigation

The mathematics teacher who is alive to the possibilities of his subject will always be on the lookout for material with which to illustrate and illuminate his lessons. There is a wide field of such material to be found in the work of the physics and geography departments, both separately and jointly. Close collaboration and correlation will, therefore, be necessary with both departments.

We have referred elsewhere to the joint responsibility of the mathematician and the physicist for the teaching of mechanics and have discussed the ways in which collaboration and correlation can be effective. We also deal at some length with the teaching of statistics and note there how the study and intelligent use of this branch of mathematics can illuminate the work in all science subjects. But both these branches are essentially mathematical studies in the sense that it has become established practice to design

at school level separate courses of instruction and to test the student's competence therein both by internal and external mathematics examinations. We propose now to refer to subjects which are not usually classed as mathematical studies in the above sense, and we have in mind surveying, in which the mathematician shares his interest with the geographer, though neither would claim direct responsibility for its teaching, and meteorology, astronomy, and navigation, in which the mathematician, the physicist and the geographer all share a common interest, but for which none would claim to be exclusively accountable.

Before proceeding, however, to say what can be done in these subjects in ordinary teaching time, we must recall that opportunities for such studies often occur in out-of-school activities. If there is a school mathematical society, even if it functions only as a branch of a larger scientific society, the subject-matter for many interesting papers can be found within the field of surveying, meteorology, astronomy and navigation. Moreover, the school scout troop or girl guide company and the school cadet force (army, navy and air force branches) may also be engaged on studies that fall within this field. Such studies can and should be co-ordinated with the work of the mathematical department and, if there is a mathematics room, should centre upon it.

Surveying. The concepts of direction, scale and relief, which are fundamental to the production of a map, are also fundamental to scale drawing in plan and elevation. In elementary work on these matters it is highly desirable that the mathematics teacher should consult and work in conjunction with the geography teacher: thus much waste of time by overlapping, not to speak of confusion in the mind of the pupil, may be avoided.

The points of the compass are learnt; the meridian can then be found by reference to the sun's shortest shadow or to the pole star. The introduction of the magnetic compass arouses the pupil's interest when he finds that the compass needle does not normally point to the true north; the ideas of compass bearing and true bearing and their use in fixing positions then follow.

The notion of scale will be introduced by practical work within the classroom, but it will generally be found that only when a pupil has completed a map or plan from his own measurements does he begin to understand what a map really means. It will, therefore, be

advantageous to do a piece of elementary practical surveying. The preparation of a simple plan of the school field by the chain and staff or by the plane table method will enable the pupils to appreciate something of the accuracy of working and attention to detail which are involved in the preparation of a map. Failing this, a plan of the classroom or of the school building may be prepared. In either case the apparatus in use and measurements obtained therefrom will have to be sufficiently accurate to give convincing results. This provides an opportunity for a discussion on degree of accuracy of measurement and the surveyor's way of distributing the closing error in a polygon. It is important that the completed plan should be used for the measurement of hitherto unknown distances and directions, thus giving a first introduction to the interpretation of maps and plans.

The concept of relief in the map and of angles of elevation and depression and front and side elevation in drawing are different aspects of the same problem—the representation of heights and distances in a vertical plane and the calculation of distances and slopes in that plane. The use of a theodolite and chain in the calculation of unknown heights and distances by drawing and measurement can often prevent work in the early stages from becoming purely academic, besides providing the pupil with a much better introduction to the fundamental ideas of numerical trigonometry.

At a more advanced level the study of map projections as part of the general topic of the representation of three dimensions in two is full of mathematical possibilities. The decision whether or not to undertake further work must remain an individual one, but we can say that where opportunity occurs for it and time permits, some very interesting exercises can be done in the use of the prismatic compass with chain or tape traverse, plane tabling, clinometer reading, levelling, simple contouring and possibly, at sixth form level, the general principles of triangulation, intersection and resection. More advanced work should only be attempted if the teacher himself has an enthusiasm for it and at times such as the last few weeks of the summer term when deviations from the ordinary school time-table may more easily be arranged.

Meteorology. There are few schools, whether in town or country, that cannot afford to establish a simple meteorological station, provided at least with thermometer, barometer and rain-gauge,

where weather phenomena may be observed and a diary kept. In this way a detailed account of the year's weather can be obtained, and it is interesting to compare results with the day-to-day observations to be found in local or other official records. The setting up of such a station and the arrangements for the collection of data will of course be the province of the geography department. Similarly the construction, understanding and use of the various instruments is taught by the physics department. But instruction in the various ways of representing the climatic data is a matter for the mathematics department. Facility in the speedy and accurate construction of a rainfall or temperature graph is a valuable aid, if not a necessity, to a pupil in his later work in geography, e.g. in illustrating the division of a country or continent into its climatic regions. Moreover, mean weekly or monthly temperatures and rainfall can be worked out and the results tabulated and graphed. This practical work can serve as an introduction not only to graphical work in general but also to the calculation of means and the beginnings of statistics.

Apart from the simple treatment outlined above, meteorology as a school subject is almost non-existent. The geographer does not appear to develop the subject further and it is left to the advanced physicist to consider the following matters: the structure of the atmosphere and vertical and horizontal distribution of temperature and wind; hygrometry; formation of clouds, fog, dew and hoar frost; meteorological optics; mirage, rainbow, haloes, coronae, colour of the sky; lightning and thunderstorms. The mathematical student cannot afford to dissociate himself from this study as he may very well take up a career as a meteorologist on leaving school.

Astronomy and Navigation. Elementary work in geography frequently includes the consideration of night and day, longitude, latitude and time, and the study of the motions of the earth and their effects. It is not possible to go far into these matters until the pupil has some acquaintance with the geometrical idea of parallel lines and some knowledge of the geometry of the circle and tangents to it. Indeed, though the geographer may introduce the idea of the world as a globe it is the mathematician who can best develop the ideas of latitude and longitude and circular movement on the earth's surface as part of his syllabus.

But provided that lessons are in progress at noon the latitude of the school can be determined, within a small degree of error, with

no more complicated apparatus than an upright pole and a measuring tape, the declination of the sun, of course, being given. This simple exercise has great practical value in clarifying and consolidating the ideas propounded in the more abstract atmosphere of the classroom. But the very mention of declination places the earth in its wider background of the universe and any explanation of it demands some study of astronomy.

Astronomy has recently been introduced into the examination syllabuses for general science at Ordinary Level. Topics to be investigated include: apparent diurnal motion of sun, moon, and stars; rotation of earth, time and longitude; the moon, its motion, phases, eclipses of sun and moon; tides and their connection with the sun and moon; apparent annual solar motion; earth as a planet; 'tilt' of axis; seasons; length of day in relation to latitude; direction between fixed stars and planets based on weekly observation of position of planet; solar system in outline; characteristics of the major planets; Newton's law of gravitation; comets and meteors; the sun as nearest star, sunspots; recognition of Great Bear, Orion, Cassiopeia, Pleiades, Pole Star, Sirius, Aldebaran, Arcturus, Vega. Simple astronomical observations such as the determination of the sun's midday altitude are possible in school time and the results can be compared with those given in some standard publication such as *Whitaker's Almanack*. The advanced student of physics may investigate the work of gravitation more fully and quantitatively, tracing the work of Copernicus, Kepler and Newton and the subsequent determination of the gravitational constant and its relation to gravitational acceleration. Much of the later work is mathematical and relates very closely to the study of mechanics at sixth form level, but the student of mathematics should take every opportunity to master the qualitative and descriptive parts of astronomy for the sake of the background it gives to his subject. Similarly the teacher of mathematics can and should use the subject-matter of astronomy to illustrate and illuminate his ordinary classroom work.

The phenomenal development of aerial transport and aerial warfare in recent years has made astro-navigation of such importance that a knowledge of elementary astronomy and its application to navigation will be as important to the next generation as knowledge of the 'Highway Code' has been to the present one. Sea navigation has long been one of the more important practical applications of

the geometry of the sphere, and has provided the mathematician with illustrative material. Perhaps it is not surprising therefore that navigation may now be studied as a separate subject at the Ordinary Level under the auspices of the Oxford and Cambridge Joint Board and that the syllabus is 'framed to be, in the main, a suitable introductory course in sea navigation or in air navigation'. It seems that this syllabus contains all that could usefully be studied at school in both astronomy and navigation and for this reason may be quoted in full:

Section I. Geometry of the sphere. Great circle. Small circle. Application to the Earth. Departure. 'Parallel-sailing'; proof and use of formula Dept. = Diff. Long. × cos (Lat.). Direction on the Earth's surface. The magnetic compass. Variation. Deviation. Use of a deviation table or curve. Relative bearings. Rhumb lines. Construction and properties of the Mercator chart. Dead reckoning. Finding the position of a ship or aircraft from observations of terrestrial objects and W/T bearings. Running fix. Application of triangle of velocities to the simpler problems. Rhumb-line formulae and their use. Use of the cosine-formula of spherical trigonometry (proof not required) in finding great circle distances and in Section II below.

Section II. The celestial sphere. Altitude and azimuth, Hour angle and declination. The first point of Aries. Sidereal Time. Mean time. Zone (or standard) time. Right ascension and sidereal hour angle. Relation between G.H.A. and Local H.A. Almanacs. Finding the local hour angle of a heavenly body. Correction of observed altitudes for refraction, parallax, and semi-diameter (descriptive only); the theory of the sextant will not be required. Theory of the astronomical position line. Determination of position at sea and in the air. (Candidates will not be expected in the paper to work out a position line in detail but it will be assumed that they are familiar with the general principles.) Sunrise and sunset. Twilight. The moon's phases. General description of the sun, moon, and planets. Constellations and the magnitudes of the stars (descriptive only).

9. Correlating Mathematics with other Subjects

The Meaning of Correlation. The word 'correlation' in education is usually taken to mean the teaching of different subjects in association with one another so that repetition and contradiction are avoided and so that knowledge is revealed as a unity and not as a bundle of separate 'subjects'. Thus, by some obvious correlation it is easy to avoid detailed exposition of the metric system in both physics and

mathematics lessons; by some less simple process it is possible to associate the teaching of mathematics and geography so that both subjects combine fruitfully in the study of latitude and longitude, map projections or the tides. Under present circumstances in which specialist teachers deal with their own subjects correlation is not easily effected, whilst the need for it is daily more apparent.

The Ideal Conditions. The ideal conditions for correlation occur when a single teacher, of enlightened mind and with sufficient knowledge, deals with two or more related subjects. This frequently occurred a generation ago in lower forms, when the form master was responsible for most of the teaching of his form. At the other end of the school it is still occasionally realized when the teaching of mathematics and physics is in the hands of the same master. Where teaching is divided between different subject specialists, two obvious conditions are necessary for any effective correlation between them: (*a*) Each specialist must be prepared to adapt his material and methods so as to reveal knowledge as a unity, and, more practically, (*b*) The schemes of work and teaching methods used must be available for all to consult in the school office or common-room. Granted these conditions, correlation may begin between heads of departments when they plan their courses, and it may continue in day-to-day discussions between colleagues teaching the same group of pupils.

General Comments. For most young children elementary mathematics is best revealed as a set of methods or tools for solving problems or investigating the world around them. Mathematics should be created *after* the need for it has become apparent, a claim which has been justified by the modification of teaching techniques in Air Training Corps instruction. This suggests that in the ideal situation the mathematics required, for instance, by the science teacher should be created by him through the need for mathematics in the study of science. The elaboration of this mathematics as a study in its own right should be the business of the mathematics teacher. Specialist teachers should not argue over *who* is to teach a particular topic in mathematics; this decision is far less important than deciding *why* and *how* and *when* a particular topic should be introduced and developed. In practice the only solution of the difficulty is for each specialist to consult with the others. One caution may be suggested. The mathematical aspects of a science or other field of study usually

belong to the later stage of development of that field, and there are good reasons for avoiding formal mathematical treatment with beginners in the initial stages. Specialists in science and in other subjects must be chary of attributing dubious values e.g. as mental training, to an early use of formal mathematics. On the other hand, teachers of mathematics should at all times be ready to widen the field of application of their subject by showing the importance of mathematics in economics, in geography or even in history.

Particular instances. Within each school there should be standardization of simple methods and notations. It is intolerable, for instance, that children should be taught different methods of multiplication by different subject specialists. The duty of the mathematical specialist must be to discuss such matters with his colleagues in the light of present-day knowledge, to reach an understanding, and subsequently to reconsider the position as conditions alter. Differences of opinion in such matters are often found to be based not on conviction so much as on lack of knowledge of current views. It may be pointed out, however, that in more advanced teaching some flexibility may be acceptable. For instance, the science teacher may set out his calculations using tables in the manner he finds most efficient. This need not be the arrangement *first* used in teaching these methods by scientist or mathematician. The general development of this final arrangement should be the business of the mathematics teacher. Likewise in the teaching of mechanics different specialists may prefer their formulae or results in different notations. So far as the teaching is concerned the important thing to attend to is that the pupils understand the reasons for these differences and agree that they are appropriate.

English and Mathematics. One of the platitudes of educational theory is that 'Every teacher is a teacher of English', but the emotive and appreciative aspects of English are sometimes overlooked by the teacher of mathematics who assumes that he is assisting the whole duty of the teacher of English by demanding precision and exactness of speech and writing on all occasions. Here again we can only advise some discussion between specialist teachers who might find, as a result of their discussions, other means of correlating their subjects—as for instance in illustrating the contention that mathematics is a language of a special kind.

10. Visual Aids

The term 'visual aids' is sometimes taken to cover a number of teaching techniques which are common to teachers of most subjects, certainly to teachers of mathematics. The use of the diagram and of the blackboard itself are so much a part of the teaching of mathematics that they need not be discussed here. It is the rarer and more specialized forms of visual aid—the model, the filmstrip, and the film with which we are concerned here. In this respect the position of mathematics is not comparable with that of other subjects, such as history and geography, where there is often considerable scope for visual aids. Moreover, in the teaching of grammar school mathematics, such are the demands of the examination system that, once the initial stages have passed, so much time is needed for the acquisition of the techniques that many teachers feel that there is no time to spare for purposes which do not show immediate results. Nevertheless we feel that the appropriate use of visual aids would stimulate interest in mathematics and would materially help in knitting ideas together without sacrificing proficiency in examinations. The inclusion of papers calling for more descriptive treatment, such as those in statistics and in the history of mathematics, in the G.C.E. examination syllabuses has made some change in this position. In other branches of mathematics, visual aids can be used most effectively at the beginning of a new topic as a foundation for the technique, or at the end of a series of topics to assist the linking-up process.

The material available for the teacher of mathematics is very limited and is likely to remain so for some time. Commercial manufacturers and producers are reluctant to enter such a hazardous field. It rests largely with the enthusiastic amateur not only to produce his own material, but also to bring it to the notice of other teachers. To assist in this work there has recently been formed an Association for Multi-sensory Aids to the Teaching of Mathematics but it is too early to say whether it will cause any notable change in the position.

The main purposes to which visual aids can be applied in the teaching of mathematics, as far as they can be determined from what has been done so far, are as follows:

(i) To present practical applications of mathematical principles, either as an introduction to a new topic, or as a sequel to the formal treatment.

(ii) By means of the animated diagram, either to introduce motion into problems involving a locus, or to analyse an intricate situation.

(iii) To show a greater variety of methods and material than is usual in the formal text-books.

(iv) To provide by means of a film-strip a convenient means of storage for intricate geometrical diagrams.

(v) To introduce by means of a film or film-strip the appearance of a third dimension into a diagram.

Models. An excellent book has been published recently (*Mathematical Models* by Cundy and Rollett) which gives a full account of a wide range of mathematical models. Most of these deal either with the borderland of mathematics and physics, or, like the regular and half-regular polyhedra, are designed to stimulate interest rather than to assist teaching. Their value in teaching appears to lie both in the mathematical thinking involved in making the model, and in the use of the finished product.

Films. Lists of mathematical and semi-mathematical films appear in the catalogues of the Educational Foundation for Visual Aids and in the report of the Visual Aids Sub-Committee of the Mathematical Association. Except for those which provide background material, the only films which call for special comment are those which use the animated diagram to introduce motion into problems involving loci. The application of this technique to mathematics is still at an early stage, but it offers much scope, especially at an advanced level, to the experimenter. Mention should be made of the pioneer work of R. A. Fairthorne in this field, and more recently of that of Nicolet in Switzerland and of T. Fletcher (Sir John Cass College) in this country.

In assessing the value of this work, it is important to recognize that a five-minute film may contain enough subject-matter to fill several chapters of a text-book, but will not, unlike the book, pause for explanation and digestion. Adequate preparation, both of teacher and class, is essential before any such film is shown, or it is likely to confuse rather than to elucidate.

These films are discussed in fuller detail in *Science in Films* (edited by Dr B. Lloyd, 1948; Sampson Low).

Film-Strips. A number of experimental film-strips have been produced from which it would appear that, at the grammar-school level, their chief value lies in their use as a storehouse of complicated diagrams. The teacher who is an amateur photographer will have little difficulty in making his own film-strips.

11. The Environmental Approach

Under this heading we have grouped together all methods and treatments which use as their starting point, not a mathematical principle or a mathematical technique, but a practical everyday interest or a group of situations. It is in this direction that the most important of modern experiments in teaching methods, especially with weaker pupils, has tended.

The normal approach in the grammar school has been to deal with each mathematical principle in its own right; having established the principle and demonstrated the associated technique, the teacher then leads the pupils through the practical applications of the technique, as far as time allows. In this way, the application is held to be subsidiary to the principle, and often escapes attention.

Most teachers have now recognized that this approach, although forming a sound basis for the potential specialist, creates many initial difficulties for the less able pupil, who loses touch because he cannot see where he is going. In algebra, for example, the older initiation, in which the first stages were often devoted entirely to bewildering manipulative work, has given way to an approach through the problem or the formula in which some purpose is at once apparent.

The environmental approach is an extension of the principle of a purposeful beginning to cover the whole range of academic work. In its extreme forms it would derive all normal subject-matter from the immediate interests or surroundings of the pupil, so that he would be required to learn no principle until he had seen its place in the outside world. Two serious disadvantages of such a method are that those preliminaries take more time than is available, and that they involve a departure from the logical sequence of topics. Against this, it is argued that the deeper understanding which results from such an approach reaps its harvest later.

The best known of these methods is the Project Method. Here the normal time-table classification into subjects is, for some part of the day, abandoned. Instead some topic of wider interest, such as railways, the aeroplane, the bicycle, a local industry, or social conditions, is studied. This involves close co-operation between the teachers, each of whom must seize on aspects of the topic which lead towards his own subject and attempt to cover, through this lead, his normal range of syllabus. The pupils are encouraged to collect for themselves the data to be used in calculations; this can stimulate enthusiasm for the work, but it is clear that very careful planning is needed in such a method if any sort of logical syllabus is to be covered and if a reasonable standard of calculation is to be achieved. Experience of this method has shown that, whatever its suitability in other subjects, it is the mathematics department which benefits least; the mathematics derived directly from a project can be very slight. Some teachers have overcome this by combining a project treatment with more formal work; for instance, a project on railways will involve the mathematics of speeds, travel graphs, and gradients, and each of these items can lead directly into formal work.

A variation of this method, which avoids some of the difficulties, is to use a 'Centre of Interest'. Here there is no departure from the normal time-table, but the work in each department is so balanced that it focuses attention on the same interest-topic. Thus, if railways were the centre of interest, the subject-matter in each classroom, and the examples or illustrations, would be in some way related to railways, so that the pupils would cease to regard the time-table subjects as watertight compartments, but would carry a common interest from one classroom to another. It is clear that many of the difficulties of the project method are still present, especially those of co-operation between departments.

Another new method, which has been the subject of recent research, is the teaching of what is known as Appreciation of Mathematics. This term is used to denote mathematics in which calculation and the solution of problems are made subsidiary both to the production of an intelligent interest in mathematics and to the interpretation of the mathematics which the ordinary citizen will meet in the course of his daily life. Some of the objectives of such a course would be: The comprehension of large (and small) numbers and measures; the reading and interpretation of graphs and diagrams;

the understanding of ideas such as ratio, periodicity and functionality; the visualization of three-dimensional figures; the checking of other people's calculations by estimation; the use of simple mathematical ideas to reduce the amount of trial and error and rule of thumb in practical situations; and the development of a critical attitude towards statistics.

The method involves the derivation of the subject-matter from the environment. Although the final subject-matter may be similar to that of the normal syllabus, the intermediate processes of formulating a problem and collecting the data are given much greater importance. The method also places less stress on facility in calculation, though some such facility is an essential preliminary to any appreciation.

The case for this type of treatment is that it is just in this way that the average person will meet such situations as call for mathematical treatment. The professional mathematician will be trained in breaking down a problem to find which techniques are needed; the person who needs some mathematics in the course of his daily occupation will soon, from experience or by a narrower training, find all that he requires in his limited field. The ordinary citizen has to equip himself to meet the mathematical needs of his daily life; appreciation of mathematics is specially designed to give him this equipment.

In practical situations there is usually a long process of formulation of problems to be solved and of collection of data before the actual calculations begin. If a person wished to build a house, he would have to choose a site, to plan the house, to find what material and labour were available and to estimate detailed costs and times before any calculations could begin. In school mathematics the pupil learns how to deal with the calculations but the preliminary stages are taken for granted. The result is that the average person after leaving school is rarely able to apply his power to calculate to any particular situation which he may encounter. Thus he comes to regard school mathematics as artificial and speedily forgets most of what he has learnt.

In this third type of environmental approach, the situation is dealt with in the classroom very much as it would be in real life. Much time is spent in discussion of method, in formulation of plans and in collection of data. The result is that only a small part of the

time is spent in training in mathematical techniques, although the value of this training is enhanced and made more lasting. Whether there is sufficient compensation depends on the pupils. It would be difficult to justify such expenditure of the limited time available for mathematics in the grammar school, except in the case of very weak sets. Experience in the secondary modern school has shown that this method can produce improvement, especially with weaker pupils, in calculation as well as in appreciation.

In general, methods of environmental approach have not found as much favour in the grammar schools as in secondary modern schools, and so long as the work is in part regulated by the necessity to reach a certain standard in mathematics in a limited time, we cannot visualize any great extension of their use, except perhaps with weaker sets.

12. The History of Mathematics

Why teach the History of Mathematics? A teacher's point of view on the teaching of the history of mathematics will depend upon his attitude to the meaning of 'Mathematics'. If it is the acquiring of certain skills, developing techniques, and practising facility in the manipulation of symbols, he is unlikely to find time for the history of its development, or to feel any necessity for it. But mathematics is much more than this. Regarded in its broadest sense it is a vital and fundamental part of human culture. No truly educated person can be without knowledge of the part it has played in the development of the material and mental progress of mankind. Finger-counting, measures of all descriptions, the reckoning of time, the decimal notation, all sprang from the needs of man in his struggle for existence; geometry is linked with the logical and philosophical inquiries of the Greeks; and so on. It is a fascinating story and should not be denied to any schoolchild.

This is not to suggest that the history of mathematics should be taught as a subject in itself. It should be brought in appropriately at various stages of the teaching of mathematics as a part of the subject as a whole. In so doing the interest of the pupil is aroused and the matter under discussion acquires added meaning and purpose.

The history of mathematics implies knowing something about the men whose names are outstanding in its development. Pythagoras

should mean more than a theorem, Archimedes more than a principle, and Newton more than a set of laws. Even if the lives of some of the men of genius who come within the range of elementary mathematics are dim in antiquity, there is usually something known, some story or legend, which will interest the pupil and humanize the subject. Inspiration can always be found in the lives of the greatest, none the less because they were men like others, with their virtues and vices, strengths and weaknesses.

From the point of view of teaching there can be much gain in considering the historical development of mathematical ideas, even if one does not go as far as to suggest that the historical approach to a new topic is always the best one. It is interesting to notice the strong historical basis of popular books on mathematics, such as Hogben's *Mathematics for the Million*, and the appeal this particular book has had for a comparatively non-mathematical public. Mach's *Science of Mechanics* and *Mechanics* by J. Cox, also written on a historical basis, are invaluable in the teaching of mechanics.

The incorporation of history in the teaching of mathematics is essential if, to quote from the introduction to this report, the ideas of 'its purpose, its structure, its wonder, its creativeness', are to be aroused in the child.

The Sources of Instruction. The teacher, must, of course, be the main source of instruction. This does not mean that he has to be an expert in the history of mathematics. The young teacher should know something of its main phases, the general idea of the contribution of the Greek and other civilizations to mathematics, and the place in history of a few outstanding mathematicians whose names the pupils are likely to come across in the course of their studies. He will probably find that, in the course of time, the extent and depth of his knowledge will increase considerably as he continues to incorporate it in his teaching. It is more effective, of course, if he can produce his knowledge as a part of normal teaching without a book, although it will be clearly necessary to refer at times to a standard history of mathematics, and it is by no means a bad thing for the class to see him do it. A number of topics suitable for incorporation in an elementary syllabus are given below.

A text-book on the history of mathematics (with a few possible exceptions such as *Number Stories of Long Ago* by D. E. Smith) is

not a suitable medium of instruction for the younger pupils. A pupil can only be interested in the history of that part of mathematics which comes within his experience; he is not capable of picking and choosing for himself from a text-book, and quickly becomes bored by it. The teacher should, therefore, be very careful in his suggestions for individual reading. Older pupils should, of course, be encouraged to read for themselves and to refer to histories of mathematics, although, even with them, guidance can be very helpful. A number of text-books contain historical notes, which can be read and discussed as they occur.

Wall charts serve a useful purpose as visual aids in the instruction of the broad outlines of mathematical history, and the construction of such charts by the pupils themselves can form an interesting exercise. Portraits of mathematicians, too, if available, are helpful in visualizing the men and the ages in which they lived.

Topics for Instructions. The following list is not exhaustive but gives examples of subjects which perhaps ought to be known to teachers of mathematics:

General phases of mathematical history throughout the ages.
Finger counting and decimal notation.
The abacus.
Difficulty of multiplying with the Roman notation.
Arabic notation (cf. numbers on present-day Egyptian and 'Iraqi postage stamps).
Famous problems of geometry: squaring the circle, duplicating the cube, trisecting an angle.
The Greek Game of Geometry. How history refutes the 'What good is it?' argument.
Eratosthenes' measurement of the Earth, and early methods of surveying.
History of Pythagoras' theorem.
Early astronomy; navigation, the astrolabe, chronometers and calendars.
Trigonometry; connection with religious observances, e.g. direction of Mecca.
Early algebra; development of notation, 'completing the square'.
Common units (yard, furlong, acre, pole); metric system.
Some idea of the men whose names are commonly met in mathematics, e.g. Pythagoras, Euclid, Archimedes, Ptolemy, Galileo, Newton; and of those people whose names are preserved in such phrases as Hooke's Law, Napierian Logarithms, Horner's Method.
The Newton-Leibniz controversy.
Derivations of the commoner mathematical terms.

How to incorporate History in the Teaching of Mathematics. Much of the history of mathematics can be taken casually as a natural part of the instruction of mathematics. The inevitable question 'Why are there 90 degrees in a right angle?' leads to a few remarks on the sexagesimal notation and the Babylonian civilization. The symbols £ *s. d.* will call for historical explanation. A name, such as Archimedes, would be an opportunity for a brief talk on that very remarkable and interesting man. The study of logarithms introduces Napier and Briggs. There is also much profit in examining the derivations of some of the commoner mathematical terms, such as *geometry*, *polygon*, *sine*, *ounce*, *pound* (as weight and money), etc. Not only does this practice form a link with other languages, but the pupils will get into the habit of inquiring into the meanings of words on their own accord.

There are also occasions when it is desirable to spend a more prolonged period on a phase of mathematical history. A talk should be given to pupils in the first form on the story of the development of number, to include prehistoric methods of counting, decimal, and other notations, early ways of symbolizing number, the Greek method by the letters of the alphabet and the Roman notations. It can be pointed out that these systems could only be used to record the results of calculations which for many centuries were performed by the abacus in one form or another. At a later stage in the pupil's course, possibly at the time of instruction on decimal fractions, this talk might be recapitulated briefly, and extended to later developments. These will include the Arabic notation (of Hindu origin), the introduction of the cipher nought, positional notation and the resultant possibility of performing arithmetical operations with numbers, and finally the comparatively late discovery of the use of the decimal point. The teacher can refer to the necessary but approximate measures of Biblical times, the multiplicity of systems of mensuration in medieval Europe; the need for, and exceeding difficulty in standardizing our own units, and the scientific metric system which required a revolution for its introduction. A consideration of the pros and cons of our own 'natural' system *vis-à-vis* the metric system is also of interest.

The Greek contribution to mathematical thought will always be in the background of geometrical teaching. Early in the deductive stage the pupils must be given some idea of the nature of proof, and this

can be illustrated by a talk on the difference between the Egyptian and Greek approach to mathematics, the Pythagoreans and the work of Euclid. Such a talk forms a basis for future historical references in the geometry course, and may lead, eventually, to a more detailed study, in the sixth form, of the Greek era.

The teaching of mechanics, too, can be vitalized by constant reference to historical origins. There is the story of the successes of the Greeks in theories of statics and hydrostatics and mechanical applications, and of their comparative failure in attempts at problems in dynamics. After long years of little progress in this respect come the pioneers of Newtonian mechanics, Simon Stevinus, Galileo, Huyghens and others. References will have to be made so frequently to our own Sir Isaac Newton that it should be found possible to find time for a brief account of his life and works.

Other topics for discussion include the Arabian contribution to algebra; the development of early astronomy in connection with navigation and the keeping of time; the theories of astronomy created by Copernicus, Galileo and Kepler.

In the sixth form the historical background of teaching should be retained. A number of names such as Descartes, Pascal, Leibniz, come within the orbit of the pupil's mathematics for the first time. The emphasis now, however, should be on individual reading rather than discussion. It is the teacher's function to see that the school library contains a suitable selection of books on the history of mathematics, and to give to his pupils encouragement and guidance in their use for reading and reference.

A bibliography will be found under the Mathematical Library section of this report.

13. The Philosophy of Mathematics

From time to time questions arise in the classroom that are not about mathematical techniques but about mathematics as a subject of study. One of the earliest is the inevitable question 'What is the use of geometry (or of algebra)?'; later questions may be 'What is mathematics? Why do we learn it?' Such questions about the status of mathematics as a subject, naïvely expressed though they may be, are questions of mathematical philosophy rather than of mathematics. How should the teacher answer them?

As a subject to be formally taught, mathematical philosophy has no place in the school curriculum. Its study requires not only a knowledge of mathematical and logical processes that few teachers would claim to possess, but also a maturity that is unlikely to be attained by pupils of school age, except possibly in the university scholarship class. Yet it is a subject that cannot altogether be shunned by the teacher of mathematics at whatever level he teaches in secondary grammar schools. At least he should admit the validity of questions about the status of mathematics, indicate that they are reasonable questions to ask, and try to answer them in terms intelligible to his pupils.

In its long history mathematics has developed in three distinct ways. There have been occasional major events, such as Thales' conception of proof, the invention by someone unknown of a symbol to represent zero, and Descartes' idea of representing points in space by co-ordinates. Between such major events there have been developments of techniques and a widening of the fields of both pure and applied mathematics. Thirdly, there have been periods, and particularly the last hundred years, where the 'inward-facing aspects' of mathematics have been studied, its foundations have been critically examined, and its structure has been strengthened. The study of the 'inward-facing aspects' of mathematics and the attempt to discover the nature of mathematics together constitute mathematical philosophy. But the distinction between ordinary mathematics and mathematical philosophy is, as Bertrand Russell emphasizes, 'not in the subject-matter, but in the state of mind of the investigator'. Thus, Euclid's systematization of geometry can be studied both as mathematics and as mathematical philosophy; if we accept the axioms and postulates and study the logical chain that can be deduced from them (as we do in school mathematics) then we are mathematicians; if we accept the logical deductions and interest ourselves in the formulation of the axioms and postulates then we are mathematical philosophers.

Pupils familiar with the outlines of the history of mathematics will be aware of the growth of the external structure of mathematics, but less aware of the fact that its internal structure has also been developed, and that even great mathematicians have asked themselves the question 'What is mathematics?' and have failed to find a wholly acceptable answer. Why is it important that

schoolchildren should be aware of the 'inward-facing aspect' of mathematics?

It must be admitted that for many children the question is not important. Even of those who are interested in mathematics many are satisfied to accept it as a collection of tools, collected because they are useful, or pretty, or amusing to play with. That mathematics has great utilitarian value is obvious to most children and is a fact that is for many a complete justification for learning it. The aesthetic value of mathematics is less frequently appreciated by schoolchildren, but it is a powerful attraction for a few. Either the utilitarian or the aesthetic appeal of mathematics may suffice in itself to lead a pupil to a deep and satisfying study of mathematics, and even to eminence as a mathematician. Mathematics also provides amusement; simple mathematical puzzles are widely popular, not least so in schools. There are, however, some intelligent children who are insensitive to the aesthetic appeal of mathematics, who despise its utility, and who scorn the trivia of puzzle-solving. Some of these may be interested, however, in another aspect of mathematics and its tools—the organization of the tool-box. These are the potential mathematical philosophers.

There may be few such children but they should not be lost to mathematics because the philosophical aspect of mathematics has never been revealed to them; their loss would be a great one. Moreover, to many other children, whether they are interested or not, mathematics must appear to be little more than an incoherent and amorphous collection of tricks; the fact that it has an internal structure and unity may be a valuable idea to them, not because they can hope to see the structure clearly and to understand it, but merely because the knowledge that it exists may encourage them to look for some structure in the chaos.

One of the attractions to schoolboys is their faith, amply justified, that their science work at school can lead them to a better understanding of the scientific discoveries, inventions, and theories that are widely publicized in the press and on the wireless. They know also that if they succeed in gaining admittance to a university faculty of science or of engineering they too can share in the excitement of research in atomic physics, radar, aerodynamics, biochemistry, or in whatever has their interest. Science is news; mathematics (apart from electronic computing machines) is not. The only living

mathematicians whose names are known to the general public are those who have applied their mathematics to scientific problems. In part this is the fault of the school (and university) teaching of mathematics; few schoolteachers know of any present developments and little is said in school to suggest that any development of mathematics is still taking place. The result is that no one expects anything new to happen in mathematics; to most people mathematics is as dead as Latin (shall we say?), though it may be admitted to be more useful. To the average schoolboy mathematics must seem to be some kind of infinite regression; topics succeed one another in steady sequence but no end appears in sight. Few boys could name a mathematical domain on the borders of knowledge where research is now being actively pursued. When children learn mathematics they have not the incentive of seeing an attainable goal before them as they have when learning science.

It may be argued that the ideas of modern mathematics so far transcend the ideas of elementary mathematics that they are inexpressible in terms intelligible to children. This is not wholly true; it is true in part because we have rarely tried to do it and therefore lack experience. But the main difficulty is that it has become traditional in English mathematical teaching to feed mathematics to children in small digestible doses—and to be certain that each is digested before the next is given. On the whole it is considered to be bad teaching to look far ahead and talk of things that may not be fully comprehended.

Mathematical philosophy has never taken firm root in English mathematics; even the *Principia Mathematica*, though much admired, is seldom studied. Few teachers of mathematics have therefore any knowledge of it and there is no traditional interest in it as there is on the Continent. Yet it can provide some topics of interest to children—at least enough to show them that all the problems of mathematics have not yet been solved.

Attempts to describe the nature of mathematics in elementary terms usually take the form of displays of the range and power of mathematical ideas. A typical and effective attempt of this kind is the book *What is Mathematics?* by Courant and Robbins. Perhaps there is no other way. If the question 'What is music?' were asked by someone untutored in music, no solely verbal description of music would be adequate; the only effective answer would be to

demonstrate various kinds of music, to explain the relations between them, and possibly to study at least one musical form or instrument. Similarly, comprehension of the nature of mathematics can be seen only by ranging broadly over its many branches and by a deeper study of one or two of them.

In answering classroom questions about the nature of mathematics the teacher can only frankly admit that such questions are difficult to answer, that the answers are still being sought, and that perhaps the only way of discovering the nature of mathematics is to learn more of it. The understanding of the organization of the tool-box can come only from a thorough knowledge of the tools it contains. At the same time the teacher can help by showing the interrelation of such fields of mathematics as are known to his pupils, and by mentioning at suitable opportunities extensions of those fields that are studied in more advanced mathematics, e.g. that there are other algebras, that there are geometries other than that of Euclid. How far expansions of such statements should be given depends on the attainment and interest of the class; but even the bare facts are rarely known, and yet to most children they are exciting news. They are exciting for two reasons. First the facts imply that mathematics is still a living and growing subject. Secondly, it is exciting to find that ideas one has long taken for granted have been successfully challenged.

In the sixth form the teacher can do rather more than mention the bare facts. His pupils will have begun to appreciate the widening of the field of mathematics that is opening before them and the teacher will have more opportunities to indicate interrelations between various topics. Further, some pupils might begin to do some reading for themselves. A short reading list, with the books in order of increasing difficulty is:

What is Mathematics? Courant and Robbins.
Introduction to Mathematical Philosophy. Bertrand Russell.
The Nature of Mathematics. Black.
Elementary Mathematics from an Advanced Standpoint. Klein.

Any book *about* mathematics rather than *of* mathematics could be added to this short list.

These books are of course intended for browsing in rather than for serious study.

Finally, occasional short talks on more advanced subjects can be given, remembering that their main aim is to stimulate. There are many possible subjects and the choice can depend on the interest of the teacher. The treatment need not be formal, and in fact the talk would probably better achieve its aim if it were possible for sixth form non-mathematicians to attend. Some suggested topics are:

The paradoxes of Zeno.
Continuity.
Is $\sqrt{2}$ a number?
Non-Euclidean geometries.
Other algebras.
The schools of mathematical philosophy.
Principia Mathematica.
Infinity.
The concept of a limit.
Topology.

—but there are many more.

IV

SIXTH FORM WORK

1. General Discussion

The Nature of Sixth Form Work. It is common to hear the term 'sixth form work' used as though it had a single definite meaning. This may have been the case in the past, when sixth form work scarcely existed outside the public schools and the older grammar schools. The sixth form of those days led to the university, very often, though by no means exclusively, by way of an open scholarship or exhibition. Today, however, the university offers only one among a large number of possibilities open to pupils in sixth forms. In fact these pupils now include many whose abilities do not fit them for a university career. One of our tasks must be to consider the needs of these various types of pupil and so decide what levels of mathematical teaching shall be provided.

Sixth form work offers a contrast to the middle-school stage of mathematical education because at that stage the pupil was receiving a general education; in the sixth form the range of his studies becomes more restricted, and a deeper knowledge of each subject becomes possible. In the study of mathematics the following points are to be noted:

(*a*) Within the range of work attempted a more detailed knowledge is acquired. There is more variety of method, and greater opportunities arise for comparing and contrasting procedures. The traditional branches of the subject fuse more closely with one another as the application of new techniques are studied, as in the application of algebraical methods to geometry.

(*b*) The interrelation of subjects with one another should be brought out even more clearly at the sixth form level. Here the points of contact with physics in particular should not be missed, and pupils studying physics should be taught to realize the value of the further mathematical knowledge that they acquire.

(*c*) Although throughout a child's education the contact between the subject of a lesson and its applications should be as close as the conditions imposed by the subject matter permit, yet, in the sixth

form, the opportunities for stressing this contact may be more fruitful than in the earlier stages.

(*d*) The greater opportunity for private study allows for a closer integration of knowledge within the pupil's mind, and permits a more thorough pursuit of the inter relations and applications mentioned in (*b*) and (*c*).

We must here emphasize that the process of acquiring a groundwork, or of learning how to handle a few essential tools for simple operations (which might be thought to belong to the middle school only) does not cease in the sixth form. For new and more complicated operations, other tools are necessary and their use has to be learnt.

Examinations and other Influences on Sixth Form Work. The strongest influence in shaping the work of sixth forms has been the syllabuses of the G.C.E., and its predecessors, the School and Higher Certificate Examinations. Associated with these are the university entrance regulations (matriculation and faculty requirements) which have tended to encourage the schools to restrict the range of studies in the sixth form. As a result pupils do not, in general, need to study more than six subjects for examination purposes. Not more than three of these need be at the Advanced Level. Two of them may be taken at the further Scholarship Level: and, for our purposes, these subjects might be Pure Mathematics and Applied Mathematics.

Another factor of importance which has controlled the aim and scope of sixth form work was the body of regulations, prescribed by the Ministry of Education and the earlier Board of Education, for recognized Advanced Courses. The basic principles of these regulations are set out in detail in Ministry circulars and may be summarized briefly as follows: The Courses were to make provision for the study of three 'main' (Advanced Level), interrelated subjects. To these subjects most of the school hours were to be given (not less than two-thirds, nor more than three-quarters). The remaining time was to be given to 'additional' subjects, some of which were to provide, as it were, a balance to the main subjects. For instance, those who chose science and mathematics as main subjects might take as an additional subject a language or literature or history. Finally, no Advanced Course was recognized unless in each year of the course there were some half dozen or more pupils. Thus there was to be partial specialization, co-ordination, balance and class instruction.

It will be seen that the Advanced Course regulations agree with the examination requirements in their insistence upon partial specialization. The examinations for open scholarships at the universities have especially encouraged far too narrow specialization.

Further Investigation of Examinations. From the sixth form point of view we may expect the Advanced and Scholarship Levels of the G.C.E. to continue to remain the most important examination. This examination is still being conducted by a number of separate examining boards, and it is disturbing to find that the syllabuses of these examining boards are so very different, For example, the Welsh and the Durham boards set papers on a more extended syllabus than that of the London board; while they in turn have a less extensive syllabus than do the Oxford and Cambridge Joint Board. It is to be hoped that the work of the Secondary Schools Examination Council may, in course of time, lead to a greater degree of uniformity in this respect.

The examinations for mathematical scholarships at the various universities show a still greater diversity than those for the General Certificate. For instance, the standard required by Newcastle, Sheffield and Nottingham is far lower than that of Oxford, Cambridge and Manchester.

Pupils competing for science scholarships who take a mathematical paper have an equally wide range of possible syllabuses. In fact a candidate who competes for a science scholarship at Cambridge takes a mathematical paper on a wider syllabus than one who competes for a mathematical scholarship at Newcastle.

A sixth form teacher must also be prepared for further complications if pupils work for special examinations such as the Civil Service (Executive Level) and the Navy, Army and Air Force examinations.

The above analysis suggests that the organization of sixth form teaching can become exceedingly complex, especially if there are pupils with a wide range of intended objectives. Such a situation becomes more serious in the smaller school, where limitations of teaching staff compel the teaching of several groups simultaneously. In circumstances like these, pupils are forced more and more towards individual work, with a little assistance now and then as the teacher can find time for them.

We admit that individual work is good, and that it should form an integral part of sixth form experience in mathematics. But some actual teaching at this stage must continue; for the break from the formal class teaching of the junior school to the almost entirely individual work of the university is not one to which the pupil becomes automatically adjusted, but one for which gradual preparation must be made.

The question to ask now is whether the complexity arising from examination requirements can be simplified. The varying standards in the Advanced and Scholarship Levels of the General Certificate may not affect a single school, as that school would be associated with one particular examining board (though there are exceptions to this). But, inasmuch as they affect the syllabus in use throughout the school, pupils who transfer from one school to another, especially late in their school career, may find difficulty in adjusting themselves to their new school.

The situation would be eased if an agreement could be reached on what is expected by universities to be the minimum knowledge at entry for mathematical and for science students. We have already put forward in our report an argument for a common level of attainment on entry to a secondary grammar school, and have suggested a certain minimum of knowledge that can fairly be expected. The grammar schools would recognize a similar right on the part of the universities in respect of their intending students, and would be prepared to equip their pupils accordingly. With such a level of attainment clearly stated one might hope that the various examining boards and the authorities responsible for university scholarship examinations, might try to use it as a basis in framing their syllabuses.

Without some such wide agreement any syllabus that we here present is in danger of being almost valueless to any particular school, since the inclusion or exclusion of any particular section is almost entirely dictated by the particular examination syllabus in use. The syllabuses we present indicate our own view on attainment at entry to a university, and provide a framework within which the organization of sixth form teaching can be made.

Analysis of possible Sixth Form Groups. We shall assume for the purposes of this section that the General Ceritficate papers of only one examining board are in use. There are then possibly five different courses required:

(*a*) A course must be provided for pupils whose main interest and ability is in mathematics itself. This course should be regarded as equivalent to two subjects at both the Advanced and Scholarship Levels, and it is strongly recommended that any boy offering this course should also offer physics at the Advanced Level.

This group may provide a sub-group consisting of candidates for university scholarships. Under present conditions, the needs of this group may be very varied.

(*b*) A course at a lower level should also be provided for pupils whose main interests are in subjects to which mathematics is applied, such as engineering, physics, and chemistry. This course would be regarded as equivalent to one subject at both the Advanced and Scholarship Levels.

There is some argument for suggesting that pupils whose main interest is in chemistry need only pure mathematics. Such pupils would then take the pure mathematics section of the (*a*) course above. It should, however, be noted that some applied mathematics is needed in physical chemistry, e.g. in kinetic theory, and also that candidates for natural science scholarships are at present usually expected to answer some questions on applied mathematics.

(*c*) A course should be provided for students whose main interest is in subjects other than the above, but for whom physics still remains an auxiliary subject, e.g. for those preparing to read medicine, dentistry, etc. This course would call for only one or two periods a week for about a year; and would, of course, be non-examinable.

(*d*) Some schools may wish to provide a course of mathematics as part of the general education of students of non-mathematical subjects (as, for example, arts students). A course can be planned to extend the scope of their previous work. An interesting course can also be planned for similar students who would like to continue mathematics after passing at the Ordinary Level.

(*e*) There may remain a medley of pupils working for miscellaneous examinations such as those for entry into government services. These can usually be absorbed into one of the above groups where each pupil is then given such individual attention as is practicable.

Organization of Teaching. Most schools arrange their time-tables on a basis of either thirty-five periods of 40 mins each, or forty periods of 35 mins each. On the assumption that between two-thirds and three-quarters of the total time is equally divided between three

main subjects, each main subject would have allotted to it from 5 to 6 hrs, per week. This may include preparation and/or private study in school time, but would not include homework time. This would be the appropriate allocation for pupils in group (*b*) above. Those in group (*a*) are counting mathematics as two main subjects, and may therefore expect to have twice as much time per week as those of group (*b*). Where pupils in either of these groups are studying for the Scholarship Level or for university scholarships, it is a great advantage, if the school staffing allows, for them to have a few teaching periods apart from other groups. For pupils in group (*c*) and non-examination pupils in group (*d*), two periods per week should be ample.

Unfortunately, except in the largest schools, the number of pupils in any one group per year will be small, and it may not be possible for groups to be taught alone for the requisite number of periods per week unless the school staffing ratio is adequate. This is a great pity, as the nature of the subject is such that pupils in different years cannot effectively be taught together. The pace of the mathematical specialist cannot be adapted to that of the science pupil who, as a rule, has less mathematical ability, and in addition has less time to devote to the subject. The nature and aims of the various groups are so diverse that it is in most cases only for a term or so of sixth form work that groups can be combined, and then only for certain sections of their work. This point has, however, been borne in mind in drawing up the syllabuses presented later.

Thus a class will often consist of a number of groups taken together; but the different groups of the same year, and possibly of different years, will have to be taught as separate units. The master will have to divide his time between them according to their needs, planning out their time and his own carefully, and devoting much time out of class to organization and supervision of their work.

The relative importance of group instruction and individual work will need to be weighed. Both have their place. It is desirable that pupils should come into intellectual contact with masters and with each other in a way that only oral work can provide. It is equally important that they should learn the valuable lesson of working for themselves. Such training is dictated by necessity in the smaller schools, but its importance can be overlooked in the larger.

This last remark is especially relevant in the case of the gifted pupils, who may easily become frustrated if sufficient attention is not paid to their needs. It is worth while to devote some thought to planning a course of work that will continue to keep them adequately extended. Such a course might provide them with a wider range of application of the work they are doing; an excursion into work not to be attempted by the remainder (as for instance, a general study of the methods of summation of series, a more thorough study of the theory of equations, etc.); a preliminary acquaintance with some of the powerful techniques or methods of mathematics (such as projections, duality, etc.). These suggestions are sufficient to show not only that the method of treatment of this type of pupil is to let him 'go ahead', but that he can obtain valuable experience in other ways.

It is here appropriate to stress the importance in sixth form work of a careful choice of text-books and an adequate reference library. The committee emphasizes the importance of the library and has devoted a separate section to this topic. It is of especial value for a master to have accessible a good range of supplementary text-books and reference books. Their additional value at the sixth form level lies not only in their supplementary sets of examples, but also in their alternative expositions of sections of theory, and pupils should acquire the habit of studying such pieces of bookwork critically.

Mathematical Attainment on Entry to Sixth Forms. The scope of the work in the middle school has an important bearing on that of the sixth form. While the syllabus which we have outlined for that stage may contain as much as many pupils can manage, others (including future specialists in the subject) would be wasting their time by confining their work within these limits. Those who have the ability to do so should proceed to more advanced work, and obtain a better idea of the nature of the subject. The specialists would be laying the foundation of their sixth form work; and indeed, unless they do so, they must be faced with a very difficult task in reaching the Advanced standard, to say nothing of the Scholarship standard, in the next two years. It is well worth while considering, if the group has been carefully selected, the advisability of omitting the Ordinary standard completely for such pupils so that they may be able to concentrate their attention entirely upon the higher level examinations.

With such a group, the work in algebra, geometry and trigonometry can be developed further and can lead to an elementary treatment of calculus and co-ordinate geometry.

In schools with an entry of more than one form per year, the necessary organization is most easily effected by regrouping pupils into parallel sets for mathematics, and allowing the abler pupils to progress more rapidly than the rest. In a one-stream school the problem is admittedly more difficult. The teaching of a form in two sections has definite disadvantages; yet it may be possible to pick out a small group of keen and able pupils, and allow them to go ahead at their own pace without seriously affecting the work of the main body of the form.

Some Notes on Sixth Form Teaching. From the traditional point of view, mathematics consists of four branches in the middle school, and of six branches (or possibly seven, according to one's method of subdivision) in the sixth form. We have tried to discourage this departmental point of view in the former case by putting forward a plea for closer integration and interrelation. The argument for such a step is even stronger at the sixth-form level, where this interrelationship is even more easily and naturally established (one such instance would be the simultaneous treatment of the logarithmic and exponential functions as a piece of work in both algebra and calculus).

If, however, it is preferred to teach according to the conventional divisions of the subject, their very number renders it rather unwise and uneconomical to study all of the branches concurrently. Better progress is made, especially when new branches are being introduced, by concentrating on two or three only, while a weekly problem paper can be used to maintain contact with other work. A start should be made with calculus, co-ordinate geometry and mechanics as early as possible, in order to allow time for assimilation, and, in the case of the calculus, because of its usefulness in developing other branches of the subject. In mechanics, an experimental basis will generally have been laid previously; but if not, time and apparatus should now be available for some work of this kind. In any case, there is much to be said for spending one of the mathematics periods each week in either a physics or mechanics laboratory, where experimental work can give a greater meaning and sense of conviction to the theoretical work.

While a sixth form course of calculus will include some introduction to modern standards of rigour, one of the first essentials will be to develop vision and reliance on intuition. In introducing a new topic, manipulative difficulties should not be allowed to obscure principles; simple functions should be used, and it should be made certain by oral questioning that pupils are not slavishly applying a rule. It is with such arguments that the applications of the calculus are introduced early by many teachers, so as to make clear the meaning and scope of the processes of differentiation and integration, while leaving until later the more complex rules for differentiation, and the systematic methods of integration. The introduction of the trigonometric and logarithmic functions is also frequently delayed for a similar reason.

We might also suggest in this section that, where the services of more than one suitably qualified master are available, the pupil will, by coming into contact with different types of teaching, gain in breadth and variety of treatment.

The Syllabus and its Background. In the next and subsequent sections, details will be given of a syllabus designed for pupils in groups (*a*) and (*b*).

Group (*a*) will study a substantial proportion of this syllabus; and, as has been suggested, they should retain physics as a subject of study. They will not, however, have time for a detailed study of chemistry.

Group (*b*), which will consist largely of science students (and we may here include the candidate for government services) will have less need for geometry, and for highly manipulative exercises. There will thus be a difference in emphasis over some sections of the course. The science specialist will benefit by acquiring, while in the sixth form, a reading knowledge of German, assuming that his French already reaches this standard.

For pupils in groups (*c*) and (*d*), thought is needed in constructing a syllabus that will save time by meeting the needs of as many individuals as possible. For some of these pupils, especially for those whose stay in the sixth form is not likely to extend beyond one year, the course need not be ambitious. Revision of fundamental processes would be obtained incidentally. There would be no attempt to push further the development of mathematical tecnhique, but the aim would be to give the non-mathematician some idea of the part played

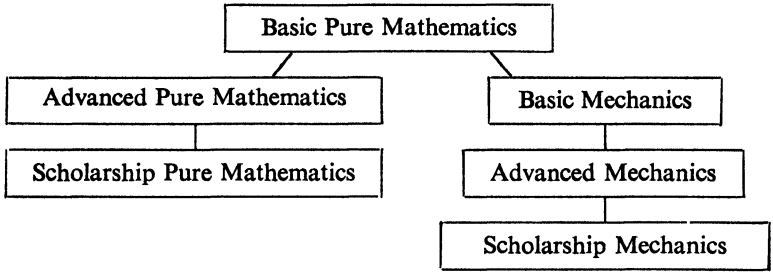
by mathematics in human life and thought. Suitable topics would be graphical work, applied especially to statistics; calculus, numerical trigonometry and practical surveying; experimental mechanics, astronomy, the history of mathematics.

For those who are capable of proceeding further, a course in elementary statistical theory, with the pure mathematics necessary and incidental thereto, is a possibility worth exploring. It is the exception rather than the rule to find a member of the classical or modern sixth finding relief in a few periods of mathematics weekly; yet, apart from any vocational value, the ability to discriminate between sound and unsound deductions from statistical data would be a useful side to a general education, now that such data enter so frequently into all kinds of activities. The biologist would probably find this more helpful than the more stereotyped mathematical work. The intending teacher will find it useful to know something of the techniques used in manipulating the mental test and other examination results. The pupil in the commercial or general sixth studying economics, economic geography etc., could for obvious reasons very profitably join this course rather then spend all of his available time on simple commercial arithmetic. An appropriate treatment of statistics is discussed elsewhere in this report.

We must not forget certain special needs for pupils in group (*c*), which would call for a separate period or so per week. They will need to be familiar with the application of differentiation and integration (more especially the latter) to physics. These are included in most text-books on the calculus, and the sections can be extracted and connected by sufficient general theory to form a unified whole. The binomial approximation, $(1+x)^n = 1+nx$, will be required, and help may be needed with some of the important principles of mechanics up to and including rigid bodies.

The Framework of the Syllabus. In the attempt to produce a syllabus suitable for groups (*a*) and (*b*), and bearing in mind the fact that in many of the smaller schools these groups would have to be taken together, the first need is to find a common core of mathematics that will meet the minimum needs of both. The starting point for both becomes therefore a course which we have labelled 'Basic Pure Mathematics'. Built on its foundation is another common-core course which we have called 'Basic Mechanics'. Starting from these

two we obtain the following framework which forms the organization of our sixth form syllabus:



Optional topics, which may be included in some of the courses, include statistics, surveying, astronomy and navigation, history and philosophy of mathematics.

It will be observed that each rectangle represents a unit of work requiring from 5 to 6 hrs of teaching time per week for a year (or 2½ to 3 hrs per week if the course is covered in two years). The uppermost rectangle, representing Basic Pure Mathematics, is the primary course. All pupils will study material from this course according to the level of their ability and the time that they have available.

Any one of the courses adjoining the primary course can be arranged to run concurrently with it; but it should be noted that a course may only be taken subsequent to, or in conjunction with, the preceding course.

The two Basic Courses may be regarded as forming a suitable syllabus for Pure and Applied Mathematics as a single subject at the Advanced Level.

2. A Syllabus of Pure Mathematics

The Basic Course. Graphical work, including the graphical representation of simple functions, graphical solution of equations (including easy transcendental equations), the graphical determination of laws.

Simple series, including the infinite geometrical series. Elementary study of the quadratic function. Simple examples on permutations and combinations. The binomial theorem for a positive integral

index. Use of the binomial expansion for other indices (excluding questions on greatest term or the sums of coefficients). Properties of roots of a quadratic equation.

Trigonometrical ratios for angles of any magnitude, and their graphs. Circular measure of an angle. The addition and subtraction formulae (without insistence on the reproduction of proofs). The function $a\cos\theta+b\sin\theta$ and its graph. Easy trigonometrical equations (excluding general solutions). The approximations $\sin\theta=\theta$ and $\cos\theta=1-\frac{1}{2}\theta^2$; the limit of $\sin\theta/\theta$ as θ approaches 0. Solution of triangles (without insistence on proofs of formulae other than the sine and cosine rules). Applications of trigonometry to easy problems in two and three dimensions.

Rectangular cartesian co-ordinates. Formulae for distance between two points and for division of a line in a given ratio. Standard forms of equation of a straight line (excluding the perpendicular form). Easy problems on loci. Elementary treatment of the circle from the equation $x^2+y^2=r^2$; and of the parabola, using the parametric form $(at^2, 2at)$.

The differentiation of algebraical and trigonometrical functions (excluding the inverse functions). Application of differentiation to tangents and normals, maxima and minima, points of inflexion, rates of change, calculation of small increments, and the sketching of graphs. Knowledge and use of the logarithmic and exponential functions and their derivatives. Logarithms to any base. Use (without proofs) of the expansions for e^x, $\log(1+x)$, $\sin x$ and $\cos x$. The definition of an integral as the limit of a sum; simple physical illustrations. Approximate numerical integration. Integration as the reverse of differentiation. Applications of integration to plane areas and volumes. Easy integration by use of partial fractions, and by substitution.

The Advanced Course. Graphs of polynomials and simple rational functions. Some study of cubic functions. Relations between roots and coefficients of a cubic equation. Miscellaneous equations. Approximate location of the roots of an equation. Partial fractions. Cyclic symmetry and its use in factorization. Solution of easy inequalities. Use of the binomial, exponential and logarithmic series (including summations). Simple instances of the method of differences applied to the summation of series. Sums of squares and cubes of the first n integers, with applications. Simple cases of elimination.

Useful applications of determinants (not exceeding the third order) to other sections of the work.

Deductions from the addition and subtraction formulae of trigonometry, including the transformation of sums and products. Identities in general, including the case when $A+B+C = 180°$. Inverse trigonometrical functions, and their principal values. Formulae for all angles having the same sine, etc.; general solution of trigonometrical equations. Formulae connected with the circumscribed, inscribed and escribed circles of a triangle; the pedal and excentral triangles. Simple examples of trigonometrical elimination.

Extension of Pythagoras' Theorem; Apollonius' Theorem. Further work on similar triangles; Ptolemy's Theorem. Circle of Apollonius. Properties of the triangle. Concurrence and collinearity; theorems of Ceva and Menelaus; Simson's line. The nine-point circle.

Elementary properties of the line and plane, of their intersections, and of the angles between them. Elementary geometry of solid figures; in particular the prism and pyramid, cylinder and cone, tetrahedron and sphere.

Complete analytical treatment of the straight line and circle. Systems of lines in the form $L+kL' = 0$. Representation of two straight lines by one equation. Elementary formulae in polar co-ordinates. General analytical geometry of the conic sections referred to their principal axes as axes of co-ordinates, and of the rectangular hyperbola referred to its asymptotes. (Free use should be made of parameters, but the work should exclude chord of contact, pole and polar, tangents from an external point, auxiliary and director circles, conjugate diameters, asymptotes in general, and the conjugate hyperbola.)

Second and higher derivatives. Applications of inverse trigonometrical functions to differentiation and integration. Integration by parts. Centres of gravity and moments of inertia as applications of integration. Simpson's rule. Hyperbolic functions; their graphs and derivatives. Approximate solution of equations. Simple reduction formulae. Use of parametric and polar co-ordinates in finding areas and in the plotting of curves.

3. Mechanics in relation to Physics

The principles of mechanics form the basis of a considerable part of physics, and in consequence some mechanics is taught in practically

all schools in the physics department. At the Ordinary Level the work is usually of a mainly experimental character, although opportunities for some simple mathematical treatment will no doubt present themselves. The scope of the work is much extended at the Advanced Level, though the general pattern of balance between experimental and mathematical treatment remains, except probably in the case of scholarship candidates, where the treatment will inevitably become more mathematical.

Unless mechanics or applied mathematics is taught as an Ordinary Level subject for examination purposes—and this probably happens only in some of the larger schools—the mathematical treatment of mechanics may be deferred until the sixth form course is reached. In consequence, the mathematical treatment may be anything up to one or two years behind the experimental treatment. This is not regarded as being necessarily a bad thing; it gives the mathematics staff some foundation on which to build the theoretical structure of the subject. The experiments which the mathematics teacher might consider essential are few in number and would include:

The parallelogram law.
Fletcher's trolley.
Determination of g by various methods.
Coefficients of limiting and sliding friction.
The simple and the compound pendulum.
Kater's pendulum.
Moment of inertia of a flywheel.
Some experiments on the impact of elastic spheres.

Experiments on these topics are regarded as fundamental, and ought to be carried out by one department or the other.

To avoid unnecessary duplication of work it is essential that there should be correlation between the physics and mathematics departments. It is most important to establish uniformity both in the symbols used for the various physical quantities and in the choice of mathematical methods. For instance, it is desirable to agree upon the symbols used to represent moments of inertia, and on the use of graphical and/or calculus methods in establishing standard formulae.

The syllabuses which follow are intended to be inclusive in the sense that they indicate work which should be undertaken by one department or the other.

4. A Syllabus of Mechanics

The Basic Course. Resultant of two forces by parallelogram law; resultant of more than two forces at a point by resolving. Graphical resultant by polygon of forces. Equilibrium of three forces by triangle of forces. Lami's theorem. Conditions of equilibrium of more than three forces at a point. Moment of a force; resultant of a number of parallel forces. Equilibrium of parallel forces. Principle of moments for two intersecting forces. Laws of statical friction; coefficient, angle, and cone of friction. The rough inclined plane. Systems of bodies in contact. Centres of gravity.

Speed, velocity and acceleration. Resultant velocities and accelerations; resolution of velocities and accelerations. Relative velocity. Motion under constant acceleration. Vertical motion under gravity; motion on an inclined plane. Newton's laws of motion. Principle of work and energy, easy examples on energy equation, conservative systems. Impulse and momentum, equation of impulse and momentum, conservation of linear momentum. Direct impact of elastic bodies, Newton's experimental law. Circular motion, radial acceleration, motion in a horizontal circle. Simple harmonic motion, equations of motion studied graphically and by calculus, simple pendulum, elastic strings. Work, energy, and power.

The Advanced Course. Couples, combination of a force and a couple. Reduction of the forces acting on a rigid body to (*a*) a force or a couple, (*b*) a force at a given point together with a couple. Single force resultant by calculation. Conditions of equilibrium of a rigid body. Jointed rods, including the reactions at the joints. Harder problems, including limiting friction, toppling and sliding. Centre of gravity and centre of mass. Centres of gravity of tetrahedron and hemisphere, compound bodies and remainders. Definition of a vector, triangle law of addition, components of a vector, resultant of $m.\overline{AB}$ and $n.\overline{AC}$. Force and link polygons for determining resultants and solving problems on equilibrium. Bow's notation, application to jointed frameworks.

Density and specific gravity. Pressure intensity, thrust on a plane area. Centres of pressure of rectangle, triangle, compound areas, and remainders. Principle of Archimedes, law of flotation, hydrometers. Problems on immersed bodies partly supported. Boyle's law.

Angular velocity and acceleration. Problems on connected

particles, relative acceleration. Oblique impact of elastic bodies, energy losses in collisions of elastic bodies. Motion of a projectile in two dimensions. Motion in a vertical circle. Simple harmonic motion. Moment of inertia, its determination for standard bodies. The compound pendulum, use of the equation of energy. Dimensions of physical quantities and their uses.

5. The Scholarship Level

Before we can discuss scholarship syllabuses in detail, certain points have to be clarified. The first, and most vital, is to define what we mean by the Scholarship Level or a scholarship course. At the moment they are vague and general terms embracing a wide range of types of pupil.

To some teachers a scholarship pupil is a pupil of above average ability who might successfully attempt harder questions on the Advanced Level syllabus at the time when he comes to take the examination. To other people he is a pupil of outstanding ability who may be subsequently entered for an open scholarship in mathematics at one of the universities. There are other conceptions of a scholarship pupil ranging between these two extremes.

We prefer to regard the Scholarship Level in mathematics as a level beyond the Advanced Level, with the obvious corollary that more work will have been covered. It will therefore be designed for those who have the time and ability to do more work in the subject; and we conceive that pupils who are capable of undertaking this section of the course are those who could, if they wished, study mathematics as a subject leading to an honours degree.

The Scholarship Level course which we propose to describe must be regarded as a full year's course, and this means that if a pupil has taken two years of his sixth form career in reaching the Advanced Level, he will need to spend a third year in the sixth form in order to extend it to Scholarship Level. Only in the case of specially gifted pupils would it be possible to accelerate this course. Such acceleration would in fact be a necessity for those who intended to compete for open scholarships at the universities. The fact that open scholarship examinations begin in December means virtually that pupils must reach the Scholarship Level in two years and one term.

The conception of a third year in the sixth form is an important one, and it is worth while discussing some of its implications, as we do not intend to justify it on mathematical grounds alone.

The first point that arises is the question of age. If the main school course is one of five years' duration, then a three-year sixth form course involves a total period of eight years, so that a pupil entering at 11+ would be 19+ on completion of the school section of his studies. On the other hand, some schools have already published organization schemes under which the more gifted pupils can by-pass part of the Ordinary Level examination and proceed direct into the sixth form after four years in the main school. Such pupils would therefore be 18+ after completing a three-year course in the sixth form. Neither of these ages seems too old for the commencement of a university career. In fact, the impression given by the universities themselves is that they do not prefer students to come into residence at too early an age.

The second point to discuss is the purpose to be served by this third year. General considerations must enter partly into this question, though we are, of course, primarily interested in the question from the point of view of the teacher of mathematics. We visualize this year principally as a year for a complete consolidation and generalization of all previous work, the school course being 'rounded off', as far as the nature of the subject allows. This does not mean to say that from time to time we may not give a foretaste of more general or more powerful modes of mathematical thought, but that they would be introduced rather with the object of whetting the appetite than with the intention of undertaking any detailed work such as might be done at the university level. The emphasis that we are making here will become clearer if our Advanced Level syllabuses are examined again a little more closely. It will be found that several items appearing in some of the present-day Advanced Level syllabuses have been omitted, and will be found below in the Scholarship Level syllabus. We admit the necessity for their inclusion from the point of view of completeness, and this is in accordance with our conception of this third year as mainly a year of consolidation.

Should our arguments be accepted, there remains the task of justifying a third year in the sixth form to a boy himself. The first argument would be the desirability of not entering a university at too young an age, but rather of obtaining the fullest preparation that a

school can give. The second argument would lie in the value of the third year course in itself, and here we ought to mention that the third-year sixth form course should comprise more than one subject. We hope that in his third year a pupil will have a little more time for studies that give balance and culture to his school education.

In the mathematical course itself there are several points to look for. The first of these is a development in the power of method. This is an affair of slow growth, possibly retarded a little in the early stages by the steady and recurrent incidence of new work that has to be mastered. But instances arising from practical experience could be multiplied of uneconomical methods of working arising from such things as a lack of appreciation of the symmetry of an expression, or from unsystematic methods of attack. Many pupils, for instance, when proving that if

$$x_r = \frac{ay_r+b}{cy_r+d}$$

then

$$\frac{x_1-x_2}{x_2-x_3} \Big/ \frac{x_1-x_4}{x_4-x_3} = \frac{y_1-y_2}{y_2-y_3} \Big/ \frac{y_1-y_4}{y_4-y_3}$$

would substitute wholesale from the first expression. Perhaps the best way to state our objective would be to say that we would look in this year for a more complete development of a mathematical approach.

The second point to look for is a growth in a pupil's ability to work efficiently on his own. His work will, of course, be mainly guided; but the breadth or narrowness of his reading should be a matter largely for his own initiative (stressing once again the importance of a good mathematical library). To some extent the pupil may be *taught* how to plan a scheme of work efficiently; many pupils, for instance, attempt to study a particular topic by picking out the appropriate chapter in some text-book and laboriously working through every single example.

6. The Scholarship Level Syllabus

As a rough guide to our general attitude we would regard the earlier section of each syllabus as the equivalent of a satisfactory Scholarship Level in the G.C.E. The later section of each (enclosed in brackets)

contains topics which, to some extent, give a foretaste of further work; and some of which might be used as material for university scholarship examinations, though in such cases we should prefer only a simple treatment.

We must, however, emphasize that this is to be regarded as a background for a teaching syllabus rather than an examination syllabus, and for this reason it is not elaborately detailed.

Pure Mathematics. Successive derivatives; Leibniz's Theorem. Taylor's and Maclaurin's series. Length of arc; area of surface of revolution. Cycloid and catenary; some familiarity with other curves. Curvature. Some of the main types of differential equations. Curve tracing.

Properties of coefficients in the binomial theorem. Further work on determinants.

Summation of simple trigonometrical series. Complex numbers. De Moivre's Theorem and simple applications.

Harmonic properties; orthogonal circles; pole and polar. Radical axis and coaxal circles. Inversion. Cross-ratio and homography. Principle of duality. Orthogonal and conical projection, with treatment of the conic.

Completion of analytical study of the conics. More general treatment of the conic. Simple treatment of homogeneous co-ordinates.

(Involutes and evolutes. Partial derivatives. More detailed study of functions and equations. Further work on series. Convergency. Inequalities. Involution. Reciprocation, homography and involution applied to the conic. Pencils and ranges of conics. Line co-ordinates.)

Mechanics. Rolling friction. Analytical treatment of equilibrium of a rigid body. Virtual work. Equilibrium by the potential energy method. General reduction of systems of forces.

Centre of pressure of a circular lamina. Total thrust on a curved surface.

Motion with variable acceleration along a straight line and curve. Systems with one degree of freedom. Projectiles; the bounding parabola. Hodograph. Resultant of S.H.M.'s General treatment of the rigid body. Motion in two dimensions.

Vector methods applied to mechanics.

(Shearing force and bending moment. Equilibrium of chains and strings. Central forces.)

7. Sixth Form Geometry

Pure Geometry. An examination of traditional syllabuses will show that pure geometry undergoes two main lines of development in the sixth form:

(*a*) An extension of work begun earlier.

(*b*) An introduction to new and often very powerful ideas.

Under the heading (*a*) will come, for instance, a more complete study of the properties of the triangle. Concurrency theorems (medians, etc.) will usually have been met in previous work; but topics such as the Euler line, the nine-point circle, Ceva's Theorem, etc. may be added. The difficulty is to decide where to stop; some, for instance, have been known to extend the work as far as symmedians, Brocard points, etc.

The circle is the other major topic in which considerable extension is made. Typical bookwork includes, for instance, work on orthogonal and coaxal circles. Again the depth and range of study varies widely.

The subject-matter indicated under (*b*) comes, however, into a different category. Whereas that of (*a*) is liable to give the impression that geometry is being slowly worked out to a final end, that in (*b*) supplies a corrective and shows that the development of the subject is only beginning. We see how, by a systematic approach, the potentialities of the subject can begin to be grasped. It is suggested that material under (*a*) be kept to a minimum for future necessities, and that as wide a range of topics as is practicable should be treated under the heading (*b*). Some of these will be mentioned, and the general lines of procedure sketched:

(i) *The idea of point-to-point correspondence.* This could be first introduced as an idea only, as for instance in connection with homothetic figures and similar figures generally (the usual constructions given here form very effective illustrations). As soon as possible, however, one should bring in instances which show clearly the value of such a form of correspondence in deducing properties of one figure from those of another. Perhaps inversion would be the first obvious illustration for many people; but it should be treated only in general outline, as its importance is not so great geometrically as in its applications (e.g. the use of the 'method of images' in electricity). Much more useful would be a study of projection—in fact, much of

the work in orthogonal and conical projection is simpler than that done in other branches of geometry, and has greater subsequent use. Much work on conics can of course be done here.

(ii) *The idea of point-to-line correspondence.* This should not really be tackled until some appreciation of the first section is obtained. When it can be seen as a natural extension of the first idea the student will be better qualified to realize its possibilities. The early introduction of the principle of duality will enable the typical dualizing statements to be justified as they arise. The beginner will deal with pole and polar with respect to a circle, the more advanced student will consider reciprocation generally.

(iii) *Properties of sets of points.* This work will be mainly concerned with the study of four collinear points, associated with the first two topics. Cross-ratio must receive some treatment, and its links with projection in (i) and with perspective must be given due prominence. Most of the work will deal with the harmonic range. The harmonic property of the quadrangle should be obtained early, and the whole theory of pole and polar with respect to a circle can soon follow. There are links with projection and perspective, e.g. the connection between inverse points and pole and polar, the use of duality in deducing properties of concurrent lines, and the pole and polar theory of the conic by means of projection. The pupil capable of further progress while still at school can now extend his work to include homographies, involutions and their applications.

These three sections should form the background of a sixth form pure geometry course, and no pupil can have obtained a full appreciation of geometry until he has become familiar with the ideas they contain. Thus, even if they are not all included in the usual examination syllabuses, time is well spent in discussing some of these ideas.

Solid Geometry. This is given a separate heading in deference to custom, and not because it is considered that it should be developed independently. In fact we wish to stress its amalgamation with other aspects of geometry.

The traditional Euclidean development of solid geometry in the form of an independent 'book' with a large number of theorems is most undesirable and fosters the very isolation that we are trying to avoid. The mathematical significance of much of this work is slight, and only one or two key-theorems are required in other work. These might well be demonstrated when the need for them arises. Perhaps

the only theorem which has much application is that a line perpendicular to two non-parallel lines in a plane is perpendicular to any other line in the plane.

We indicate three ways in which the work in solid geometry can be amalgamated with other geometry.

(i) There are many two-dimensional problems which can be extended to three dimensions, and many text-books include such among their sets of examples on plane geometry theorems, e.g. $ABCD$ is a rectangle and P is any point, not necessarily in the plane of the rectangle; prove that $PA^2+PC^2 = PB^2+PD^2$. An alert teacher can in fact go a step further and 'generalize' many of his earlier riders (e.g. the usual rider on the lines joining the mid-points of adjacent sides of a quadrilateral, in which the four corners need not be coplanar). If this procedure could be regularly adopted with junior forms, much of the fear of three-dimensional geometry would be removed.

(ii) Much work on solid figures can be done by considering plane sections only. Obvious instances are those associated with a sphere (especially inversion with respect to a sphere, which is most naturally treated with inversion generally). Less obvious is the fact that many of the well-known properties of the tetrahedron can be developed at the same time as the corresponding properties of the triangle.

(iii) A place must be found in the pure geometry section for some study of the elements of three-dimensional space, and their relationships to one another—otherwise the study of projection cannot be adequately carried out. It is suggested that in teaching projection, the introduction should discuss the relations of points and lines in two-dimensional space, and the results be then extended to three dimensions. Some deductions can be made from the three-dimensional propositions themselves; for instance, the principle of duality can be extended to three dimensions.

Analytical Geometry. Again it is not desirable to isolate this topic, though, until the necessary skills have been mastered, there is bound to be some separation. Its position as a 'new and powerful idea' in geometry emerges at the very beginning, and the teacher needs to bring out as early as possible the correspondence between a locus and an algebraical equation. The power of the subject then becomes immediately apparent, and the pupil can easily appreciate a development in which we study either types of loci (line, circle, conic) or types of equation (first degree, second degree).

To many pupils analytical geometry involves little more than a study of rectangular cartesian co-ordinates. Other co-ordinate systems may well be discussed informally, e.g. oblique cartesians (their use and limitations), polar co-ordinates (their connection with the definition of the trigonometrical ratios of any angle is well known; the polar equation of a conic with its focus at the pole is well worth including because of its application in mechanics to central orbits), bipolar co-ordinates (perhaps just worth mentioning in connection with magnetism), and, of course, for the specialist, general homogeneous co-ordinates and line co-ordinates. The power obtained by choosing a suitable triangle of reference (especially in connection with conics) must be demonstrated.

We also plead for the early introduction and illustration of parametric methods. Opportunities can easily be found to illustrate the use of a parameter when the axes (or in later stages, triangle of reference) have been suitably chosen; and abundant illustration should be given of the simplicity of using a single variable parameter in preference to a set of co-ordinates.

These three ideas—of using algebra, of stressing flexibility in the choice of co-ordinate system, and the use of the parameter—are the essence of analytical geometry. With them a pupil can extend his studies to curves of higher order, and to the geometry of three dimensions.

As the study of analytical geometry progresses there is value in comparing and contrasting the power of its methods with those of pure geometry, e.g. the theory of coaxal circles can be developed by a blend of pure and analytical methods. Properties first developed analytically can often be used to deduce further properties by non-analytical methods so that a separate treatment of geometrical conics becomes unnecessary.

A plea may be made for a systematic training in conciseness of method. It is common, especially in the earlier stages, to see unnaturally long answers to problems which involve several steps. A few of the ways in which greater conciseness may be achieved are:

(*a*) A careful treatment of the method of obtaining the equation of a line through a given point with a given gradient. The form of equation $y-mx = y_1-mx_1$ has much to recommend it in comparison with the more usual form $y-y_1 = m(x-x_1)$. If the left-hand side is thought of mentally, and cleared of fractions if necessary

before being written down, a boy can soon learn to write down the required equation immediately.

(*b*) The equation of the perpendicular to a given line through a given point should also be written down immediately by remembering the rule for obtaining from $ax+by+c = 0$ the left-hand side of the standard equation $bx-ay = bx_1-ay_1$.

(*c*) Tangents and normals to circles and conics can be obtained quickly by memorizing only the transformations $x^2 \rightarrow xx_1$, etc., which give the equation of the tangent at (x_1, y_1). The tangent at a particular point (e.g. one of given parameter) can then be obtained immediately by substituting the co-ordinates of the point, and the normal by using (*b*) above. There is thus no need to memorize any formulae for normals; and, further, any calculation of gradients by calculus or other methods is unnecessary.

(*d*) The point of intersection of two lines is so frequently required that it is well worth while teaching the method of solution of two simultaneous equations by means of determinants. In fact, there is much to be said for extending this work to three-row determinants and showing some of the useful applications of these in analytical geometry, e.g. in expressing the area of a triangle (many of the mnemonics given for memorizing the usual formula are in fact nothing more than a disguised form of determinant notation), the condition for the collinearity of points and the concurrency of lines, and the condition for the general second degree equation to represent two straight lines.

(*e*) In homogeneous co-ordinates the determinant form is the natural one to use for writing down the equation of the line joining two given points, but, in writing down the equation of a line through a vertex, say X, of the triangle of reference the pupil should be taught to start with the form $y = kz$ rather than to use a determinant.

(*f*) Plenty of practice should be given in the use of equations of the form $S+kS' = 0$, where $S = 0$ and $S' = 0$ are the equations of two given loci.

A Study of Fundamentals. It is assumed that in all of the preceding work no deliberate attempt will have been made to avoid metrical methods of proof of theorems and results: at some stage, however, time should be found to assess the geometrical work done as a whole. A logical examination of the fundamentals of Euclidean geometry should be made, bringing out clearly the fact that it is based on

certain definite assumptions. This can be done by discussing other possible sets of assumptions which lead to other and self-consistent geometries. A brief discussion of non-Euclidean geometries and their significance is worth while. Such work as this may also be suitable for non-mathematical students who may be receiving one or two 'General Mathematics' periods per week.

The specialist pupil however, ought not to let the matter rest at this stage. He should appreciate the need for some 'comprehensive' geometry which would include the others as special cases. For him, some study of non-metrical geometry is possible, and he can soon cover enough ground to realize its comprehensive character. There are two possible approaches: the purely geometrical, as in Faulkner's *Projective Geometry*, or in O'Hara and Wood's *Projective Geometry* (the former is three-dimensional, which is preferred, and the latter two-dimensional); or an algebraical approach, as in Maxwell's *General Homogeneous Co-ordinates.*

It is recommended that, if the book is available, no pupil should miss reading Young's *Fundamental Concepts of Algebra and Geometry.*

Conclusion. In terms of syllabuses, these suggestions would seem to cover much ground. The future specialist should study as much of it as he can and obtain some acquaintance with most of the ideas. The time required for this depends not so much on the scope as on the detail of the work that is attempted. If our point of view is accepted, most pupils would gain a wider appreciation of geometry than would be possible within the limits of the traditional syllabuses.

VII

THE MATHEMATICAL LIBRARY

1. General Remarks

A mathematical library is now recognized as indispensable to a broad mathematical education. In some schools the library may consist of a number of books, for consultation, in a cupboard in a classroom; in others it may form part of a general school library; and in yet others it may be a combination of both. But however it may be housed, its potentialities cannot be realized if there is an insufficiency of suitable books, or if the pupils lack guidance in the use of the material available. Every master in charge of mathematics should aim at acquiring as comprehensive a selection of books as possible. There are many sources available to him to keep abreast of the times. At the end of this chapter we have appended a list of suggested books. It will be appreciated that some of them may not be immediately available. The *List of Books suitable for School Libraries* issued by the Mathematical Association should also be consulted, as well as reviews of recent publications in the *Mathematical Gazette*, the *Times Educational Supplement*, and other educational journals. When funds for the purchase of books are not sufficiently generous there can be scope for fruitful collaboration with a local public free library. Many library committees would indeed welcome suggestions from schools and other local educational institutions.

The mathematical library can be divided roughly into three parts, one for the staff, one providing text-and reference books for the use of more advanced pupils, and the third for general use.

In the staff section should be found a variety of books to which the teacher may turn for reference, refreshment, and inspiration and in which he can find information on the latest teaching methods in this country and abroad. Here, too, should be found a place for the various reports on the teaching of mathematics which are issued from time to time by the Mathematical Association.

Sixth form pupils, and particularly the mathematical specialists, will require a wide selection of books, not included in ordinary stock,

and these will probably be the first charge on the library. It may be convenient to have a number of such books kept in a mathematical classroom for ready accessibility.

It is perhaps in the general section that careful selection of books can affect the majority of pupils using the library. Ideally this section should contain books of interest to the youngest members of the school as well as to the eldest. Although this may not always be easy in practice, books can be found which are suitable for younger pupils. It has to be remembered that there are three main reasons why a pupil will consult a library:

(*a*) because he will find in it books which will interest him on their own account;

(*b*) because it contains books which will assist him in his work or his hobbies; and

(*c*) in order to further his mathematical studies.

Bearing this in mind we will, for convenience, classify this part of the library into sub-sections.

History. A pupil's first introduction to the library may well be through the historical section. There is a wide choice of books from which a selection may be made. Some, like the works of W. W. R. Ball and Sir T. L. Heath, will be used mainly by senior boys for reference. Others, such as *Number Stories of Long Ago* and *Number Puzzles before the Log Fire* by D. E. Smith; *A Short History of Mathematics* by Vera Sandford; and *The Story of Reckoning in the Middle Ages* by F. A. Yeldham, are suitable for pupils in the lower or middle school. As far as possible, books in this section should be graded for different stages in the school. Some biographies of mathematicians are also suitable, and one, at least, of Sir Isaac Newton should be in the library.

Books of Puzzles and Amusements in Mathematics. There are many attractive books of this type. Most libraries will contain W. W. R. Ball's standard work *Mathematical Recreations and Essays;* other books of a lighter nature are not to be despised. Many pupils are attracted by mathematical problems in fancy dress, and the interest in mathematics stimulated in this way may provide much enjoyment in later years.

Astronomy and Relativity. Few schools find time in the curriculum to teach Astronomy, although occasional reference to it can be made throughout the school course in mathematics. Properly handled,

references in the classroom may encourage many pupils to read further for themselves. There is an excellent range of books available, and some are of considerable interest even to those who are not mathematically inclined; as, for example, several of the late Sir J. Jeans.

Relativity is a subject difficult of comprehension even to mathematical specialists in the sixth form. The popular explanations of the subject by C. V. Durell, Sir P. Nunn, and by Einstein himself can, however, be recommended.

Books presenting Mathematics from an Unusual or Popular Standpoint. Such books excite curiosity and interest, and a number well worth a place in the library have been published in recent years. L. Hogben's *Mathematics for the Million* is a very popular example of a book of this type. Others are *Companion to Elementary School Mathematics* by F. C. Boon; *Flatland* by E. A. Abbott; *Number, The Language of Science* by Dantzig; *Mathematical Recreations* by M. Kraitchik; *Mathematician's Delight* by W. W. Sawyer. The reader can doubtless add to this list for himself.

Links with Other Subjects, Occupations, and Hobbies. Books which widen the contacts of mathematics should always be sought. Links can be made, for example, with biology, physics, chemistry, architecture, aeronautics, engineering, electricity and wireless, surveying, geography, etc.

Foreign Languages. Pupils in grammar schools normally have some knowledge of at least one modern foreign language, and those studying mathematics in the sixth form should be encouraged to do a little reading in mathematics outside their own language. They will at least learn something of mathematical terminology in another language, and may also lay the foundations of an appreciation of the work of mathematicians in other countries.

Finally a place should be found for copies of the *Nautical Almanac, Whitaker's Almanack* and other statistical tables.

Use of the Library. Even though it may be suitably stocked with books the mathematical library will still be neglected unless pupils are instructed in its use. It has no obvious appeal to most pupils, as have certain other sections of the school library, and the more inquiring who explore on their own account, and without guidance, may be discouraged by an early choice of unsuitable books. The following suggestions may be of help:

(i) The library should be kept in mind by the teacher in his class-work at all times and publicized accordingly. Even the youngest pupils can be made aware of its existence. The mention of a famous mathematician in the classroom, or the discussion of some matter of mathematical interest, may be made an opportunity to obtain a book from the library and to read out appropriate sections.

(ii) Pupils in their fourth or fifth year can be taken, as a class, to the library. Books which may interest them can be pointed out and described briefly. A list of such books might then be posted for reference.

(iii) A further talk on books, and an extended list, should be given to students entering the sixth form, and they should be given every encouragement to support their mathematical studies by further reading in the library. Text-books useful in their work can be referred to as occasions arise.

(iv) At all stages of classroom instruction pupils should be made aware of the broader mathematical background, of its links in history, and of its contributions to man's development.

(v) New acquisitions to the library can be publicized on a notice board by a display of their dust covers and possibly by short notes on their contents.

(vi) It is suggested that at certain periods books be made accessible to pupils and particularly that they should be made available to sixth-formers during private study periods.

(vii) If liaison has been made with a local free library mathematical books available there for scholars can also be discussed and publicized.

2. Some Suggestions for Books for the Mathematical Library

The following books are selected from the very many which could be regarded as suitable for a mathematical library in a grammar school. The omission of any book does not imply any criticism of its value, or, indeed, that it is any less suitable than other books included in the list.

Books which might be regarded as normal text-books for classroom use have not been included.

The Committee cannot guarantee that all the books are in print at the time of going to press.

History

BELL, E. T. (i) *Men of Mathematics* (Gollancz) (reprint by Pelican in two volumes); (ii) *Development of Mathematics* (McGraw-Hill).
HEATH, SIR T. L. (i) *History of Greek Mathematics* (2 vols.) (Oxford); (ii) *Manual of Greek Mathematics* (Oxford).
HEIBERG, J. L. *Mathematics and Physical Science in Classical Antiquity* (trans. by D. C. MacGregor) (Oxford).
HOOPER, A. *Makers of Mathematics* (Faber).
KLINE, M. *Mathematics in Western Culture* (Allen and Unwin).
SANFORD, VERA. *Short History of Mathematics* (Harrap).
SMITH, D. E. (i) *History of Mathematics*, vol. I. 'General Survey of the History of Elementary Mathematics'; vol. II. 'Special Topics of Elementary Mathematics' (Ginn); (ii) *Source Book in Mathematics* (McGraw-Hill).
STRUIK, D. *Concise History of Mathematics* (Bell).
SULLIVAN, J. W. N. *History of Mathematics in Europe* (Oxford).
TURNBULL, H. W. *The Great Mathematicians* (Methuen).
YELDHAM, F. A. *The Story of Reckoning in the Middle Ages* (Harrap).

Mathematical Recreations

BALL, W. W. R. and COXETER, H. S. M. *Mathematical Recreations and Essays* (Macmillan).
BOON, F. C. *Puzzle Papers in Arithmetic* (Bell).
CUNDY, H. M. and ROLLETT, A. P. *Mathematical Models* (Oxford).
DUDENEY, H. W. *Amusements in Mathematics* (Nelson).
KRAITCHIK, M. *Mathematical Recreations* (Allen and Unwin).
NORTHROP, R. P. *Riddles in Mathematics* (Hodder and Stoughton).

Astronomy and Relativity

BURNS, P. F. *First Steps in Astronomy* (Ginn).
EDDINGTON, SIR A. (i) *Nature of the Physical World* (Dent); (ii) *Space, Time and Gravitation* (Cambridge); (iii) *The Expanding Universe* (Cambridge).
EINSTEIN, A. *Relativity, The Special and the General Theory* (Methuen).
HARGREAVES, F. J. *The Size of the Universe* (Pelican).
IONIDES, S. A. and M. L. *One Day Telleth Another* (Arnold).
JEANS, SIR J. (i) *The Universe around Us* (Cambridge); (ii) *The Stars in their Courses* (Cambridge); (iii) *The Mysterious Universe* (Cambridge).
JONES, SIR H. S. *General Astronomy* (Arnold).
ROBSON, A. *The Earth and the Sky* (Bell).
WATERFIELD, R. L. *A Hundred Years of Astronomy* (Duckworth).

Statistics

ALLEN, R. G. D. *Statistics for Economists* (Hutchinson).
TIPPETT, L. H. C. *Statistics* (Home University Library).
WEATHERBURN, C. E. *Introduction to Mathematical Statistics* (Cambridge).
YULE, G. U. and KENDALL, M. G. *Introduction to the Theory of Statistics* (Griffin).

Technological Mathematics

ABBOTT, W. *Practical Geometry and Engineering Graphics* (Blackie).
BRODETSKY, S. *First Course in Nomography* (Bell).
GEARY, A., LOWRY, H. V. and HAYDEN, H.A. *Advanced Mathematics for Technical Students* (Longmans).
RICHARDSON, W. A. *Surveying for Schools and Scouts* (G. Philip).
SAWYER, W. W. *Mathematics in Theory and Practice* (Odhams).
STEERS, G. A. *Introduction to the Study of Map Projections* (U.L.P.).
SUTTON, O. G. *Mathematics in Action* (Bell).

Books Suitable for more Advanced Pupils

BAKER, H. F. *An Introduction to Plane Geometry* (Cambridge).
FROST, P. *Curve Tracing* (Macmillan).
GILLESPIE, R. P. *Integration* (Oliver and Boyd).
HARDY, G. H. *Pure Mathematics* (Cambridge).
HARDY, G. H. and WRIGHT, E.M. *Introduction to the Theory of Numbers* (Oxford).
HILBERT, D. *Foundations of Geometry* (Open Court).
HYSLOP, J. M. *Infinite Series* (Oliver and Boyd).
INCE, E. L. *Integration of Ordinary Differential Equations* (Oliver and Boyd).
KLEIN, F. *Famous Problems of Elementary Geometry* (Ginn).
KNOPP, K. *Theory and Application of Infinite Series* (Blackie).
MCCREA, W. H. *Analytical Geometry of Three Dimensions* (Oliver and Boyd).
MACH, E. *Science of Mechanics* (Open Court).
PHILLIPS, E. G. *Functions of a Complex Variable* (Oliver and Boyd).
PIAGGIO, H. T. H. *Differential Equations* (Bell).
RUTHERFORD, D. E. *Vector Methods* (Oliver and Boyd).
SOMMERVILLE, D. M. Y. *The Elements of Non-Euclidean Geometry* (Bell).
STEWART, C. A. *Advanced Calculus* (Methuen).
TANNERY, J. *Leçons d'Arithmétique Théorique et Pratique* (Armand Colin).
TURNBULL, H. W. *Theory of Equations* (Oliver and Boyd).

Miscellaneous

ABBOTT, E. A. *Flatland* (Blackwell).
BAKST, A. A. *Mathematics, Its Magic and Mystery* (Van Nostrand, New York).
BELL, E. T. (i) *Magic of Numbers* (McGraw-Hill); (ii) *Mathematics, Queen and Servant of Science* (Bell).
BOON, F. C. *A Companion to Elementary School Mathematics* (Longmans).
CLIFFORD, W. K. *The Commonsense of the Exact Sciences* (Kegan Paul).
COURANT, R. and ROBBINS, H. *What is Mathematics?* (Oxford).
DANTZIG, T. *Number, the Language of Science* (Allen and Unwin).
HARDY, G. H. *A Mathematician's Apology* (Cambridge).
HOGBEN, L. *Mathematics for the Million* (Allen and Unwin).
KASNER, E. and NEWMAN, J. *Mathematics and the Imagination* (Bell).
LITTLEWOOD, J. E. *A Mathematician's Miscellany* (Methuen).
MCKAY, H. (i) *Odd Numbers* (Cambridge); (ii) *The World of Numbers* (Cambridge).
POINCARÉ, H. (i) *La Science et la Méthode* (Flammarion); (ii) *La Science et l'Hypothèse* (Flammarion); (Both books are available in translated form from Dover Co., New York).
POLYA, G. *How to Solve It* (Princeton Univ. Press).
RUSSELL, B. *Introduction to Mathematical Philosophy* (Allen and Unwin).
SAWYER, W. W. *Mathematician's Delight* (Pelican).
STEINHAUS, H. *Mathematical Snapshots* (Oxford).
WHITEHEAD, A. N. *Introduction to Mathematics* (Oxford).
WHITTAKER, E. T. *From Euclid to Eddington* (Cambridge).
YOUNG, J. W. (i) *Lectures on the Fundamental Concepts of Algebra and Geometry* (Macmillan); (ii) *Monographs on Topics of Mathematics relevant to the Elementary Field* (Longmans).

Books for Teachers

BALLARD, P. B. *Teaching the Essentials of Arithmetic* (U.L.P.).
BLACK, M. *The Teaching of Mathematics: a Bibliography* (Christophers).
BROOKES, B. C. *Notes on the Teaching of Statistics in Schools* (Heinemann).
GODFREY, C. and SIDDONS, A. W. *The Teaching of Elementary Mathematics* (Cambridge).
KLEIN, F. *Elementary Mathematics from an Advanced Standpoint*, vol. I. 'Arithmetic and Algebra'; vol. II. 'Geometry' (Macmillan).
NUNN, SIR T. P. *Teaching of Algebra (and Exercises in Algebra)* (2 vols.) (Longmans).
SUMNER, W. L. *The Teaching of Arithmetic and Elementary Mathematics* (Blackwell).
WESTAWAY, F. W. *Craftsmanship in the Teaching of Elementary Mathematics* (Blackie).

The Year Books of the National Council of Teachers (Bureau of Publications, Teachers' College, Columbia Univ., New York):

The Nature of Proof
The Training of Mathematical Teachers
The Place of Mathematics in Secondary Education
Arithmetic in General Education
A Source Book of Mathematical Applications
Multi-sensory Aids in the Teaching of Mathematics

Mathematical Association Reports on Teaching in Schools (all published by Bell):

Arithmetic (1932)
Algebra (1934)
Geometry—First Report (1924); *Second Report* (1938)
Mechanics (1930)
Trigonometry (1950)
Calculus (1951)
Higher Geometry (1953)
Mathematics in Secondary Modern Schools (1950)
Mathematics in Secondary Technical Schools (1950)
Visual Methods in Teaching Mathematics (1953)

VIII

THE MATHEMATICS ROOM

1. Introduction

Although we have sought throughout the country for evidence of the success of mathematics rooms or laboratories and have appealed through the press for information, very little material has reached us. We have, therefore, been forced to the conclusion that, though much may have been said or thought about developing such rooms in schools, very little work has actually been done. Certainly no effort has yet been made to collate the experience gained in the few ventures which have been at least partially successful. Too often the experiments have been short-lived; they have been started by enthusiastic individuals who have been translated to other spheres before their work has reached maturity. Usually no efforts have been made to appoint successors committed to develop the projects, and so through lack of continuity they have fallen into disuse. More generally, however, there has been such keen competition for the development of specialist rooms in other subjects that teachers of mathematics have all too often found that their pleas for mathematics rooms or laboratories have fallen on deaf ears; there have been neither rooms nor funds available.

The reader may, therefore, well ask why we devote a whole chapter to the mathematics room. We have three good reasons. First, there is evidence that schools which have devoted a room specially to mathematical work have found that the resulting stimulus has amply justified the experiment. Secondly, though these notes have in mind mathematics rooms in grammar schools it is probable that they could be of greater use in modern schools where the time given to practical work is not limited by the exigencies of a public examination. Moreover, under the new G.C.E. the mathematical course need not lead to a public examination for all grammar-school pupils. In the future, therefore, we anticipate that more pupils will have time for practical work in mathematics. Thirdly, we as a committee believe that this kind of development in teaching is to be encouraged and we hope that our words may give some of our readers the desire to experiment.

In our first investigations we confined our attention to the idea of a mathematics laboratory designed solely for experimental or practical work, but it has gradually been made clear to us that there are advantages in pressing for a room which is set aside solely for mathematical studies, but in which practical work, though still important, is but one of several forms of activity carried on therein. We have therefore taken as our title 'The Mathematics Room' in preference to 'The Mathematics Laboratory'.

2. The Purpose of the Mathematics Room

The organizer of the mathematics room will want it to be the centre of all mathematical activity within the school. It will be but one of the many rooms in which mathematical work is done, but its influence must be made to radiate to any part of the school where mathematics is studied. This will happen only if it is accessible at some time during the week, either in teaching time or in out-of-school activity, to all pupils; to the boys in the lower forms and the more backward streams as well as to the average pupils, to the mathematical specialists and to all the other members of the staff who teach mathematics. It must be looked upon by all, from the youngest pupil to the senior mathematics master, as the headquarters of the mathematical department.

The mathematics room will have many and varied uses but its prime function during school hours will be to give as many pupils as possible the opportunity to engage in the sort of practical work which cannot be undertaken in an ordinary classroom. It will, therefore, be equipped for such experimental work at all levels in mathematics, mechanics and possibly statistics as has not already been catered for in the physics laboratory.

The main aim of practical work will be to perfect and clarify the skills and ideas that are being taught in the classroom: it will be closely correlated with the syllabus of classroom work. It will be devised by the master in charge to give in the first place more significance to mathematical skills and eventually to elucidate mathematical ideas, particularly abstract ones such as 'functionality'. Since the newer examination syllabuses show greater concern for mathematical ideas than syllabuses of the past, a fairly comprehensive course of practical work these days should enable at least

the better pupils to acquire such a feeling for mathematics or mathematical sense as came to previous generations only by persistent study of mathematical techniques over long periods—and sometimes not at all.

If the main aim of practical work seems ambitious, we hasten to add that it has other and more utilitarian aims. Not the least among these is the desire to give some reality to the subject, i.e. to exhibit the relatedness of mathematics to everyday life. Too often the pupil thinks of mathematics as a set of tricks to solve problems of recognized patterns, and the longer he is allowed to retain this false conception the more difficult does it become to dispel it. Practical work from an early stage can and should be used to place mathematics in its proper setting—against a background of real life.

No less utilitarian is the desire to promote interest and sustained concentration on the part of the pupil. Practical work permits the use of more of a pupil's faculties than does classroom teaching, which relies mainly on hearing and seeing. In bringing more of his personality to bear on the subject he achieves a greater degree of concentration and, one hopes, a greater degree of success thereby. Moreover, since practical work usually takes longer to do, it can hold his attention to an idea for a longer time than a theoretical solution might require: this gives him time for assimilation. Indeed, if his aptitude for the subject is limited, he may find, through this different medium, explanation and clarification of so much that previously remained a mystery that his boredom or indifference may be replaced by a thirst for knowledge.

Though our contention is that practical work has its place in mathematical studies and that the mathematics room must provide the facilities, we believe that its use is limited and that it does not replace but only supplements classroom teaching. We shall discuss the use of the mathematical room for teaching purposes at a later stage. Here we offer two reminders:

(1) In classroom teaching pupils should still be encouraged to see the mathematical significance of common objects and to reason mathematically without needing special apparatus. The main work on solid geometry and inclined planes, for example, will still be done in the classroom with the edges of furniture, desk lids, the walls of the room and rulers. It will be supplemented by practical work in

the mathematics room in particular examples that are difficult to visualize.

(2) In practical work in the mathematics room great care should be taken by the teacher to prevent the misconception that mathematics depends only upon axioms which are established by measurement.

The mathematics room will also be the focal point for practical work in such topics as surveying, astronomy, navigation and statistics. Indoor practical work in any of these branches will be done here. Instruments and apparatus that have been bought or made within the school, either for indoor or outdoor work, will be kept in the mathematics room. Apparatus or results that are suitable for display will be on show here and will be available at stated times for close scrutiny. There will be facilities for displays of mathematical material from newspapers or periodicals such as *Mathemtical Pie*, data relating to the history of mathematics or sketches of functions and curves. These displays must be changed from time to time.

The mathematics room will provide a meeting place for the Mathematical Society and an instruction centre for some of the work of the combined cadet force and the scout troop or girl guide company. There is much mathematical work that would give pleasure to senior pupils as an out-of-school activity, or that occurs in the syllabuses of the uniformed organizations; what better place for them to be studied in than the mathematical room, which presumably will already have the appropriate atmosphere and equipment.

Finally, the mathematics room will be the administrative centre of the whole department and in this way it will fulfil its primary aim. It is to this room that the mathematics staff will naturally come for their meetings, whether formal or informal. Here one may expect to find the reference library of mathematics books suitable for teachers and specialist pupils. The room may also contain, as far as accommodation permits, the stock of surplus mathematics text-books; it may also be the store from which other mathematical supplies, e.g. blackboard compasses, drawing boards, instruments and exercise books, are issued and to which they are returned. But in spite of all this activity of administration, the mathematics room will not really have performed its function as the headquarters of the mathematics department nor fulfilled its primary aim of radiating

a mathematical influence to all corners of the school until on 'Open Day' it houses an exhibition of mathematical work worthy of the department.

3. Description of the Mathematics Room

The mathematics room should have about twice the floor space of a normal classroom or be of a size comparable with the main laboratories. As in modern school construction widths of rooms tend to remain standard throughout the one school, we shall presume that the extra space will be achieved by lengthening the room and that windows and doors (except those to anterooms) will in general be on the side of the room and not on the ends. The room will have features in common with a demonstration room, an art room, a woodwork and metalwork shop and a physics laboratory, with some elements from a builder's yard thrown in. It may be divided roughly into four parts; the teaching end, the class area, the laboratory or workshop end, and an anteroom or store room.

At the teaching end there should be a large wall blackboard. Mathematics teachers need a large amount of blackboard space because they often need to refer back to previous work.

A part of the blackboard should have ruled lines for graphical work. A roller blackboard with three plain surfaces and one graphed surface, each about five feet square, may be found convenient. On the other hand, if fixed wall blackboards are in use, it is an advantage to have a single *detachable* blackboard with graphed surface which can be hinged to the fixture with the long side horizontal or vertical as required. It will be easier to change the position of the graph board if the lower hinge pin is slightly longer than the upper one. Whatever type of graph board is adopted, however, it is desirable that it should bear markings similar to the graph paper in normal use. It will then be seen that the detachable graph board offers an advantage in that the teacher may place his board with its long side horizontal or vertical whilst each member of the class does the same with his piece of graph paper, and may demonstrate convincingly the size, shape and position of the graph under discussion. Moreover, once it has been established that there is a definite scale relationship between the graph work on the board and that on paper, the same scale relationship can be used for geometrical drawing on the board and, of course, measurements can be read off by the

teacher to the same degree of accuracy as that obtained by the pupils. All that is required is a blackboard ruler for the teacher's use marked out to the appropriate scales in inches on one side and centimetres on the other; two corresponding scales screwed to the wooden supports of the blackboard are also needed so that the class may judge the accuracy of the teacher's work. Where there are facilities for detachable boards, the teacher may later construct a graph board for polar co-ordinates to correspond with the polar paper in normal sixth form use.

On each side of the blackboard low cupboards up to 4 ft. high may be installed either as fixtures or as standard-type furniture. In either case they should vary in dimension from front to back, from narrow ones suitable for books to those wide enough to take drawing boards, and some of them should be glass-fronted. These cupboards will be used primarily to house personal necessities and equipment useful to the teacher, including the mathematical reference library, blackboard instruments, demonstration models, exhibition material and visual aids (but not the film or film strip projector). The space on the wall above the cupboards may be used for display purposes, for example a fairly large notice board and a display cabinet could be fixed on opposite sides. The notice board would be used to display the results of experiments or statistical investigations, mathematical material from newspapers and periodicals, data relating to the history of mathematics, and sketches of most of the elementary functions and curves. The cabinet on the other hand would be used to display models and pieces of interesting apparatus where they will be free both from dust and meddlesome fingers. Material displayed either on the notice board or in the cabinet must be changed from time to time.

It is unnecessary to specify whether the teacher should have for his use in front of the blackboard a desk, as in a classroom, or a demonstration bench as in a laboratory. The final decision in this matter may depend on the extent to which the mathematics room is to be used as a laboratory and it may also depend on the availability of gas, electricity and water—for a demonstration bench would be of little use without these facilities. If a desk is provided it should be large and it should have not only a flat top on which to set up apparatus for demonstration purposes but also an ample supply of drawers. It might also be useful to have a beam above the teacher's

desk from which he could suspend demonstration models of the pulley type. It will be seen in a later paragraph that we have made such provision at the workshop end of the room.

The class, which occupies the middle of the room, may be supplied with flat-topped tables and chairs. Tables 2 ft. by 3 ft. 6 in. by 2 ft. 3 in. high, with two drawers, suitable for two pupils, may be found convenient and at least fifteen tables would be required. If the room has to be used as a form-room, which is quite likely in these days of overcrowded schools, the form's personal effects can be kept in lockers which would be placed for convenience by the side walls between the windows. On the other hand, if the room can be used less as a form-room and more as a laboratory, each pair of pupils could work not at a table, but at a specially designed 'bench storage-unit' about 2 ft. by 3 ft. 6 in. by 3 ft. high. Each unit would be equipped with a selected range of numbered components rather like a 'Meccano' set from which the pupil could build up apparatus for a wide range of experiments. In the unit would be numbered compartments, slots or other form of storage for each component, thus enabling the teacher to make a rapid and effective check on apparatus.

Whatever alternative is adopted for the middle of the room, the laboratory or workshop end of the room should contain one or more work benches of the type used in woodwork and metalwork shops. It is on these benches that most of the mathematics department's new ideas will be developed; here the experimental work will be prepared, here the models both for display and class-teaching will be made, and here the keen mathematicians will have their play-ground for developing new techniques.

It is not anticipated that there will be any very general demand for the use of heavy apparatus, though some schools use industrial models of apparatus such as Weston's differential pulley block, but it is necessary to have a clear idea of the amount of heavy apparatus that will be used before planning the room's lay-out. It will generally be found sufficient to have one girder support parallel to the work-shop end of the room, with clear space for suspended apparatus. The ends of the girder may be embedded in two opposite walls or the girder might be attached to the ceiling or form part of the framework of an open roof. It will be useful to have a bench at least a foot wide, fixed against the wall and under the beam so that

apparatus fixed on one side can rest on the bench and on the other side hang clear of it. Normally the bench could be used for storing balances or other equipment and it is suggested that as this may be the only fixed bench in the room it should run the full width of the room, be fitted with cupboards and drawers below and have taps for gas and water and a point for electricity above.

One of the main purposes of the electric point is to provide power for a film or film-strip projector. There are two consequences of this intention to use a projector; first, the designers must make arrangements for the windows to be blacked out and secondly either a white screen must be fitted so as to fall in front of the blackboard or the teacher's desk or bench must be large enough to support a portable screen. We offer the suggestion that one of the surfaces of a roller blackboard or the reverse side of a detachable blackboard could be silvered for this purpose.

For such apparatus as pulleys, pendulums, ballistic pendulums and longitudinal coiled springs, a wooden beam fixed to the walls at intervals by brackets and built into the wall at each end is useful. Any supports can be screwed into the wooden beam. Most class experiments on pulleys and inclined planes can be performed on the bench using a framework of heavy retort-stands and cross-pieces supplied by all manufacturers of apparatus, although many of these systems are expensive. The advantages of arrangements which utilize clamps attached to the bench will be obvious to all teachers who have had experience of this kind of work. Apparatus is easier to arrange if geometrically true clamps and boss heads are used. Many elaborate fittings fixed to ceilings or walls have from time to time been suggested, but it is doubtful if some of these justify the expense.

It should be pointed out that arrangements which enable pulley, parallelogram of forces, pendulum and similar experiments to be carried out by pupils around the walls of the room have the effect of increasing the room's useful area. Arrangements that have been found useful include:

(1) Wooden beams supported on girders built into the walls, at right angles to the beams.

(2) A girder across the end of the room, 3 ft. from the wall, 10 ft. from the ground, with movable supports for apparatus.

(3) Bolts built into walls at suitable intervals to which either vertical or horizontal bars can be fixed.

(4) Tapped sockets embedded in a concrete ceiling; supports can be screwed into these sockets.

(5) Two-inch boards arranged parallel to the wall and 4 in. from it.

(6) Various arrangements of vertical and horizontal wooden strips in sizes such as 4 in. × 2 in., 2 in. × 2 in.

(7) Arrangements of $\frac{1}{2}$ in. steel rods parallel to and 4 in. from the wall surface.

(All supports parallel to a wall should be fixed clear of the wall.)

Finally, opening out of one end of the room, preferably the workshop end, there should be a small store-room with accommodation for the stock of surplus mathematical text-books, for the stock of mathematical exercise books and other expendable stores and for the storage of partly erected experimental equipment which must be locked away but cannot be conveniently placed in cupboards. It may even be found that such an anteroom could be adapted for the teaching of small sixth form classes, say those containing a maximum of six pupils; there would, of course, need to be good light, proper seating accommodation and room for a blackboard.

Though it has been presumed in the above description that the mathematics room is to be a feature of a new school building it may also be found convenient to convert to this purpose a large classroom in an existing school. In this case steps should be taken to incorporate as many of the above features as possible. In all cases of proposed overhead girders or beams for suspending apparatus the architect must be consulted and the arrangement adopted must depend largely on the design and lay-out of the room.

4. Apparatus

Just as the design of the mathematics room depends so much on local conditions that it is impossible to give more than suggestions of features to be included, so the use of the room depends so much on the intentions of the staff who organize it that it is impossible to give more than suggestions for its initial stock. This minimum of apparatus will allow the work to go forward, so that as the teacher develops his own syllabus he can add to it to suit his needs. Some articles are needed for frequent use in elementary work, and are thus

required in sufficient numbers to provide each pair of pupils in a full form, or each member of a half-form, with a complete set. The first essential is to obtain these. Purchase of apparatus which is too expensive or too cumbersome to be issued to members of a class but useful for sixth form work and for demonstration purposes or for larger groups of pupils can be deferred until later.

Among items to be acquired immediately are the following:

Metal vessels such as calorimeters and tins; paste-pots or jam-jars; tin boilers for raising steam.

Metre and half-metre rules—preferably marked also in inches and tenths.

Measuring cylinders—100 c.c. (by 1 c.c.), 200 c.c., 500 c.c.

Density bottles—25 c.c. or 50 c.c.

U-tubes, Hare's tubes—50 cm. leg.

Boyle's Law tubes (glass only); barometer tubes—not so narrow that mercury cannot be poured down them.

Retort stands—large (24 or 30 in. high) with good bases, three rings (stainless steel is recommended), clamps and bosses to take smaller as well as wider articles.

Tripods of suitable height to use with Bunsen burners.

Bridge and glass jar for weighing immersed solids.

Inclined plane and brass roller.

Single pulleys (3), double (2), made of aluminium or bakelite.

Spring balances (3), e.g. up to 1 kg. or to 7 lb., or a light set in metric weights and a heavier set in lb. For general purposes 8 lb. by $\frac{1}{4}$ lb. and 3 kg. by 1/10th kg. (tubular) will serve.

Set of iron weights with rings—100, 200 (2), 500 and 1,000 gm.

Set of 2 cm. cubes of various materials for density experiments.

Balance to read up to 250 gm. at least.

Note. Sets of apparatus are very useful and very efficient, but usually rather expensive.

The smooth running of the mathematics room will often depend upon the supply of a number of 'oddments'. These include such things as string and thread, sealing wax and soft red wax, sellotape, needles and pins, drawing pins (good quality), sandpaper, pliers, screwdrivers, rubber bands, gummed paper and labels, matches, squared paper, white cardboard, scissors, corks, aluminium paint, white and dead-black paint, iron wire, cotton wool, modelling wax for filling up holes round tubes or wires, or for supporting a plumb line from the edge of the bench, etc. Small tins which have contained cigarettes, pastilles, etc., are very useful as containers, so are pill-

boxes to be obtained from chemists, preferably used with glass lids so as to avoid much labelling.

A good supply of small metal parts suitable for building can be obtained by the purchase of 'Meccano'-type components and model railway parts. These, together with the appropriate miniature nuts, bolts and screws, may form the basis of some good model-making. There should also be available supplies of small timber, primarily for model-making, and at least one full set of woodworkers' tools. A motor, electric or clockwork, is most useful.

It is not anticipated that the mathematics room will have a film or film-strip projector for its own use, but that both will be available in the school. The mathematics room should, however, contain a small library of film-strips and possibly a few films, or there should be an arrangement by which these may be borrowed as required. The above, and any other visual aid material purchased in advance (and the reader is reminded that the majority of visual aid models will of course be made in the mathematics room) may be classed as teaching apparatus. Other items of teaching apparatus required include a set of blackboard instruments.

5. The Use of the Mathematics Room in Teaching

In discussing the aim of the mathematics room, we have indicated ways in which the room may be used for purposes other than class-teaching, but as its main function is to give opportunities for practical work within the framework of the school time-table, it is necessary to outline the ways in which this may be done.

It will be seen that for the first five years of the school course every form can have the opportunity of using the room for at least one period a week of normal teaching time each year, that the sixth form may have several periods allotted, and that this still leaves time for additional special courses. One such course might consist of a double period a week in the fourth year (or one period a week in the fourth and fifth years) to be devoted very largely to mechanical experiments, the emphasis being mathematical rather than physical. Such a course could be made a suitable introduction to the ideas of dependent and independent variables, functionality and the calculus, thus giving reality to processes that might otherwise become a series of special tricks. In a four-stream school, for example, the periods

of a 35-period week could be allocated as follows: 1 period to each of 20 forms (5 years of 4 streams), 8 periods to a mechanics course as outlined above (4 forms at 2 periods each or 8 forms at 1 period each) and 7 periods to sixth form work.

Lessons in the mathematics room can be of three types: (*a*) practical lessons where work is done individually or in small groups; (*b*) demonstrations by the teacher; (*c*) ordinary lessons which gain by being held in a mathematical setting.

Ideally, practical work should be done individually or in pairs so that every member of the class is taking an active part in it. In practice, however, apparatus for some pieces of practical work is in limited supply and in these cases one of two alternatives can be adopted. In the first method, a number of experiments can be amalgamated into a group and conducted on a rota system so that only one or two pairs are doing the same one at the same time. Alternatively pupils may occasionally work in groups of four or even more. This is not in general to be recommended and should be allowed only when other arrangements are impossible and in cases where each pupil in turn may perform the operations of the experiment. Experiments on the parallelogram law and on the determination of g fall into this category.

There are times, however, when apparatus is not available in sufficient quantities even for group work. In such situations it is appropriate for the teacher to conduct a demonstration lesson. He may also decide to adopt this course at any point in the syllabus when he feels that there is not sufficient time for general practical work but that a swiftly and neatly executed practical display will illuminate underlying principles.

Some mathematics lessons gain in effectiveness by being held in a mathematical setting. It would obviously be impossible to give an exhaustive list of such types of lessons, especially as their effectiveness in any case depends very largely on the way in which they are handled by the teacher, but for purposes of illustration we have in mind the following examples:

1. Lessons in constructional geometry done by draughtsman's methods with drawing boards, T-squares, etc.
2. Lessons in solid geometry with the solid objects and their sections available for inspection and reference.
3. Lessons illustrated by film or film-strip projection.

4. Lessons on graphs and curves, especially if prepared diagrams can be displayed on the notice board and if the mathematics room contains the only available graph board.

5. Lessons on statistics for which pictorial information can be displayed around the room beforehand or in which sampling experiments are performed.

The organizer of a mathematics room should have a sure grasp of the essential principles which he wants to illustrate; he should have some ability at display work even to the point of showmanship; he should be good with his fingers and be able to improvise; he should be imaginative and enthusiastic. He will know how and when to use ready-made apparatus but he will recognize that there is value of a different kind in building up his own laboratory material. He will not value the accumulation of specimens for their own sake so much as for the benefits which his pupils derive from making and collecting them. It would perhaps be too hard to say that at the end of every school year the organizer should dispose of all specimens, yet the mathematics room should share with the art room the feature that it is mainly current work that is exhibited; it should not become a museum.

There are several dangers which the organizer of a mathematics room should do his best to avoid. Three of these deserve special mention:

(1) The work of the mathematics room may become wasteful in the sense that time is not being employed to the best mathematical advantage; attention must be continually focused on the mathematical value of what is being done. The teacher should handle and be familiar with the difficulties and limitations of any apparatus before using it in class. He should be assured that it does not take too long to set up and he should have an idea of the standard of accuracy of the results which may be obtained from it. Only by this kind of preparation can he leave adequate time to impress on the class the mathematical value of the exercise.

(2) The work of the mathematics room may easily become stereotyped and limited and so cease to be stimulating. It is true that work will have to be devised so that it is as simple as possible and so that it is capable of being performed within one period and recorded either for homework or in a subsequent period. In this sense it can become so much a routine operation that the fact that the work was originally designed to make some diagram or problem come

alive or to verify some principle is gradually forgotten. The work then ceases to be stimulating.

(3) Experiments must be related not only to the pupil's experience, but also to his stage of mental development. It would be so easy, for example, to leave a pupil with the idea that a moment is something connected with the balancing of weights on a metre rule, instead of convincing him by comparison with other examples with which he is already familiar that it is the measure of the turning effect of a force about a point.

In outlining the use of the mathematics room for teaching purposes we do not feel it to be part of our duty to suggest courses of work to be undertaken therein. In the first place the development of the idea of a mathematics room is itself in such an experimental state that we are in no position to say that certain courses have proved satisfactory; we can but offer suggestions. Secondly, we have made it clear elsewhere that the development of any one mathematics room will depend so much on the organization of the school in which it functions and on the teacher who organizes it, that, were we to offer more than suggestions, these would probably fit neither the intention of the organizing teachers nor the arrangements within the school. We think that the mathematics room will stimulate the organizing teacher to illustrate practically, and so enliven, the ideas he wishes to stress. Apart from the already well-known series of experiments in mechanics which we have mentioned elsewhere and which he would obviously arrange to do unless they are already being satisfactorily dealt with in the physics laboratory, he might care to undertake some or all of the following pieces of practical work, which, we repeat, are suggestions only and are not intended as a comprehensive list:

(1) The making and use of a cross-staff.

(2) The making and use of a simple clinometer.

(3) The making of a demonstration model for setting out a right angle using two isosceles triangles.

(4) The making of solids from their nets or otherwise.

(5) The making of skeleton solids with $\frac{3}{16}$ in. square section wood and balsa wood cement.

(6) The setting up of a projector and a 45° mirror so that by shadows and the skeleton solids made above, plans and elevations can be demonstrated simultaneously.

(7) The use of the plane table.

(8) The construction and use of parallelogram mechanisms including letter balances, kitchen scales, locomotive coupling rods to ensure four- or six-wheel drive, etc.

(9) The construction and use of a Watt's 'parallel motion' mechanism for illustrating locus. (Three rods and two rubber suckers to attach it to the blackboard are all that is required.)

(10) The making and use of slide rules, pantographs and verniers.

6. Conclusion

Though we have described the mathematics room in detail and though we have advocated greater experimentation along these lines, it must be made clear in conclusion that we have not advocated a mathematics room for every school. In some schools many of the facilities described above will have been ensured by very close co-operation between departments and so the advantages to be gained by a reorganization might not justify the time, trouble and expense. In others there is no mathematician of the right temperament and training to undertake the work of organization, in which case the experiment would be doomed to failure before ever it began. However, given the right circumstances, we believe that there is great advantage in the creation of a mathematics room.

IX

THE TEACHING OF STATISTICS

1. Statistics up to the Ordinary Level

What is Statistics? Provisionally, statistics may be defined as the collection, analysis and interpretation of numerical data. Boys and girls at school are confronted with numbers not only in the mathematical classroom but also in their reading of newspapers and in their other school subjects such as physics, chemistry, biology, geography and history: for the Ordinary Level pupil statistics is the critical assessment of such numbers.

Many teachers of mathematics would claim that if 'statistics' is used in this sense then they already teach the subject, and, with limitations, this claim is justified. The teaching of this kind of statistics may even begin in the primary school, though it is neither necessary nor desirable to call it statistics at that stage. In the pre-Ordinary Level stages, however, it is suggested that the teaching of the 'interpretation of numerical data' is likely to become more purposeful and more effective if the subject is given a label: 'statistics' is the most appropriate.

Statistics is sometimes confused with arithmetic; in fact it is sometimes thought of as being a particularly futile form of arithmetic. But there is a difference between statistics and arithmetic even though they are usually inextricably mixed. In arithmetic we are concerned mainly with the *manipulation* of numbers, in statistics mainly with their interpretation in physical terms. The arithmetician accepts the data with which he is provided, or abstracts them from their physical occurrence, performs his calculations and, when he has produced his answer, the task is finished. The statistician, on the other hand, challenges the validity of any numerical data proffered for his inspection, then he satisfies himself that they have some physical meaning and are what they purport to be before he turns arithmetician. And when the arithmetical answer is obtained he does not simply write 'Answer: 306,259' (or whatever the answer may be) but interprets the answer in physical terms. To be able to challenge

the validity of the data and to interpret the results of his arithmetic the statistician must of course have some knowledge of the field from which the data is drawn. To the statistician, arithmetic is a necessary evil which he takes some trouble to avoid or to minimize. Those, for example, who add up marks which have been taken in a variety of ways from a variety of subjects to produce totals and averages for form orders are arithmeticians; the statistician would find it difficult either to accept the marks as meaningful measures of comparable quantities or qualities or to attach any precise significance to the process of their summation.

Up to Ordinary Level, statistics can therefore be thought of simply as applied arithmetic. When we recommend, as we do here, the wider teaching of statistics in this sense we are of course implying that, in general, the teaching of arithmetic in secondary grammar schools is at present too 'pure', that is, too remote from practical application within the understanding and experience of schoolchildren. But though statistics can provide a direct link between the arithmetic of the classroom and that of everyday life, it must be remembered that schoolchildren spend much of their everyday lives, and find many of their main interests, in the school itself. Most of the numerical data with which they are confronted arise out of lessons in science, history and geography, from their marks and from their games. The teacher, by interesting his pupils in this kind of data, can make mathematics more realistic for them. Statistics can help to link mathematics with other school subjects and other school activities.

Statistics as applied arithmetic requires the learning of no techniques or arithmetical processes beyond those ordinarily found in syllabuses of arithmetic, but only their wider and more intelligent application. It does however require an understanding of arithmetical process deeper than that normally required to 'do sums' (e.g. Divide 3·071 by 0·036); it also requires both the ability to abstract the essential numerical data from a practical problem and the ability to express lucidly in words the significance of the result obtained. The teaching of statistics can be an interesting and enjoyable way of encouraging the development of these useful skills. The subject could well displace some formal geometry as a training in logical analysis and expression because it is more closely and more frequently related to matters of the everyday life of the schoolchild.

Topics Suitable for Discussion. The items listed below are grouped together only for convenience of discussion. It is not suggested that they constitute a logical or comprehensive course in statistics: they are intended to show only how the usual course in arithmetic can be supplemented. In the classroom most of the items would arise naturally out of the usual work in arithmetic, though the main sections could reasonably form occasional topics in themselves.

Some suggestions for the teaching approach to the subject are also offered. Statistics can become as academic and 'pure' as any other branch of mathematics if the teacher is more interested in his subject than in his pupils. It must be realized that a critical attitude to numbers cannot be taught in a few lessons on 'statistics'; if the critical attitude of the statistician is turned on like a tap only when the teacher expounds statistics it is not likely to be acquired or respected by his pupils. If the teacher has no academic knowledge of statistics then some reading may be necessary. The Home University Library's volume on *Statistics* by L. H. C. Tippett is a readable introduction to the subject: some further suggestions for reading will be found in the Library List in Ch. VII, p. 176.

Suggested topics are:

(1) Graphical representation of numerical data. The use of bar-charts, 'pie'-charts, graphs of time-series, isotype. Misleading forms of representation.

Comments:

(*a*) The data to be represented should be derived from other school work or social activities in which the children are interested so that graphical representation of such data is looked upon, not as an end in itself, but as an incidental though necessary part of the work of understanding and interpreting the data.

(*b*) The most effective means of representing the data should be considered. Pictorial representation is intended to help the appreciation of the data; it must therefore be simple and clear and more readily intelligible than the numerical data it represents. For display work this topic offers an opportunity of collaboration between teachers of mathematics and teachers of art.

(*c*) The representation should give a truthful picture of the data. Representation of quantities by areas or volumes, and especially by diagrams which *suggest* volumes can be very misleading; they are

sometimes used in advertisements. If the origin of a bar-chart or graph is not the zero of the quantities measured, then this fact must be emphasized on the chart. Pupils should be encouraged to bring for discussion in class any misleading examples they find in newspapers and magazines.

(2) Significant figures. Practical limits of measurement of length, speeds, weights, etc. Experimental error in science; the 'tolerances' of the engineer.

Comments:

(*a*) Many pupils leave secondary schools with too great a respect for mathematical exactness; it is, for example, all too readily assumed that a measurement, however obtained, is *the* measure of the quantity measured. But all practical measurements are approximations which are not exactly repeatable; exactitude in practice is impossible. The figures which are 'significant' in practice are determined by the method of measurement used and by the purpose for which the measurement is required, but these considerations are usually ignored even in the teaching of 'significant figures'. For example, some measurements on the school field (e.g. finding the area of the cricket pitches for returfing) might be adequately made by pacing out yards, whereas others (e.g. marking off the end of the quarter-mile or 100 yd. running tracks) would need the use of some measuring instrument. What instrument would be appropriate and how much doubt about the length of the 100 yd. track would it be reasonable to accept?

(*b*) Discussion of experimental error requires the collaboration of the teachers of physics and chemistry—but so much the better. In a physics experiment one boy may find that the specific heat of copper is 0·0912, while another obtains 0·0897 for his result. Differences of this kind are usually ignored, but discussion of them is the best way of leading pupils to understand the principles and technique of experimental science. The teacher of mathematics can help by using class results of this kind to illustrate the various measures of location (means, modes, medians) and the dispersion or spread of frequency distribution, and to indicate which measurements in the practical work are the most critical (e.g. the weight of about a 100 gm. of water may easily be determined correct to the nearest milligram but the rise of about 5° C. in temperature may be

measurable only to the nearest fifth of a degree on the thermometers available).

(*c*) When an engineer designs a steel shaft to be, say, 2·000 in. diameter, he does not expect the lathe operator to produce a shaft of precisely that diameter. He always provides the operator with a 'tolerance', i.e. a range within which the diameter must lie, e.g. $2{\cdot}000 \pm 0{\cdot}005$ in. One of the design problems of the engineer is to allow as great a tolerance as is compatible with the precision of his design requirements. But difficulties arise when tolerances interfere with each other, e.g. in the design of a two- or three-pin plug to fit into a socket where tolerances permitted between the *centre* of the pins and the centre of the sockets interfere with the tolerances permitted on the pins and sockets themselves. Here is a source of problems which require careful analysis, which are related to practical work, and which emphasize the difficulties of making things 'exactly'.

(3) *Averages.* The arithmetic mean as a 'statistic' for comparing groups of numerical results, methods of reducing the labour of computing the average of large groups of numbers. Other 'statistics' for comparing groups, i.e. the mode and the median, and their advantages and disadvantages. The geometric mean and its use in the calculation of index numbers. The harmonic mean and its use in averaging rates. The weighted mean and its application to mechanics, co-ordinate geometry, etc. The moving average as used for smoothing irregular time-series to show 'trends' or to eliminate periodic changes.

Comments:

(*a*) Suitable data can be found in marks, experimental science results, height and weight measurements; for index numbers, tuckshop prices; for moving averages, the local meteorological data, absentee numbers, figures of outputs of local industries, etc.

(*b*) Pupils are always interested in 'short cuts' to finding averages.

(*c*) The arithmetic mean is only one of several means that are used; its use is arbitrary and sometimes misleading, so that other available 'measures of location' should be examined. Some of them are interrelated and these interrelations provide problems in algebra, e.g. the relations between the arithmetic, geometric and harmonic means.

(4) *Dispersion.* The inadequacy of the average as a statistic for comparing graphs of numerical results; spread, range, precision of

measurements, consistency and variability; normal and skew distribution descriptively treated.

Comments:

(*a*) The average is a simple and popular statistic for comparing results but it gives an incomplete picture; consistency of results may be more satisfying than having a higher average with greater variability (e.g. in mass-production processes) and so a measure of spread or dispersion is useful.

(*b*) With the co-operation of the science staff the results of class science experiments can be analysed. For this purpose the mean range of samples of four results (as used in the quality control of mass-production processes) is a suitable measure of the spread. Details of the use of the range for this purpose can be found in most elementary text-books of statistics.

(5) *Sampling.* Elementary ideas on sampling and randomness.

Comments:

(*a*) Practical illustration of sampling methods can most rapidly and certainly dispel the widespread belief in the possibility of exact results.

(*b*) The use of random numbers for removing the possibility of personal bias (even when the sampler is aware of the possibility) is easily understood.

(*c*) The constructing of a set of random numbers for class use is itself an exercise in random sampling.

General Comments. It is more important that the statistical ideas outlined above should permeate school arithmetical work generally than that they should be taught formally in classes on 'statistics'. Since the main purpose of introducing these ideas is to make school arithmetic more realistic for schoolchildren it is better to discuss the statistical aspect of any subject as and when the opportunity arises rather than to delay discussion for a more formal treatment. Nevertheless, an occasional review of a major topic such as dispersion or sampling will be needed to integrate such discussions.

It must be admitted that the external examination papers in mathematics do not encourage the teaching of this kind of applied arithmetic. Written examination papers necessarily flatter the pupil who is good and quick at mechanical computation for which the

data are provided and for which only a numerical answer is required, and give little opportunity to those who, less quick and able at straightforward computation, are yet more capable of analysing and solving a practical problem of arithmetic in everyday life. The teaching of elementary statistics in the way outlined should help the pupil who is simply a good computer to become more practical and, at the same time, encourage the more practical pupil to improve his computing by demonstrating that the arithmetical processes have useful application.

2. Statistics in the Sixth Form

The Place of Statistics in Sixth Form work. If statistics is taught in the sixth form it may, but need not, take a more formal place than it does in the work at the Ordinary Level. It may, at one extreme, be treated formally as a branch of applied mathematics up to Scholarship Level; on the other hand it may be readily adapted for those non-specialists who wish to continue their mathematics after passing the Ordinary Level.

Statistics in the sixth form is no longer applied arithmetic; it now becomes the branch of applied mathematics which 'deals with the data obtained by counting or measuring the properties of populations of natural phenomena'. Time for statistics can be found only at the expense of mechanics because competence in mathematical statistics is heavily dependent on a wide and sound knowledge of pure mathematics—including geometry as well as analysis.

It is emphasized that there is no demand from any quarter for schools to encourage early specialization in statistics. If, on leaving school, a pupil wishes to read for an honours degree in statistics (or in mathematics including statistics) then the best course for him to follow in school is to acquire as wide and sound a training in pure mathematics as he can. No qualifications in statistics are at present required of any entrant to any university department of statistics, nor are any such qualifications likely to be required until the teaching of mathematical statistics has become generally established. At the same time the sixth form mathematical specialist can find in statistics some new and interesting fields of application for pure mathematics, e.g. in an extension of the limited and rather sterile treatment of probability usually found in text-books of algebra.

Pupils who propose, on leaving school, to specialize in any of the physical, biological or social sciences, or in engineering, will, however, find a school course of statistics very helpful. It is usually possible for them later to attend special university or other academic courses of statistics when the need arises, but a school course which can be spread over a year or two is likely to be more effective than a short and intensive lecture course. As the content of special lecture courses for engineers, for biologists, or for economists is basically the same, it seems reasonable to consider statistics as a basic subject which could be better taught in schools than in universities. It is this idea of statistics as a necessary ancillary to all subjects dealing with quantitative data which places it in the sixth form mathematical course.

Statistics in the G.C.E. Examination. Most present-day teachers of mathematics had no opportunity of studying statistics in their formal training and they will naturally feel reluctant to begin the teaching of a subject new to them by committing a whole class to a complete statistics paper in the G.C.E. examinations. At the same time, under present-day pressure of teaching, there is little opportunity of acquiring the necessary knowledge and teaching techniques by teaching the subject outside examination requirements. It is therefore important that the G.C.E. examining bodies should devise syllabuses and examination papers which will permit teachers to introduce statistics gradually. Teachers could then feel their way at a convenient pace, and extend the range of their teaching as they gain experience and confidence.

Further, it should also be possible for pupils to include statistics at the Ordinary Level, and in a way that will encourage the teaching of statistics to pupils who are not specializing in mathematics or the exact sciences. Biologists and economists usually do not study mathematics beyond the Ordinary Level and their needs and capabilities deserve consideration.

A perusal of the present G.C.E. syllabuses shows that, except for the Cambridge Local Syndicate, these two requirements have not been appreciated by those examining bodies that now include statistics as part of their mathematics syllabuses. (The relevant parts of these syllabuses are given in the Appendix to the Royal Statistical Society's Memorandum on the *Teaching of Statistics in Schools.*)

In the Cambridge Local Syndicate scheme the two requirements are met by a paper at Ordinary Level, one of three alternatives, which consists of two sections. Section A contains six questions on pure mathematics; Section B contains four questions on mechanics, four on statistics and four on further pure mathematics. Candidates are asked to answer not more than four questions from Section B, with complete freedom of choice. Thus it would be possible for a teacher, if he so wishes, to begin with only part of the statistics syllabus, and to extend it year by year as he wished. In this way he can acquire experience without committing his pupils to the serious risk of taking a complete paper in an untried subject. At the same time, the Cambridge syllabus provides a useful introduction to statistics for boys who are not mathematical specialists, and its content of pure and applied mathematics is well balanced.

The Ordinary syllabus of the University of London School Examinations Council also includes statistics, but in this examination the pupil must take three subjects in three separate papers, one of which is wholly statistics. With this scheme the teacher has to commit himself and his class to 'do or die', and he may well hesitate to run the risk of sacrificing his innocent pupils. The Joint Matriculation Board of the Northern Universities has introduced a statistics paper as one of two papers on two separate subjects at the Ordinary Level. The same objection will therefore apply to the Northern as to the London scheme. The Oxford Local Examinations authorities also provided an Ordinary Level paper in Statistics as a separate subject. The syllabus, however, was long, and in 1953 it was re-classified as an Advanced Level paper.

It can be seen that of these diverse schemes the Cambridge scheme will appeal most to teachers who would like to introduce statistics in a modest way; the other schemes will in general appeal to the small proportion of mathematics teachers who have had some training in, or some practical experience of, statistics.

At Advanced and Scholarship Levels, the syllabuses and schemes are of less immediate importance. The syllabuses are all suitable for boys and girls who are reading mathematics, physics or chemistry. Few teachers, however, will be inclined to tackle them until they have tried their hand at the Ordinary Level. But even at Advanced Level, the papers and selections of questions should be arranged to allow a gradual introduction of statistics, rather than to demand an

'all or nothing' commitment. The Advanced papers of the Cambridge and London authorities meet this requirement.

Some General Teaching Points. In offering some comments on teaching statistics in the sixth form it has to be assumed that the reader already has some knowledge of the subject.

(*a*) *Selecting Examples.* A special difficulty that arises in teaching descriptive statistics is that exercises involving the computation of the mean, the standard deviation, the coefficient of correlation, etc., all involve much arithmetic and take much longer to complete than the usual exercises in pure mathematics or mechanics. It is therefore useful in planning a course to select exercises each of which can be used to illustrate successive items in the syllabus, i.e., before the course begins the teacher should select exercises primarily to illustrate the later parts of the course, but use them also to illustrate the earlier topics. For example, the testing of the significance of the difference of two means requires two sets of similar data; these two sets can be used earlier as an example and an exercise on the computation of the mean, the median and the standard deviation. When the time comes to discuss the test of significance the necessary basic results for an exercise are already available if the pupils have retained all examples and exercises in a loose-leaf file; time is saved, and the class is not distracted from the point at issue by having to settle down to a tedious preliminary computation.

Numerical problems from text-books should be used only to illustrate computing techniques—and then only sparingly. There is little need for drill exercises in statistics since all the early computations such as that of the arithmetic mean are constantly used in later work. Whenever possible the data on which the exercises are based should be collected by the pupils themselves from local sources, e.g. examination marks, laboratory results, meteorological data. The class will then learn not only some of the difficulties of collecting and classifying data, but will also be able to discuss and interpret the results of statistical analysis in real terms.

(*b*) *Discrete and Continuous Variables.* Some of the difficulties that schoolboys find in elementary statistics arise from the fact that the statistical variables are mostly discrete, whereas the variables of the concurrent pure mathematics course are mostly continuous. It will often help the class familiar with calculus notations if, wherever possible, new ideas concerning corresponding discrete variables and

their expression in terms of Σ are repeated in the corresponding form concerning continuous variables in terms of the integral sign.

Some Particular Teaching Points: (*a*) *Histograms.* If only data with equal class-intervals are used in exercises it is easy to overlook the fact that the principle of the histogram is that the *areas* of the columns and not their heights are proportional to the corresponding frequencies. This is already an example of a case in which the introduction of a continuous frequency distribution will make the concept clearer for the discrete variable.

(*b*) *Measures of Location.* Though the *arithmetic mean* is the best-known measure for locating the 'middle' of a frequency distribution, it is quite arbitrary since the 'middle' can be satisfactorily defined in other ways, each of which gives a different, but useful, measure. The other common measures, the *mode* and the *median* have some advantage over the mean (the mode needs no computing; the median can be determined graphically) and the *geometric* and *harmonic means* also have their special applications.

The simple algebraic properties of the arithmetic mean are worth exploring because they not only give practice in using the Σ notation but because they establish simple identities which will be used in later work. Wherever possible the symbols should be interpreted verbally to make pupils familiar with statistical jargon. Thus if $m = \Sigma x/n$, then $\Sigma (x-m) = 0$, or 'the sum of the deviations about the mean is zero'. Some algebraic manipulation exercises may be useful in preparing the class for future applications, e.g. to verify that $\Sigma (x-m)^2 = \Sigma x^2 - nm^2$.

For pupils who can appreciate it, the analogy between the determination of the arithmetic mean of a frequency distribution and the determination of the mean centre of a plane lamina should be mentioned.

The graphical method of determining the median introduces the important concept of the *cumulative frequency distribution.* In plotting the cumulative frequency curve it has to be emphasized that the point corresponding to a given class-interval must be placed on the ordinate at the upper end of the class-interval, and not at its centre.

The *weighted mean* of two quantities x_1, x_2, with weights w_1, w_2, respectively is $\frac{w_1x_1+w_2x_2}{w_1+w_2}$, a formula which, with its extensions, has applications in co-ordinate geometry and mechanics.

Data providing material for topical examples on *index numbers* will be found in any issue of the *Annual Digest of Statistics* (H.M.S.O.). Both arithmetic and geometric means are used.

(*c*) *Measures of Dispersion.* The second measure required in describing a frequency distribution is that of the spread or dispersion of the distribution. Again, the various measures are arbitrary; and all have their special uses. The *range* is very simple, so simple that it is often overlooked as a serious measure of dispersion. Though it has obvious defects it has important practical applications in engineering in the inspection of mass-produced articles (quality control), where the mean range of *small* samples is used as a measure of dispersion.

The *quartiles* are of decreasing importance, but they can be used to introduce *percentiles*, which still have some practical applications (e.g. to examination marks, anthropometric data for the design and planning of machines, clothes, etc.).

The *standard deviations* and the *variance* are mathematically the most important measures of dispersion. If the class is familiar with the mechanics of the rotation of a rigid body, then the analogy between the standard deviation and the radius of gyration of a plane lamina about an axis in its plane should be pointed out. The analogy with the Principle of Parallel Axes helps in the understanding of the correction usually applied when computing the variance from a value of the variable, which is more convenient than the true mean. Note that the standard deviation is a special case, the minimum value, of the more general 'root mean square deviation'.

Conversion of 'raw' examination marks to some standard measure with a given mean and standard deviation is a useful exercise.

The determination of the standard deviation of a frequency distribution is required in comparing one frequency distribution with others; a single standard deviation unrelated to any other is of no value or interest.

(*d*) *Probability.* Though some exercises on the rules of compound and total probability, similar to those found in standard text-books of algebra, are needed to establish the application of the rules, the exercises need only be simple. Problems about urns, dice and cards can easily become elaborate; they should, however, be regarded merely as convenient models in terms of which more practical problems of probability and sampling can be discussed or

demonstrated. The solution of such problems should not become an end in itself.

In addition to the rules of compound and total probability, the 'at least one' rule deserves emphasis, e.g. the probability of *not* throwing a six at dice in four throws is $(\frac{5}{6})^4$, hence the probability of throwing *at least one* six in four throws in $1-(\frac{5}{6})^4$.

It is important with beginners to stress the difference between answers given in terms of probability and the exact answers obtained in arithmetical problems, e.g. what fraction of a foot is two inches? What is the probability of throwing a five at dice in one throw? The answer to both questions is $\frac{1}{6}$, but what this answer means as a probability usually needs to be demonstrated by recording the results of many throws. The popular belief that a five 'should' appear precisely once in each six throws dies hard.

In solving combination and permutation problems about dice, coins and other mass-produced objects, confusion can arise about the question of distinguishable cases. Authors of some text-books, well aware of this difficulty, take pains to distinguish dice, for example, by specifying that the dice are of different colours, and are therefore distinguishable. The question of distinguishability is important because we usually assume that the distinguishable cases are equi-probable. The practice of identifying similar objects by allotting them colours or numbers is useful whether or not the objects actually need to be distinguished. All objects of everyday life, e.g. dice, ball-bearings, peas, etc., however closely similar they may be, are distinguishable in the sense that we can always distinguish them by marking or colouring, if we go to sufficient trouble. As a simple example, consider the problem of the currants in a bun. Two buns of equal size, marked A and B are made from dough into which two currants are mixed; what is the probability that the bun marked A contains the two currants? If we mentally label the currants c_1 and c_2 the following distinguishable arrangements are possible:

A	c_1	c_2	c_1c_2	—
B	c_2	c_1	—	c_1c_2

If we assume that these four cases are equi-probable, then the probability of finding both currants in the bun marked A is $\frac{1}{4}$. If, however, we argue that one currant (for all cookery purposes) is as good as another, we have to drop the suffixes and label both currants c. We may then be led to reason that there are only three distinguishable cases:

A	*c*	*cc*	—
B	*c*	—	*cc*

On the assumption that each distinguishable case is equi-probable the required probability is therefore $\frac{1}{3}$. The first of these two answers is the one that agrees with practical cookery results, and is, therefore, the correct answer.

On the other hand, if we recast the problem of the buns and currants into a similar problem about atomic nuclei and electrons it is sometimes found that the first model of distinguishable arrangements gives results at variance with experimental results while the second model agrees with them. The fundamental particles of physics sometimes behave as though they were not distinguishable in the sense defined above; electrons refuse to be identified like currants, and, therefore, problems about currants and problems about electrons may need different probability models. They have been provided for the physicist by the Bose-Einstein and Fermi-Dirac 'statistics'. A given probability model is justified, not by *a priori* arguments, but by its success in predicting results in agreement with practice. Outside the field of atomic physics the model that gives results in accord with practice is the one in which the objects being arranged are distinguishable in the sense that we can identify them if we wish to. The confusions about distinguishable cases that can arise in classroom discussion about the throwing of dice can be dispelled if the dice are identified by giving them marks or colours, but the reason for this procedure—that 'it works'—should also be given.

(*e*) *Probability Distribution.* Statistical data are obtained in two distinct ways, by *counting*, e.g. a census of population or of traffic, and by *measuring*, e.g. heights of schoolchildren or prices of commodities. We count *attributes* and measure *variables*. The counting of attributes gives rise to *discrete* distributions; the measuring of

variables gives rise, at least ideally, to *continuous* distributions. In practical measurements, however, the variable is usually treated as though it were discrete; we never measure lengths exactly, but only to the nearest unit of practical significance, which may range from millions of light-years in astronomy to millionths of an inch in metrology. The distinction between measuring and counting may thus become obscured in the mind of a beginner. If a teacher hears any member of his class ask 'Do we use the Normal or the Binomial distribution here?' he can be sure that at least one member of the class does not appreciate the distinction.

A probability distribution can always be represented by a function $y = f(x)$, where y is the 'probability density' at x and is always positive or zero, where x can range between any two limits, where $f(x)$ may or may not be continuous, and where the total area 'under the curve' is equal to unity. The main property of the function is that $\int_a^b f(x)\,dx$ is equal to the probability that a random value of the variate x lies between a and b; it is thus analogous to the area property of the histogram.

A reading of syllabuses, and even of text-books, may suggest to the beginner that there are only three probability distributions, the Binomial, the Normal (or Gaussian), and the Poisson. These distributions are of course of basic importance in statistical theory, but there are several others of practical importance, and there is no limit to the number that can arise from particular probability problems. It is, therefore, desirable to introduce the concept of a probability distribution in a general way, and to use some examples which do not relate to the basic types. The rectangular distribution is of no special interest, but because of its simplicity it can provide an introduction to the concept of the probability distribution which is not obscured by mathematical difficulties. Examples of discrete and continuous rectangular distributions are given by one throw at dice, by the selection of digits from a table of random numbers, and by the time indicated by the clocks and watches in a watchmaker's window.

Of the standard distributions usually considered in a school course, the Binomial is the fundamental one from which the others are derived. It arises from a sequence of trials in which the probability of success, p, remains constant for each trial (such sequences are

sometimes called 'Bernoulli trials'). Work on the Binomial distribution is unlikely to be profitable unless the class is already familiar with the Binomial Theorem for a positive integral exponent and with the simpler properties of the coefficients of the expansion. Further, in addition to the theoretical work, some practical demonstration of the occurrence of the Binomial distribution must be given. Here are three examples which may suggest experiments appropriate for a class:

(i) Tables of random numbers can be used to give trials over a wide range of values of p, the probability of success, and of n, the sample size. Some examples of distributions that can be demonstrated are:

$(\frac{1}{2}+\frac{1}{2})^{10}$, by counting the numbers of odd digits which occur in sequences of ten random numbers;

$(\frac{1}{3}+\frac{2}{3})^{10}$, by counting the multiples of three which occur in sequences of ten random numbers;

$(\frac{1}{10}+\frac{9}{10})^{5}$, by counting the frequency of occurrence of any given digit per sequence of five random numbers.

(ii) A mixture of sterile and viable seeds of cabbage or mustard is made, and ten (or multiples of ten) seeds are distributed to each member of the class to grow on damp blotting paper. The probability distribution of numbers of seeds germinating in each group of ten is given by the expansion of $(p+q)^{10}$ where p is the proportion of viable seeds.

(iii) A simple apparatus which can be used for quick sampling consists of a sealed glass bulb half-full of beads of two colours, say red and white. From the bulb there extends a short sealed projecting tube which can hold a small exact number, e.g. five or six, of the beads. A random sample is obtained merely by turning the bulb to empty the tube and again to fill it. The probability distribution of the numbers of red beads per sample is Binomial (but only approximately since the number of beads in the bulb is finite).

In all the experiments described above, and in any others of the same kind, the fit between the theoretical and practical distributions will be at best approximate; the fit becomes convincing to the beginner only if the total number of observations is large. It is therefore desirable whenever possible to arrange the experimental details so that the results of individual members of the class can be added together.

As $n \to \infty$ with p constant, the Binomial distribution, which is discrete, approximates more and more closely to the Normal distribution, which is continuous. The mathematical derivation of the Normal from the Binomial distribution is unlikely to be required for examination purposes, but in any case a derivation appropriate to the mathematical attainment of the class should be selected from the several methods given in the standard text-books. Even if a mathematical derivation is given, however, most pupils will find it difficult to accept the transition, as n increases, from the sometimes very asymmetrical histogram of the Binomial distribution to the smooth symmetry of the Normal curve. Some visual demonstration of the following facts will therefore be needed:

(i) That asymmetrical distributions such as $(0{\cdot}9+0{\cdot}1)^n$ become less asymmetrical as n increases.

(ii) That the Normal curve 'fits' the corresponding Binomial histogram well, even for small value of n if p is equal to $\frac{1}{2}$ or approximates to it.

The simplest way of demonstrating the first point is by using the mechanical apparatus devised by Galton and modified by Karl Pearson to give Binomial distributions of cabbage seed or ball-bearings. If this is not available then the histograms of the distributions for which $n = 5$, 20 and 50 or some such series must be drawn. The tedium of computing the terms of $(0{\cdot}9+0{\cdot}1)^{50}$ is reduced by using Stirling's approximation for factorials, $n! = (2n\pi)^{\frac{1}{2}}n^n e^{-n}$, and the work can be shared among the members of the class. The second point can be demonstrated by plotting on the same axes the histogram of the Binomial $(\frac{1}{2}+\frac{1}{2})^9$ for which $\sigma = (npq)^{\frac{1}{2}} = \frac{3}{2}$, and the corresponding Normal curve $y = \dfrac{1}{\sigma\sqrt{(2\pi)}}\, e^{-x^2/2\sigma^2}$ with σ also equal to $\frac{3}{2}$ taking the mean of the Binomial as the origin of x. The Normal curve is thus introduced as an approximation to the Binomial histogram, which is especially useful when n is large. The advantage of the approximation is that the tedious computations and summations of sequences of binomial terms are replaced by the tabulated values of areas under the Normal curve. The usefulness and accuracy of the approximation should be made apparent by one or two suitable examples.

Even if tables of the Normal probability function are available it helps pupils to understand their origin and application if they use

their own graphs instead. A carefully drawn graph of $\frac{1}{\sqrt{(2\pi)}} e^{-x^2/2}$ is first required; from it can be constructed, by counting squares, the graph of $\frac{1}{\sqrt{(2\pi)}} \int_0^x e^{-x^2/2}\, dx$. The accuracy of values obtained from the graph is sufficient for most examples. A table of the integral can also be prepared as a class exercise by expanding the integrand in series and integrating the series term by term. Some methods of obtaining rapidly converging series for large values of x are given in Whittaker and Watson, *The Calculus of Observations*.

For the construction or checking of exercises on the Normal distribution, printed sheets of specially prepared Arithmetical Normal Probability graph paper are useful. On this paper the plotted points of a cumulative frequency curve are exactly collinear if the distribution is Normal. From the line the mean and standard deviation of the distribution can be determined graphically.

If, as $n \to \infty$, $p \to 0$ with $pn = a$, a constant, then the successive terms of a Binomial distribution approximate to an exponential series—the Poisson distribution. The advantage of the Poisson approximation is that the two parameters of the Binomial, n and p, are replaced by the one parameter a, for the given limiting conditions. This approximation is usually more acceptable to the class than the Normal because there is a one-to-one correspondence between the terms of the Binomial and those of the Poisson and its accuracy for simple cases is easily verified by one or two numerical examples.

The Poisson distribution also needs to be demonstrated practically. Some suggestions are:

(i) Scatter sand very thinly over a sheet of graph paper to give a random distribution of particles. The number of particles in each centimetre (or $\frac{1}{2}$ in.) square is counted; the probability distribution of this number conforms to the Poisson law.

(ii) Mark off in pairs the digits of a table of random numbers. Note the frequency of any specified pair, e.g. 00 or 37, per 100 digits.

(iii) Count the number of people who pass through a shop door or gateway, or the number of cars or buses which pass a fixed point in a street in successive short intervals of time. The movement of people

or traffic must, of course, be random, i.e. not affected by any kind of control such as that imposed by queues, traffic lights or traffic jams.

(iv) Expose sheets of graph paper for a few minutes to light rain and count the number of rain-drops in unit areas of appropriate size. (For convenience in computing the Poisson series and for comparing practical and theoretical results the unit 'cell'—of area or of time etc., as the case may be—should contain on average not more than five or six particles or 'events'.)

(*f*) *Sampling.* The purpose of sampling is to provide an estimate of some particular property of a population by examining the members of only a fraction of the population. The statistician has two main objects concerned with sampling; the first is to devise sampling methods for any given purpose which are as 'fair' as possible—'unbiased' is the technical term—and the second is to say what confidence can be placed in the estimate provided by a sample taken in any given circumstances.

The subject can be introduced by discussion of sampling methods that are obviously biased, e.g. the sampling of milk for fat content without shaking the bottle, or the taking of a public opinion poll on a political issue by telephoning the names listed in a trade directory, etc. The difficulty of avoiding personal bias, even when the sampler is aware of the difficulty, can usually be illustrated by asking a member of the class to select ten members of the school whom he thinks would be a fair sample; the remainder of the class can usually provide the necessary criticisms. The usefulness of tables of random numbers in helping to avoid personal bias—or even fear of personal bias—can then be explained; the effectiveness of the tables in giving random samples can be demonstrated or criticized by using the method, for example, to provide ten names from the school list. If tables of random numbers are not available, simple methods of constructing tables can be discussed. One of the methods can be used to provide the class with its own table—their first exercise in the taking of a random sample and a useful demonstration of the concept of randomness.

Some experimental work is desirable because only practical problems can give reality to the difficulties of random sampling. It is especially important to make thoroughly understood the fact that no sampling method, however elaborate, can be devised to *ensure*

that a sample of exactly correct proportions can be drawn from a population that is not homogeneous. Pupils new to the subject find it difficult to understand that a sample provides only an *estimate*, and they tend to think of estimates as being 'right' or 'wrong'. Experimental sampling should therefore be performed with populations whose statistical parameters are unknown, at least to the pupils, so that equal weight is given to every random sample fairly and similarly taken.

For experiments on the sampling of attributes bags or bottles of beads or beans of two colours are convenient; for experiments on the sampling of variables the most convenient apparatus for mathematical classrooms consists of bottles of numbered discs, the discs of each bottle representing a distribution whose mean and standard deviation are known to the teacher. Sets of numbers (500 to 1000) to give approximately Normal distribution with given means and standard deviations can be derived from tables of the ordinates of the normal curve.

In experiments on the sampling of attributes the main points to demonstrate are:

(i) That the frequency distribution of sample counts for a number of samples of a given size is approximately Binomial.

(ii) That the variance of the frequency distribution of sample counts varies inversely as the size of the sample.

In experiments on the sampling of variables the main points to demonstrate are:

(i) That the means of samples of equal size drawn from a Normal population are Normally distributed.

(ii) That the variance of the distribution of means varies inversely as the size of the sample.

(iii) That the distribution of means of samples of equal size drawn from almost any population has a form which is more nearly Normal than that of the original.

(iv) That the occurrence of a mean of an n-sample which lies outside the value $m+3s$, where m is the mean of the n-samples and s is the standard deviation of the means, is very rare.

A useful convention in statistical theory, which has to be borne in mind from the beginning of the course, is that parameters of population distributions are designated by Greek letters, e.g. μ and σ, and parameters of sampling and frequency distributions are

designated by the corresponding Roman letters, e.g. m (sometimes x) and s.

Some of the confusion that occurs with beginners about the 'correctness' of estimates obtained from sampling may arise from the terminology of statistics, and especially from the use of the term 'standard error' as shorthand for the 'standard deviation of a sampling distribution'. The term is now well established and must be used in teaching, but the teacher must explain that the use of the term does not imply that there is necessarily an error in the sampling result, though there is always a doubt. The term 'probable error' is also misleading, but since it is now little used there is no need to introduce it in a first course.

The arbitrary basis of significance testing needs to be explained very fully. The usual probability levels of 5 per cent and 1 per cent, or the cruder one based on three times the standard error, merely provide convenient criteria which have become conventionalized in statistical practice. A statement about the significance of a result is incomplete unless the probability level is specified. In written problems of significance testing it is essential to insist on explicit statements of the 'null hypotheses' and full and exact formulation of the conclusion.

(*g*) *Regression and Correlation.* Some preliminary discussion of bivariate distributions is necessary because schoolchildren have usually experienced variable number-pairs (x, y), only as two variables related by a mathematical equation or by an exact physical law. Such terms as 'linearly associated variables', 'unassociated variables', and 'variables associated non-linearly', need to be explained and exemplified. Methods of representing bivariate frequency distributions in visual form, e.g. by 'solid histograms' or by 'scatter diagrams' should be demonstrated.

In introducing the concept of regression it is probably wiser to demonstrate it graphically rather than algebraically. The lines of regression are the loci of the means of the two sets of arrays (i.e. of the rows and of the columns) of a grouped frequency distribution. In graphical methods these means are computed separately and are marked on the graph so that the occurrence of two distinct lines of regression is made clear.

Before deriving the general formula for a regression line for the first time it is helpful to have a preliminary discussion about the

covariance of two variables, to define it and to demonstrate its algebraic properties that will be needed in later work. Using only elementary methods it is difficult and tedious to prove satisfactorily that the ratio of the covariance to the product of the standard deviations is zero if the two variables are mutually independent, but most teachers of mathematics will consider that a proof is desirable. Using the concept of *expectation*, however, the proof is simple. The expectation, $E(x)$, of the probability distribution of a random variable x is defined as the mean value of the variable, assuming that a mean exists. Two theorems are useful at this stage:

(i) That $E(x+y) = E(x)+E(y)$ where x and y are any two random variables.

(ii) That $E(xy) = E(x)E(y)$ if x and y are *mutually independent* random variables.

From the second theorem it follows at once that the expectation of the covariance of two independent random variables measured from their means is zero.

In deriving the general formula for a regression line it should be emphasized that *two* criteria are required to fix a line passing 'through' an experimental linear series of points, and that the introduction of a new criterion—here the Principle of Least Squares—is necessary. It needs to be emphasized also that regression lines are merely convenient mathematical fictions which, in spite of the possibly heavy computation they may demand, are no more 'real' than the lines drawn by eye, and which in any case are subject to errors of sampling.

The product-moment coefficient of correlation is a convenient measure of the degree of *linear* association of two variables; it has to be emphasized that a significant value of a correlation coefficient does not imply that there is any causal relationship between the variables. The interpretation of a correlation coefficient demands a thorough knowledge of the field from which the data are drawn, and exercises should, therefore, be set whenever possible on data with which the pupils are familiar, e.g. examination marks, their own practical science results, etc. The sampling error of the coefficient for small samples is large, so that in experimental work large samples must be taken. (For a sample of ten pairs of values drawn from a normal bivariate population the coefficient is significant at the

5 per cent level only if its absolute value exceeds 0·63; for 100 pairs the corresponding value is 0·20.)

The Spearman rank correlation coefficient is best explained as the product-moment coefficient of correlation of the bivariate distribution obtained by substituting ranks for the original crude data.

Books of Reference. Until statistics is more widely established as a school subject the teacher will need to refer to some of the standard works for ideas and examples. The following books are recommended for this purpose:

YULE and KENDALL. *An Introduction to the Theory of Statistics.* (Very readable; fully descriptive; useful examples; calculus methods avoided, some proofs and derivations omitted.)

WEATHERBURN. *A First Course in Mathematical Statistics.* (Concise but clear; mathematical methods appropriate for sixth-form work; useful examples and tables; more concerned with theory than with practice or computation.)

KENDALL. *The Advanced Theory of Statistics.* Vols. I and II. (Fully descriptive and explanatory; mathematically readable and sound; many examples; not as difficult as the title might suggest.)

Further books are recommended in the Statistics section of the Library List.

3. CONCLUSION

It is hoped that these comments on the teaching of statistics in secondary grammar schools will encourage more teachers to interest themselves in its possibilities, both as a subject of general interest and wider application and also as a new and interesting field of application for pure mathematics in the sixth form. There is as yet little experience of teaching statistics in schools and there are therefore great opportunities for the enterprising teacher to find new ways of presenting the subject at all stages of school work.

There is especially a need for devising suitable experimental work or practical tests in sampling which can be carried out in the classroom or mathematics room. Teachers of mathematics may feel reluctant to embark on practical work; it can be maintained, for example, that mathematics should stand above such mundane processes. But, except for gifted pupils (and they are few), mathematics is usually a dry and inaccessible subject with little relevance to the everyday life of the school child. It is suggested that statistics, *so*

long as it is taught as a practical subject, can help to make school mathematics more realistic for many who now find little interest in the subject or derive any benefit from it. This claim may be disputed on *a priori* grounds, but it is becoming strong enough to demand that it be tested by practical experience.

The teacher who would like to embark on the teaching of statistics may hesitate to do so because he can find no text-book suitable for his pupils; he should not hesitate on this account, but rejoice in the freedom this situation gives him to develop for himself a statistics course which both satisfies his own inclinations and is appropriate for his pupils.

APPENDIX

THE JEFFERY REPORT

School Certificate Mathematics

Report of a Conference of Representatives of Examining Bodies and Teachers' Associations, with a suggested Alternative Syllabus and Specimen Papers

The Conference was convened on the initiative of the Cambridge Local Examinations Syndicate to consider School Certificate Mathematics with special reference to an alternative syllabus in Geometry which had been prepared.

The Conference met in London on 29 September, 1943, when the following bodies were represented:

The University of Bristol. The Cambridge Local Examinations Syndicate. The Central Welsh Board. The University of Durham. The University of London. The Joint Matriculation Board of the Northern Universities. The Oxford Local Examinations Delegacy. The Oxford and Cambridge Schools Examination Board. The Incorporated Association of Head Masters. The Association of Head Mistresses. The Association of Assistant Masters. The Association of Assistant Mistresses. The Mathematical Association.

All the representatives appointed (four from the Mathematical Association and two from each other body) were present at the meeting. The Conference appointed Professor G. B. Jeffery as its Chairman and Mr J. L. Brereton as its Secretary.

A wide measure of agreement was apparent that the Cambridge proposals in geometry were in the right direction and represented a good alternative line of approach to mathematics at the School Certificate stage; that the principles underlying the proposals were equally applicable to the whole subject; and that the traditional divisions of mathematics at this stage were harmful and restrictive in teaching practice.

It was decided to prepare and to submit for the consideration of the Examining Bodies an alternative type of syllabus covering the whole subject of Elementary Mathematics and to illustrate this syllabus by specimen papers.

For this purpose a Committee was appointed to prepare proposals for consideration at a later meeting of the Conference. The Committee consisted of:

Miss K. S. Chamberlain, Mr C. T. Daltry, Professor G. B. Jeffery (*Chairman*), Mr G. L. Parsons (*Secretary*), Mr A. Robson, Mrs E. Shuttleworth, Dr G. I. Sinclair.

APPENDIX

The Committee held a number of meetings and was assisted by letters and memoranda from teachers, and by reports of discussions in branch meetings of the Mathematical Association. It presented its report to a second meeting of the Conference held in London on 4 April 1944, all the representatives again being present.

After full consideration of the Report of the Committee the Conference decided to issue the appended Syllabus for the consideration of Examining Bodies and others with the suggestion that Examining Bodies should consider the desirability of adopting a Syllabus in Elementary Mathematics on these general lines as *an alternative* to the existing syllabuses.

It is proposed that such a syllabus should be alternative to the present syllabuses because it is desired to give freedom to the considerable body of teachers who desire to develop the subject in this way without attempting to use the syllabus of the examination to impose change upon teachers who may prefer to follow the more traditional line of approach. It should be emphasized, however, that the object is to propose a syllabus of equal weight with the present syllabus; it is not to provide a syllabus of lower standard for weaker candidates. Some apparently weaker pupils would find the suggested alternative syllabus more congenial, and their work would be correspondingly more profitable, but this is not the main reason for advancing these proposals.

It was fully realized that the Conference had no power to anticipate the views which might be formed by the Examining Bodies after consideration of these proposals, nor was it expected or desired that any Examining Body would adopt them in the precise form in which they are here put forward. There is room for differences of opinion on the detail of the syllabus, and indeed such differences were felt within the Conference itself. The Conference, was, however, united in its approval of the general character and structure of the proposed alternative syllabus.

The Conference was not in a position to give detailed consideration to the syllabus in Additional Mathematics. Suitable pupils would be encouraged to pursue their interest beyond the point represented by Elementary Mathematics by offering Additional Mathematics, Mechanics, or Geometrical Drawing. On a suitably designed syllabus the proportion of candidates taking Additional Mathematics should be greater than it is at present. The syllabus in Additional Mathematics should be, as it is not invariably at present, a natural and continuous extension of the syllabus in Elementary Mathematics. An alternative syllabus in Elementary Mathematics therefore requires an alternative syllabus in Additional Mathematics, but the details of the latter cannot usefully be proposed until those of the former have been settled.

In relation to Elementary Mathematics an overwhelming case can be made out for the fusion of mathematical subjects and in particular for a closer association of geometry and trigonometry. To encourage this, each examination paper should be ' mixed ', i.e. it should be a paper on mathematics and not on some section of it such as algebra or arithmetic. In

these papers complete freedom of method should be allowed, save in exceptional questions in which a special process is demanded by the examiner. The freedom should extend to the use of mathematical tables and instruments. If special books of tables are required in a particular examination, these should be ample tables which will not force the use of particular methods.

For reasons which will appear, papers should be divided into two parts: a short section to be attempted by all candidates, followed by a longer section in which a choice of questions is permitted. With such papers an examining body could secure, if necessary, that no important section of mathematics should be entirely neglected; and the second part of the paper could include a variety of questions, some of which might not be immediately acceptable to all teachers. The first part of the paper might well be used to enforce a higher standard of numerical accuracy in the simplest work.

The Cambridge proposals, besides encouraging the fusion of geometry and trigonometry, remove much of the emphasis from formal work, but the list of theorems is still too long. Since the fundamental theorems about congruence, similarity, elementary trigonometry, and elementary solid geometry ought to be presented to pupils on an informal basis, no proofs of these results should be demanded in examinations. Outside this basic work, there are certain key-theorems and a number of other theorems that can be derived from them. Proofs of the key-theorems, and deductions of the others, form an important part of the geometry teaching, but it is doubtful whether any proofs of theorems should be required in examinations. If all the key-theorems and derived theorems are liable to occur, too much teaching time will be sacrificed to their preparation. The theorem-work in examinations might well be limited to the key-theorems: or a few other theorems might be included to illustrate diverse methods of proof. The proposed list of 'italicized' theorems is not necessarily the best possible selection, but it is of about the right length if there are to be theorems at all. There is room for discussion among teachers about the ideal list of 'italicized' theorems, and it might be useful to compile a list of derived theorems (to be available for rider work, but of which proofs would not be required in the examination). It is unlikely that all the examining bodies will make precisely the same classification of theorems. There should not be more than one theorem in any paper, and it should come in the non-compulsory section. This restriction would prevent undue concentration on this part of the work.

The most striking feature of the proposed syllabus is the inclusion of the beginning of calculus in the optional parts of the papers, and this is a proposal which the Conference has made only after the most careful deliberation. The pitfalls are obvious, for there are few branches of the subject which can more easily afford the opportunity for blind manipulation of a notation or the mechanical application of rules. But these pitfalls can be avoided and the intrinsic importance of the subject is very great.

The transition from the static mathematics of the formula, which enables one quantity to be calculated when another in known, to the dynamic mathematics of the function, which considers how one thing changes with another, is one of the chief ways in which mathematics has adapted itself to the consideration of practical problems. The ideas underlying differentiation and integration are not difficult to grasp if presented in appropriately simple circumstances. They grow naturally and easily out of the consideration of graphs, which rightly now occupy so important a place in the elementary teaching of the subject.

The whole syllabus is inspired by the desire to bring mathematics more closely into relation with the life and experience of the pupil. The flat geometry of the blackboard should be linked up with the solid geometry of the real world by the study of plans and elevations and the consideration of the simpler solid figures. The simple geometry of the surface of a sphere has an obvious interest for the earth-dweller. The interests of the pupil in air and sea navigation, and in handicrafts, should be used as opportunities to demonstrate the power of elementary mathematics as an aid to understanding.

If all this is to be accomplished it is clear that there must be a drastic pruning of the present syllabus. Some time will be saved by the reduction in the amount of formal geometry. More should be saved by the exclusion from the arithmetic and algebra of heavy calculations and manipulations. Fractions in arithmetic and algebra should be limited to those likely to arise naturally in applications. Examples like

$$(2\tfrac{3}{14}+1\tfrac{5}{7}) \div (1\tfrac{1}{51}+1\tfrac{5}{78})$$

or

$$\frac{a+b}{a}+\frac{a-b}{b}-\frac{b^2-a^2}{3ab}$$

should not appear.

Factors like those of a^3+b^3 and $a^2-2ab+b^2-c^2$ and awkward examples of $ax+ay+bx+by$ and px^2+qx+r such as $a^2b+b-ab^2-a$ and $8x^2-6x-27$ should be excluded so should; literal equations and equations for three unknown numbers. To test the solution of quadratic equations an example like $2x^2-6x = 1$ is quite sufficient: there is no need at the same time to test ability to juggle with ugly fractions. In general, the development of processes should proceed only as far as is necessary to illustrate the principles involved. Although there is a traditional type of variation question which should be discouraged, the study of relations such as $y = kx^2$ and $y = k/x$ should be retained but merged in the wider idea of functionality.

If these omissions and simplifications are made, the emphasis will be removed from a relatively unimportant development of algebra to the all-important ideas connected with functionality which begin with graphs and should lead to rates of change and gradients.

We assume that the teacher will aim at providing a suitable background to the work, giving due weight for example to the historical development

of the subject, and to its unity as a body of related ideas and results whose truth is independent of individual preferences, as well as indicating its applications in practical affairs. If the papers seem to emphasize the last aspect, the teaching should not ignore the others.

The Conference is indebted to certain members of the Committee for the preparation of the specimen papers, which were not considered in detail by the Conference itself.

No copyright is claimed in the syllabus or specimen papers.

Signed on behalf of the Conference,
G. B. JEFFERY,
Chairman

April 1944

SUGGESTED ALTERNATIVE SYLLABUS FOR ELEMENTARY MATHEMATICS

There will be three papers of 2½ hours. Each paper may contain questions on any part of the syllabus, and the solution of any question may require knowledge of more than one branch of the syllabus.

Each paper will consist of four questions to be attempted by all candidates and of a second section in which a choice will be allowed. The first four questions will be elementary in type and may consist of two or three short parts.

It is not expected that all candidates will have covered the whole syllabus.

Unless the terms of the questions impose specific limitations,

(*a*) A candidate may use any appropriate method.

(*b*) Tables of logarithms, trigonometrical functions, squares and square roots, compound interest, etc., may be used wherever they give the required degree of accuracy.

(*c*) T-squares, set squares, graduated rulers, diagonal scales, protractors and compasses may be used.

1. NUMBERS

The ordinary processes of Arithmetic.

The commoner systems of weights, measures, and money, including metric units.

Fractions, decimals, ratio, percentage.

Use of common logarithm and square-root tables.

Significant figures.

(Questions may be set on the applications of these processes to problems of everyday life in the home and community, but such questions will not involve complicated operations or the knowledge of uncommon technical terms. It is not intended that the 'long rules' for the extraction of square roots, the determination of H.C.F., etc., should be included.)

2. Mensuration

The rectangle, triangle, and figures derived from them, including easy extensions to three dimensions.

The circle, cylinder, cone and sphere (but formulae for the last two need not be memorized).

(Questions may involve knowledge of the altitude and centre of an equilateral triangle and the ratio of the sides of the right-angled isosceles triangle and the 30°, 60°, 90° triangle.)

3. Formulae and Equations

Construction of a formula through symbolical expression of a functional relation (e.g. V is proportional to x^2), or through generalization of an arithmetical result.

Interpretation, evaluation, and very easy manipulation of a formula.

The use of indices.

(Only simple examples of fractional and negative indices will be set, and questions involving manipulation of such indices or of surds will not be set; but candidates may be expected to understand the use of indices to express such numbers as $3{\cdot}74 \times 10^8$ or $1{\cdot}35 \times 10^{-6}$.)

The use of suffixes.

Common factors; factors of (a^2-b^2) and of such extensions as occur in mensuration.

Easy trinomial factors.

Fractions whose denominators are single terms or linear expressions.

(Simple manipulations should not include more than two such fractions.)

Simple equations, quadratic equations, and linear simultaneous equations in two variables.

4. Graphs, Variation, Functionality

Graphs from statistical data.

The idea of a function of a variable.

Translation into symbols of relations such as 'y is inversely proportional to x', 'V varies as X^3', and their illustration by sketch-graphs.

Simple cases of the function

$$y = Ax^3+Bx^2+Cx+D+\frac{E}{x}+\frac{F}{x^2},$$

where the constants are numerical and at least three of them are zero.

Graphical treatment of these functions.

The gradients of these graphs, by drawing or by calculation.

(Questions will not involve theoretical treatment of limits.)

Application of gradients to (*a*) rates of increase, (*b*) easy linear kinematics including the distance-time and speed-time curves, (*c*) maxima and minima.

The determination of a function from its gradient. The area 'under' a graph.

Application to areas, volumes of revolution, linear kinematics.

(Questions on Calculus will not be set in the compulsory part of the paper. The use of the formal notation of the calculus will not be required, but candidates may be expected to understand the use of the terms 'differentiate' and 'integrate'. Questions on rates of change will not involve the use of the formula

$$\frac{dF(x)}{dt} = \frac{dF(x)}{dx} \cdot \frac{dx}{dt}.\Big)$$

5. Two-Dimensional Figures

A sound appreciation of the properties set out below will be expected. Proofs of the items in *italics* may be required. Proofs of the other properties will not be required. Riders may be set which can be solved by the use of these properties, but they themselves will not be set as riders. In solving riders candidates may use any knowledge they possess. Candidates will be expected to understand the relation between a theorem and its converse.

Properties of angles at a point and angles made with parallel lines.

The exterior angle property and angle-sum of a triangle.

Angle-sum properties of polygons.

Congruency of triangles: similarity of triangles.

Symmetry about a point or line.

The isosceles triangle: the parallelogram, rectangle, square.

Parallelograms on the same base and between the same parallels are equal in area.

The straight line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side. (By congruence or areas or similarity.)

Other area properties of rectangles, parallelograms, triangles and trapezia, including the formulae

$$\tfrac{1}{2}bc \sin A \quad \text{and} \quad \sqrt{\{s(s-a)\,(s-b)\,(s-c)\}}.$$

Connections between Algebra and Geometry, e.g. $(a+b)^2$, use of co-ordinates, including negative co-ordinates.

The sine, cosine, and tangent of an angle, acute or obtuse.

(Trigonometry of obtuse angles will not occur in the first section of the paper.)

The sine-rule for a triangle.

(With numerical applications involving the use of four-figure tables but not including the ambiguous case.)

The Theorem of Pythagoras. (By proving that the perpendicular drawn to the hypotenuse divides the triangle into similar triangles; or otherwise.)

$$\sin^2 A + \cos^2 A = 1, \qquad a^2 = b^2 + c^2 - 2bc \cos A.$$

(This replaces the 'extensions of Pythagoras'; easy numerical applications using four-figure tables will be set.)

The symmetrical properties of chords of a circle.

An angle at the centre of a circle is twice any angle at the circumference standing on the same arc.

The other 'angle properties' of the circle.

The perpendicularity of tangent and radius; the distance between centres of circles in contact; the equality of tangents from an external point.

The 'alternate segment' theorem.

The 'intersecting chord' theorem for an external point.

$$(OP.OQ = OR.OS = OT^2.)$$

The analogous property for an internal point.

The relationship between the areas of similar triangles.

Corresponding results for similar figures and extension to volumes of similar solids.

The bisector of any angle of a triangle divides the opposite side in the ratio of the sides containing the angle.

The anolagous property for an exterior angle.

Knowledge of simple loci, with easy extensions to three dimensions. The method of intersecting loci. Plotting of simple loci other than the circle, e.g. parabola or ellipse.

Knowledge of the following 'ruler and compasses' constructions will be assumed; set squares will be allowed for the construction of parallel lines.

Bisection of angles and straight lines.

Construction of perpendiculars to a given line and of angles equal to a given angle.

Construction of angles of 30°, 45°, 60°.

Construction of triangles, quadrilaterals, and circles from simple data, including the inscribed and circumscribed circles of a triangle.

Division of a straight line into a given number of parts or in a given ratio.

Construction of a triangle equal in area to a quadrilateral or pentagon and of a square equal in area to a given rectangle.

Construction of tangents from an external point.

Construction of a segment containing a given angle.

6. Three-Dimensional Figures

The properties of parallel planes: the normal to a plane, the angle between two planes, and the angle between a plane and a straight line.

The forms of the cube, rectangular block, pyramid, tetrahedron, prism, circular cylinder, circular cone, and sphere (including calculations of lengths and angles).

APPENDIX

7. Practical Applications

In addition to ordinary riders, questions involving drawing, trigonometry or geometrical reasoning may be set on the following topics:

(*a*) Simple map problems, scales, contour lines, slopes.

(*b*) Determination of positions by two bearings; the nautical mile and knot; the triangle of velocities.

(*c*) Heights and distances.

(*d*) Simple plan and elevation problems.

(*e*) Latitude and longitude; great and small circles on a sphere.

(*f*) The length of an arc in terms of the radius and the measure (in degrees) of the angle at the centre.

(*g*) Three-dimensional problems which can be solved by analysis into plane figures.

(The specimen papers are not printed here. A number of Examining Bodies now examine on an 'Alternative Syllabus' based on the Jeffery Report, so that actual examination papers are available for study.)

INDEX

Abbreviations, 67, 99
Accuracy, 37, 69, 70, 73
  degrees of 41, 45, 47, 51, 81–5, 89, 91, 106, 125, 197–8
  of results from given data, 83, 90
Addition, 20
  reliable figures in 83, 84
Aims in the teaching of mathematics, 8–16
  of the first two years, 36–7
Algebra, in the first two years, 42
  in the third year, 47
  in the fourth year, 50
  in the fifth year, 51
  approach to through problem or formula, 133
Alternative syllabuses at 'O' Level, 3, 13, 16, 94, 110, 115, 219–26
Analytical geometry, 167–9
Answers in text-books, 78
Anti-logarithms, 88
Apparatus in the mathematics room, 187–9
Appreciation of mathematics, 134
Approximations, *see* Accuracy
Area, the teaching of, 104–9
Arithmetic, 13
  in the primary school, 17–35
  in the first two years, 41, 42
  in the third year, 47
  in the fourth year, 50
  connection with statistics, 194, 195
Arrangement of written work, 70
Arts students, mathematics for, 150
Assessment of work, 59, 72, 195
Astronomy, 111, 126–8, 138, 140, 172
  in the sixth form, 155, 156
  practical work in, 182
Attainment on entry to a grammar school, 30–5
Axioms, 97, 141

Basic concepts, 48, 52
Blackboard, use of, 67–9
  instruments, 68
  in the mathematics room, 183, 184

Calculus, in the fifth form, 51, 115–23
Centre of interest, 134
Checking of answers, 69, 88
Civil Service syllabuses and examinations, 93, 110, 148
Classification of pupils, 74
Coloured chalk, use of, 68, 108
Coloured pencils, 58, 68, 95, 99
Compound units, 14
Concentric plan, 14, 51
Confidence in pupils, 36, 55
Congruent triangle complex, 101
Contracted methods, 85, 89
Converse theorems, 96
Correction of work, 73
Correlation with other subjects, 123, 124, 126, 128–30, 146, 159
Courses, framing of, 15

Decimals, 14, 18, 27, 41, 44, 45
Deduction in geometry, 91–104
Definitions in geometry, 96, 98
Division, methods of, 23, 24
  accuracy of results in, 84
  tests of divisibility, 41, 45, 46
Drill, routine, 13, 61

Emotional response of pupils, 55
Ends, immediate and ultimate, 8–16
Environmental approach to teaching mathematics, 133–6
Euclid, 138, 140
Euclidean tradition, 109, 111, 169
Euclid's *Elements*, 91, 92, 98, 109, 113, 114, 141, 169
Examinations, in the sixth form, 147–50
  Civil Service, 93, 110, 148
  see also G.C.E. examinations
Examples in mathematics, 69, 77
Exposition of new work, 65, 66, 101, 102, 107

Facts and formulae required to 'O' level, 79, 80
Film and filmstrip, 131–3, 189, 190

Foreign languages, mathematical books in, 173
Freehand sketches, 43, 45, 70
Furniture and fittings in the mathematics room, 185–7

G.C.E. examination, 2, 93, 95, 113
  omission of 'O' level, 152, 162
  varying standards at 'A' and 'S' levels, 149
Geography, correlation of mathematics with, 95, 123, 124, 126
Geometry, in the first two years, 42, 43
  in the third year, 47
  in the fourth year, 50
  in the fifth year, 51
  in the sixth form, 165–70
  axioms and axiomatic development, 97, 99, 141
  converse theorems, 96
  definitions, 96, 98
  general figures in, 71, 100
  non-euclidean, 11, 144, 145, 170
  practical, 92
  proof, 94–104
  riders, 58, 70, 93, 95, 96, 98–104
  stages A, B, and C, 93
  style in, 101–4
  three dimensional, 10, 42, 43, 45, 92, 94, 97, 109–14, 158, 166, 167
Graphs, in the first two years, 43, 45
  in the third year, 47
  in the fourth year, 50
  in the fifth year, 51
  interpretation of, 45

History of mathematics, 136–40
Homework, 71, 75, 76

Incommensurables, 107
Individual work, 147, 148, 149, 151, 163
Inductive reasoning, 6, 15
Infants' school, 17, 18, 19
Instruments, blackboard, 68
Integrated syllabus, 44, 76
Interest as a motive for learning, 4, 11, 13, 36
Intuition, use of, 6, 58, 95, 114, 154

Jeffery Committee and Report, 3, 16, 110, 115, 218–26
Junior school, 17, 18, 19, 37

Kinematical approach to the calculus, 116

Laboratory, mathematics, 59, 179–93
Lecturing, 66
Library, mathematical, 59, 171–8
  in mathematics room, 182
  and sixth formers, 78, 152
Logarithms, the teaching of, 85–9

Marking, 59, 72, 195
Mathematical intuition, 6, 58, 95, 114, 154
  library, 59, 171–8; and sixth formers, 78, 152
  logic, 6
  philosophy, 140–5
  societies, 124, 182
  structure, building, 62
  teaching, ends of 8, 9; danger of premature formalization in, 104; by Project Method, use of Centre of Interest and method of Appreciation, 104
  tools, 12, 129, 142
Mathematics, 'abstract' and 'interest' approaches to, 7
  and mental training, 3, 4, 9, 11, 38, 97, 130
  as a creative and dynamic activity, 15
  as a living and growing subject, 144
  as an aesthetic experience, 55, 58, 142
  as a set of methods, or tools, 129, 142, 147
  as a vital part of human culture, 136
  associated with the experience of the pupil, 37
  curriculum in, 3, 4, 8, 11
  environmental approach to, 133–6
  experimental work in, 189–93
  for new grammar school entrants, 38, 39
  for the non-specialist in the sixth form, 150
  historical development of, 8, 137
  in school and classroom, 54–80, 181

in the sixth form, 146–70
logical coherence of subject matter in, 8, 133, 141
love of, 12, 36, 54–9
modern developments in, 143
motive in learning, 6, 7, 8, 142
nature of, 4, 5, 141, 144
room (or laboratory), 59, 179–93
specialist teachers of, 56, 74, 75, 111, 129
two-fold aspect of, 6, 8, 15, 141, 142
utilitarian value of, 141
Mechanics, at 'O' and 'A' levels, 159
at sixth form level, 127, 153
essential experiments in, 159
experimental basis of, 153, 155
history of, 140
in relation to physical chemistry, 150
in relation to physics, 158, 159
notations, 130
sixth form syllabuses in, 160, 161, 164
Memory work, 61
Mental work, 60
Meteorology, 125, 126
Models, construction of, 13, 19, 111, 113, 132
Money, four rules in, 24
Multiplication, degree of accuracy in, 84
methods of, 21, 22, 130
tables, 33, 38
Music and mathematics, 6

Navigation, 51, 95, 126–8, 138, 140
O. and C. Joint Board 'O' level syllabus in, 128
practical work in, 182
Neatness, 70
Non-euclidean geometry, 11, 113, 144, 145, 170
Notation in geometry, 99
Number sense, 17, 61
Nunn, Sir T. P., 5, 6, 8, 9, 108, 173

Organization of the work of a school, 74

Paradoxes, 58
Parametric methods in analytical geometry, 157, 158, 168
Pencils, coloured, 58, 68, 95, 99
Periods, division of subject among, 76
Perspective drawing, 97, 111, 112
Philosophy of mathematics, 140–5
Physics, correlation with, 123, 126, 146, 159
in relation to mechanics, 158, 159
Plans and elevations, 13, 19, 43, 44, 45, 47, 48, 50, 52, 110, 111, 112, 113, 125
Private study, 147, 148, 149, 151
Probability, 205–7
distribution, 207–12
Problems, 12, 14, 15
in the grammar school, 39
in the junior school, 18, 28, 29
involving area, 45
involving fractions, 41
solved by equations, 42
speed, 48
Projects, and project methods, 39, 40, 134
Proof, deductive, 45, 94, 139
mature conception of, 104
Psychological approach, 12, 65, 105, 107, 109
Puzzles, 39, 58, 59, 95, 142

Questioning, 64, 65

Reductio ad absurdum, 96, 104
Reports, *Arithmetic and Spatial Knowledge in Primary Schools*, 13
Hadow, 2, 7
issued by the Mathematical Association, 178
Jeffery, 218–26
Scottish Advisory Council on Secondary Education, 2, 4
Spens, 8, 10
Teaching of Elementary Geometry (I.A.A.M.) 92
Teaching in Primary Schools (A.T.C.D.E.), 38
Teaching of Statistics in Schools, 201
Revision, 63
for grammar school entrants, 38
Riders, 58, 93, 95, 98–104
grading of, 100
in solid geometry, 110

one-step and two-step, 98, 100, 104
style in the solution of, 101
trial attempts at, 99
written proofs of, 96–8
Rough working, 67, 70
Routine drill, 13, 61

Sampling, 199, 212–14
Scale drawing, 18, 41, 46, 124
and areas, 106
Scholarship examinations in the sixth form, 147–9
work in the sixth form, 161–4
Science lessons and degrees of accuracy, 81
Sets, mathematical, 74, 153
Significant figures, 82, 83, 197
definition of, 83
Sixth form, analysis of possible groups in, 149, 150
choice of text-books in, 152
courses for non-specialists in, 150
influences on work in, 147
integration of subject in, 153
introduction to mathematical philosophy in, 144, 145
mathematical attainment on entry to, 152
notes on teaching of, 153, 154
organizing of teaching in, 150-2
private study in, 147, 148, 149
reference library in, 152
specialization in, 147, 148
statistics in, 200–16
syllabuses, 154–70
the mathematical specialist, and the gifted boy, in, 151, 152
use of text-books in, 78
work in, 146–70
Sketches, 43, 45, 70
Slide rule, 15, 50, 52, 91, 193
Solids, and three-dimensional geometry, 42, 43, 45, 92, 94, 97, 109–14, 192
Solid geometry, formal, 113
in the sixth form, 158, 166, 167
Space, knowledge, 18, 19
measurement, 42, 44
perception, 97
sense, 17
study, 111
Specialist teachers of mathematics, 56, 74, 75, 111, 129
Specialization, in the sixth form, 147, 148
Standard deviation, 203, 205, 211
error, 214
Standardization of methods and notations, 130
Statistics, 123, 194–217
and meteorolgical readings, 126, 203
in the sixth form, 155, 156, 200–16
practical work in, 182, 191
to 'O' level, 194–200
Status of mathematics, 140, 141
Subtraction, methods of, 20
accuracy of results in, 84
Suffixes, 47, 99
Surveying, 15, 19, 43, 51, 124, 125
early methods of, 138
in the sixth form, 155, 156
practical work in, 182
Syllabus, 16, 149
for the first two years, 40–3, and (integrated) 44–6
for the first five years, in a grammar school, 36–53
for the third year, 47–9
for the fourth and fifth years, 50–3
for the sixth form, 154–70
of basic course in mechanics for the sixth form, 160
of advanced course in mechanics for the sixth form, 160, 161
of basic course in pure mathematics for the sixth form, 156, 157
of advanced course in pure mathematics for the sixth form, 157, 158
of scholarship pure mathematics and mechanics, 164

Tables, 18, 19, 89–91
Teachers, allocation of, 74, 75
Teaching periods, division of subject among, 76
Techniques, necessary, 13, 135, 136
needless, 16
new, 146
Tests, 64.

Text-books, answers in, 70, 78
choice of, 76–8, 152
examples in, 69, 77
influence of, 2
single copies of, 77
in the sixth form, 78, 152
subsidiary to teacher, 78
Theorems, 62, 63, 65, 96, 97
agreed sequence of, 92
converse, 96
groups of, 96
key, 96, 98
sequence of, 97, 104, and (in solid geometry), 114
use of, 96
Third year in the sixth form, 161, 162
Time allocation for mathematics, 75
in the sixth form, 150–2
Traditional syllabuses, 16
Transfer of training, 9, 10, 11
Trigonometry, in the first two years, 43, 46
in the third year, 47, 48
in the fourth year, 50
in the fifth year, 51
in the sixth form, 157, 158, 164
T-squares, use of, 112
Twofold aspect of mathematics, 6, 8, 15

Unification of school mathematics, 3
University entrance regulations, 147
need for reform of, 149
University scholarship examinations, 148, 149, 164

Variance, 205, 213
Visual aids, 131–3

Wall charts, 138
Written work in class, 69