A

Source Book

of

Mathematical

Applications

The National Council of Teachers of Mathematics

SEVENTEENTH YEARBOOK

A SOURCE BOOK *of* MATHEMATICAL APPLICATIONS

COMPILED BY
A COMMITTEE OF THE NATIONAL COUNCIL
OF TEACHERS OF MATHEMATICS

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EDITOR'S PREFACE

This is the seventeenth of the series of Yearbooks started in 1926 by the National Council of Teachers of Mathematics. The titles of the preceding Yearbooks are as follows:

1. A Survey of Progress in the Past Twenty-Five Years.
2. Curriculum Problems in Teaching Mathematics.
3. Selected Topics in the Teaching of Mathematics.
4. Significant Changes and Trends in the Teaching of Mathematics Throughout the World Since 1910.
5. The Teaching of Geometry.
6. Mathematics in Modern Life.
7. The Teaching of Algebra.
8. The Teaching of Mathematics in the Secondary School.
9. Relational and Functional Thinking in Mathematics.
10. The Teaching of Arithmetic
11. The Place of Mathematics in Modern Education.
12. Approximate Computation.
13. The Nature of Proof.
14. The Training of Mathematics Teachers.
15. The Place of Mathematics in Secondary Education.
16. Arithmetic in General Education.

Since the present Yearbook is a Source Book of Mathematical Applications, it represents a most important addition to the volumes that precede it. As editor of the series I wish to express my personal appreciation to the Committee who have compiled this Yearbook and to their associates and the many others who cooperated in making it possible.

W. D. Reeve

of the study of mathematics would constitute a grievous error. To fail to recognize the educational importance of examining the logical structure of mathematics would reveal a condition of irretrievable intellectual blindness.

The work of compiling and arranging the material for this book has been done by a central committee appointed by the National Council of Teachers of Mathematics. One member of the central committee, assisted by a subcommittee, has taken the responsibility for each of the sections. The membership of the several committees has been as follows:

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* At the beginning of the work, Dr. Lazar acted as chairman for the Geometry section, but owing to circumstances beyond his control he was forced to resign. Dr. Boyer, at considerable personal sacrifice, then assumed the responsibility for the completion of this section.

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While the members of the various committees have taken the initiative in collecting material, a large number of other individuals have given unselfish support to the undertaking. Many of them have contributed valuable applications; others have rendered service by their suggestions and constructive criticism. As far as possible the names of all these people have been included in the list of contributors at the close of this Preface. So many have helped in various ways, however, that it is feared that this list is by no means complete; an apology is hereby offered to anyone whose name has been omitted.

The authors and publishers of many textbooks and treatises on mathematics have been very generous in granting permission for the quotation or adaptation of a considerable amount of printed material. Wherever the source could be determined, such permission has been requested. However, with such a large number of contributions, it is possible that there are a few for which the original reference has been omitted inadvertently. General acknowledgment of any such material is hereby given.

In 1938 the mathematics section of the Society for Promotion of Engineering Education authorized the appointment of the following committee to collect engineering applications of elementary mathematics: R. S. Burington, R. V. Churchill, A. Hazeltine, D. L. Holl, W. C. Krathwohl, B. R. Teare, J. H. Weaver, J. W. Cell, *Chairman*. From the preliminary report of this committee a number of applications were selected for inclusion in the algebra and trigonometry sections. Special acknowledgment is due this committee for its generosity in making its findings freely available.

In conclusion, appreciation should be expressed to the Board of Directors of the National Council of Teachers of Mathematics for authorizing the subvention which made this book possible; and to President Mary A. Potter, of the National Council, for her helpful advice and heartening encouragement; and to Chairman W. D. Reeve and his editorial committee for their kindly guidance and help in preparing the manuscript for publication. E. G. O.

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EXPLANATORY NOTES

A careful reading of the following notes will give the user of this book an insight into the arrangement of material and enable him to find more rapidly the type of application he desires.

General plan. The main body of the text is divided into four sections: Arithmetic, Algebra, Geometry, and Trigonometry. In each section, mathematical topics are listed alphabetically, and under each topic there appears a list of applications. Each application carries a caption, indicating the life activity where it proves useful. All the applications are numbered, with the integral part of the number belonging to the topic and the decimal part distinguishing the problem. For example, to find application for arithmetical fractions, turn to the Arithmetic Section and locate the word **Fractions** in its proper alphabetical position (or turn to the Table of Contents and locate it by page number). On the same line appears **10.01–10.21**. This indicates that Fractions is the tenth topic under Arithmetic and that twenty-one applications are listed under this topic. Again, these applications are arranged alphabetically according to fields, i.e., ***Aviation*, 10.01**; ***Medicine*, 10.16**; etc.

To discover what mathematical topics have application in some particular field of human endeavor, such as Aviation, it is necessary to consult the Index of Uses of Mathematics, pp. 285 ff. Following *Aviation* we find numbers corresponding to all the applications to Aviation which are listed in the various sections. In case one particular name is not listed, it is advisable to try other roughly synonymous names before concluding that no applications are available.

Form of applications. In some cases applications appear in the form of a specific problem. Some of these merely are stated, others have hints as to method of attack, and still others have a complete solution. In some cases only the setting for the applications are supplied and the methods of utilization are optional. With a knowledge of the general background, it is possible to build a large number of problems of various types and degrees of difficulty. For example, using the following specifications for a tractor (as given in **Ar. 4.01**), a teacher might build the problems listed below, as well as many others.

Horsepower
- Belt
  - Maximum 23.87
  - Rated (85% of maximum) 20.29
- Drawbar
  - Maximum 16.90
  - Rated (75% of maximum) 12.68

Transmission ratios

Gear	*Gear ratio*	*Speed at 1400 r.p.m.*
Low	73.3 to 1	2.51 m.p.h.
Intermediate	57.0 to 1	3.23 m.p.h.
High	24.6 to 1	7.48 m.p.h.

SUGGESTED PROBLEMS

1. Is the maximum belt horsepower 115% of the rated belt horsepower?
2. What is the ratio of the rated drawbar horsepower to the rated belt horsepower?
3. How many revolutions per minute does the motor make at top governed speed?
4. How fast does the tractor travel in low gear when the motor is making 1000 revolutions per minute?
5. If the low gear ratio were increased by 50%, how fast would the tractor travel when the motor was making 1500 revolutions per minute?

6. Compute the motor speed when the tractor is travelling 10 miles per hour in high gear.
7. How many feet per second does the tractor travel when moving in intermediate gear at top governed speed?
8. If the tractor pulls two 18″ plows (each plow turns a swath 18″ wide) and runs in intermediate gear at a motor speed of 1400 revolutions per minute, how many acres will it plow in 6 hours?

Duplication and overlapping. A number of topics appear in each of several sections. For example, we meet graphs in Arithmetic and Algebra, and angles in Arithmetic, Geometry, and Trigonometry. There may be instances, therefore, when the teacher of one branch will find it useful to consult one or more of the sections belonging to the other branches.

Omitted topics. The user may be surprised at the omission of certain topics, particularly logarithms and logic in Geometry. The theory of logarithms provides a convenient device for certain types of computation. It has no particular applications apart from those it acquires by virtue of its use in connection with some other topic. As regards applications of geometric methods of proof to non-geometric situations, the position is taken that geometry itself is an application of logic. On this basis, such applications do not fall within the scope of this volume.

CONTENTS

ARITHMETIC

ARITHMETIC

ANGLE 1.01–1.02

1.01. Mechanics. A 1940 automobile owner's manual* gives directions for advancing and retarding the ignition timing to adjust the ignition to varying grades of "regular" or "premium" gasolines. This discussion uses *degrees* as follows: ". . . one degree of distributor movement is equal to two degrees of flywheel travel. One degree is approximately ⅛ in. at flywheel timing mark. This mark is 4° before center on Series 40–50, and 6° before center on Series 60–70–80–90." This manual explains the use of the rear-vision mirror as follows: "The ball socket on the mirror is off-set, up and down, so that if the mirror is swung around (180°) it can be raised or lowered for tall or short operators."

Buick cars for 1940 feature a mark on the steering wheel which must be at a certain position when the wheels are pointing straight ahead. If through accident any part of the steering mechanism is bent, a slight bend may not show in a casual check but the position of the mark on the steering wheel can be used to detect bend "because a bend as slight as 1° in some parts will cause a 20° movement at the wheel."

1.02. Mining. "The following table can be used by people desiring to use wire rope on slopes, inclined planes, etc., to determine the strain produced by any load.

"The table gives the strain produced on a rope by a load of one ton of two thousand pounds, an allowance for rolling friction being made. An additional allowance for the weight of the rope will have to be made.

"*Example:* For an inclination of twenty-five feet in one hundred feet, corresponding to an angle of 14½°, a load of 2000 pounds

* See 1940 *Buick Owner's Manual*, p. 63.

will produce a strain on the rope of 497 pounds; and for a load of 8000 pounds, the strain on the rope will be

$$\frac{497 \times 8000}{2000} = 1988 \text{ pounds.}$$

Elevation in 100 Ft.	*Corresponding Angle of Inclination*	*Strain in Lb. on Rope from a Load of 2000 Lb.*	*Elevation in 100 Ft.*	*Corresponding Angle of Inclination*	*Strain in Lb. on Rope from a Load of 2000 Lb.*
5	$2\frac{7}{8}°$	112	95	$43\frac{1}{2}°$	1385
10	$5\frac{1}{2}°$	211	100	$45°$	1419
15	$8\frac{1}{3}°$	308	105	$46\frac{1}{2}°$	1457
20	$11\frac{1}{5}°$	404	110	$47\frac{3}{4}°$	1487
25	$14\frac{1}{2}°$	497	115	$49°$	1516
30	$16\frac{3}{4}°$	586	120	$50\frac{1}{4}°$	1544
35	$19\frac{1}{5}°$	673	125	$51\frac{1}{2}°$	1570
40	$21\frac{5}{6}°$	754	130	$52\frac{1}{2}°$	1592
45	$24\frac{1}{4}°$	832	135	$53\frac{1}{2}°$	1614
50	$26\frac{1}{3}°$	905	140	$54\frac{1}{2}°$	1633
55	$28\frac{5}{6}°$	975	145	$55\frac{1}{2}°$	1653
60	$31°$	1040	150	$56\frac{1}{4}°$	1671
65	$33\frac{1}{12}°$	1100	155	$57\frac{1}{4}°$	1689
70	$35°$	1156	160	$58°$	1703
75	$37°$	1210	165	$58\frac{4}{5}°$	1717
80	$38\frac{2}{3}°$	1260	170	$59\frac{1}{2}°$	1729
85	$40\frac{1}{2}°$	1304	175	$60\frac{1}{4}°$	1742
90	$42°$	1347			

"A factor of safety of five to seven times should be taken—that is, the working load on the rope should only be one fifth to one seventh of its breaking strength. As a rule, rope for shafts should have a factor of safety of five, and on inclined planes, where the wear is much greater, the factor of safety should be seven."*

Problem: A mine shaft 600 ft. deep drops 5 ft. for every 10 ft. of horizontal distance. The "buggy," loaded with coal, weighs 1500 lb. If we wish to use a safety factor of five, what strain should the cable used to hoist the buggy be able to withstand (disregard the weight of the cable)?

* W. G. Beaver, *A Text Book for Stationary Engineers*, pp. 145–146. David McKay Company, Philadelphia, Revised 1939.

AVERAGE 2.01–2.06

2.01. Agriculture. The average is often used to gain a comprehensive view of some problem; to see trends, as in the following illustration:* "The American people have a 57-billion-dollar farm investment in land, buildings, implements, and livestock. This investment is spread over almost seven million farms occupied by approximately 32 million people. Today agriculture is sick. In order that her case may be cured we must study her case history.

"To begin, we need to examine price trends of the last seventy years. These will show the average prices to the average farmer over each period as calculated from the statistical reports of the U. S. Department of Agriculture:

Period	*Average Price of Wheat to the Farmer*
1870–1881	$1.07 per bushel
1882–1915	0.79 " "
1916–1919	2.09 " "
1920–1929	1.19 " "
1930–1939	0.72 " "

We also study the *average* yields for the nation as a whole.

Decade	*Average Yield per Acre of Wheat*
1870–1879	12.46 bushels
1880–1889	13.13 "
1890–1899	13.72 "
1900–1909	14.39 "
1910–1919	14.12 "
1920–1929	13.99 "
1930–1939	13.22 " "

Problem: Plot the *average* yield per acre of wheat from 1870 to 1939.

2.02. Agriculture.† Whether animal manure or commercial fertilizers should be purchased for plant food is a question for farmers and vegetable growers. For guidance purposes the following fig-

* Adapted from Bliss Isely, "The Case History of Wheat." *The Atlantic Monthly*, Vol. CLXV, pp. 632–638, May, 1940.

† See The Pennsylvania State College School of Agriculture and Experiment Station, Division of Agriculture Extension, Circular 138—Revised May, 1932, *Fertilizing Vegetable Crops*, p. 4. State College, Pennsylvania.

ures are given: The average prices for nitrogen, phosphoric acid, and potash per pound, in commercial fertilizers, are 15, 5, and 5 cents respectively. With these figures as a basis and a knowledge of the average composition of fresh manure per ton a table as follows has been drawn up:

Source	*Nitrogen*	*Phosphoric Acid*	*Potash*	*Value*
Hen	20 lb.	16 lb.	10 lb.	$4.30
Sheep	20	9	17	4.30
Hog	13	7	10	2.50
Horse	11	6	8	2.05

2.03. Agriculture. The farmer often sells his cotton at different prices during the gathering season. At the end of the season he wants to know the average price per pound that he has received for his cotton.

Problem: Find the average price per pound of the yearly yield of cotton as follows:

Number of Bales	*Price per Pound (in Cents)*	*Weight (in Pounds)*	*Value (in Dollars)*
28	10.70¢	12,860	$1376.020
32	9.54	15,240	1433.896
5	9.70	2,470	239.590
17	9.50	8,395	797.525
67	8.50	33,710	2865.350
14	8.80	5,990	527.120
24	9.00	12,165	1094.850
93	7.75	45,945	3560.837

2.04. Agriculture.* "If a farmer plans to buy a farm, he should first investigate its average gross receipts for the preceding years," says J. E. McCord, professor of farm management and agricultural economics at the Pennsylvania State College. Professor McCord says further that the receipts should average at least 25 per cent of the capital investment.

Problem: If the gross receipts of a farm average $4000, how much, according to Professor McCord's theory, may be invested in it?

2.05. Daily Life.† With respect to many characteristics of life

* See *The Patriot*, Harrisburg, Pa., November 28, 1940.

† Adapted from Progressive Education Association, *Mathematics in General Education*, pp. 122–123. Appleton-Century Company, New York, 1940.

there is an implicit awareness, on the part of nearly everyone, of the central tendency of the particular characteristic. This is manifested in such expressions of judgment as: "He is tall for his age"; "She is rich"; "It is a cold day"; "India is far away." It is interesting to note that the range of each characteristic is divided into two parts: one above the average; the other below the average—thus, tall and short, rich and poor, etc. In scientific work the concept of central tendency is expressed in terms of several more or less standardized units: mean, median, and mode.

2.06. EDUCATION. Many colleges attempt to stimulate scholarship by using a "quality point marking system." Under this system a semester grade of *A* is awarded three quality points per semester-hour course credit; a semester grade of *B*, two quality points per semester-hour course credit; a grade of *C*, one quality point per semester-hour course credit; a grade of *D*, zero quality points per semester-hour course credit; and a grade of *F* (failure), a minus one quality point per semester-hour course credit. At the end of each semester, the Quality Point Average is computed for each student by dividing the total number of his quality points by the number of semester hours of work he carried that semester. College regulations demand that a certain average be achieved for a student to become eligible for practice teaching, for the Dean's list, for graduation, etc. At some colleges, honors are assigned upon the quality point averages, as follows:

Quality Point Average	*Honor*
2.75 or better	summa cum laude
2.50 to 2.75	magna cum laude
2.25 to 2.50	cum laude

Problem: Student *S* carries the following schedule and is awarded the following grades:

Course	*Semester Hours' Credit*	*Grade*
Intro. to Teaching	3	F
English II	3	D
Music Appreciation	2	C
Health	1	A
Trigonometry	3	B
Physics I	4	A

What will be his Quality Point Average for this semester?

Also see Fractions Ar. 10.14; Ratio Ar. 21.02.

COMPOUND NUMBERS 3.01–3.02

3.01. Daily Life. A fully equipped soldier in World War II carries the following equipment:

Shelter half (stakes, rope, pole)	2 lb. 8 oz.
Blanket	4 " 0 "
Haversack	2 " 8 "
Trench tool and carrier	2 " 0 "
Toilet articles	1 " 5 "
Clothing (incl. overcoat)	19 " 8 "
Bayonet and scabbard	1 " 8 "
Rifle	9 " 0 "
Mess kit	1 " 0 "
Helmet	2 " 0 "
Raincoat	2 " 4 "
Canteen (filled) and cup	3 " 8 "
Cartridges and belt	6 " 0 "
Gas mask	5 " 0 "
First aid kit	0 " 4 "

Problem: How much is the total weight of this soldier's equipment?

3.02. Daily Life. When the Columbia River was dammed at Grand Coulee, it began to form the third largest reservoir of water in the United States. To express the growth of the lake a rarely used denominate number appeared in the press. It was the acre-foot, a concept which expresses area as well as volume. During 1940 this man-made lake increased by 2,590,000 acre-feet. Ultimately the maximum capacity of the dam will be 9,517,000 acre-feet.

Problems: (a) If one acre-foot is amount of water needed to cover one acre to a depth of one foot, how many families of five persons each could be supplied with water for one year with this amount? Allow 30 gallons of water per person per day.

(b) How large a town could be supplied for one year with the water that constitutes the maximum capacity of the Grand Coulee Dam?

DECIMALS 4.01–4.17

4.01. AGRICULTURE. Specifications of tractors merit scrutiny and comparison. The specifications for 1940 Ford tractors feature:

Horsepower
Belt
Maximum 23.87
Rated (85% of maximum) 20.29
Drawbar
Maximum 16.90
Rated (75% of maximum) 12.68

Transmission ratios

Gear	*Gear Ratio*	*Speed at 1400 r.p.m.*
Low	73.3 to 1	2.51 m.p.h.
Intermediate	57.0 to 1	3.23 m.p.h.
High	24.6 to 1	7.48 m.p.h.

At top governed speed, the tractor can be operated at 3.94 m.p.h. in low gear, at 5.10 m.p.h. in intermediate, and at 11.75 m.p.h. in high.

Problem: If a tractor pulls two 14 in. plows (each plow turns a swath 14 in. wide) and if intermediate gear ratio 57.0 to 1 (motor makes 57.0 revolutions to one revolution of rear wheel which has an over-all diameter of 48 in.) is used, how many revolutions will the motor make in plowing a strip equal in area to one acre? If the farmer uses this tractor to plow nine acres in one day, how many revolutions will the tractor make this day? Do you think changing crankcase oil after 100 hours of operation is unreasonable?

4.02. AGRICULTURE. In many instances a farm is not equipped with wagon scales, and to measure ear corn the farmer is forced to rely upon another means of measuring which is known as gauging and which gives a result very close to the actual weight measure. To gauge a wagon box or corn crib, the number of cubic feet is found and multiplied by 0.4, or, in formula form:

No. of bushels
of corn on cob = (length in feet × width in feet × depth in feet) × .4

The farmer can estimate quite readily the capacity of the standard wagon bed, for it holds approximately one bushel of ear corn per inch in length.

To find the number of bushels of *old* corn in a crib, the number of cubic feet of corn is multiplied by 0.45.

Problem: Find the number of bushels of old corn in a crib, the dimensions of which are 16 ft. × 8 ft. × 4 ft. Since the crib is not evenly filled with corn, regard average depth of corn as 7 ft.

4.03. Astronomy.* Mean distances of planets from the sun are concisely expressed in terms of decimal fractions of astronomical units. Thus, the mean distances are as follows:

From Sun to Mercury	0.387 astr. units
" " " Venus	0.723 " "
" " " Earth	1.000 " "
" " " Mars	1.524 " "
" " " Jupiter	5.203 " "

Problem: If one astronomical unit is 92,900,000 miles (the mean distance of the earth from the sun), find the mean distance between Mars and the sun.

4.04. Astronomy. The periodic time of a planet is the time it requires to revolve around the sun. This is often most meaningful when compared with the periodic time of the earth. Thus, taking the earth's year as the unit, the periodic times of the planets are:

Mercury	*Venus*	*Earth*	*Mars*	*Jupiter*	*Saturn*	*Uranus*	*Neptune*
.2408	.6154	1	1.881	11.86	29.46	84.02	164.8

The distances from the earth to the planets are also meaningfully expressed by the use of decimals if the earth's distance from the sun is used as a unit, thus:

Mercury	*Venus*	*Earth*	*Mars*	*Jupiter*	*Saturn*	*Uranus*	*Neptune*
.387	.723	1	1.52	5.20	9.54	19.19	30.07

4.05. Automobiles. In an automobile owner's manual (Buick, 1940) the function of the transmission is explained. By allowing the crankshaft to make more turns than the driveshaft, extra motive power is provided. Since the ratio of the number of times the crankshaft turns to a single turn of the driveshaft is seldom an integer (because of size of gears used), decimals must be used to

* Adapted from *Van Nostrand's Scientific Encyclopedia*, p. 865. By courtesy of D. Van Nostrand Company, New York, 1938.

give this description. (Thus, in low gear the crankshaft turns 2.39 times as often as the driveshaft; in second gear this ratio is 1.53 to 1.)

In this same manual, optional gear ratios for the rear axle are explained. Standard gear ratios provide a good balance of speed and power with brilliant acceleration. This ratio for the Series 40 Buick is 4.4 to 1 and means that the motor crankshaft makes 4.4 revolutions to the rear wheels' single revolution. The optional gear ratio provides greater fuel economy, slightly greater top speed, and less engine wear, but sacrifices some brilliance of acceleration and a certain amount of power on hills. This ratio, for the above-mentioned car, is 3.9 to 1.

4.06. AUTOMOBILES. The Illinois Highway Department has put up many new signs. One reads PICNIC TABLE 1000 FEET AHEAD. That distance is approximately what fraction of a mile? How will the reading on the speedometer change between the sign and the table? Other signs read SAFE DRINKING WATER 2000 FEET AHEAD, NO PASSING FOR 700 FEET, DIP IN THE PAVEMENT 500 FEET.

4.07. DAILY LIFE. The principles for constructing graphs can be used to illustrate: (a) The all-year-round climate of New Orleans. (b) The monthly stages of the Mississippi River at New Orleans. The following are representative problems.

a. Climate

Monthly Mean Temperature of New Orleans for 1939

Month	*Jan.*	*Feb.*	*Mar.*	*Apr.*	*May*	*June*	*July*	*Aug.*	*Sept.*	*Oct.*	*Nov.*	*Dec.*
Temp. in degrees	59.0	60.2	66.0	68.4	74.9	81.9	83.6	82.7	81.4	72.5	60.2	58.6

b. River Stages

Monthly Mississippi River Stages for 1939

Month	*Jan.*	*Feb.*	*Mar.*	*Apr.*	*May*	*June*	*July*	*Aug.*	*Sept.*	*Oct.*	*Nov.*	*Dec.*
Height in feet	2.5	9.3	16.4	16.0	14.2	6.2	5.0	2.0	1.8	1.8	1.5	1.1

4.08. Daily Life.* The standard diameter of match sticks in Japan is expressed in terms of decimals. Because of shortages of raw materials, Japan reduced the standard diameter of match sticks from .072 to .06 of an inch.

Problem: If a match stick is two inches long, how much wood was saved in the manufacture of 1,000,000 matches by making the above reduction in diameter?

4.09. Daily Life. Whether or not automobile drivers who meet with accidents are under the influence of liquor is often a controversial question. The state of New York has passed a law (effective July 1, 1941) to the effect that the results of chemical analysis of the driver's blood, taken within two hours of arrest, will be admitted as evidence in court. The presence in the blood of 0.15 per cent or more alcohol by weight will be admitted as *prima facie* evidence that the defendant was in an intoxicated condition.

4.10. Engineering.† Pressure of wind (in lb. per sq. ft.) against a vertical surface is approximately $.004V^2$, in which V, the velocity, is given in miles per hour.

Problem: A three-inch pipe supports a circular area devoted to advertising a certain brand of tobacco. If the pipe safely stands a breaking stress of 300 lb., will it be strong enough to weather a gale of 75 miles per hour if the circular area is 4 ft. in diameter?

4.11. Engineering. To determine the correct length of V-belt to use with two pulleys in a given situation is a problem demanding the use of arithmetic. One company gives these instructions: Add the diameter of the machine pulley to the diameter of the engine pulley. Multiply this sum by 1.57. Add to this sum twice the distance from center to center. The result is the outside circumference of the V-belt needed. This result is an approximation which improves as the two pulleys approach equality in size.

4.12. Housewife. A common package of Kellogg's All-Bran features the following analysis:

* See *Time*, Vol. 36, No. 17, p. 30, October 21, 1940.

† Adapted from *Van Nostrand's Scientific Encyclopedia*, p. 965. By courtesy of D. Van Nostrand Company, New York, 1938.

Nutritional	*Grams per Ounce*
Protein	3.3
Carbohydrates	18.6
Fiber	2.4
Mineral matter	2.6
Calcium	0.0289
Phosphorus	0.3700
Iron	0.00164
Copper	0.000454
Cereal oils	0.8
Water	0.6
Vitamin B^1	23 I. U. per ounce

Ingredient	*Per Cent*
Wheat bran	83.0
Sugar	12.5
Salt	3.5
Malt flavoring	1.0

4.13. MECHANICS.* In a study of the use of decimals by employees in the Boston office of the Otis Elevator Co., Buckley lists the following:

.0087	a percentage used for purposes of analysis in a financial statement.
.00625	inch, measurement of a hatchway and car.
.00315	inch, clearance of one part to another.
.0021	inch, reference to machine alignment.
2500.0325	revolutions per minute.
4.8763	inches, measurement of a shaft.
1.003	inches, micrometer reading referring to the size of a wire.

4.14. MECHANICS.† The amount of expansion of substances when they are heated is expressed decimally under the heading "coefficients of expansion." Certain practices are followed, depending upon the size of the coefficients.

Since the coefficient of nickel steel (36% nickel) is a mere .0000003, this material is used to manufacture measuring instruments required in situations where expansion of the instrument would introduce serious errors.

* See Joseph E. Buckley, "The Uses of Decimals in Business." Master's thesis, Boston University, 1935.

† Adapted from E. B. Norris and K. G. Smith, *Shop Arithmetic*, pp. 216–218. McGraw-Hill Book Company, New York, 1913.

In laying out steam pipe lines, expansion must be taken into account, as the following example shows.

Example: What will be the expansion of a steam pipe (steel) 200 ft. long when subjected to a temperature of 300° F., if erected when the temperature is 60° F.? Solution:

Coefficient of expansion = .0000065 (from table)
Temperature change = 300° − 60° = 240°
Expansion per degree = 200 × .0000065 = .00130 ft.
Total expansion = 240 × .0013 = .312 ft. *Answer.*

4.15. Medicine.* Practically all medical prescriptions are given in English units and in metric units. The ability to convert a prescription from one system to another is needed by all physicians. To facilitate such conversions the following rules are given:

To convert drams into grams, multiply the number of drams by 3.9, the number of grams in 1 dram.

To convert grains into the corresponding metric quantity, multiply the number of grains by .065, the metric equivalent of 1 grain.

To convert grams into drams, divide the number of grams by 3.9. To convert grams into grains, divide the number of grams by .065.

4.16. Meteorology.† In recording barometric pressure the barometer scale is read and recorded in inches, tenths and hundredths, thus: 29.62, 30.00, 26.30, etc.

Problem: What difference in interpretation would be given to the following recorded pressures (a) 29.6, (b) 29.60, (c) 32, and (d) 32.00?

4.17. Railways.‡ In making shrink fits it is often necessary to determine the number of degrees of temperature rise needed to shrink a band onto a shaft, e.g., a steel tire onto a locomotive wheel.

Problem: If we follow the rule that .001 in. per inch of diameter is used to achieve a shrink fit, to what temperature must a steel

* See A. A. Stevens, *A Textbook of Therapeutics*, p. 740. W. B. Saunders Company, Philadelphia, 1924.

† See *Commercial Aeronautics—16*, p. 22. American Technical Society, Chicago.

‡ Adapted from E. B. Norris and K. G. Smith, *Shop Arithmetic*, pp. 219-220. McGraw-Hill Book Company, New York, 1913.

tire be raised to slip over a locomotive wheel? Initial temperature of wheel and tire is 70° F. Solution:

Coefficient of expansion = .0000065 (from table)
.001 ÷ .0000065 = 154°
Final temperature = 154° + 70° = 224°. *Answer.*

In actual practice the tire would be heated to a slightly higher temperature to expand it enough to slip it on easily and quickly before it had time to cool off or warm the wheel.

Problem: What should be the space left between the ends of a 96-foot section of railroad railing when laid during a temperature of +45° F. in a locality where the temperature range is from −24° to +110° F.?

Also see Average Ar. 2.06; Exponents Ar. 8.01; Fractions Ar. 10.07, 10.13; Fundamental operations Ar. 11.08, 11.31; Graphs Ar. 12.03; Mensuration Ar. 16.13, 16.15; Numbers Ar. 18.01, 18.11, 18.12; Proportion Ar. 20.02, 20.11.

DECIMAL POINT 5.01–5.02

5.01. Banking. Liberty Bonds are quoted with a decimal point, viz., 95.15. The fractional portion indicates the number of thirty-seconds; e.g., 95.15 means $95\frac{15}{32}$. These are the only bonds thus quoted.

5.02. Psychology.* Mathematical expressions frequently are characterized as "shorthand." That is, mathematicians have invented many symbols which are easily written, occupy little space, and lend themselves readily to mathematical operations. To know the meaning of the symbols often means ability to interpret the mathematical statement. On the other hand, lack of acquaintance with the symbols makes for misinterpretation of fact. An example of the use of simple mathematical symbolism is found in the following:

Ages of children are frequently given to the nearest month. But to write the age as "14 years 9 months" or even as "14 yrs. 9 mos." consumes time and space. To simplify writing ages the decimal point has come into use. Thus, 14 years 9 months is written 14.9; 12 years 4 months, 12.4.

* See Progressive Education Association, *Mathematics in General Education*, pp. 297 and 313. D. Appleton-Century Company, New York, 1940.

DENOMINATE NUMBERS 6.01–6.22

6.01. AGRICULTURE. A cubic foot is 1728 cu. in. A Winchester bushel is 2150.42 cu. in. Hence the ratio of a cubic foot to a bushel is about .8. However, the error in using .8 for this ratio is nearly $\frac{1}{2}$ of 1% and the ratio is too small by that per cent. For example, a bin whose inside contents equal 800 cu. ft. will hold $800 \times .8 + \frac{1}{2}$ of 1% of this product in bushels. Thus $800 \times .8 \times 1.005 = 643$. The ratio .8 is as precise, however, as the measurements of the bin could be made. But the measurements might be too small or they might be too large. On the other hand, the ratio .8 would always be too small. For this reason it seems wise to add $\frac{1}{2}$ of 1% to the product when the contents in cubic feet are multiplied by .8 to change the contents to bushels.

6.02. AGRICULTURE.* The Agricultural Market Service of the Department of Agriculture has developed standards for grading eggs. Four government grades are:

U. S. Special; very few on market	Grade AA
U. S. Extra; good breakfast eggs	Grade A
U. S. Standard; satisfactory for cooking	Grade B
U. S. Trade; suitable for preparing dishes where egg flavor does not count much.	

"Large" eggs must weigh 24 ounces to the dozen.
"Medium" eggs must weigh $20\frac{1}{2}$ ounces to the dozen.
"Small" eggs must weigh 17 ounces to the dozen.

Problem: Approximately what should one egg weigh in each case? Weigh some eggs for comparison with standards.

6.03. AGRICULTURE.† It is estimated that 400 billion tons of topsoil are carried yearly into the Gulf of Mexico by the Mississippi River.

In 1939 the capacity of the average railroad-owned freight car was 49.8 tons.

Problem: How many such freight cars could be filled with the topsoil deposited annually into the Gulf of Mexico?

* See U. S. Department of Agriculture, *Consumer's Guide*, December 2, 1940.

† See H. B. Bruner and C. B. Smith, *Social Studies, Book Three*, p. 365. Charles E. Merrill Company, New York, 1938.

6.04. Agriculture. Laborers in the cotton lands are often paid on a per-acre basis. To determine the pay per day, it is necessary to know how many rows in the field constitute an acre.

Problem: A field of cotton is one quarter of a mile long. If the rows are 3 feet 8 inches apart, how many rows make an acre? Solution:

$$5280 \text{ ft.} = 1 \text{ mi.}$$
$$\tfrac{1}{4} \text{ of } 5280 \text{ ft.} = 1320 \text{ ft., length of rows in feet}$$
$$1320 \text{ ft.} \times 3\tfrac{2}{3} \text{ ft.} = 4840 \text{ sq. ft. in 1 row}$$
$$43{,}560 \text{ sq. ft.} \div 4840 = 9 \text{ rows}$$

6.05. Aviation. A plane with an air speed of 200 miles per hour, and loaded with 250 gallons of gasoline, leaves City A and travels a distance of 400 miles to City B. If the plane burns $4\frac{1}{2}$ pounds of gasoline per minute (1 gal. of gas weighs 6 lb.), how much gas will be left upon arrival at City B?

6.06. City Manager.* The Berkeley, Calif., costs for cleaning residential streets are expressed by using denominate numbers as follows:

Fiscal Year	*Linear Miles*	*Square Yards*	*Total Cost*
1934–35	2,214.32	15,589,863	$25,160.01
1935–36	2,249.65	15,839,196	24,323.56
1936–37	2,523.99	17,767,196	25,184.95
1937–38	2,462.75	17,137,862	26,431.66
1938–39	2,501.24	17,155,119	26,100.36
1939–40	1,800.90	12,678,336	25,115.65

Problem: On this basis, the cost per linear mile in 1934–35 was $11.36 and the cost per square yard, .0016¢. Find these costs for other years.

6.07. Clothing. The Bureau of Standards of the United States Department of Commerce has issued a set of measurements, given in inches, which will be the basis upon which pattern makers will produce their patterns. They are as follows:

* See H. Goodridge, "Street Cleaning Practice in Berkeley, Calif." *The American City*, Vol. 56, No. 4, p. 58, April, 1941.

Women's Sizes

Bust	34	36	38	40	42	44	46	48	50
Waist	28	30	32	34	36	38	40	42	44
Hip	37	39	41	43	45	47	50	53	56

Misses Sizes

Size	14	16	18	20
Bust	32	34	36	38
Waist	27	28	30	32
Hip	35	37	39	41

Problem: What are the minimum and maximum differences between hip and waist measurements?

6.08. Daily Life. A Winchester liquid gallon contains 231 cu. in. Hence the approximation of $7\frac{1}{2}$ gal. to the cubic foot is nearly $\frac{1}{4}$ of 1% too large, since $1728 \div 231 = 7.48$. Thus a vat that contains 800 cu. ft. will hold $800 \times 7\frac{1}{2} - \frac{1}{4}$ of 1% of this product, or 5985 gal. While the error of using $7\frac{1}{2}$ as the ratio of the cubic foot to the gallon results in an over-estimate of $\frac{1}{4}$ of 1%, the error is probably less than that involved in the measurement of the vat.

6.09. Daily Life. An acre is 43,560 sq. ft. Hence a square tract of land 208.7 ft. on the side is an acre. A football playing field, less one ten-yard strip across, is a good illustration of about an acre:

$$270 \times 160 = 43{,}200.$$

6.10. Daily Life. Denominate numbers can be used to determine the number of hours to be worked in the case of a company observing a certain hour law.

Problem: One company allowed its employees to work forty hours per week. If Mr. Johnson worked a total of 28 hr. 10 min. from Monday through Wednesday of a particular week, what would be an average of the time that he could work on both Thursday and Friday of that week?

Solution: A total of 28 hr. 10 min. from the maximum of 39 hr. 60 min. would leave 11 hr. 50 min. to be worked. An average of 11 hr. 50 min. would be 5 hr. 55 min. to be worked on both Thursday and Friday of that particular week.

6.11. Daily Life. The deepest point in the oceans sounded thus far is near the Philippine Islands and is more than 35,000 ft. in depth.* The greatest depth in the Atlantic Ocean is just north of Puerto Rico and is 30,000 ft. The latter is called the Milwaukee depth. It was sounded in 1939.

6.12. Daily Life. The facts learned in the study of denominate numbers can be used to satisfy a growing interest of today's air-minded children.

Problem: Consider the following problem and its solution: John sent a letter by air mail to a friend in New York. The letter left a Texas airport at 1:05 p.m. and arrived in New York at 2:55 a.m. How long did it take to get there? Solution:

11:60
~~12~~:~~00~~ p.m.
−1:05 p.m.
10:55 = 10 hr. 55 min.

10 hr. 55 min.
+2 hr. 55 min.
12 hr. 110 min. = 13 hr. 50 min.

6.13. Electricity. In spacing lights in a paneled ceiling, consideration must be given to equal distribution of light in the room and the centering of fixtures in the panels. For equal distribution of light, distance from wall to end-fixture should be one-half the distance between fixtures.

Consider this problem: Given room dimensions of 45′ × 80′ and ceiling panels 16″ × 16″ (common sizes 12″ × 12″, 16″ × 16″, 16″ × 32″). Find the correct spacing for 18 lights, 3 across the width and 6 along the length. Solution: 45 × 12″ = 540″, width of room in inches. 540″ ÷ 16 = $33\frac{3}{4}$ blocks. $\frac{3}{4} \div 2 = \frac{3}{8}$ block on each side before we meet first whole block. That is, across room we will use 33 full blocks and $\frac{3}{8}$ of a block on each side. 33 blocks ÷ 3 = 11 blocks between fixtures going from side to side. 11 × 16″ = 176″ = 14′ 8″ distance between fixtures from side to side. 176″ ÷ 2 = 88″ = 7′ 4″ + 6″ = 7′ 10″ distance between side wall and first fixture. 80 × 12″ = 960″; 960″ ÷ 16 = 60 blocks. 60 blocks ÷ 6 = 10 blocks between

* See *World Almanac*, 1941.

fixtures. 10 × 16″ = 160″. 160″ = 13′ 4″ between fixtures from end to end. 160″ ÷ 2 = 80″ = 6′ 8″ between end wall and first fixture.

6.14. Engineering. Before a highway is built (or rebuilt) the roadbed is surveyed and stakes called "stations" are driven into the earth along the side of the road. These stations form reference points for measurements of ditches, pavements, drain pipes, etc. Furthermore, the stations are numbered in an unusual way to increase visibility and facilitate ease of reading. They are numbered thus: 11 + 00, 12 + 50, 7 + 32, etc. 11 + 00 means 1100 and 12 + 50 means 1250.

Problem: A 5-foot sidewalk begins at station 7 + 32 and ends at station 11 + 16. If the contractor receives 50¢ per square yard for removing the sidewalk, what is the total amount he will receive?

6.15. Engineering.* The area of a safety valve for a boiler is determined by allowing one square inch of valve area for every two square feet of grate surface.

Problem: What should be the area in square inches of a safety valve used on a boiler whose grate is 6 ft. × 18 in.?

The size of a safety valve for a boiler is its diameter in inches, viz., a 3-inch valve, a 5-inch valve, etc. If D = diameter of valve in inches and A = area of valve in square inches, then

$$D = \sqrt{\frac{A}{.7854}}.$$

Problem: What size of safety valve should be used with the boiler above described?

6.16. Engineering.† During a recent 80-mile-per-hour windstorm it was found that the top of the Empire State Building in New York swayed through an eight-inch arc at the rate of 7.85 times per minute. At one end of the sway, the building reached $6\frac{3}{4}$ inches out of plumb away from the wind and at the other end it returned $1\frac{1}{4}$ inches beyond plumb toward the wind.

* Adapted from H. W. Marsh, *Industrial Mathematics*, p. 52. John Wiley and Sons, New York, 1912.

† Adapted from *Scientific American*, Vol. 163, No. 12, p. 315, December, 1940.

Problem: How far would the top of the building sway during twelve hours of such a storm?

6.17. Home Economics. To determine the number of inches of material to curtain one window 72 in. long and 31 in. wide. For 100 per cent fullness the following material would be required:

Two lengths 36 in. material	144 in.
Double hem at bottom, $1\frac{1}{2}$ in. deep	6 in.
Heading and casing, $3\frac{3}{4}$ in. each	$7\frac{1}{2}$ in.
Shrinkage, 2 in. or more	4 in.
Total	162 in.

6.18. Housewife. A housewife wishes to carpet a stairway and a hall with new stair carpet that sells at $1.65 a yard. The stairway has 24 steps, each 7 in. high and 9 in. wide. The hall is 32 ft. long.

Problem: What will it cost to carpet the stairway and the hall?

6.19. Housewife. Every standard electric sweeper or iron is tagged with a plate upon which is stated the number of watts that are consumed by the machine in one hour. This enables the owner to determine the cost of operating the machine as well as to compare the consumption of one machine with another.

Problem: What will be the cost of operating, for two hours, a 120-volt Westinghouse electric iron which is stamped "600 Watts" when electric current costs $7\frac{1}{2}$ cents per K.W.H.? Solution:

$$\frac{\overset{3}{\cancel{600}} \times 2 \times \overset{.015}{\cancel{.075}}}{\underset{5}{\cancel{1000}}} = .09, \text{ or } 9\text{¢}.$$

6.20. Mining. Anthracite coal is graded as to size. It is sized by passing the coal over screens that have square meshes. The following table explains the method of grading:

Name of Size of Coal	*Will Pass Through a Square Mesh of:*	*Will Not Pass Through a Square Mesh of:*
Broken	$4\frac{3}{8}''$	$3\frac{1}{4}''$
Egg	$3\frac{1}{4}''$	$2\frac{7}{16}''$
Stove	$2\frac{7}{16}''$	$1\frac{5}{8}''$
Chestnut	$1\frac{5}{8}''$	$\frac{13}{16}''$
Pea	$\frac{13}{16}''$	$\frac{9}{16}''$
Buckwheat	$\frac{9}{16}''$	$\frac{5}{16}''$
Rice	$\frac{5}{16}''$	$\frac{3}{16}''$
Barley	$\frac{3}{16}''$	$\frac{3}{32}''$

6.21. PAPER DEALER.* "A paper manufacturer sells paper by standard size and weight.

Standard book paper is 25″ by 38″.
Standard writing paper is 17″ by 22″.
Standard wrapping paper is 24″ by 36″.

The paper manufacturer sells paper by the ream. Book papers and writing papers have 500 sheets to the ream, and wrapping paper has 480 sheets."

The weight of the paper refers to the weight of one ream of the standard size of the paper. Thus, if one ream of writing paper 17″ by 22″ weighs 24 lb., this is known as 24-lb. paper and any paper cut from it is known as 24-lb. paper, although its actual weight is less or more.

6.22. POULTRY: EGGS. Eggs are graded in city markets. The basis of grading is the weight (in ounces) per dozen eggs. Thus eggs weighing 24 oz. and up are known as *standard* eggs; between 22 and 24 oz., *medium* eggs; between 20 and 22 oz., *pullet* eggs; below 20 oz., *pewee* eggs. The prices per dozen vary according to classification--about three cents per division. Standard eggs are, of course, the highest priced.

ALSO SEE Fundamental operations Ar. 11.08; Mensuration Ar. 16.12; Per cent Ar. 19.02; Proportion Ar. 20.09; Scale drawing Ar. 22.01.

EQUATIONS 7.00

SEE Algebra.

EXPONENTS 8.01

8.01. DAILY LIFE.† Exponents are sometimes used to shorten large numbers having zeros. Thus the estimated age of the earth is 694,000,000,000 days. This number can be written 6.94×10^{11}. Or, the diameter of the orbit of an electron of a hydrogen atom

* Adapted from Edward Reich and Carlton J. Siegler, *Consumer Goods, How to Know and Use Them*, pp. 271–272. American Book Company, New York, 1937.

† Adapted from H. R. Cooley, D. Gans, M. Kline, H. E. Wahlert, *Introduction to Mathematics*, pp. 103–105. Houghton Mifflin Company, Boston, 1937.

is supposed to be .000,000,000,53 millimeters. This can be written 5.3×10^{-10}.

Problem: Write these magnitudes without the use of exponents:

1. The diameter of the average red corpuscle is 8×10^{-5} cm.
2. The length of time that a motion-picture image is on the screen is approximately 6.4×10^{-2} sec.

ALSO SEE Fundamental operations Ar. 11.33.

FORMULAS 9.00

SEE Algebra.

FRACTIONS 10.01–10.21

10.01. AVIATION.* Aviators have occasion to read both maps and charts. (Charts represent sections of the earth's surface which show more water than land.) Since the unit of measurement on maps is the mile and the unit of measurement on ocean charts is the nautical mile, aviators must learn to use the following rules:

1. To change statute miles (5280 ft.) to nautical miles, multiply by $\frac{7}{8}$.
2. To change nautical miles to statute miles, multiply by $\frac{8}{7}$.
3. To change miles per hour to knots (one knot equals a speed of one nautical mile per hour) per hour, multiply by $\frac{7}{8}$.
4. To change knots per hour to miles per hour, multiply by $\frac{8}{7}$.

10.02. AVIATION.† The size of identification letters and numbers on airships and balloons is controlled by law and governed by the size of the circumference of the gas bag. The height of the letters must be at least $\frac{1}{12}$ the largest circumference of the bag but need not exceed 8 ft. The width of the letters and figures must be $\frac{2}{3}$ the height, and the width of the stroke must be at least $\frac{1}{6}$ the height. A space equal to at least $\frac{1}{2}$ the width of a letter must be left between each letter or figure.

Problem: Determine for an airship whose gas bag's maximum circumference is 78 ft., the height of identification letters of minimum size, their width of stroke, and the space between the letters.

* Adapted from *Commercial Aeronautics—26*, p. 5. American Technical Society, Chicago.
† Adapted from *Commercial Aeronautics—50*, p. 16. American Technical Society, Chicago.

10.03. Botany.* The arrangement of leaves on plants is described by means of "divergencies." These "divergencies" are stated in fraction form. For example, a twig having opposite leaves has a divergency of $\frac{1}{2}$ since from any given leaf we need to go halfway round the stem to meet the next leaf. Twigs featuring alternate leaf arrangement may have divergencies of $\frac{1}{2}$, as in the beech and the sycamore, or $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{8}$, $\frac{5}{13}$, $\frac{8}{21}$, . . . Not only are the divergencies expressible in fraction form but there exists a definite relationship between numerators and denominators, for it is seen that the numerators and denominators of every fraction after the second are the sums of the numerators and denominators of the two preceding fractions.

10.04. Daily Life. After Oct. 24, 1940 the Wage Hour Law requires that employers and employees coming under the law take account of "straight time" and "overtime." "Straight time" means the first 40 hours of work in the work week. Time above 40 hours is called "overtime." The law provides that for overtime the employee be paid "time and a half." Example: What should a man whose regular hourly rate of pay is 50 cents receive for working 50 hours? Solution: For the first 40 hours he should receive $.50 \times 40 = \$20.00$. For the $50 - 40$ or 10 hours of overtime he should receive $1\frac{1}{2} \times .50 \times 10$ or \$7.50. Therefore his total weekly wages should be $\$20.00 + 7.50 = \$27.50$.

Problem: What should be the weekly pay for a man working 54 hours at a regular hourly pay of 62 cents?

10.05. Daily Life. Engineers have defined horsepower as the ability to do 33,000 foot-pounds of work per minute. The horsepower required to do a certain job is found by dividing the foot-pounds done per minute by 33,000.

Problem: If a boy weighing 125 lb. climbs 4 flights of stairs or 48 feet in 2 minutes, what is the horsepower equivalent? Solution:

$$\frac{\cancel{125} \times \overset{24}{\cancel{48}}}{\cancel{2} \times \underset{\underset{11}{\cancel{264}}}{\cancel{33{,}000}}} = \frac{1}{11}\text{ h.p.} \quad \textit{Answer.}$$

* See National Council of Teachers of Mathematics, *Eleventh Yearbook*, pp. 221–222. Bureau of Publications, Teachers College, Columbia University, New York, 1936.

10.06. Dentist.* A dentist must know how many grains of a drug are necessary to make one pint of a 1 to 2000 solution. Here is one method of finding the answer to this problem:

The weight of a pint of water is 7291.2 grains. One to 2000 indicates 1 gr. in 2000 gr. of water, or $\frac{1}{2000}$ gr. in 1 gr. of water. Therefore:

$$\frac{7291}{1} \times \frac{1}{2000} = 3.6456 \text{ gr. in 1 pt. of water.}$$

10.07. Engineering. If drills are manufactured in $\frac{1}{64}''$ size, what drill should be purchased to drill holes for pins 0.112″ in diameter? For pins 0.840″ in diameter?

10.08. Engineering.† The safe working pressure for a boiler is commonly taken as $\frac{1}{5}$ of the bursting pressure.

If P_b = bursting pressure in lb. per sq. in.
T = tensile strength of the boiler plate in lb. per sq. in.
t = thickness of the boiler plate in inches
R = radius of boiler in inches
C = a constant (.56 for single-riveted boilers)

$$P_b = \frac{T \times t \times C}{R}.$$

10.09. Fruit Grower. Mr. Smith was informed by the County Farm Agent that the correct amount of fertilizer for young fruit trees was $\frac{1}{4}$ lb. for each year of the tree's age.

Problem: Mr. Smith owns 12 pear trees which are 6 years old. How much fertilizer should he purchase?

10.10. Housewife. Prices per ounce or per pound often vary considerably, depending upon the amount purchased. Examples:

2 lb. flour	12¢
5 lb. flour	23¢
15 oz. can cherries	13¢
1 lb. 14 oz. can cherries	23¢
$\frac{1}{5}$ lb. can cocoa	5¢
$\frac{1}{2}$ lb. can cocoa	8¢

* Adapted from R. M. Goepp, *Dental State Board Questions and Answers*, p. 264. W. B. Saunders Company, Philadelphia, 1924.

† Adapted from H. W. Marsh, *Industrial Mathematics*, pp. 39–40. John Wiley and Sons, New York, 1912.

If five dollars' worth of each of these items were purchased, what difference would it make, as to quantity, if it were purchased in large containers instead of small? Is there any reason why the small containers should ever be purchased?

Problem: If, over a period of three months, a family uses 30 lb. of flour, 30 lb. of cherries, and 2.5 lb. of cocoa, what per cent of the cost of the small cans would be saved if the purchases were all made in the large containers? Would using small cans cause more or less waste?

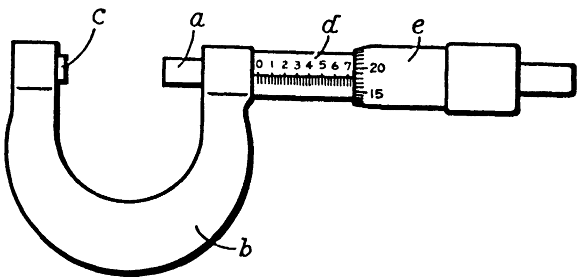
10.11. Housewife. The cook book accompanying an electric range gives the following as a guide to the housewife in preparing meats:

Kind of Meat	*Time Required per Lb.*
Beef	
Rare	15 min.
Medium	20 min.
Well done	25 min.
Pork	30 min.
Turkey	22 min.

Problem: How long will it require to cook (well done) an $8\frac{3}{4}$-lb. beef roast?; a $16\frac{1}{2}$-lb. turkey?; a $5\frac{1}{2}$-lb. pork roast?

10.12. Lumbering. A board rule is a graduated extension device used to determine, by matching the width with a mark on the rule, the number of board feet in a board, or in a number of boards lying side by side on a pile of a given length. Usually board rules are graduated to use with 12-ft., 14-ft., and 16-ft. lengths. To understand completely the use of the rule as marked or to construct a rule to use with boards of other lengths, the lumberman needs to figure as follows, using fractions and the definition of a board foot of lumber: Since a board $1' \times 1' \times 1''$ contains one board foot, a board 18′ long and 1″ wide would be considered $18' \times \frac{1}{12}' \times 1''$ or $1\frac{1}{2}$ bd. ft. Therefore the graduation on a board rule, to be used with 18-foot boards, for 1 bd. ft. would be $\frac{2}{3}$ of an inch from the zero mark. For 2 bd. ft. graduation would be $2 \times \frac{2}{3}$ or $\frac{4}{3}$ of an inch from zero mark; for 3 bd. ft., 2 inches from zero mark, etc.

10.13. Mechanics.



MICROMETER CALIPER

"The principle of the micrometer depends on the fact that $\frac{1}{25} \times \frac{1}{40} = \frac{1}{1000}$. The micrometer, as shown above, is made up of the frame or yoke *b*, the anvil *c*, the screw or spindle *a*, the barrel *d*, and the thimble *e*. The spindle *a* is threaded inside of *d*. The thimble *e* is attached to the end of spindle *a*. The piece to be measured is inserted between *c* and *a*, and the caliper closed on it by screwing *a* against it. The screw on *a* has 40 threads to the inch, so if it is open one turn, it is open $\frac{1}{40}$ in., or $\frac{25}{1000}$, or .025. Along the barrel *d* are marks to indicate the number of turns or the number of fortieths inch that the caliper is open. Four of these divisions ($\frac{4}{40}$) represent one-tenth of an inch, so the tenths of an inch are marked by marking every fourth division on the barrel. Around the thimble *e* are 25 equal divisions to indicate parts of a turn. One of these divisions on *e* will, therefore, indicate $\frac{1}{25}$ of a turn, and the distance represented will be $\frac{1}{25} \times \frac{1}{40} = \frac{1}{1000}$ in. Let us read the micrometer shown above. First, we find the figure 7 exposed on the barrel, indicating more than $\frac{7}{10}$ in. This we put down as a decimal. In addition, there is one of the smaller divisions uncovered. This is .025 in. more. And on the thimble we find it is 3 divisions beyond the 15 mark toward the 20 mark. This would be 18, and indicates .018 in. more. Adding the three, $.7 + .025 + .018 = .743$ in. *Answer.*"*

10.14. Mechanics. A machine shop manager wishes to check on the consumption of tool bits for an engine lathe. He uses the

* M. E. Norris and K. G. Smith, *Shop Arithmetic*, pp. 50–51. McGraw-Hill Book Company, New York, 1913.

established facts that for this particular bit a repointing is necessary, on the average, every day and that each repointing necessitates grinding away, on the average, $\frac{3}{32}$ in.

Problem: If a new bit measures $2\frac{3}{4}$ in. in length and if a bit of this kind must be at least $1\frac{1}{2}$ in. long, how many days can it be used if no accident occurs?

10.15. MECHANICS. Bits, used in drilling machines, are numbered in terms of sixteenths of an inch. Thus, a half-inch bit is numbered "8" and a three-quarter-inch bit, "12." In this way the shop worker finds an application of the rule, "You may multiply both the numerator and the denominator of a fraction without changing its value."

10.16. MEDICINE.* The effect of a drug is modified by many conditions. The age of the patient is one. Two rules are in common use for determining the proper dose for children of various ages:

Young's rule is as follows: Add 12 to the age and divide by the age, and the quotient will be the denominator of a fraction the numerator of which is 1. Thus, for a child of four years, $\frac{4 + 12}{4} = 4$, and the dose is $\frac{1}{4}$ of the adult dose.

Cowling's rule is to divide the age of the child at its next birthday by 24. Thus, for a child 3 years old, the dose would be $\frac{4}{24} = \frac{1}{6}$ of the adult dose.

Problem: What would be the dose for a ten-year-old child according to each rule?

10.17. PATTERN MAKING. Many workmen in foundries have occasion to use shrink rules. These rules are usually 24 inches long and are made of steel or boxwood. The graduations on the rules are longer than those of the ordinary foot rule since the shrinkage allowance is included in all measurements. The patterns must be oversize so that after the molten metal cools and shrinks the hardened casting will be the desired size. Example: Shrinkages per foot for various metals are: $\frac{1}{16}$, $\frac{1}{12}$, $\frac{1}{10}$, $\frac{3}{32}$ of an inch. A $\frac{1}{16}$ shrink

* Adapted from A. A Stevens, *A Textbook of Therapeutics*, pp. 33–34. W. B. Saunders Company, Philadelphia, 1924.

rule which is marked 24 inches is actually $24 \times \frac{192}{191}$ inches long.

Problem: If a $\frac{1}{12}$ shrink rule is marked 24 inches, actually how long is it?

10.18. Poultry. A farmer who raises turkeys wished to mix his own "starter" feed. The government bulletin gives the following recipe:

120 lb. yellow corn meal
65 lb. standard wheat bran
60 lb. wheat middlings
50 lb. finely ground oats
60 lb. dried milk
25 lb. alfalfa meal
55 lb. meat scrap
55 lb. fish meal
5 lb. salt
5 lb. cod-liver oil

Problem: The above recipe gives a total mixture of 500 lb. Mr. Jones has a flock of twenty-five turkeys and thinks a mixture of 200 lb. will be sufficient for his needs. What fractional part of the ingredients must he use to make the 200-lb. mixture? How many pounds of each ingredient will he need to make the desired mixture?

10.19. Real Estate.* The cost of steam, hot water, and vapor heating systems is generally computed from the number of square feet of radiation required. One of the common rules for computing the number of square feet of radiation for a low pressure steam system is the "200–20–2" rule, which is

$$R = \frac{V}{200} + \frac{W}{20} + \frac{P}{2},$$

in which R = square feet of radiation required; V = volume of room in cubic feet; W = exterior wall area minus door and window openings in square feet; P = exterior door and window openings in square feet.

* Adapted from W. L. Prouty, W. W Collins, and F H. Prouty, *Appraisers and Assessors Manual*, pp 248–249 McGraw-Hill Book Company, New York, 1930.

Example: Determine the number of square feet of radiation required for a room 16′ × 12′ × 9′, which has one door 3′ × 7′, two windows 2.5′ × 4.5′, and two exterior walls.

$$R = \frac{16 \times 12 \times 9}{200} + \frac{(16 \times 9) + (12 \times 9) - (3 \times 7) - (2 \times 2.5 \times 4.5)}{20} +$$

$$\frac{(3 \times 7) + (2 \times 2.5 \times 4.5)}{2} = 8.64 + 10.43 + 21.75 = 40.81 \text{ sq. ft.}$$

10.20. Student.* "The quarter-day by which each year exceeds an even 365 days has caused a great deal of trouble. No calendar has been made that has an exact number of days in it. There just are not 365 or 366 full days in a year, and there is nothing we can do about it short of altering the earth's motion.

"Therefore our calendars have to be conventionalized along with our clocks. Ordinarily, we have 365 days in a year, but that makes each year about five hours, 48 minutes, 46 seconds too short. This discrepancy has been known, too, for many centuries. The Julian calendar, adopted by a decree of Julius Caesar in 45 B.C., put in an extra day every four years to make up the difference, which was then supposed to be an even quarter day. Therefore, the Julian year is too long by $\frac{8}{1000}$ of a day—11½ minutes. In the course of 1,000 years the calendar was out by about eight days.

"By 1582, the error was about ten days and a correction was applied by Pope Gregory. Thus in the Gregorian calendar a leap year is omitted every century in the last year of the century, the one ending in "00" (not in "99"), but every 400 years this is put back as a leap year. Therefore 1600 and 2000 are leap years, but 1700, 1800, and 1900 are not."

Problems: (a) Had the Julian calendar continued, how far wrong would we now be? Will any further corrections be necessary?

(b) The year consists of 365 days, 5 hours, 48 minutes, and 46 seconds. How can this year be made perpetual, every year being the same, each divided into 12 months and having equal quarters and half years, and yet begin the year always on a Sunday and each quarter on a Sunday?

* W. H. Barton, Jr., "Sky Clocks and Calendars." *Journal of Calendar Reform,* p. 170. 4th quarter, 1940. Permission for reproduction also secured from *The Sky,* Hayden Planetarium, 81st Street and Central Park West, New York.

10.21. TREE SURGERY. Trees need to be fed adequately to thrive and present a healthy appearance. The most satisfactory tree food will contain 10% available nitrogen, 8% available phosphorus, and 6% available potassium. A tree should be fed once a year. The serving should consist of three-fourths pound of food, of the above analysis, per inch of trunk circumference. To feed the tree drill holes 2 inches in diameter, 18 inches deep, and 2½ feet apart in *concentric circles* around the tree. The outer circle should start with the outer limits of the branch spread and the next one should be 2½ feet nearer the trunk of the tree.

Problem: A tree surgeon prepares his own tree food by mixing the necessary amounts of nitrate of soda, acid phosphate, muriate of potash, and sawdust as a filler. The nitrate of soda contains 15% available nitrogen and costs $3.00 per hundred pounds. The phosphorus is secured from acid phosphate, which features 16% available phosphorus and costs $1.50 per hundred pounds. The sawdust costs 25 cents per hundred pounds.

If the tree surgeon charges a profit of 25% on the fertilizer materials to cover storage, left-overs, etc., how much should he charge the owner of an estate for the materials to feed his trees if their circumferences total 650 inches.

ALSO SEE Angle Ar. 1.02; Denominate numbers Ar. 6.17, 6.20; Fundamental operations Ar. 11.13; Metric units Ar. 17.01; Numbers Ar. 18.06; Proportion Ar. 20.02; Ratio Ar. 21.06, 21.07.

FUNDAMENTAL OPERATIONS 11.01–11.37

11.01. AGRICULTURE. Find the cost of filling a silo, using the following facts: Size of silo—diameter, 16 ft.; height, 35 ft. One ton of ensilage equals 75 cu. ft. Labor necessary, 7 men at $2.50 per day; 3 trucks and drivers at $10.60 per day each. Cost of fodder, $1.00 per acre. Amount of fodder, 5 tons per acre. Cost of binding fodder, $1.00 per acre. Cost of twine, $7.50 per 100 pounds; amount necessary, 200 pounds per five acres. Cost of cutting up fodder and blowing into silo, 40¢ per ton.

11.02. AGRICULTURE. Find the cost of plowing an acre of ground, using the following facts: Gasoline, 8.4¢ per gallon. Oil, 50¢

per gallon. Grease, 10¢ per pound. Cost of tractor, $1,091.00; cost of plow, $135.00. Life of tractor, 5000 acres; life of plow, 2500 acres. Labor, $2.00 per day. Plowing speed, 12½ acres per day. Gas consumption, 28 gallons per day. Oil consumption, ½ gallon per day. Grease consumption, 2½ pounds per day.

11.03. Agriculture.* The United States Soil Conservation Service made a study of soil erosion. It found that over a certain period of time a certain type of land having a certain slope will lose the following amounts of soil when planted as follows:

Crop	*Topsoil Loss per Acre through Erosion*
Grass	almost none
Wheat	10 tons
Corn	40 tons
Fallow (plowed but not planted)	60 tons

When the lost topsoil was analyzed to learn how much plant food was wasted, it was found that each ton of topsoil washed away contained plant food that would cost one dollar if replaced by commercial fertilizer.

Problem: A certain university experimenting with soil erosion found that a piece of forested land lost 115 lb. of soil in one year. A piece of uncovered or fallow land of the same area lost 112,316 lb. If the plant food contained in the 115 lb. of topsoil was valued at 5¢ per pound, how much plant food was lost when the 112,316 lb. of topsoil were washed away?

11.04. Agriculture. In order to make up weekly pay rolls correctly a cane farmer in Louisiana must be able to figure out the total number of hours each employee has worked and then find total wages earned. Although, in general, the Louisiana cane farmers have no fixed wages or hours for any other work, they use the U. S. Scale of Wages for Cane Farmers and work only nine hours a day.

* Adapted from H. B. Bruner and C. B. Smith, *Social Studies, Book Three,* pp. 366–369. Charles E. Merrill Company, New York, 1938.

The farmer pays his employees thus:

1. For cane cultivation, a man is paid $1.20 per hour.
2. For cane cutting (harvest), a man is paid $1.50 per hour.
3. For cane cutting and loading, a man is paid $1.60 per hour.
4. For cane loading, a man is paid $1.80 per hour.
5. For truck driving, a man is paid $1.75 per hour.

The scale for paying by the ton is:

1. To cut and load small barrel (small cane), a man is paid 95¢.
2. To cut and load large barrel (large cane), a man is paid 80¢.
3. To load only, a man is paid 45¢ per barrel.

11.05. Automobiles. Gasoline costs 20¢ per gallon. A car makes 20 mi. per gallon at 40 mi. per hour and 16 mi. per gallon at 60 mi. per hour.

Mr. Smith leaves home at 7:00 a.m. and begins work at once on arriving at his job 50 miles away. What hourly wage would make the above two rates of travel give equal returns in wages and saving of gasoline?

11.06. Automobiles. "If you are driving 50 miles per hour at night and you are blinded for three seconds by the lights of an approaching car, how many feet will your automobile travel while you are blind?"*

11.07. Automobiles.† A man had his car fitted with four tires guaranteed for 12,000 miles at a cost of $37.80. He might have paid $48.00 for four with a guarantee of 20,000 miles. Which purchase would have given him the greater value and how much?

11.08. Automobiles.‡ Arithmetic is used constantly by auto research workers to solve problems like the following:

Determine, mathematically, the probable relative durability of pistons in two vehicles having varying stroke, tire size, or axle ratio. Given:

* A. L. O'Toole, "Mathematics in Safety Education." *Safety Education*, May, 1941, p. 412.

† Adapted from V. S. Mallory, *Mathematics for Everyday Affairs*, p 300. Benj. H. Sanborn and Company, New York, 1941

‡ See poster entitled "The Application of Mathematics in the Design of Chevrolet Cars and Trucks," procurable from Chevrolet Motor Car Company.

Vehicle #1 Piston travels 2043 ft. per car mile.

Vehicle #2 $\begin{cases} \text{Stroke, } S = 3\frac{3}{4}'' \\ \text{Axle ratio, } R = 4.222:1 \\ \text{Tire revolutions per car mile, } T = 746 \end{cases}$

Formula: $\frac{2SRT}{12}$ = Piston travel (in ft.) per car mile.

Substituting above given values we find 1968.507 = Piston travel per car mile, Vehicle #2. Dividing 2043 by 1968.507 we get 1.0378.

Conclusion: The piston travel is 3.78% more for vehicle #1 than for vehicle #2 and the piston for #1 will wear out 3.78% more quickly.

11.09. Automobiles. The seriousness of an automobile accident often depends upon the speed of the car at the time of the accident. The relationship between the force of the impact, the weight of the car, and its speed is mathematically expressed by the formula

$$KE = \frac{mv^2}{2},$$

where KE represents kinetic energy, m the mass, and v the velocity.

If the units are in the foot-pound-second system, kinetic energy will be expressed in foot-poundals, convertible to foot-pounds by dividing by 32. Thus, in foot-pounds,

$$\text{Force of crash} = \frac{mv^2}{64}.$$

Illustration: If a car weighing 3200 pounds and traveling 60 m.p.h. strikes a telephone pole, then

$$\text{Force of crash} = \frac{\overset{50}{\cancel{3200}} \times 3600}{\cancel{64}} = 180{,}000 \text{ foot-pounds.}$$

This force would be similar to that produced by a 1000-pound (half ton) steel safe falling from a height of 180 feet.

Problems: (a) How much greater is the force of crash at 60 m.p.h. than at 30 m.p.h.?

(b) What will be the force of the crash of two cars each weighing 3000 lb. if they collide head-on when one is traveling 40 m.p.h.

and the other 50 m.p.h. Hint: Consider one automobile at a standstill and the other moving at 90 (i.e., 40 + 50) m.p.h.

11.10. Aviation. A plane travels from point *A* to point *B* at an air speed of 200 m.p.h. The distance from *A* to *B* is 100 miles. How long will it take the plane to travel the distance if: (a) there is no wind; (b) there is a 20 m.p.h. headwind; (c) there is a 20 m.p.h. tailwind.

11.11. Aviation. In air-line offices a detailed record is kept of the earnings for each leg of the route-trip.

Problem: The United States Government pays an air line about 28¢ per mile to carry mail. Passenger fare is 5¢ per mile. Find the total revenue (mail pay plus passenger fares) from the following trip from City A to City F.

City	A to	City	B	is 90 mi.	Carry	9	passengers
"	B to	"	C	" 210 mi.	"	8	"
"	C to	"	D	" 146 mi.	"	9	"
"	D to	"	E	" 168 mi.	"	6	"
"	E to	"	F	" 180 mi.	"	4	"

11.12. Aviation. Keeping the weight of a plane within licensed gross weight is required by the United States Government. Persons who sell tickets are responsible for not overloading plane.

Problems: (a) The maximum gross weight of a certain plane is 10,500 lb. If empty weight is 6,900 lb., gas capacity is 228 gal. (6 lb. per gal.), oil capacity is 48 qt. (7 lb. per gal.), and crew allowance is 170 lb. each for 2 pilots, what is the pay load (the remaining weight which may be used for mail, passengers, baggage and express)?

(b) If 200 lb. of mail and 50 lb. of express are to be carried, how many passengers can be carried, allowing 200 lb. for 1 passenger and his baggage?

(c) If ten passengers at 200 lb. each are to be carried, how much gas will have to be drained in order not to exceed the licensed gross weight?

11.13. Aviation.* The size of the Venturi tube in the carburetor

* Adapted from *Commercial Aeronautics—30*, p. 33. American Technical Society, Chicago.

of airplane motors depends upon the volume of a cylinder (displacement) and the revolutions per minute at which the motor is to be operated. The size of the tube is equivalent to its area and is found by using the following formula which holds for cases where one carburetor feeds three or fewer cylinders:

$$\text{Venturi area} = \frac{\text{displacement of one cylinder} \times \text{r.p.m.}}{133{,}000} + \text{discharge nozzle area.}$$

If one carburetor feeds four or more cylinders,

$$\text{Venturi area} = \frac{\text{displacement of all cylinders} \times \text{r.p.m.}}{480{,}000} + \text{discharge nozzle area.}$$

11.14. Daily Life.* High wattage electric light bulbs may be better buys than low wattage bulbs. One 100-watt bulb gives as much light as two of 60 watts or six of 25 watts. The original cost of the 100-watter is approximately 15¢, while the two 60-watters will cost about 26¢, and the six 25-watters will cost 60¢. That is one saving.

To operate the 100-watter for 1000 hours in a community where electricity costs 4¢ per kilowatt hour would cost $4.00. The cost of operating the two 60-watters would be $5.00 and of the six 25-watters, $6.00.

Problem: What would each type cost in your community?

11.15. Daily Life. One brand of liver vitamin capsules sells for $2.50 per package of 100. Each capsule contains 10,000 units of vitamin A, 250 units of vitamin B, 1000 units of vitamin D, and 100 units of vitamin G. A second brand sells in a package of 30 for 50¢. In the case of this brand each capsule contains 6500 units of A, 200 units of B, 400 units of D, and 100 units of G. The latter brand claims to have all the elements contained in the more expensive brands, but at the same time is much cheaper.

A comparison of the two costs shows a ratio of 2 to 3. The more expensive brand costs 5¢ for two capsules, the other 5¢ for three capsules.

* See U. S. Department of Agriculture, *Consumers' Guide*, November 15, 1940.

A comparison of the actual vitamin content and the relative costs of these contents is shown in the following table:

I. Cost: 100 for \$2.50 2 for 5¢	II. Cost: 30 for 50¢ 3 for 5¢

Vitamin units per 5¢ of cost:

I. A.	20,000	II. A.	19,500
B.	500	B.	600
D.	2,000	D.	1,200
G.	200	G.	300

Summarizing: The two costs rank very close together. A dose of two capsules of I or three capsules of II would cost the same. This dosage in I would be slightly higher in A and D content; in II the ratio of B and G is somewhat greater. If high content of A and D is desirable, then I is the better choice. Or, if high dosage of B and G has been prescribed, then II is preferable.

11.16. Daily Life. The small sized package of a certain salve weighs 4 oz., and costs 43¢; the large sized package weighs 20 oz., and costs \$1.19. At the price charged for the small package, the large size would cost \$2.15. There is therefore a possible saving of 96¢ in purchasing the larger size. However, a number of factors may prevent one from really saving as much as 96¢ by buying the larger size. Can you name several such factors?

11.17. Daily Life. A newspaper reported that during a single 20-inch snowfall about 2,000,000 tons of snow fell on the city of Hull, England. This statement can easily be checked by finding the area of Hull (9042 acres) and remembering the following arithmetic facts:

1 acre equals 4840 sq. yd.
1 cu. ft. of water weighs about $62\frac{1}{2}$ lb.
10 in. of snow equals 1 in. of water.

Problem: A 16-inch snowfall yields how many tons of water to the acre?

11.18. Daily Life. *Consumer Tips* (U. S. Department of Agriculture) suggests that buyers of cod-liver oil compare costs and vitamin potency of various brands of oil before purchasing. Since doses are usually in terms of teaspoonfuls, it is suggested that comparisons be made on this basis. The following arithmetical techniques are suggested:

Cost per teaspoonful: Write down cost of bottle and number of ounces (1 pint equals 16 ounces).

To figure cost per teaspoonful: First, divide cost of bottle by number of ounces, to get cost per ounce. Second, divide cost per ounce by 7, to get cost per teaspoonful (1 ounce equals about 7 teaspoonfuls).

Example: Brand X: Cost, 57¢ per bottle; bottle contents, 1 pint; cost per tsp.: $57 \div 16 = 3.6$¢; 3.6¢ $\div 7 = 0.5$¢, or $\frac{1}{2}$¢.

Brand Y: Cost, 37¢ per bottle; bottle contents, 10 ounces; cost per tsp.: $37 \div 10 = 3.7$¢; 3.7¢ $\div 7 = 0.5$¢, or $\frac{1}{2}$¢.

Vitamin D potency per teaspoonful: Write down U.S.P. (United States Pharmacopoeia) units per gram (see label).

To figure potency: Multiply U.S.P. units per gram by 4 (1 teaspoonful equals about 4 grams).

Example: Brand X: U.S.P. units vitamin D per gram, 250; potency per tsp.: $250 \times 4 = 1000$ U.S.P. units.

Brand Y: U.S.P. units vitamin D per gram, 95; potency per tsp.: $95 \times 4 = 380$ U.S.P. units.

Comparison: Cost per teaspoonful of both Brand X and Brand Y is about the same, but Brand X is well over twice as potent in vitamin D as Brand Y. Many authorities recommend from 400 to 800 U.S.P. units of D daily for adolescents and younger children. One teaspoonful of Brand Y would not give quite the minimum dose of 400 units, while one teaspoonful of Brand X would give well over 800 units.

11.19. Daily Life. Arithmetical calculations serve to express distances and volumes in ways that make possible vivid comparisons. This is especially true of star-distances. Example: "It takes seven years for light to travel across the top of the Big Dipper, at the rate of 186,300 miles per second. The distance is over 41,000,000,000,000 miles. An express train traveling at 60 m.p.h. would require 78,120,000 years to make the trip. Allowing 40 years for an engineer's functioning ability and providing for an 8-hour shift, the trip would take 5,859,000 different engineers. At the rate of three cents a mile, the fare would be $1,231,796,160,000. If a person should have that amount of money, this would be a useless way to part with it, for, relatively speaking, his threescore and ten years would be over before the train had pulled out of the station. An observer at the end of the handle of the dipper, with a telescope powerful enough so that he could see the opposite side of the dipper, would be able to witness the start of the trip nearly 17 years after it took place.

"If this huge dipper were of the size and shape that the cross section indicates, it would hold approximately 1,220,541,302,774,-293,635,320,000,000,000,000,000,000 cubic miles of water. A giant who would care to use a dipper of that size would be so large that the highest balloon ascension ever made would not clear the epidermis on the bottom of his foot. The distance to the top of his toe would be thousands of times the distance from the earth to the sun. If our earth should fall into his dipper he would swallow it and never realize our presence in his system. The Big Dipper, however, is only a very small group of stars in the universe."*

11.20. Dairying. "For $50 a man can buy a scrub cow that gives 14 pounds of milk per day, or for $120 a Jersey cow that gives 25 pounds of milk per day. If the first cow's milk is 2 per cent butterfat and the Jersey's $4\frac{1}{2}$ per cent butterfat, if one pound of butterfat will make about $1\frac{1}{8}$ pounds of butter, and the price of butter is $.70 a pound, which cow will yield the greater per cent on the investment the first year?"†

11.21. Dietitian.‡ The Oppenheimer formula for the coefficient of nutrition is 100 times the arm girth (taken midway between elbow and shoulder) divided by the chest girth (average between girth expanded and girth deflated). For normal nutrition this coefficient should be at least 30.

11.22. Economics. Filling the yearly income tax blank intelligently offers rich opportunities to apply arithmetic. It illustrates

* Adapted from O. Caldwell, C. Skinner, and J. Tietz, *Biological Foundations of Education*, pp. 3–4. Ginn and Company, Boston, 1931.

† E. H. Barker, *Applied Mathematics*, p. 203. Allyn and Bacon, New York, 1920.

‡ Adapted from O. Caldwell, C. Skinner, and J. Tietz, *op. cit.*, p. 340.

the fact that the job of handling real life problems often requires careful reading, precise records, rather extended use of the fundamental operations and per cent, and care in checking the answer.

Problem: Secure several income tax blanks, state the case of a hypothetical earner, and calculate the amount of tax he would have to pay. Check for correctness.

11.23. Engineer. Railroad locomotive engineers must do a great deal of calculation to keep on time, as is suggested by the following: A railroad locomotive engineer finds that he is fifteen minutes behind scheduled time because an area on his route was struck by a heavy snowfall. From the point he is at present, outside the snow area, to another point 30 miles away the track is relatively straight and he can increase his speed to almost any limit without danger. He therefore decides to make up ten minutes on this 30-mile run. If his normal speed is 60 m.p.h., what must his increased speed average to make up the ten minutes? Following this 30-mile stretch of good road curves will be met and the maximum speed he may travel with safety is 70 m.p.h. How many miles must he travel to reach an on-time position, assuming he uses the maximum speed?

11.24. Engineering.* In computing the cost of electricity to users in a community, it is necessary to figure the distance the electricity is transmitted from the main line to the house. When the average length of wire per customer (i.e., total feet of distribution divided by number of customers) differs from 500 feet, the cost for wires and poles will be greater or less than $55 per customer by the sum of change in the number of feet times six cents. For example, if the average length of conductor (including overhead and ground conductors) is 600 feet, the investment per average residence customer will be increased by 100 × $.06 = $6.00, and the total investment will be $55.00 + $6.00 = $61.00 per customer. If the average length of conductors is only 300 feet, the investment will be reduced by 200 × $.06 = $12.00. Thus, if the average feet per customer is 300, the investment for an average residence customer is reduced from $55.00 to $43.00.

Problem: Figure investment costs at 250 and 350 feet.

* Adapted from M L. Cooke, *What Electricity Costs*, p. 99. New Republic, Inc., New York, 1933.

11.25. HOME OWNER. One seed company recommends that a pound of grass seed be used for every 1000 sq. ft. of surface area in seeding a new lawn. What is the cost of seeding a lot 70 ft. by 100 ft. if the house on it is 27 ft. by 37 ft. and the seed costs 30¢ a pound?

11.26. HOUSEWIFE. Every electric light bulb bears a stamp showing how many watts it consumes in one hour. If the housewife remembers that "kilo" means thousands, she can easily understand the rule, "To find cost of operating a lamp (or any other electric appliance) multiply watts × hours in operation × cost per K.W.H. in cents and point off five." Thus, to operate a 60-watt bulb for 10 hours with current costing 6¢ per K.W.H., we find the cost as follows: $60 \times 10 \times 6 = \$0.036$.

The cost of operating an electric clock that is stamped 112V—60 cycle—2W for 24 hours with current costing 6¢ per K.W.H. is $2 \times 24 \times 6 = \$0.00288$.

11.27. HOUSEWIFE. "If a kitchen poorly arranged requires the housekeeper to take 100 more steps, 20 inches to a step, each day in preparation of the meals, than she would in a well-arranged kitchen, how many miles of unnecessary travel does this amount to in the course of a year?"*

11.28. HOUSEWIFE. To be able to check the readings of electric and gas meters and to check the computation that yields the month's bill is an extremely valuable ability to possess. This ability is easily acquired if it is remembered that the indicators on the four clock-like dials on the meter indicate the digits that should be placed in the thousands, hundreds, tens, and units positions of a four-place number. Thus, a reading might be 8604. The monthly amount of current or gas used is, of course, found by subtracting the reading at the beginning of the month from the reading at the end of the month. Computation of the bill is based on an understanding of a Monthly Rate Statement, of which this is typical:

75¢ which includes 7 K.W.H.
5.0¢ per K.W.H. for the next 60 K.W.H.
3.0¢ per K.W.H. for the next 83 K.W.H.
2.0¢ per K.W.H. for the next 100 K.W.H.
1.5¢ per K.W.H. for all additional K.W.H.

* E. H. Barker, *Applied Mathematics for Junior High Schools*, p. 202. Allyn and Bacon, New York, 1920.

To find the cost of 345 K.W.H. we calculate as follows:

For first	7 K.W.H.	price is		.75
For next	60 K.W.H.	price is	@ 5.0¢	3.00
" "	83 K.W.H.	" "	@ 3.0¢	2.49
" "	100 K.W.H.	" "	@ 2.0¢	2.00
" "	95 K.W.H.	" "	@ 1.5¢	1.42
	Total for 345 K.W.H. is			$9.66

Problem: Find the cost of 175 K.W.H.; 260 K.W.H.

11.29. MECHANICS. If a relatively small cogwheel drives a relatively large cogwheel, the small driver wheel is called a "pinion" and the larger wheel is simply referred to as a "gear." To determine the relationships between the revolutions made by the gear and the pinion long division is used. For example, if the number of teeth on the flywheel (gear) of an automobile is 120 and the number of teeth on the starter gear wheel (pinion) is 8, how many revolutions does the starter motor make to turn the engine once?

Problem: If, for quick starting, the engine should be turned at the rate of 120 r.p.m., how many r.p.m. must the starter motor make?

11.30. MECHANICS. The horsepower a belt will transmit is given by

$$H = \frac{P \times W \times V}{33{,}000},$$

where H = horsepower; P = effective pull allowed per inch of width (this is usually 35 lb. in a single belt per inch of width); W = width of belt in inches; V = velocity of belt in feet per minute.

Problem: Find the width of a single leather belt to transmit 12 h.p. at a speed of 2400 ft. per minute. Solution: Substituting given values in formula above,

$$12 = \frac{35 \times W \times 2400}{33{,}000}.$$

Canceling,

$$12 = \frac{28W}{11}$$

$$W = 4\tfrac{5}{7} \text{ in. Use 5 in. belt.}$$

11.31. MECHANICS.* Safe cutting speeds for a certain tool bit are:

Material	*Speed in R.P.M.*	*Feed in Inches per Minute*
Cast iron	300	0.125
Steel casting	140	00.033
SAE 3140 (300 Brinell)	100	00.025
Cold rolled	250	0.033
Copper	300	0.030
Stainless steel	250	0.015

How long would it take to drill through a two-inch section of each of the above materials?

11.32. PAINTING.† A painter usually mixes his materials for each job. If he uses 100 lb. of white lead, he will add 5 gal. of raw linseed oil, 1 gal. of pure turpentine, 1 pt. of drying japan. This will bulk to make about 9½ gal. of paint ready for use.

Problem: If a gallon covers 750 sq. ft. of surface, how much of each material must a painter mix to do a paint job covering 11,625 sq. ft.?

11.33. PHYSICS. Newton's Law of Gravitation between two bodies is given as

$$\text{Attraction} = \frac{GM_1M_2}{d^2},$$

in which G is the gravitational constant .000000000033, written $3.3/10^{11}$, M_1, mass of one body, M_2, mass of other body, and d, distance between the bodies' centers of gravity.

Problem: What is the attraction between two ships, one weighing 20,000 tons and the other 40,000 tons, when the distance between their centers of gravity is 100 ft.?

$$\text{Attraction in pounds} = \frac{3.3 \times (20{,}000 \times 2{,}000) \times (40{,}000 \times 2000)}{10^{11} \times (100)^2}$$

$$= \frac{3.3 \times 2^2 \times 10^7 \times 2^3 \times 10^7}{10^{11} \times 10^4}$$

$$= 3.3 \times 2^5 \times 10^{-1} = \text{about } 10.56 \text{ lb. weight}$$

* See "Tool Bit Test Results." *Business Week*, p. 36, August 12, 1939.

† Adapted from A. S. Kelly, *The Expert Paint Mixer*, pp. 121–122. David McKay Company, Philadelphia, 1920.

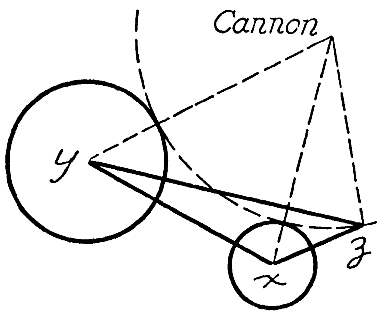
11.34. Radio. It has been stated that when a radio fan tunes to his favorite band 600 miles away, he hears the music one half second before the spectators listening just 600 feet from the bandstand. This can be checked arithmetically if one knows that sound travels through air at the rate of 1120 ft. per second and that radio waves travel at the rate of 186,000 mi. per second.

Problem: If the radio station were strong enough, how many times would the radio waves encircle the world during the time the band's music is reaching the spectators 600 feet away?

11.35. Science. Since sound travels at a definite speed in a given medium many distances can be determined by calculation. For example, if a ship is approaching a cliff its distance from the cliff can be determined by noting how many seconds intervene between the first sound of the whistle and the first sound of the whistle's echo.

Problem: If sound travels at the rate of 1120 ft. per second, how close is a ship to the cliff if $9\frac{1}{2}$ seconds intervene between the blowing of the whistle and the hearing of the echo?

Although the speed of sound in water is 4700 ft. per second, much faster than it is in air, the same principle as above is used to determine the depth of the ocean; or to guide lost ships, provided the depth of the ocean is known at various points.



Sound recordings are also used to locate enemy cannons. At several stations records are kept of the instants at which the cannon's explosion is heard. Let us call these stations x, y, and z. Further, let us suppose station y hears the explosion 10 seconds later than z, and x 5 seconds later than z. In 10 seconds sound travels in air 11,200 ft.; in 5 seconds, 5600 ft. If we describe a circle with a radius of 11,200 units about y and another of a radius of 5600 units about x, the location of the cannon can be determined by finding the center of the circle that passes through z and is tangent to the circles about y and x.*

11.36. SCIENCE.† When a light ray passes from air to glass the light ray is bent. The amount of bending for a light ray passing from air to another transparent substance is described by the refractive index, symbolized by R. Tables furnish the refractive indices of transparent substances with reference to air, but sometimes we wish to know the refractive index R, when the ray of light passes from one transparent substance other than air to another, such as, for example, from water to flint glass. If the data found in the tables is supplemented by *division*, the desired data can be found. The rule for finding this relationship is expressed as follows:

$$R(B \text{ to } C) = R(A \text{ to } C) \div R(A \text{ to } B),$$

where R represents refractive index, A represents air, B and C, two other substances.

Problem: What is the refractive index for a ray of light passing from water to flint glass? Solution: Here we will let water be represented by B in the formula and we will let flint glass be represented by C in the formula. From tables: $R(A \text{ to } C) = 1.65$ and $R(A \text{ to } B) = 1.33$; thus:

$$R(B \text{ to } C) = \frac{1.65}{1.33} = 1.24,$$

the refractive index of a ray of light passing from water to flint glass.

11.37. STUDENT. Silver Springs at Ocala, Florida, probably has the largest flow of any spring in the world, namely, 22,134,780 gallons per hour. Water system engineers figure that the average person uses about thirty gallons of water per day.

* For a discussion of cannon problem, see L. Hogben, *Science for the Citizen*, pp. 309-311. Alfred A. Knopf, New York, 1938.

† Adapted from L. Hogben, *Science for the Citizen*, p. 137. Alfred A. Knopf, New York 1938.

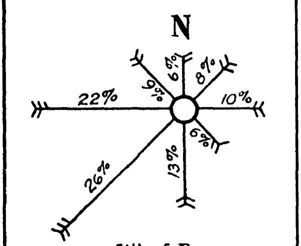
Problems: (a) From these data and population data of towns in your own state, find a town or city whose water needs could be cared for by the volume of water Silver Springs furnishes.

(b) How large in diameter would a flume have to be to carry away all the water issued by Silver Springs if the flow through the flume is at the same rate as the water issues from spring?

Also see Angle Ar. 1.02; Compound numbers Ar. 3.02; Decimals Ar. 4.10; Denominate numbers Ar. 6.03; Fractions Ar. 10.04, 10.06; Mensuration Ar. 16.17; Proportion Ar. 20.13.

GRAPHS 12.01–12.04

12.01. Aviation.*



Wind Rose

The Wind Rose, appearing on sketches in the airway bulletins of the U. S. Department of Commerce, is a graphic device used to indicate relative wind frequencies and average velocities at each landing field.

The lines radiating from the center show wind directions toward the center, the length indicating the relative frequency of the wind from the various directions.

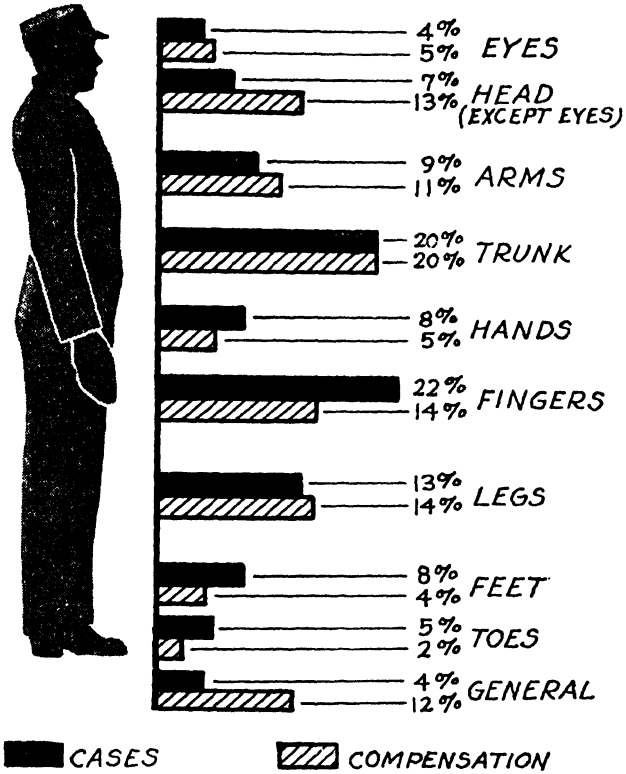
The number of barbs on the lines represents a number in the Beaufort Scale (see "Numbers Ar. 18.03") for ascertaining the average velocity in any particular direction.

In the wind rose shown in the figure, the highest prevailing wind is from the southwest. The wind blows from this direction 26% of the time. The five barbs on the line indicate a velocity of from 19 to 24 miles per hour. The lightest winds are from the southeast and from the north, their velocities being 8 to 12 miles per hour. These winds prevail 6% of the time.

12.02. Daily Life. The type of injuries that constitute the industrial injuries of the United States is a matter of concern to labor

* Adapted from *Commercial Aeronautics*—22, pp. 8–10. American Technical Society, Chicago.

departments and industrial commissions. *Accident Facts,** 1941 edition, p. 21, gives this information by using *graphs* and *per cents*, as indicated in the following figure.†

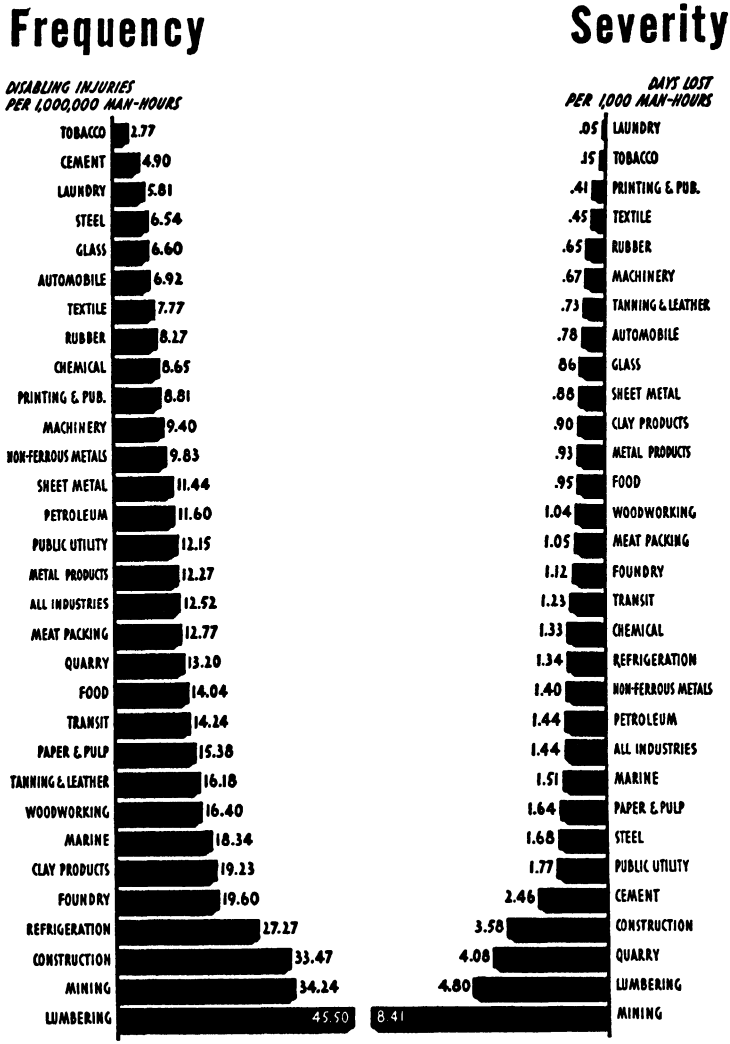


12.03. Daily Life. Life insurance companies and social security and compensation committees are vitally interested in learning about the frequencies and severity of injuries sustained by industrial workers. This information for 1940 is given by *Accident Facts,** 1941 edition, p. 15, by means of *graphs*, *ratio*, and *decimal numbers* (see figure on following page).

* *Accident Facts* is published yearly by National Safety Council, Inc., 20 N. Wacker Drive, Chicago.

† Adapted from *Accident Facts*, 1941, p. 21. The pictorial original is in two colors. Source: Reports from State Labor Departments or Industrial Commissions.

1940 Injury Rates By Industry



Note: Adapted from *Accident Facts*, 1941, p. 15. The pictorial original is in two colors.
Source: Reports of industrial establishments to the National Safety Council.

12.04. Economics. In the University (of Chicago) High School course in Economic Society a unit is offered on "The Exchange of Goods." This unit "calls for diagrams explaining the law of diminishing utility and the law of supply and demand, based upon schedules of original data. For example, the student is asked to make line graphs of supply and demand schedules on the basis of the following data:

Supply Schedule

At 5¢ a doz., sellers offer 2 doz.
At 10¢ a doz., sellers offer 5 doz.
At 15¢ a doz., sellers offer 10 doz.
At 20¢ a doz., sellers offer 15 doz.
At 25¢ a doz., sellers offer 20 doz.
At 30¢ a doz., sellers offer 30 doz.

Demand Schedule

At 30¢, buyers take 3 doz.
At 25¢, buyers take 5 doz.
At 20¢, buyers take 10 doz.
At 15¢, buyers take 15 doz.
At 10¢, buyers take 25 doz.
At 5¢, buyers take 35 doz.

"After the graphs are completed, the student uses them in answering the following questions:

1. Assuming free competitive conditions
   (a) Give the market price ....
   (b) Give the quantity exchange ....
2. Are production costs increasing, decreasing, or constant? ....
3. Is demand elastic or inelastic? ....
4. Indicate
   (a) Producer's surplus ....
   (b) Consumer's surplus ....
5. What price would a monopolist set? ...."*

ALSO SEE Decimals Ar. 4.07; Numbers Ar. 18.09, 18.10.

INSURANCE 13.01

13.01. DAILY LIFE. On August 30, 1940, the Associated Press in London announced that a Briton's chance of getting killed or injured by a Nazi airplane bomb was 1,000 to 1 according to statistics compiled by Lloyds, insurance brokers. These brokers of-

* *Monograph No. 8, Mathematics Instruction in the University High School,* p. 155. University of Chicago Press, Chicago, 1940.

fered insurance policies to civilians for which the premium rate would be one shilling (about 20 cents) per month for every 50 pound sterling (approximately $200) of insurance.

Problem: At these rates, what would be the approximate cost of $5000 worth of insurance for one year?

INTEREST 14.01–14.09

14.01. Daily Life. The principles for computing simple interest can be used to evaluate the desirability of installment buying. Consider the following problem and its solution:

Mr. S. learned that he could purchase an electric motor for $24.75 cash, or $4.75 down and $8.00 a month for three months. What rate of interest would he pay using the installment plan? Solution: For the first month Mr. S. paid interest on $20.00; the second month, $12.00; the third month, $4.00.

Thus, by using the installment plan, Mr. S. would pay interest on $36.00 for one month. The difference between $28.75 and $24.75 represents the cost to Mr. S. for borrowing $36.00 for one month.

Since $r = \frac{i}{pt}$ we have by substituting,

$$r = \frac{\overset{4}{\cancel{4.00}} \times \cancel{12}}{\underset{\underset{3}{\cancel{3.00}}}{\cancel{36.00}}} = 1.33\tfrac{1}{3}, \text{ or } 133\tfrac{1}{3}\%. \quad \textit{Answer.}$$

Problems: (a) A thrift and loan association seeks loans on the following basis: "The loan is repayable in fifty-six weekly installments. For example, a loan of $100.00 is repaid $2.09 a week for 56 weeks; a loan of $300.00 is repaid $6.25 a week for 56 weeks. At the time of the last payment, for every weekly installment paid on time, a refund is made at the rate of 9 cents per $100.00 loan per week."

If installments were promptly met by Mr. Jones who borrowed $100.00 from the firm, how much did he pay as interest? What rate of interest would this be?

(b) An automobile finance corporation lends money as follows:

If \$100.00 is forwarded, a note for \$106.00 is issued to cover the loan. The note is repaid by depositing monthly \$8.83. Is this a high rate or a low rate of interest to pay?

(c) At one time a personal finance company lent money as follows: \$150.00 cash loan could be repaid by making eighteen monthly payments of \$9.57 each. Was this a "good" or a "bad" proposition?

14.02. Daily Life. Some installment buying problems feature a carrying charge. The actual yearly rate of interest paid on money actually lent in these cases can be determined by means of the following formula:

$$r = \frac{12f}{Pn},$$

where r = rate of interest in per cent; f = financing charge; P = cash price less the down payment; n = number of payments.

Example: A washing machine sells for \$45 cash, or \$49.50, \$4.50 down and \$5 per month for nine months.

$$f = \$4.50;\ P = \$40.50;\ n = 9.$$

$$r = \frac{12 \times \$4.50}{\$40.50 \times 9} = 15\%.$$

14.03. Daily Life. Which is the better way to save money: buying building and loan stock or starting a savings account in a local bank at 2% interest compounded semi-annually? As one basis for this judgment we compare Mr. Jones's experience with that of Mr. Smith. Mr. Jones purchased 1 double share of Atglen Building and Loan stock. He paid \$2.00 each month for seven years and then received \$207.09. Mr. Smith deposited \$24.00 to his savings account every year for seven years. The bank paid interest at the rate of 2% compounded semi-annually. Note: As suggested above, the results of this calculation may serve only as a partial answer to the question in mind. Questions such as relative safety and ease of withdrawing money if unforeseen events (such as accidents) make withdrawal advisable must also be considered.

14.04. Daily Life. You can buy U. S. Defense Savings Bonds for \$18.75, \$37.50, \$75.00, and \$375.00 which in ten years will be worth $25.00, $50.00, $100.00, and $500.00. You can also buy Defense Postal Savings Stamps for 10¢, 25¢, 50¢, $1.00, and $5.00. When a sufficient number have been purchased you can exchange them for U. S. Defense Savings Bonds.

Problem: If the interest is calculated as being compounded annually, what rate of interest is paid on the bonds?

14.05. Daily Life.* Long ago the Dutch bought Manhattan Island from the Indians for $24.00. A banking expert figured out that $24.00 invested at compounded interest since the date of purchase would have a present worth of a little more than $4,000,000,000, whereas the Island was recently valued at only $3,800,000,000. The purchasers, or their unlucky heirs, are about $200,000,000 in the hole. It is felt the Dutch paid a dollar or fifty cents too much for the Island and that the Indians should return that amount.

14.06. Daily Life. "Compound interest has been called Aladdin's magic lamp of money. Someone has figured that if $1 had been placed on interest at 3%, compounded annually, when Jesus was born, the amount in 1906 would have been approximately $3,000,000,000,000,000,000,000,000. At simple interest, the amount would have been only $58.18."†

14.07. Daily Life. The principles of compound interest can be used to arouse a desire to stop smoking and bank the money.

Problem: If a man who smokes a package of cigarettes daily at approximately 20¢ a package would save this amount and deposit it on January 1 of each year in a bank that pays 2% interest compounded annually, how much would his savings be at the end of 20 years? Solution:

$.20 × 365 = $73.00 yearly amount to be deposited.
($1 deposited annually for 20 years at 2% compounded annually will grow to $24.783.)
73 × $24.783 = $1,809.16, savings at the end of 20 years.

14.08. Home Owner. Mr. Blair purchased a home for $6000. He had available $2400 to pay on the home, leaving $3600 to be

* See *The Mathematics Teacher*, Vol. 33, No. 2, p. 59, February, 1940.

† G. M. Wilson, M. B. Stone, and C. O. Dalrymple, *Teaching the New Arithmetic*, p. 272. McGraw-Hill Book Company, New York, 1939.

borrowed. Mr. Blair had two alternatives: He could borrow the $3600 from a local bank and pay 6% interest, with the provision of paying $300 per year to reduce the principal. Or, upon inquiry, he learned he could get $3600 from the local Building and Loan Association and pay it back at the rate of $36.00 per month for 142 months. Which plan would be the better for Mr. Blair?

14.09. Housewife. Electric Home and Farm Authority (E.H.F.A.) made loans to persons who wished to buy gas and electric appliances. The payments were made monthly when the bills for gas and electric current were paid. Following are sample contracts:

	Electric Refrigerator	*Radio*	*Refrigerator and Gas Range*
Purchase price	$149.50	$59.95	$294.45
Down payment	29.50	14.95	49.45
Unpaid balance	120.00	45.00	?
Time charge	17.88	2.76	36.88
Amount of contract	137.88	47.76	?
Cash per month	$3.83 for 36 mo.	? for 12 mo.	$7.83 for 36 mo.

Problem: Find value of ?'s. Are the interest rates charged too high?

MEASUREMENT 15.01–15.02

15.01. Calculator. "The accuracy achieved by medieval astronomers who worked like Tycho Brahe without telescopic instruments was due to the large scale of their instruments. Thus Tycho Brahe used a quadrant 19 feet across. A circle of radius 19 feet has a circumference (360 degrees) of 19 × 2 × 3.14 feet, i.e., roughly 120 feet, or 4 inches per degree. Since it is quite easy to distinguish half a millimeter (one-fiftieth of an inch), an accuracy of from one one-hundredth to one two-hundredth of a degree could be obtained if the scale was accurately calibrated."*

15.02. Daily Life. "Mathematics is a way of thinking that has been developed by the race in answer to the fundamental need for

* Adapted from L. Hogben, *Science for the Citizen*, p. 225. Alfred A. Knopf, New York, 1938.

a way to make reflective thinking precise and accurate," says O'Toole. "Mathematical thinking profits greatly by employing symbols and by manipulating them according to mathematical laws. This technique allows man's mind to march far from its physical camp."

The foregoing description of the power of mathematics is nicely illustrated by a somewhat common riddle as follows:

If a steel tape which fits snugly around the earth at the equator were lengthened 6.00 feet, how far from the equator would the lengthened tape rest if it were everywhere equidistant from the equator and remained in the equator's plane?

Difficulties are encountered if one tries to solve this problem by using numbers only. First, the earth's circumference should be known precisely enough so that a change of 6.00 feet can be recognized. Second, the calculations are long and cumbersome. However, if a bit of mathematical symbolism is employed the solution is quite simple, short, and as precise as can be desired, as follows:

Let d = diameter of earth

Then πd = circumference of earth

$\pi d + 6.00$ = length of lengthened steel tape

$\dfrac{\pi d + 6.00}{\pi}$ = diameter of circle formed by lengthened tape

$$\frac{1}{2}\left(\frac{\pi d + 6.00}{\pi} - d\right) = \frac{\pi d + 6.00 - \pi d}{2\pi} = \frac{6.00}{2\pi}, \text{ or}$$

0.95 ft. = distance lengthened tape is from equator

Problem: Solve the same problem by using a water tower 30.00 feet in diameter instead of the earth. What conclusion would we have if we used a tub 3.00 feet in diameter instead of the water tower?

MENSURATION 16.01–16.32

16.01. Agriculture. An approximate measure of grain hauled loose in open wagon boxes of standard size can easily be determined if it is known that in a box of standard size one inch of grain in depth makes two bushels. As grain is sold by weight and not by measure, and as grain varies greatly in weight, this calculation is not exact. However, it is much used in estimating approximate number of bushels when moving or storing grain, if exact number is not important.

Problem: In hauling his feed from the granary to the feeding bin a farmer found that he had two loads and part of a load six inches deep in the wagon box which was twenty-eight inches deep. How many bushels did he have?

16.02. AGRICULTURE. "After a heavy summer rain the water of a small stream was found to contain 1 pound of sediment in every 500 gallons of water. If the rainfall was one inch, the area of the basin drained 4 square miles, and the amount of water that ran off $\frac{1}{4}$ of that which fell on the ground, how much soil did the rain carry away?"*

16.03. AGRICULTURE. The number of acres in a field on the prevailing size of quarter section in mid-United States is often determined by finding the width of a field in rods when its length is the length of the quarter section, or one-half mile. Each rod in width makes one acre of land. When this field is planted to corn of the standard width of row made by American corn planters, every nine rows make two rods. By counting the width of the field in rows the acreage may be determined.

A field nine rows wide and half mile long has two acres. Fields less than half mile long will be of proportional size. A field 80 rods long and nine rows wide contains one acre.

16.04. AGRICULTURE. One way of increasing the traction of light-weight tractors is to fill the tires with water to which calcium chloride has been added to prevent freezing. The inside diameter of the tire fitting a 1940 Ford tractor is approximately 7 inches and the circumference of the tire (through the center of the tube) is approximately 125 inches.

Problem: If the weight of one pint of the water-calcium-chloride solution is one pound, how much weight can be added by filling the tires half full with the solution? Notes: One pint of water occupies a space of approximately 28.85 cu. in. Regard the tire as

* E. H. Barker, *Applied Mathematics*, p. 204. Allyn and Bacon, New York, 1920.

equivalent to a cylinder 125 inches high with a diameter of 7 inches.

16.05. Agriculture. John, age 12, signed up with the County Agent as a 4-H Club member. He decided to grow tomatoes for his project. The plot of ground which his father gave him measured 240 ft. by 180 ft. The County Agent said the tomato plants should be planted in squares so that there would be $1\frac{1}{2}$ feet between any two of them.

Problem: How many tomato plants should John buy in order to have no "left overs"?

Mary, age 14, chose a rose garden for her project. The County Agent suggested a circular flower bed 14 feet in diameter with the rose bushes planted in concentric circles two feet apart. On the circles the rose bushes should also be spaced two feet apart.

Problem: If Mary allows one foot around the edge of the bed for mulching, how many rose bushes should she purchase to plant the bed?

16.06. Agriculture.* A silo, properly filled, contains one ton of silage for every 50 cu. ft. of space. A ton of hay stored in a barn requires 400 cu. ft. of space. Some common dimensions of silos are:

Diameter (in feet)	*Depth (in feet)*	*Diameter (in feet)*	*Depth (in feet)*
10	20	14	30
12	20	16	24
12	24	16	26
12	28	16	30
14	22	18	30
14	24	18	36
14	28		

16.07. Agriculture. A farmer has a pasture 180 rods wide and 200 rods long that he wishes to fence with barbed wire. The fence is to be five wires high. The local warehouse sells barbed wire on spools containing 80 rods.

Problem: How many spools of wire should the farmer buy in order to have a minimum of wire left over?

* See *Farmer's Pocket Ledger.* John Deere Co., Moline, Ill., 1938.

16.08. AVIATION.* Since Lindbergh's epochal flight to Paris, the air-cooled radial type of airplane engine has been so satisfactory that its use is almost universal. The drives of war have given rise to the V-type liquid-cooled Allison engine. One feature of superiority of the latter is a reduction in air resistance. This is due to the fact that the Allison engine has a frontal area of $6\frac{1}{4}$ sq. ft. as against an area of over 16 sq. ft. in the radial engine of the same power. This difference in frontal area, which is a big factor in producing "drag," becomes gigantic when it is remembered that "drag" increases as the square of the speed.

16.09. AVIATION. Find the speed of the tip of a 8′ 6″ propeller running 1500 r.p.m.? 2400 r.p.m.? 36 r.p.m.?

16.10. CARPENTRY. "A 4-in. shingle laid 5 inches to the weather covers 4 by 5 or 20 sq. in. An area of roof of 100 sq. ft. is called a square. The number of shingles to cover a square is $100 \times \frac{144}{20} = 720$. Hence, to find the exact number of shingles required for a roof, find the number of squares and multiply by 720."†

16.11. DAILY LIFE. The Pittsburgh Equitable Meter Company gives the following statistics on waste of water:

Leak through Opening of Given Diameter at 60 lb. Pressure	*Waste per Quarter in Gallons*	*Waste per Quarter in Cu. Ft.*
$\frac{1}{4}$ in.	1,200,000	160,000
$\frac{3}{16}$ "	675,000	90,000
$\frac{1}{8}$ "	300,000	40,000
$\frac{1}{16}$ "	75,000	10,000
$\frac{1}{32}$ "	19,000	2,500

Problem: A leak $\frac{1}{4}$ inch in diameter wastes four times as much water as a leak $\frac{1}{8}$ inch in diameter. Make other comparisons. Since the volume here depends on the area of the circular opening you can fill in the following blanks: If the diameter is doubled the area is .... times as large. If the diameter is tripled the area is .... times as large.

* See "1,090 h.p. and up—100 a Week." *Fortune*, Vol. XXII, No. 1, pp. 50–51, July, 1940.
† Reprinted by permission from *Mathematics for Technical and Vocational Schools*, by Samuel Slade and Louis Margolis, published by John Wiley and Sons, Inc., 1922.

16.12. Daily Life. Village X was installing a water system. After the cast-iron water pipes were joined but before the ditch was filled in the water line was tested for possible leaks. In order that even a small leak would be noticeable or any weakness in the line shown, the test standard was set at 100 lb. per square inch.

Problem: How tall would an open standpipe need to be to give rise to a water pressure equal to the test pressure? Note: Find weight of 1 cu. ft. of water and find how tall a column of water is needed to produce 200 lb. pressure on 1 sq. ft. of base area. Since base area has no effect upon pressure produced, this height will be the necessary height for standpipe.

16.13. Dentistry.* A question sometimes asked of dentists is the following: Express the equivalent in the metric system of the following: 1, 5, 10, 15.5 grains, 1 ounce, a teacupful, a wineglassful, a tablespoonful, a teaspoonful, and a minim.

1 grain	=	0.065 gm.
5 grains	=	0.324 gm.
10 grains	=	0.650 gm.
15.5 grains	=	1.000 gm.
1 ounce	=	31.000 gm.
a teacupful	=	120 c.c.
a wineglassful	=	60 c.c.
a tablespoonful	=	15 c.c.
a teaspoonful	=	3.75 c.c.
a minim	=	0.061 c.c.

16.14. Engineering. Frequently pipes fastened to roof joists are used to hold up ceilings over garages, auditoriums, etc. The weight a certain pipe will sustain safely depends upon its tensile strength, its own weight, and its cross-section area.

Problem: What weight will a cylindrical pipe made of steel hold up if its outside diameter is 6.00 in. and its inside diameter, 5.00 in.? (a) If the weight of the pipe is neglected? (b) If the pipe is 10.00 ft. long and its weight is considered? (c) What length pipe would be required to just hold up its own weight?

Take the tensile strength of the steel to be 70,000 lb. per square inch. Take steel as weighing 0.29 lb. per cubic inch.

* See R. M. Goepp, *Dental Board Questions and Answers*, p. 19. W. B. Saunders Company, Philadelphia, 1924.

16.15. ENGINEERING. Frequently it is necessary to determine the number of feet of belting when the belting (or belt) is rolled in a coil. This is accomplished by using the formula:

$$\text{Length of belt in coil (in feet)} = .131\ (D + d)N,$$

where D equals outside diameter of coil in inches, d the small inside diameter in inches, and N the number of laps in entire coil.

To check the validity of this formula, the engineer needs to reason that the circumference of the *average* lap in the coil, in feet, is

$$\frac{3.1416\ (D + d)}{24} = .131\ (D + d).$$

16.16. ENGINEERING. Find the total bearing area of a motor that has five bearings. The two end bearings are 3.00 in. long. The three center bearings are 3.50 in. long. The diameter of the crankshaft is 2.00 in.

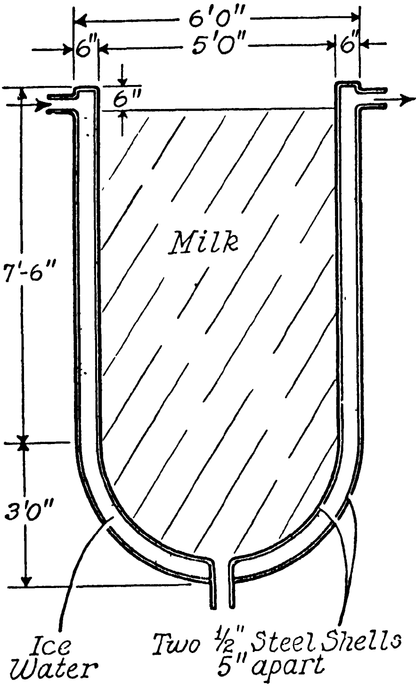
16.17. ENGINEERING. If a locomotive is running at a speed of 50 m.p.h., what is the surface speed of the driving wheels in feet per second, assuming that the wheels do not slip on the rails?

16.18. ENGINEERING. An engineer was hired to draw floor plans for a milk station. It was proposed that the floor carry several cylindrical water-jacketed vats with hemispherical bases, as shown in the figure (page 58). One calculation required was to find the total weight of the tank shown in the figure. This includes weight of the tank itself, the ice water in the jacket to within 6 in. of top of milk, and the milk. For this purpose it was considered that

Ice water weighs	56.0 lb. per cu. ft.
Milk weighs	64.3 lb. per cu. ft.
$\frac{1}{2}$ in. steel plate weighs	20.4 lb. per sq. ft.

For the purpose for which the information is to be used, it can be assumed that all piping welded to or through outside and inside shells of the vat would weigh the equivalent of the $\frac{1}{2}$ in. thick plate it replaces.

Problems: (a) How many gallons of milk will the vat hold if it is filled to within 12 inches of the top? (b) If the ice water jacket is changing every five minutes, how many gallons of water will be used in one hour?



16.19. Engineering. In order to have the proper bearing area a steel pin that is 4.00 in. long must be 0.75 in. in diameter. To reduce the weight, a $\frac{3}{8}$ in. hole is drilled lengthwise through the entire length. How much will the pin weigh if 1 cu. in. of the steel used weighs 0.29 lb.?

16.20. Foundry. Since liquids (melted iron) always seek their own level and since pressure in a liquid is exerted equally in all directions, the flasks containing the mold must be securely clamped or weighted. In order to clamp or weight the flask properly the foundryman must calculate the downward, side, and upward pressures. This is done by calculating the volume of the casting in cubic inches and multiplying by 0.26 lb., the weight of cast iron per cubic inch. The pressure of the sand used to form the mold must also be calculated by finding its volume in cubic inches and multiplying by 0.06 lb., the weight of a cubic inch of sand. The volumes calculated are in many varied forms having bases that are rectangular, circular, triangular, elliptical, etc.

In addition to determining volumes for purposes of meeting pressures, the foundryman must know the volume of the casting to be cast so that he can decide how much metal he needs to melt to make the casting.

Problem: A foundryman wishes to cast a rectangular solid of cast iron $3\frac{1}{2}' \times 18'' \times 6''$. He proposes to use a flask whose inside dimensions are $4' \times 24'' \times 12''$. Since the flask is in two parts, each six inches deep, what pressure must the clamps be able to withstand so that the melted iron will not be forced out between the two parts of the flask?

Note: A 3-in. layer of sand will surround casting. Only half of the volume of casting will create pressure where the flask is joined. Cast iron weighs 0.26 lb. per cubic inch; sand, 0.06 lb. per cubic inch. The difference between the weight of half the casting and the weight of the sand in the upper half of the flask yields the desired answer.*

16.21. Home Owner. A firm advertises that a gallon of outside paint will cover 400 sq. ft. of surface if the buildings are in reasonably good condition. Usually a 5% loss must be allowed for evaporation and waste.

Problem: Choose a building in your locality, estimate its painted area, find the cost per gallon of medium priced paint, and estimate the cost of painting the building.

16.22. Home Owner. In some Pennsylvania communities anthracite coal can be purchased from various distributors who deliver their goods by auto truck. Whether or not the truck load of coal contains the weight indicated on the waybill is very often a significant question in the purchaser's mind. A convenient and valid answer to this question can be found by determining the volume in cubic feet of either the bin or the truck and dividing by 35, the number of cubic feet occupied by 1 ton of anthracite coal.

* See W. C. Stimpson, B. L. Gray, and J. Grennan, *Foundry Work*, pp. 88–91. American Technical Society, Chicago, 1940.

16.23. Mechanics.* The following rule is often used to find the heating surface of any number of tubes in a steam boiler: Multiply the number of tubes by the diameter of one tube in inches, then this product by its length in feet, and then by 0.2618. The final product is the number of square feet of heating surface.

The heating surface of 66 3-in. tubes each 18 ft. long would be:

$$66 \times 3 \times 18 \times 0.2618 = 933 \text{ sq. ft.}$$

The factor, 0.2618, is determined by the fact that the surface area of a pipe in square feet is equivalent to that of a rectangle of the same length and with width equal to the circumference of the pipe in feet.

$$\pi d \text{ in inches} = \frac{\pi d}{12} \text{ in feet.}$$

Thus, in feet the circumference of the pipe in question is

$$\frac{\pi}{12} \times 3 \quad \text{or} \quad 0.2618 \times 3.$$

16.24. Mechanics.† A certain "rule of thumb" for finding the volume in gallons of a cylindrical tank is:

$$\text{Volume in gals.} = (\text{diameter in ft.})^2 \times 5\tfrac{7}{8} \times \text{height in ft.}$$

The validity of this rule is determined by comparing it with the arithmetic rule $V = B \times h$ and remembering the arithmetic fact that 231 cu. in. = 1 gal. Thus the above rule must be equivalent to

$$V = \frac{\pi \times 144 \times (\text{dia. in ft.})^2 \times 12 \times \text{ht. in ft.}}{4 \times 231}$$

To the extent that $\dfrac{\pi \times 144 \times 12}{4 \times 231} = 5\tfrac{7}{8}$, the rule is correct. By arithmetic calculations it is found that the rule gives a result 0.003+% too small.

* Adapted from C. I. Palmer, *Practical Mathematics, Part II, Geometry*, p. 135. McGraw-Hill Book Company, New York, 1912. Publishers announce a revised edition of this work in 1941.

† Adapted from C. I. Palmer, *Practical Mathematics, Part II, Geometry*, p 133. McGraw-Hill Book Company, New York, 1912.

16.25. MECHANICS.* Because the body of a bolt is greater in diameter than the threaded part, the two parts will not stretch uniformly when the bolt is under strain. For this reason the bolt is most likely to break where the threaded part joins the other part. To prevent this, a hole is sometimes drilled from the center of the head to the beginning of the threaded part. This hole is made of such size that the cross-sectional area of the body is the same as that at the root of the thread.

Problem: Find the diameter of the hole to be drilled in the following bolt in accordance with the preceding statement. Diameter of bolt is $\frac{3}{4}$ in., with 10 U.S.S. threads to 1 in. (Depth of various threads are stated in tables. Depth of U.S.S. thread with 10 threads to the inch is .06495.)

16.26. MECHANICS. "Piston displacement" is a term commonly used in connection with gasoline engines. It refers to the volume in cubic inches of all the cylinders of an engine. To find the piston displacement of a four-cylinder engine whose bore (cylinder diameter) is 4 in. and whose stroke (effective length of cylinder) is 6 in. we multiply $2^2 \times 3.1416 \times 6 \times 4$, and obtain 301.6 cu. in., approximately.

Problem: What is the piston displacement of a six-cylinder engine whose bore is $4\frac{1}{2}$ in. and whose stroke is $5\frac{1}{2}$ in.?

16.27. MECHANICS. Holes are punched in thick sheets of metal or in heavy plates by means of great pressure exerted by a hydraulic press. This pressure, in pounds, is usually about 60,000 times the area (expressed in square inches) of the cylindrical surface cut out, or, in other words, about 60,000 lb. per square inch of the cylindrical area.

Example: How much pressure will it be necessary for a hydraulic press to exert on a sheet of boiler plate $\frac{7}{16}$ in. thick, in order to punch holes $\frac{1}{2}$ in. in diameter? Solution: Pressure is equal to $60{,}000 \times \pi \times \frac{7}{16}$.

Problem: How much pressure would be required if the thickness of the plate and the diameter of the hole, in the above example, were doubled?

* Adapted from C. I. Palmer, *Practical Mathematics, Part II, Geometry*, p 137. McGraw-Hill Book Company, New York, 1912.

16.28. Mechanics. In building modern large heating plants the cylindrical boilers are lined with asbestos sheeting to prevent excessive radiation.

Problem: How much sheet asbestos will be needed to line a boiler 10 ft. in diameter and 12 ft. in length?

16.29. Mechanics.* Cutting speeds on lathe and boring mill work can be calculated. The life of a lathe tool depends upon the rate at which it cuts the metal.

Problems: (a) An iron casting is 30 in. in diameter. Find the r.p.m. necessary for a cutting speed of 40 r.p.m.

$$\text{Circumference (in ft.)} = \frac{3.14 \times 30}{12} = 7.8 \text{ ft.}$$

$$\text{r.p.m.} = \frac{40}{7.8} = 5.1. \quad \textit{Answer.}$$

(b) How many r.p.m. should a 50-in. Huron stone be run if it is to be used for rough grinding? (For rough grinding Huron stones should be run at a surface speed of between 3,000 and 3,400 feet per minute.)

16.30. Medicine.† In disinfecting rooms the amount of drug used depends upon the air space of the apartment. As a disinfectant in the control of malaria, yellow fever, and other mosquito-borne diseases 4 lb. of sulphur should be burned for every 100 cu. ft. of air space. Another disinfectant formula demanding calculation to meet the needs of any specific situation is: 20 oz. of liquid formaldehyde and 8½ oz. of permanganate per 1000 cu. ft. of room space requiring disinfection.

16.31. Photography. Camera shutter adjustments are marked *f*4, *f*6.3, *f*8, etc. These *f* markings are known as the focal ratios. They are the ratios between the focal distance of the camera and the

* Adapted from E. B. Norris and K. G. Smith, *Shop Arithmetic*, pp. 78–81. McGraw-Hill Book Company, New York, 1913.

† See A. A. Stevens, *A Textbook of Therapeutics*, pp. 434, 437. W. B. Saunders Company, Philadelphia, 1924.

areas of diaphragm openings. Thus, if the focal distance of a candid camera is 2 in. and the diameter of the diaphragm opening (effective lens opening) is $\frac{1}{8}$ in., the f rating is $\frac{2}{\pi/16} = 32/\pi$, or 10.2. If the photographer wishes twice as much light for his next exposure, he must set the diaphragm stop at the f $\sqrt{2} \times (10.2)$, or f 14.2 reading. If he wishes only half as much light for his next exposure he must set the stop at f $\sqrt{\frac{1}{2}} \times (10.2)$, or f 7.2.

Problem: If the focal distance of a candid camera is 2.5 in. and if the diaphragm is opened .75 in., what f reading is produced? Suppose for the next exposure the user of the camera wishes three times as much light; to what f opening should he set his camera? If he wishes half as much light for the second exposure, what f opening should be used?

16.32. STUDENT.* Two children, Tom and Mary, were riding a merry-go-round; Tom was sitting on a horse 16 ft. from the center while Mary was riding one on the outer edge which was 20 ft. from the center. How much farther and faster did Mary go than Tom?

METRIC UNITS 17.01

17.01. MEDICINE. "The air capacity of the sacs of the lungs is about 3500 c.c., of which only 500 c.c. is changed at each ordinary breath. Some 1500 c.c. more can be expelled with special effort, while the remaining 1500 c.c. is residual air, which remains stationary."†

Problem: What fractional part of our lung capacity do we ordinarily use? About what per cent of our total lung capacity can we use with special effort?

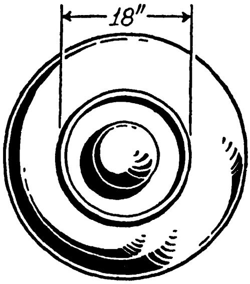
ALSO SEE Fundamental operations Ar. 11.26; Numbers Ar. 18.02.

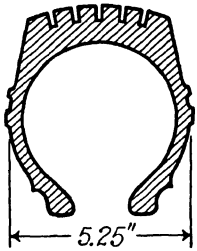
* Adapted from Guy T. Buswell, William A. Brownell, and Lenore John, *Daily Life Arithmetic, Book Three*, p. 255. Ginn and Company, Boston, 1935.

† See O. Caldwell, C. Skinner, and J. Tietz, *Biological Foundations of Education*, p. 351. Ginn and Company, Boston, 1931.

NUMBERS 18.01–18.12

18.01. Automobiles. Tire sizes are described by using numbers; thus: 5.25 × 18. The 18 refers to the diameter from bead to bead. The 5.25 refers to the outside diameter of the casing. Some tire





sizes are given in three numbers, viz., 28 × 5.25 – 18. The 28 refers to the outside over-all diameter, the 5.25 refers to the diameter of the air space, and the 18 refers to the diameter from bead to bead, as before explained.

Problem: How many cubic feet (approximately) of air space does a 6.50 × 18 tire afford?

18.02. Automobiles. At one time the viscosity of crankcase oils was described by using such words as "light," "medium," "heavy," etc. These descriptions of viscosity have been replaced by a more objective and scientific nomenclature which uses viscosity numbers, viz., S.A.E. 10, S.A.E. 20, etc.

The Society of Automobile Engineers has defined the S.A.E. viscosity number 20, for example, as the correct representation of an oil of which 60 mm. ± 0.05 mm. will flow through an orifice 0.1765 cm. in diameter within a time range of 120 to 185 secs. at 130° F.

18.03. Aviation.* Aviators have frequent need for expressing wind force, and for this purpose have established the Beaufort Scale of Wind Force, which assigns a single number to represent a certain range of wind velocity. Thus:

* See *Commercial Aeronautics—15*, p. 21. American Technical Society, Chicago.

Beaufort Number	*Represents Wind Velocity in*	*Miles per Hour*
0		less than 1
1		1–3
2		4–7
3		8–12
4		13–18
5		19–24
6		25–31
7		32–38
8		39–46
9		47–54
10		55–63
11		64–75
12		above 75

18.04. Button Dealers.* Most people do not know button sizes; they buy buttons by matching. Button dealers designate button sizes by numbers. Buttons are measured by a button gauge having 40 parts to an inch. A 40-gauge button is 1 in. across and a 20-gauge button is $\frac{1}{2}$ in. across.

Thread sizes, such as No. 8 or No. 40, are additional applications of numbers to needs of our daily lives.

18.05. Daily Life. "About a hundred years ago, an English monk worked out a plan for shoe sizes. He began with a small shoe, only 3 inches long, for a baby. He called this 'size 1.' Then size 2 was $\frac{1}{3}$ of an inch longer, or $3\frac{1}{3}$ inches long. Thus, sizes went up and up, each higher size being $\frac{1}{3}$ inch longer than the size below."†

The lengths of men's and women's shoes also increase $\frac{1}{3}$ inch for each increase in size.

18.06. Daily Life. Men's hat sizes are expressed by using integers and fractions. The smallest size is $6\frac{3}{4}$. This hat fits a head measuring $21\frac{1}{4}$ inches. The next size, $6\frac{7}{8}$, fits a head measuring $21\frac{5}{8}$

* Adapted from G. M. Wilson, M. B. Stone, and C. O. Dalrymple, *Teaching the New Arithmetic*, p. 250. McGraw-Hill Book Company, New York, 1939.

† G. M. Wilson, M. B. Stone, and C. O. Dalrymple, *Teaching the New Arithmetic*, p. 250. McGraw-Hill Book Company, New York, 1939.

inches. Men's hat sizes increase by $\frac{1}{8}$. The head measurements corresponding to hat sizes increase by $\frac{3}{8}$ of an inch.

18.07. Daily Life. Night latches, Yale locks, automobile locks, and all other locks requiring flat-grooved keys are systematically numbered. If, for example, an automobile owner loses his car's key but knows the make of the lock (viz., GM, Ford, etc.) and the key number, any certified locksmith can manufacture a new key without having ever seen the old key or the likeness thereof. Not only are many different keys of any particular style or make of lock numbered but the groove pattern of each key is expressed by numbers. Thus, if the auto owner remembers his key to be a GM 8129, he gives this information to a certified locksmith. The locksmith consults his "dope" book and there finds the ridge-and-valley pattern of this lock to be, perhaps, 332241. Accordingly a skeleton key, of correct basic design, is placed in a grinding machine and "valleys" ground into it as directed by the number 332241. In this latter number, 1 stands for a shallow cut or "valley," 2 for a deeper cut, 3 for a still deeper cut, and 4 for a very deep cut. Since there are six digits in this number pattern the key's edge will feature six valleys; two deep valleys adjacent to each other, two moderately deep valleys also adjacent to each other, one very deep valley, and finally a shallow valley near the end of the key. The valley represented by a 1, 2, or 3, of course, is a standard depth known to all certified locksmiths.

18.08. Daily Life. Wherever numbers are used order, system, design, and pattern become evident. A city street address like "19th and Chestnut" is generally comprehensible as far as the "19" is concerned but "Chestnut" suggests no helpful order, system, design, or pattern. The layout of the city of Miami, Fla., is easy to comprehend because all north-south roads are called Avenues and are numbered consecutively beginning with the one farthest east along Biscayne Bay. All the east-west roads are called Streets, which are also numbered consecutively beginning with the one farthest south. Using numbers in this way clearly defines an address, such as "14th Ave. and 163rd St."

18.09. Furrier. The durability of furs is indicated by numbers.

All furs are rated relative to the strongest fur, otter, which has a durability rating of 100. Examples of the durability of various furs are:*

Name	*Durability*
American racoon	65
Black bear	94
Persian lamb	65
Beaver	90
American muskrat	46
Silver fox	40
Ermine	25

Problem: Illustrate the comparative durability of the above furs by using a bar graph.

18.10. Leather Dealer.† The strengths of various leathers are expressed numerically. For this purpose kid is arbitrarily rated as 100. Thus:

Leather	*Relative Strength*
Kangaroo	117
Kid	100
Glazed horse	98
Calf	86
Cow	72
Buckskin	32

Problem: Illustrate by means of a bar graph the relative strengths of the above kinds of leather.

18.11. Library. "In the Dewey Decimal classification there are ten main groups. The numbers run from 10 to 999, and each hundred is used for a main topic. For instance, the numbers 500 to 599 stand for classes in science; and 800 to 899, for literature . . . each hundred is divided into ten groups, shown by the second figure of the class number. For instance, in the number 940, 9 means history, and 4 means Europe. The ten main groups, each with ten subdivisions, make one hundred classes. In the full Dec-

* Adapted from Edward Reich and Carlton J. Siegler, *Consumer Goods*, pp. 179–186. American Book Company, New York, 1937.

† Adapted from Edward Reich and Carlton J. Siegler, *Consumer Goods*, p. 207. American Book Company, New York, 1937.

imal classification, each of these hundred classes has ten subdivisions. . . . For instance, under 940, each country has a number. In the number 942, 9 means history, 4 means Europe, and 2 means England.

"In some libraries, the numbers are further subdivided by decimal fractions. For instance, the number 973, meaning United States history, is subdivided to show the periods of history. In the full scheme, there are period subdivisions for most countries. To illustrate:

973.3 American Revolution.
[9 means history; 7, America; 3, United States.]

973.31 Political history; causes, results
973.311 Causes
973.3111 Stamp act, 1765
973.3112 Navigation acts
973.3113 Boston Massacre

(Numbers for other causes follow here)

973.312 Political history, 1775–83
973.313 Declaration of Independence"*

18.12. Mineralogy. A characteristic that should be known about any mineral before much scientific use can be made of it is its hardness, i.e., resistance to surface abrasion. In order to indicate conveniently the hardness of minerals, mineralogists have set up and use the Mohs Scale of Hardness, as follows: 1. Talc (softest), 2. Gypsum, 3. Calcite, 4. Fluorite, 5. Apatite, 6. Orthoclase, 7. Quartz, 8. Topaz, 9. Sapphire, 10. Diamond (hardest). Example: If a mineral can be scratched by orthoclase and will not scratch apatite, its hardness is between 5 and 6. Thus, olivenite has a hardness of 3; galena, a hardness of 2.5; and copper a hardness of 2.5–3.0.

Also see Average Ar. 2.06; Fundamental operations Ar. 11.28.

PER CENT 19.01–19.32

19.01. Automobiles.† Before trucks are purchased by the United States Government for use in the Army they must meet rigid and

* Quoted from Zaidee Brown, *The Library Key*, pp. 2-4. The H. W. Wilson Company, New York, 1939.

† See "Did you Know." *Lancaster New Era*, p. 10, July 7, 1940.

specific performance standards. One of these standards requires that the trucks be able to ascend a 60% grade with a full load. Diagram such a grade. What is grade of steepest hill near school?

19.02. Automobiles. A study made by B. F. Goodrich Co. shows that tire wear increases some 13% as the temperature increases from 40° F. to 60° F. When the temperature soars to 80° F., tire wear is about three times that at 40° F. This accounts for the tremendous difference in tire wear between summer and winter, and also for poor mileage obtained by car owners in the South.

The same study shows that as car speed increases from between 20 and 30 m.p.h. to between 30 and 40 m.p.h. tire wear increases around 25%. Above 40 m.p.h. the rate of wear jumps up even more rapidly until at 50 m.p.h. it is almost double that at 20 m.p.h.

In other words, a tire that would give 15,000 miles of service on a car driven at an average of 30 m.p.h. will give less than 9,000 miles at an average speed of 50 m.p.h.

19.03. Automobiles. A garage gives the following table in connection with a certain kind of anti-freeze, X:

Capacity of Radiator in Quarts	*Quarts of X Required to Protect against Given Temperatures*		
	3	4	5
10	−5°	−27°	−53°
11	0	−18	−40
12	3	−12	−31
13	7	−7	−23
14	9	−3	−17
15	11	1	−12
16	13	3	−8
17	15	6	−4
18	17	8	−1
19	17	9	2
20	18	11	4

Problems: (a) About what per cent of anti-freeze must be used to protect against zero weather? (b) Find out from neighboring garages what per cent of alcohol will protect a car against zero temperatures, and make a comparison between cost of using X and using alcohol.

19.04. Automobiles. The per cent concept allows one to make precise comparisons easily. The preliminary edition of *Accident Facts** for 1941 (p. 1) states that traffic deaths increased from 32,386 in 1939 to 34,400 in 1940. This is an increase of 6%. However, motor vehicle mileage also increased 6%. It is thus seen by comparing percentages that there were relatively no more traffic deaths in 1940 than in 1939.

19.05. Aviation.† The quantity of engine oil in an airplane's oil system needs to be scientifically determined. Thus, for varying sizes of motors installed in planes for trips of varying lengths we find that the standards set for the Curtiss Challenger engines demand a minimum oil capacity of five gallons. If unusually long flights are to be made, the oil tank capacity is determined by taking 8% of the total gasoline capacity.

19.06. Banking. A man from Winnipeg purchased $2.65 worth of gasoline at a filling station in North Dakota. At that time there was a 20% discount on Canadian money in the United States. The filling station attendant collected $3.18 in Canadian money. Had he computed the bill correctly? Answer: No. He should have collected $3.31. The $2.65 represented only $\frac{4}{5}$ or 80% of the correct amount. Hence, the attendant should have collected the $2.65 plus $\frac{1}{4}$ of $2.65, or $3.31 in all.

19.07. Banking. Knowledge of the principles of interest as figured on money borrowed from banks and finance companies is essential in making a wise choice in such matters.

Problem: Billy's father wishes to borrow $50.00 from the Bank of Commerce and Trust Co. He finds that the bank charges 8% interest on all money lent. When he gives them the note for the money, the bank offers him the sum of $50.00 less the interest that is charged for the loan. Billy's father refuses this and demands the full sum of $50.00 with the privilege of paying the interest when the principal is due. Why is it sound business practice for Billy's father to do this?

* *Accident Facts* is published yearly by National Safety Council, Inc., 20 N. Wacker Drive, Chicago, Ill.

† See *Commercial Aeronautics—41*, p. 29. American Technical Society, Chicago.

19.08. Daily Life.* In Massachusetts, approximately one out of every four persons over 65 years of age is receiving old age assistance at a total cost of $2,550,000. In all 87,000 persons are benefiting.

Problem: If the payment in Massachusetts is 43% above the national average, what is the average? It is estimated that the number of beneficiaries will increase about 45% in the next 20 years; what will the total tax cost be then?

19.09. Daily Life. A mail-order house gives the following table of terms for buying on credit:

If Cash Price Amounts to	*Down Payment Is*	*Monthly Payment Is*
$10.00–15.00	$2.00	
15.01–17.50	2.50	$2.00
17.51–20.00	2.50	
20.01–23.00	3.00	$3.00
23.01–28.00	3.00	
28.01–30.00	3.00	$4.00
30.01–40.00	4.00	
40.01–55.00	5.00	$5.00

The carrying charge is 10% of the unpaid balance after the down payment and is included in the total amount of the monthly payments; that is:

Cash price of order	$35.00	
Down payment	4.00	
Unpaid balance	$31.00	
10% carrying charge	3.10	
Due	$34.10	6 weeks at $5.00
		1 week at 4.10

Problem: Find monthly payments and length of time to pay on an item whose cash price is $18.75.

19.10. Daily Life.† A man wishes to lend you $100, saying that you can return it in 12 monthly installments at $10 each, thus

* See *Taxtalk*, March, 1941, p. 2, issued by Massachusetts Federal Taxpayers' Association.

† Adapted from R. S. Underwood and F. W. Sparks, *Living Mathematics*, p. 170. McGraw-Hill Book Company, New York, 1940.

paying him $20, or 20% on his money. You may think that his rate of interest, while rather high, is not altogether unreasonable in view of the smallness of the loan. Remember, however, that you start paying back the loan one month after it is made, so that the total interest of $20 is charged on an outstanding debt averaging approximately $50 during the year. The actual interest you pay is in the neighborhood of 40 per cent.

19.11. Daily Life. A loan company advertised the following rates on loans from $25 to $300:

Choose Your Monthly Payment Here

	2 payments	4 payments	6 payments	8 payments	10 payments	12 payments	16 payments	18 payments
$ 25	$ 12.97	6.65	4.54	3.49	2.86	2.44		
50	25.94	13.29	9.08	6.97	5.71	4.87		
75	38.91	19.94	13.62	10.46	8.57	7.31	5.74	5.23
100	51.88	26.58	18.15	13.95	11.43	9.75	7.66	6.97
125	64.79	33.20	22.67	17.41	14.26	12.16	9.55	8.69
150	77.70	39.79	27.16	20.85	17.07	14.56	11.43	10.39
200	103.51	52.97	36.13	27.72	22.68	19.33	15.15	13.76
250	129.26	66.11	45.08	34.57	28.27	24.08	18.85	17.11
300	155.02	79.26	54.02	41.41	33.85	28.82	22.54	20.46

What amount of interest was paid on a loan of $100 if payments were made in 12 monthly payments? What rate of interest was paid? Are there laws in your state concerning the rate of interest charged on small loans? Are the rates of this loan company legal in your state? What changes have recently been made?

19.12. Daily Life. Mr. D. has several men working on commission in his automobile repair shop. Up to $50.00 labor, he pays them 50%; over $50.00, he pays them 40%.

Problem: Consider the following problem and its solution. Mr. K. did $184.63 worth of work last week. All of this was labor. Up to $50.00 he gets 50%; over $50.00 he gets 40%. During the week Mr. K. had borrowed $16.75 from Mr. D. This is taken out after his salary has been figured. What will Mr. K.'s check be for the week?

19.13. Daily Life. In every field involving workmen's compensation known as Social Security the employer needs to be able to calculate with per cents. For example, Mr. S. paid a total of $1247.23 in wages to his employees during the past quarter. The law requires him to pay 2.7% of this amount to the Social Security Fund.

Problem: What amount did Mr. S. owe to the Social Security Fund for this quarter's business?

19.14. Daily Life. 4-H members take samples of milk from cows at their homes and send them to the creamery to be tested. At meetings they figure out the production in terms of cash for the month. Harry brought the following statistics:

Butter-fat	3.8%
Daily production of milk	42 lb.
Price of butter-fat in March	34¢ per pound

Problem: What was the value of the butter-fat for the month of March? Harry said that milk which tests 3.5% butter-fat is ranked as fair, 4 to 4.5% as good, and 6.5% as high.

19.15. Engineering.* Percentage is used to state the efficiencies of engines or machinery. The efficiency of a machine is that part of the power supplied to it that the machine delivers. If it requires 100 horsepower to run a dynamo which generates only 92 horsepower of electricity, then the efficiency of the dynamo is 92%.

Examples of machine efficiencies: (a) The efficiency of the average jackscrew is between 10% and 20%, and for a rough estimate an efficiency of 15% might be assumed. (b) The efficiency of differential hoists (hoists featuring wheels of different sizes) is very low, an average figure being about 30%. Such a hoist will lift only about 30% of the theoretical load that could be lifted were it without friction. (c) The efficiency of a gas engine is around 80%, while the efficiency of a steam engine is generally a little better than 90%.

19.16. Foundry.† Castings are formed in sand molds. The sand

* See E. B. Norris and K. G. Smith, *Shop Arithmetic*, pp. 67, 163, 176, 171, 198. McGraw-Hill Book Company, New York, 1913.

† See W. C. Stimpson, B. L. Gray, and J. Grennan, *Foundry Work*, pp. 12-13. American Technical Society, Chicago, 1940.

molds are made in forms or containers called flasks. These flasks consist of two or more boxes—the lower box is called the drag and the upper one, the cope. The flasks must be built sturdily since they contain heavy materials, are subjected to great heat, and receive rough usage when the mold is shaken out. To guarantee sufficiently strong flasks standard dimensions are given for flasks six inches deep. Thus for a flask 18″ to 24″ wide and up to 5′ long, the sides shall be built of 2″ materials and the crossbars shall be 1″ in diameter. If the flask is deeper, 25% shall be added to the thickness of the materials for every additional 6-inch depth.

Problem: What should be the thickness of the side materials and crossbars of a flask 4′ long, between 18″ and 24″ wide, and 18″ deep?

19.17. HOME OWNER. A certain home insulation firm guaranteed Mr. A. that if he allowed them to insulate his house at a charge of $225.00 he could save 15% of his coal bill.

Problem: If Mr. A. uses, on the average, 10 tons of coal a winter, for which he pays $8.50 per ton, in how many years could he save the expense of insulating his home, provided he does not figure interest on the cost of insulation? If Mr. A. does not have the money to pay for the insulation but can borrow it from the bank at 6 per cent interest, would it be to his advantage to borrow the money and have his house insulated?

19.18. HOME OWNER. The principles of percentage can aid one in purchasing real estate.

Problem: Mr. Dee can buy at Union a nice piece of property assessed at $15,000. He can buy a similarly good property at Franklin, where it is assessed only $12,500. The property tax at Union is 13 mills; at Franklin there are two taxes, one of 28 mills, and another town tax of 9 mills. Other things being equal, which property would you buy?

Solution: Tax on $15,000 property at .013 = $195.00 per year. Taxes on $12,500 property at .028 and .009 = $462.50 per year. Franklin property would cost $267.50 more per year in taxes, though assessed at a cheaper rate. In the long run the Union property would be the much cheaper buy.

19.19. HOUSEWIFE. In a certain state the sales tax is 2%. A grocery store sells a book of coupons for \$10.00. The grocer has been tearing out 20¢ worth of coupons for the tax when the book is sold. A customer insisted on paying the 20¢ tax in cash. The grocer insisted there was no difference. How much profit does the grocer make on 100 books by his system? Solution: For \$9.80 worth of groceries the tax should be 19.6¢ instead of 20¢. The grocer's profit on one book is 4 mills; on 100 books, 40¢.

19.20. INDUSTRY. The principles of simple interest can be used to determine whether it pays to install safety devices for employees since many concerns are required to pay insurance premiums on their employees.

Problem: A company spent \$4250 in installing safety devices and finds that it reduces losses due to accidents by 40%, thus reducing the premium paid on its employees by \$678 per year. If the devices are guaranteed for 10 years, and the money invested is worth 6%, does the project pay? If so, how much? Solution: Since the \$4250 will not be the same each year, the unpaid balance will have to be found to determine the interest for each year:

Year	*Amount Paid*	*Interest*	*Unpaid Balance*
			\$4250
1	678	255.00	3572
2	678	214.32	2894
3	678	173.64	2216
4	678	132.96	1538
5	678	92.28	860
6	678	51.60	182
		10.92	000

Thus the amount is paid out within 6 years, making a total paid out plus interest equal to \$5180.72. At the end of 10 years the company will have saved \$6780. Therefore \$6780 – \$5180.72 equals \$1599.28 plus interest on human welfare.

19.21. INTERIOR DECORATOR. The color of the walls and ceiling of a room—and, to some extent, of curtains and large articles of furniture—is an important item in determining how much electricity will be required to light a room adequately. This table gives the approximate percentage of light reflected by various colors:

Color	*Per Cent*
White	80%
Light ivory or cream	70
Light beige	65
Yellow	65
Light buff or pink	55
Pale green	50
Gray medium	45
Rose	12
Dark green or blue	10

19.22. Medicine.* Various compounds contain different percentages of a certain drug. For example, Aristol contains about 45% of iodine, Europhen contains about 28% of iodine, Nosophen contains about 60% of iodine, and Iodol contains about 90 per cent of iodine. If the dose for a certain purpose is to be 4 grains of iodine, the physician needs to calculate the precise amount of any one of the several compounds (containing iodine in different amounts) needed for this particular purpose.

19.23. Merchant. A certain chain store calculates the per cent of total business done by each department of the store. For example, the candy department may do 20% of the store's business; the tobacco department, 15%; the drug department, 40%; etc. Similarly the per cent of business done by each department is calculated, using the entire chain of stores as a basis. Comparisons of these percentages guide the individual stores in their advertising practices. For example, one store's per cent of sales in the tobacco department was higher than the per cent of sales in the tobacco departments of all stores, while the per cent of sales in the candy department was below the per cent of sales for candy departments of all stores.

From this it was concluded that the tobacco appetites of the people passing the store windows were relatively better satisfied (and thus less susceptible to temptation) than would be the candy appetites. Therefore, the front window display of tobacco products was removed to a smaller side window and the larger front window was used for a prominent display of candy products.

* See A. A. Stevens, *A Textbook of Therapeutics*, p. 432. W. B. Saunders Company, Philadelphia, 1924.

19.24. Merchant: Mark-up.* Purchase mark-up (initial mark-up or mark-on) is the difference between cost and retail price when the merchandise is first priced. The mark-up in dollars is the amount that is available to cover mark-downs, expenses, and inventory shortages, and leave a net profit on the operations.

Cumulated mark-up, or mark-up to date, is the difference between the total cost and total retail of merchandise handled to date, including the inventory at the beginning of the period.

Percentage of mark-up. The percentage of mark-up is determined by dividing the mark-up in dollars by the total retail. Unless otherwise stated, the mark-up percentage is based upon the retail price. This is the practice recommended for all stores. Since all expenses are paid out of the sales dollar and expense percentages are figured on the selling price, it is highly desirable that buyers compute their mark-up percentage on the retail price.

Problems: (a) Find the percentage of mark-up when the cost and retail price are known. First, subtract the cost from the retail price to find the amount of mark-up in dollars. Then, since we are to compare the amount of mark-up with the retail price to find the per cent of mark-up, divide the mark-up amount by the retail price, and express the result in hundredths.

Example: Retail price = $24.00
Cost price = 15.00
Find the percentage of mark-up.

Solution: $24 (retail) − $15 (cost) = $9.00 (mark-up)
$9.00 ÷ $24.00 = .375 or 37.5%

(b) Knowing the cost and the mark-up per cent, find the retail. Since we make all of our comparisons with the retail price, 100 per cent is always represented by the retail price. Therefore, if we know the mark-up per cent, its complement is the percentage of retail represented by the cost (cost plus mark-up = retail or 100%). Knowing now that the cost (given in dollars) represents a specified per cent of the retail, we find the retail by dividing the cost in dollars by the cost per cent of retail.

Example: Mark-up desired = 35%
Cost price = $130.00
Find the retail price.

* See "Merchandise Managers' Division." *The Buyer's Manual* (Second Edition), pp. 32–33. National Retail Dry Goods Association, New York.

Solution: Cost plus mark-up = 100% (retail). Since mark-up per cent = 35%, 100% minus 35% = 65% or the cost. Therefore, the cost ($130.00) = 65% of retail. If 65% = $130.00, 1% of retail = $\frac{1}{65}$th of $130.00 or $2.00. 100% (retail) = 100 times $2 or $200.00.

$$\text{Formula: Retail} = \frac{\text{Cost Price}}{100\% - \text{mark-up } \%} \text{ or } \frac{\text{Cost Price}}{\text{Cost } \%}.$$

19.25. Newsboy. A certain boy was approached to sell a weekly magazine. He was offered a weekly salary of $6.00 or a commission of 25%.

Problem: What amount must the total weekly sale of magazines exceed in order to make the commission proposition attractive?

19.26. Physics.* Foundry coke or so-called 72-hour coke consists of the following materials, given in per cents:

Moisture	.49
Volatile matter	1.31
Fixed carbon	87.46
Sulphur	.72
Ash	10.02

This type of coke delivers about 12,937 B.T.U.'s per pound.

Problem: Given one long ton of coke, under normal consumption (a) how much carbon is burned, (b) what will be amount of ashes, and (c) how much total heat is expected?

19.27. Physics. The degree of accuracy of measurements is very often stated by using the per cent concept. Example: "Tiny pellets that look like medicinal pills are being used in place of costly heat-measuring instruments. . . . Various pills melt at definitely known temperatures from 200 to 1500 degrees Fahrenheit and are accurate within three per cent."†

19.28. Religion, Church Finance. A church congregation had 235 members, 26% of whom were inactive. The annual church

* See H. Burton Gray, *Foundry Work*, p. 126. American Technical Society, Chicago, 1919.

† "Melting Pills Tell Temperature of Heated Metals." *Popular Mechanics Magazine*, Vol. 74, No. 2, p. 214, August, 1940.

budget was $3450.00, of which 86.6% was for local expenses and 13.4% for missions. What was the per-capita local expense giving of the church as a whole? Of the active members? What was the per-capita missionary giving of the church as a whole? Of the active members? What was the active members' average weekly gift for local expenses and missions?

19.29. Religion, Church Finance. A church accustomed to various money-raising methods—suppers, plays, etc.—was giving in their regular Sunday offerings at the rate of $3.20 per capita. The membership was 238. The local budget was $2000; missionary budget, $300. After a serious study of the principles of Christian stewardship the church decided to drop all other methods and rely entirely upon consecrated sharing of means. What per cent of increase would be necessary in their per-capita giving to carry the entire local budget and also increase the missionary budget 10%?

19.30. Sociology. In studying the forces influencing rural life in a central Pennsylvania community some description of church membership seemed necessary. Accordingly a bar graph was made showing the per cents of men, women, children 6 to 15, etc., among the membership. Such a per cent bar graph not only presented an impressive picture of church membership in the community studied but also made comparisons with other communities of different sizes easily possible.

19.31. Sugar Manufacturer. The principles of computing simple interest are used to determine the price of sugar cane. For example, on Nov. 14 of a certain year a Louisiana sugar house was making 177 lb. of sugar per ton of cane ground. On Nov. 15 there occurred a severe frost which killed the cane left in the field. After cane is killed by frost it begins to lose sugar. On Nov. 28 the factory was making only 133 lb. of sugar per ton of cane ground. Since the price for cane paid to the farmers depends upon the sugar content, by how much per cent should the price of Nov. 14 cane be cut so that the per cent reduction in price for Nov. 28 would equal the drop in sugar content?

19.32. Taxes. The rate of taxation on property is based upon two factors, amount to be raised and the assessed valuation of property to be taxed. Taxes in individual cases are computed at a rate of so many mills per dollar assessment.

Problem: A school district wishing to erect a gymnasium at a cost of $60,000 determines that it will require $3900 per year to retire the bonds and interest on this amount over a 20-year period. The average assessment for the district is $1,500,000. What tax rate should be fixed in order that sufficient money can be obtained annually?

ALSO SEE Average Ar. 2.04; Decimals Ar. 4.01, 4.09; Fractions Ar. 10.10, 10.21; Fundamental operations Ar. 11.22; Graphs Ar. 12.01, 12.02; Mensuration Ar. 16.21; Scale drawing Ar. 22.02.

PROPORTION 20.01–20.13

20.01. DAILY LIFE. If 8 machines produce 100 bolts in 6 hours, 2 machines will produce 100 bolts in (8 ÷ 2) times 6 hours, or 24 hours.

Problem: How many machines must be used to manufacture 1000 bolts in 10 hours?

Puzzle problem: If one man requires 12 minutes to polish a certain key, how long will it require 60 men to polish another identical key?

20.02. DAILY LIFE. "On Flag Day, June 14, 1923, a National Flag Conference was held in Washington to adopt a code for displaying the flag. The code grew out of the need for uniformity and a common understanding throughout the nation in the use of the flag. A year later the code was revised by the Second National Flag Conference and is now generally accepted. The proportions of the flag are:

Hoist (width) of flag	1.
Fly (length) of flag	1.9
Hoist (width) of union	$\frac{7}{13}$
Fly (length) of union	0.76
Width of each stripe	$\frac{1}{13}$
Diameter of each star (i.e., of circle in which it is drawn)	.0616"*

* Quoted from "The United States Flag Code." *Journal of National Education Association*, Vol. 30, No 2, p 37, February, 1941.

Problem: If a flag is 18 in. wide, calculate its length, the dimensions of the union, the width of each stripe, and the diameter of each star.

A somewhat standardized rule for determining the size of flag to use on a pole is that the length of the flag is to be about one fourth the length of the pole.

20.03. Daily Life. It is a common and striking practice to represent statistics by pictures or graphs. These pictorial representations may take the form of lines, circles, or cubes. However, when the representations assume more than one dimension their use must be carefully watched if the representations shall be true. Thus in representing quantities by circles the radii of the circles should be in proportion to the *square roots* of the quantities represented. If cubes are used, their edges should be in proportion to the *cube roots* of the quantities represented.

Thus if we wish to represent pictorially 36 and 729, the proportions would be:

Using lines	36:729	= 1:20.3
Using circles	6:27	= 1:4.5
Using cubes	3.3:9	= 1:2.7

20.04. Daily Life. The expression 3.674 miles is relatively as accurate as the measurement 3.674 inches. That this is true is seen by applying the principles of proportion. In each case the greatest possible error is .0005 of a unit since each quantity is measured to the nearest thousandth. The proportion of the first number that may be error is $\frac{.0005}{3.674} = \frac{1}{7348}$. The proportion of the second number that may be error is also $\frac{1}{7348}$.

Problem: Is the proportion of error (i.e., the degree of accuracy) of the following numbers identical? 36.74 miles, 367.4 inches, 3674 feet?

20.05. Dentistry.* Dentists have occasion to change gold from higher to lower carat. This is accomplished by adding sufficient

* See R M. Goepp, *Dental State Board Questions and Answers*, pp. 92-93. W. B. Saunders Company, Philadelphia, 1924.

silver or copper or both according to the following formula: "As the required carat is to the carat used, so is the weight used to the weight of the alloyed mass when reduced. Then the weight of the mass used subtracted from this will give the quantity of the alloy to be added." Example: Reduce 4 oz. of 20 carat gold to 16 carat gold. Solution: 16 : 20 :: 4 oz.: x; whence $x = 5$ oz. 5 oz. − 4 oz. = 1 oz., alloy to be added.

Problem: How many ounces of alloy must be added to 6 oz. of 18 carat gold to reduce it to 12 carat gold?

Note: By adding pure gold according to the following formula dentists can raise gold from lower to higher carat: "As the alloy in the required carat is to the alloy in the given carat, so is the weight of the alloyed gold to the weight of the reduced alloy required. Then the weight of the alloyed gold used subtracted from this gives the amount of pure gold to be added." Example: Raise one dwt. of 16 carat gold to 18 carat. Solution: Subtract 16 and 18 from 24 (pure gold is 24 carat) to find alloy in each carat. By rule 6:8::1:x; $x = 1\frac{1}{3}$ dwt. $1\frac{1}{3}$ less 1 = $\frac{1}{3}$ dwt., pure gold to be added.

20.06. Engineering. It has been found that air resistance in ventilating systems is proportional to the square of the velocity. A certain department store uses a 10-horsepower motor to supply 100,000 cu. ft. of air per minute. How powerful must a motor be to deliver twice as much air in the building?

20.07. Geology. "On a certain map the linear scale is 1 inch to 10 miles. How many acres are represented by a square $\frac{3}{4}$ inches on a side?"*

20.08. Home Owner. A family which pays $45 per month rent wishes to change the due date from the 25th to the 1st. How much should they pay for this portion of the month if the rent is paid on a 30-day month basis?

20.09. Housewife. A group of students found the following prices of materials in varying sized containers at a local store:

	Amount	*Cost*
Beans (canned)	12 oz.	10¢
	1 lb. 2 oz.	2 for 25¢

* E. H. Barker, *Applied Mathematics*, p. 242. Allyn and Bacon, New York, 1920.

	Amount	*Cost*
Peaches (sliced)	1 lb. 13 oz.	19¢
	8 oz.	3 for 25¢
Tomatoes (canned)	15 oz.	10¢
	1 lb. 12 oz.	15¢
Ketchup	8 oz.	12¢
	14 oz.	18¢

What should the amounts in the large containers cost if bought at the small container prices?

20.10. PHOTOGRAPHY. The use of proportion is valuable in showing comparisons of enlarged pictures.

Problem: Jack's camera makes pictures 6″ long and 3″ wide. He wishes to enlarge one of his pictures so that it will be 15″ long. How wide will the enlarged picture be? Solution:

Let $x =$ width of enlarged picture.

Then, $\frac{x}{3} = \frac{15}{6}$

$$x = \frac{15}{\cancel{6}_{2}} \times \cancel{3} = \frac{15}{2} = 7\tfrac{1}{2}''$$

20.11. PHYSICS. In problems involving attraction between heavenly bodies it is convenient to use a table in which the masses and radii of the sun, moon, and planets are expressed as fractional parts of the mass and the radius of the earth. This eliminates the work of using very large numbers. The following table gives the masses of the sun, moon, and planets, taking the earth's mass and radius as 1.

Body	*Mass*	*Radius*
Sun	329,000.	109.
Moon	0.012	0.27
Mercury	0.04	0.35
Venus	0.82	0.96
Earth	1.	1.
Mars	0.11	0.53
Jupiter	314.5	11.
Saturn	94.1	9.6
Uranus	14.4	3.9
Neptune	16.7	4.2

Tables like the above are used in comparing the weights of a certain object on earth with its weight on another planet. For example, how much more will a man weigh on the sun (with its greater mass) than he weighs on the earth? And, what will a 150-pound man weigh on the sun?

$$\text{Weight of a 150-pound man on sun} = \frac{G \times 150 \times 329{,}000}{109^2}$$

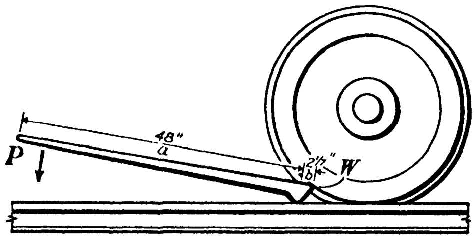
$$\text{Weight of 150-pound man on earth} = \frac{G \times 150 \times 1}{1^2}$$

$$\text{Comparison: } \frac{\dfrac{G \times 150 \times 329{,}000}{109^2}}{\dfrac{G \times 150 \times 1}{1}} = \frac{329{,}000}{109^2} = 28$$

That is, the man would weigh 28 times as heavy on the sun as on the earth. Actual weight on sun would be $150 \times 28 = 4200$ lbs.

Problem: How much would a 150-pound man weigh on Mars? On the moon?

20.12. Railways. A simple type of lever called a "pinch bar" is illustrated below.



As shown here it is used to move railway cars. A force of 2800 lb. is required at the point *W* of the bar to move the car. A force of 125 lb. applied at *P* is just sufficient to move the car. How long is the weight arm *b*?

From the law of the lever, $P:W::b:a$. Substituting given values and solving we have $b = 2\frac{1}{7}''$.

20.13. Student. A scale model of the solar system is to be constructed in a field adjoining the school grounds. The sun is to be represented by a sphere 2 feet in diameter. An encyclopedia yields the following facts:

Body	*Distance from Sun (Millions of Miles)*	*Diameter (Miles)*
Sun		866,000
Mercury	36	2,800
Venus	67	7,500
Earth	93	8,000
Mars	142	4,200
Jupiter	483	88,000
Saturn	886	76,000
Uranus	1783	31,000
Neptune	2794	33,000

Using scale adopted (866,000 miles = 24 inches) the diameter of the earth would be

$$\frac{24 \times 8000}{866,000} = .22 \text{ in. (about } \tfrac{1}{4} \text{ in.)}$$

The distance from sun to earth would be

$$\frac{2 \times 93,000,000}{866,000 \times 3} = \text{about 70 yds.}$$

The scale model would therefore feature the earth as represented by a small sphere about $\frac{1}{4}$ inch in diameter, at a distance of about 70 yards from the two-foot globe which represents the sun.

Problem: Calculate the diameters of the other planets and their distances from the sun.

Also see Per cent Ar. 19.30.

RATIO 21.01–21.09

21.01. Agriculture. The U. S. Soil Conservation Service estimates that the ratio of corn yield on non-eroded land to corn yield on badly eroded land is 7 to 1.

Problem: Find the loss per acre on badly eroded land if the market price of corn is 60 cents per bushel and non-eroded land will produce 100 bushels of corn per acre.

21.02. Biology.* The study of inheritable characteristics among the offspring of plants and animals was suggested as early as 1771. Mendel, about a century later, was the first to establish the numerical constancy of the various types of progeny. When the parents differed with respect to a single characteristic, he found that intercrossing the first hybrid generation gave "dominants" and "recessives" in the ratio 3:1; that is, three times as many of the offspring had the characteristic of the one grandparent as had the characteristic of the other.

That Mendel's suspicion of the ratio 3:1 was sound is evidenced by findings of subsequent experiments from which the *average* of all experiments can be found and thus the pattern or import of all of the experiments.

Hogben gives the following table. In this table "Yellow" is the characteristic of one grandparent (sometimes called *dominant*) while "Green" is the characteristic of the other grandparent (sometimes called the *recessive*).

	Yellow		*Green*		
Investigator	*Number*	*Per Cent*	*Number*	*Per Cent*	*Total*
Mendel, 1865	6,022	75.05	2,001	24.95	8,023
Correns, 1900	1,394	75.47	453	24.53	1,847
Tschermak, 1900	3,580	75.05	1,190	24.95	4,770
Hurst, 1904	1,310	74.64	445	25.36	1,755
Bateson, 1905	11,902	75.30	3,903	24.70	15,806
Lock, 1905	1,438	73.67	514	26.33	1,952
Darbishire, 1909	109,090	75.09	36,186	24.91	145,246
Totals	134,736	75.09	44,692	24.91	179,399

Using the concept of the average we can summarize the findings of the seven investigators by noting that with respect to number the ratio was 134,736:44,692 = 3.01:1 = 3:1; with respect to per cent the ratio was 75.09:24.91 = 3.01:1 = 3:1.

Later study of inheritable characteristics gave rise to what is called the chromosome hypothesis. This theory rests upon the existence and position of material particles, associated with chromosomes, called *genes*. The various possible combinations in breeding animals featuring chromosomes with genes placed in various regions also permit prediction of the numerical amounts of in-

* Adapted from L. Hogben, *Science for the Citizen*, pp. 959–987. Alfred A. Knopf, New York, 1938.

herited characteristics with stated amounts of possible variation, i.e., within what limits the results are certain to be. Along this line Hogben reports that a certain inheritable characteristic was predicted mathematically to be present in 19.9 ± 8.2 per cent of the offspring.

21.03. Carpentry. It is said that a window gives the most pleasing effect if the ratio of its height to its width is as 7:5.

Problem: What should be the height of a window if its width is 2′ 9″?

21.04. Daily Life. A geographic mile is the length of one minute of arc on a great circle of the earth. The average circumference of the earth is about 24,860 statute, or common, miles. It is also 360 × 60, or 21,600, geographic miles. Hence the ratio of the geographic mile to the common mile is $\frac{24860}{21600}$ or 1.151. For example, a speed of 1 geographic mile per hour, called a knot, is a speed of 1.151 common miles per hour. Hence a speed of 35 knots is equal to about 40 miles per hour.

21.05. Daily Life. To find the date of Easter in any year in this century, proceed as follows:

$$\frac{\text{Year}}{19} = a + \frac{A}{19}$$

$$\frac{\text{Year}}{4} = b + \frac{B}{4}$$

$$\frac{\text{Year}}{7} = c + \frac{C}{7}$$

$$\frac{19A + 24}{30} = d + \frac{D}{30}$$

$$\frac{2B + 4C + 6D + 5}{7} = 1 + \frac{E}{7}$$

Now Mar. 22 (the earliest date upon which Easter can fall) + D + E = the date of Easter.

In the year 1942, A = 4, B = 2, C = 3, D = 10, E = 4. Hence Easter is Mar. 22 + 14 = Apr. 5.

21.06. Music.* "The musical effect of two tones sounded together depends upon the ratio of the frequency of their vibrations. The major scale, which consists of the eight notes given here, may be used to illustrate this point.

	Middle C	*D*	*E*	*F*	*G*	*A*	*B*	*C*
Ratio of vibrations compared with middle C	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
Vibrations per second	256							

"Two tones are harmonious if the ratio of the frequencies of their vibrations may be expressed by two of the simple numbers 1, 2, 3, 4, 5, or 6. Thus, a note and its octave when sounded together give a pleasing effect, and their ratio of frequency is 1:2. In the same way, C and B sounded together would not be harmonious, for the ratio of their frequencies is either 15:8 or 15:16.

"*Problems:* (a) If middle C makes 256 vibrations per second, determine the number of vibrations made by each of the other notes of the octave.

(b) What pairs of notes within the octave produce harmony when sounded together? Example: The ratio of the frequencies of E and G is 5:6 or 6:5 and they are thus harmonious. Sound them on piano.

(c) A major triad in the diatonic scale consists of any three notes whose vibrations have the ratio 4:5:6. What major triads can be found in the table above?

(d) If there were 384 vibrations in the first note of a major triad, how many vibrations would there be in the other two notes of the triad?

(e) How many vibrations will there be in each of the other two notes if there are 288 in the first note of a triad?"

21.07. Music. The vibration frequencies for each octave of the different notes of our heptatonic scale of music are in the following ratios: $1 : \frac{9}{8} : \frac{5}{4} : \frac{4}{3} : \frac{3}{2} : \frac{5}{3} : \frac{15}{8} : 2$. In our orchestral scale, middle

* Adapted from A. Leonhardy, M Joseph, and R. D. McLeary, *New Trend Geometry*, pp. 300–301. Charles E. Merrill Co., New York, 1940.

A has been taken as 440 vibrations. Hence the following table expresses the frequency in each key of orchestral music:

Vibration Frequency per Second of Pitches

Key	*C*	*D*	*E*	*F*	*G*	*A*	*B*	*C′*
C	264	297	330	352	396	440	496	528
D	278	297	334	371	396	445	495	557
E	275	309	330	371	412	440	495	550
F	264	293	330	352	396	440	469	528
G	264	297	330	371	396	445	495	528
A	275	293	330	367	412	440	495	550
B	279	309	330	371	412	464	495	557

Rearranged with sharps included the above table follows:

Vibration Rates per Second

Key	*C*	*C#*	*D*	*D#*	*E*	*F*	*F#*	*G*	*G#*	*A*	*A#* *B♭*	*B*
C	264		297		330	352		396		440		495
D		278	297		334		300	396		445		495
E		275		309	330		371		412	440		495
F	264		293		330	352		396		440	469	
G	264		297		330		367	396		445		495
A		275	293		330		367		412	440		495
B		279		309	330		371		412		464	495

The foregoing table shows:

Key of C has no sharps or flats					
D has two sharps	C	F			
E has four sharps	C	D	F	G	
F has one flat	B				
G has one sharp	F				
A has three sharps	C	F	G		
B has five sharps	C	D	F	G	A

21.08. Music. The ratios in the proposed "even-tempered" scale are the following:

$$1 : 2^{1/12} : 2^{1/6} : 2^{1/4} : 2^{1/3} : 2^{5/12} : 2^{1/2} : 2^{7/12} :$$
$$2^{2/3} : 2^{3/4} : 2^{5/6} : 2^{11/12} : 2$$

On the assumption that C = 264 vibrations, the frequency of the tones of the even-tempered scale and the corresponding present frequency follow:

Even-tempered:	264, 280, 297, 314, 332, 352, 373, 395, 421, 444, 471, 498, 528
Present:	264, 277, 297, 308, 330, 352, 370, 396, 412, 440, 469, 495, 528

21.09. PHYSICS. "The ratio of the weight of a certain volume of a substance to the weight of the same volume of water is the *specific gravity* of the substance.

"Thus, one cubic foot of cork weighs 15 lb. while one cubic foot of water weighs 62.5 lb. Hence the specific gravity of cork is $\frac{15}{62.5} = .24$ (approximately)."*

Problem: Find the weight of one cubic foot of (a) iron if its specific gravity is 7.7; (b) blood if its specific gravity is 1.05; (c) kerosene if its specific gravity is 0.8.

ALSO SEE Decimals Ar. 4.01, 4.05; Fundamental operations Ar. 11.08, 11.15; Graphs Ar. 12.03; Insurance Ar. 13.01; Mensuration 16.31.

SCALE DRAWING 22.01–22.02

22.01. AVIATION.† Wind direction and velocity affect the course of an airplane. Distances actually covered and angles of steering are determined by a graphic application of the parallelogram and triangle of velocities. This graphic work must be *drawn to scale.* A commonly used scale is $\frac{1}{4}'' = 10$ miles.

Example: Suppose a pilot whose plane's speed in still air is 80 m.p.h. wishes to travel from point *A* (in drawings below) to point *B* while a wind is blowing from 330°. To find the angle at which the plane must be steered the following scale drawings have been made.

Plot true north line through point of departure, *A*. Lay off, in

* C. W. Sutton and N. J. Lennes, *Economic Mathematics*, p. 265. Allyn and Bacon, New York, 1940.

† Drawings and text adapted from *Commercial Aeronautics—26*, pp 10–15. American Technical Society, Chicago.

direction of point of destination, a line representing the distance covered by the plane in one hour's flying in still air (Figure 1).

Add a line showing direction of wind and distance traveled by wind in one hour (Figure 2). Draw parallel line (Figure 3).

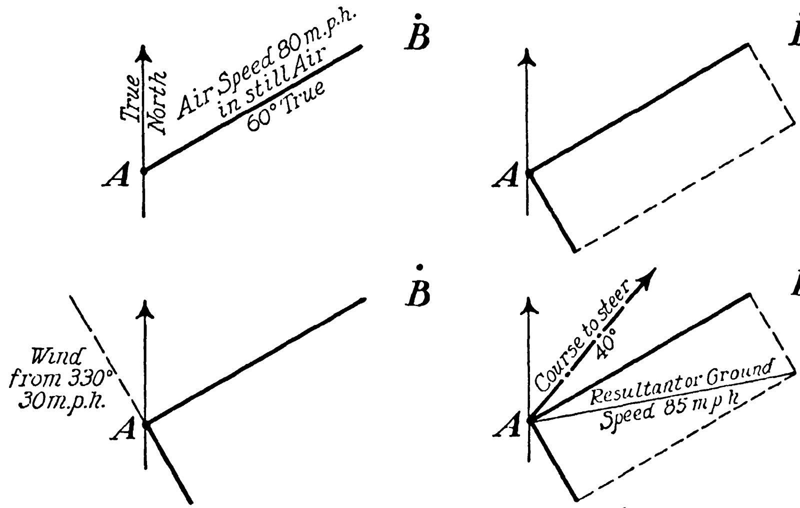


FIGURE 1 *FIGURE 3*

FIGURE 2 *FIGURE 4*

The diagonal represents actual ground speed per hour. The angle made by the diagonal and the longer side of the parallelogram represents the tendency of the wind to throw the pilot off his desired course. This angle is measured and reconstructed on the opposite side of the point of destination's true bearing line. This line indicates the course the pilot must steer to reach the desired destination (Figure 4).

22.02. Aviation.* The job of meeting aircraft and ships at sea, searching for lost parties, and photographic mapping demands calculation to determine the distances a given aircraft can fly and return with safety. This distance is called the Radius of Action. In the calculations it is customary to make a 25 per cent allowance in figuring the fuel supply to provide for such contingencies as a

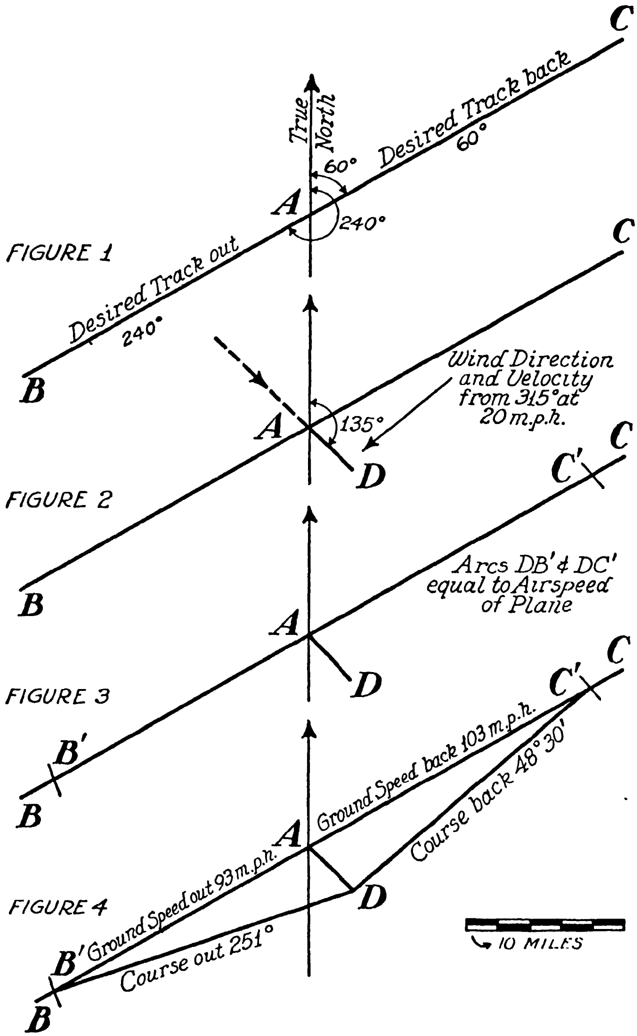
* Drawings and text adapted from *Commercial Aeronautics—26*, pp. 24–27. American Technical Society, Chicago.

loss in speed in climbing and a change in the velocity and direction of the wind.

To determine the Radius of Action the following formula is used:

$$\text{Radius} = \frac{\text{Speed out} \times \text{Speed in} \times \text{Fuel hours}}{\text{Speed out} + \text{Speed in}}$$

The Speed Out and Speed In, as well as courses out and in, are determined by scale drawing. (See illustrations below.)



Problems: Required to find how far a plane may safely journey and return to its base, with the following conditions given:

1. Air speed of ship 100 m.p.h. (in still air).
2. Wind velocity 20 m.p.h.
3. Wind direction 315° (measured clockwise from true north line).
4. Fuel supply four hours; 25 per cent allowance— three hours.
5. Track out 240° (measured clockwise from true north line).

Solution: (See Figures 1 to 4, page 92.)

As shown in Fig. 1, let *A* be point of departure. Lay off *AB* to represent desired track out from *A* at 240°. Extend line in opposite direction, *AC*, to represent desired track back.

As shown in Fig. 2, lay off at 135° to scale, the velocity of the wind for one hour, *AD*.

As shown in Fig. 3, with point of compass at *D* as center, describe an arc the radius of which represents the air speed of the plane, cutting lines *AB* and *AC* at *B′* and *C′* respectively.

As shown in Fig. 4, draw lines *DB′* and *DC′*.

DB′ is the true course out	251°
AB′ is the ground speed out	93 m.p.h.
DC′ is the true course in	48° 30′
AC′ is the ground speed back	103 m.p.h.

Substituting these values in formula above for Radius of Action,

$$\text{Radius} = \frac{93 \times 103 \times 3}{93 + 103} = 147 \text{ miles, approximately.}$$

SQUARING A NUMBER 23.01

23.01. Daily Life. Many people know that two-digit numbers ending in 5 can be squared easily by setting down 25 and prefixing the product of the tens digit times itself increased by unity. Thus, $25^2 = 625$ (set down 25 and prefix $2 \times (2 + 1) = 6$. And, $75^2 = 5625$ (set down 25 and prefix $7 \times (7 + 1) = 56$.

Fewer people realize that this rule can be illustrated in a general way by using this formula: Any two-digit number ending in 5 can be written as $10n + 5$ where n represents any single integer.

Then, $(10n + 5)^2 = 100n^2 + 100n + 25 = n(n + 1)\,100 + 25$, or, 25 prefixed by $n(n + 1)$.

This rule can be extended to find the product of two two-digit numbers with the same tens digit and the sum of the units digits 10. Thus:

72	64			77	88
78	66	etc.	Also	82	73
5616	4224			6314	6424

Problem: Develop a formula similar to the above to illustrate the generality of the practice illustrated by these last few examples.

ALSO SEE Decimals Ar. 4.10.

ALGEBRA

ALGEBRA

FORMULAS 1.01–1.108

1.01. Automobile. "How soon can a car be stopped in the face of danger?" The answer to this question depends upon the speed at which the car is being driven, the length of time necessary to start applying the brakes, and the rate at which the car slows down after the brakes are applied. (For safety this rate should, on the average, not exceed 22 feet per second. It will vary with road and brake conditions.) A car traveling at the rate of 60 miles per hour is traveling 88 feet per second. The average reaction time necessary before a driver can start applying the brakes is .75 second. This means that a car being driven at the rate of 60 miles per hour will travel 66 feet before the driver can get his foot on the brake. For a car traveling at the rate of r miles per hour, the minimum number of feet, d, necessary for stopping the car after danger is realized, is given approximately by the formula

$$d = .045r^2 + 1.1r.$$

This is for a dry road with the best of braking conditions. At 60 miles per hour a car would travel 90 feet farther before stopping on a wet road, 450 feet farther in mud or snow, and 1050 feet farther on sleet or ice.

When traveling at night, the maximum distance for clear vision is 150 feet (one-half the length of a football field). After a driver meets a car with bright headlights, he is practically blinded for 2 seconds. If he is traveling at 60 miles per hour, he will travel 176 feet without clear vision; at 30 miles per hour he will travel 88 feet.

1.02. Automobile. The average life of an automobile tire has been given as 11,000,000 revolutions. If d is the diameter of the tire in feet and m is the distance the tire travels in miles, the average expected mileage for any tire is

$$m = \frac{11{,}000{,}000\pi d}{5280} = 6545d.$$

1.03. AUTOMOBILE. If a moving automobile hits a stationary object, such as a solid wall, a tree, or a telephone post, the force of the impact is equivalent to that force with which it would strike the ground when falling from a certain height. If s represents this height in feet and r is the speed of the car in miles per hour, then

$$s = .0336r^2.$$

Example: If a car is traveling at the rate of 60 miles per hour

$$s = .0336(60)^2 = 121 \text{ ft.}$$

If the car should run into a solid wall, it would hit it with the same force as it would strike the ground if it should fall from the top of a building 121 feet high.

1.04. AUTOMOBILE. A car weighing w pounds, traveling at the rate of s miles per hour around a curve whose radius of curvature is r feet, is acted upon by a centrifugal force, $F = \frac{.067ws^2}{r}$, measured in pounds.

1.05. AUTOMOBILE. A formula which may be used to find the horsepower of an automobile engine is $H = 1.6nr^2$, where H is the horsepower, n the number of cylinders, and r the diameter of each cylinder in inches.

1.06. AUTOMOBILE. Another formula used for determining the horsepower of an automobile engine is

$$H = \frac{Planc}{792{,}000},$$

where H is horsepower, P pressure in pounds per square inch, l length of the piston in square inches, n number of revolutions per minute of the flywheel, and c number of cylinders.

1.07. AUTOMOBILE. The formula used to determine the perimeter of an elliptical section through a hand-brake lever is

$$p = \pi\sqrt{2(a^2 + b^2) - \frac{(a - b)^2}{2.2}},$$

where p is the perimeter and a and b are the semi-major and semi-minor axes, respectively, of the ellipse.*

1.08. AVIATION. The formula for the lift of an airplane wing is $L = L_cAV^2$, where L is the lift in pounds, L_c lift coefficient, A wing area in square feet, and V speed of plane in miles per hour.

1.09. AVIATION. The maximum ceiling for an airplane is given by the formula

$$H = 40{,}000 \log_{10} \frac{\max P_a}{\min P_r},$$

where H is height of ceiling in feet, P_a horsepower of engine available at sea level, and P_r horsepower of engine required to fly plane at sea level.

1.10. AVIATION. The power which an airplane engine might be expected to develop at any given altitude is

$$H = H_0\left(\frac{D}{D_0}\right)^{1.4},$$

where H is horsepower at the given altitude, H_0 horsepower at sea level, D air density at given altitude, and D_0 air density at sea level.

1.11. AVIATION. The pressure in pounds of the air upon the wings of an airplane is given by the formula $p = .0005v^2$, where v is the speed of the plane in miles per hour.

1.12. AVIATION. When a mail bag, a bomb, or any other object is dropped from an airplane, it passes through a path somewhat like a jet of water projected horizontally from a hose. Its own inertia in virtue of the motion of the plane, which we shall assume to be moving parallel to the ground, gives it a drift forward as it falls.

* Used by permission of the Chevrolet Division of General Motors Corporation.

In order to know where to release the object so it will reach its proper destination, it is necessary to calculate the distance, d, between the position of the airplane at the correct time of dropping the object and a point directly above the place where the object is supposed to land. If we neglect all forces except that due to the speed of the plane and that due to gravity, we can derive a very simple formula for an approximate value for d.

If the airplane is cruising at a height of h feet with a speed of r feet per second, then the time, t, in seconds, necessary for the object to travel the distance, d, is $t = \frac{d}{r}\cdot$ In this same time, the object will drop, because of gravity, through a height $s = 16.1t^2$. If it sinks to height y while traveling through distance d, then

$$y = h - s = h - 16.1t^2 = h - 16.1\frac{d^2}{r^2}\cdot$$

When the object reaches the ground, $y = 0$, therefore

$$d = r\sqrt{\frac{h}{16.1}} = .249r\sqrt{h}.$$

If v = speed of the airplane in miles per hour, then $r = 1.47v$, whence

$$d = .366v\sqrt{h}.$$

Example: If a mail bag is dropped from an airplane cruising at a speed of 60 miles per hour at a height of 640 feet above the ground

$$d = .366(60)\sqrt{640} = 556 \text{ feet}.$$

The mail bag should be dropped 556 feet before the plane is directly over the place where the bag is supposed to land.

1.13. BALLISTICS. The depth, h, in inches, of a cylindrical shell whose diameter is d inches and capacity v quarts is given by the formula

$$h = 73.53\,\frac{v}{d^2}\cdot$$

1.14. BALLISTICS. If an object is thrown downward with a velocity of v feet per second, the distance, d, it will fall in t seconds is $d = 16.1t^2 + vt$.

1.15. BALLISTICS. If v_0 is the velocity with which a shell d inches in diameter and weighing w pounds is projected from the muzzle of a gun, then the velocity, v, at which the shell is traveling t seconds after leaving the gun is

$$\frac{1}{v} = \frac{1}{v_0} + \frac{td^2}{7000w}.$$

1.16. BIOLOGY. A type of formula which is very important in many phases of biology is the following which gives the total number, n, of bacteria in a given culture at any time, t, measured in hours:

$$n = 1000e^{4t}. \qquad (e = 2.71828)$$

1.17. BUILDING. Contractors frequently must estimate the number of bricks to be used in a projected construction. A formula that is sometimes used in making such an estimate is $n = 22\,lwh$, where n is the number of bricks, l the length in feet, w the width, and h the height of the object to be constructed.

1.18. BUILDING. If shingles are to be laid so that 4 inches of each shingle is to be the exposed portion, then the formula that carpenters use to estimate the number, n, of shingles needed to cover an area of A square feet is $n = 10A$.

If the shingles are to be laid so that 5 inches of each shingle is to be the exposed portion, then the formula is $n = 7.5A$.

1.19. BUILDING. To have a well-designed chimney it is necessary to take into consideration symmetry of appearance as well as performance of function. If q is the amount of coal used in pounds per hour, and h the height of the chimney, then the area, A, in square feet, of the top of the chimney should be determined by the formula $A = \frac{.03q}{\sqrt{h}}$. The effective area, E, of the chimney is then given by the formula $E = A - .6\sqrt{A}$.

1.20. BUILDING. The maximum horsepower, p, of the boiler which can be effectively served by a chimney h feet high and with a cross section of A square feet is

$$p = 3.33\,(A - .6\sqrt{A})\,\sqrt{h}.$$

1.21. Building. A formula known as Fuller's rule for the estimation of the number of barrels, n, of Portland cement to be used in mixing one cubic yard of concrete is $n = \frac{11}{c + s + g}$, where c is number of parts of concrete, s number of parts of sand, and g number of parts of gravel or crushed stone.

1.22. Business. Profit is figured most frequently by merchants to be a certain per cent of the selling price. Hence a problem they have to consider is: "At what price must an article which cost c dollars be sold in order to make a profit of p per cent of the selling price s?" The answer to this question is found by the formula

$$s = \frac{c}{1 - p},$$

where p is expressed as a decimal fraction. If the profit is to be p per cent of the cost, then the formula for the selling price is

$$s = c(1 + p).$$

1.23. Business. If a merchant wishes to mark the price of his goods so that he can give the purchaser a discount of d per cent from the marked price, m, and still make p per cent profit on the selling price s, the formula is

$$m = \frac{c}{1 - (p + d) + pd},$$

where c is the cost, and p and d are expressed as decimal fractions.

If the profit is to be p per cent of the cost, then the marked price, m, would be obtained by the formula

$$m = \frac{c(1 + p)}{1 - d}.$$

1.24. Business. An original debt of P dollars is to be discharged in equal monthly payments of p dollars each. The interest is i per cent per payment period and is to be calculated at each payment period on the unpaid balance. Each payment is to be used first to take care of the interest on the unpaid balance and then to reduce the principal. As the principal is reduced, the interest due with each payment declines and the balance available for reducing the principal increases.

The amount, A, due after n equal payments is given by the formula

$$A = P - (p - Pi)\left[\frac{(1 + i)^n - 1}{i}\right].$$

Example: Mr. Green bought a house for \$1850. He paid \$150 cash. He was to pay the unpaid balance of \$1700 in equal monthly payments of \$17.00 each. The rate of interest was to be 6% simple interest on the unpaid balance. How much did he still owe after he had made 12 payments?

Since each payment period is one month, $i = \frac{.06}{12} = .005$. Of the \$17.00 paid at the end of the first month \$8.50 was the interest; so \$17.00 − \$8.50 = \$8.50 was used to reduce the principal, etc.

$$A = 1700 - [17.00 - (1700)(.005)]\left[\frac{(1.005)^{12} - 1}{.005}\right]$$

$$= 1700 - (8.50)\left[\frac{(1.005)^{12} - 1}{.005}\right]$$

By logarithms, $(1.005)^{12} = 1.062$

$$A = 1700 - 8.50\left(\frac{.062}{.005}\right) = \$1594.60.$$

1.25. Business. Using the same notation of Al. 1.24, the number of payment periods necessary to discharge such a debt is given by the formula

$$n = \frac{\log\left(\frac{p}{p - Pi}\right)}{\log(1 + i)}.$$

Example: In the example of Al. 1.24, how many months will it take Mr. Green to take care of the entire debt?

$$n = \frac{\log\left(\frac{17}{17 - 1700(.005)}\right)}{\log(1.005)} = \frac{\log 2}{\log(1.005)}$$

$$n = \frac{.30103}{.00217} = 138.72 \text{ months.}$$

1.26. Business. To find the amount of each equal payment, p, on an original debt of P dollars when the number of payments is n and the rate is i per cent per payment period,

$$p = \frac{Pi}{1 - (1 + i)^{-n}}.$$

Examples: (a) Mr. Jones borrowed \$5000 for 3 years at 6% simple interest on the unpaid balance. What is the amount of the 3 equal yearly installments necessary to discharge this debt?

$$p = \frac{\$5000 \times .06}{1 - (1.06)^{-3}} = \frac{\$300}{1 - (1.06)^{-3}}$$

By logarithms $(1.06)^{-3} = .8396$, hence

$$p = \frac{\$300}{.1604} = \$1870.33.$$

(b) In the example of Al. 1.24, what is the amount of each equal monthly payment necessary to discharge the debt of \$1700 at 6% simple interest on the unpaid balance in 138.72 months?

$$i = \frac{.06}{12} = .005 \text{ per month}$$

$$p = \frac{\$1700 \times .005}{1 - (1.005)^{-138.72}} = \frac{\$8.50}{1 - (1.005)^{-138.72}}$$

By logarithms $(1.005)^{-138.72} = .50$

$$p = \frac{\$8.50}{.50} = \$17.00.$$

1.27. Business. To find the true rate, r, of interest when interest is charged on the initial debt rather than on the unpaid balance, $r = \frac{2pc}{b(n + 1)}$, where p is number of payment periods in one year, c carrying charge or interest, b unpaid balance at the beginning of the credit period (cash price—down payment), and n number of payments after down payment has been made.

1.28. Business. When a government or a corporation issues a bond, it promises to pay the sum of S dollars, specified in the bond, at an established maturity date. Furthermore, it promises to pay, at stated intervals, a certain amount of interest on the face value of the bond and at an agreed rate of interest, r. This rate of interest is seldom the same as that prevailing in the money market at the time of the purchase of the bond, thus the market value of the bond usually differs from its face value. The interest rate specified in the bond is that used to compute the value B of each interest coupon.

If, in purchasing a certain bond, an investor wishes a specified investment rate of i per cent, then the price, P, which he should be willing to pay is given by the formula

$$P = S\left[1 - (i - r)\frac{1 - (1 + i)^{-n}}{i}\right].$$

Example: A $500 city bond, maturing in 10 years, bears interest at 3%, payable annually. What price should a buyer be willing to pay for this bond if he wants to make 5% compounded annually?

$$P = \$500\left[1 - (.05 - .03)\frac{1 - (1.05)^{-10}}{.05}\right]$$

$$P = \$500\left[1 - (.02)\frac{1 - (1.05)^{-10}}{.05}\right]$$

By logarithms $(1.05)^{-10} = .6139$

$$P = \$500\left[1 - (.02)\frac{1 - .6139}{.05}\right] = \$500[1 - (.02)(7.72)]$$

$$P = \$500[.8456] = \$422.80.$$

1.29. Calendar.* A method of finding the day of the week of any given date is as follows:

Let p = the day of the month of the given date.

q = the number of the month in the year. When the date occurs in January or February, $q = 13$ for January or 14 for February, that is, they are counted as the 13th and 14th months, respectively, of the previous year.

N = the year.

* Adapted from H. L. Rietz and A. R. Crathorne, *College Algebra*, p. 21. Henry Holt and Company, New York, 1939.

$$P = p + 2q + \left[\frac{3(q+1)}{5}\right] + N + \left[\frac{N}{4}\right] - \left[\frac{N}{100}\right] + \left[\frac{N}{400}\right] + 2.$$

(Any expression in brackets means the largest integer contained in the enclosed number, for example $\left[\frac{15}{4}\right] = 3$, since 3 is the largest integer in the quotient $\frac{15}{4}$.)

With the above notation the number of the day of the week of any given date, always counting Sunday as the first day of the week, is the remainder obtained by dividing P by 7.

Example: On what day of the week was Christmas Day, 1941, December 25, 1941?

$$p = 25,\ q = 12,\ N = 1941$$

$$P = 25 + 24 + \left[\frac{3\,(12+1)}{5}\right] + 1941 + \left[\frac{1941}{4}\right] - \left[\frac{1941}{100}\right] + \left[\frac{1941}{400}\right] + 2.$$

$$= 25 + 24 + 7 + 1941 + 485 - 19 + 4 + 2 = 2469$$

$$\frac{2469}{7} = 352, \text{ with a remainder of } 5.$$

Hence Christmas Day, 1941, was on Thursday.

1.30. CHEMISTRY. A small body moving through a fluid so slowly that no eddies are formed is subject to a certain retarding force, F. For a smooth sphere of radius r this force is given by the formula

$$F = 6\pi\mu r v,$$

where μ is absolute viscosity of the containing fluid and v the velocity of the body relative to the fluid.

1.31. CHEMISTRY. The intensity, I, of the activity of radium after t minutes of time is given by the formula

$$I = I_0(2.718)^{-Kt}$$

where I_0 is the intensity at the beginning of the experiment, and K is a constant.

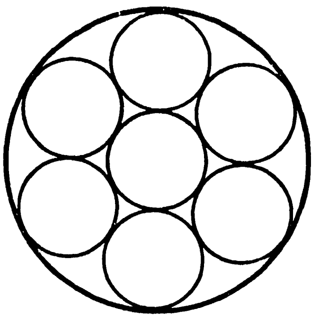
1.32. CHEMISTRY. If radiant energy E is expressed in British Thermal Units per hour per square foot and temperature T in degrees Fahrenheit absolute, then

$$E = .173\left(\frac{T}{100}\right)^4.$$

1.33. COMMUNICATION. The number of telephone connections possible through a switchboard to which n telephones are connected is

$$C = \frac{n(n-1)}{2}.$$

1.34. COMMUNICATION. The approximate number, n, of smaller circles of the same diameter which may be inscribed in a larger circle is given by the formula below.



$$n = .907\left(\frac{D}{d} - .94\right)^2 + 3.7,$$

where D is the diameter of the large circle and d is the common diameter of the small circles.

Example: How many wires .20 inch in diameter can be placed in a cable which is $1\frac{1}{2}$ inches in diameter?

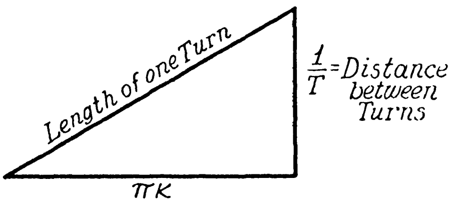
1.35. COMMUNICATION.* In designing a single helical coil filament, usually the following factors are independent variables:

* Used by permission of R. C. A. Manufacturing Co.

d = diameter of wire
L = cut length of wire
D = outside diameter of helical coil
S = wound length

From these it is desired to determine the turns per inch, T, for the purpose of designing the winding equipment.

Let $K = D - d$. Then the length of the undeveloped turn of the coil is the hypotenuse of a right triangle whose two legs are πK and $\frac{1}{T}$ (the distance between turns).



The length of one turn $= \sqrt{(\pi K)^2 + (1/T)^2}$

and the length of N turns $= N\sqrt{(\pi K)^2 + (1/T)^2} = L.$

But $N = ST.$

Simplifying $L = ST\sqrt{\frac{(\pi KT)^2 + 1}{T^2}}$

$$= S\sqrt{(\pi KT)^2 + 1}.$$

Squaring both sides of equation:

$$L^2 = S^2(\pi^2K^2T^2 + 1)$$

$$L^2/S^2 = \pi^2K^2T^2 + 1$$

$$T^2 = \frac{L^2/S^2 - 1}{\pi^2K^2}$$

$$T = \frac{\sqrt{(L/S)^2 - 1}}{\pi K}.$$

Frequently, $(L/S)^2$ is very much greater than 1,

whence, $T = L/\pi KS$ (approx.).

1.36. ELECTRICITY (Ohm's Law). Ohm discovered that the steady flow of an electrical current produced by a constant e.m.f. (electromotive force) is directly proportional to the e.m.f. Hence, in any circuit

$$E = IR,$$

where E is constant e.m.f. (voltage), I the sustained current, and R a constant, called the resistance of the circuit.

1.37. ELECTRICITY. The amount of energy or power expended when one volt causes a current of one ampere to flow in a circuit is called one watt of power.

$$W = EI \quad \text{or} \quad W = I^2R,$$

where W is power in watts, E pressure in volts, I current in amperes, and R resistance in ohms.

1.38. ELECTRICITY. Copper wire is most frequently used in electrical wiring. The resistance, R, for copper wire at any given temperature, t, is

$$R = .02057(1 + .00387t + .00000597t^2).$$

1.39. ELECTRICITY. In electrical engineering it is very convenient to use complex numbers to designate voltage, E, current, I, and resistance, R. Find the third term in the formula $E = IR$ for each of the given sets of values:

(a) $E = 50 + 30j$ volts; $I = 4 + 2j$ amperes $(j = \sqrt{-1})$
(b) $I = 2 + j$ amperes; $R = 25 - 10j$ ohms
(c) $E = 150$ volts; $R = 15 - 15j$ ohms.

1.40. ELECTRICITY. The total amount of voltage E (electromotive force) developed by a generator is

$$E = \frac{N\phi S}{10^8},$$

where N is the number of armature conductors which are active, ϕ the flux per pole, and S the number of revolutions per second made by the armature.

1.41. ELECTRICITY. The voltage, E, of a one-cell battery is given by the formula $E = I(r + R)$, where I is the current in the circuit, r the resistance within the cell itself, and R the resistance of the circuit outside the battery.

This same formula holds for a group of cells connected either in series or in parallel. It must be remembered that the total voltage and the total internal resistance of a group of batteries connected in series is found by multiplying the V and r for one cell by the total number of cells. If the cells are connected in parallel, the voltage for the group will be that of one cell, and the internal resistance will be that of one cell divided by the number of cells.

Examples: (a) Six cells, each having an e.m.f. (electromotive force or voltage) of 3.25 volts and an internal resistance of .054 ohm, are connected in series. How much current will this battery send through a circuit whose resistance is 1.75 ohms?

$$E = 3.25 \times 6 = 19.50 \text{ volts}$$
$$r = .054 \times 6 = .324 \text{ ohm}$$
$$R = 1.75 \text{ ohms.}$$

Hence
$$19.50 = I(.324 + 1.75)$$
$$I = 9.4 \text{ amperes.}$$

(b) What would be the current sent out by the battery if the cells were connected in parallel?

$$E = 3.25 \text{ volts}$$
$$r = .054 \div 6 = .009 \text{ ohm}$$
$$R = 1.75 \text{ ohms.}$$

Hence
$$3.25 = I(.009 + 1.75)$$
$$I = 1.8 \text{ amperes.}$$

1.42. ELECTRICITY. An electrician can use the formula $E = IR$ to locate trouble, such as wires blown down by a storm. If he knows the voltage of the electromotive force generating the current and the resistance of the wires, he can measure the current and thus locate the trouble.

Example: Suppose an electromotive force of 110 volts is sending a current over a wire which has a resistance of 1 ohm for every 75 feet of wire. At the time of trouble the current measures $2\frac{1}{2}$ amperes; then from the formula, $110 = 2\frac{1}{2}R$, or $R = 44$ ohms.

Hence the trouble is at a distance of $75 \times 44 = 3300$ feet from the source of power.

1.43. ELECTRICITY. The quantity of electricity stored in a condenser, called the charge, is proportional to the applied voltage, $q = ce$, where q is charge in coulombs, e applied voltage, c a constant called the capacity of the condenser, expressed in farads.

1.44. ELECTRICITY. To determine the size, S, of a wire in circular mils necessary to carry a current of I amperes a distance of d feet from the source of distribution with a drop of e volts:

$$S = \frac{21.6dI}{e}.$$

(A round wire .001 inch in diameter has a cross-sectional area of one circular mil.)

1.45. ENGINEERING. The approximate velocity, v, of a water jet escaping through a sharp-edged frictionless opening is given by the formula

$$v = 8\sqrt{h},$$

where h is the distance the opening is below the level of water.

1.46. ENGINEERING. A jet of water of cross-sectional area A square feet, flowing with a velocity of v feet per second, will generate P foot-pounds of power, where

$$P = \frac{wv^3A}{64},$$

if one cubic foot of water weighs w pounds.

1.47. ENGINEERING. The number of seconds necessary to discharge from a vessel of cross-sectional area of A_1 square inches into one of A_2 square inches in cross section, through an opening whose area is A square inches, is

$$t = \frac{.116A_1A_2(F_1^{\frac{1}{2}} - F_2^{\frac{1}{2}})}{(A_1 + A_2)A}.$$

F_1 and F_2 are the differences in the heights of the water in the two vessels at the beginning and end of the discharge; these measurements are in inches.

1.48. Engineering. Water falling f feet and generating h horsepower will turn a waterwheel at the rate of n revolutions per minute, where

$$n = \frac{213\, f^{\frac{5}{4}}}{4h^{\frac{1}{2}}}.$$

1.49. Engineering. Sewers which allow an average water depth of m feet and have a slope of s will discharge at the rate of v feet per second, where

$$v = 124\, m^{\frac{2}{3}} s^{\frac{1}{2}}.$$

1.50. Engineering. The formula for a safely distributed load for a pine beam of length l, width w, and depth d is

$$L = \frac{180wd^2}{l}.$$

1.51. Engineering. The Rankine formula for the allowable span, l feet, for corrugated steel roofing is

$$l = \left(.178\, \frac{bhst}{w}\right)^{\frac{1}{2}},$$

where b is the width of the sheet in inches, h the depth of the corrugations in inches, s the working stress in pounds per square inch, t the thickness of the sheet in inches, and w the safe load in pounds per square foot.

1.52. Engineering. The pressure in pounds on a dam across a river is given by $p = wd^2$, where w and d are the width and depth, respectively, of the river, each measured in feet.

1.53. Engineering. The diameter, d, of a pipe which will carry the same amount of water as two pipes whose diameters are respectively d_1 and d_2 is

$$d = \sqrt{d_1^2 + d_2^2}.$$

1.54. Engineering. When a steel tape is held off the ground it sags. The magnitude of the error, E, involved in any measurement made is

$$E = \frac{w^2 l}{24p^2},$$

where w is the weight in pounds of the tape between the end supports, l the length between these supports, and p the tension in pounds on the tape.

1.55. Engineering. The approximate formula for the length, l, of a flexible cable which carries a load uniformly distributed along the horizontal is

$$l = a\left[1 + 2.7\left(\frac{f}{a}\right)^2 - 6.2\left(\frac{f}{a}\right)^4\right],$$

where a is the length in feet of the span and f the sag of the cable measured in feet.

1.56. Engineering. A formula that is used for finding the weight, w, of steel roof trusses in pounds per square foot of horizontal covered area is

$$w = \frac{P}{45}\left(1 + \frac{L}{5\sqrt{d}}\right),$$

where P is the capacity of the trusses in pounds per square foot of horizontal covered area, L the length of the span in feet, and d the distance between the centers of the trusses in feet.

1.57. Engineering. The pressure, p, in pounds per square foot of a wind blowing with a speed of v miles per hour is $p = .003v^2$.

1.58. Engineering. A locomotive can pull with a tractive power of T pounds,

$$T = \frac{.85d^2LP}{w},$$

where d is the diameter of the cylinder, L the length of the piston stroke, w the diameter of the driving wheel (each measured in inches), and P the mean steam pressure in pounds per square inch.

1.59. Engineering. The velocity of water as it passes through a freshly cleaned slow sand filter-bed is computed by the following formula:

$$V = 850\, d^2 \frac{h}{a}\left(\frac{t + 10}{60}\right),$$

in which d is the effective size of the sand in millimeters, h head of water on the filter-bed in feet, a the thickness of the sand-bed in feet, t temperature in degrees Fahrenheit. The formula gives V in meters per day.

1.60. Engineering. In riveting two steel boiler plates of thickness t inches with a single row of rivets, the distance D, in inches, between the centers of the rivets should be

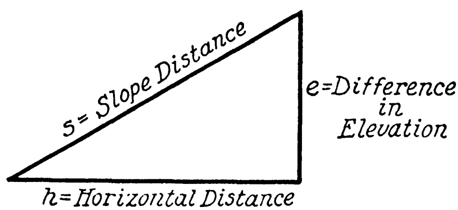
$$D = .56\frac{d^2}{t} + d,$$

where d is the diameter of the rivet holes.

1.61. Engineering. The flat end plate of a boiler, whose area is a square inches and whose thickness is t sixteenths of an inch, can safely sustain a pressure of p pounds per square inch, where

$$p = \frac{200\,(t + 1)^2}{a - 6}.$$

1.62. Engineering. An approximate correction to be made in order to measure horizontal distance from slope distance is $\frac{e^2}{2s}$, where e is the difference in elevation of the two end-points and s the slope distance between them.



If s is slope distance between two objects of different elevations, h horizontal distance, and e difference in elevation,

$$s^2 - h^2 = e^2 \quad \text{or} \quad (s-h)(s+h) = e^2,$$

whence
$$s - h = \frac{e^2}{s+h} = \frac{e^2}{2s} \text{ (approximately).}$$

Therefore
$$h = s - \frac{e^2}{2s}.$$

1.63. Engineering. When an irregular area is bounded by straight lines, we have a formula which will give this area regardless of the number of sides provided the vertices are known relative to a set of rectangular coordinate axes taken anywhere in the plane. If the coordinates are read in counter-clockwise direction we have:

$$\text{Area} = \tfrac{1}{2}\,[y_1(x_5 - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_4) + y_4(x_3 - x_5) + y_5(x_4 - x_1)]$$

when there are five sides.

In general, for an area bounded by n lines,

$$\text{Area} = \tfrac{1}{2}\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \ldots & x_n & x_1 \\ y_1 & y_2 & y_3 & y_4 & \ldots & y_n & y_1 \end{pmatrix}$$
$$= \tfrac{1}{2}\,(x_1y_2 + x_2y_3 + \ldots - x_2y_1 - x_3y_2 - \ldots)$$

1.64. Engineering. To estimate the number of cubic yards of dirt to be removed from a cut, or needed to fill in a sunken portion of a road under construction, the engineer gets the areas, A_1 and A_2, in square feet, of two cross sections which are L feet apart. The approximate volume is then

$$V = \frac{L(A_1 + A_2)}{54}.$$

A more exact formula would be

$$V = \frac{L}{27}\left(\frac{A_1 + 4A_m + A_2}{6}\right),$$

where A_m is the area of a cross section midway between A_1 and A_2.

1.65. Engineering.* Computation of the critical load for a slen-

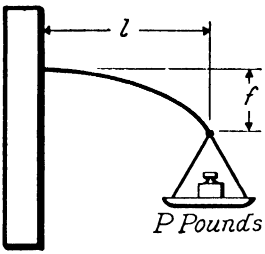
* Used by permission of the Chevrolet Division of General Motors Corporation.

der vertical column, with one end fixed, is made by this formula:

$$P = \frac{\pi^2 EI}{4l^2},$$

where P is critical load, E modulus of elasticity, I the smallest equational moment of inertia, and l length of the column.

1.66. ENGINEERING. The vertical deflection of a cantilever formed by a horizontal spring leaf loaded at its free end by a weight is:



$$f = \frac{Pl^3}{3EI},$$

where f is deflection, P weight, l length of the cantilever, E modulus of elasticity, and I moment of inertia with reference to the neutral axis.*

1.67. FARM. If l is length, w width, and h height of a bin in the shape of a rectangular solid, then the number of bushels of corn, C, and the number of bushels of wheat, W, which the bin will hold are given by the respective formulas:

$$C = \frac{lwh}{2.5}, \qquad W = \frac{lwh}{1.25}.$$

1.68. FARM. A simple method for approximating the average velocity, V, at which a stream of water is flowing is the following:

Select a place on the stream where the bed is smooth and of uniform depth and width, and then determine the time it takes for a floating stick to cover a given measured distance of S feet. Make several such tests from bank to bank and take the average

* The above formula is used with permission of the Chevrolet Division of General Motors Corporation.

time. If t is the average number of seconds for the stick to float through distance S, then

$$V = \frac{.80S}{t} \text{ feet per second}$$

or $$V = \frac{.80S}{t}(60) = \frac{48S}{t} \text{ feet per minute}$$

or $$V = \frac{48S}{t} \cdot \frac{60}{5280} = \frac{6S}{11t} \text{ miles per hour.}$$

To get the approximate volume of the stream flow multiply the rate of flow by the computed cross-sectional area, A square feet, of water in the stream bed. If C is the number of cubic feet per unit of time, with the above notation,

$$C = VA.$$

Whence $$C = \frac{.80SA}{t} \text{ cu. ft. per second}$$

$$C = \frac{48SA}{t} \text{ cu. ft. per minute}$$

$$C = \frac{6SA}{11t} \text{ cu. ft. per hour.}$$

Each of these results should be multiplied by $7\frac{1}{2}$ to get the volume of stream flow expressed in gallons per unit of time.

The cross-sectional area may be computed by multiplying the stream width at the point where stream flow is being observed by the average of several measurements of depth taken from bank to bank.

1.69. FARM. Theoretical water power is estimated from the volume of water flowing and its vertical fall (head) from water surface to water surface at two points on the stream. The theoretical amount of horsepower, H, generated by a stream with a volume of c cubic feet per minute and a head of h feet is

$$H = \frac{ch}{528}.$$

The theoretical water power expressed in kilowatts, K, is given by

$$K = \frac{ch}{708}.$$

1.70. FARM. The terracing of sloping ground is important as an aid to landscaping and in the prevention of soil erosion. Simple rules have been developed for approximating the number of feet for each vertical interval between successive terraces. If V represents the recommended number of feet in each vertical interval and s the per cent of slope, then:

For northern states	$V = s/3 + 2$
For southern states	$V = s/2 + 2$

1.71. FARM. One formula for determining the amount of solids, which are not fat, in milk, known as Babcock's modified formula, is

$$S = .25l + .2f + .14,$$

where S is solids not fat, l the lactometer reading, and f the per cent of fat.

1.72. HOME. A simple formula for approximating the number of tons, T, of coal in a bin whose dimensions in feet are l, w, and h, is

$$T = \frac{3lwh}{94}.$$

1.73. HOME. The cost, C, of using any electrical appliance is given by the formula

$$C = \frac{IEtr}{1000},$$

where I is the current used expressed in amperes, E the voltage of the current (in most homes this is 110 volts), t the time in hours the appliance is in use, and r the rate per kilowatt hour.

Sometimes it is more direct to use the formula

$$C = \frac{Wtr}{1000},$$

where W is the number of watts used. Since $W = EI$, the two formulas are seen to be the same.

1.74. Home. The number of units of heat, H, given out in t seconds by an electric heater which uses I amperes of current at E volts is

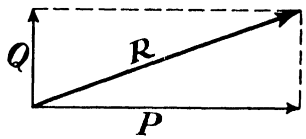
$$H = \frac{IEt}{1050}.$$

1.75. Mechanics. The formula used in the computation of the contact surface of a spherical steel ball and a flat steel plate is

$$d = 1.76\sqrt[3]{\frac{PD}{E}},$$

where d is the diameter of the minute contact circle, P the pressing force, E the modulus of elasticity of steel, and D the diameter of the ball.*

1.76. Mechanics. The Pythagorean Theorem is brought into use when expressing the resultant of two forces at right angles to each other. This is a special case, but in other more general cases the trigonometric relations are involved. In the figure if P is a force acting in the horizontal direction and Q is a force acting in the vertical direction, the resultant, R, is determined by the relation



$$R = \sqrt{P^2 + Q^2}.$$

1.77. Mechanics. If several forces are acting in a plane, it is sometimes desirable to replace them by their resultant force, R. This may be done by resolving them into two components, one along the x-axis and the other along the y-axis. Then the resultant is

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}.$$

1.78. Mechanics. To combine a system of non-coplanar forces each should be resolved into its x-, y-, and z-components. These

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components may be added algebraically, and the resultant found:

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}.$$

1.79. Mechanics. The formula used for finding the horsepower of a steam engine is

$$H = \frac{PLAN}{33{,}000},$$

where P is the pressure on the piston in pounds per square inch, L the length of the piston stroke in feet, A the area of the piston in square inches, and N the number of strokes per minute.

1.80. Mechanics. The indicated horsepower, HP, of a steam engine is given by the formula

$$HP = \frac{fPLAN}{33{,}000},$$

where f is a constant depending on the type of engine, P mean effective pressure in pounds per square inch, L length of stroke in feet, A area of piston in square inches, and N number of strokes per minute.

1.81. Mechanics. The formula for the length of a steel bar after expansion due to a rise in temperature is

$$L' = Le + (T' - T) + L,$$

where L is length at temperature T before expansion, L' length at temperature T' after expansion, and e the coefficient of expansion.

1.82. Mechanics. The crushing weight in pounds for cylindrical columns of diameter d inches and length l feet is:

wrought iron $\quad P = 299{,}600 \dfrac{d^{3.55}}{l^2}$

cast iron $\quad P = 98{,}920 \dfrac{d^{3.55}}{l^{1.7}}.$

1.83. Mechanics. In Diesel engines the oil is burned by a heat that is produced through compression of air in the cylinders. If

P_1 and V_1 are the pressure and the volume, respectively, before compression; and P_2 and V_2 the pressure and the volume after compression, then

$$\frac{P_2}{P_1} = \left(\frac{V_1}{V_2}\right)^{1.32}.$$

1.84. Mechanics. The moment of force, M, about a point is given by the formula

$$M = Fd,$$

where F is the force and d the length of the perpendicular from the point to the line of action of the force.

1.85. Mechanics. The approximate length, L, of a crossed belt which passes over two wheels whose radii are r_1 feet and r_2 feet, respectively, and which are l feet apart is

$$L = \frac{27}{8}(r_1 + r_2) + 2l.$$

1.86. Mechanics. The safe working strength, S, of a leather belt which is w inches wide and t inches thick is

$$S = 300wt \text{ pounds}.$$

1.87. Mechanics. The speed, S, of a belt in feet per minute is

$$S = \frac{\pi dr}{12},$$

where d is the diameter of the pulley in inches and r the number of revolutions it makes per minute.

1.88. Mechanics. When w is the angular velocity of a particle moving in a circular path of radius r, then the straight line, linear, velocity v is given by $v = rw$.

1.89. Mechanics. The power, P, in foot-pounds per minute done by an engine in t minutes, when working at a constant rate, is

$$P = \frac{w}{t},$$

where w is the amount of work done.

1.90. MECHANICS. A steel tape which has been standardized at a given temperature will remain that length at that temperature. Because of expansion the tape will change length for other temperatures, and consequently corrections must be made. The temperature corrections in feet for a steel tape of length l feet standardized at t_s degrees Fahrenheit and used at t degrees Fahrenheit is $.00000645\ l(t - t_s)$.

1.91. MECHANICS. Steel tapes are also standardized to stand a certain amount of pull. When the pull is heavier, a correction will have to be made in the length measured. For a steel tape of length l feet and cross-sectional area A square inches, standardized for a pull of p_s pounds along a floor and used under a pull of p pounds, the pull correction is equal to

$$\frac{l(p - p_s)}{28{,}000{,}000A}.$$

1.92. MECHANICS. The depth, d, in feet, of the sag of an overhead trolley wire of length l_1 feet which connects two poles l_2 feet apart is

$$d = \sqrt{\frac{3l_2(l_1 - l_2)}{8}}.$$

1.93. MEDICINE. A close approximation to the normal weight of an individual over 5 feet in height is given by the formula

$$w = 5.5(h - 60) + 110,$$

where w represents the number of pounds in the individual's weight, and h his height in inches.

For individuals less than 5 feet tall the formula becomes

$$w = 110 - 5.5(60 - h).$$

1.94. MEDICINE. The total number of hours, h, of sleep which a child should have in one day is

$$h = 8 + \frac{18 - a}{2},$$

where a is the child's age in years.

1.95. Medicine. The blood pressure, P, of a person should be

$$P = 110 + \frac{a}{2},$$

where a is his age in years.

1.96. Medicine. The healing of a certain type of wound follows the formula

$$A = 107 \times 10^{-\,0221t}$$

where A is the area in square centimeters of the wound after t days have elapsed.

1.97. Physics. The focal distance f of a lens may be determined by measuring the distance d from some object to the lens and the distance i of its image from the lens, and substituting in the formula

$$\frac{1}{f} = \frac{1}{d} + \frac{1}{i}.$$

1.98. Physics. The steam pressure per square inch allowed by the U. S. Steamboat Inspection Rules on Boilers is given by the formula

$$P = \frac{Tt}{6r},$$

where P is steam pressure in pounds per square inch, T tensile strength of the material of the boiler in pounds per square inch of cross section, t thickness of the material in inches, and r radius of the boiler in inches.

1.99. Physics. The number of degrees, T, which the sun loses in heat for every t million years is $T = 12{,}000e^{-.0868t}$.

1.100. Physics. The formula for the intensity, I, of light at d feet from a source of c candlepower is $I = \frac{c}{d^2}$.

1.101. Physics. The velocity of sound is approximately 1093 feet per second. Most watches tick at the rate of about 5 ticks per second. Hence, in approximately 25 ticks of your watch sound will travel one mile. You can use these data to find the distance away of a flash of lightning or an explosion. When you see a flash of lightning, count the number of ticks your watch makes before you hear the thunder. If n is the number of ticks and d is the distance in miles, then $d = \frac{n}{25}$.

1.102. Physics. The formula for the balance of a lever is

$$L_1 W_1 = L_2 W_2,$$

where W_1 and W_2 are two suspended weights and L_1 and L_2 are their distances from the fulcrum.

1.103. Physics. The formula used in radio work to change kilocycles to wave length is

$$W = \frac{300{,}000}{k},$$

where W is the wave length in meters and k the number of kilocycles.

1.104. Physics. Concrete expands when it gets hot and contracts when it cools. This must be taken into consideration in the construction of highways, sidewalks, streets, etc. The increase in length, I, for which provision must be made in the construction of a given length, l, of highway, for example, at a given temperature, t, is

$$I = .000012l(T - t),$$

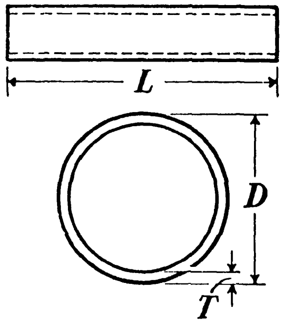
where T is the maximum temperature likely to occur at any time in that particular locality.

Example: For how much increase in length must provision be made in constructing one mile of highway in a temperature of 70° if the maximum likely temperature is 110°?

$$\begin{aligned} I &= .000012(5280)(110 - 70) \\ &= 2.54 \text{ feet.} \end{aligned}$$

1.105. Physics. Glass tubing is usually purchased in certain fixed lengths for various values of outside diameter. In order to check quickly the wall thickness of the tubing, the following relationship is utilized, involving the use of the weight of the glass (which can be easily obtained). It is assumed that the outside diameter has been checked to satisfaction.

Volume of tubing:



$$V = \pi(D/2)^2L - \pi\frac{(D-2T)^2}{2}L$$
$$= \frac{\pi LD^2}{4} - \frac{\pi L(D^2 - 4DT + 4T^2)}{4}$$
$$= \frac{\pi L}{4}(D^2 - D^2 + 4DT - 4T^2)$$
$$= \pi LT(D - T)$$

Weight of tubing: The density of pyrex glass,* e.g., is 36.7 grams per cubic inch; therefore, the weight of 1 inch of pyrex tubing is

$$W = 36.7V$$
$$W = 36.7\pi T(D - T)$$
$$W = 115T(D - T) \text{ (in grams)}$$

Therefore, if outside diameter and length are known, the wall thickness can be checked by comparing the actual weight with that determined in the above equation.†

1.106. Radio. The inductance, L, of a radio circuit is

$$L = \frac{1}{4\pi^2N^2C},$$

where N is frequency and C capacity.

1.107. Radio. The length, w, of a radio wave is

$$w = 1885\sqrt{LC},$$

where L is inductance and C capacity.

* The density of glass varies from 36.7 grams per cubic inch for pyrex glass to 49.7 grams per cubic inch for lead glass.

† Used by permission of R. C. A. Manufacturing Co.

FRACTIONAL EQUATIONS 2.01–2.05

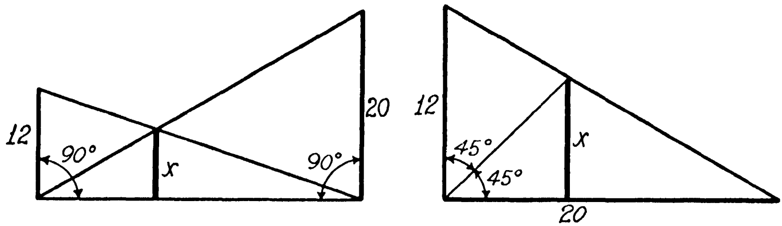
2.01. Electricity. When two resistances, of r_1 and r_2 ohms each, are connected in parallel, the combined resistance, R, is given by

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}.$$

Find the combined resistance of two resistances of 12 ohms and 20 ohms, respectively, when connected in parallel. The fractional equation to be solved is

$$\frac{1}{R} = \frac{1}{12} + \frac{1}{20}.$$

Two graphical methods of solving an equation of this type are presented in the accompanying figure.



2.02. Electricity. Two condensers of 12 microfarads and 20 microfarads, respectively, are connected in series. Find the resulting capacity. This problem is the same as Al. 2.01.

2.03. Physics. If a lens has an image distance of 12 feet and an object distance of 20 feet, find its focal length. This problem is the same as Al. 2.01. (See also Al. 1.98.)

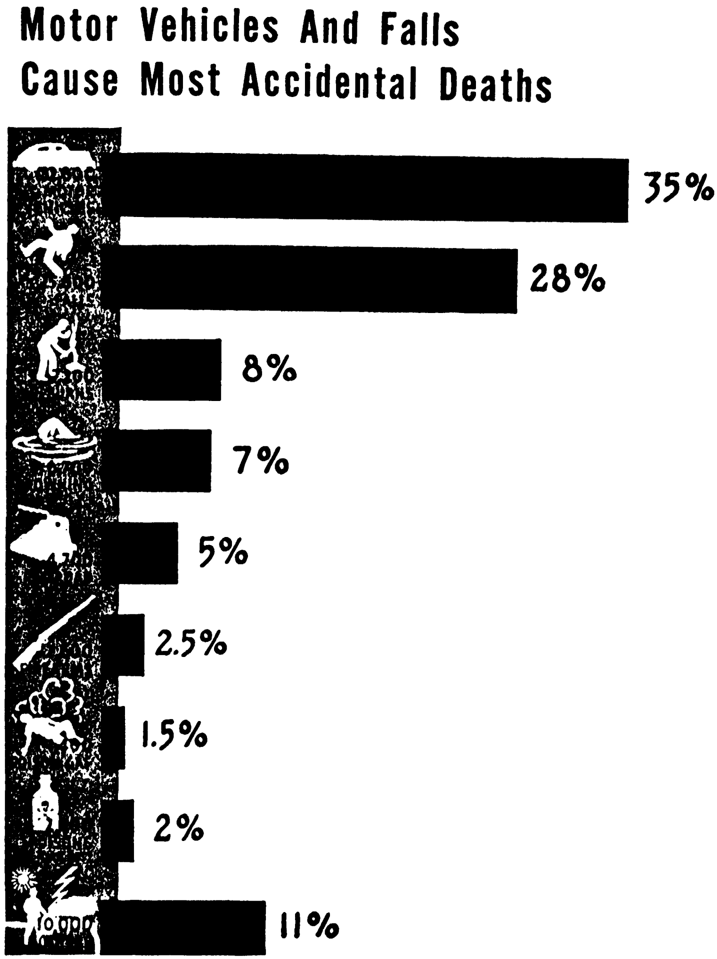
2.04. Work. All work problems which call for the results of the combined efforts of two laborers are of the type of Al. 2.01.

2.05. Work. Extensions of the problem may be made to include the combined efforts of any number of working units.

GRAPHS 3.01–3.26

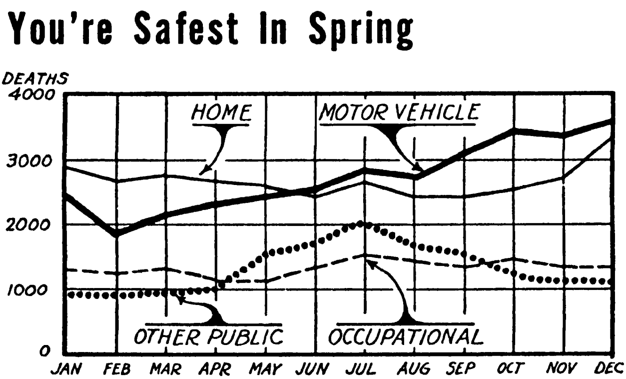
Graphs are very effective in presenting facts in a concise but informational way. An important illustration of the use of graphs is the pamphlet *Accident Facts* published in June of each year by the National Safety Council, Inc. Material for applications Al. 3.01 through Al. 3.09 is taken from the 1940 edition of *Accident Facts* and is used with the permission of the National Safety Council.

3.01. Accident Facts. The bar graph shows the number and per cent of accidental deaths due to each different cause during the year 1939.*



*Adapted from *Accident Facts, 1940*. The pictorial original is in two colors.

3.02. Accident Facts. An interesting multiple-line graph is the one following, which shows the monthly distribution of accidental deaths. From this graph can be made a comparison of the total number of accidental deaths from month to month, and also a comparison between the different classes of accidental deaths. How many different significant comparisons can you read from the graph?*

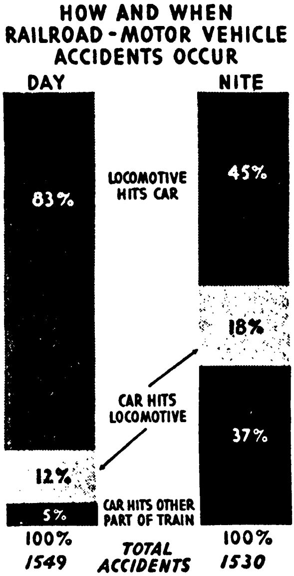


3.03. Accident Facts. An effective use of the 100% bar graph for comparison purposes is shown on page 127. Note how the parallel use of the two bars enables one to make additional comparisons.

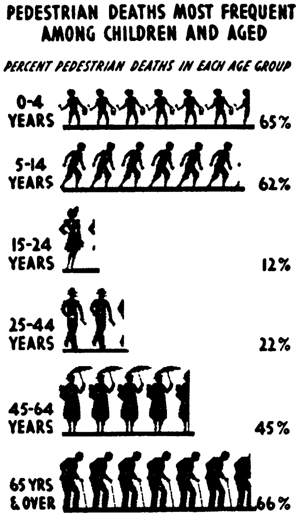
3.04. Accident Facts. In 1939 the distribution of accidental deaths among the different industries was as follows: agriculture, 27%; trade and service, 23%; construction, 17%; transportation and public utilities, $11\frac{1}{2}$%; manufacturing, $11\frac{1}{2}$%; mining, quarrying, oil and gas wells, 10%. Draw a 100% bar graph of this information.

3.05. Accident Facts. The total number of accidental deaths in the different industries was 15,500. Use the per cents given in Al. 3.04, above, to find the number of deaths in each industry. Draw a bar graph to represent these facts. Do you think this bar graph or the 100% bar graph is better for comparison purposes? What are the main advantages and disadvantages of each type of graph?

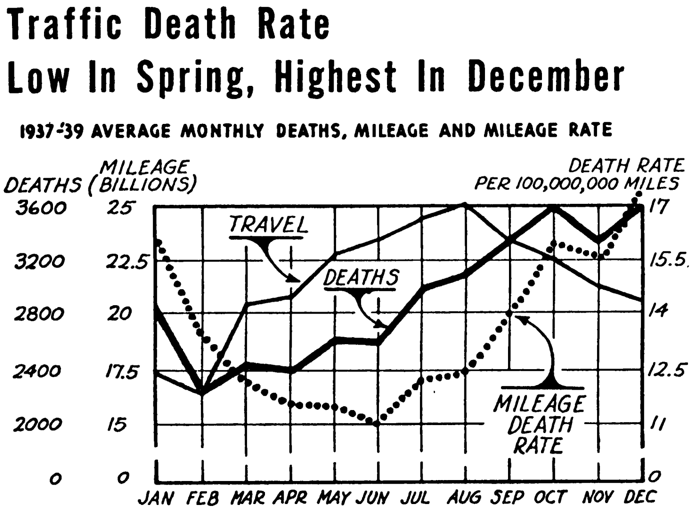
* Adapted from *Accident Facts, 1940*. The original is in two colors.



Al. 3.03*



Al. 3.06*



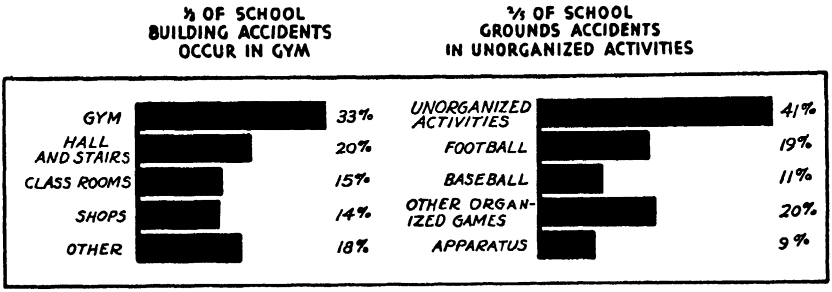
Al. 3.07*

* Graphs adapted from *Accident Facts, 1940*. The pictorial originals are in two colors.

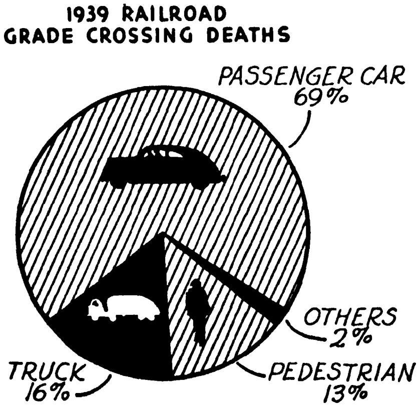
3.06. Accident Facts. A good example of the pictorial graph is the one in the figure on page 127, which shows the distribution of pedestrian deaths by age groups.

3.07. Accident Facts. Another interesting use of the multiple-line graph is shown on page 127. Note the three different vertical scales and how they are indicated. Note also the break in the graph to call attention to the fact that the zero- or base-line is out of relative position.

3.08. Accident Facts. Valuable information concerning accidents among school children is shown in this figure.*



3.09. Accident Facts. An excellent example of the use of the circle graph is given in this figure.*



* Graphs adapted from *Accident Facts, 1940*. The pictorial originals are in two colors.

3.10. Automobile. The minimum stopping distance, d, for an automobile traveling at the rate of r miles per hour is given by the formula $d = .045r^2 + 1.1r$. (See Al. 1.01.) Read from the graph on page 130 the minimum distances for any speed up to 100 miles per hour.*

3.11. Automobile. A car moving at the rate of r miles per hour hits a stationary object with a force equivalent to that which it would have upon falling from a building s feet high, where $s = .034r^2$. (See Al. 1.03.) Draw a graph of this formula.

3.12. Automobile. On the average, a pedestrian can walk about 5 feet in one second, while an automobile traveling at the rate of 25 miles per hour will go 37 feet in one second; at 30 m.p.h., 44 ft. per sec.; at 35 m.p.h., 51 ft. per sec.; at 40 m.p.h., 59 ft. per sec.; at 45 m.p.h., 66 ft. per sec.; at 50 m.p.h., 74 ft. per sec.; at 55 m.p.h., 81 ft. per sec.; at 60 m.p.h., 88 ft. per sec. Draw a bar graph to get a clear comparison of these rates of travel.

3.13. Aviation. Draw a graph of the formula $p = .0005v^2$ given in Al. 1.11.

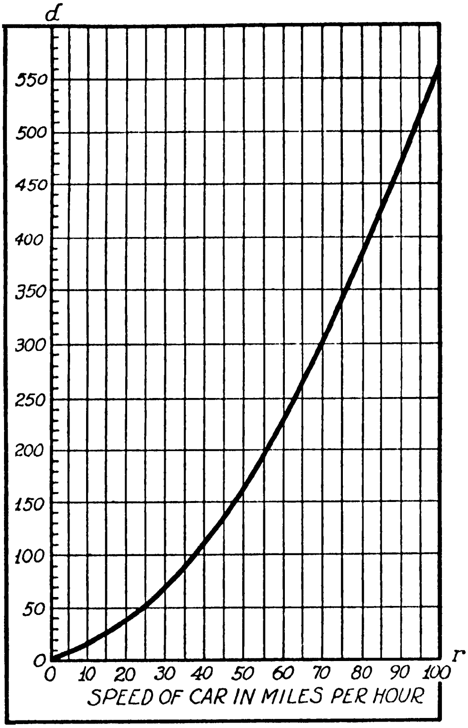
3.14. Aviation. The records for airplane speed in ten-year periods since 1903, the date of the first airplane flight, are: 1903, 30 miles per hour; 1913, 126 m.p.h.; 1923, 266 m.p.h.; 1933, 424 m.p.h.; 1941, 458 m.p.h. Draw a bar graph showing these record speeds.

3.15. Aviation. Information concerning airplane accidents can be made into an interesting pictorial graph from the following data from *Accident Facts, 1940*.

Types of Airplane Accidents

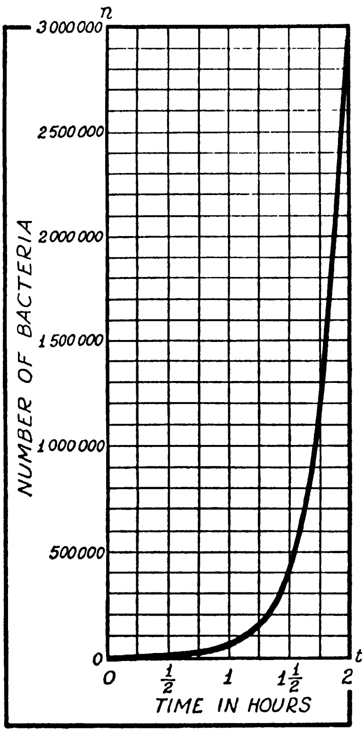
	scheduled domestic flying	*private flying*
Collisions	9%	5%
Take Off (Inc. taxying)	31%	26%
Landing	36%	38%
Forced landing	12%	15%
Spin or stall	1%	11%
Other types	11%	5%

* From the formula: 10 miles per hour, 15.5 feet stopping distance; 20—40; 30—73.5; 40—116; 50—167.5; 60—228; 70—297.5; 80—376; 90—463.5; 100—560.


d
550
500
450
400
350
300
250
200
150
100
50
0
0 10 20 30 40 50 60 70 80 90 100
r
SPEED OF CAR IN MILES PER HOUR

Al. 3.10

Al. 3.16


n
3 000 000
2 500 000
2 000 000
1 500 000
1 000 000
500 000
0
NUMBER OF BACTERIA
0 ½ 1 1½ 2
t
TIME IN HOURS

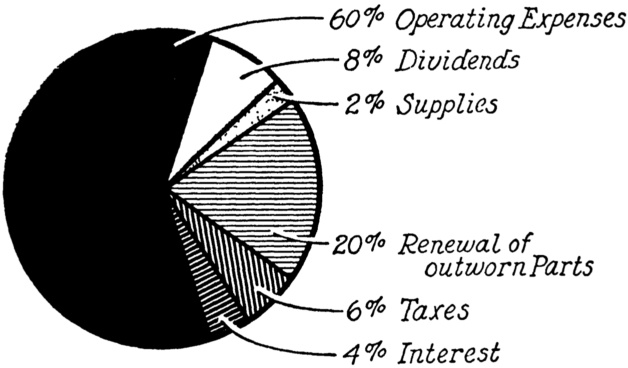
Causes of Airplane Accidents

	scheduled domestic flying	*private flying*
Personnel	35%	55%
Engine and plane	30%	25%
Weather	15%	7%
Airport and terrain	12%	8%
Others	6%	4%
Undetermined or doubtful	2%	1%

3.16. BIOLOGY. Compare the values of n given for different values of t in the graph in the figure on page 130, with those given by the formula of Al. 1.16.*

3.17. BUILDING. Draw graphs of the formulas given in Al. 1.18. Show how they might be used for reference by carpenters.

3.18. BUSINESS. Large corporations usually issue statements showing how their income is spent. A circular graph showing how one corporation spent its income is reproduced below.



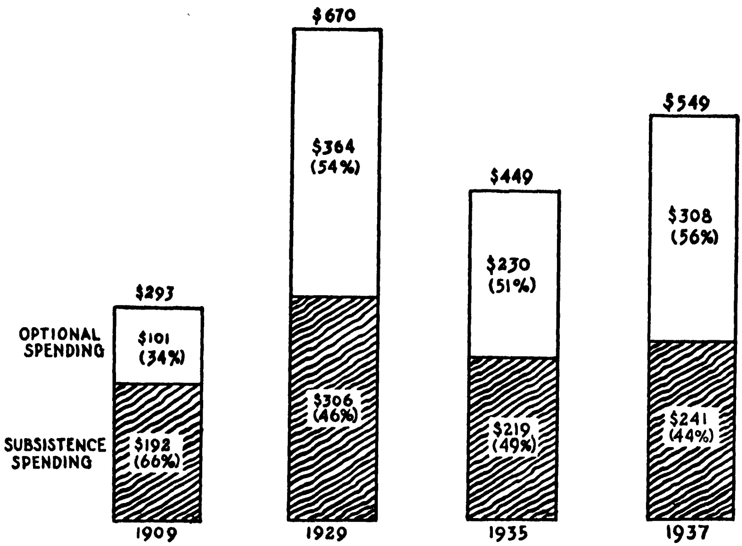
3.19. BUSINESS. Draw a bar graph of the distribution, shown in the accompanying table, of men's shirts sold in one year's time by a department store.

Size	*No. of Shirts*	*Size*	*No. of Shirts*
13.5	10	16	70
14	18	16.5	23
14.5	50	17	12
15	75	17.5	10
15.5	90		

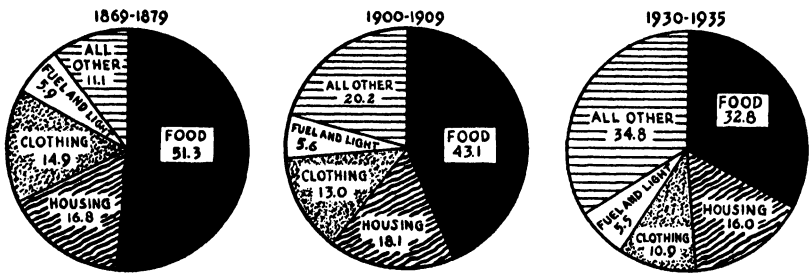
* From the formula: ¼ hr.—2718; ½ hr.—7389; ¾ hr.—20,090; 1 hr.—54,600; 1¼ hr.—148,400, 1½ hr.—403,400; 1¾ hr.—1,097,000; 2 hr.—2,981,000.

Consumption and Living Standards

CONSUMPTION EXPENDITURES PER CAPITA POPULATION
IN DOLLARS



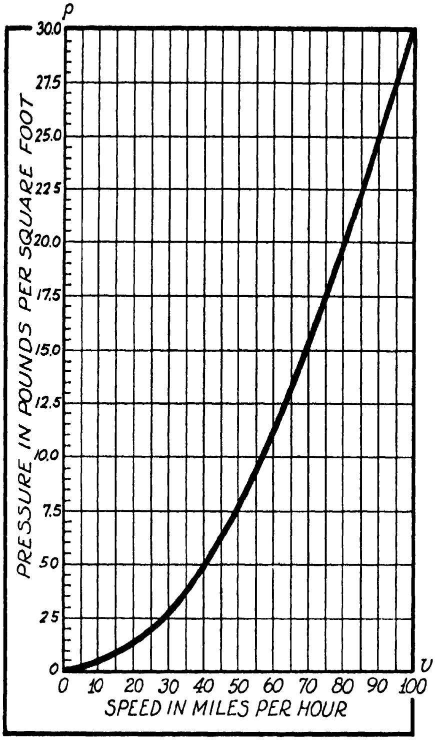
CHANGES IN EXPENDITURES OF WAGE EARNERS' FAMILIES
PERCENTAGE DISTRIBUTION



"Subsistence Spending" indicates the typical expenditures of low-income groups. "Optional Spending" is that portion of the yearly spending per capita for items whose choice is less restricted and which usually are not necessaries. The chart shows the variations by percentages in such spending in the different eras.

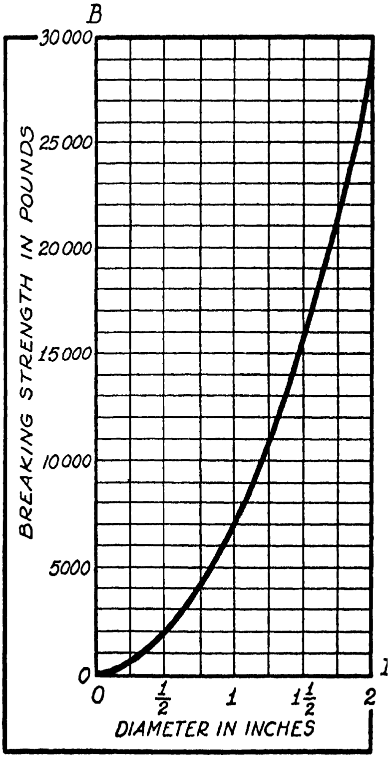
National Industrial Conference Board chart (redrawn)

Al. 3.20


p
30.0
27.5
25.0
22.5
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0
v
0 10 20 30 40 50 60 70 80 90 100
PRESSURE IN POUNDS PER SQUARE FOOT
SPEED IN MILES PER HOUR

Al. 3.21

Al. 3.24


B
30 000
25 000
20 000
15 000
10 000
5000
0
l
0 1/2 1 1 1/2 2
BREAKING STRENGTH IN POUNDS
DIAMETER IN INCHES

3.20. Business. A combined use of the 100% bar graph and the circle graph is shown on page 132.

3.21. Engineering. Use the graph on page 133 to find the pressure in pounds per square foot of a wind blowing with a speed of v miles per hour. Compare the pressure for different speeds with that given by the formula in Al. 1.57.*

3.22. Engineering. When engineers build bridges across, or tunnels under, rivers, or when navigators steer boats along the rivers, they need to know the shallow parts and the deep parts. Profile charts of the river bed and contiguous shore lines are helpful in recording such information. The figure on page 135 shows a profile chart for a river bed and its neighboring shore. The horizontal axis, or zero-line, is the water level; the vertical axis shows feet above and below the water level.

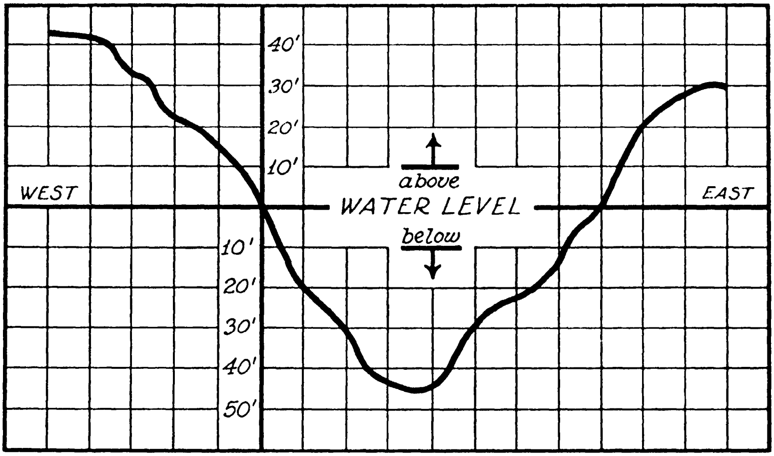
3.23. Engineering. In many engineering problems it is necessary to find irregular areas and volumes, centroids, moments of inertia, etc. A graphic method frequently used is that illustrated on page 135. The problem here was to represent the area lying between the curve $y^2 = 4x$, and the line $y = x$.

3.24. Mechanics. The graph on page 133 shows the breaking strength B in pounds for manila rope of diameter D inches. Write the formula which expresses B as a function of D.†

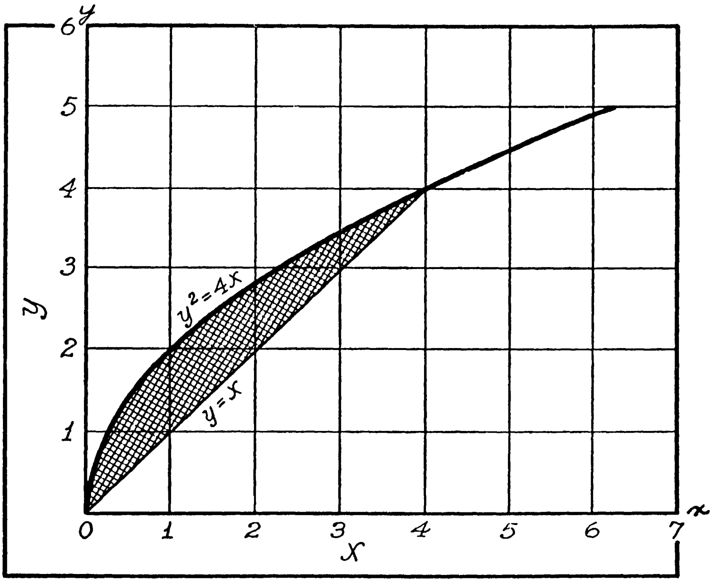
3.25. Mechanics. To locate a hidden gun, take two points at a known distance apart and determine the time at which the sound from the firing of the gun is heard at each. Do the same for two other points at a known distance apart, not necessarily the same distance as in the first case. Since the difference in time is constant for each pair of points, two hyperbolas can be determined with each pair of points as foci. The intersection of the hyperbolas will give the location of the gun.

* From the formula: 5 miles—.1 lb. pressure; 10—.3; 15—.7; 20—1.2; 25—1.9; 30—2.7; 35—3.7; 40—4 8; 45—6.1; 50—7.5; 55—9.1; 60—10.8; 65—12 75; 70—14.7; 75—16.9; 80—19.2; 85—21.7; 90—24.3; 95—27.1; 100—30.

† From the formula: $\frac{1}{8}$ in diameter—111 lb.; $\frac{1}{4}$ in.—444; $\frac{3}{8}$ in.—999, $\frac{1}{2}$ in.—1775, $\frac{5}{8}$ in.—2774; $\frac{3}{4}$ in.—3994, $\frac{7}{8}$ in.—5439; 1 in.—7100; $1\frac{1}{8}$ in.—8986, $1\frac{1}{4}$ in.—11,094; $1\frac{3}{8}$ in.—13,424; $1\frac{1}{2}$ in.—15,975; $1\frac{5}{8}$ in.—18,740; $1\frac{3}{4}$ in.—21,744; $1\frac{7}{8}$ in.—24,949, 2 in.—28,400.



Al. 3.22



Al. 3.23

3.26. Mechanics. Draw the graph for the data given in the table:

Volume	200	160	100	80	50	40	25	20
Pressure	20	25	40	50	80	100	160	200

These data represent a law known as Boyle's Law. Using v for volume and p for pressure, write the formula which expresses Boyle's Law for this particular set of data.

LINEAR EQUATIONS 4.01

4.01. Many of the formulas listed in applications Al. 1.01 to Al. 1.108 involve at least one variable linearly. Substitution in these formulas for the purpose of finding such variables results in the solution of linear equations. These are suggestive of the varied applications of formulas and equations to science, industry, and other life activities.

PROGRESSIONS 5.01–5.26*

5.01. AUTOMOBILE. Mr. Jones estimates that, on the average, his automobile depreciates in value at the rate of 25% per year. If his automobile was worth $1100 when new, find its value at the end of 5 years.

5.02. BIOLOGY. "Galton's Law of Heredity states that the influence of ancestors on an individual may be compared in the following way. The parents' influence may be said to be $\frac{1}{2}$; that of all grandparents, $\frac{1}{4}$; that of all great-grandparents, $\frac{1}{8}$; etc. What is the comparative influence of the eighth generation back? What is the comparative influence due to the last eight generations?"†

5.03. BIOLOGY. It is said that the number of bacteria in milk doubles every three hours. What is the formula for finding the number of bacteria in milk after n hours?

5.04. BUSINESS. If P dollars is invested at the rate of r per period for n periods, then the compound amount $A = P(1 + r)^n$.

5.05. BUSINESS. The present value, P, of the compound amount, A, for n periods at the rate, r, per period is

$$P = \frac{A}{(1 + r)^n} = A(1 + r)^{-n}.$$

* Items 5.09–5.11, items 5.14 to 5.18, and item 5.26 are arithmetic; the others in this section are geometric.

† H. R. Cooley, D. Gans, M. Kline, and H. E. Wahlert, *An Introduction to Mathematics*, p. 349. Houghton Mifflin Company, Boston, 1937.

5.06. Business. If a man invests P dollars at the beginning of each year for t years at $r\%$ interest, how much money will he have at the end of the t years?

5.07. Business. A person begins on January 1 to save the price of a package of cigarettes each day and then deposits the amount in a savings bank on the first day of each subsequent month. Assuming that there are 30 days to the month and that the bank pays 2% interest compounded semi-annually, how much money will this person have to his credit at the end of 10 years?

5.08. Business. In the constant-percentage method of calculating depreciation, the amount of depreciation at the end of any period is $i\%$ of the initial value at the beginning of the period. If D_n represents the depreciation value at the end of n periods of an initial value, V, when the depreciation rate per period is $i\%$, then $D_n = V(1 - i)^n$.

5.09. Business. In a certain school system the beginning salary is $1200 and there is an increase of $110 a year until the maximum salary of $2740 a year is reached. How long will it take a teacher to reach the maximum salary schedule?

5.10. Business. A man makes a purchase for $400, agreeing to pay $100 down and the balance in 10 monthly payments with interest on the unpaid balance at the rate of 6%. Find the total amount paid.

5.11. Business. In the variable-percentage method of calculating depreciation, each depreciation period is assigned a serial number. The ratio of each serial number to the sum of the serial numbers is then expressed as a rate per cent. These rates are then reversed in order to give the variable rate of depreciation.

Example: A piece of machinery costing $10,000 has a scrap value of $500 after 10 years. Make a table showing the variable rates of depreciation, if depreciation is figured on a semi-annual basis.

5.12. Engineering. Preferred numbers are series of numbers which are to be used for the standardization of sizes, ratings, etc., of technical products. The first number of any series is taken as 10, and then the other numbers of the series are obtained by multiplying 10 by consecutive integral powers of the constant factor for the series. The constants adopted are as follows:

For the "5" series, constant = 1.60 = $\sqrt[5]{10}$, approx.

For the "10" series, constant = 1.25 = $\sqrt[10]{10}$, "

For the "20" series, constant = 1.12 = $\sqrt[20]{10}$, "

For the "40" series, constant = 1.06 = $\sqrt[40]{10}$, "

For the "80" series, constant = 1.03 = $\sqrt[80]{10}$, "

Thus the "20" series is (in round numbers):

10, 11.2, 12.5, 14, 16, 18, 20, 22.4, 25, etc.*

5.13. Mechanics. A certain pump will remove at each stroke $\frac{1}{4}$ of the air from a container. How much air will remain in the container after six strokes of the pump?

5.14. Miscellaneous. What is the sum of the first *n* integers?

5.15. Miscellaneous. What is the sum of the first *n* even integers?

5.16. Miscellaneous. What is the sum of the first *n* odd integers?

5.17. Miscellaneous. How many feet of string will be required to wind around a cone if 50 turns are required and the first three turns are $\frac{1}{2}$ in., 1 in., and $1\frac{1}{2}$ in. long, respectively?

5.18. Miscellaneous. How many strokes are struck by a clock in striking the hours 1 through 12?

5.19. Miscellaneous. How many ancestors does an individual have in the six generations which immediately precede him?

5.20. Miscellaneous. A certain child has one great-great-great-grandparent living. If all of his great-great-great-grandparents were living, how many would there be?

5.21. Miscellaneous. A person wrote the number 1 on a chain letter which he sent to each of five friends. He asked each friend to send five copies of the letter after writing the number 2 on each copy of the letter. Each person receiving a copy of the letter was

* Adapted from B. Lester, *Applied Economics for Engineers*, pp. 128–132. John Wiley and Sons, New York, 1939.

asked to follow the same directions. Assuming that the procedure were legal and that the chain would remain unbroken until every individual in the United States had received a copy of the letter, what would be the number written on the last set of letters? How many letters in this last set?

5.22. Miscellaneous. It is related that an Indian prince offered the inventor of the game of chess any reward he desired. The inventor replied that he would merely ask for 1 grain of wheat for the first square of the chess board, 2 for the second, 4 for the third, 8 for the fourth, and so on for the 64 squares. Counting 10,000 grains as approximately a pint and 64 pints to one bushel, how many bushels of wheat are required for the prize?*

523. Miscellaneous. It is related that in 1932 two men made a bet of $2.50 that the capitol building of the state of Louisiana would not be standing in the year 2432. They placed the $2.50 in a bank to draw interest at 4% compounded semi-annually, and signed a contract which specified that the bank should pay the compound amount to the heirs of the winner. How much will be due these heirs at the end of the 500 years?

5.24. Music. "The fundamental frequencies of the 12 notes of the equally tempered piano scale are in geometric progression. Find the common ratio if the frequency of the 13th note is double that of the first."†

5.25. Music. "In the equally tempered scale of 12 notes used on the modern piano, the ratio of the fundamental frequencies of any two consecutive notes is 1.059. Find the fundamental frequency of A, which is the ninth note after C, if the frequency of C is 256 vibrations per second."†

5.26. Physics. When a body falls from rest, it drops approximately 16.1 ft. the first second, 48.3 ft. the second second, 80.5 ft. the third second, and so on. A baseball was dropped from the top of Washington Monument, 550 ft. high; about how many feet per second was the baseball traveling when it hit the ground?

* Adapted from Harold T. Davis, *College Algebra*, p. 75. Prentice-Hall, New York, 1940.

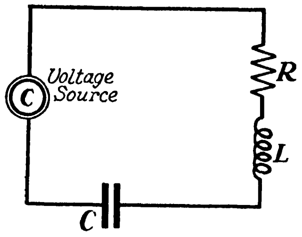
† H. R. Cooley, D. Gans, M. Kline, and H. E. Wahlert, *Introduction to Mathematics*, p. 349. Houghton Mifflin Co., Boston, 1937.

QUADRATIC AND OTHER EQUATIONS 6.01–6.05

6.01. Many of the formulas of Al. 1.01 to Al. 1.108 involve variables raised to the second and higher powers, as well as variables raised to fractional powers. Substitution and solution for different variables will give rise to higher degree equations of many types.

6.02. Aviation. The greatest distance in miles a balloonist or an aviator can see when at a height of h miles is given by the formula $d^2 = h^2 + 8000h$.

6.03. Electricity. The adjoining electric circuit shows a resistance of R ohms, an inductance coil of L henries, and a condenser of C farads connected in series. In the electrical problem of the determination of the amount of current in this circuit, it is necessary to solve the equation



$$Lm^2 + Rm + 1/c = 0$$

for m. Express m as a function of L, R, and C.

6.04. Engineering. Cox's formula for measuring the velocity of water escaping from a reservoir through a horizontal pipe is:

$$4v^2 + 5v - 2 = \frac{1200hd}{l},$$

where v is the velocity of the water in feet per second, h is the height in feet of the water in the reservoir, d is the diameter of the pipe in inches, and l is the length of the pipe in feet.

6.05. Engineering. For sandy soil with a grade angle of 20°, the thickness t and the height h, each measured in feet, of a rectangular retaining wall are connected by the homogeneous quadratic equation

$$t^2 + .19th - .18h^2 = 0.$$

For loam with the same grade angle the relation is

$$t^2 + .14th - .13h^2 = 0.^*$$

VARIATION AND PROPORTION 7.01–7.21

7.01. Astronomy. The magnitudes of the stars are rated according to measurements of light intensities which vary according to a geometric progression whose ratio is $\sqrt[5]{100}$.

Example: If a sixth magnitude star shines with intensity I, then the intensity of a star of fifth magnitude is $\sqrt[5]{100}I$, fourth magnitude $\sqrt[5]{(100)^2}I$, etc.

7.02. Electricity. The number of amperes, I, of electric current required to melt a fuse wire of diameter d inches varies as $d^{\frac{3}{2}}$, or $I = kd^{\frac{3}{2}}$.

7.03. Electricity. Transformers change the voltage and current in the approximate ratio of

$$\frac{I_1}{I_2} = \frac{N_2}{N_1} = \frac{E_2}{E_1},$$

where N_1 and N_2 represent the number of turns of wire on the primary and secondary coils, respectively.

7.04. Electricity. When an electric current is turned off, it does not vanish instantly but decreases according to the formula $I = I_0e^{-kt}$, where I_0 is the light intensity at the time current is cut off, I is the intensity after t seconds, and k is a constant.

7.05. Mechanics. The wind pressure in pounds per square foot on any plane surface varies approximately as the square of the wind velocity in miles per hour, or $p = kv^2$.

7.06. Mechanics. The velocity in feet per second of a jet of liquid flowing through an orifice into the atmosphere varies as the square root of the height of the orifice measured in feet, or $v = k\sqrt{h}$.

7.07. Mechanics. If P_1, V_1, and T_1 represent, respectively, pressure, volume, and absolute temperature of a mass of air at sea

* Adapted from H. L. Rietz and A. R. Crathorne, *College Algebra*, p. 86. Henry Holt and Company, New York, 1939.

level and P_2, V_2, T_2, the corresponding values of the same mass of air at any given height, then

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2}.$$

This relation may be used to find the density of any given mass of air at any given height for which the pressure and temperature are known.

7.08. MECHANICS. If a fluid with velocity V_1 flows from a pipe of diameter D_1 into a second pipe of diameter D_2 where it flows with velocity V_2, then $D_1^2 V_1 = D_2^2 V_2$.

7.09. MECHANICS. The maximum friction possible between two bodies sliding or tending to slide, one on the other, varies as the normal pressure acting at right angles to the surface of contact of the two bodies, or $F = kN$.

7.10. MECHANICS. The horsepower necessary to propel a ship varies as the cube of the speed of the ship measured in knots, or $p = kv^3$.

7.11. MECHANICS. The number of revolutions per minute necessary to keep the balls of a ball governor h inches below the point of suspension varies inversely as the square root of h, or $n = kh^{-1/2}$.

7.12. MECHANICS. The force necessary to keep the motion of a particle following a circular path varies directly as the square of the velocity, v, and inversely as the radius, r, of the circle. $F = \frac{kv^2}{r}$.

7.13. MECHANICS. The crushing load of a square pillar of solid oak, whose cross-sectional area is a^2 and length is l, varies directly as a^4 and inversely as l^2. $L = \frac{ka^4}{l^2}$.

7.14. MUSIC. The three major chords in music are composed of tones whose vibration numbers are to each other as 4 : 5 : 6. The three minor chords are composed of tones whose vibration numbers are to each other as 10 : 12 : 15.

7.15. PHOTOGRAPHY. The length of exposure, t seconds, necessary to photograph a given subject varies as the square of the distance, d, in feet, the subject is from the source of light, i.e., $t = kd^2$.

7.16. PHOTOGRAPHY. The time, t seconds, necessary to make an enlargement of a given photographic negative varies as the area, A, of the proposed enlargement, i.e., $t = kA$.

7.17. PHOTOGRAPHY. In projecting pictures the brightness, b, of the projecting light varies directly as the distance, d, of the desired projection, i.e., $b = kd$.

7.18. PHYSICS. Atmospheric pressure, P, in pounds per square inch at a height of h feet above sea level is given by the formula

$$P = P_0 e^{-kh},$$

where P_0 is atmospheric pressure at sea level and k is a constant.

7.19. PHYSICS. A radioactive substance decomposes at such a rate that if N_1 is the initial number of atoms of the substance and N_2 the number remaining after t hours, then

$$N_2 = N_1 e^{-kt},$$

where k is a constant.

7.20. PHYSICS. Newton's law for the cooling of a hot body is

$$T = T_a + (T_b - T_a)\, e^{-kt},$$

where T_b is the temperature of body at the time it begins to cool, T_a the temperature of the surrounding cool air, T the resultant temperature after t minutes, and k a constant.

7.21. PHYSICS. The intensity, I, of a beam of light after it passes through t centimeters of a liquid which absorbs light is

$$I = I_0 e^{-kt},$$

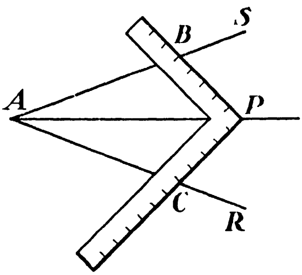
where I_0 is the intensity of the light as it enters the liquid, and k a constant.

GEOMETRY

GEOMETRY

ANGLE, BISECTOR 1.01

1.01. Carpentry.* To bisect an angle *RAS* with a carpenter's square, lay off *AB* = *AC* and place the square so that *PB* = *PC*. Then *AP* bisects the angle.



Problem: Explain. Would the method be correct if ∠*BPC* were not a right angle?

Also see Circles G. 5.07.

ANGLE, POLYHEDRAL 2.01

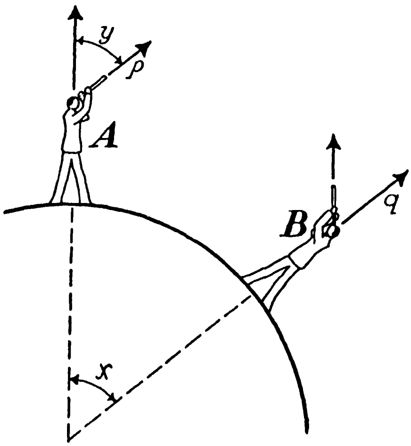
2.01. Daily Life. An excellent illustration of the use of polyhedral angles is the revolving winter door at hotels and store entrances.

ANGLES 3.01–3.21

3.01. Astronomy. The earth's circumference may be found by several methods. One method is to station two men at various parts of the earth—say, Chicago (*A*) and New Orleans (*B*)—allowing them to communicate with each other by long distance

* Adapted from F. E. Seymour, *Plane Geometry*, p. 61. American Book Company, New York, 1925.

telephone. By previous agreement both observe stars directly overhead at the same time of night. "Observer *A* at Chicago sees one group of stars directly overhead and observer *B* at New Orleans another group, for, because of the curvature of the earth, what is directly overhead for the one is not directly overhead for the other. A line drawn from *A* to the star that is directly overhead for *B*



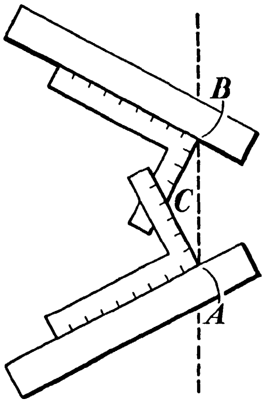
would be inclined as much to *A* as is *B*, i.e., angle x is equal to angle y, as shown in the accompanying figure. (You will note that the lines p and q are parallel, the reason being that any star observed is so many billions of times greater than terrestrial distance that the departure from parallelism defies detection.) In this way, without seeing *B*, *A* can find *B*'s inclination to himself by measuring the angle y between the star directly overhead at Chicago and the star reported to him as being directly overhead at New Orleans. He will find that this angle is approximately 12°. Along the most direct route, the distance between Chicago and New Orleans is about 830 miles."*

Since the total number of degrees in the earth's circumference is 360°, the circumference of the earth in miles by this experiment is $\frac{360}{12} \times 830$ mi. $= 24{,}900$ mi., approximately.

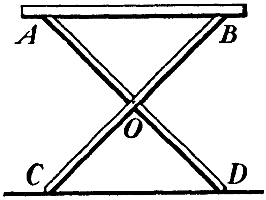
* Quotation from W. Bartky, *Highlights of Astronomy*, pp. 4–5. The University of Chicago Press, Chicago, 1935.

3.02. Carpentry.* If a carpenter wishes to mark a line that will cut two converging boards at equal angles, he places two steel squares against the boards, as shown in the drawing, and adjusts them so that $AC = BC$.

Problem: Prove that the line AB then makes equal angles with the edges of the two boards.



G. 3.02



G. 3.04

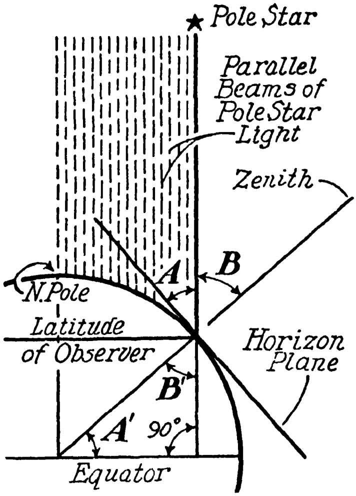
3.03. Daily Life. We regard a positive angle as being formed by the counter-clockwise rotation of the generating line. The logic in this idea is the fact that the earth, all planets, and all satellites except a very few have their revolution about the sun and their rotation on their axis in a counter-clockwise direction.

3.04. Daily Life. John built a folding table as illustrated in the figure, with $AO = BO = CO = DO$. If John wishes to lower the height of the table, can he do so merely by shortening CO and DO equal amounts, leaving the bolt through O as it was originally? Prove conclusion by geometric reasoning.

3.05. Daily Life.† Geometry students living in the northern hemisphere can determine their latitude by measuring the elevation of the pole star. That this is a fact can be demonstrated by

* Adapted from J. C. Stone and V. S. Mallory, *New Plane Geometry*, p. 100. Benj. H. Sanborn, Chicago, 1937.

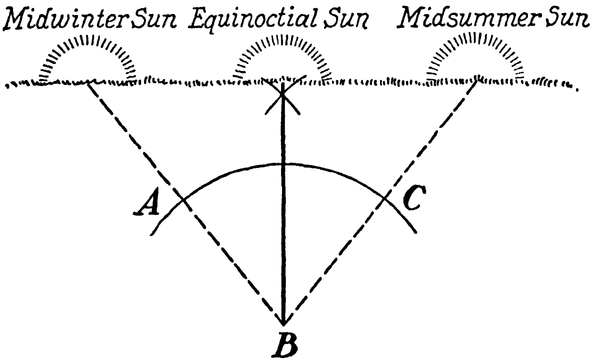
† Adapted from L. Hogben, *Mathematics for the Million*, pp. 167–168. W. W. Norton and Co., 1937.



applying theorems of plane geometry to the accompanying figure. (Prove angle A equal to angle A'.)

The latitude of students living in the southern hemisphere can be determined, by plane geometry reasoning, to correspond to the complement of the angle of elevation of the sun at noon on one of the equinoxes.

3.06. Daily Life.* Early man noticed that the sun rose at different points on the horizon during different seasons. By erecting crude stone monuments in line with the rising sun on the day the sun rose farthest north and on the day it set farthest south (we call these days the solstices) man was enabled to solve one of his first geometric problems—that of determining an east-west line. This was done by bisecting angle ABC in the figure below.



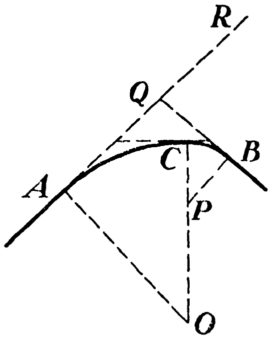
3.07. Daily Life. The fact that the earth's axis makes an angle of $23\frac{1}{2}$ degrees with the line perpendicular to the plane of the earth's orbit is basic to the explanation of our change of seasons.

* Adapted from L. Hogben, *Mathematics for the Million*, p. 51. W. W. Norton and Co., New York, 1937.

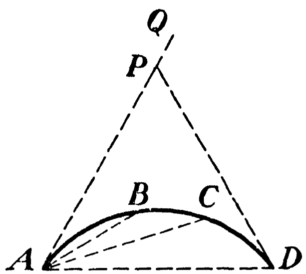
3.08. Daily Life. When a light ray passes from a less dense to a more dense medium, its path is deflected. The amount of deflection is called the angle of refraction, which is determined for various transparent mediums. The cutting of a diamond is done very carefully so that the angularity of the faces, together with the angle of refraction, will make possible the greatest amount of light being emitted from the diamond. This produces the sparkle.

3.09. Engineering.* The railroad curve *ACB* is a compound curve, composed of $\overset{\frown}{AC}$ and $\overset{\frown}{CB}$, with centers at *O* and *P*, respectively. Point *P* is on *OC*. See the figure below.

Problem: Prove that the intersection angle *BQR* equals the sum of the central angles, $\angle COA$ and $\angle BPC$. Suggestion: Draw the common tangent at *C*.



G. 3.09



G. 3.10

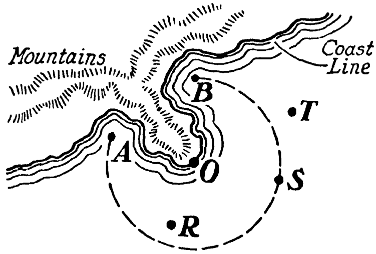
3.10. Engineering.† In railroad surveying, curves are laid out by turning off equal angles and setting stakes every 100 ft. If the curve begins at *A*, $\angle BAP$ is turned off from the tangent *AP* and *AB* is measured 100 ft.; then $\angle CAB$ is turned off and *BC* is measured 100 ft., etc., the process being continued until the curve ends in the tangent *DP* at *D*. See the figure above.

Problem: Since the curve is to be an arc of a circle, prove that $\angle BAP$, $\angle CAB$, etc., must be made equal, and that each must equal one half the central angle of a 100-ft. chord.

* Adapted from J. C. Stone and V. S. Mallory, *New Plane Geometry*, p. 270. Benj. H. Sanborn, Chicago, 1937.

† Adapted from J. C. Stone and V. S. Mallory, *New Plane Geometry*, p. 269. Benj. H. Sanborn, Chicago, 1937.

3.11. Navigation. Along seacoasts sharp promontories are often found. To safeguard shipping, buoys are placed at the sides of the promontory and a circular arc is drawn on maps indicating the limit of safety.



In the above case the arc *AB* (of circle *O*) equals 270°.

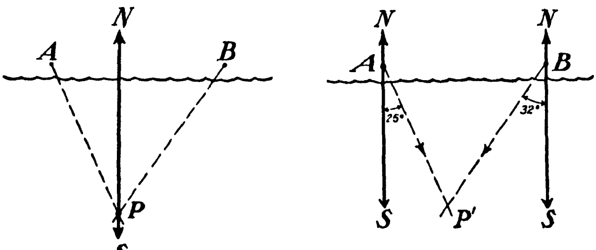
Problems: (a) If point *S* is on arc *AB*, how many degrees are in angle *ASB*?

(b) If point *T* is outside arc *AB*, is angle *ATB* less than or greater than angle *ASB*?

(c) If point *R* is inside arc *AB*, what can be said of angle *ARB*?

(d) Can you make a general statement concerning safe positions? Unsafe ones?

3.12. Navigation.* The radio compass is a device used by navigators to receive radio signals from radio stations at known locations. The compass features a dial so that the angle between the north-south line and the radio signal sending station can be determined. With this device and a map listing the radio stations, navigators can easily determine their position at sea as follows:

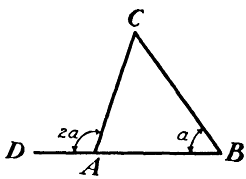


* Adapted from W. W. Strader and L. D. Rhoads, *Modern Trend Geometry*, pp. 71–72. The John C. Winston Co., Philadelphia, 1940.

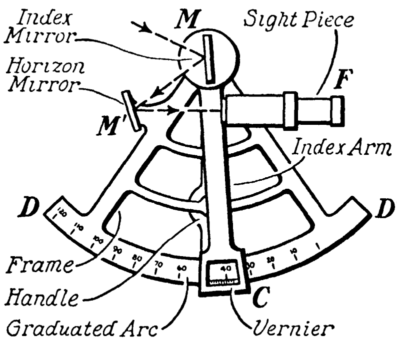
Suppose a ship is fog-bound at P. By radio compass $\angle APN = 25°$ and $\angle BPN = 32°$. The navigator takes map featuring stations A and B. Draws north-south lines through points A and B. Constructs angles of 25° and 32° as in figure, equal to $\angle APN$ and $\angle BPN$, respectively. Extends these lines until they intersect at P'. P' shows the position on the map of the fog-bound ship!

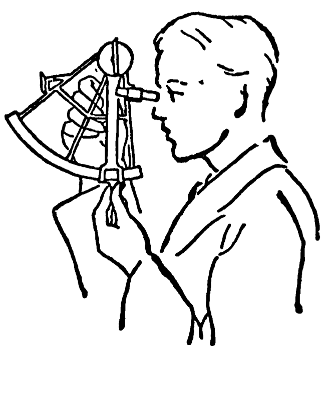
Problem: Validate this practice by geometric reasoning.

3.13. NAVIGATION. A navigator may calculate his distance from a light-house as follows: Let C represent light-house and BD course of ship. With instruments determine $\angle a$, when ship is at point B. Let ship sail on until $\angle CAD$ is twice $\angle a$; the navigator can then find length of CA, for he can obtain length of AB from his log.



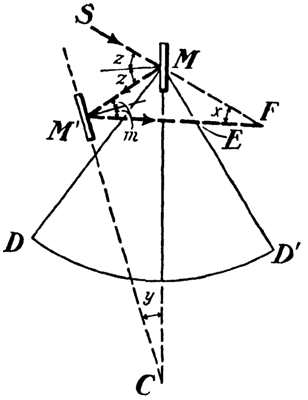
3.14. NAVIGATION. "A convenient hand instrument used for measuring angles in any plane and an indispensable aid in determining





the latitude of a ship at sea is a sextant. Its name is derived from the fact that it contains a 60° arc of a circle.

"*DD'* is a 60° arc of the circle whose center is *M*; this arc is graduated from 0° to 120°. *MC* is a movable arm, which has a mirror attached at *M* in line with *MC* and perpendicular to the plane of the sextant. Fixed at *M'* is another mirror, the 'horizon glass,' also perpendicular to the plane of the sextant and parallel to *MD'*. *E* is a small telescope fixed across *MD'* and pointed toward the mirror *M'*. Since only the lower half of this mirror is silvered, if the observer wishes to find the angle of elevation of the sun, he first sights the horizon through the unsilvered portion. Then the arm *MC* is moved so that the image of the sun is reflected from mirror *M* to mirror *M'* and thence through the telescope to the eye. The observation consists in bringing the horizon and the image of the sun into coincidence. The angle of elevation of the sun is read on the graduated arc.



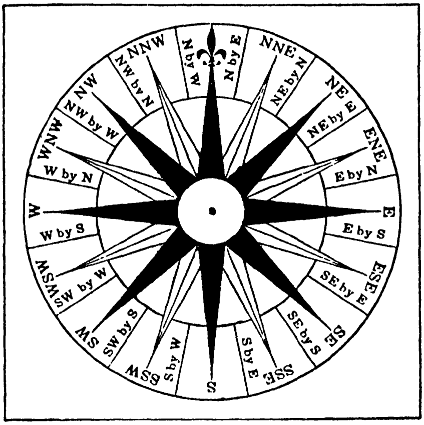
"Because it is often difficult to see the horizon, a bubble sextant is usually used. A small spirit level is attached to the tube of the telescope, and when the bubble is centered, the effect is the same as that which is produced when the horizon is sighted."*

Problem: Prove $\angle MFM' = 2\ \angle D'MC$, and hence arc *DD'* must be graduated from 0° to 120° instead of from 0° to 60°. Hint: Apply to the exterior angle of $\triangle MM'F$ at *M* and that of $\triangle MM'C$ at *M'* the theorem that an exterior angle equals the sum of opposite interior angles.

3.15. Navigation.† The Mariner's Compass used in all navigation is divided into 32 equal angles each equal to $11\frac{1}{4}°$. All bearings of points are given in terms of these commonly used angle divisions of the compass.

* A. Leonhardy, M. Joseph, and R. D. McLeary, *New Trend Geometry*, pp. 231–233. Charles E. Merrill Co., New York, 1940.

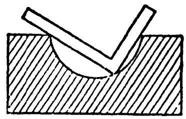
† From J. C. Stone and V. S. Mallory, *New Plane Geometry*, p. 29. Benj. H. Sanborn, Chicago, 1937.



MARINER'S COMPASS

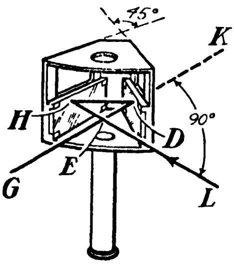
Problem: What is the angle between a buoy and a light-house if the light-house bears N.E. by N. from a ship and if the buoy has a bearing of E. by S. from the ship?

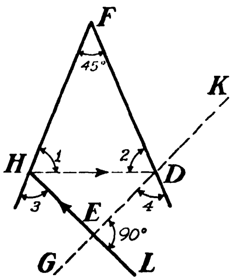
3.16. PATTERN MAKERS. Pattern makers use a try-square in testing a core-box to see whether it is a true semicircle.



Problem: What theorem in plane geometry justifies this practice?

3.17. STUDENT. "The angle mirror is a simple instrument used in laying out certain angles, and consists of two mirrors held in a





supporting frame. If the angle of the frame is 45°, then it may be used in laying out right angles.

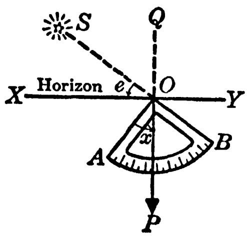
"If a perpendicular is to be constructed to line *KE* at *E*, sight from *G* through a window above the mirror at *D* to stake *K* at a convenient point in line *KE*. Then have a second stake *L* set up by an assistant so that its image is reflected twice, as indicated by the arrows in the diagram, and appears in the mirror at *D*, directly under and in line with the stake *K* seen through the window. Then the line *EL* is perpendicular to *KE* as desired.

"*Problems:* (a) Prove that $\angle KEL = 90°$.

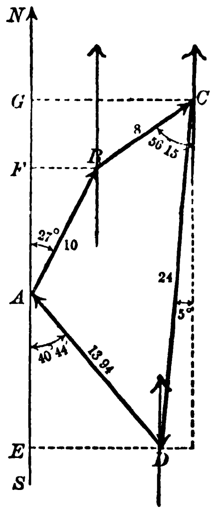
(b) If $\angle HFD = x°$, prove that $\angle KEL = 2x°$."*

3.18. STUDENT.† The figure below shows how an instrument called a quadrant (a quarter of a circle) may be used to measure the angle of elevation *e* of a star *S*. The side *BO* of $\angle AOB$, which is a right angle, is sighted at the star, and the angle of elevation is found from *x*, the angle between *AO* and the plumb line *OP* which hangs vertically from *O*.

Problems: Of what is *e* the complement? Of what is *x* the complement? Why are these angles equal?



G. 3.18



G. 3.19

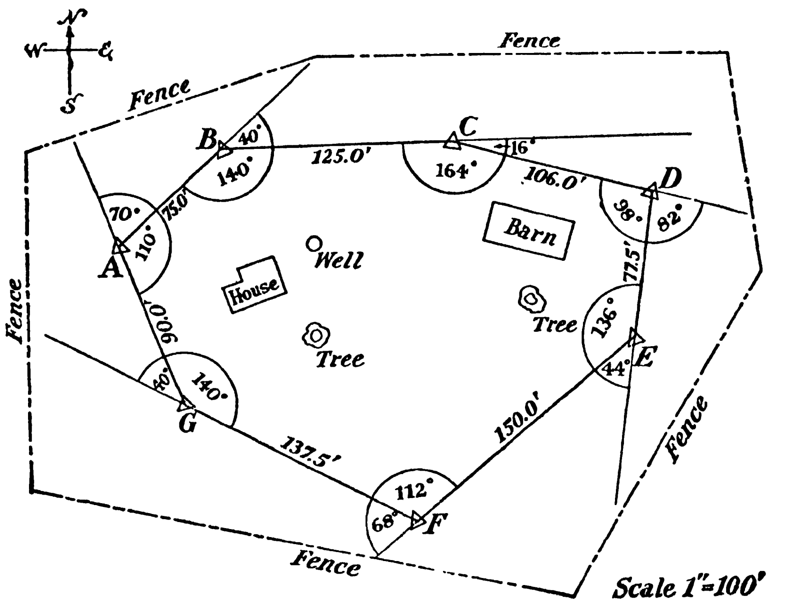
* From A. Leonhardy, M. Joseph, and R. D. McLeary, *New Trend Geometry*, p. 147. Charles E. Merrill Co., New York, 1940.

† Adapted from D. E. Smith, W. D. Reeve, and E. L. Morss, *Text and Tests in Plane Geometry*, p. 57. Ginn and Co., Boston, 1933.

3.19. Surveying. When land is surveyed the bearing, or direction angle of points, is measured from the north-south line. By the time a surveyor has completed the circuit of the plot, numerous angle calculations can be made by using the ideas of complementary, alternate-interior, and supplementary angles, as seen in the figure on page 154.

In this case the surveyor's work was as follows: He started from point A running N. 27° E. 10 chains to B, thence N.E. by E. 8 chains to C, and thence S. 5° W. 24 chains to D, and finally N. 40° 44′ W. 13.94 chains to A, the starting point.

3.20. Surveying.* The sum of the interior and exterior angles of a polygon can be used as a convenient check by surveyors. The figure shows a survey map of a piece of country property. The



basis of the survey is the polygon $ABCDEFG$. The polygon is formed by the base lines between transit stations, which are indicated by small triangles at each vertex.

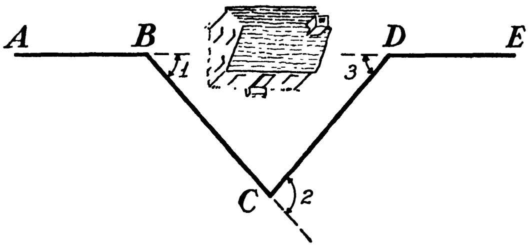
* Adapted from D. Reichgott and L. R. Spiller, *Today's Geometry*, p. 179. Prentice-Hall, Inc., New York, 1938.

The number of degrees in the exterior and interior angles at each station is indicated on the map.

Problem: What should be the sum of all the exterior angles in the polygon? What should be the sum of all the interior angles?

As a final check, what should be the sum of each pair of exterior and interior angles located at each station?

3.21. Surveying. In surveying it is often necessary to run a line, such as *AB* in the figure, so that it will be in the same straight



line as *DE* but on the other side of some obstruction, such as a house. The rule for this procedure follows: Lay off ∠1 so as to clear the house. Take *BC*, a convenient distance. Lay off ∠2 = 2 ∠1. Make *CD* = *BC*. Lay off ∠3 = ∠1.

Problem: Prove by geometry that this is a valid procedure.

Also see Circles G. 5.09; Geometry construction G. 11.01; Great circles G. 13.02; Parallel lines G. 15.01, 15.05; Spheres G. 22.01.

AREAS 4.01–4.08

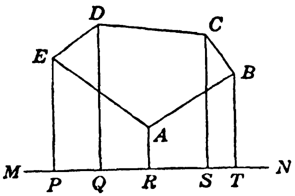
4.01. Engineering. If a steam locomotive is to be designed to haul a certain maximum load up a certain maximum grade, it is necessary to build the boiler to withstand a certain pressure and equip the engine with a certain sized cylinder in order to have a certain total pressure applied to the piston.

Problem: If the highest safety pressure of a boiler is 200 lb. and if the engine is equipped with two 18-in. cylinders, what is the total pressure against both pistons?

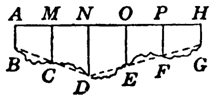
4.02. Engineering.* Frequently it is necessary to calculate areas

* Adapted from J. C. Stone and V. S. Mallory, *New Plane Geometry*, p. 359. Benj. H. Sanborn, Chicago, 1937.

whose shapes are irregular, and therefore none of the standard area formulas can be applied directly. Various ingenious procedures are resorted to, depending upon the local conditions.



No. 1



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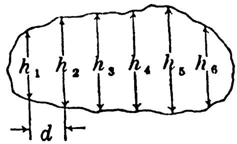


No. 3

1. Surveyors sometimes find the area of a tract of land thus: A base line MN is staked off, and from the various points in the boundary the distances to this line are then measured, as EP, DQ, etc., and the distances PQ, QR, etc., are also measured.

Problems: (a) Show how to compute the area of $ABCDE$.

(b) If $EP = 2000$ ft., $DQ = 2500$ ft., $CS = 2300$ ft., $BT = 1800$ ft., $AR = 800$ ft., $PQ = 800$ ft., $QS = 1900$ ft., $ST = 500$ ft., $TR = 1700$ ft., compute the area.

2. In order to determine the flow of water in a stream, the area of a cross section $ABCDEFGH$ of the stream, at right angles to the current, is first found as follows: Soundings are taken at A, M, N, etc., and the areas of trapezoids $ABCM$, $MCDN$, etc., are computed and added.

Problem: If $AM = MN = NO = OP = PH = 10$ ft., $AB = 5$ ft., $MC = 9$ ft., $ND = 12$ ft., $OE = 9$ ft., $PF = 7$ ft., and $HG = 5$ ft., find the area of the cross section. This is known as the trapezoidal rule for finding an area. It is used in measuring the area of land bounded on one side by an irregular line.

3. A fairly accurate method used in computing the area bounded by an irregular curve, known as the mean ordinate method, is as follows: At equal distances d measure the widths h_1, h_2, h_3, etc., of the area inclosed.

Problem: Show that if a large number of widths are measured, a close approximation to the area is given by $d(h_1 + h_2 + h_3 + \ldots)$.

4.03. ENGINEERING. Find the area of a rubbing surface in a steam cylinder 91.5 in. in diameter, stroke of piston 6 ft. 8 in.

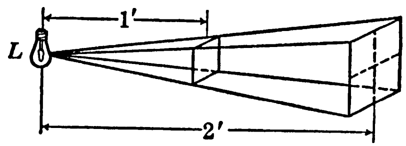
4.04. Engineering. A cast-iron bar has an elliptical cross section with axes 6 in. and 4 in. Find the pull per square inch of the cross section under a tensile load of 125,000 lb. (Note: the area of an ellipse is πab, where a and b are the semi-axes.)

4.05. Painting. Painters must be able to calculate the surface areas of surfaces to be painted in order to determine the amount of paint they will need.

Problems: (a) Find the surface area of a church steeple 60 ft. tall and having a hexagonal base, each side of which is 3.5 ft.

(b) Find the surface area of a dome in the shape of a half-sphere 60 ft. in diameter.

4.06. Physics.* The accompanying figure illustrates the law of physics which states that the intensity of light on a surface is inversely proportional to the square of the distance from the source.



Problems: (a) Explain the geometric principles involved, as illustrated by the figure. Hint: Since light rays travel in straight lines, over how large a surface must the same amount of light received by a 3-inch square 1 foot from L be distributed when the surface is 2 feet from L? Why does the intensity of the light—the amount of light per unit of area—decrease as the distance from L increases?

(b) A boy reading 5 feet from an electric light moves 2 feet nearer the light. The intensity of light on his book is how many times as great as it was at first?

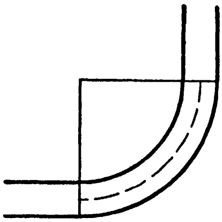
4.07. Plumbing. How many 1-inch water pipes can be attached to a 6-inch pipe so that if all the 1-inch pipes are opened the 6-inch pipe will keep all the smaller pipes filled?

* Adapted from D. E. Smith, W. D. Reeve, and E. L. Morss, *Text and Tests in Plane Geometry*, p. 227. Ginn and Co., Boston, 1933.

4.08. Steamfitting. How many 3-inch steampipes can be connected to an 18-inch steampipe?

CIRCLES 5.01–5.11

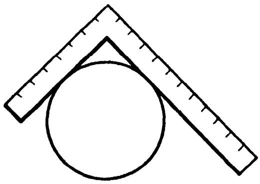
5.01. Builder. If a concrete builder builds a curb or wall in which two straight stretches are connected by a curved channel, as shown in the figure, and he wishes to determine the cubic yards of concrete needed for the curved section, he is instructed by manuals on the subject to regard the curved section as a straight section whose length equals the arc of the circle which lies in the middle of the two bounding circles. He thus is using the concept of the arc of a circle as taught in geometry.



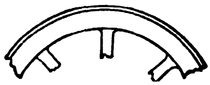
G. 5.01

5.02. Carpentry.* Carpenters frequently check the roundness of pillars and the like by using a carpenter's square.

Problem: Show that if a circular block of wood is placed in the angle of a square and the two are laid flat on a table, the edges of the square will touch the block at equal distances from the heel. Could this procedure be used to test whether the block is a true circle?



G. 5.02



G. 5.03

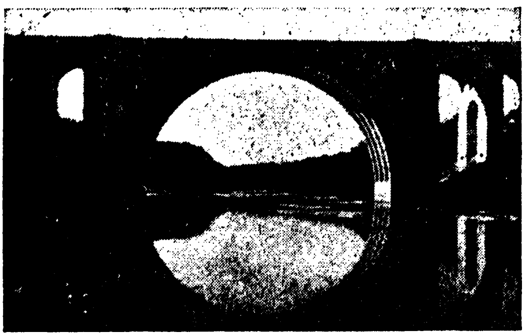
5.03. Curator.† The accompanying figure shows a piece of a broken chariot wheel found by a group of archaeologists.

* Adapted from F. E. Seymour, *Plane Geometry*, p. 152. American Book Company, New York, 1925.

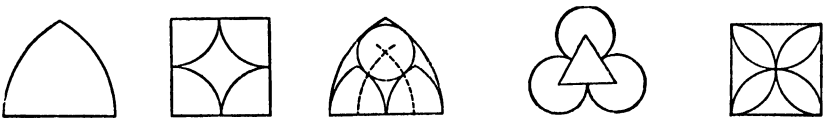
† Adapted from D. E. Smith, W. D. Reeve, and E. L. Morss, *Text and Tests in Plane Geometry*, p. 131. Ginn and Co , Boston, 1933.

Problem: How could the radius of the original wheel be found from this piece in order to determine the size of the wheel?

5.04. DAILY LIFE.* While the circle is a very useful form for wheels, rings, links, and the like, it is also a very beautiful figure. In one form or another it is frequently used in architecture as in the bridge below.



The circle is basic in many window designs as suggested below.



1. 2, 3. 4. 5.

The *trefoil* is formed by drawing circles as in figure 4 above.

Problem: If the side of an equilateral triangle is 12 in., find the area of the trefoil, that is, of the entire surface enclosed by the the three arcs. Also, find its perimeter.

The *quatrefoil* is also formed by arcs of circles. See figure 5.

Problem: If the side of a square is 4 in., find the area and perimeter of the quatrefoil.

The circle is also cleverly used in handwork, as in tatting.

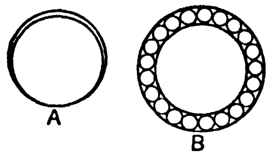
5.05. DAILY LIFE.† Friction is the factor that accounts for much

* From D. E. Smith, W. D. Reeve and E. L. Morss, ***Text and Tests in Plane Geometry***, p. 133. Ginn and Co., Boston, 1933.

† Adapted from W. W. Strader and L. D. Rhoads, ***Modern Trend Geometry***, p. 186. The John C. Winston Co., Philadelphia, 1940.

of the inefficiency of machines in which shafts must turn in bearings.

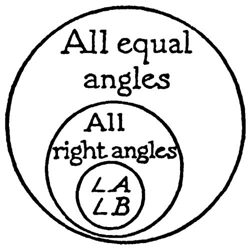
Problem: Figure A shows a shaft turning in a plain bearing. Figure B shows a shaft turning in a roller bearing. What is the advantage of the roller bearing over the plain bearing?



5.06. Daily Life. Leonhard Euler, a brilliant Swiss mathematician of the eighteenth century, noticed that "the kind of deductive reasoning used in our geometry could be visualized by means of circles.

"(1) For instance, he reasoned that: *If all right angles are equal angles and angles* A *and* B *are right angles, then angles* A *and* B *are equal angles.*

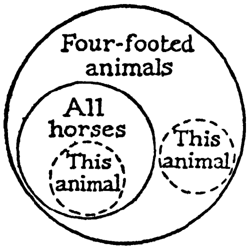
"If we use a large circle to represent all equal angles, we may have within it a smaller circle representing all right angles. We may do this because our first statement tells us that all right angles are included within the class of all equal angles. Then, within this smaller circle we may have a still smaller circle representing angles *A* and *B*, for they are a part of all right angles.

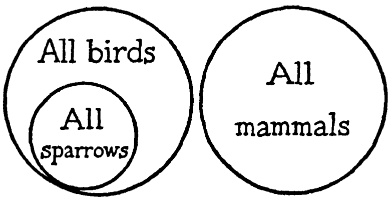


"It is clear that the smallest circle is entirely within the medium-sized circle and the medium-sized circle is entirely within the largest circle. It is obvious, then, that the smallest circle is entirely within the largest circle. The argument becomes clearer as we study the arrangement of circles. Circles used in this way are called Euler's Circles.

"(2) Errors in thinking may often be detected by a similar representation. Suppose we have the argument that: (a) All horses have four feet. (b) This animal has four feet. (c) Therefore it is a horse.

"The circle representing all horses is entirely within the larger circle representing all four-footed animals. But the circle representing this animal is not necessarily within the smaller circle representing all horses (we are told only that the animal has four feet); it may be either within this circle or outside it. If we represent it in these two possible positions, we can readily see that this animal may or may not be a horse.





"(3) As the reasoning becomes more complex, the circles become more useful. Consider the following argument: (a) No bird is a mammal. (b) All sparrows are birds. (c) Hence no sparrow is a mammal.

"*Problems:* 1. Explain the graphical representation.

"2. Use Euler's circles to represent the following arguments:

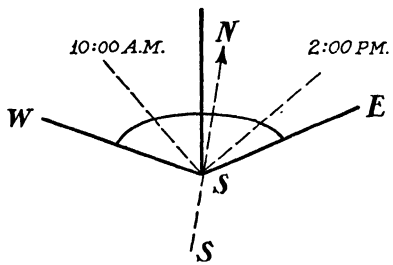
(a) 1. The opposite sides of a parallelogram are equal.
    2. *AB* and *CD* are opposite sides of a parallelogram.
    3. Hence *AB* and *CD* are equal.

(b) 1. All rectangles are parallelograms.
    2. *ABCD* is a parallelogram.
    3. But *ABCD* is not necessarily a rectangle."*

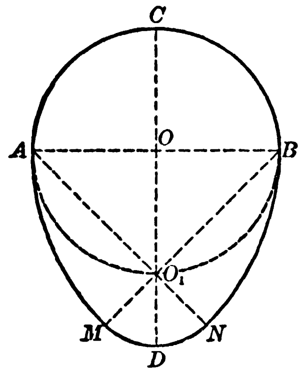
5.07. Daily Life. Establishing a true north-south line (the meridian) is basic to any problem solving based on directions, viz., star study. This can be done by suspending a plumb line over some

*J. R. Clark, R. R. Smith, and R. Schorling, *Modern School Geometry*, pp. 435–436. World Book Co., Yonkers-on-Hudson, New York, 1938. Reproduced by written permission of the publisher.

point, e.g., S in the figure below. Draw a circle with a radius of several feet, using S as center. As the sun journeys through the sky on any clear day, the shadow will move from SW to SE. At two equal intervals before and after noon, viz., 10:00 a.m. and 2:00 p.m., mark shadows of the plumb line on the circle. Bisect the angle formed by these points at S. This will be north-south line. The shadow at noon will coincide with this bisector, but often it is so short that its direction is difficult to ascertain.



G. 5.07



G. 5.08

5.08. ENGINEERING. Experience has shown that a sewer is most efficient if it is constructed in the form of an oval. It is designed as follows: Construct circle O. Erect CD a $\perp$ bisector of diameter AB cutting circle at O_1. With AB as radius and A and B as centers, describe arcs AM and BN which are terminated by extended chords AO_1 and BO_1. The arc MDN has the center O_1 and O_1N as radius.

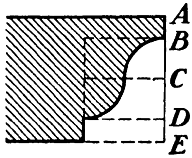
Problems: (a) Construct such an oval. (b) Is arc ACB tangent to arc AM? Why? (c) Is arc AM tangent to arc MD? Why? (d) If $AB = 8'$, find BO_1, O_1M, and CD.

Thus, if the sewer is 8 ft. wide, what is its depth?

5.09. ENGINEERING. Two pulleys, with diameters 16 in. and 24 in., respectively, are belted together at such a distance apart that the arc of contact on the smaller pulley is 120°. Find the arc of contact on the larger pulley, the distance between the centers of the pulleys, and the length of the belt.

5.10. GEOGRAPHY. North-south distances on the earth's surface are expressed in terms of degrees of latitude. The "parallels" of geography are circles of varying sizes lying in planes parallel to the plane of the equator and indicate the extent of latitude.

5.11. HOME OWNER. Many styles of molding are basically arcs of circles. Explain how the molding design may be constructed if $BC = CD = 2AB = 2DE$.



Problem: If $AB = \frac{1}{2}$ in., find the length of the curved portion of the molding.

ALSO SEE Geometric forms G. 10.01, 10.07; Geometry construction G. 11.04, 11.08, 11.09; Locus G. 14.01.

CONES 6.01

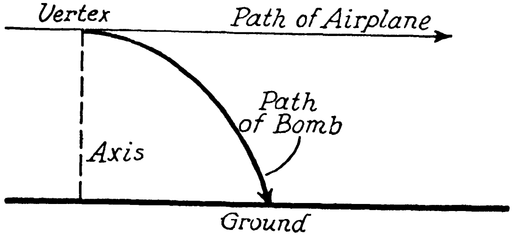
6.01. NAVIGATION. Some buoys have the shape of a double cone. If the largest diameter of a certain buoy is 3 ft. 4 in. and the length from the tip of one cone to the tip of the other is 5 ft. 6 in., find the weight of the buoy. The sheet iron used to build the buoy is $\frac{3}{8}$ in. thick, and one cubic inch of iron weighs $\frac{1}{4}$ lb. If one cubic foot of water weighs 63 lb., how deep will the buoy sink if it is kept in a vertical position? It is understood that the weight of the buoy equals the weight of the water it displaces.

CONIC SECTIONS 7.01–7.07

7.01. ART. A crescent, such as the crescent moon, is bounded by a semicircle and a semi-ellipse.

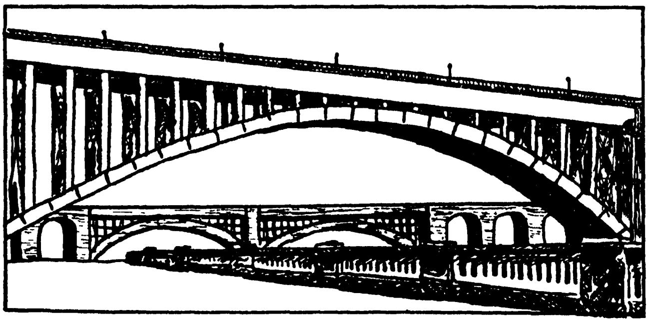
7.02. ART. If a circle is to be represented in a drawing as viewed from a point which is not directly overhead, an ellipse should be drawn.

7.03. DAILY LIFE. "The path of a projectile shot from a gun is parabolic, except that for high velocities air resistance alters the path considerably. Knowledge of the muzzle velocity of the projectile and of the angle of elevation of the gun permits the determination of the almost parabolic path of the projectile. From this it is possible to know in advance how far the projectile can go, how long it will take to go that distance, what angle of elevation will send the projectile furthest, and other information useful to



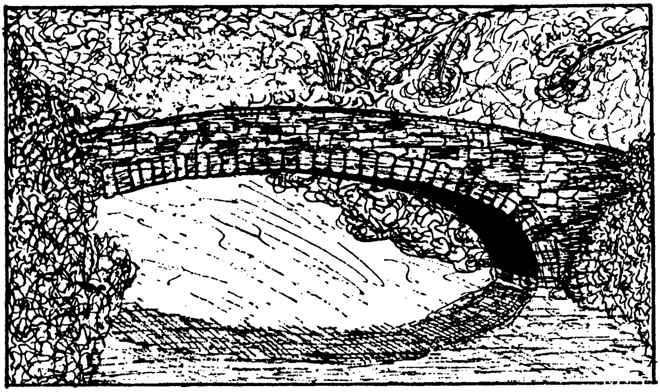
armies. A bomb dropped from an airplane flying horizontally will descend in a parabolic path. The axis of this parabola is vertical and its vertex is the position of the bomb at the instant it is released. (See above.)

"In addition to being useful in studying the paths of moving objects, the conic sections are useful in the construction of many physical objects. The arches used in some of the best modern constructions are elliptic, or parabolic. A parabolic arch is stronger



Sketch by Romaine E. Boyer
from original photograph

than any other and hence is used when the strongest possible arch is necessary, as in steel construction. On the other hand, an elliptic arch is generally considered more beautiful than the parabolic and hence is often used for massive structures of masonry desirable chiefly for ornamental purposes.



Sketch by Dolores E. Boyer
from original photograph

"Besides being used for ornamental arches, ellipses are frequently used for other decorative purposes, such as for flower beds, curved walks and drives, mirrors, paneling in furniture, and so on. The supporting cable of a suspension bridge is parabolic if the weight of the cable is negligible in comparison with the weight of the road and if the road weighs the same for each foot of its length. Both of these conditions are practically realized in the case of the George Washington Bridge and the Brooklyn Bridge, in New York."*

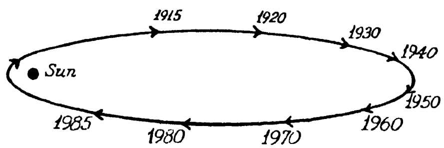
7.04. Daily Life. The early astronomers who studied the movements of the planets postulated that the planets moved in circular orbits. The entire Copernican theory rested on the belief that the behavior of the planets could be predicted by charting their circular courses. There were, of course, a number of signs that suggested the validity of the assumption of circular orbits, notably the way the moons of Jupiter behaved. However, astronomers noted rather significant differences between the theoretically predicated paths and the actual paths as observed by telescope. Finally Johannes Kepler, a German astronomer, after discarding nineteen

* Quotations from H. R. Cooley, D. Gans, M. Kline, and H. E. Wahlert, *Introduction to Mathematics*, pp. 147–148. Houghton Mifflin Co., Boston, 1937.

other hypotheses investigated the possibilities of elliptical orbits for the planets. Perhaps with surprise, for he had worked nearly twenty years on the job, he learned that by using elliptical paths he could account for all of the behavior patterns of the planets. Had the previously developed knowledge of the conic sections not been available for use the chances are very great that the science of astronomy would have been delayed a long time.

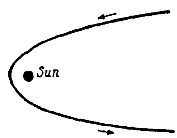
Furthermore, once Kepler explained planetary motion scientifically, Newton was enabled to formulate his Law of Gravitation.

"The path in which a comet belonging to our solar system travels is either an ellipse, a parabola, or a hyperbola, having the sun situated at one focus. A comet is visible only when it is near the sun, and it is from observations made during this short period that its orbit is computed. The orbit of a comet which travels in an ellipse is long and narrow, with the sun situated near one end of the ellipse. On leaving the neighborhood of the sun such a comet recedes to a very great distance and after many years returns toward the sun. Knowledge of its orbit enables astronomers to



Orbit of Halley's Comet

predict when the comet will reappear and in what part of the heavens it will be visible.



Parabolic Orbit

"Thus in 1682, Halley observed the comet which now bears his name, calculated an elliptic orb for it, and successfully predicted its return in 1757. It has returned approximately every 75 years since, and should return next in 1985. This is the first comet for which an elliptic orbit has been computed and the first case in which the reappearance of a comet has been successfully predicted. Comets which travel in parabolic and hyperbolic orbits come near to the sun once and then recede never to return again."*

7.05. Daily Life. A stream of water gushing from an elevated horizontal pipe or hose describes a parabola as it rushes forward and falls downward toward the ground.

7.06. Daily Life. The following list illustrates additional uses of the parabola:

1. If a parabola is rotated about its axis, a parabolic surface is formed. If a source of light is placed at the focus, all the rays from this light which meet the surface will be reflected parallel to the axis. It is for this reason that parabolic reflectors are used in automobile headlights and searchlights.
2. The large reflector in a reflecting telescope is usually parabolic.
3. The equation $S = \frac{1}{2}gt^2$ is the equation of a parabola.
4. The equation expressing the relation between the period and the length of a pendulum is the equation of a parabola.
5. The equation expressing the relation between the bending moment at a point on a uniformly loaded beam and the distance from the point of support is the equation of a parabola.
6. If a cylindrical vessel partly filled with water is whirled about the axis of the cylinder, a plane through the axis will cut the surface of the water in a parabola.

The following list illustrates additional uses of the ellipse:

1. Elliptic gears are used in machines to obtain a slow, powerful movement with a quick return, as in power punches.
2. Whispering galleries usually have elliptical ceilings arranged so that one may stand at a focus. Thus situated he can hear a slight noise made at the other focus, while an individual standing between the foci hears nothing.

The following list illustrates uses of the hyperbola:

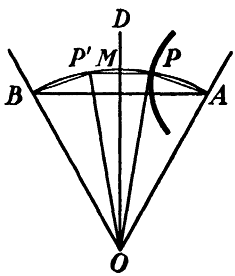
* Quotations from H. R. Cooley, D. Gans, M. Kline, and H. E. Wahlert, *Introduction to Mathematics*, p. 146. Houghton Mifflin Co., Boston, 1937.

1. The equation expressing Boyle's law for a perfect gas under pressure at a constant temperature is of the form of the equation of an equilateral hyperbola referred to its asymptotes as axes.

2. The hyperbola may be used to locate an invisible source of sound, as in locating an enemy's guns or in range-finding. Two listening posts are established and the difference in time of arrival of the report of the gun at the two posts is measured as accurately as possible. The difference in time multiplied by the rate at which sound travels gives the difference in the distances from the two listening posts to the gun. If the two listening posts are used as foci and the difference in distances from the gun as the transverse axis of a hyperbola, then the position of the gun will lie on the hyperbola. A second observation from new positions will give a second hyperbola through the position of the gun. The simultaneous solution of the equations of the two hyperbolas will locate the gun at one of the points of intersection of the two curves. How can one tell at which point of intersection the gun is located?

3. A type of reflecting telescope uses a hyperbolic mirror as the small reflector to reflect the image to the eyepiece.*

7.07. Student.† The hyperbola can be used to trisect any angle, such as AOB. Let OD bisect AOB. Then, using OD as the directrix and A as a focus, construct one branch of an hyperbola of eccentricity 2. Let this cut the arc of the circle AB at P. From the definition of the hyperbola $\frac{AP}{MP} = 2$, or $AP = 2MP$. Now fold the figure $OAPM$ over line OD, letting P fall on P' and A on B. Chords $AP = PP' = BP'$ and arcs $AP = PP' = BP'$ (Why?), and $\angle BOP' = \angle P'OP = PAO$ (Why?).



DIAGONALS 8.01–8.03

8.01. Carpentry. To find the center of the ceiling of a room (for locating a light fixture) or the center of a rectangular house (for

* All of the above, except 1 under "parabola," is quoted from T. E. Mason and C. T. Hazard, *Brief Analytic Geometry*, pp. 87–101. Ginn and Co., Boston, 1935.

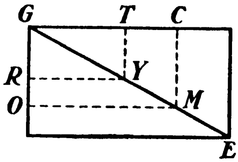
† See H. E. Buchannan and G. E. Wahlin, *The Elements of Analytic Geometry*, p 92. Farra and Rinehart, New York, 1937.

locating a chimney), a carpenter can stretch two lines diagonally and mark their intersection. The same method is used in centering the chuck and tailstock ends of wood of square or rectangular cross section to be turned in a lathe.

8.02. Daily Life. Explain geometrically how a door frame or a window frame may be "squared" by measuring and comparing distances between opposite corners.

8.03. Photography. Engravers and photographers frequently have occasion to enlarge or reduce the size of pictures or cartoons. The method they commonly use is described below.

Let *GTYR* represent a photograph $3\frac{1}{2}$ in. by $4\frac{1}{4}$ in., which is to be enlarged so that the width of the enlarged picture will be 9 in.



A proportion could be stated which would easily determine the width, but this method is seldom employed. Instead, the given photograph *GTYR* is placed on a larger rectangle, whose diagonal is *GE*, so that the diagonal of the picture coincides with the diagonal of the large rectangle. Count down 9 in. from *G* and call this line *GO*. A line from *O* || to *GT* and intersecting *GE* at *M* will be the required length of the enlargement.

Problems: (a) Prove that rectangles *GY*, *GM*, and *GE* are all similar.

(b) Newspaper cartoons usually are made quite large and then reduced for printing. If rectangle *GE* represents a cartoon which has 2 columns or 4 inches of space allotted to it, what would the depth of the printed cartoon be if the original was 9 in. by 14 in.?

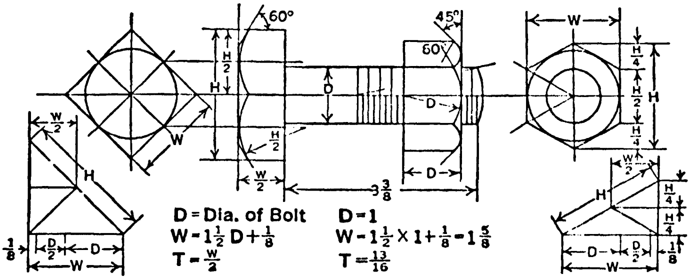
Also see Geometric forms G. 10.02; Golden section G. 12.01.

GEOMETRIC DRAWINGS 9.01

9.01. Daily Life.* Draftsmen and all who use their drawings, viz., carpenters, bricklayers, plumbers, home owners, etc., must be able to understand geometric figures and drawings.

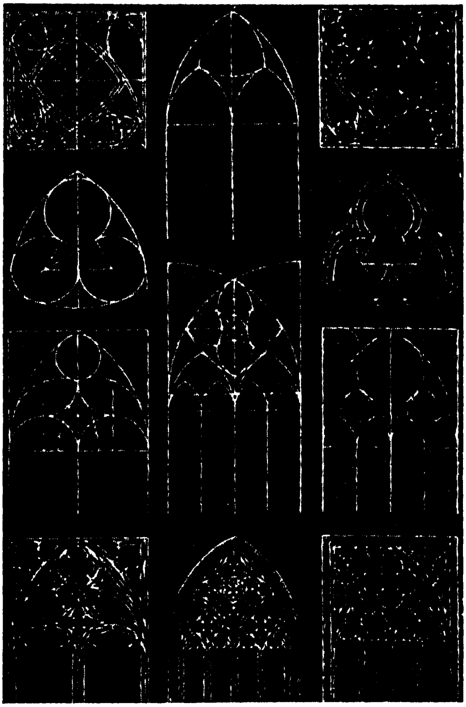
* From W. W. Strader and L. D. Rhoads, *Plane Geometry*, p. 33. The John C. Winston Co., Philadelphia, 1934.

The accompanying figure illustrates the geometric methods used to illustrate the size of a simple bolt.



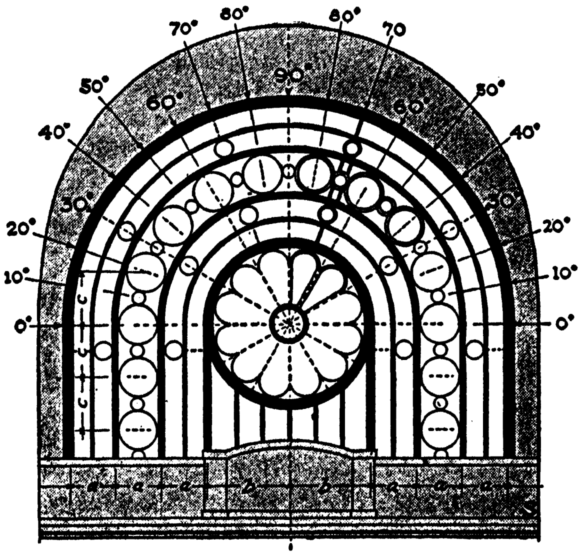
GEOMETRIC FORMS 10.01–10.07

10.01. Architecture.* Geometric forms abound in architecture: isosceles triangles in gable ends, rectangular and circular windows, cylindrical silos, hemispherical domes, conical church spires, etc.



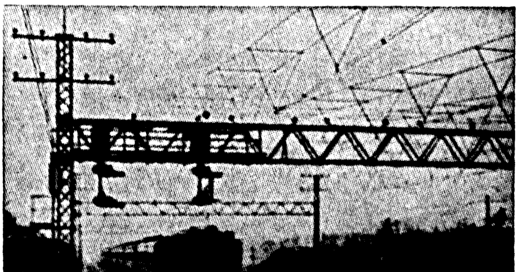
* From J. C. Stone and V. S. Mallory, *Modern Plane Geometry*, p. 154. Benj. H. Sanborn, Chicago, 1939.

Basic to all such designs are many geometric concepts, viz., centers, radii, circles, angles, parallel lines, symmetry, etc. These ideas are illustrated in the figure below:*



Problem: Can you tell from the diagram how the centers of the twelve arcs surrounding the central angle were determined?

Geometric forms also abound in industrial constructions, as here illustrated.†



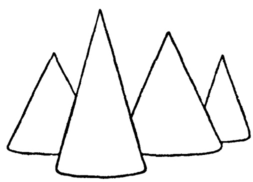
* From J. R. Clark, R. R. Smith, and R. Schorling, *Modern-School Geometry*, p. 524. World Book Co., Yonkers-on-Hudson, N. Y., 1938. Reproduced by written permission of the publisher.

† From D. E. Smith, W. D. Reeve, and E. L. Morss, *Text and Tests in Plane Geometry*, p. 82. Ginn and Co., Boston, 1933.

10.02. ART.* Geometric figures are very commonly used in painting to accomplish various purposes. The idea of orderliness is often achieved by grouping the objects according to a geometric figure. For example, the pyramidal form is supposed to give to a painting a sense of rest, strength, and dignity. We thus find the character (in the case of "Mona Lisa," by Leonardo da Vinci) or the characters (in the case of "Madonna and Child, with St. Francis and St. George," by Giorgione) arranged in the form of a pyramid or triangle resting on a base. Sometimes the center of interest in the painting is located at the mathematical center (determined by the intersection of the diagonals) of the painting. This is nicely illustrated in the painting, "Shepherds of Arcadia," by Poussin. Often the diagonals furnish a line of concentration, as it were, for the characters. View a number of pictures and note how the characters seem clustered around the diagonals.

Cubism is a modern style of painting that freely uses geometric figures: parallelopipeds, cones, cylinders, etc. This style is illustrated by Picasso in "Figure."

Cubism gives the artist great freedom in representing physical objects. He is not bound by the conventional outward appearance of things, but can represent them to suit his purpose. For example, he might wish to fix the viewer's attention on the bulk, weight, and essential shape of a mountain, and not on its foliage, snow, and other surface features. In this case he might represent the mountain as a cone or combination of cones.



10.03. DAILY LIFE. Geometric forms exist almost everywhere. Spherical raindrops fall in straight-line paths which strike the windowpane of a moving train at an angle related to the speed of the train. The beautifully colored rainbow spans the sky in the form of a circular arc. Sedimentary rocks, deposited many years ago, can be seen in parallel layers along nearly every highway cut. If a stone is dropped into a calm pond or lake many waves are sent toward the bank in the pattern of concentric circles. Waves formed

* Adapted from an excellent discussion of geometry in art. *Introduction to Mathematics*, by Cooley, Gans, Kline, and Wahlert, pp. 151–159. Houghton Mifflin Co., Boston, 1937.

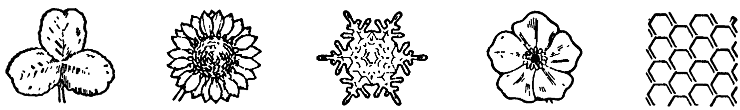
in the wake of a boat arrange themselves in the pattern of an angle extending far behind the boat. Snowgeese in their long migratory flights arrange themselves in a similar geometric pattern sometimes called a V-formation. Heavenly bodies, like the sun and the moon, are introduced to the child as "balls" but in reality they are geometric spheres. It is quite likely that because of these frequent occurrences of geometric forms an early mathematician once said, "God eternally geometrizes."

Geometric forms are often used in building designs, of which the following are suggestive. These cuts illustrate simple, possibly primitive, forms, yet comparison with the world's most famous buildings reveals that the same geometric forms are basic.

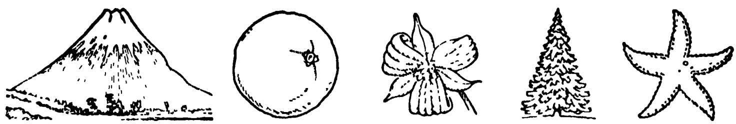
Indian Tent **Eskimo House** **Pueblo Home** **Water Tank** **Church Steeple**

Exercise: Can you name the geometric forms which these pictures suggest? Write the names, etc.

In nature an abundance of geometric forms are also found, as the following illustrate.



Clover **Sunflower** **Snowflake** **Wild Rose** **Honeycomb**

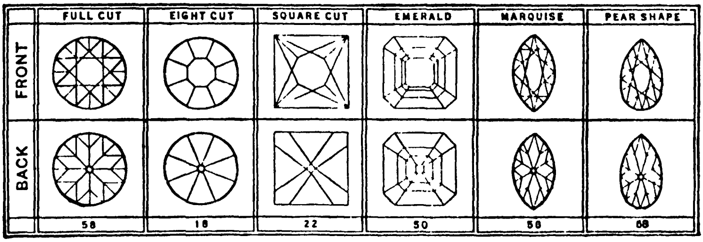


Mountain **Orange** **Trillium** **Spruce Tree** **Starfish**

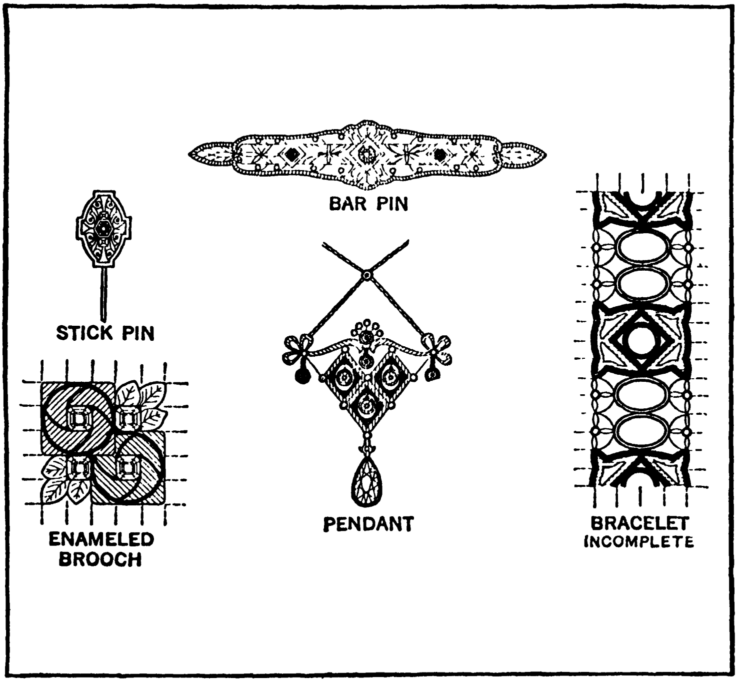
Problem: Can you name geometric forms which these pictures suggest?*

* Adapted from D. E. Smith, W. D. Reeve, and E. L. Morss, *Text and Tests in Plane Geometry*, pp. 2-3. Ginn and Co., Boston, 1933.

10.04. Daily Life.* Geometric forms can be seen in precious stones, as the following illustrates. The small figures show the total number of faces or "facets."

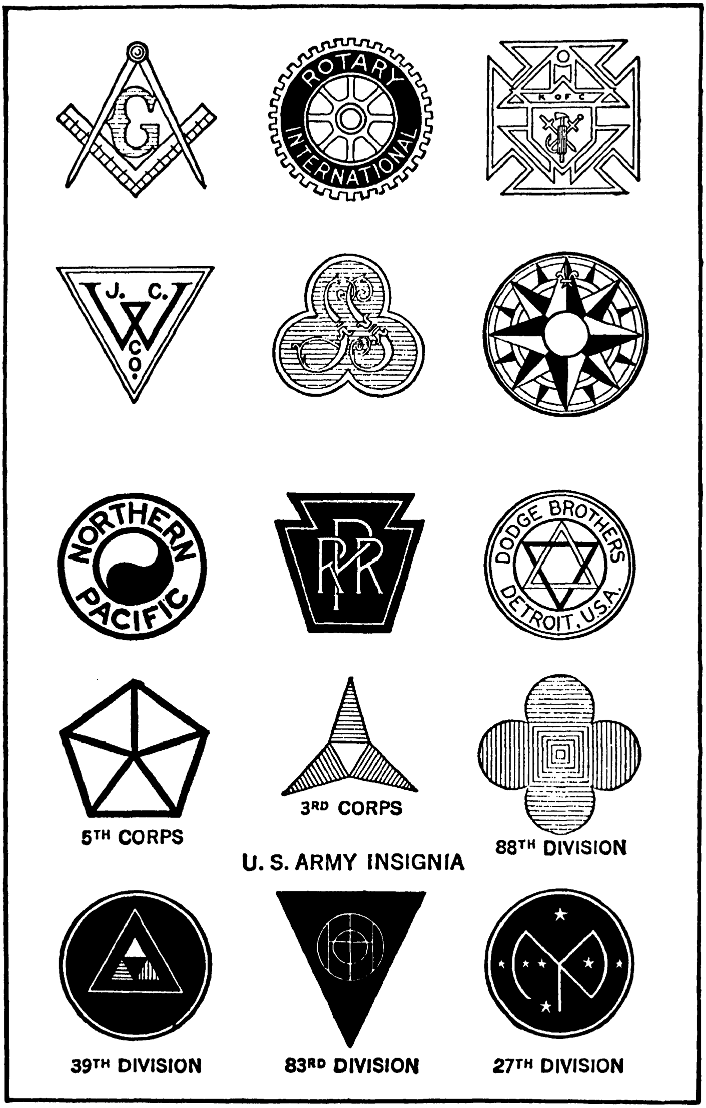


Geometry in jewelry is illustrated in the designs below.



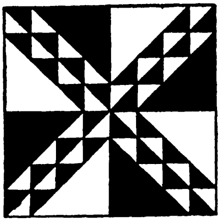
* From W. W. Strader and L. D. Rhoads, *Plane Geometry*, p. 192. The John C. Winston Co., Philadelphia, 1934.

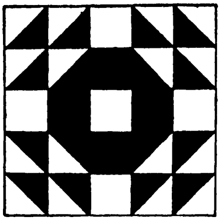
10.05. Daily Life.* Geometric figures form the foundation for many well known trade marks, emblems, insignia, etc.

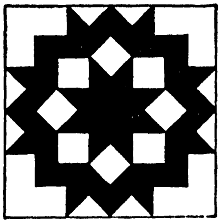


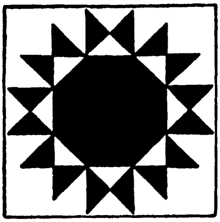
* From W. W. Strader and L. D. Rhoads, *Plane Geometry*, p. 264. The John C. Winston Co., Philadelphia, 1934.

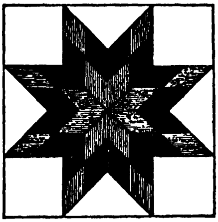
10.06. Housewife. Geometric designs are used in old-fashioned quilts, as the following illustrate.*

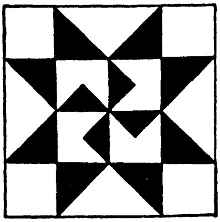

No. 1. Path through Woods

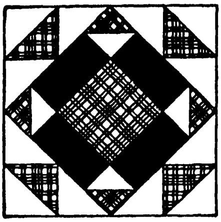

No. 2. Wedding Ring

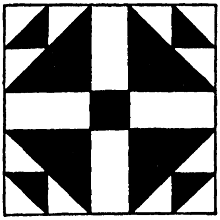

No. 3. Dutch Rose

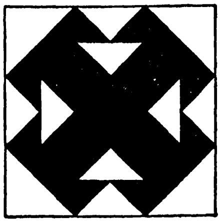

No. 4. Wheel of Fortune

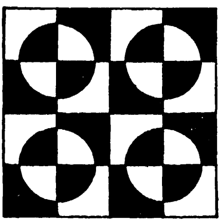

No. 5. Star of the East

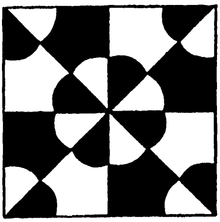

No. 6. Octagonal Star

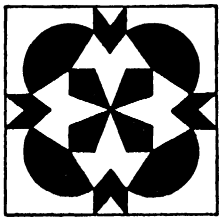

No. 7. Scotch Plaid


No. 8. Wild Goose Chase


No. 9. Capital T


No. 10. Indiana Puzzle


No. 11. Dutch Windmill


No. 12. Farmer's Wife

* From M. Sykes, C. E. Comstock, and C. M. Austin, *Plane Geometry*, p. 322. Rand, McNally Co., New York, 1933.

10.07. Photography. "One of the most interesting pastimes in photography is to analyze a good picture—take it apart, as it were, and try to find what makes it 'tick.' By doing this, one can learn the things that are essential to a good picture—and often discover what features his own shots lack.

"For example, observe the picture which accompanies this article. It illustrates, to a marked degree, one feature that is common to most well-composed pictures. That is, a definite geometrical plan of composition.

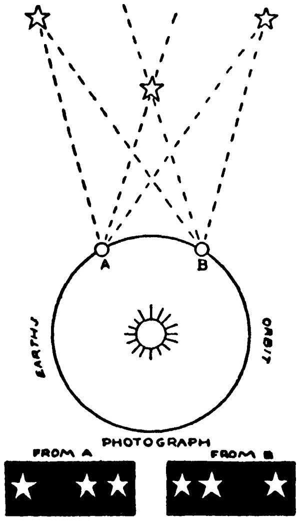
"Don't let the reference to geometry frighten you. It simply means that most good pictures are based on a pleasing grouping of one or more circles, triangles, ovals, or other conventional shapes. Usually this underlying arrangement is well concealed by the subject and seldom is obvious.

"Note, in this picture, that each boy forms a triangle or pyramid, and that their heads are inclined so that together the two figures form a larger pyramid. The pyramidal form is always suggestive of strength and stability; and you feel that stability without the need of analyzing its source.

"Observe that two smaller triangles are formed by the spacing of the legs; that right legs are parallel and left legs are parallel; that the arms are rhythmically arranged, and that both boys' eyes are turned in the same direction. All these details are part of the general geometrical plan. The two hoops, finally, are purely geometrical, and so placed that each duplicates the other in a rhythmic manner."*

GEOMETRIC CONSTRUCTION 11.01–11.09

11.01. Astronomy.† Early man supposed the earth was stationary and that the sun, stars, and planets revolved around it. As early



as 300 years before Christ a few men, like Aristarchus of Samos, dared to suggest that the sun was stationary and that the earth and stars revolved around it. This gave rise to a heated controversy which needed to be proved geometrically. Even Aristarchus believed that if careful observation were made, the closer stars would move back and forth relative to the more distant ones. Men looked

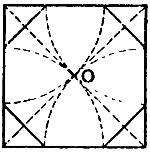
* *Pittsburgh Press*, April 14, 1940.

† Adapted from W. Bartky, *Highlights of Astronomy*, p. 14. The University of Chicago Press, Chicago, 1935.

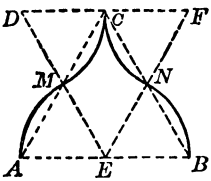
for this shift in the relative position of the near-by stars for many years. Finally, in 1840 telescopes, microscopes, and photography were sufficiently developed so that the back and forth movement (*annual parallox*) was clearly established. A geometric drawing illustrating the findings is reproduced on page 179.

11.02. Carpentry.* The accompanying cut illustrates a geometric method by which a carpenter may cut a regular octagon for a table top from a square board.

Problems: (a) Explain how it is done. (b) Can you prove the figure is a regular octagon? (c) If a side of the square is 12 in., find a side of the octagon.



G. 11.02



G. 11.04

11.03. Daily Life. The swastika, a mysterious symbol which our Aryan ancestors brought from the plateaus of central Asia in prehistoric times, is a pure geometric figure, i. e., it can be made with compass and straight-edge only.

11.04. Daily Life.* The Persian arch, shown in the figure above, is freely used in designs. It can be constructed with the tools of geometry.

Problem: (a) Construct it. (b) Prove that the area of the arch equals the area of $\triangle ABC$.

11.05. Drafting. To determine the radius of a steel ball which touches a piece of metal at A is tangent to the metal at B, and is tangent to an imaginary line which is an extension of the end of the metal, CD. (See figure on next page.)

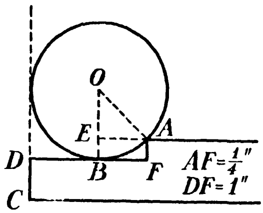
* Adapted from J. C. Stone and V. S. Mallory, *New Plane Geometry*. Benj. H. Sanborn, Chicago, 1939.

Let $AO = r$

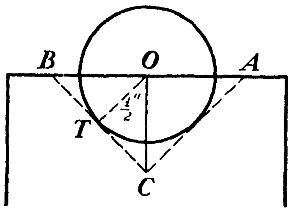
and $OE = a$

Then $OB = a + \frac{1}{4} = r$

$$AE = DF - DB$$
$$= 1 - r$$
$$= 1 - (a + \tfrac{1}{4})$$
$$= \tfrac{3}{4} - a$$
$$\overline{OE}^2 + \overline{EA}^2 = \overline{OA}^2$$
$$a^2 + (\tfrac{3}{4} - a)^2 = (a + \tfrac{1}{4})^2$$
$$a = .293''$$
$$r = .543''$$



G. 11.05



G. 11.06

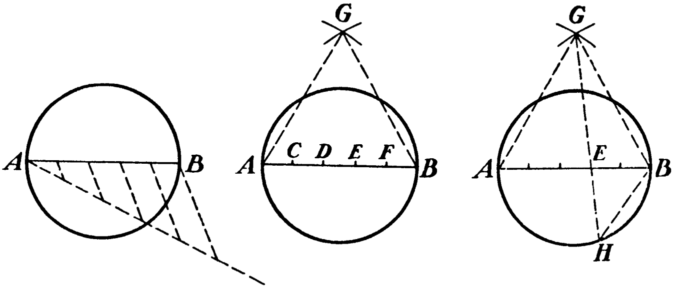
11.06. Drafting. To make a right-triangle cut in the end of a bar of metal into which a steel ball one inch in diameter will rest and be half submerged below the end of the bar. (See figure above.)

$$\triangle ABC \text{ shall be rt. isosceles } \triangle$$
$$\triangle OBC = \text{rt. isosceles } \triangle \text{ (why?)}$$
$$LB = 45^\circ$$
$$OT = \tfrac{1}{2}''$$
$$BT = OT = TC = \tfrac{1}{2}''$$
$$\overline{OT}^2 + \overline{TC}^2 = \overline{OC}^2$$
$$OC = \tfrac{1}{2}\sqrt{2}$$

Therefore $AB = \sqrt{2}$

11.07. Drafting. A geometric method used in mechanical drawing to inscribe regular pentagons in a circle is illustrated as follows:

(a) Divide the diameter of the circle into as many equal parts as there are sides of the required pentagon. (b) Upon the diameter as a side construct an equilateral triangle ($\triangle ABG$). (c) Draw a line from the vertex of the triangle to the second point of division on the diameter from *B* and produce it until it cuts the circle (Line



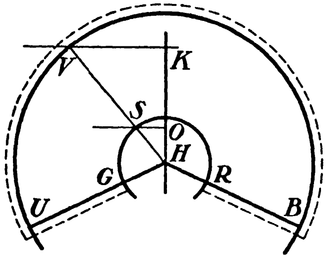
GEH). (d) Join *H* and *B*. Line *HB* is taken as the side of the required regular pentagon inscribed in the given circle.

11.08. Tinsmith. The tinsmith must lay off many patterns for the variously shaped articles he builds. In practically all of these the principles and constructions are basic. For example, note the following procedure for drawing a pattern for a frustum of a cone—useful to build funnels, hoppers, brooder stove shields, etc. Lay off a right angle *HKV*, as shown on page 133. Lay off altitude *KO*. Draw *OS* || *KV*. On *KV* lay off half the diameter of the large end (in this case *KV*), and on *OS*, half the diameter of the small end (in this case *OS*). Through *V* and *S* draw a line cutting *KH* at *H*. With *H* as center describe circles *BU* and *RG*. From *KV* and *OS* circumferences can be calculated. Lay off larger circumference *BU* and draw lines to *H*. Allow for seams.

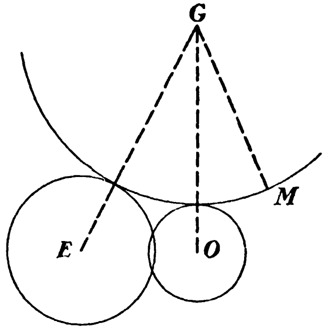
11.09. Warfare. The method described below was used during World War I to determine with a high degree of accuracy the position on guns many miles away. (See figure on page 133.)

Suppose *E*, *O*, and *M* represent the positions of three stations where observers have accurate recording apparatus. Each one notes the exact time that the instrument records the sound of the gun and telephones it to a central station. (This station is not shown on this drawing for its position does not affect the solution.)

If the instrument at *O* registers reception $\frac{1}{2}$ second after it is received at *M*, and the instrument at *E* records at 1 second after it is received at M, and conditions are such that sound travels at 1100 ft. per second, then O may be said to be 550 ft. and E 1100 ft. further from the gun than M is.



G. 11.08



G. 11.09

The officer at the central station has the exact location of stations E, O, and M on a scale drawing. With the same scale and O as a center, he draws a circle whose radius is 550 ft. With E as a center, he draws a circle whose radius is 1100 ft. To determine the position of the gun, the officer must draw a circle which is tangent to the two circles just drawn and which will pass through M.

Looking once more at the drawing, you can understand from the explanation given above that O is 550 ft. and E is 1100 ft. further away from G than M is.

Problem: Instead of O registering reception $\frac{1}{2}$ second later, use $\frac{3}{4}$ second, and instead of E registering 1 second later, suppose it were $1\frac{1}{4}$ seconds. Now make a careful scale drawing showing all construction lines necessary to construct the circle whose center shows the position of the gun.

GOLDEN SECTION 12.01

12.01. Daily Life. One of the constructions of plane geometry is the problem of dividing a given line segment into extreme and mean ratio. A segment thus divided is called the ***golden section.*** The Greeks considered this proportion to have mystical significance and referred to it as the ***divine proportion.***

A rectangle formed by using the longer part of the golden section as length and the shorter part as width is considered to be proportioned so as to be most pleasing to the eye.

Problems: (a) Select a picture frame which you think is well proportioned. Determine its semi-perimeter. Divide this into extreme and mean ratio and see how closely these divisions correspond to the actual length and width of the picture.

(b) If a picture frame is 6 in. long, how wide should it be in order that its proportion may be most pleasing to the eye? Note: Let w = width. Then $6 + w : 6 : : 6 : w$.

(c) The proportions of a rectangle that present the most pleasing aspect can be written $\frac{w + l}{l} = \frac{l}{w}$. If $l = 1$, find value of w.

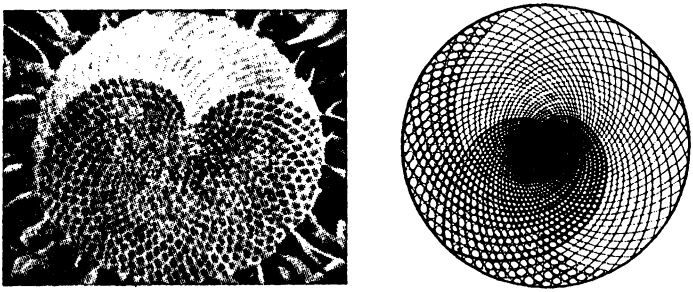
(d) Find the dimensions of some of the common sizes of films used in cameras. See how closely their width is .618 times their length.

Nature has used the golden section in vegetable and animal life, viz., in leaf arrangement on twigs, petal arrangement on flowers, seed arrangement on a sunflower, and the five-pointed starfish. The connection between these arrangements and the golden section depends upon the fact that these arrangements can be represented by a fraction in the series, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{8}$, $\frac{5}{13}$, $\frac{8}{21}$, $\frac{13}{34}$, etc. The denominators of these fractions form the series 2, 3, 5, 8, 13, 21, 34, ... As the numbers in this series increase in size, the quotient obtained by dividing any number by the next consecutive one becomes nearer and nearer .618+, which is the ratio of the larger segment to the whole when a line segment is divided into the golden segment.

"In the diagram of an oak twig at the right, notice that leaf bud 6 is directly above bud 1 on the stem. If 1 is the first bud of one cycle, then 6 is the first bud of the next cycle. In this figure a complete cycle consists of five leaves arranged equally distant apart in two revolutions of the spiral. We can represent this arrangement by the fraction $\frac{2}{5}$. The elm tree has the $\frac{1}{2}$ arrangement and the beech tree has the $\frac{1}{3}$ arrangement. Some trees have the $\frac{3}{8}$ arrangement, and some, including certain bushes, have the $\frac{5}{13}$ arrangement.

"The sunflower illustrates the use of the golden section in two ways. The fruit sockets of a sunflower head form a series of intersecting curves which seem to be logarithmic spirals. A very small sunflower head has 21 curves crossing 34 curves, another has 34 curves crossing 55 curves, and a large head has 55 curves crossing 89 curves. The numbers of these curves are also of the series 1, 2, 3, 5, 8, 13, 21, 34, . . .

"The pine cone and the common teasel have a similar arrangement of seed pods, except that their heads are not so flat as those of the sunflower."



PHOTOGRAPH AND GEOMETRIC DRAWING OF THE SUNFLOWER*

"The most beautiful rectangle is constructed as follows (see first figure on page 186):

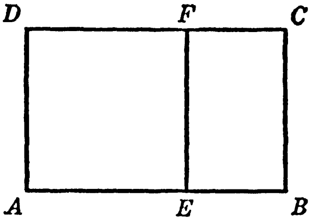
"Divide the base *AB* into extreme and mean ratio so that

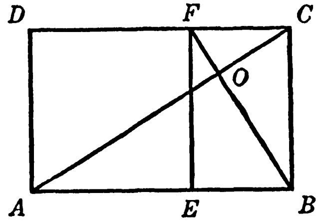
$$\frac{AB}{AE} = \frac{AE}{EB}.$$

Then construct the rectangle *ABCD* having the height $AD = AE$. This rectangle has many interesting properties. It consists of the similar rectangle *EBCF* and the square *AEFD*. Its diagonal *AC* is perpendicular to *BF*, the diagonal of *EBCF*. It is sometimes called the rectangle of whirling squares. Since its numerical properties are related to plant and animal life, including the human skeleton, and since it has a property of dynamic symmetry, its use helps to give life to painting and sculpture. See how many

* Picture from J. C. Stone and V. S. Mallory, *New Plane Geometry*, p. 417. Benj. H. Sanborn & Co., Chicago, 1937.

of the following properties you can prove, and determine whether or not they apply to any rectangle."





"*Problems:* (a) The square AF is equal to a rectangle whose base is AB and whose altitude is equal to EB.

(b) Rectangle $AC \sim$ rectangle EC. Suggestions: $\frac{AB}{AE} = \frac{AE}{EB}$.
Then $\frac{AB}{BC} = \frac{BC}{EB}$.

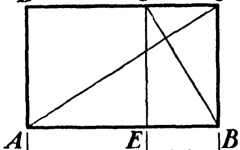
(c) $\triangle ABC \sim \triangle BCF$.

(d) $BF \perp AC$.

(e) AB is the mean proportional between BC and $AB + BC$.

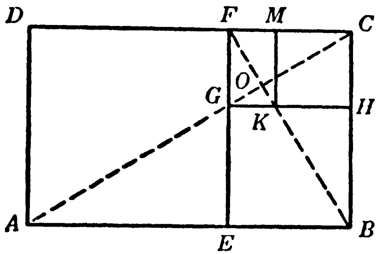
(f) $\frac{AO}{OB} = \frac{OB}{OC} = \frac{OC}{OF}$.

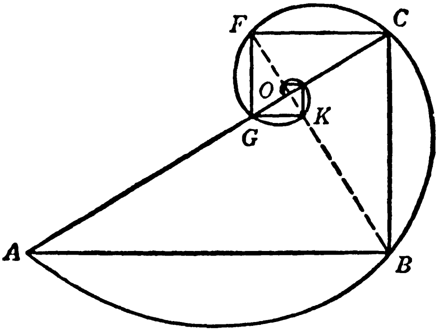
(g) If $AD = 1$, show that rectangle $EBCF$ is the reciprocal of rectangle $ABCD$, or that $EBCF = \frac{1}{ABCD}$.



"If the length of the most beautiful rectangle $ABCD$ is represented by $\frac{\sqrt{5}+1}{2}$ units or 1.618, its width is represented by 1 unit. The rectangle $ABCD$ is composed of the reciprocal rectangle $EBCF$ and the square $AEFD$.

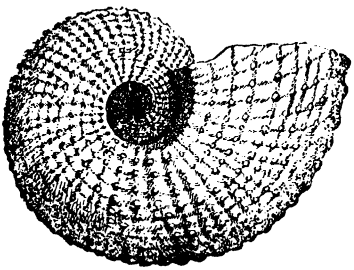
"The reciprocal rectangle $EBCF$ can be constructed by drawing $BF \perp AC$ and then drawing $EF \perp DC$. The reciprocal rectangle $EBCF$ cuts off the square $AEFD$. Likewise the rectangle $GHCF$ is the reciprocal of the rectangle $EBCF$ and cuts the square $EBHG$ from rectangle $EBCF$. Again, the rectangle $GKMF$ cuts the square $KHCM$ from rectangle $GHCF$. If this process is continued in-definitely, a series of squares is formed which continually get smaller and arrange themselves about 0 at intervals of 90°. (See figure at left below.)

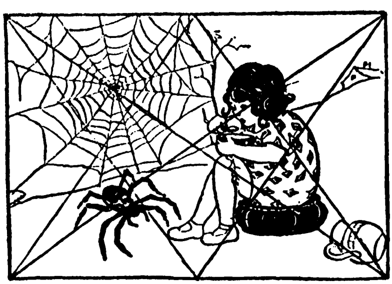




"If the longer side of each rectangle is drawn as shown above, a broken line is formed which winds itself about the point O, but never reaches O. The points $ABCFGK$ . . . lie on a curve called the logarithmic spiral." This design is often found in various shells, as in the Nautilus.

It has been pointed out that the reciprocal rectangle of the most beautiful triangle can be constructed by a perpendicular line from a vertex to the diagonal. These lines (diagonals and perpendiculars) are often used to locate the points of interest in a painting, drawing, or photograph. This is illustrated in the picture of Little Miss Muffet.* †





* Picture from J. C. Stone and V. S. Mallory, *New Plane Geometry*, p. 417. Benj. H. Sanborn and Co., Chicago, 1937.

† Quotations from A. M. Welchons and W. R. Krickenberger, *Plane Geometry*, Revised Edition, pp. 425–428. Ginn and Co., Boston, 1940.

GREAT CIRCLES 13.01–13.02

13.01. ASTRONOMY. The *meridian* of any point on the earth's surface is the great circle which passes through that point and the earth's two poles.

13.02. GEOGRAPHY. Locations on the earth's surface are described by reference to two great circles, the equator and the prime meridian. Terrestrial distances between cities and other points as well as journeys are often stated in terms of degrees, or miles on an arc, of a great circle passing through the end points. These distances on an east-west line are referred to as differences in longitude; on a north-south line as differences in latitude.

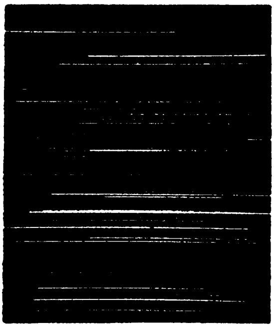
The shortest distance between any two points on the earth's surface is the length of the shorter arc of a great circle passing through the two points—thus the phrase "great-circle sailing."

A nautical mile (geographical mile or knot) is the length of one minute of arc of a great circle of the earth.

ALSO SEE Spheres G. 22.01: Spherical angles G. 23.01; Spherical triangles 24.01.

LOCUS 14.01

14.01. ASTRONOMY.* Star paths are often studied by photographing their positions at various intervals with a fixed camera. Stars



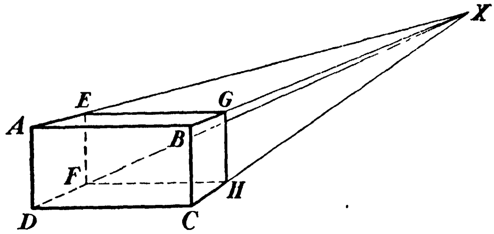
near the poles trace circular paths on the photographic plate, while stars near the equator trace straight lines.

* Adapted from W. Bartky, *Highlights of Astronomy*, p. 7. The University of Chicago Press, Chicago, 1935.

PARALLEL LINES 15.01–15.05

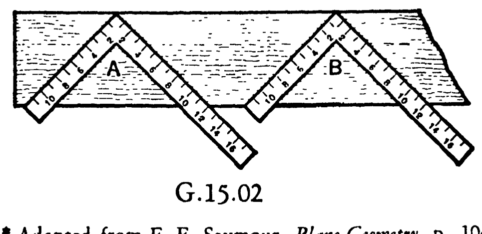
15.01. Architecture. In solid geometry students are taught to visualize and comprehend three-dimensional objects as portrayed by two-dimensional drawings. The rules which guide the artist in making these representations are called the laws of perspective, and they are entirely geometric. Architects, interior decorators, costume designers, and engineers use these laws when they make sketches to show their ideas to their customers. People who make blueprints observe these laws in their drawings so that carpenters, bricklayers, masons, and plumbers can carry out the plans of the architect.

Good perspective is secured by representing parallel lines by non-parallel lines and right angles by obtuse or acute angles. In the figure below, a rectangular solid is shown. Can you tell which parallel edges are represented by non-parallel lines in the drawing? Which right angles of the solid are represented by oblique angles?

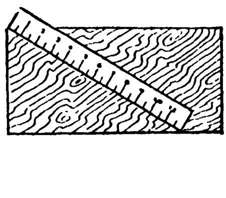


Problem: Find in some journal an architect's design of a house and answer the above questions with respect to it.

15.02. Carpentry.* If a plank is to be sawed into three strips of equal width, place the square as in *A* and make marks at the 4-inch



G.15.02



G.15.03

* Adapted from F. E. Seymour, *Plane Geometry*, p. 104. American Book Co., New York, 1925.

and 8-inch divisions. Repeat the operation with the square similarly placed at *B*. Lines joining corresponding marks will be parallel and divide the plank into three equal parts.

Problem: Explain the geometric principle basic to this process.

15.03. Carpentry.* A board is to be ripped into three strips of equal width. A carpenter places his 12-inch ruler as shown in the drawing on the preceding page, and marks points on the board at the 4-inch and 8-inch marks. He then rips the board by sawing lengthwise through these points.

Problem: Prove that the strips are of equal width.

15.04. Daily Life. The concept of parallelism must have been developed early in the life of the race. But when students of the universe, noting that all sun rays entering a room at the same time are parallel, concluded that for practical purposes the light rays from *any* star (including the sun) to *any* place on the earth's surface at *any* moment are parallel, a tremendously significant fact was established.

With this fact it was possible to determine the height of the pyramids, inaccessible cliffs, the diameter of the earth (and in turn the extent of the earth's unexplored areas), the fact that on the equinoxes the sun rises due east from every point on the earth's surface, etc.

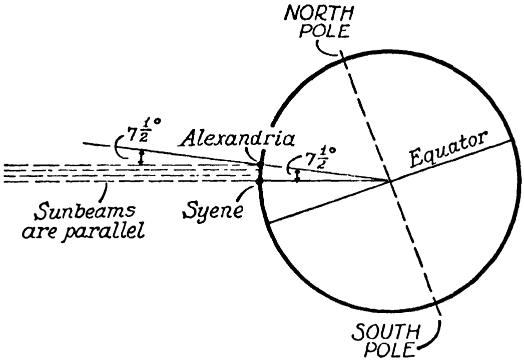
Problem: Find the diameters of both the earth and the sun and show by scale drawing why it is reasonable to regard the light rays from the sun to the north and south poles as parallel.

15.05. Daily Life.† As early as 200 years before the birth of Christ scientific men wondered how large the earth was. Eratosthenes, the librarian at Alexandria and a man who knew something about plane geometry, set the circumference of the earth at 25,000 miles. He arrived at this figure by applying plane geometry to these two facts which were recorded in the library as events of importance in connection with a certain calendar festival: At noon on this day

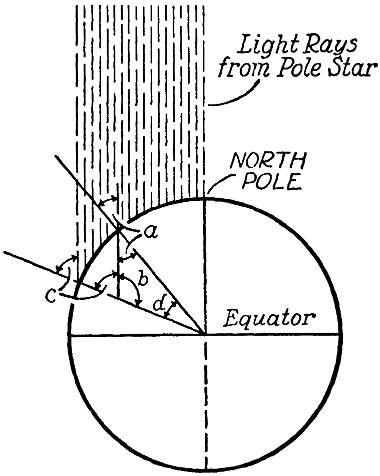
* Adapted from J. C. Stone and V. S. Mallory, *New Plane Geometry*, pp. 148–149. Benj. H. Sanborn, Chicago, 1937.

† Adapted from L. Hogben, *Mathematics for the Million*, pp. 233–236. W. W. Norton and Co., New York, 1937.

the sun was reflected in a deep well near Syene (now Assouan) at the first cataract of the Nile. On the same day at Alexandria, five hundred miles due north of Syene, the shadow of a pillar at noon showed the sun seven and one half degrees south from the vertical.



A somewhat similar exercise can be engaged in by present-day high school pupils by finding the difference between the horizon angles of the North Star at two places, one due north or south of the other, such as Pittsburgh and Miami. (Prove d equal to $c - a$, in figure below.)

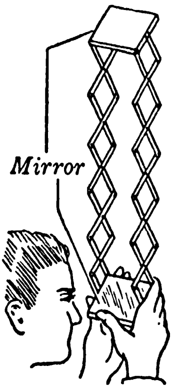


Also see Angles G. 3.04, 3.05, 3.19; Circles G. 5.10; Diagonals G. 8.01; Geometric forms G. 10.01; Spheres G. 22.02; Triangles G. 28.05.

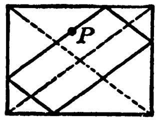
PARALLELOGRAMS 16.01–16.05

16.01. Daily Life. The collapsible mirror, shown here, is useful to see over the heads of a crowd at a parade or an athletic contest. No matter to what extent the mirror is extended, it is always in "working position."

Problem: Explain how it is constructed and what geometric principles were used in its design.



G. 16.01



G. 16.02

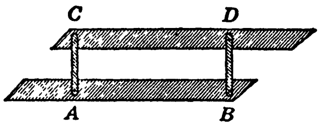
16.02. Daily Life. If a billiard ball is placed at P, in what direction must it be shot to return to the same point after hitting all four sides? Note: Consider starting ball parallel to a diagonal of the billiard table. Is there a relationship between the length of the path traveled by the ball and the length of the diagonals of the table?

16.03. Drafting.* The figure below shows an instrument called a parallel ruler, used for drawing parallel lines. It consists of two rulers, AB and CD, which are connected by two cross-pieces, AC and BD. The four parts work on pivots at A, B, C, and D, so that by revolving AC and BD the rulers may be brought closer together or placed farther apart. $AB = DC$ and $AC = BD$.

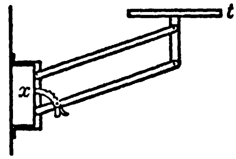
Problem: Prove that for all positions, $AB \parallel CD$.

* Adapted from J. C. Stone and V. S. Mallory, *New Plane Geometry*, p. 139. Benj. A. Sanborn, Chicago, 1937.

16.04. Housewife.* A store offers an adjustable wall bracket for supporting a small working table. The salesman claims that the table top *t* will always be horizontal provided *x* is fixed to a



G. 16.03

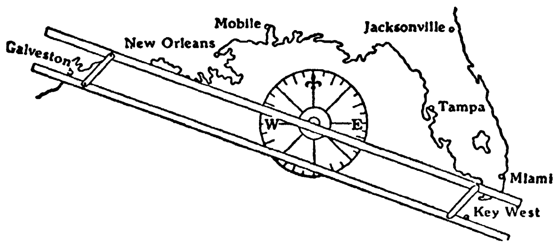


G. 16.04

vertical wall. The housewife could not see why this should be so when the table is raised or lowered. The salesman assured her that the facts of plane geometry prove his claims.

Problem: Can you furnish the proof?

16.05. Navigation.† The parallel ruler is used by sailors in determining the courses of their ships in sailing from one port to another. Thus, to determine the course or direction from Galveston to Key West, in the Gulf of Mexico, the parallel ruler is placed so that one of the rulers connects these two points on a navigator's map.



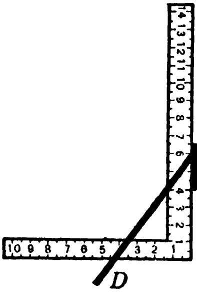
Problem: Explain how the other ruler may then be placed in order to read the required direction on the mariner's compass which is printed on the map.

* Adapted from D. E. Smith, W. D. Reeve, and E. L. Morss, *Text and Tests in Plane Geometry*, p. 109. Ginn and Co., Boston, 1933.

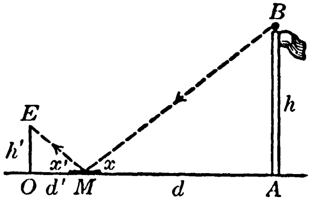
† Adapted from J. C. Stone and V. S. Mallory, *New Plane Geometry*, p. 141. Benj. A. Sanborn, Chicago, 1937.

PROPORTION 17.01–17.04

17.01. Carpentry.* A carpenter's steel square may be used to solve problems in proportion. The cut below shows a square graduated to half-inches. *ACD* is a frame made of two pieces of wood hinged at *C*. *AC* can slide on the long arm of the square. Find a number, x, so that $\frac{12}{16} = \frac{9}{x}$. Place *ACD* so that the outer edge of *CD* is on 6 on the long arm (12 half-inches) and $4\frac{1}{2}$ on the short arm. Without changing the angle of adjustment, move the frame higher until the outer edge of *CD* passes through 8 on the long arm. How is the value of x found? Why?



G. 17.01



G. 17.02

17.02. Daily Life.† The figure above shows how height h of an object can be measured by means of a mirror placed on the ground at M, d' being the distance from M to the point O at which an observer can see the top of the object in the mirror, and h' being the height of the observer's eye E above the ground.

This idea can also be used to find the height of objects reflected in pools of water.

Problem: Why does $x = x'$? What proportion would you use in finding h?

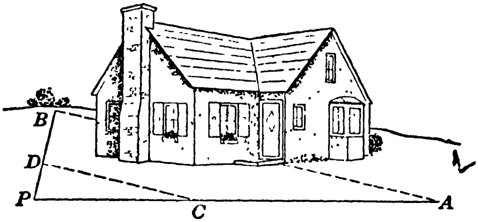
17.03. Engineering. A brakeman pulls with a force of 125 pounds on a brake wheel 20 inches in diameter. The force is communicated

* Adapted from M. Sykes, C. E. Comstock, and C. M. Austin, *Plane Geometry*, p. 285. Rand, McNally and Co., New York, 1933.

† Adapted from D. E. Smith, W. D. Reeve, and E. L. Morss, *Text and Tests in Plane Geometry*, p. 187. Ginn and Co., Boston, 1933.

to the brake chain by means of an axle $2\frac{1}{2}$ inches in diameter. Find the pull on the brake chain.

17.04. SURVEYING.* The distance between two accessible points *A* and *B* which are separated by an obstacle may be measured as follows: From a convenient point *P* the distances *PA* and *PB* are



measured. Then in these lines points *C* and *D*, respectively, are located so that $PC : PA = PD : PB$. Finally, *CD* is measured.

Problems: (a) Prove the proportion by means of which *AB* may now be computed.

(b) If $PA = 240$ ft., $PC = 40$ ft., $CD = 20$ ft., find *AB*.

ALSO SEE Diagonals G. 8.01; Similar triangles G. 21.01.

PYTHAGOREAN THEOREM 18.01

18.01. SURGERY. The following news was carried by the Associated Press on Nov. 28, 1940. "Chicago, Nov. 28. A professor of physics and a surgeon who hitched together the X-ray and a geometrical theorem described the result today as a new and more exact method of finding foreign objects—such as bullets, needles or glass—in the human body.

"Dr. A. D. Hummell, of Eastern Kentucky State Teachers College, told the American Physical Society how he and his co-worker, Dr. O. F. Hume, placed patients between two sets of right-angle rods, notched for measurement and fixed a known distance apart. Then they took two X-ray pictures from points a few centimeters apart.

"Dr. Hummell said that by use of the Pythagorean theorem—that the square of the long side of a right triangle is equal to the

* Adapted from J. C. Stone and V. S. Mallory, *New Plane Geometry*, p. 332. Benj. A. Sanborn, Chicago, 1937.

sum of the squares of the short sides—they could determine the exact position of the object from the right triangle formed by its shadow and the shadow of the rods."

ALSO SEE Geometry construction G. 11.05, 11.06.

RECTANGULAR SOLIDS 19.01

19.01. CARPENTRY. Sand will run down the sloping bottom of a bin if the angle of the slope of the bottom planes is 45° or more.

A carpenter wishes to build a square bin with the planes of the pyramid-shaped bottom sloping toward the center at an angle of 45°. He wishes to support the "valleys" where the sloping surfaces intersect but does not know the size of the angle which the "valley" makes with the horizontal.

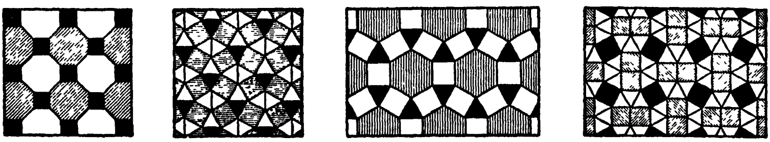
Problem: How could geometry aid him?

ALSO SEE Geometric forms G. 10.02; Parallel lines G. 15.01; Volumes G. 30.03, 30.04.

REGULAR POLYGONS 20.01–20.02

20.01. AUTO MECHANICS. Straight-eight engines in automobiles are equipped with a single distributor, which means that the platinum points must close and open eight times for each revolution of the distributor rotor. The cam is therefore in the shape of an octagon. Similarly a "six" is equipped with a hexagon cam and a "four" with a square cam.

20.02. DAILY LIFE.* Tile floors, parquet flooring, linoleum patterns, etc., are frequently built up by using *regular* polygons, as illustrated below.



* From D. E. Smith, W. D. Reeve, and E. L. Morss, *Text and Tests in Plane Geometry*, p. 232. Ginn and Co., Boston, 1933.

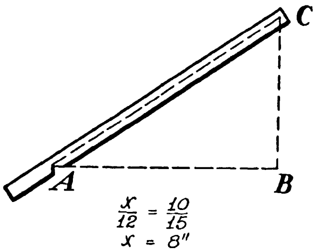
Notice that in each case the polygons are so chosen that the vertices that meet at a common point fill the space around the point.

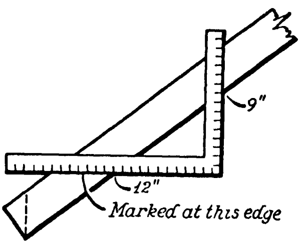
Problems: (a) Can a tile floor be laid entirely of tiles that are regular pentagons? Rectangular octagons?

(b) Using only regular polygons of the same number of sides in each case, which of the regular polygons of 3, 4, 5 . . . 16 sides can be fitted together in making tile floors?

SIMILAR TRIANGLES 21.01–21.14

21.01. Carpentry. The carpenter uses similar triangles and proportion in marking cuts on rafters, as illustrated below. If *BC* is 9 ft. and *AB* is 12 ft., the carpenter would use the 9-inch and 12-inch marks on his square to get the *A*, *C*, and *D* cuts on the rafter. The square might be placed at *A* as illustrated, and the rafter would be marked along the outside edge of the long arm. For convenience the 12-inch mark on the long arm is used whenever possible.

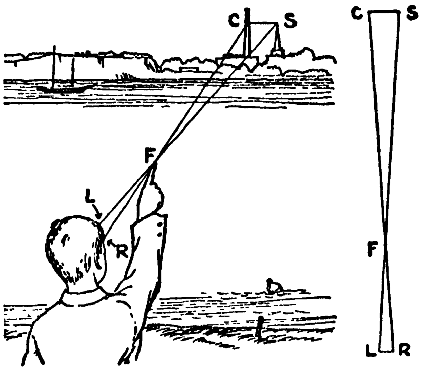




Problem: What mark would the carpenter use on the short arm of his square if *BC* in the figure were 10 ft. and *AB* 15 ft.?

21.02. Daily Life. "Tom wants to know how far it is to the tall chimney, *C*, across the river. He estimates this distance as follows:

"Stretching out his right arm at full length and shutting his left eye, he points at the chimney, *C*, so that the line of sight from his right eye, *R*, to the chimney passes over the tip of his finger, *F*. Then, without moving his arm or finger, he shuts his right eye and opens his left eye, sighting with the left eye, L, across the tip of his finger, as before. He now finds that his finger is pointing at a steeple, S, instead of at the chimney. Tom estimates that the distance, CS, from the chimney to the steeple is about 60 ft. He then multiplies 60 ft. by 10 and gets 600 ft., which he says is the



approximate distance from him to the chimney." In rule form: *To estimate a distance straight ahead of you, estimate distance CS, as described above, and multiply the result by 10.*

"*Problem:* Prove geometrically that this procedure is a valid one. The method of solution must accept this fact, established by numerous measurements: The distance, RF, from the right eye to the tip of outstretched right arm is about ten times the distance, RL, beween the pupils of the eyes. The procedure also assumes that one can estimate a *short* distance (like CS) which is nearly perpendicular to the line of sight much more reliably than a long distance (like FC) which is straight ahead."*

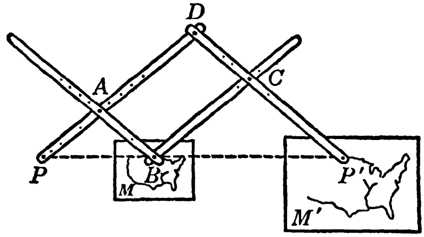
21.03. Drafting.† The figure below illustrates the principle of the pantograph, an instrument used to enlarge or reduce maps and drawings. The bars, which are pivoted at B and D, can be adjusted at A and C to form the ▭$ABCD$ having its sides in different

* Quotation from G. D. Strayer and C. B. Upton, *Strayer-Upton Practical Arithmetics, Book Six*, (Copyright) p. 495. American Book Co., New York, 1934. Used by permission of the publishers.

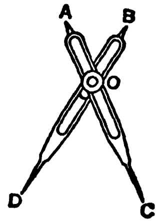
† Adapted from D. E. Smith, W. D. Reeve, and E. L. Morss, *Text and Tests in Plane Geometry*, p. 187. Ginn and Co., Boston, 1933.

ratios. A sharp point is fitted at P so as to hold P stationary, a tracing point is placed at B, and a pencil point is placed at P'.

Problem: Why are $\triangle PBA$ and $BP'C$ always similar? As B is moved over the outline of a figure, such as the map M, why does P' trace a similar figure?



G. 21.03



G. 21.04

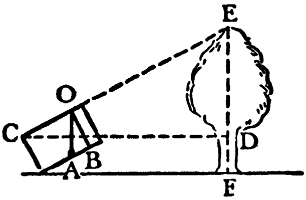
21.04. Drafting. If a draftsman wishes to increase or decrease the size of a copied drawing, he uses a pair of proportional compasses. Two equal bars are joined by an adjustable thumbscrew at O, making $OA = OB$. (See figure above.)

Problem: Why is $\triangle OAB$ similar to $\triangle OCD$? (Triangles are formed by thumb screw and points of compasses.) Is $\frac{CD}{AB} = \frac{OC}{OA}$? Why? If the compasses are adjusted so that $\frac{OC}{OA} = \frac{4}{3}$, how will any measurement made with the points C and D compare with a measurement made with A and B?

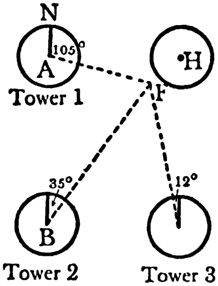
21.05. Forestry.* "A simple form of height-measuring instrument, the hypsometer, is often used by lumbermen and forest rangers to determine the height of trees. A hypsometer can be made of graph paper fastened to a board, along the top of which are two sights. A plumb bob is suspended at point O. To find the height of a tree, the observer may hold the instrument to his eye at C and sight along CO toward the top of the tree. He should note the lengths of OB and AB and measure the length of CD." (See figure on page 200.)

* A. Leonhardy, M. Joseph, and R. D. McLeary, *New Trend Geometry*, p. 342. Charles E. Merrill Co., New York, 1940.

Problem: Prove that $\triangle AOB \sim \triangle CED$. What proportion must then be used to compute DE? How may the length of EF be found if the tree and the observer are on level ground?



G. 21.05



G. 21.06

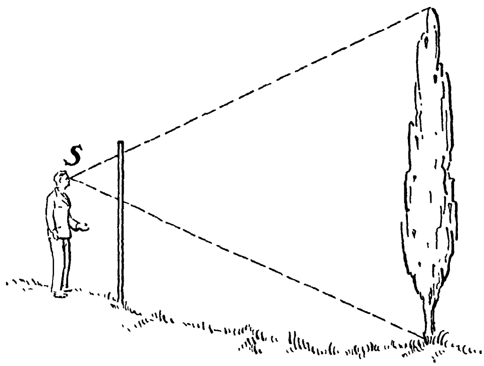
21.06. Forestry.* "Fire towers are used in the forest areas of our country for the purpose of locating forest fires. Each fire tower is the center of an imaginary circle with a radius pointing north in it, from which all angles are measured. When the ranger in Tower 1 sees a fire, he sights toward it with his alidade, notices the angle, and reports it to headquarters by telephone. Suppose that this reading is 105° east of north, and that from Tower 2 the direction of the same fire is 35° east of north. Then the point where these two lines intersect on the map determines the exact position of the fire, and the fire fighters may be sent to the proper place without delay. The report from a third tower is used as a check on the accuracy of the other two."

Problems: "(a) Prove that triangle ABF is similar to the triangle on the ground formed by the two towers and the point at which the fire is located.

(b) "If Tower 1 is 50 miles north of Tower 2 and 40 miles west of the headquarters station H, use the angles reported above to make a scale drawing, showing the location of the fire. From the scale determine its distance and direction from headquarters."

* A. Leonhardy, M. Joseph, and R. D. McLeary, *New Trend Geometry*, pp. 341-342. Charles E. Merrill Co., New York, 1940.

21.07. Forestry. When shadows cannot be used to determine the height of trees, the following procedure is sometimes used. A staff is planted upright in the ground. A forester sights from S to the top and the foot of tree. His assistant notes where his line of sight crosses the staff.



Problem: What measurements does the forester need to make? State the geometric reasoning for the validity of this technique.

Sometimes a question arises whether a certain tree is tall enough to furnish a long piece of lumber. It is of great importance to the forester to learn this without first cutting the tree. The question is answered by the following procedure, if the tree is on reasonably level ground: On a stake the forester marks the height of his eyes from the ground. He then measures a distance from the base of the tree equal to the length of log desired but returns toward the tree the distance his eyes are from the ground. At this point the stake is driven into the ground so that the height above the ground equals the height his eyes are from the ground. He then lies on the ground with his feet against the stake and sights over the stake. His line of sight will indicate the height needed to furnish the desired length of log.

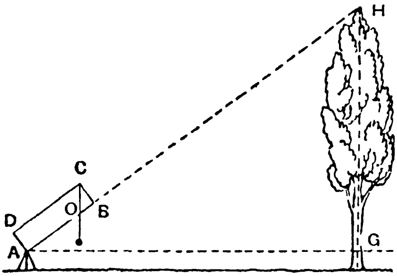
Problem: Prove by geometry that this is a valid procedure.

21.08. Forestry.* A forester used the hypsometer illustrated to measure the heights of trees. He attached a plumb line to a

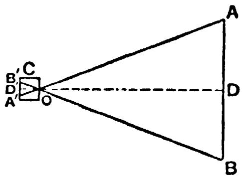
* Adapted from W. W. Strader and L. D. Rhoads, *Modern Trend Geometry*, p. 326. The John C. Winston Co., Philadelphia, 1940.

corner (*C*) of a rectangular board *ABCD* and made the board adjustable to various positions on an upright stand. The width of the board (*BC*) was 6 in. and *AB* was marked in inches. To measure the height of a certain tree (*GH*), the forester stood at a point so that his distance (*AG*) from the tree was 120 ft. As he sighted along *AB* to the top of the tree, the plumb line swung so that *OB* was 8 in.

Problem: Prove $\triangle OBC$ similar to $\triangle AGH$. Find the height of the tree *GH*, allowing 5 ft. from *A* to the ground. What is the reason for this allowance?



G. 21.08



G. 21.09

21.09. Photography.* Lucy has a camera (*C*) with a fixed focus. The lens at *O* is always 4 in. from the film at *D*. The width of the film *A′B′* is 2 in. Her range finder is broken and she wants to take a picture of her house (*AB*), which is 60 ft. long.

Problem: Find the length of *OD* for her so that she may know how far to stand from the center of her house to include its whole length on her picture.

21.10. Student.† The cross-staff may be used in measuring either vertical or horizontal distances. The horizontal bar *AD* is free to move on the vertical bar *EG* and may be adjusted so that *A*, *E*, and *C* are in a straight line.

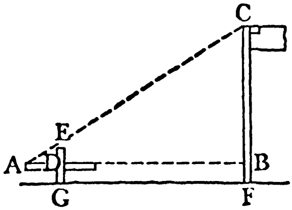
* Adapted from W. W. Strader and L. D. Rhoads, *Modern Trend Geometry*, p. 319. The John C. Winston Co., Philadelphia, 1940.

† Adapted from A. Leonhardy, M. Joseph, and R. D. McLeary, *New Trend Geometry*, p. 337. Charles E. Merrill Co., New York, 1940.

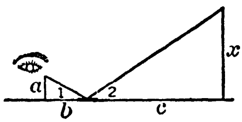
Problems: (a) What proportion will you use in order to find CB?

(b) How may CF be found, if points G and F are both on level ground?

(c) If the figure is so drawn that $DE = 12$ in., $AD = 15$ in., $GF = 50$ ft., and $DG = 4$ ft., what is the length of CF?



G. 21.10

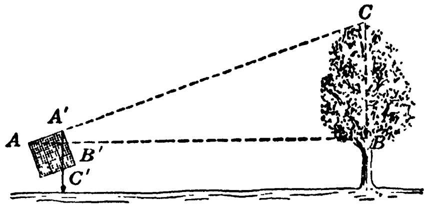


G. 21.11

21.11. Student.* Jack placed a mirror horizontally on the ground and stood at a distance b from the mirror so that the top of his school building (x) was just visible in the mirror. The height of Jack's eye was 5 ft., the length of b was 3 ft., the length of c was 45 ft., and $\angle 1 = \angle 2$ by the laws of physics.

Problem: Find the height of the school building (x).

This method may also be applied to find the height of a tree or a building if a reflection pool is near-by.



G. 21.12

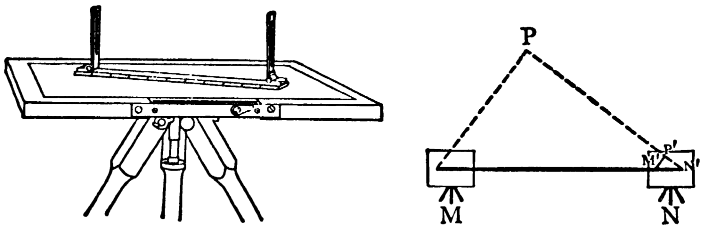
21.12. Student.† A modern form of the geometric square (or hysometer) can easily be made by tacking a piece of graph paper onto a board and attaching sights and a plumb bob.

* Adapted from W. W. Strader, and L. D. Rhoads, *Modern Trend Geometry*, p. 319. The John C. Winston Co., Philadelphia, 1940.

† Adapted from J. C. Stone and V. S. Mallory, *New Plane Geometry*, p. 333. Benj. A. Sanborn, Chicago, 1937.

Problem: Show that $\triangle A'B'C' \sim \triangle ABC$ and that the length of BC can be computed from $B'C'$.

21.13. Surveying. "The plane table is a simple surveying instrument used in finding the distance to an inaccessible point or in mapping where a high degree of accuracy is not necessary. It is a drawing board mounted on a tripod, free to rotate, but fitted with a clamping device so that it may be fixed in position.



"The sighting device, called an alidade, is a metal ruler graduated on one edge and having a vertical sight at each end.

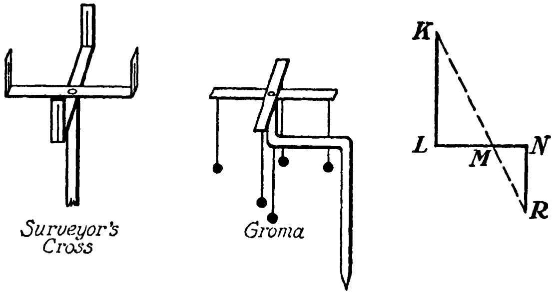
"The method of intersections may be used in finding the distance to an inaccessible point, such as P. On the drawing board $\triangle M'N'P'$ is drawn similar to $\triangle MNP$ on the ground, and MP may be found by proportion or by means of a scale. The method is as follows: A convenient base line MN is measured off on the ground, and the board is set in a level position with point M' on the board over point M on the ground. Stake N is sighted with the alidade, and $M'N'$ is drawn on the paper, fastened to the board, along the edge of the alidade and of proper length to represent MN drawn to scale. The alidade is then used to sight and draw a line toward P. The table is moved to N. After sighting back along the base line to M so that $M'N'$ is again in line with MN, point P is sighted and a line drawn toward it, intersecting the previous line at P'."*

"*Problems:* (a) Prove $\triangle MNP$ similar to $\triangle M'N'P'$. How may MP be found by proportion? By scale?

(b) If, in the above figure, $MN = 100$ ft., $M'N' = 10$ in., and $M'P' = 16\frac{1}{2}$ in., what is the distance from M to P?"

* A. Leonhardy, M. Joseph, and R. D. McLeary, *New Trend Geometry*, p. 340. Charles E. Merrill Co., New York, 1940.

21.14. Surveying. "The Surveyor's Cross is a modern adaptation of the groma of Roman times. The former consists of a double alidade with the two parts rigidly set at right angles on a tripod. The purpose of this instrument is to construct right angles, and it is used particularly in laying out streets and highways.



"*Problem:* In the above figure, a situation is shown in which the groma may be used in indirect measurement of line *KL*. If angles *L* and *N* are made right angles by the use of the surveyor's cross, if *LMN* and *KMR* are straight lines, and if $LM = 50$ ft., $MN = 30$ ft., and $NR = 66$ ft., compute *KL*."*

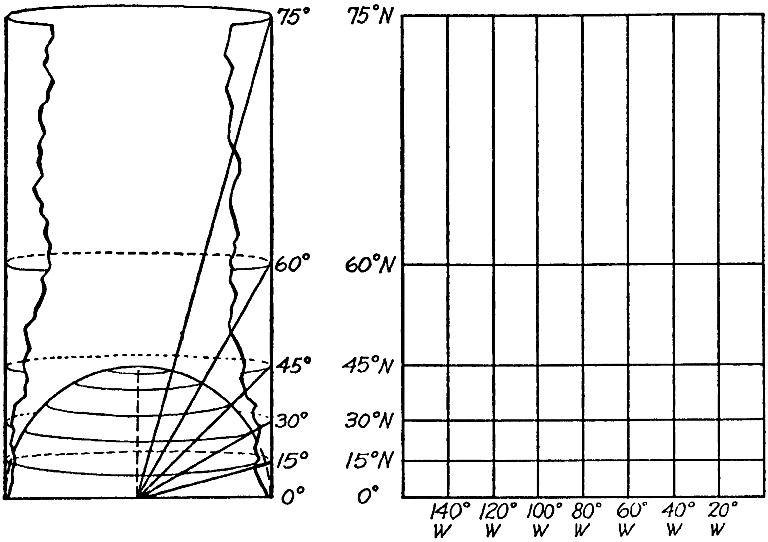
SPHERES 22.01–22.02

22.01. Astronomy. The imaginary sphere on which the heavenly bodies seem to be projected is called the celestial sphere. The great circles passing through the poles of this sphere are called ***hour angles***. ***Declination*** is also measured on a great circle, the celestial meridian. The astronomical, or celestial, location of all stars is given in terms of hour angle (***right ascension***) and declination, while the terrestrial location of stars is given in terms of ***altitude*** and ***azimuth***. Both altitude and azimuth are stated in terms of degrees representing distances on great circles.

22.02. Daily Life. A geometric drawing and practice in interpreting it aid tremendously in understanding the principles used by Mercator to make maps of various sections of the earth. The accompanying sketch, showing a globe enclosed by a cylindrical

* A. Leonhardy, M. Joseph, and R. D. McLeary, *New Trend Geometry*, p. 341. Charles E. Merrill Co., New York, 1940.

surface tangent along the equator, illustrates how he used parallel lines to represent the meridians and parallels of latitude. Furthermore, the sketch also reveals how a Mercator projection represents rather faithfully those regions near the equator but distorts pretty badly the regions near the pole.



SPHERICAL ANGLES 23.01

23.01. Astronomy. The *bearing* of one place from a second place is the angle between the arc of a great circle drawn from the second place to the first place, and the meridian of the second place.

SPHERICAL TRIANGLES 24.01

24.01. Astronomy. The *astronomical triangle* is the basis for many astronomical calculations. The sides of this triangle are arcs of great circles passing through the zenith, pole, and star.

STRAIGHT LINES 25.01

25.01. Daily Life. The geometric fact that two points determine a straight line is applied when a gun barrel is equipped with two sights.

The plowman, beginning to "plow together" a strip of land, knows that if he "marches upon two points" the first furrow will be straight. This will result in all points of the plowed plot reaching the boundary fence at about the same time. The phrase "marches upon two points" means that the plowman selects two points beyond the end point of the furrow and then drives the team so as to keep these two points in line.

The same practice may be used by a person walking across a field if he wishes to reach his destination with the least expenditure of energy.

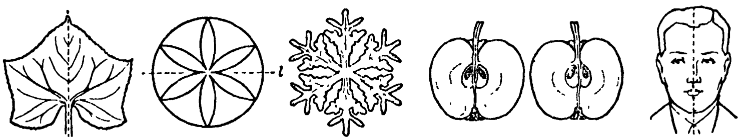
Also see Geometry construction G. 11.08; Locus G. 14.01.

SYMMETRY 26.01–26.02

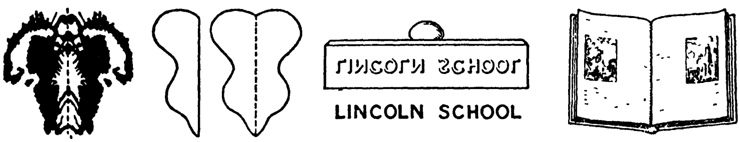
26.01. Architecture. Symmetry is basic in architectural beauty. In the opinion of many, the most beautiful building in the world is the Taj Mahal at Agra, India. It was built by Shah Jahan as a tomb for his favorite wife, Mumtaz-i-Mahall, at a reputed cost of around $15,000,000.

Problem: Examine a picture of the Taj Mahal and locate the axes of symmetry.

26.02. Daily Life.* Symmetry is a geometric concept freely ob-



(1) Melon Leaf (2) Design (3) Snow Crystal (4) Apple (5) Human Head



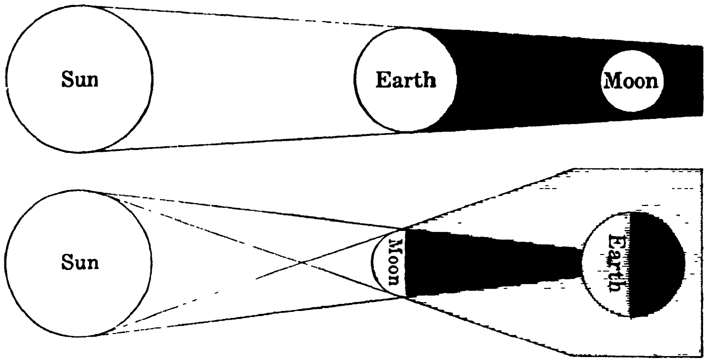
(3) Ink Blot (4) Cut Paper (5) Rubber Stamp (6) Pages of a Book

* From D. E. Smith, W. D. Reeve, and E. L. Morss, *Text and Tests in Plane Geometry*, pp. 26 and 27. Ginn and Co., Boston, 1933.

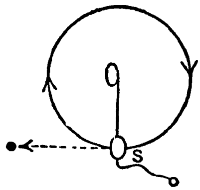
served in nature and in the works of man. It is an important characteristic of beauty in designs.

TANGENTS 27.01–27.02

27.01. Astronomy.* Geometry is used in the study of eclipses. Tangent lines indicate the extent of the eclipse, as indicated in sketch below.



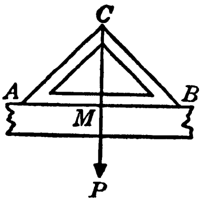
27.02. Daily Life.† If a body is rotated rapidly in a circle and then suddenly released, it tends to fly off at a tangent to the path of the circular motion. This is the principle on which the sling (*S*) operates.



Problem: What other illustrations can you give of particles from a rapidly revolving wheel flying off at a tangent to the path of circular motion?

Also see Circles G. 5.08, 5.09.

TRIANGLES 28.01–28.07



28.01. Carpentry.‡ The figure at right shows how a plumb line *CP* and a triangular frame *ABC*

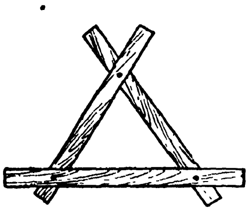
* From J. C. Stone and V. S. Mallory, *New Plane Geometry*, p. 25. Benj. H. Sanborn and Co., Chicago, 1937.

† Adapted from W. W. Strader and L. D. Rhoads, *Modern Trend Geometry*, p. 205. The John C. Winston Co., Philadelphia, 1940.

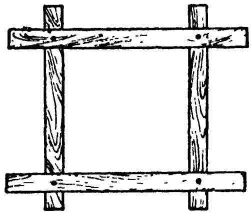
‡ Adapted from D. E. Smith, W. D. Reeve, and E. L. Morss, *Text and Tests in Plane Geometry*, p. 84. Ginn and Co., Boston, 1933.

having $AC = BC$ can be used by the carpenter instead of a spirit level to tell when the beam AB is horizontal.

28.02. Daily Life. If we construct a triangle such as figure 1, it is seen that its shape cannot be changed unless one or more of the

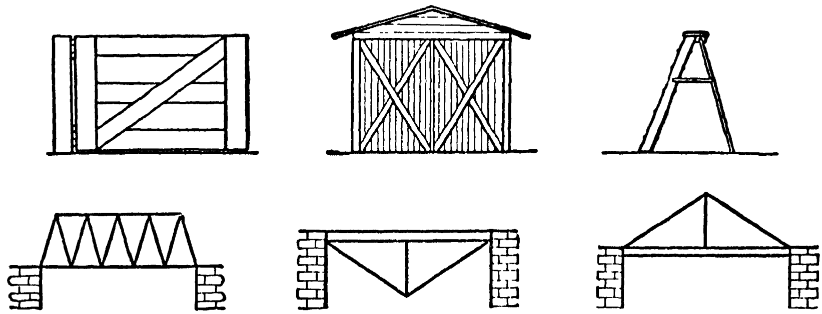


No .1

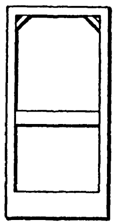


No .2

sides are broken. This is *not* the case with a quadrilateral such as figure 2. For this reason the triangular frame is known as a *rigid*



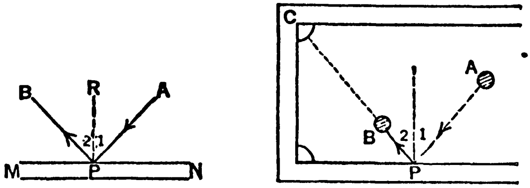
frame. Its design is incorporated in many structures, viz., gates, stepladders, bridges, airplane wing supports, lifting arms of cranes, derricks and steam shovels, towers of broadcasting stations, automobile frames. In the screen door one broad horizontal brace and two narrower triangular braces are noticed. Which brace contributes most to door's rigidity?



28.03. Daily Life.* It is a law of physics that when a ray of light strikes a reflecting surface it will be reflected

* Adapted from W. W. Strader and L. D. Rhoads, *Modern Trend Geometry*, p. 177. The John C. Winston Co., Philadelphia, 1940.

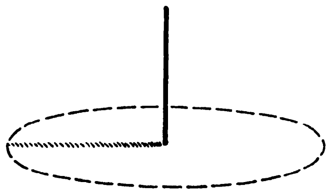
so that the angle of incidence equals the angle of reflection. In the figure, the angle of incidence, ∠1, equals the angle of reflection, ∠2. (*RP* is a perpendicular to the reflecting surface *MN*.) The same law



applies to hard bodies rebounding from an elastic surface. The accompanying drawing represents a pool table with balls at *A* and *B*. The problem is to make *A* strike *B* in such a manner that *B* will roll into the corner pocket at *C*. This can be done if *A* strikes the cushion at *P*, making ∠1 = ∠2.

Problem: How may such a point *P* be found?

28.04. Daily Life. When the altitude of the sun is 45° can be determined in various ways, but no instruments are necessary if one drives a stake into the ground vertically, describes a circle on the ground around the stake, using the height of stake as radius, and then waits until the shadow of the stake, cast by the sun, just touches the circle as in the figure.



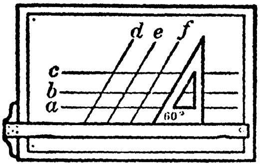
Problem: Prove that the sun's altitude at this moment is 45°.

When the sun is at this altitude, the height of any flag pole, tree, or building standing on level terrain can be determined merely by measuring the length of the shadow.

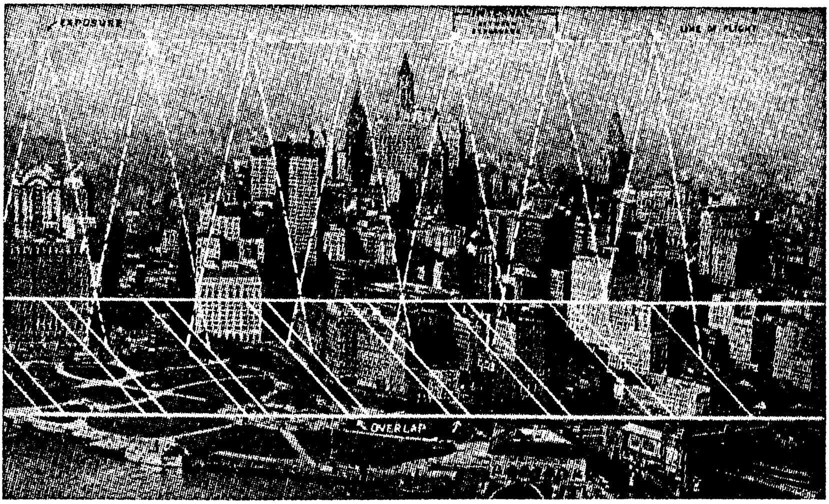
Problem: Prove this statement geometrically.

28.05. Drafting. The 30°–60° right triangle is used often in mechanical drawing to draw parallel lines.

Problem: In the figure directly below, why is $a \parallel b \parallel c$? Why is $d \parallel e \parallel f$?



28.06. Photography.* To take an aerial photograph from an airplane, the line of flight, speed of airplane, interval between exposures, and distances from the ground are calculated by the aid of geometry, as indicated below.

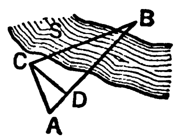


Courtesy International Newsreel

Problems: (a) What lines are parallel or equal? (b) What triangles are congruent, similar, or equal?

* From W. W. Strader and L. D. Rhoads, *Plane Geometry*, p. 214. The John C. Winston Co., Philadelphia, 1934.

28.07. Surveying.* To find distance *DB* across a river *S*, one observer stands at *D*, another observer walks along line *DA*, while a third walks to the left at right angles to *AB* for a convenient distance *DC*. The right angle *BCA* is observed, and the observer walking along *DA* is signaled to stop. *DC* and *DA* are then measured.

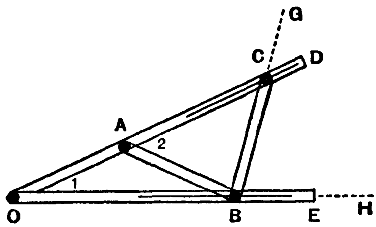


Problems: (a) Show that $DB = (DC)^2 \div DA$. (b) If $DC = 75$ yd. and $DA = 25$ yd., find the distance *DB*.

Also see Angles G. 3.05, 3.13; Geometric forms G. 10.02, 10.07; Proportion G. 17.04; Pythagorean theorem G. 18.01; Rectangular solids G. 19.01.

TRISECTION OF AN ANGLE 29.01

29.01. Student.† The trisection of an angle has attracted the interest of geometry students since antiquity. While this task cannot be accomplished with the tools of plane geometry, the device shown in the accompanying figure may be used to trisect mechanically an angle such as $\angle HBG$. The mechanism is jointed at *O*, *A*, *B*, and *C*; *B* slides along *OE*, and *C* along *OD*; $OA = AB = BC$.



Problem: Show that when $\angle EBC$ is adjusted to coincide with $\angle HBG$, $\angle O = \frac{1}{3} \angle HGB$.

Note: There are other ways of trisecting an angle. For example, one may solve the problem approximately by the use of a protractor, and accurate solutions are possible by certain curves studied in higher mathematics.

* Adapted from F. E. Seymour, *Plane Geometry*, p. 223. American Book Co., New York, 1925.

† Adapted from W. W. Strader and L. D. Rhoads, *Modern Trend Geometry*, p. 117. The John C. Winston Co , Philadelphia, 1940.

VOLUMES 30.01–30.06

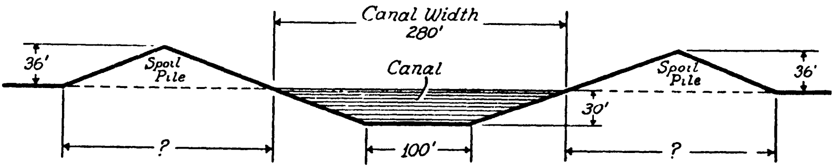
30.01. COAL DEALER. Hard coal dumped in a pile lies at an angle of 30° with the horizontal. Estimate the number of tons in a conical shaped pile 10 ft. high, if the coal is large egg size weighing 38 lb. per cubic foot.

30.02. ENGINEERING. When bids are sought for construction jobs, such as the removal of earth, the amount of earth to be moved must be stipulated by the authority seeking the bids. Also, with this amount in mind the contractors submit bids, for payment is based upon the quantity of materials moved, expressed in cubic units. Engineers must therefore be able to find the volume of geometric solids represented by the following problem.

Problem: Find the number of cubic yards of earth removed in making a cut 200 ft. long, if one end is 18 ft. deep, 75 ft. wide at the top, and 24 ft. wide at the bottom; the other end is 14 ft. deep, 66 ft. wide at the top, and 24 ft. wide at the bottom?

30.03. ENGINEERING.

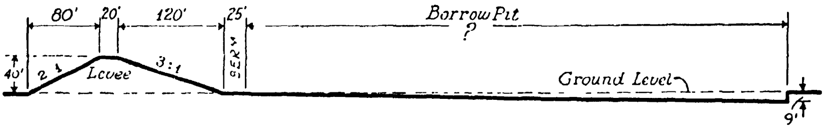
Work to be done: To excavate a canal of the finished dimensions shown, putting one-half of the excavated earth into each of the spoil piles. When excavated the earth becomes loose and its volume increases 10%; this increase is called "swellage." The spoil piles must therefore be of a combined area of 10% more than that of the cross-sectional area of the canal.



Problem: Find width of bases of the spoil piles.

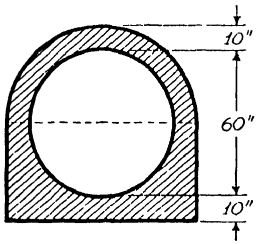
30.04. ENGINEERING.

Work to be done: To build levee from earth taken from borrow pit. The earth as removed from the borrow pit and put into the levee is loose. Consequently after a period of time the levee shrinks and settles, and to allow for this, 10% more earth must be put into the levee when it is built than is indicated by the final area of its cross section after this shrink has taken place. The dimensions of the levee shown are the final dimensions after shrinkage.



Problem: Find the width of the borrow pit, of shape shown and 9 ft. deep at outer end, required to furnish sufficient earth to build this levee.

30.05. Engineering. A conduit made of concrete has a cross section as shown. How many cubic yards are used in making 500 yd. of this conduit?



30.06 Engineering. A hollow copper sphere used as a float weighs 1 lb. and is 10 in. in diameter. How heavy a weight will it support? Note: Water weighs 63 lb. per cubic foot.

TRIGONOMETRY

TRIGONOMETRY

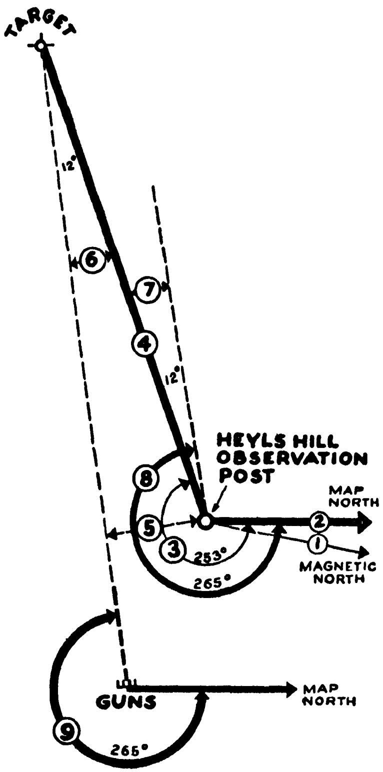
In order that junior high school teachers can readily locate suitable material, appropriate problems in this section are given under the headings "Cosine Ratio," "Cotangent Ratio," "Sine Ratio," and "Tangent Ratio." Under the heading "Trigonometric Functions" it will be found that most of the problems are more involved. The problems listed under "Trigonometric Manipulation" in general require more use of arithmetic and algebra. While problems in many vocations, for example electrical engineering, require much use of trigonometry along with calculus, their treatment is beyond the scope of this book; a few, more general, problems of this nature, however, are included under the heading "Trigonometry and Calculus."

ANGLES 1.01–1.03

1.01. Field Artillery. "Guns Are Aimed by Trigonometry."—"Aiming and firing a 75 can be done in one of three ways. The simplest method is to handle it like a rifle and aim it with the gunsight. This is effective in anti-tank work. Another method, if there is sufficient time to make surveys, secure data and aerial maps, is by unobserved fire. In this method the battery's position and the position of the target are plotted on the map, and the range and direction in which the gun should be fired are calculated from the map.

"Usually, however, sufficient time and adequate maps are not available. Then, observed fire must be used. For this, in addition to the gun position, an observation post must be established. It should have a full view of the target. Because it will probably be the first objective of enemy batteries, it must be well hidden. In the example here, the main observation post has been placed on the forward slope of Heyls Hill, in a position where it can be well camouflaged by trees and bushes. As an added advantage, a forward observation post will be established right up in the front lines of infantry. From there observers will be able to advise the battery as to locations where fire is advantageous.

"As soon as the location of the guns has been chosen and the observation posts established, calculations to determine how and where the guns should be fired are begun. The officer at the main observation post, with his men disposed, determines the range and direction of the target from his position. Then by the simple trigonometric problem, they determine the range and direction of the target from the gun battery. This information is telephoned to the executive officer at the guns by the battery commander. . . .



"Data for Fire

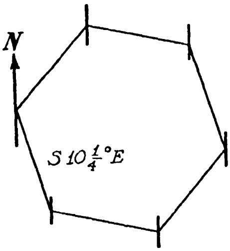
(1) Magnetic north is found on compass of aiming circle.
(2) Correction for map north is set on aiming circle.
(3) Aiming circle is aimed at target, and angle between map north and target found to be 253°.
(4) Range finder gives distance to target as 3,300 yd.
(5) Battery commander estimates shortest distance from observation post to line between guns and target as 700 yd.
(6) Artillery men know 700 yd. at 3,300 yd. range subtend an angle of 12°.

(7) 12° angle is added to aiming circle angle (253°) giving observation post a reference line parallel to the line between guns and target and

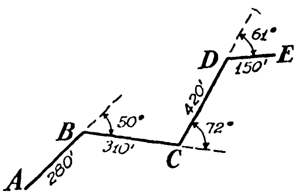
(8) A reference angle (265°) equal to angle between gun-target line and map north.

(9) Observation post sends guns data that target bears 265° from map north."*

1.02. SURVEYING.† The bearing of one side of a field in the shape of a regular hexagon is S10¼°E, proceeding around the field in the left-hand (counter-clockwise) direction. Find the bearings of the other sides, taken around the field in order. Bearings are taken each time from the north-south line.



T. 1.02



T. 1.03

1.03. SURVEYING. Plotting Traverses.—Running a traverse with a transit, a compass, or a plane table means running a series of connected lines and locating the stations with respect to one another by a combination of angular and linear measurements. Traversing is a favorite method of the scout or the explorer; by it he can conveniently map his route of march as he proceeds. It is a method extensively used by military map makers.

An *open traverse* is a path determined by directions and distances that do not form a closed polygon. Such a path is shown in the figure above. A traverse may be plotted in any one of a number of ways. One way is by *deflection angles*. Each distance is measured. The deflection angle is measured to the right or left of the extension of the preceding line.

* Quoted from article in *Life*, February 10, 1941.

† See C. B. Breed and G L. Hosmer, *Elementary Surveying*, Vol. 1, p. 32. John Wiley and Sons, New York, Seventh Edition, 1938.

By deflection angles, plot the traverse determined by these data:

	Distance	*Deflection Angle*
AB	320 ft.	
BC	200 ft.	Angle that *BC* makes with *AB* = 60° to the left
CD	210 ft.	Angle that *CD* makes with *BC* = 150° to the right
DE	190 ft.	Angle that *DE* makes with *CD* = 100° to the right

COSINE FUNCTION OF TWICE AN ANGLE 2.01

2.01. Latitude and Longitude.* Helmert's formula is a formula for finding g, the acceleration due to gravity, for a given latitude and altitude. It is:

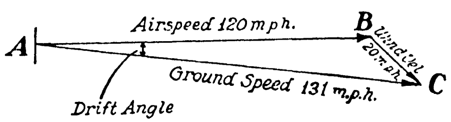
$$g = 980.616 - 2.5928 \cos 2\lambda + 0.0069 \cos^2 2\lambda - 0.0003086H,$$

where λ is the latitude, H is the height in meters above sea level, and g is expressed in cm. per sec. per sec.

Problem: Acceleration due to gravity varies, the variation depending on where on the earth you are. At Iowa City, the official latitude is given as 41° 39′ and the altitude as 655.78 ft. Find g.

COSINES, LAW OF 3.01–3.08

3.01. Aviation. "Compass course: the true course plus or minus variation and deviation, but without allowance for wind effect. Compass heading: the true course plus or minus variation and deviation, and including allowance for wind; the direction by compass in which the plane is pointed. The compass heading is found graphically by applying the principle of resultants.



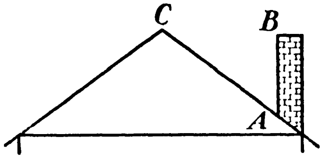
"*Problem:* A plane is headed due east, flying at an air speed of 120 m.p.h. and should reach its destination in 1 hour. However, it is subject to a 20 m.p.h. wind from 300°. Find its true course and its ground speed.

"The plane takes off from *A* headed due east. In still air it would arrive at *B*, 120 miles away, in 1 hour (line *AB* is drawn

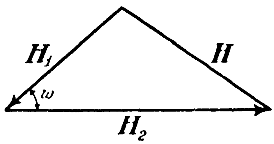
* See *Chamber's Technical Dictionary*, p. 409. The Macmillan Company, New York, 1940.

to scale). Through B, and from 300°, draw the wind vector to scale (BC). Connect A with C. Line AC is the path actually followed by the plane and also represents the ground speed of the plane. Thus, in 1 hour's time the plane would be past, and to the south of, its destination. Angle BAC is the drift angle."*

3.02. Buildings. Below is a sketch of a vertical section of a house roof, with a chimney near the edge. Since the chimney did not draw very well, it was suspected that the top was below the peak of the house and therefore the wind, coming across the ridge, created a down draft. In order to determine if this were true, the distances AB, AC, and BC were measured and the computation was made without tables. Having measured $AB = 10$ ft., $BC = 8$ ft., and $AC = 13$ ft., it was found that C was 3 in. higher than B.



T. 3.02



T. 3.03

3.03. Electricity. A magnetic field H is the resultant of two fields H_1 and H_2. Each one of the components is proportional to the current I in amperes and inversely proportional to the distance.

H_1 = magnetic field due to current I in first conductor
H_2 = magnetic field due to current I in second conductor
d_1 = distance from first conductor
d_2 = distance from second conductor
w = angle between H_1 and H_2 in triangle H_1H_2H.

It is known that $H_1 = \frac{.2I}{d_1}$, and $H_2 = \frac{.2I}{d_2}$. Formulate H_2 in terms of d_1, d_2, I, and w.

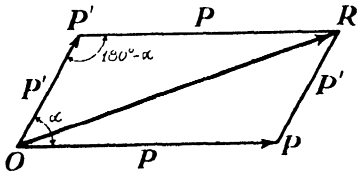
$$H^2 = H_1{}^2 + H_2{}^2 - 2H_1H_2 \cos w = 0.04I^2\left(\frac{1}{d_1{}^2} + \frac{1}{d_2{}^2} - \frac{2 \cos w}{d_1d_2}\right)$$

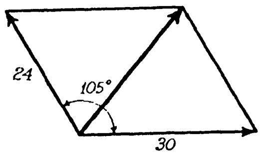
$$H = \frac{0.2I}{d_1d_2}\sqrt{(d_1{}^2 + d_2{}^2 - 2\, d_1d_2 \cos w)}.$$

* Commander A. E. Downer, *Practical Mathematics of Aviation*, p. 79. Pitman Publishing Corporation, New York, Second Edition, 1940.

3.04. Forces. Let two forces P and P', acting in a plane on point O, be represented in magnitude and direction by OP and OP', and their resultant by R (OR). Let the angle between the forces be α. Then in $\triangle OP'R$ by the law of cosines,

$$R^2 = P^2 + P'^2 - 2\,PP'\,(\cos 180 - \alpha)$$





Example: Two forces of 24 lb. and 30 lb., respectively, are acting at an angle of 105°. What is the intensity of the resultant?

Solution:

$$R = \sqrt{30^2 + 24^2 - 2 \cdot 30 \cdot 24 \cos 75°}$$
$$R = \sqrt{900 + 576 - 1440 \cdot .2588}$$
$$R = \sqrt{1103.328} = 33.2\text{, approx.}$$

3.05. Forces. Two forces P and Q, whose intensities are 5 and 12, have a resultant of 13. Required the angle between them.

Solution: $$R^2 = P^2 + Q^2 - 2\,PQ \cos \alpha,$$

in which α is the angle between the forces. By substituting:

$$13^2 = 5^2 + 12^2 - 2 \cdot 5 \cdot 12 \cos \alpha$$
$$\therefore \cos \alpha = 5^2 + 12^2 - 13^2 = 0$$
$$\therefore \alpha = 90° \text{ as arc cos } 0 = 90.$$

3.06. Navigation. A ship steams N 35° E from a point A at a rate of 18 miles per hour. After 1.5 hours it alters its course to N 10° W. Find its distance from A at the end of 4 hours.

3.07. Railways. From the crossing of two railroads, at an angle of 52°, the distances to two bridges over the same river are $5\frac{1}{2}$ miles and 8 miles. How far apart are the bridges?

3.08. Surveying.* In a legal document, surveyor's measurements may be recorded in chains and links. A link is 7.92 inches. One hundred links make one chain. Ten square chains equal one acre.

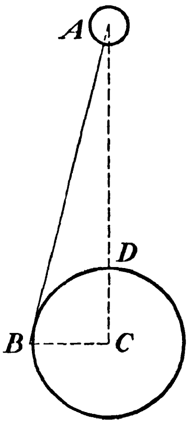
* Adapted from B Partridge, *Country Lawyer*, pp. 114–116. McGraw-Hill Book Company, New York, 1939.

In a surveyor's description courses indicate direction. The first course starts from a fixed point, located with reference to a permanent monument or bench-mark. The last course returns to the original point. The direction of any course is given with regard to its bearing from a north-south line. For example: if a survey starts from point *A*, and runs the first course 10 chains N 30° E, the second course 12 chains S 60° E, and the third course back to *A*, it will have enclosed 6 acres of land.

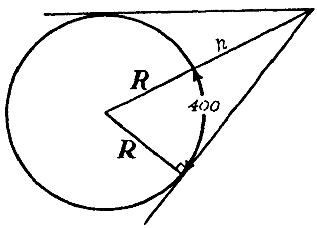
No trigonometry is needed for this problem, but if the second course were S 40° E, the law of cosines would be needed.

COSINE RATIO 4.01–4.08

4.01. Astronomy.* "An astronomer at *D* observes that the moon is directly overhead. Six hours later, when the earth has turned so that he is at *B*, he finds $\angle CBA$ to be 89°. If the radius of the earth is about 4000 miles, how far is the moon from *B*?"



T. 4.01



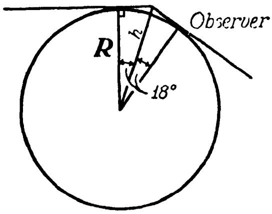
T. 4.02

4.02. Astronomy. A meteor is observed to burst, just as it apparently hits the ground, by a person who has an unobstructed horizon and who is 400 miles away from the real end point of the meteor. Find the height of the meteor on bursting. Take one degree on the earth's surface to be 69 miles. (See figure above.)

$$\frac{R}{R+h} = \text{cosine}\left(\frac{400}{69}\right).$$

* C. H. Mergendahl, and T G Walters, *Intermediate Algebra*, p 133. D Appleton-Century Company, New York, 1941.

4.03. Astronomy. Twilight lasts until the sun is 18° below the horizon. From this find the height of the atmosphere.

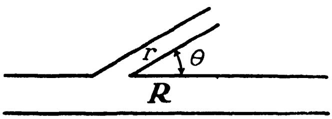


$$\frac{R}{R+h} = \text{cosine } 9^\circ$$

R = radius of the earth = 3960 miles
h = height of the atmosphere

4.04. Biology. "One further point making for the economical working of the inland transport service owes its enunciation to John Hunter. He wrote, 'To keep up a circulation sufficient for the part, and no more, Nature has varied the angle of origin of the arteries accordingly.'

"That is, the angle of origin required is such that its cosine is numerically equal to the radius of the branch divided by the radius of the main trunk.



"The size of the angle of origin is governed neither by the radius of the branch vessel nor by the radius of the main vessel, *but* by the ratio of these two quantities. For any particular value of the ratio $\frac{r}{R}$, we have therefore a constant value of θ; that is, all of the branches of equal radius will be equally inclined to the main artery.

"(1) In particular, if the artery bifurcates into two equal branches, the angles of bifurcation will be equal.

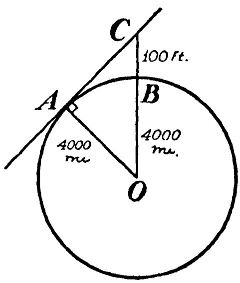
"(2) If r is so small compared with R that the amount of blood going to the branch is almost negligible, then $\cos\theta = \frac{r}{R}$ tends to be infinitely small, i.e., angle θ will be almost 90°.

"(3) If r differs but slightly from R, it is obvious that $\cos\theta$ tends toward the limiting value = 1, i.e., θ will be very small."*

4.05. Buildings. From the edge of an excavation 33 feet wide the angle of elevation of the top of a wall on the opposite edge of the excavation is 59°. Compute the length of the wire required to reach from the point of observation to the top of the wall.

4.06. Latitude and Longitude. Find the length of the circle of latitude that passes through Chicago, 41°50′ N, if the earth is a sphere of radius 3959 miles.

4.07. Navigation. A boat is sailing on the earth's surface and has a mast 100 feet high. How far away can you see it?



Solution: We desire $\overset{\frown}{AB}$. From $\triangle AOC$ find $\angle AOC$ by trigonometric ratio and hence arc length $\overset{\frown}{AB}$.

4.08. Surveying. Three towns, A, B, and C, are situated so that B is directly north of C, and A is 25.6 miles directly west of C. If the bearing of B from A is N54°E, how far is it from A?

* Quoted from David Burns, *Introduction to Biophysics*, p. 380. The Macmillan Company, New York, 1929.

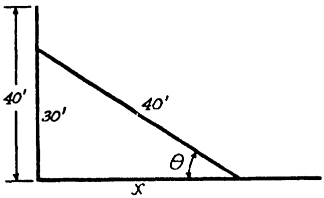
COTANGENT RATIO 5.01–5.05

5.01. Aviation. The pilot of an airplane wanted to estimate his distance from a beacon on the ground. His altimeter showed that he was 1200 feet from the ground, and he found that his line of sight to the beacon made an angle of 15° with the horizontal. How far was he from the beacon?

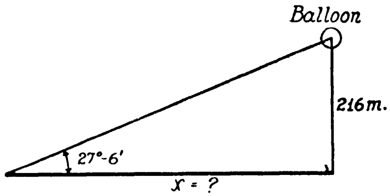
5.02. Aviation. An aviator, flying at an altitude of 3680 feet, observes the angle of depression of an airport to be 24°. How far from the airport is the point on the ground directly below him? How far is he from the airport?

5.03. Navigation. An observer on a boat measures the angle between the horizontal and his line of sight to the top of a lighthouse and finds it is 12°. His chart tells him that the top of the lighthouse is 95 feet above the level of the observer's eye. How far is the boat from the lighthouse?

5.04. Pole-bracing. In bracing telephone poles a practice is to attach one end of the wire brace three quarters of the way up a 40-foot pole. If the wire brace is 40 feet long, how far from the foot of the pole should the anchor be placed?



T. 5.04



T. 5.05

5.05. Weather Bureau. Four times each day balloon flights are used by the United States Weather Bureau to determine the speed and direction of the wind. A balloon is inflated to give it a free lift of 154 grams. The upward motion then proceeds at a definite rate per minute. The balloon reaches the following heights: 216 meters from the earth the first minute; 414 meters at the end of the second minute; 612 meters at the end of the third minute, etc. At the end of each minute the azimuth* angle and the angle of elevation of the balloon are read by the use of a theodolite. The

* The angle measured between the north line and the line to the balloon.

horizontal distance covered is figured for each minute, and from this the horizontal speed is determined. (x = horizontal distance the first minute.)

RADIANS 6.01–6.05

6.01. Aviation. The "pitch" of a screw or propeller is the distance it would advance in one revolution if there were no slipping. An airplane propeller is to have a pitch of 10 ft. at a radius of 42 in. from the center. What should be the angle of the blades with the plane of rotation at that point?*

6.02. Highways. A culvert is built in the form of an arc of a circle, the chord being 24 ft. and the maximum height above this, 8 ft. If water flows at the rate of 10 ft. per second, find the volume of water passing through in one minute.

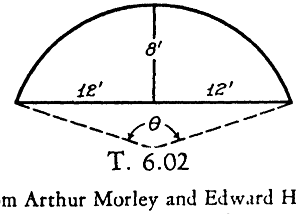
Solution: Find the area of the segment and multiply this area (in square feet) by 10 and by 60.

Let O be the center of the circle and d the diameter.

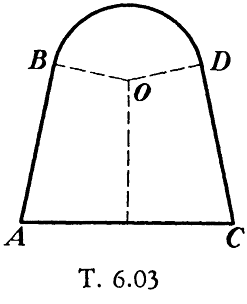
$$8(d - 8) = 144 \text{ (Equal products of intersecting chords).}$$
$$d = 26$$
$$\theta = 2 \text{ arc tan } 12/5 = 134° \, 46' = 2.3521 \text{ radians.}$$

$$\text{Area of segment} = \frac{169}{2}(\theta - \sin \theta).$$

Volume of water per second $= 84.5 \times (2.3516 - .7100) \times 10 \times 60$ cu. ft.
$= 83{,}254$ cu. ft. (This represents the maximum possible amount, making no allowance for friction.)



T. 6.02



T. 6.03

* From Arthur Morley and Edward Hughes, *Elementary Engineering Science*, p. 31. Longmans, Green and Co., New York, 1937.

6.03. Highways. The head of a construction company whose mathematical training had been limited (secured principally without the help of a teacher) brought the following problem to his state university mathematics department:

The steel sides and roof of a cattle underpass through an embankment are to be made of three pieces of standard sheet metal, 50 inches wide, put together as shown in the figure (see page 225), which represents a vertical cross section of the underpass, cut by a plane parallel to the direction of the highway overhead.

AB and *CD* are tangent to the arc, *BD*, of a circle. Each of the three is 50 in., the width of the sheet metal used. The pathway, *AC*, is to be 6 ft. wide. The problem is to find angle *BOD*, and consequently the radius *OB*, in order to fabricate the curved roof so that it will fit on *AB* and *CD*.

Solution: Let $2\theta = \angle BOD$ and $r = OB$. Then we have the equations:

$$r\theta = 25 \text{ in.} \qquad (1)$$
$$r \sin\theta + 50 \cos\theta = 36 \text{ in.} \qquad (2)$$

where θ must of course be expressed in radians.

To get an approximate solution, plot on polar coordinate paper with the same origin and pole.

$$r = 25/\theta \qquad (1')$$
$$r = 36 \csc\theta - 50 \cot\theta \qquad (2')$$

The values of r and θ at the point of intersection in the first quadrant satisfy the equation (approximately, of course). We find the approximate values to be

$$\theta = 1.23 \text{ radians } (70.5°)$$
$$\angle BOD = 141°$$
$$r = 20.3 \text{ inches.}$$

6.04. Machines. An 8-pole synchronous motor is running at 900.0 revolutions per minute. (a) Calculate (to four figures) the angular velocity in radians per second. (b) What is the linear velocity of a strap on the armature at a distance of 0.82 feet from the axis?

6.05. Railways. If a train travels with a speed of 17 miles per hour on a curve of 2000 feet radius, through what angle does it turn in one minute?

Solution: Let x = the angle (in radians) turned per minute.

$$x \cdot 2000 = \frac{17 \cdot 5280}{60}$$

$$x \approx .75 \text{ radians per minute.}$$

Apply the above method of solution to the following problem: If the driving wheels of a locomotive are 16 feet in circumference, how many revolutions must they make per minute so that the locomotive can attain a speed of 70 miles per hour?

SINE CURVE 7.01–7.04

7.01. Cosmetology. "Sinusodial currents are mechanically made to conform to various combinations of sines, and to have the time of each cycle longer or shorter; to have a pause between each alternation; or to have the current held for an appreciable period when it is at its highest voltage. So we have rapid sinusoidal currents, slow sines, interrupted rapid or slow sines, etc. (See diagram.)

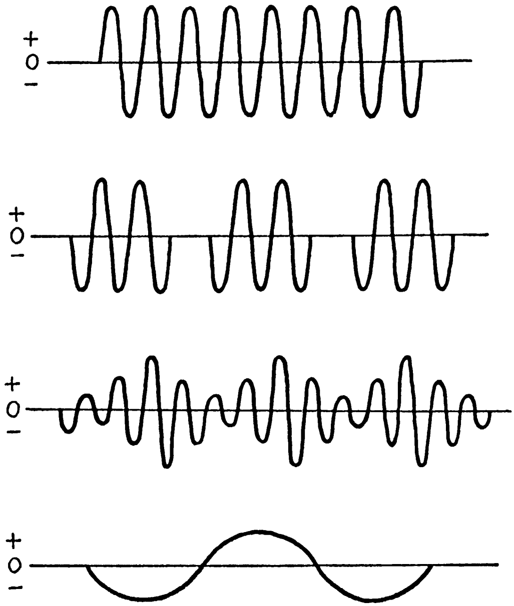
"It is unnecessary for the beauticians to know all of these different forms, but just as with medical men these currents have displaced the faradic current, so in beauty and scalp work they are surely going to supplant faradism, and those who are first to recognize this and become familiar with sinusoidal currents will be first to benefit.

"The currents are applied through well-saturated pad electrodes, or in the same manner as faradic currents.

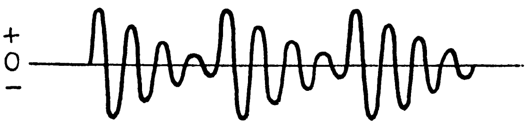
"A number of special face and scalp electrodes have been devised in addition to the standard forms.

"In all electrical applications where pads or metal electrodes are used, complete and perfect electrical contact is necessary to avoid unpleasant sensations.

"The sine currents are usually currents of medium or low frequency. When the frequency is increased we usually say rapid frequency. If the frequency of alternation is greatly increased a point is reached where there is not enough time for a perfect alternation, and although the current swings alternately to the positive



and negative sides, it does so with dwindling waves which we call oscillations, and we say the current is an oscillating instead of an alternating current. (Note the diagram below.) This type is



known as the high frequency current and occurs when the frequency is around or above 10,000 cycles per second.

"The current tracing is shown in the diagram and it may be seen that it reaches a maximum point and then dwindles to zero and then repeats.

"In high frequency currents there is a loss of sensation to the passing of the current because it is vibrating too fast to affect the sensory nerves. The same is true of motor nerves, so there is no contraction of muscles."*

7.02. NAVIGATION. The assumed tide curve of Cape Cod Bay is

$$h = h_m \sin (2\pi\, t/T),$$

where $h_m = 6$ feet, and $T = 44715$ seconds. Plot h as a function of time t seconds.

7.03. MACHINES. The piston of an engine transmits a force of

$$F = 100 \sin 30\pi t \text{ pounds.}$$

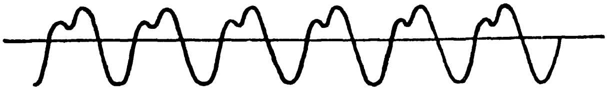
Plot a graph of force as a function of time.

7.04. MUSIC. "The fact that sin x has the same value at intervals of 360° is expressed by saying that the function $y = \sin x$ is periodic (with period 360°). Periodic functions are extremely important in a large variety of physical applications. In fact, the earliest scientific observations made by the human race were doubtless certain periodic phenomena like the alternation of day and night, the cycle of the seasons, the recurrent phases of the moon, the recurrent patterns of stars in the sky, etc. Similarly, many modern scientific phenomena are periodic in character, like sound waves, the vibrations of a violin string, the oscillation of a pendulum, etc. The graph of sin x suggests wave motion, and, in fact, the trigonometric functions (which are the simplest of all periodic functions) are very useful in the description of such periodic phenomena.

"The achievements of indirect measurement are quite remarkable, although they may seem pale beside such wonders as radio, television, etc. However, the student should remember that he now understands the more ancient miracles of trigonometrical indirect measurement, whereas he is doubtless merely familiar with the conveniences of the more modern miracles. The elementary applications of the trigonometric functions to the measurement of in-

* Noble M. Eberhart, "The Sinusoidal Currents." *The American Hairdresser*, pp 53, 86, December, 1931.

accessible distances are interesting and important, but many of the most important and interesting applications of the trigonometric functions of more advanced character are due to the periodicity of these functions. Any book on the advanced physics of sound, light, mechanics, electro-magnetics, etc., will be found to make frequent use of the trigonometric functions."



"Shown here is a photograph of a sound from a violin. An approximate equation of this graph is:

$$y = 151 \sin x - 67 \cos x + 24 \sin 2x + 55 \cos 2x + 27 \sin 3x + 5 \cos 3x."*$$

SINES, LAW OF 8.01–8.09

8.01. Aviation. According to a news item: "A new aid to flying safety - a device that tells ground crews the geographical position of an airliner in flight—has been developed by United Air Lines.

"The device will be placed in operation on the United system during 1941, J. R. Cunningham, director of communications, announced yesterday.

"By means of the device an airline dispatcher at Chicago, for example, could have a running record of a liner's flight across the nation. His chart would indicate any deviation of the plane from its true course.

"Cunningham said that the system involves the use of shortwave radio. Signals transmitted automatically by the plane are picked up by two or more receiving stations equipped with a large metallic frame antenna rotated by an electric motor.

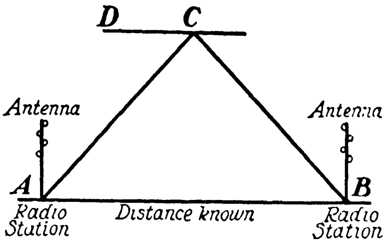
"Telephone wires carry the signals to the dispatcher's office where the plane's position is determined by triangulation."†

* M. Richardson, *Fundamentals of Mathematics*, pp. 367-368. The Macmillan Company, New York, 1941.

† Article in *New York Post*, January 3, 1941.

TRIANGULATION BY DIRECTIONAL ANTENNAE

The determination of the position of an airplane or a ship is effected by the following method:



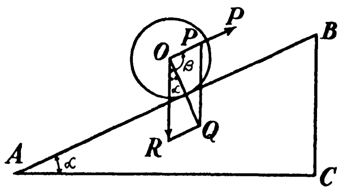
When the radio station's antenna picks up the loudest signal from the airplane the $\angle CAB$ (and $\angle CBA$) is determined, where C is the position of the ship. The angle that AB makes with the true north is known. Since $\angle DCA$ has been determined the angle that the ship makes with the true north may be found. By use of the law of sines, both AC and BC may also be found.

8.02. Field Artillery. Two artillery officers want to locate the position of the enemy's battery by the flash of their guns. One officer is at C and another officer at B, 480 feet from him. If there is a gun flash at A, and angle BCA is observed to be $88°$ and angle ABC is observed to be $83° \, 15'$, how far away is battery A from point C?

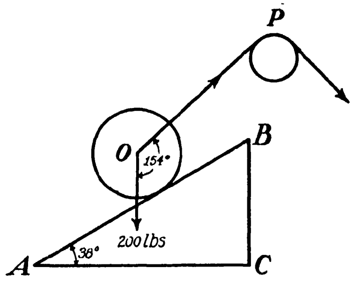
8.03. Forces.* "Let AB be an inclined plane making an angle α with the horizontal line AC. Let OR or R represent the forces of gravity in direction and intensity. Let OP or P represent the force holding the body in equilibrium. Then the resultant will pass through the point of support where O touches the plane AB. Completing the parallelogram $PORQ$, the diagonal OQ will represent

* Adapted from G. W. Peck, *Analytical Mechanics*, pp. 91–92. A. S Barnes and Company, New York, 1887.

the resultant, and OP will represent the force P necessary to prevent rolling down the plane."



(a)



(b)

In $\triangle ORQ$, $RQ = P$, $OR = R$, $\angle RQO = \beta - \alpha$.

Then by the law of sines:

$$\frac{OR}{QR} = \frac{\sin(\beta - \alpha)}{\sin \alpha} \quad \text{or} \quad \frac{R}{P} = \frac{\sin(\beta - \alpha)}{\sin \alpha}$$

$$\therefore P = \frac{\sin \alpha}{\sin(\beta - \alpha)} \times R$$

Problem: An inclined plane makes an angle of 35° with the horizontal. A barrel weighing 200 lb. is to be rolled up the plane. What force will just hold the barrel from rolling down the plane if the force is applied parallel to the plane?

Solution: In diagram (a), if OP is parallel to AB, $(\beta - \alpha)$ or $\angle QOP = 90°$. But $\sin 90° = 1$.

$$\therefore P = \frac{\sin \alpha}{\sin(\beta - \alpha)} \times R = \frac{\sin 35°}{1} \times 200 \text{ lb.}$$

Or $\qquad P = .5736 \times 200 \text{ lb.} = 114.72 \text{ lb.}$ *Answer.*

Note: Since in this case $P = \frac{\sin \alpha}{1} \times R$ and $\sin \alpha = \frac{BC}{AB}$, we may say $\frac{P}{R} = \frac{BC}{AB}$, or:

The force is to the resistance to be balanced as the height of the plane is to the length of the plane, when the force acts parallel to the inclined plane.

Problem: "For the purpose of rolling a body up an inclined plane a rope passing over a pulley P was used. When the body was at O, shown in diagram (b), the angle between the gravity line and the pulley rope was 154°. The inclined plane made an angle α of 38° with the horizontal. What force will just maintain O in position under these conditions?"

Solution: The functional relation is:

$$P = \frac{\sin \alpha}{\sin (\beta - \alpha)} \times R.$$

In our problem: $\beta = 154°$, $\alpha = 38°$, $R = 200$ lb.

$$\therefore P = \frac{\sin 38°}{\sin 116°} \times 200 \text{ lb.} = \frac{.6157}{.8988} \times 200 \text{ lb.} = 137.0 \text{ lb.}$$

8.04. Forestry. In order to determine the height of a tree on sloping ground, a base line 50 feet long was run from the foot of the tree, directly up the hill, the slope of which was .134. From the end of this base line the angle of elevation of the top of the tree was 38°. Compute the height of the tree.

8.05. Forestry. A timber cruiser standing on a slope at a distance of 125 feet from the foot of a tree wishes to determine its height. By using an Abney Level he finds the angle of elevation of the top of the tree to be +43° and the angle of depression to the base to be −3° 30′. What is the height of the tree? (See figure on page 234.)

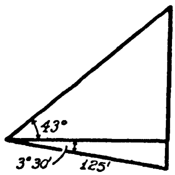
h = height of the tree
d = slope distance to foot of tree
x = angle to top of tree
y = angle to bottom

$$h = \frac{d[\sin (x + y)]}{\cos x}$$

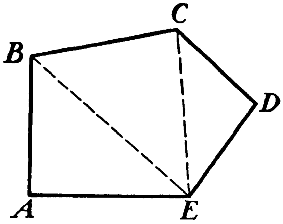
$$h = \frac{125 [\sin (43° + 3° 30')]}{\cos 43°}$$

$$h = 124.0 \text{ feet.}$$

8.06. Navigation. A boat is steaming N 40° E at a rate of 18 miles per hour. At ten o'clock a radio tower bears N 10° W, and at twelve o'clock it bears S 70° W. How far was the boat from the tower at ten o'clock?



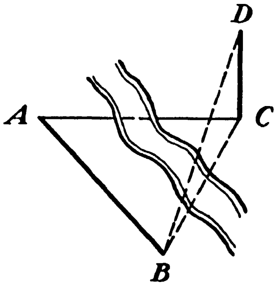
T. 8.05



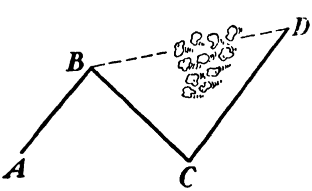
T. 8.07

8.07. Surveying. In the figure shown above, the area *ABCDE* represents the boundary of a tract surveyed by a simple transit survey, each of the angles of the three triangles into which the figure is divided being measured, but only the distance *AB* being determined in the field. In order to determine the length of the unknown boundaries, it is necessary to solve in succession the triangles *ABE*, *BEC*, and *ECD*, the lengths of all sides being determined by the relation that the length of the side of a triangle is proportional to the sine of the opposite angle.

8.08. Surveying. Being on one side of a river, and wanting to know the height of a cliff on the other side, I measured 500 yards, *AB*, along the side of the river, and found angle $ABC = 74° 14'$, and angle $BAC = 49° 23'$, and angle of elevation $CBD = 11° 15'$. Required the height of the cliff. (*Answer:* 272.0 feet.)



T. 8.08



T. 9.01

SINES AND COSINES, LAWS OF 9.01–9.03

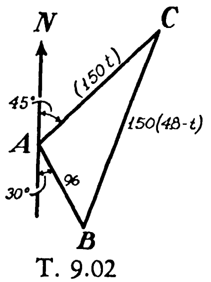
9.01. Surveying. In surveying a field a thick wood prevents the measurement of the angle *ABD* and of the distance *BD*. A random line BCD is run. The angle ABC is found to be 70° 14'.6, line BC is 743.86 ft.; the angle BCD is found to be 62° 14'.4 and the distance CD to be 912.82 ft.

$$\therefore BD = 868.34 \text{ to } 868.38 \text{ ft.};$$
$$\angle CBD = 68° 28'.1; \angle CDB = 49° 17'.5;$$
$$\angle ABD = 138° 42'.7.$$

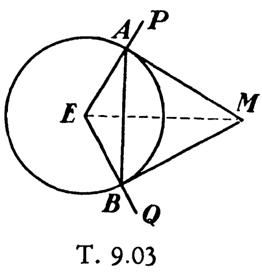
9.02. AVIATION. A plane has 4.8 hours of fuel. The path of the carrier is S 30° E from A and the path of the plane is northeast from A. The rate of the plane is 150 m.p.h. and the rate of the carrier is 20 m.p.h. How far out can the plane go before turning back to the carrier and at what angle will it fly back? See figure.

Solution: Let t = number of hours before the plane turns back

$$[150(4.8 - t)]^2 = (150t)^2 + 96^2 - 2(96)(150t)\cos 105°$$
$$t = 2.278 \text{ hours on } AC$$
$$4.8 - t = 2.522 \text{ hours on } BC$$
$$AC = 341.7 \text{ miles}$$
$$BC = 378.3 \text{ miles}$$
$$\frac{\sin C}{96} = \frac{\sin A}{378.3}$$
$$\sin C = \frac{96(.96593)}{378.3} = .2541$$
$$\angle C = 14° 11'.$$



T. 9.02



T. 9.03

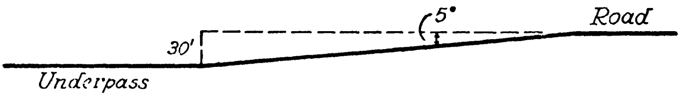
9.03. ASTRONOMY. Distance to the moon or a near-by planet.— Two observers, A and B, on the earth know their locations by latitude and longitude. From the known figure of the earth (not quite a perfect sphere) they know the distances EA and EB and the angle AEB. See figure above.

They can compute the distance AB and the angles EAB and EBA

by the law of cosines and the law of sines. The angles PAM and QBM can be measured. Then in $\triangle ABM$, the angles at A and B, and the side AB are known, whence either AM or BM can be had and, consequently, EM. (The law of tangents is better adapted to the use of logarithms than the law of cosines.)

SINE RATIO 10.01–10.03

10.01. Highways. In the construction of the New York City Belt Parkway System, many underpasses were built, the clearance in some cases being 30 feet. If the road is level under the bridge and if a safe angle of descent for the road is considered to be 5°, how long are the approaches to the underpass?



10.02. Highway.* "An overpass on highway U. S. 2 must clear the railroad tracks by 42 feet, vertical distance. If the approaches to the overpass proper must be at an angle of 10° to the horizontal, how long must the approaches be made?"

10.03. Tools. The legs of a pair of dividers are set at an angle of 30° 16′. If the legs are $5\frac{1}{4}$ inches long, determine the distance between the points.

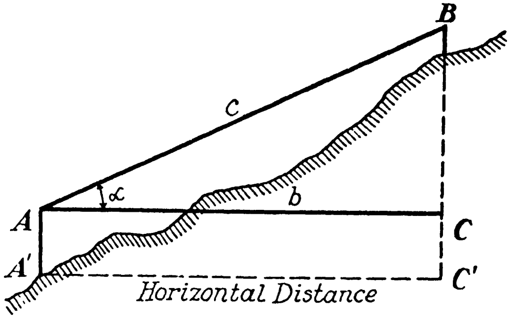
SINE, VERSED 11.01–11.02

11.01. Surveying. A problem in surveying is the horizontal measurement on sloping ground with a tape. An accurate method is to stretch the tape from one stake to another and determine the slope of the tape, and then to calculate the horizontal distance. When the angle ($\angle\alpha$) between the sloping line (c) and the horizontal line (b) is small, it is common practice to calculate first the difference in length between the slope distance and the horizontal distance, this being a more accurate method than using the cosine.

$$b = c \cos \alpha$$
$$\cos \alpha = 1 - \operatorname{vers} \alpha \text{ (versed sine } \alpha = \operatorname{vers} \alpha = 1 - \cos \alpha)$$
$$c - b = c \operatorname{vers} \alpha$$

* C. H Mergendahl and T. G. Walters, *Intermediate Algebra*, p. 134. D. Appleton-Century Company, New York, 1941.

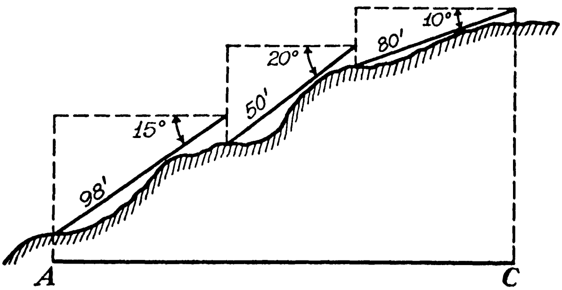
Example: Find the horizontal distance $A'C'$ ($= AC$) when the length of the tape between stake A and stake B is 100 ft. and $\angle\alpha = 4°$.



Solution: Looking in a table of natural versed sines* we find

$$\begin{aligned} \text{vers } 4° &= .00244 \\ c \text{ vers } 4° &= (100)\,(.00244) \\ &= .244' \\ c - b &= c \text{ vers } 4° \\ b &= 100' - .244' \\ &= 99.76'^{*} \end{aligned}$$

11.02. SURVEYING.† If it is not practicable to sight directly at the stake on top of the slope, measurement may be taken to any point



on the string of a plumb bob suspended over the mark, and the actual slope of the tape then determined by sighting the telescope

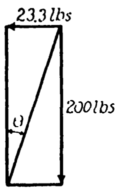
* Surveying often requires the use of $(1 - \cos A)$, hence surveyors have tables of this kind and name the quantity *versed sine*.

† See C. B. Breed and G. L. Hosmer, *Elementary Surveying*, Vol. 2, p. 10. John Wiley and Sons, New York, 1938.

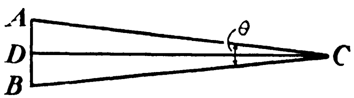
at the point to which the measurement was made. In this way a slope measurement which is longer than the length of the tape itself may be broken into parts, each intermediate point being taken at the proper line and grade; the whole distance is then reduced to the horizontal as a single measurement."

TANGENT RATIO 12.01–12.17

12.01. Athletics. Because of "centrifugal force" a man on a bicycle going around a curve will tend to lean at an angle which depends on his weight, his speed, and the radius of the circular curve around which he is moving. If the combined weight of the man and bicycle is 200 lb., and if he is going around a curve of 60 feet radius at a speed of 15 ft. per second, then the "centrifugal force" is 23.3 lb. Use the figure below to determine the angle θ at which he should lean.



T. 12.01



T. 12.02

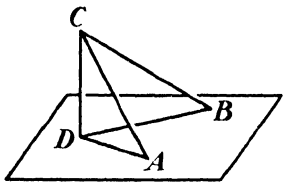
12.02. Astronomy. A problem in astronomy is to calculate the diameter of the sun or the moon, given the focal length of the lens and the diameter of the image formed. In the figure, AB represents the image of the moon and θ the angle formed with its vertex at the lens, C. If the focal length of the lens is 110 inches and the diameter of the image of the moon is one inch, θ can be determined.

$$\frac{AD}{DC} = \frac{.5}{110} = \tan \tfrac{1}{2}\,\theta.$$

Knowing the distance of the moon (mean distance about 238,800 miles), its diameter can be determined.

12.03. Astronomy. When the height of a meteor is calculated, the height is the average of the heights found from the data of many observers, or is checked by those data. From the data furnished by an observer, the angle of elevation of the position of a bursting meteor is determined. The distance from an observer to whêre the meteor strikes the ground is found from a map.

From the given data calculate the height of the meteor when it burst and check your result. What is the difference in the calculated heights?



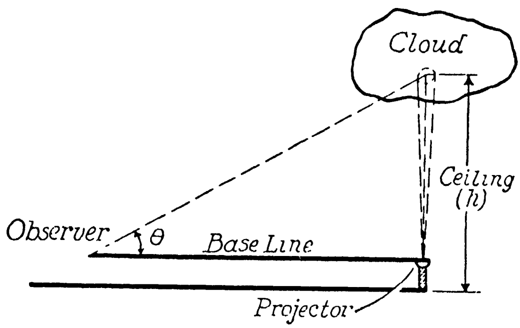
The meteor bursts at C and is seen by one observer at A, angle of elevation, $35° 40'$, and by another observer at B, angle of elevation $20°$. A is 50 miles from where the meteor fell, and B is 100 miles.

12.04. Aviation. From an airplane directly above a town, the angle of depression of another town 20 miles away on level ground was $17°$. How high was the airplane at the moment that the observation was taken?

12.05. Aviation. A pilot at an altitude of 520 ft. wants to land at an aviation field 3000 ft. distant from his present position. If a safe angle of glide is $10°$, may he descend to the field in a straight line, or must he first circle it?

12.06. Aviation. Each weather bureau station which is situated at an airport makes observations on the height of the ceiling at night. A known base line is established. At the Spokane station it is 1190 ft. and extends from top of the weather bureau to a ceiling-light projector at the top of a hangar. This light is aimed vertically and makes a bright spot on the clouds. The angle of elevation of this spot is read with a clinometer, and the height of the ceiling is calculated by the tangent ratio. The height which can be found depends on atmospheric conditions and the lighting at the field, but may run to 19,000 feet. The base line at the Seattle station is 1000 feet and an effort is being made to have base lines at all stations 1000 feet as the computing is simpler.*

12.07. AVIATION.† "For accurate determinations of the height of ceiling at night, so important for safe flying, ceiling-light projectors, a form of electric searchlight, are employed at night to throw a spot of light on the under side of the cloud layer. The projector throwing a vertical beam is located at a horizontal distance of 500 to 1000 feet or more from the point of observation of the light spot. Knowing this fixed distance, it is only necessary to measure the angular elevation of the spot of light from the observing point to compute the height of the light spot or the ceiling. This is accomplished by means of a clinometer." (Figure given schematically.)



If the point of observation is 1000 feet from the projector, what angle of elevation must the observer have found, if at one time he reported, "Ceiling 3000 feet"? Again, "Ceiling 500 feet"? And again, "Ceiling zero"?

$$h = \text{base line (feet)} \times \tan \theta.$$

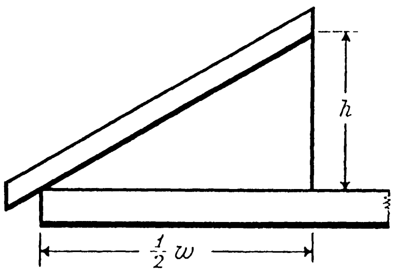
* Source of information: U. S. Weather Bureau through airport stations at Spokane and Seattle (Data as of 1941)

† B. C. Haynes, *Meteorology for Pilots*. U. S. Department of Commerce, Civil Aeronautics Administration, Washington, D. C.

12.08. Buildings. A building 107.8 ft. high stands on the same level with a monument. The angles of depression of the top and the base of the monument as read from the top of the building are 5° 15′ and 6° 41′, respectively. Compute height of monument.

12.09. Buildings. The ridge pole of a roof is $15\frac{1}{2}$ ft. above the center of the attic floor and the attic is 62 ft. wide. What is the slope of the roof? (The slope is the tangent of the angle of inclination to the horizontal.)

12.10. Buildings. The figure shows the gable end of a building with one side of a roof having both sides of equal pitch. The pitch of a roof of this type equals rise h, divided by w, the entire width of the building.



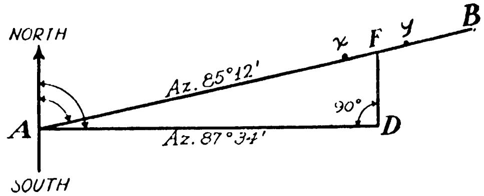
Formulate and compute the pitch and the angle of elevation of the roof when $h = 8' \ 4''$ and $w = 24' \ 5''$.

12.11. Forestry.* To assist the fire fighter in locating a reported fire, the lookout finds the line of sight on the fire and also the line of sight of some prominent landmark in the vicinity. He then figures the tangent offset of the line of fire from the landmark.

Let A be the lookout station, AB the lookout line of sight through the fire (azimuth 85° 12′), AD the lookout's line of sight on landmark D (azimuth 87° 34′). The angular distance between the lines is 2° 22′ or 142′. The distance from A to landmark read from map is 8 miles. The tangent for 1′ at 1 mile is $1\frac{1}{2}$ ft. The tangent offset for 142′ at 8 miles is 1704 ft.

* Adapted from W. B. Osborne, *The Western Fire Fighter's Manual*, Chap. V, "The Lookout System." Western Forestry and Conservation Association, 949 Henry Building, Seattle, Sixth Edition, 1934

Assume the fire fighter has been unable to find the fire but can locate D. He takes a backsight with his compass on A, turns an angle of 90°, and paces 1704 ft., which will place him approximately on the line of fire, at F. He knows what his backsight on A should be, and using this he follows a compass course on AB until he finds the fire. It might be at x, or it might be at y.

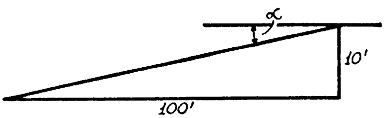


It is important to report the size of fires. Lookouts are instructed to measure precisely the angle between the extremities of the fire and compute its tangent.

Example: Assume that a fire is 12 miles away. The azimuth on the base at the right extremity is 154° 38′ and on the left extremity 154° 34′; then the angle between the extremities is 4′. The tangent for 1′ at 1 mile is $1\frac{1}{2}$ ft. The tangent for 4′ at 12 miles is 72 ft. The width of the fire from the lookout's point of view is 72 ft.

12.12. Highways. A road rises 2 feet for every 70 feet of roadway. What angle does the road make with the horizontal?

12.13. Highways. In the construction of the New York City Belt Parkway it was found that at one point there was a ten-foot drop to the continuation of the right of way. If the engineers considered 100 ft. to be a sufficient distance to absorb this drop, what was the angle of depression from the top to the bottom of the drop?



12.14. Machines. "If a line is drawn around a circular cylinder so that it advances a certain distance along the cylinder for each revolution, the curve thus formed is a spiral of a helix.

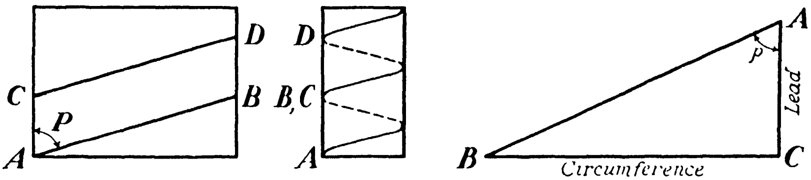
"If a piece of paper is cut as shown in figure (a), and lines *AB* and *CD* are drawn, this piece of paper can be rolled into the cylinder (b), where the lines *AB* and *CD* of (a) form the spiral running from *A* to *D* of (b).

"The advance along the cylinder for each turn of the spiral is the lead of the spiral, or the spiral lead. In the figure the distance *AC* is the lead. It is customary to give the lead of the spiral as so many inches per one turn; for example, a spiral that advances 8 inches in one turn is called an 8-inch spiral. The angle α that the spiral makes with an element of the cylinder is the angle of the spiral. It is seen that:

$$\text{Tan } p = \frac{\text{circumference of cylinder}}{\text{lead of the spiral}},$$

or, in figure (a)

$$\text{Tan } p = \frac{CB}{AC}.$$



(a) (b) (c)

"In setting milling machines for cutting spiral, such as worms, spiral gears, counter bores, and twist drills, it is often necessary to know the angle of the spiral. To find the angle of the spiral for the cutters in cutting a spiral, make a drawing as shown in figure (c); the angle *C* being a right angle, *CB* the circumference, and *AC* the lead. Angle *A* is the angle required, and may be measured with a protractor, or it may be found by finding tangent *A* equal to *CB* divided by *AC* and using the table of tangents. For ready reference the following rules are given:

"*Angle:* Divide the circumference of the spiral by the lead (advance to one turn), and the quotient is the tangent of the angle of the spiral.

"*Lead:* Divide the circumference of the spiral by the tangent of the angle, and the quotient is the lead of the spiral.

"*Circumference:* Multiply the tangent of the angle by the lead of the spiral, and the product will be the circumference."*

12.15. SURVEYING. A civil engineer wished to find the height of a cliff. He measured off 139 ft. from the cliff, and then found his line of sight to the top to be 41°. How high was the cliff?

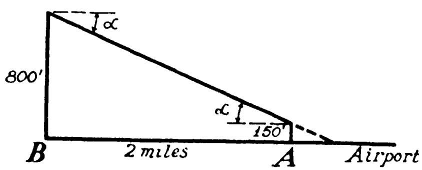
12.16. SURVEYING. To find the height of a city office building a surveyor tapes out horizontally from the wall 198.75 ft. At the end of this distance he sets his transit and observes the angle of elevation of the top to be 27° 25′ 30″. He also sets the telescope level and notes its line of sight; then he strikes the building 5.12 ft. above the bottom. How high is the building?

12.17. TOOLS. A wedge measuring 15 inches along the sides is 4 inches thick. Determine the angle at the point of the wedge.

TRIGONOMETRIC FUNCTIONS 13.01–13.42

13.01. ASTRONOMY. The distance from the earth to the sun (93,000,000 miles) subtends at the nearest fixed star an angle of approximately .76″. Find the distance to the star.

13.02. AVIATION. According to the United States Army Air Corps system of blind landing, on his final approach an aviator must be in the line made by two stations (*B* and *A*) and 800 feet over sta-



tion *B*. He then noses his ship down toward station *A* so that when he reaches the vicinity of *A*, he will be 150 feet off the ground.

* C. I. Palmer, *Practical Mathematics for Home Study*, p. 439. McGraw-Hill Book Company, First Edition, New York, 1919.

*Problem:** Disregarding the effects of weather conditions, what must be the angle of depression for all airplanes using this system?

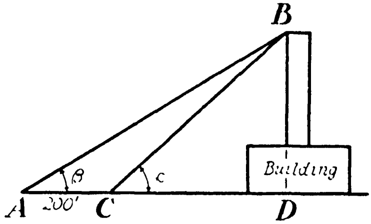
Solution:

$$\tan \alpha = \frac{800 - 150}{2 \times 5280} = \frac{650}{10,560}$$

$$\tan \alpha = .0616, \text{ approximately}$$

$$\angle\alpha = 3^\circ\ 31'. \textit{ Answer.}$$

13.03. Buildings. It is desired to measure the height of a chimney, but it is impossible to measure on the ground from a point outside to the foot of the chimney. The following procedure may be used. A sight is taken from A to the top of chimney B, and $\angle CAB$ is determined. At point C, 200 feet closer to the building, a sight is taken to B and $\angle DCB$ is determined.



$$DB = DC \tan \alpha$$

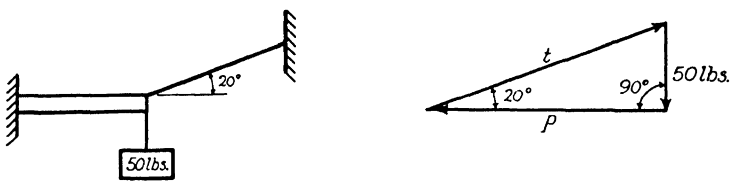
$$DB = [200 + CD] \tan \beta$$

$$DB = 200 \tan \beta + \frac{DB}{\tan \alpha} \tan \beta$$

$$DB\,(1 - \cot \alpha \tan \beta) = 200 \tan \beta$$

$$DB = \frac{200 \tan \beta}{1 - \cot \alpha \tan \beta}.$$

13.04. Forces. A weight of 50 lb. is suspended at the free end of a horizontal bar held in position by a cord which makes an angle

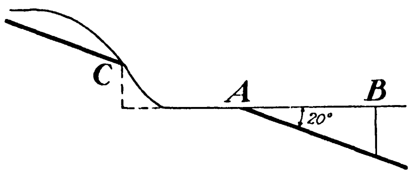


* See Van Nostrand's *Scientific Encyclopedia*, p. 147, 1938.

of 20° with the horizontal. Determine the tension in the cord (T lb.) and the thrust in the bar (P lb.) from the triangle of forces.

13.05. GEOLOGY.* Rock formations are seldom in horizontal strata. Very frequently the geologist wishes to know the depth of a bed of coal, water, salt, petroleum, gold, silver, or other mineral. By measuring the angle of the same rock formation at another point at or near the surface, the geologist finds the depth of the particular rock formation at a given point.

Example: Coal cropped out on the surface at A. A land-owner near-by wanted to find the distance to the coal vein under his own land and chose point B. If B were $1\frac{1}{4}$ miles from A, and the angle of depression of the vein of coal at A were 20°, how long a shaft would need to be sunk at B to reach coal? (It was found that the strata of other rock formation near the coal bed at A also were directed at a 20° angle from the horizontal, and so it was assumed that the coal formation continued at that angle for some distance.)



Someone might also wish to know whether the hill at the left in the diagram were high enough for the same vein of coal to be found there. Point C could be determined by the surveyor, who would use a trigonometric ratio.

When near-by rock formations leave the surface of the earth at a different angle, the problem of locating the vein desired at a particular point becomes more complicated.

13.06. HIGHWAYS: BRIDGE CONSTRUCTION. In the construction of a certain bridge, it is necessary to locate a driven pile, forming part of a coffer dam, at a point D upstream from the bridge. A base line AC = 500.0 ft. long is laid out on shore. The center line of the bridge, BB', makes a measured angle of 74° 37′ with the

* Adapted from Frederic H. Lahee, *Field Geology*, pp. 627–642. McGraw-Hill Book Company, New York, Third Edition, 1931.

base line. $AB = 207.1$ ft., $BB' = 305.4$ ft., and $B'D$ is to be 250.0 ft. in length in a direction perpendicular to the bridge line. The driving of the pile at D will be controlled by sighting a transit along AD and CD. At what angle with the base line must the instrument be set, sighting from A? (Hint: AD may be considered to be a vector, which is given by the vector sum of AB, BB', and $B'D$. Find its components along the base line, and perpendicular to it.

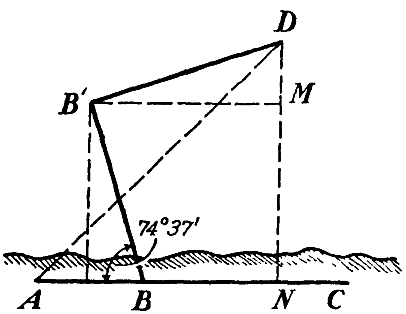
Calculations by logarithms:

$$\begin{aligned}
\text{Proj}_{ac}\, AB &= & 207.100\\
-\text{Proj}_{ac}\, BB' &= 305.4 \cos 74°37' & -\ 81.016\\
\text{Proj}_{ac}\, B'D &= 250.0 \sin 74°37' & 241.040\\
\hline
\text{Proj}_{ac}\, AD &= & 367.124'
\end{aligned}$$

$$\begin{aligned}
MN &= 305.4 \sin 74°37' & 294.45\\
MD &= 250.0 \cos 74°37' & 66.32\\
\hline
DN &= & 360.77'
\end{aligned}$$

$$\text{Tan } A = \frac{DN}{\text{Proj}_{ac}\, AD} = \frac{360.77}{367.124}$$

$A = 44°30'$ *Answer.*



$$\begin{aligned}
\log 305.4 &= 2.48487\\
&\ \ \underline{9.42370 - 10}\\
&\ \ 1.90857
\end{aligned}$$

$\text{Proj}_{ac}\, BB' = 81.02'$

$$\begin{aligned}
\log 250.0 &= 2.39794\\
\log \sin 74°37' &= \underline{9.98415 - 10}\\
&\ \ 2.38209
\end{aligned}$$

$\text{Proj}_{ac}\, B'D = 241.04'$

$$\begin{aligned}
\log 305.4 &= 2.48487\\
\log \sin 74°37' &= \underline{9.98415 - 10}\\
&\ \ 2.46902\\
MN &= 294.45'
\end{aligned}$$

$MD = 250.0 \cos 74°37'$

$$\begin{aligned}
\log 250.0 &= 2.39794\\
\log \cos 74°37' &= \underline{9.42370 - 10}\\
&\ \ 1.82164
\end{aligned}$$

$$\begin{aligned}
MD &= 66.32'\\
DN &= 360.77
\end{aligned}$$

$$\begin{aligned}
\log 360.77 &= 2.55723\\
\log 367.124 &= \underline{2.56481}\\
\log \tan A &= 9.99242 - 10
\end{aligned}$$

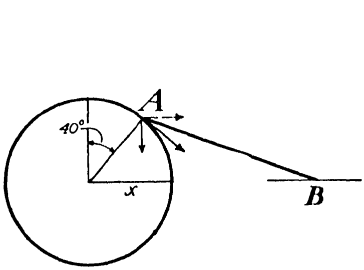
$A = 44°30'$

13.07. Machines. The drive wheel on a gas engine is 13 in. in diameter and the pulley on a rotary pump is 5 in. in diameter. If the shafts of the drive wheel and the pulley are placed 2 ft. apart, find the length of the belt required when uncrossed.

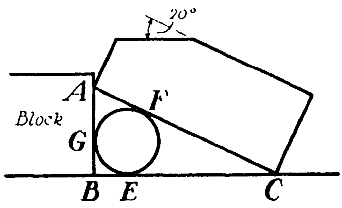
13.08. Machines. In a certain machine a sloping slide is to be made at an angle of 15° with the horizontal. The slide is to be 2500 ft. long and raised 6.00 in. above the horizontal bed at the lower end. The work is to be figured to the nearest thousandth of an inch. How high is the upper end, and how long is the bed under it?

13.09. Machines. "The piston of a pump is run by a wheel, as shown in the figure. When the piston is in the position shown, the motion of the point *A* can be resolved into two motions, one parallel to the motion of the piston *B*, and the other at right angles. If the motion of the point *A* is 75 ft. per minute, in the direction of the tangent to the circle, find how fast the point *A* is falling and how fast it is traveling in the direction of the line *B*.

"When the angle *x*, in the figure, becomes 25°, how fast will point *A* be moving?"*



T. 13.09



T. 13.10

13.10. Machines. See drawing at right above.

"Required: To cut a rectangular block forming a surface at a 20° angle with the top of the block which is two units in length.

"The block to be cut is held on a magnetic table by means of another rectangular block and a cylinder of diameter *D*. The upper edge is then ground off in a plane parallel to the magnetic table.

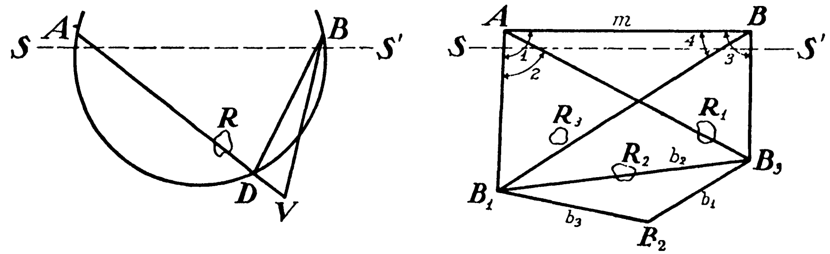
* Reprinted by permission from *Technical Mathematics*, Vol. III, p. 16, by H. M. Keal and C. J. Leonard. John Wiley and Sons, New York, 1923.

Find the diameter of the cylinder.

$\angle C = 20°$		Superimposition
$AB = 2 \sin 20° = .6840$		Right triangulation
$BC = 2 \cos 20° = 1.8794$		
$AG = AF, \quad CE = CF$		Two tangents to a ⊙ are equal
$AB + BC - AC = GB + BE$		
$GB + BE = .6840 + 1.8794 - 2 = .5634$		
$GB =$ radius of the cylinder		
$BE =$ radius of the cylinder		
$GB + BE =$ diameter $= .5634$*		

13.11. Navigation. A tug that can steam 27 miles per hour is at point A. It wishes to intercept as soon as possible a steamer that is due east at a point B and making 21 miles per hour in a direction N 24° W. Find the direction the tug must take and the time it will take if B is 3.5 miles from A.

13.12. Navigation.† Danger Angle Defined: SS' is a shore, R a rock or other obstruction dangerous to navigation. Figure (a). If two prominent objects A and B on shore are sighted from any point on the circumference ADB, the angle will be equal to angle D. If the ship stays outside the circle of danger, the angle



(a) (b)

between VA and VB will be *less* than angle D. A and B will be noted on the chart, and the danger angle D is known. The navigator must see that his angle, subtending AB, is less than D.

Problem: To locate A and B on the danger circumference and fix the danger angle. This problem is broken down into parts as follows: See figure (b).

* C. J. Leonard, *Mathematics in Industry*, Vol. 29, pp. 249-255.

† Adapted from Nathaniel Bowditch, *American Practical Navigator*, p. 64. D. Appleton-Century Company, New York, 1918.

(1) To find the radius of the danger circle. Let SS' be the shore and R_1, R_2, R_3, rocks dangerous to navigation. B_1, B_2, B_3 are buoys temporarily anchored at safe distances from the obstructions, through which the danger circle is to pass.

Select a base line on shore and two observation stations on it, A and B.

Problem: Find the radius of the circle passing through B_1, B_2, B_3. Measure the distance m and the angles 1, 2, 3, and 4 shown on the diagram. Then

(a) Solve $\triangle ABB_1$ (a side and 2 adj $\angle$s given), which gives line AB_1, etc.

(b) Solve $\triangle ABB_3$, giving side AB_3.

(c) Now solve $\triangle AB_1B_3$ (2 sides and included $\angle$ now known). This gives b_2 or $B_1\,B_3$. Similarly, sighting to B_1 and B_2 from A and B and solving three triangles, we find b_3 or $B_1\,B_2$, and by sighting to B_2 and B_3, measuring two angles at A and two at B each time, we find b_1 or $B_2\,B_3$.

(d) Find the diameter of the circle passing through $B_1\,B_2$ and B_3 by use of

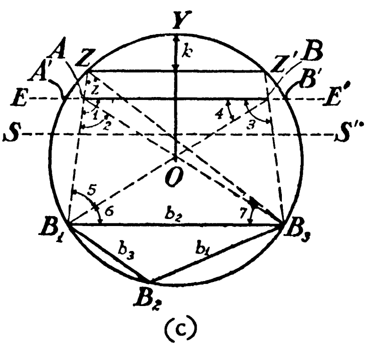
$$d = \frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}},$$

in which a, b, c are the lengths of the sides of a $\triangle ABC$. Here

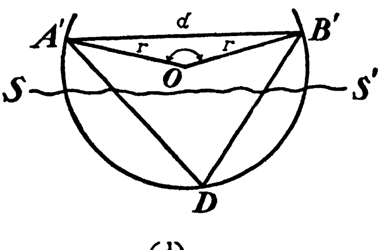
$$d = \frac{b_1b_2b_3}{2\sqrt{s(s-b_1)(s-b_2)(s-b_3)}}$$

and

$$r = \tfrac{1}{2}d.$$



(c)



(d)

(2) To locate lights on AB (extended) so that they will be on the circumference passing through B_1, B_2, B_3, and be the objectives from which to ascertain the safety angle; i.e., to find A' and B' on the diagram. See figure (c).

Our problem now is to locate A' and B' on AB extended. Evidently since we have solved $\triangle AB_1B$ and AB_3B completely, we know angles 5, 6; and the similar solutions giving the sides of $\triangle B_1 B_2 B_3$ gave us data for finding $\angle B_2$ since

$$\tan \tfrac{1}{2} B_2 = \sqrt{\frac{(s - b_1)(s - b_3)}{s(s - b_2)}}.$$

As $\angle Z$ is the supplement of $\angle B_2$, we now know B_1B, $\angle B_1$ (or $5 + 6$) and $\angle Z$ in $\triangle ZB_1B_3$ or two angles and a side opposite one of them, or also B_1B_3 and $\angle B_1$ $(5 + 6)$ and $\angle B_3$ or $\angle 7$. Therefore, we can solve for B_1Z, and, subtracting B_1A we have point Z, by measuring AZ equal to this difference. Similarly Z' is located.

Now since we know the radius r, and ZZ', we can find k on the diagram since

$$k(2r - k) = (\tfrac{1}{2} ZZ')^2.$$

Now we have three points Z, Y, and Z' of the circle on shore, and can easily locate successive points on the circumference on shore until A' and B' are reached.

(3) We now find the danger angle thus: See figure (d).
Call $A'B'$, found above, d or the danger base; r, the radius, has been found.

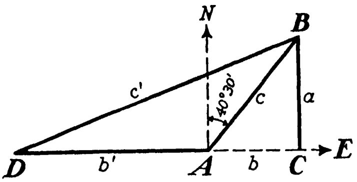
$$\therefore \tan \tfrac{1}{2} \angle O = \sqrt{\frac{(s - d)(s - r)}{s(s - r)}} = \sqrt{\frac{(s - d)}{s}}.$$

But

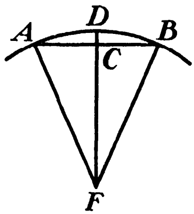
$$\angle D = \tfrac{1}{2} \angle O.$$

13.13. Navigation. A ship leaves a dock D at 9 o'clock and steams due east at a rate of 20 miles per hour. At 11 o'clock its course is changed to N 40° 30′ E. Find its distance and bearing from the dock at 12:30 o'clock. (See figure below.)

$$b' = 40, \qquad c = 30,$$
$$a = c \cos 40° \, 30'$$
$$b = c \sin 40° \, 30'$$
$$\tan D = \frac{a}{b + b'}.$$



T. 13.13



T. 13 15

13.14. Pyramids. If when complete one of the Egyptian pyramids was 354 ft. on a side at the base and 212 ft. high, how much material did it contain? Neglect the vaults and passages in it. What angle did the edges make with the horizontal?

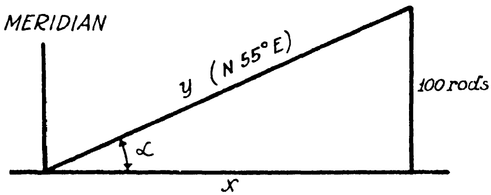
13.15. Railways. "A railroad engineer needs to know the radius of a curve so that he will be able to raise the outside rail the right amount to keep the train from jumping the track. He cannot measure it directly, as the track passes between a cliff and a river, so he determines that an arc of 20° has a chord, AB, 430 ft. long. Find FB, the radius of the curve ($\angle F = 10°$ and $BC = \frac{1}{2} AB$)."* (See figure above.)

13.16. Surveying. To find the width of a river a line AB is measured on one side parallel to the bank. A stake C is set on the opposite bank. If AB is 352 ft., angle CAB is 37° 18′, and angle CBA is 31° 14′, find the width of the river.

13.17. Surveying. *Bearing* is a surveying term used to define the

* C. H. Mergendahl and T. G. Walters, *Intermediate Algebra*, p. 134. D. Appleton-Century Company, New York, 1941.

direction of a line in terms of the acute horizontal angle which it makes with a given line called the *meridian.*

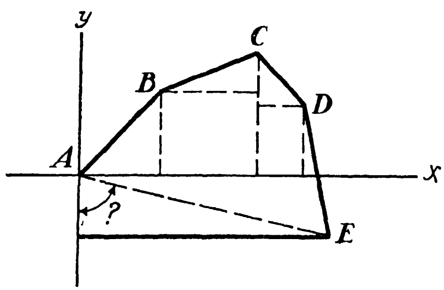


Problem: If the bearing is N 55° E for a boundary line on a farm 100 rods long, one end of which is due east, find how far the observer is from each end of the boundary. (Any desired modifications may be appended.) $x = ?$ $y = ?$

13.18. Surveying. A straight level road leads to the foot of a hill 165 ft. high. From the top of the hill the angles of depression of two objects in direct line on the road are 15° and 12°, respectively. Compute the distance between the two objects.

13.19. Surveying. A man surveying a mine measures a line *AB*, equal to 170 feet, from the mouth *A* of the mine due east at a dip of 14° 20′ into the mine. From *B* he follows a tunnel *BC* 220 feet along a line running due south at a dip of 24° 17′. How far is *C* below the level of *A*? If *D* is the point directly above *C* in the horizontal plane with *A*, what is the direction from *A* to *D* and how long is *AD*?

13.20. Surveying. A tunnel is to be constructed through a mountain. A survey is run around the mountain. A sketch of the plan of work is shown.



$$x_1 = 423.62 \sin 27^\circ 15' 10'' = 193.99$$
$$x_2 = 376.21 \sin 81^\circ 19' 40'' = 371.90$$
$$x_3 = 192.07 \sin 47^\circ 26' 00'' = 141.46$$
$$x_4 = 321.74 \sin 15^\circ 45' 50'' = 87.41 \qquad 794.76$$

$$y_1 = 423.62 \cos 27^\circ 15' 10'' = 376.60$$
$$y_2 = 376.21 \cos 81^\circ 19' 40'' = 56.73$$
$$y_3 = 192.07 \cos 47^\circ 26' 00'' = -129.92$$
$$y_4 = 321.74 \cos 15^\circ 45' 50'' = -309.64 \qquad -6.23$$

Point	*Distance*	*Bearing*
A	423.62 ft.	N 27° 15′ 10″ E
B	376.21 ft.	N 81° 19′ 40″ E
C	192.07 ft.	S 47° 26′ 00″ E
D	321.74 ft.	S 15° 45′ 50″ E

What is the distance and bearing of point E from A?

$$\text{Length} = \sqrt{(794.76)^2 + (6.23)^2} = \sqrt{631682.3} = 794.79 \text{ ft.}$$

Answer.

$$\tan L = \frac{794.76}{6.23} = 127.57$$

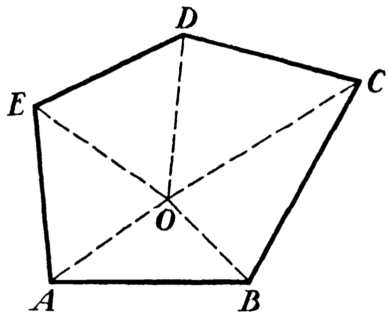
$$L = 89^\circ 33' 03'' \quad \textit{Answer.}$$

13.21. Surveying. A lake with an altitude of 1262 ft. lies between two mountains with altitudes of 2086 ft. and 2504 ft. As observed from the summit of one mountain, the angle of depression of the reflection of the summit of the other in the lake is 15° 30′. Find the distance between the mountain tops.

13.22. Surveying. From a position 185 feet above the surface of a lake, the angle of elevation of a mountain top is 22° 30′, and the angle of depression of its reflection in the lake is 30° 45′. Find the height of the mountain above the surface of the lake.

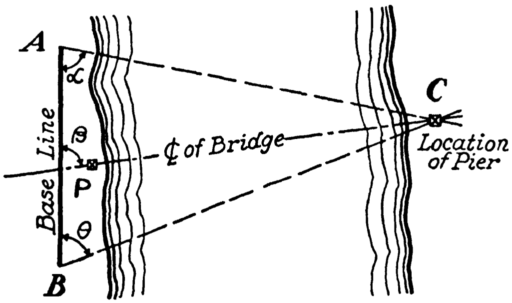
13.23. Surveying. The angle of elevation of the top of a mountain from a point at its base is 40° 21′. At a point 1000 feet up the slope of the mountain inclined at a constant angle of 17° 31′, the angle of elevation of the top is 55° 38′. Find the height of the mountain.

13.24. Surveying. The figure shown below illustrates a survey made by a single set-up of the transit at O, such as might be used for a small lot when the property lines $ABCDE$ were obstructed or the transit could not be set up at the corners. Under these circumstances the angles about O and the distances OA, OB, OC, OD, and OE are measured in the field. There are then known in each triangle, two sides and the included angle. The area of each triangle may be determined by means of the following expression:



$$\text{Area} = \tfrac{1}{2}\, ab \sin C.$$

13.25. Surveying. If a bridge is to be constructed over a river where tapes cannot be stretched across, the location of the pier can be determined so that it is the desired distance from the river's edge and on the line desired for the direction of the bridge.



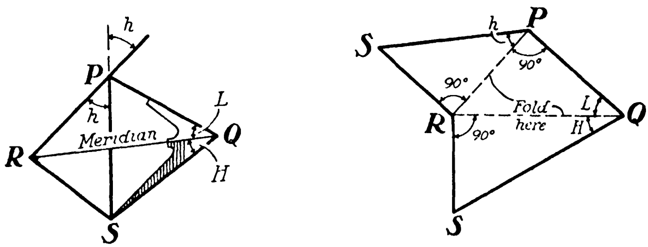
In the figure above the direction and length for the bridge is known, and it has been decided that a pier will be located at C.

In order that the bridge will fit on the piers on each side of the river, the location of pier C must be located very accurately. Any base line AB is established and its length determined by means of a tape. From the direction of the base line and the direction line for the bridge $\angle B$ is also known.

The location of pier P has already been established. Knowing the length of AB and the distance along the line of the bridge from the base line to point C, the angles α and θ can be calculated. The engineers then set up their instruments at points A and B and "turn off" the angles α and θ.

The rodman, near C, then proceeds to locate his rod until sighted by each of the instruments at A and B.

13.26. Time. "How to Calibrate the Sundial. A solid model of the sunbeam, style, and shadow, can be made by folding the figure $SPQSR$. The angle H is the shadow angle with the meridian; L the



latitude of the place and the inclination of the style to the horizon plane; and h the sun's hour angle. The figure shows that

$$\tan H = \sin L \tan h.$$

Thus to mark off the angle H corresponding to two o'clock (p.m.) or ten o'clock (a.m.), i.e., when $h = \pm(15 \times 2)^\circ = \pm 30^\circ$, at latitude 51° we have

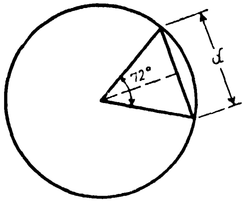
$$\tan H = \sin 51^\circ \tan 30^\circ$$
$$\tan H = 0.7771 \times 0.5774 = 0.4487$$
$$\therefore \text{ from tables of tangents, } H = 24.15^\circ.\text{"*}$$

* L Hogben, *Science for the Citizen*, p. 109. Alfred A. Knopf, Inc., New York, 1938.

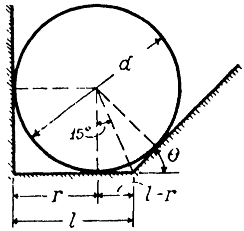
13.27. Tool and Die Work. Five holes are to be spaced equally about a circle twelve inches in diameter. What must be the setting of the dividers to space these holes?

$$\text{Central } \angle \text{ between two successive holes} = \frac{360^\circ}{5} = 72^\circ.$$

$$d = 2\,(6 \sin 36^\circ) = 7.05. \quad \textit{Answer.}$$



T. 13.27



T. 13.28

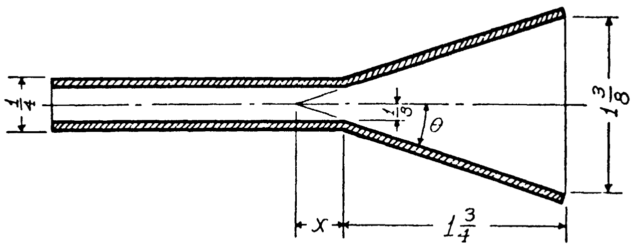
13.28. Tool and Die Work. It is required that a measuring wire be inserted in a machine part so that the three sides are tangent to the wire at one time. Determine the diameter of the wire if $l = 3.5$, and $\theta = 30^\circ$.

$$\frac{l - r}{r} = \tan 15^\circ$$

$$0.268r = 3.5 - r$$

$$d = 2r = \frac{(3.5)(2)}{1.268} = 5.52. \quad \textit{Answer.}$$

13.29. Tool and Die Work. At what angle θ should a cutting tool be fed to bore out the mouthpiece or funnel shown in the figure below?

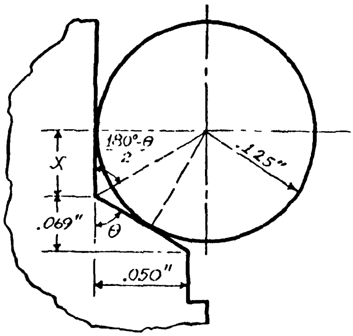


$$\frac{x}{\frac{1}{8}} = \frac{x + \frac{7}{4}}{\frac{11}{16}}$$

$$\tan\theta = \frac{\frac{11}{16}}{\frac{7}{4} + \frac{7}{18}} = 0.321+$$

$$\theta = 17^\circ\ 50'. \quad \textit{Answer.}$$

13.30. Tool and Die Work. The sketch shows part of an intricate die punch. Find the distance x when the other dimensions are as designated.

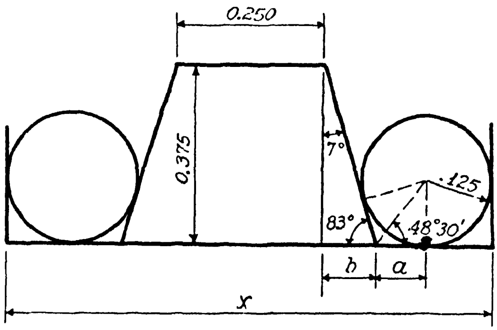


$$\tan\theta = \frac{0.050}{0.069} = 0.725$$

$$x = \frac{0.125}{\tan\frac{180 - 36}{2}}$$

$$= \frac{0.125}{3.078} = 0.041. \quad \textit{Answer.}$$

13.31. Tool and Die Work. What is the total length of this machine when it is made to the specifications given on the diagram?



$$a = \frac{0.125}{\tan 48^\circ\ 30'} = \frac{0.125}{1.13} = 0.111$$

$$b = 0.375 \tan 7^\circ = (0.375)(0.123) = 0.046$$

$$x = (2)\ (0.111) + (2)(0.046 + (2)(0.125) + (0.250)$$

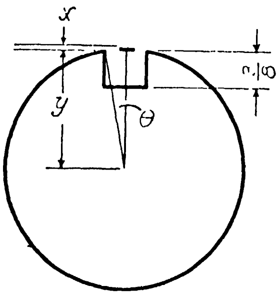
$$x = 0.814. \quad \textit{Answer.}$$

13.32. Tool and Die Work. A shaft of diameter $3\frac{5}{16}$ in. is to have a square keyway $\frac{3}{8}$ in. high cut along it. What is the additional distance which must be added to $\frac{3}{8}$ in. to give the total depth from the point where the cutting tool first begins to cut? (Answer to nearest thousandth of an inch. Use trigonometric functions.)

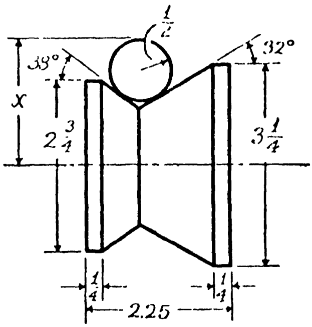
$$\sin\theta = \frac{\frac{3}{16}}{\frac{53}{32}} = 0.1132 \qquad \tan 6^\circ 30' = \frac{0.1875}{y}$$

$$\theta = 6^\circ 30' \qquad y = \frac{0.1875}{0.1139} = 1.646$$

$$x = r - y = 1.656 - 1.646 = 0.010 \text{ in.} \quad \textit{Answer.}$$



T. 13.32

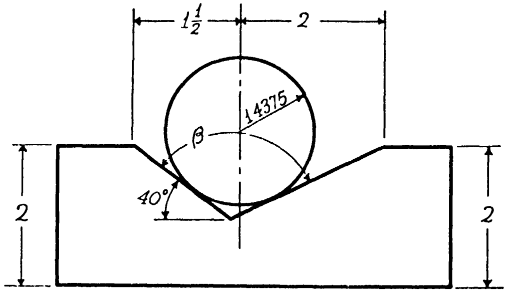


T. 13.33

13.33. Tool and Die Work. Tool and die work consists of the making of tools and dies used to produce other machines and tools. Large factories employ both designers and makers of tools and dies. Both groups would encounter a problem of the type presented above. The applications of trigonometry are very numerous.

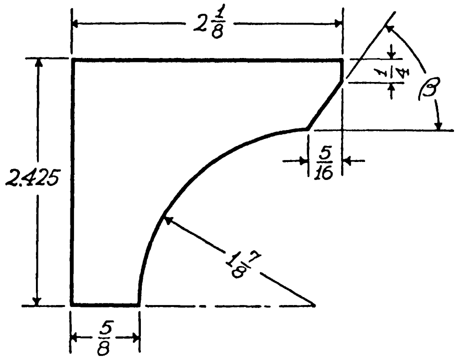
In the design of a form-rolling machine (rolling pieces to desired shapes) it was necessary to determine the distance from the center of the roller to the top of the piece to be formed. In the figure, which shows this arrangement, determine the distance x.

13.34. Tool and Die Work. The dimensions in figure 13.34 below were taken in checking a V-block. What is the value of the angle β?

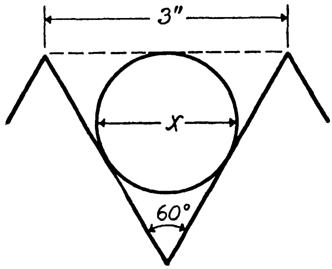


T. 13.34

13.35. Tool and Die Work. Determine the angle β in the die part illustrated below.

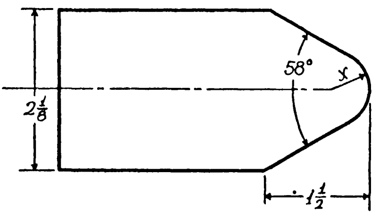


T. 13.35

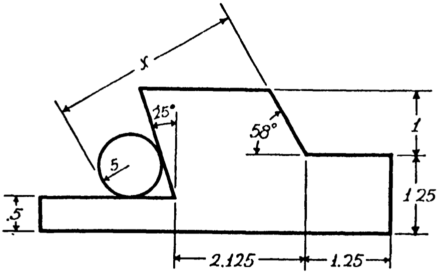


T. 13.36

13.36. Tool and Die Work. The figure (right above) illustrates a method of checking screw threads and gages. Determine the diameter of the circular piece to be used in checking the part illustrated.



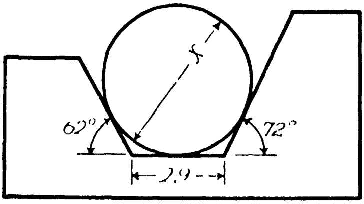
T. 13.37



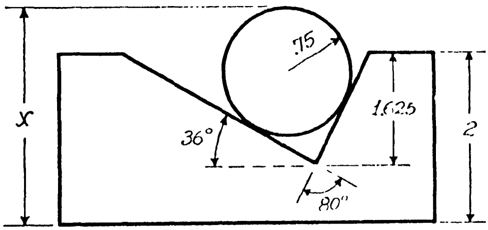
T. 13.38

13.37. Tool and Die Work. In the die punch on page 260, determine the radius x. This particular die punch is used to produce an impression upon a piece of steel, giving it the shape seen in the tapered end in the drawing. This is a problem used in shop mathematics.

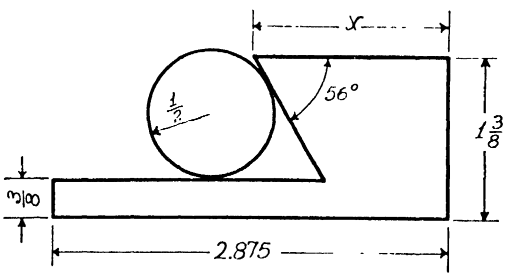
13.38. Tool and Die Work. Figure T. 13.28 above illustrates the method used to check a certain machine part. Determine the distance x.



13.39. Tool and Die Work. What size (diameter) gage piece must be used in order to check the piece illustrated above?

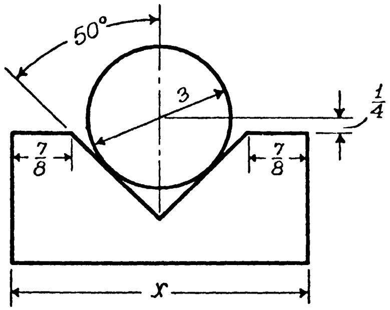


13.40. Tool and Die Work. The figure above illustrates the method used to check the V-block. Determine the distance x which the designer would need to calculate.



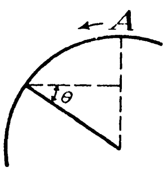
13.41. Tool and Die Work. In checking the dovetail on a machine tool the method illustrated on page 261 was employed. Determine the distance x, which the designer would need to calculate.

13.42. Tool and Die Work. The figure below illustrates a V-block used in a certain machine tool. Determine the distance x. A V-block is a piece of steel with a V-shaped groove in it. The particular one in the drawing is used as a guide for the steel ball shown. This design is used in many types of machines.



TRIGONOMETRIC MANIPULATION 14.01–14.23

14.01. Athletics. A ski jumper starts down a hill from the point marked A. The cross section of the hill is a circle of radius R. It can be shown, by methods of physics and mechanics, that the radius to the point at which he will leave the surface of the hill (neglecting friction) will make an angle θ with the horizontal, where $\sin\theta = 2(1 - \sin\theta)$. Determine this angle.



14.02. Athletics. "When a heavy body leaves the ground, the distance it will go in a horizontal direction is given by the formula:

$$d = \frac{V^2 \sin A \sin B}{16}.$$

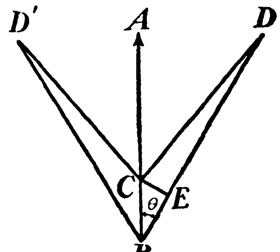
Where V is its starting velocity; A, its angle of elevation; and $B = 90° - A$. In the Olympic contests, the winner in the 16 lb. hammer-throwing contest could throw the hammer with a velocity of 76 feet a second. If he threw it at the most advantageous angle (45°, theoretically), what is his record? What would the distance have been if the angle had been 38°?"*

14.03. BIOMATHEMATICS. "(a) Find the work done by the contraction of a penniform muscle. (b) The angle made by the fibres of the mylohyoid with the central raphé is 45°. If the longest fibre contracts by $\frac{1}{10}$ inch, how far will the middle point of the hyoid bone be drawn up?

"(a) Work is measured by force multiplied by the distance through which the force acts in its own direction.

"Now fixing our attention on the right side of the penniform muscle, let DB represent one fibre, which by contraction pulls the point B up to C. The fibre DB will therefore have contracted to DC.

"Now, if CE is drawn perpendicular to DB, then BE is equal to the amount of shortening of DB (since DE is very nearly equal to DC), and BC is the distance through which B has been moved.



"But $$BC = \frac{BE}{\cos\theta} = BE \sec\theta,$$

and the total force exerted by both sides of the muscle in pulling B in the direction of BC is the resultant $2F\cos\theta$

$$\therefore \text{Work done} = (2F\cos\theta)(BE\sec\theta)$$
$$\text{Work done} = (2F)(BE).$$

But $(2F)(BE)$ is the work inherent in all muscular fibres of both sides of the muscle if arranged in a prismatic manner.

"Work done by penniform muscle equals that done by the prismatic muscle of the same length of fibre.

"(b) From formula

$$BC = BE\sec\theta$$
$$\text{we have } BC = \tfrac{1}{10}\sec 45° = \tfrac{1}{10}\sqrt{2}$$
$$BC = 0.14 \text{ inch.}\text{"}\dagger$$

* C. H. Mergendahl and T. G. Walters, *Intermediate Algebra*, p. 137. D. Appleton-Century Company, New York, 1941.

† W. M. Feldman, *Biomathematics*, p. 47. Griffin & Company, London, 1935.

14.04. Business. By trigonometry only, it can be shown that if h is the depth of the oil, r the radius of the cross section (circular) of the tank, and s the length, all expressed in feet, then the number of gallons in the tank is given by

$$7.481\,[r^2 \text{ arc cos } (r - h)/r - (r - h)(2rh - h^2)^{1/2}].$$

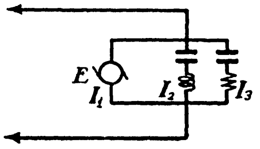
14.05. Business. The following chart was supplied by a large business concern. It has to do with the calibration of gage for a 100-gallon gasoline tank in ***horizontal position***, 28 in. in diameter and 40 in. long.

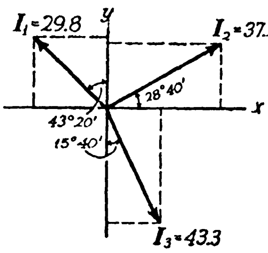
$$\text{Gallons} = 40\left(\frac{616}{360}\cdot 2\cos^{-1}\cdot\frac{\text{alt}}{14} - 14\cdot \text{alt}\cdot \sin x\right)\left(\frac{1}{231}\right)$$

Depth in inches	$x = \cos^{-1}\dfrac{alt}{14}$	*Sq. in.* $\dfrac{616}{360}(2x)$	*sin x*	14 *sin x*	*Sq. in. triangle*	*Net area*	*Approx. gallons*
1″	21°40′	74.1	.37	5.18	67.3	6.8	1.13
2″	30°50′	105.3	.513	7.19	86.5	18.8	3.26
3″	38°10′	130 7	.618	8.65	95.2	35.5	6.15
4″	44°20′	151.8	.699	9.79	97.9	53.9	9.34
5″	50°00′	171.0	.766	10.72	96.5	74.5	12.90
6″	55°10′	189.0	.821	11.50	91.9	97.1	16.80

Continuing with this, the approximate gallons for depth in inches are: 7″, 20.80; 8″, 25.20; 9″, 30.00; 10″, 34.35; 11″, 38.80; 12″, 43.70; 13″, 48.50; 14″, 53.40; 15″, 58.30; 16″, 63.10; 17″, 68.00; 18″, 72.45; 19″, 76.80; 20″, 81.60; 21″, 86.00; 22″, 90.00; 23″, 93.90; 24″, 97.46; 25″, 100.65; 26″, 103.54; 27″, 105.67; 28″, 106.80.

14.06. Electricity. A preliminary calculation shows that the currents through a divided (||) circuit may be represented by the





vectors as shown. Find the resultant current. (vector sum of I_1, I_2, I_3)

$$I_R \text{ on } x = 37\ 2 \cos 28°40' + 43\ 3 \sin 15°40' - 29.8 \sin 43°20'$$
$$x = 32\ 64 + 11\ 69 - 20.45 = 23\ 88$$
$$.I_R \text{ on } y = 37.2 \sin 28°40' + 29.8 \cos 43°20' - 43.3 \cos 15°40'$$
$$y = 17.85 + 21\ 68 - 41\ 691 = -2\ 16$$
$$I_R = \sqrt{(23\ 88)^2 + (-2\ 16)^2} = \sqrt{576\ 84}$$
$$\doteq 24 \text{ amps}$$
$$\angle \text{ with } x - \text{axis} = \tan^{-1}\left(\frac{-2\ 16}{23\ 88}\right) = -5°10'$$

14.07. FORCES. "Simple mechanics:

"*Vector:* A vector is a straight line with an arrow showing direction. In mathematical solutions the vectors are usually not drawn to scale. In graphical solutions they are drawn to scale.

"*Force:* A force, in simple words, is a push or a pull. It is a pushing force when there is a compression. If it is a pulling force, it is said to be a tensile force and the body on which it acts is said to be in tension. When power is transmitted through a revolving shaft, such as a crankshaft in a motor, a twisting, or torsional force is set up. Still another force is that of shear, the act of cutting as with a pair of shears. This type of force is common in rivets. ·

"*Representation of Forces:* Three things are necessary to represent properly a force: magnitude, direction, and point of application. By using a vector all three things can be shown.

"*Concurrent Forces:* When the vectors representing two or more forces meet at a common point, or will meet at a common point if extended, they are said to be concurrent forces.

"*Resultant Force:* If two or more concurrent forces act upon a body there is a single force that could replace them and produce the same result. This force is known as the resultant force.

"*Resolution of Force:* A single force may be replaced by two forces that will produce the same result. These are called components of the force. The components most commonly used are the horizontal component (H_c), and the vertical component (V_c).

"*Mathematical Method for Finding the Resultant for a System of Concurrent Forces:*

"Break each force up into its horizontal and vertical components. If the horizontal component is to the right of the vertical line it is indicated by the + sign; if the force is to the left, by the − sign. If the vertical component is above the horizontal line it is indicated by the + sign, and if below, by the − sign. The algebraic sum of the horizontal components is found, as is the sum of the vertical components. The resultant is found by the formula:

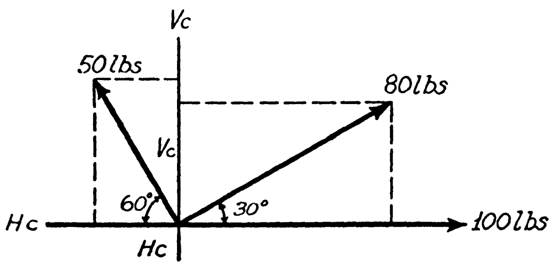
$$R = \sqrt{\Sigma V_c^2 + \Sigma H_c^2},$$

where R = resultant, ΣV_c = algebraic sum of the vertical components, ΣH_c = algebraic sum of the horizontal components. The angle the resultant makes with the horizontal is found:

$$\tan A = \frac{\Sigma V_c}{\Sigma H_c}.$$

"*Example:* Find the resultant of the following system of concurrent forces:

100# force
$H_c = +100$
$V_c = 0$
80# force
$H_c = 80 \cdot \cos 30° = +69.282$
$V_c = 80 \cdot \sin 30° = +40.00$
50# force
$H_c = 50 \cdot \cos 60° = -25.00$
$V_c = 50 \cdot \sin 60° = +43.30$



"The results are tabulated as follows:

Force	V_c	H_c
100	0	+100
80	+40	+ 69.28
50	+43.30	− 25.00
	ΣV_c +83.30	ΣH_c +144.28

$$R = \sqrt{83.3^2 + 144.28^2}$$

$$R = 166.6\#$$

$$\tan A = \frac{83.3}{144.28} = 0.57735$$

$$A = 30^\circ\ 0' +\text{"*}$$

14.08. Forces. "If a force acts at any angle to the plane as X in figure 2 and the angle of the incline makes Y degrees with the horizontal, then

$$\frac{P}{W} = \frac{\sin Y}{\cos X},$$

from which

$$P = \frac{W \sin Y}{\cos X} \qquad W = \frac{P \cos X}{\sin Y} \qquad \cos X = \frac{W \sin Y}{P}$$

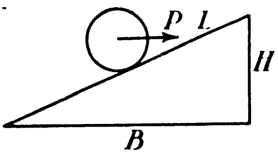


FIGURE 1

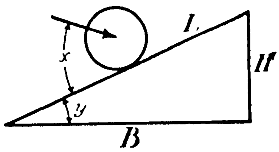


FIGURE 2

Problem: "A boiler drum weighing one ton is to be rolled up a 10-degree incline. What force will be required (ignoring friction) if X is 20 degrees?

$$\sin Y = \sin 10^\circ = 0.1736$$

$$\cos X = \cos 20^\circ = 0.9397.$$

Then,

$$P = \frac{W \sin Y}{\cos X} = \frac{(2000)(0.1736)}{0.9397} = \frac{347.2}{0.9397} = 369.5 \text{ lb. } \textit{Answer.}\text{"}\dagger$$

* Commander A. E. Downer, *Practical Mathematics of Aviation*, pp. 74-75. Pitman Publishing Corporation, New York, Second Edition, 1940.

† M. E. Jansson and M. D. Harper, *Handbook of Applied Mathematics*, p. 213. D. Van Nostrand and Co., New York, Second Edition, 1936

14.09. Highways. Area of an irregular piece of land.

Problem: Find the area of the curvilinear, triangular corner between the outer edge of a 66-foot highway that curves in the arc of a circle from a northerly direction to a westerly direction at the corner of a section, the curve being a complete quadrant.

Solution: The required area CDE is the corner of a square, whose side (AB) is 433 ft., cut off by the arc of a circle whose radius is 466 ft. and whose center is at the opposite corner of the square. By using the Pythagorean Theorem, we get

$$BC = 172.2 \text{ ft.}$$

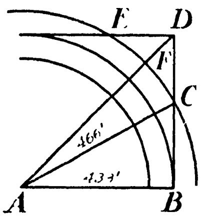
$$\text{Angle } CAD = 45^\circ - \text{arc tan} \frac{172.2}{433} = 23^\circ 19' = .4069 \text{ radians}$$

$$\text{Sector } CAF = \tfrac{1}{2}(466)^2(.4069)$$

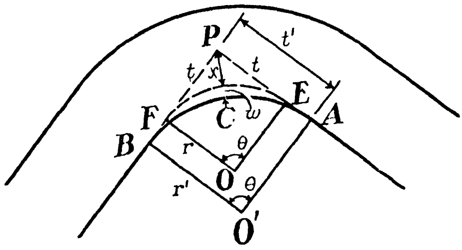
$$\text{Area } CDE = (433)^2 - 2(ABC) - 2(CAF)$$

$$CDE = (433)^2 - 433(172.2) - (466)^2\,(.4069)$$

$$CDE = 24{,}564 \text{ sq. ft.} \quad \textit{Answer.}$$



T. 14.09



T. 14.10

14.10. Highways. "The tendency of a motorist to 'cut the corners' is due to his unconscious desire to give the path of his car around a turn the longest possible radius. Many highway engineers recognize this tendency by widening the pavement on the inside of the curve as shown in the figure above. The practice adds to the attractive appearance of the highway. If the pavement is the same width around the curve as on the tangents, the curved section appears narrower than the normal width; whereas if the curved section is widened gradually to the mid-point of G of the turn, the pavement appears to have a uniform width all the way round.

"In order that the part added may fit the curve properly, it is necessary to have the curve of the inner edge a true arc of a circle, tangent to the edge of the straightaway sections, and therefore it must start before the point E of the curve is reached. The part added may be easily staked out on the ground with transit and tape, by means of data derived from the radius r, the central angle θ of the curve, and the width w. In practice the width w is taken from 2 feet to 8 feet, according to the value of r. The width added can be readily computed when values for r, w, and θ are given.

"*Widening of pavement on curves:*

$$x = r \sec \tfrac{1}{2}\,\theta - r = \frac{r}{\cos \frac{1}{2}\,\theta} - r$$

$$x + w = r' \sec \tfrac{1}{2}\,\theta - r' = \frac{r'}{\cos \frac{1}{2}\,\theta} - r'$$

$$\therefore r' = \frac{x + w}{\sec \frac{1}{2}\,\theta - 1} = \frac{(x + w)\cos \frac{1}{2}\,\theta}{1 - \cos \frac{1}{2}\,\theta}$$

$$t = r \tan \tfrac{1}{2}\,\theta$$

$$t' = r' \tan \tfrac{1}{2}\,\theta.\text{''*}$$

14.11. Highways.† "Mathematics of the Traffiscope."

"The Traffiscope is a device aimed at eliminating dangerous highway hazards on the crests of hilltops without incurring great expenses for land condemnation and regrading. A steel frame, arching the highway, supports a number of glass prisms which refract light rays in such a way that an auto driver ascending the hill is enabled to see traffic approaching the hill crest on the opposite side of the hill.

"Since not all hill crests have the same sloped approaches, prisms of different sizes are needed for various locations. Also the cumulative height of the total number of prisms used in one Traffiscope is not the same for all situations. This, in turn, requires a steel supporting frame of a different size. In order to know the type of prisms needed, the height of the lens, and the size of steel frame

* C. I Palmer, *Practical Mathematics*, p. 439. McGraw-Hill Book Company, First Edition, New York, 1919.

† Quoted from L. E. Boyer, *Mathematics of the Traffiscope*. L. E. Boyer, Millersville, Pa., 1941. The information and pictures basic to his explanations were secured from Mr. R. M. Cooley, Vice President and Chief Engineer of Traffiscope, Inc., 2500 Rand Tower, Minneapolis, Minn.

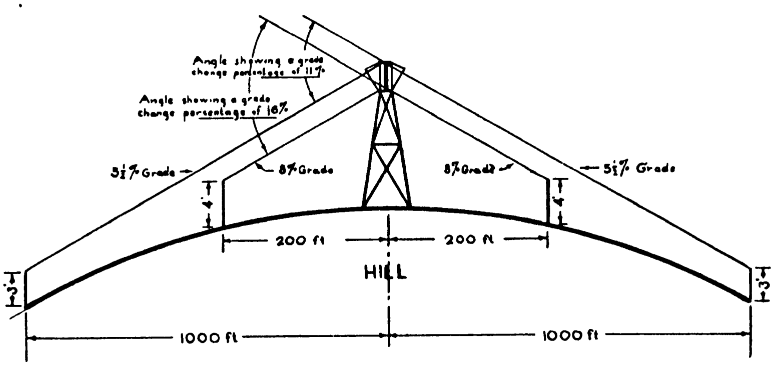
needed to support the prisms for any one situation, a Dimension Chart is used by the Company's home office. *It should be noted that the Dimension Chart is first established by applying considerable mathematical theory and calculation; thereafter it can be readily and easily used for any and all situations without any of the previous hard work.*

"The Dimension Chart showing the use of decimals, fractions, denominate numbers, angles, and sine ratios is given [in part] below.

Prism lite designation 1	Height in inches measured on face 2	Top thickness in inches 3	Bottom thickness in inches 4	Difference between top and bottom in inches 5	Angle between surfaces 6
1	4	.751	1¼	0.499	7° 10′ 00″
2	4	.726	1¼	0 524	7° 31′ 30″
3	4	.701	1¼	0 549	7° 53′ 00″
4	4	.677	1¼	0 573	8° 14′ 30″
5	4	.652	1¼	0.598	8° 36′ 00″
15	4	406	1¼	0 844	12° 11′ 00″
16	4	.381	1¼	0 869	12° 32′ 30″
17	4	.357	1¼	0 893	12° 54′ 00″
18	4	.333	1¼	0 917	13° 15′ 30″
19	4	308	1¼	0 942	13° 37′ 00″
25	3½	.306	1¼	0 944	15° 39′ 17″
26	3½	287	1¼	0 963	15° 58′ 06″
27	3½	.269	1¼	0 981	16° 16′ 54″
28	3½	253	1¼	0 997	16° 35′ 43″
29	3¼	.306	1¼	0 944	16° 53′ 11″
30	3¼	.290	1¼	0 960	17° 10′ 39″
50	2½	.304	1¼	0 946	22° 14′ 20″
51	2½	.295	1¼	0.955	22° 27′ 47″
52	2½	.286	1¼	0 964	22° 41′ 13″
53	2½	.277	1¼	0.973	22° 54′ 39″
54	2½	.268	1¼	0 982	23° 08′ 05″
55	2½	.259	1¼	0 991	23° 21′ 32″
85	2	.278	1¼	0 972	29° 05′ 31″
86	2	.272	1¼	0 978	29° 16′ 16″
87	2	.267	1¼	0 983	29° 27′ 01″
88	2	.261	1¼	0 989	29° 37′ 46″
89	2	.255	1¼	0 995	29° 48′ 31″
90	2	.250	1¼	1.000	30° 00′ 00″

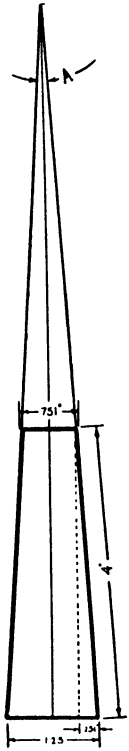
Prism lite designation 1	Sine of angle between surfaces 7	Sine of refraction angle 8	Angle of refraction 9	Grade change percentage 10	Cumulative height 11
1	.12476	.06537	3° 45′	6 54	21′ 6¾″
2	.13094	.06861	3° 56′	6 86	21′ 2¾″
3	.13716	.07187	4° 07′	7.19	20′ 10¾″
4	.14334	.07511	4° 19′	7 51	20′ 6¾″
5	.14954	.07836	4° 30′	7.84	20′ 2¾″
15	21104	.11058	6° 21′	11 06	16′ 10¾″
16	.21715	11379	6° 32′	11 38	16′ 6¾″
17	.22325	.11698	6° 43′	11.70	16′ 2¾″
18	22934	12017	6° 54′	12 02	15′ 10¾″
19	23542	.12336	7° 05′	12.34	15′ 6¾″
25	.26985	.14140	8° 08′	14 14	13′ 7½″
26	27511	.14415	8° 17′	14 42	13′ 4″
27	28036	14691	8° 27′	14 69	13′ ½″
28	28560	14965	8° 36′	14 97	12′ 9″
29	.29048	.15221	8° 45′	15 22	12′ 5½″
30	29535	.15476	8° 54′	15 48	12′ 2¼″
50	.37847	.19832	11° 26′	19 83	7′ 5″
51	.38211	.20023	11° 33′	20 02	7′ 2½″
52	.38570	.20211	11° 40′	20 21	7′ 00″
53	.38902	.20384	11° 46′	20 38	6′ 9½″
54	.39289	.20587	11° 53′	20.59	6′ 7″
55	.39650	.20777	11° 59′	20.78	6′ 4½″
85	.48621	.25477	14° 46′	25.48	1′ 00″
86	.48895	.25621	14° 51′	25.62	0′ 10″
87	.49166	.25763	14° 56′	25.76	0′ 08″
88	.49438	.25905	15° 01′	25.91	0′ 06″
89	.49735	.25061	15° 06′	26 06	0′ 04″
90	.50000	.26200	15° 11′	26 20	0′ 02″

"Before the Dimension Chart can be used, a profile of the hill crest where the Traffiscope is to be used is made somewhat as follows:



SKETCH SHOWING PRINCIPLES OF INSTALLING TRAFFISCOPE ON HILL

"From this sketch it is seen that the grade change for the top is 11% and the bottom grade change is 16%. Therefore by consulting the Dimension Chart it is evident that prisms numbered 32 to 15 inclusive would be selected for the prisms in a Traffiscope for this particular hill crest. The height of the lens would be 5′ 7″, as found from numbers in extreme right hand column (by subtracting 11′ 7.75″ from 16′ 10.75″.) Column 1, along the left margin, represents an assignment of numbers to the various sized prisms. This simple association of numbers to the prisms of various sizes is of immense help in designing and manufacturing Traffiscopes because the numbers make possible a concept of order and a means of easy reference. This is an application of the fundamental mathematical idea of one-to-one correspondence. The numbers in columns 2, 3, 4, and 5 are easily determined and understood. The numbers in column 6 are determined as follows: In the drawing of prism No. 1 it is seen that the *ratio* of the side opposite angle *a* to the hypotenuse is .250/4 or .0625. This ratio is called the sine function of angle *a* and by using a Table of Natural Functions angle *a* is found to be 3° 35′, to the nearest minute. The angle be-



PRISM No 1

tween the surfaces of the prism, A, equals twice 3° 35′ or 7° 10′, the first number found in column 6. (That angle A = twice a can be proved by dropping a perpendicular from the vertex of A to base of prism. This line bisects angle A and each of the halves of angle $A = a$ because corresponding angles of parallel lines are equal.) From the Table of Natural Functions it is found that the sine of 7° 10′ is .12476, the first number in column 7. To find .06537, the first number in column 8, the following formula

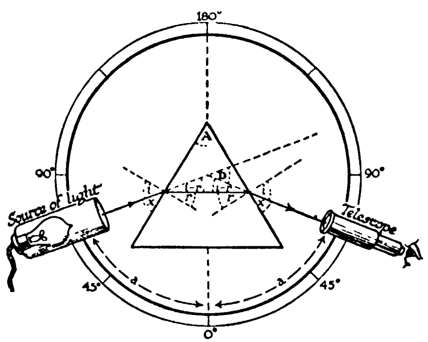
$$R \times \text{sine } A - \text{sine } A = \text{sine } D$$

is used. In this formula R, equal to 1.524, is the index of refraction of the kind of glass used to manufacture the prisms.* A is the angle between the surfaces of the prism and D is the angle of total grade change. In the case of prism No. 1, .524 sine A = sine D. That is, .524 × .12476 = .06537 = sine D.

"It may be noticed that the formula used above is not identical with that found in some books that treat the theory of light waves passing through

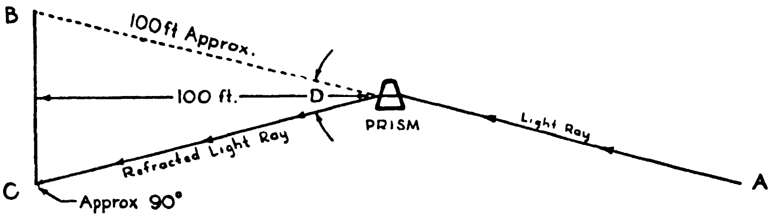
* Again one is tempted to ask, How is the index of refraction determined? This is another interesting application of mathematics. Lancelot Hogben in his book *Science for the Citizen* (Alfred A. Knopf, New York, 1938, p. 140) answers this question clearly. Therefore we reproduce, with the publisher's permission, his explanation. Note the frequent use of theorems from plane geometry and concepts from trigonometry.

"Accurate Determination of Refractive Index Diagram of Spectrometer.



a glass prism. The formula used here is a convenient approximate formula that yields useful results in this application.

"The angles given in column 9 are found by using the Table of Natural Functions. Thus, the angle whose sine is .06537 (in column 8) is found to be 3° 45′. The "Grade Change Percentage" is found as follows. *CBD*, in the drawing which follows, is regarded



as an approximate right triangle with a right angle at *C*, a hypotenuse *BD* equal to 100 feet and angle *D* (the angle of refraction or the total amount the light ray is deflected by passing through the prism) equal to 3° 54′. Solving this right triangle for *BC* (using the relationship sine $D = \frac{BC}{BD}$) we have, upon substituting given values, .06537 × 100 = *BC*. That is, *BC* = 6.54 ft. and it can be said that the amount of grade change in the line of vision per 100 feet is approximately 6.54 ft.

"The numbers in column 11 merely represent the cumulative height of prisms numbered 90 to 1 inclusive and furnish a ready

"The prism (upper figure) is set so that the two faces which enclose *A* are equally inclined to the line joining 0° to 180° on the scale. The source of light and the telescope are moved till the telescope receives the beam while they are inclined at equal angles *a* to this line. The angle of deviation (*D*) is then 180° − 2*a*.

"The beams entering and leaving the prism then make equal angles (*x*) with the two faces which enclose the angle *A*, and $i = 90° - x$.

"Since $\angle D$ = the sum of the two interior opposite angles, each $(i - r)$

$$D = 2(i - r) \therefore i = \tfrac{1}{2}D + r.$$

"Since $\angle A$ and the two base angles, each equal to $x + (i - r)$, make up two right angles

$$\therefore A + 2(x + i - r) = 2(90°)$$
$$\therefore A + 2(90° - i + i - r) = 2(90°)$$
$$\therefore r = \tfrac{1}{2} A$$
$$\therefore R = \frac{\sin i}{\sin r} = \frac{\sin \frac{1}{2}(A + D)}{\sin \frac{1}{2} A}."$$

means of determining the approximate height of the lens for any given hill crest, as used earlier in this application.

"The above Dimension Chart, like many other works of mathematics, was produced for Traffiscope, Inc., by its engineers. As seen in this application, it is the work of the mathematician that furnishes the design and the basic calculations that guide the manufacturing process from the beginning to the end. Without a body of knowledge concerning decimals, fractions, denominate numbers, angles and ratios, the mathematical engineer would be helpless; or, maybe, he would be a nonentity!

"On the other hand, with an extensive body of mathematics, a group of willing workers and the will to do good, man can fashion his environment for his welfare. The story of the cleverly built Traffiscope is an example of how the application of mathematics may save pain, broken bones and even precious lives."

14.12. MACHINES. The axial force P of a cone clutch is $2\pi c(r_2 - r_1)$, and the moment that can be transmitted is $M = \dfrac{\pi uc}{\sin\alpha}(r_2{}^2 - r_1{}^2)$, where r_1 and r_2 are the inner and outer radii of a conical disc, and c is a constant depending on the normal wear of the friction surfaces. Express M in terms of P, u, $\sin\alpha$, and D, which is the mean diameter $(r_1 + r_2)$.

$$r_2 - r_1 = \frac{P}{2\pi c}$$

$$r_2 + r_1 = D$$

$$M = \frac{\pi uc}{\sin\alpha}\cdot\frac{P}{2\pi c}\cdot D = \frac{uPD}{2\sin\alpha}. \qquad \textit{Answer.}$$

14.13. MACHINES. The follower on a cam* at a time t seconds has for its abscissa $x = 3\sin 2\pi wt$ inches and for its ordinate $y = 4\cos 2\pi wt$ inches. The quantity w is a constant. Show that $x^2/25 + y^2/16$ always has the value $+1$, irrespective of time.

*A cam is a moving part having a surface which transmits a predetermined irregular motion to a machine part called the follower. A cam may be designed to give any desired motion no matter how complicated it may be. Very often the marvelous features of automatic machinery which may seem to operate with human intelligence are obtained by the use of large numbers of specially designed cams.

14.14. Machines. A tapered keyway is to be ground into a machine housing. To check the sizes, a wire is inserted and the distance x is measured. Before this is done, it is necessary to compute the r of the wire, tangent to the three sides.

Then $x = b - 2r,\ 2a = 1.68,\ b = 1.50$

$\alpha = 7°$ Find r and x.

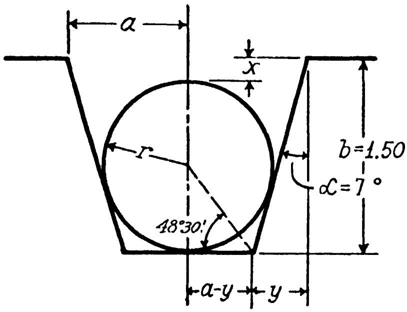
$$\frac{y}{1.50} = \tan 7° \quad y = 1.50 \tan 7°$$

$$2a = 1.68 \qquad a = 0.84$$

$$a - y = 0.84 - 1.50 \tan 7° = 0.84 - 0.184 = 0.656$$

$$r = 0.656 \tan 48° 30' = (0.656)(1.13) = 0.742. \textit{ Answer.}$$

$$x = 1.50 - 1.484 = 0.016. \quad \textit{Answer.}$$



In the figure, if a, b, and α are given, show that

$$r = \frac{a \cos \alpha - b \sin \alpha}{1 - \sin \alpha}$$

Solution: $r = (a - y) \tan (45 + \frac{1}{2} d)$

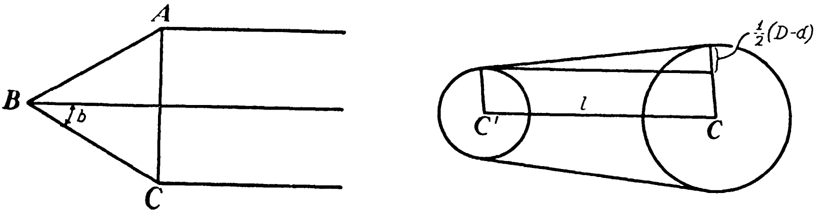
$$r \tan \left(45° - \frac{\alpha}{2}\right) = a - y = a - b \tan \alpha$$

$$r \tan \frac{(90° - \alpha)}{2} = \frac{a \cos \alpha - b \sin \alpha}{\cos \alpha}$$

$$r \frac{1 - \cos (90° - \alpha)}{\sin (90° - \alpha)} = \frac{a \cos \alpha - b \sin \alpha}{\cos \alpha}$$

$$r (1 - \sin \alpha) = a \cos \alpha - b \sin \alpha.$$

14.15. Machines.* A *steep taper* is usually referred to as an angle. Angles up to 10° are commonly designated as tapers, while a larger angle is stated either as the included angle or as the angle with the center line. In figure (a) ABC is the included angle and the angle b is the angle with the center line.



(a) (b)

The following formulas may be used to calculate b, the angle with the center line:

(a) When the taper per foot ($T.P.F.$) is known,

$$\tan b = \frac{T.P.F.}{24}.$$

(b) When the diameters and length of the taper are known,

$$\tan b = \frac{D - d}{2l}.$$

Problem: If the taper per foot is $\frac{1}{2}$ inch, find the angle with the center line and the included angle.

$$\tan b = \frac{T.P.F.}{24} = \frac{\frac{1}{2}}{24} = \frac{1}{48} = 0.02083.$$

From the table of tangents, $0.02083 = 1° \, 11' \, 35''$. Therefore, b, the angle with the center line, is $1° \, 11' \, 35''$ and the included angle is $2b$, or $2° \, 23' \, 10''$.

Problem: In figure (b) if $D = 1\frac{1}{8}$ in., $d = \frac{1}{2}$ in., and $l = 1\frac{1}{4}$ in., find the angle with the center line and the included angle.

$$\tan b = \frac{D - d}{2l} = \frac{1\frac{1}{8} - \frac{1}{2}}{(2)(1\frac{1}{4})} = \frac{1}{4} = 0.25000.$$

From the table of tangents, the $\tan^{-1} 0.25$ is $14° \, 2'$ and the included angle is $2b$ or $28° \, 4'$.

* Adapted from M. E. Jansson and Harper, H. D., *Handbook of Applied Mathematics*, pp. 531–532. D. Van Nostrand Co., New York, Second Edition, 1936.

14.16. Machines.* Taper means the difference in diameter or width in one foot of length. Tapered pins, much used for fastening cylindrical parts and for doweling, have a standard taper of one-fourth inch per foot.

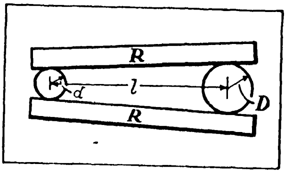
Find the diameter of the large end of a standard taper pin, the diameter of whose small end is .2837 inches and whose length is two and three-fourths inches.

How many degrees in the angle of a standard taper?

Measuring tapers with discs. One way of measuring the angle of a taper is by means of two discs of unequal diameters.

The discs are placed as shown in the figure, and the straight edges R, R, which are made of hardened steel and carefully ground, are adjusted so that the tangent lines form the taper.

Taking a as the angle with the center axis, D as the larger diameter, d as the smaller diameter, and l as the distance between the centers, as shown in the figure, we have



$$\sin a = \frac{\frac{1}{2}(D - d)}{(l)} = \frac{D - d}{2(l)}.$$

Angle a can then be found from a table of sines, and from it we can find $2a$, the included angle of the taper.

Furthermore, from the formula for the sin a we have

$$l = \frac{D - d}{2 \sin a}$$

so that, given D, d, and the angle with the axis, we can find l.

What should be the difference in height of the discs for testing a taper which is to have an included angle of 26° 30′, if the distance between the center of the discs is three inches?

14.17. Optics.† In a Hufner's spectrophotometer, the extinction

* Adapted from M. E Jansson and H. D Harper, *Handbook of Applied Mathematics*, pp. 554–555. D. Van Nostrand Co , New York, Second Edition, 1936.

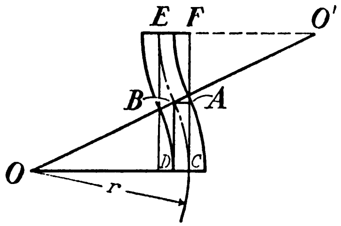
† Adapted from W. M. Feldman, *Biomathematics*, p. 48. Griffin & Co., London, 1935.

coefficient is given by the formula $e = -\log \cos^2 \phi$, where ϕ is the angle through which a Nicol's prism has to be rotated to restore equality of spectra. Find e, when $\phi = 61° 52'$.

$$\begin{aligned}
\text{Log} \cos^2 \phi &= 2 \log \cos \phi = 2 \log \cos 61° 52' \\
\text{Log} \cos^2 \phi &= 2(.6735 - 1) \\
\text{Log} \cos^2 \phi &= 1.347 - 2 \\
\text{Log} \cos^2 \phi &= -0.653 \\
\therefore e &= -2 \log \cos \phi = 0.653.
\end{aligned}$$

14.18. Plumbing. Given: Two eighth elbows ($11\frac{1}{8}°$) each having an inside arc 14 inches in length. These elbows are put together as shown in the figure.

Required: To find the amount of "off-set" of the centers of the opposite ends of the two elbows. (C is the center of lower elbow and E the center of npper.)



Given $\angle O = \angle O' = 11\frac{1}{8}°$
Arc BC = Arc BE = 14 in.
To find: EF or $2AB$

Formulae: Circumference = Arc $BC \cdot (32.4)$

$$r = \frac{C}{2\pi}$$

$$AB = DC = 1 - \cos 0 \text{ (in a unit circle)}$$
$$\therefore AB = r(1 - \cos 0)$$

Solution:

$$C = 14\,(32.4)$$
$$\therefore r = \frac{14\,(32.4)}{2(3\frac{1}{7})}$$
$$r = \frac{(14)(32.4)\,(7)}{2\,(22)} = \frac{1587}{22}$$
$$2AB = \frac{1587}{11}(.01896) = 2.74.$$

14.19. Projectiles. The range of a certain gun varies with the elevation θ, thus:

$$R = 40000 \sin \theta \cos \theta.$$

What elevation gives a range of 10,000 feet?

$$\sin \theta \cos \theta = \frac{\sin 2\theta}{2}$$

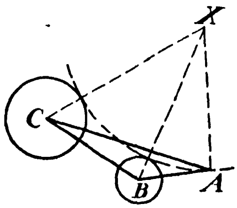
$$10000 = 20000 \sin 2\theta \qquad \text{Max. range when arc } \sin 2\theta = 1$$

$$\sin 2\theta = \tfrac{1}{2} \qquad \text{or } \theta = 45^\circ \textit{ Answer.}$$

$$2\theta = 30^\circ \qquad R = 20000 \sin 90^\circ = 20{,}000 \text{ ft.}$$

$$\theta = 15^\circ.$$

14.20. Projectiles.* "*Sound Ranging in Artillery. A, B, C* are stations with microphones at known distances apart. The microphones record simultaneously with an electric signal and timekeepers at headquarters. X is the hidden field battery. The sound is heard first at A. At B it is heard t seconds, at C, T seconds after A. If S is the speed of sound, it travels St feet during t seconds and ST feet during T seconds. Hence:



$$XB - XA = St$$
$$XC - XA = ST$$

Describe circles about B and C of radii $XB - XA$ and $XC - XA$, respectively. The location of X is now the trigonometrical problem of finding the centre of a circle which just grazes the circumference of the first two circles and also passes through A."

In 10 seconds sound travels in the air 11,200 ft.; in 5 seconds, 5,600 ft. If we describe a circle with a radius of 11,200 units around C and another of a radius 5,600 units about B, the location of the cannon is determined by finding the center of the circle that passes through A and is tangent to the circles about C and B.

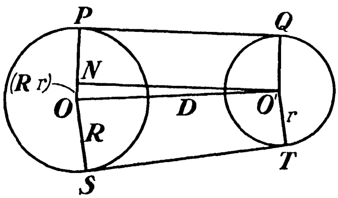
* Adapted from L. Hogben, *Science for the Citizen*, p. 311. Alfred A. Knopf, New York, 1938.

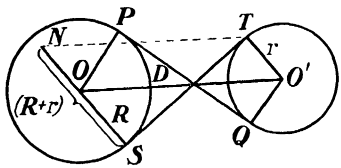
14.21. Pulleys. "If R and r are the radii of two pulleys, D the distance between the centers, and L the length of the belt, show that when the belt is not crossed the length is given by the following formula where the angle is taken in radians:

$$L = 2\sqrt{D^2 - (R + r)^2} + \pi(R + r) + 2(R - r) \sin^{-1} \frac{R - r}{D};$$

show that when the belt is crossed, the length is given by the following formula:

$$L = 2\sqrt{D^2 - (R + r)^2} + (R + r)\left(\pi + 2 \sin^{-1} \frac{R + r}{D}\right).$$



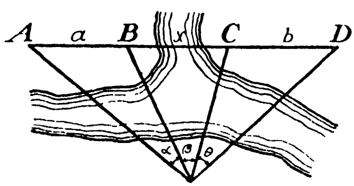


"Note: These formulas would seldom be used in practice. An approximate formula would be more convenient, or the length would be measured with a tape line.

"A rule often given for finding the length of uncrossed belts is: Add two times the distance between the centers of the shafts to half the sum of the circumferences of the two pulleys."*

(Also $L = 2D + \frac{13}{4}(R + r)$, where D is distance between centers.)

14.22. Surveying.



Data: $a, b, \beta, \alpha, \theta$
Required: BC or x

* C. I. Palmer, *Practical Mathematics for Home Study*, p. 433. McGraw-Hill Book Company. First Edition, New York, 1919.

$$\frac{a}{\sin\alpha} = \frac{EB}{\sin A}; \quad \frac{a+x}{\sin(\alpha+\beta)} = \frac{EC}{\sin A}$$

$$\frac{EB}{EC} = \frac{a}{a+x} \cdot \frac{\sin(\alpha+\beta)}{\sin\alpha}$$

$$\frac{b}{\sin\theta} = \frac{EC}{\sin D}; \quad \frac{b+x}{\sin(\beta+\theta)} = \frac{EB}{\sin D}$$

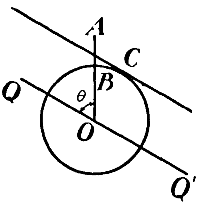
$$\frac{EC}{EB} = \frac{b}{b+x} \cdot \frac{\sin(\beta+\theta)}{\sin\theta}$$

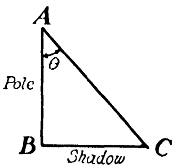
$$\therefore (a+x)(b+x)\sin\alpha\sin\theta = ab\sin(\alpha+\beta)\sin(\beta+\theta)$$

14.23. Surveying.* "The 'hypotenusal allowance' is the distance added to each chain length, when chaining along sloping ground, in order to give a length whose horizontal projection shall be exactly one chain. For the 100-link chain the hypotenusal allowance is $100(\sec\theta - 1)$, where θ is the angle of slope of the ground from the horizontal."

TRIGONOMETRIC TABLES 15.01–15.04

15.01. Astronomy. The latitude of a place may be easily calculated at the equinox at noon.† If the latitude is desired at point B we set up a pole AB and observe the length of the shadow cast as in the accompanying diagram.





$$\theta = \angle AOQ = \angle OAC$$

$$\theta = \tan^{-1}\frac{BC}{AB} = \text{latitude.}$$

* *Chambers' Technical Dictionary.* The Macmillan Company, New York, 1940.

† Equinox occurs about March 21 and September 22. One half the earth being always in the sunlight, the circle of illumination is a great circle. The vertical ray marks the center of the lighted half of the surface of the earth. At the equinoxes the vertical ray is at the equator, and the circle of illumination extends from pole to pole, bisecting every parallel. Since at this time any given parallel is cut into two equal parts by the circle of illumination, one half of it is in the sunlight, and one half of it is in darkness, and during one rotation a point on a parallel will have had twelve hours of day and twelve hours of night. (No allowance is made for refraction at twilight.

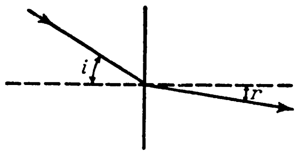
15.02. Friction. One of the laws of friction is given by the formula:

$$\tan i = k,$$

where i is the inclination of the plane and k is the coefficient of friction. For iron and stone, k varies from 0.3 to 0.7. Between what angles does i vary?

15.03. Optics. In symbols one of the laws of refraction is given by the relation:

$$\frac{\sin i}{\sin r} = k,$$



where i is the angle of incidence, r is the angle of refraction, and k is the index of refraction.

For rock salt $k = 1.54$; find r if $i = 31° 30'$.
For water $k = 1.33$; find i if $r = 17° 30'$.
For potassium chloride $k = 1.49$; find r if $i = 67°$.
Find k if $i = 77° 30'$ and $r = 65°$.

15.04. Time. The period of vibration of a pendulum is given by the nearly exact formula

$$T = 2\pi\sqrt{L/G}\,(1 + 1/4 \sin^2 \theta/2 - 9/64 \sin^4 \theta/2),$$

where L is the length of the pendulum, g is approximately 32.2 ft./sec^2, and θ is the half-angle of swing.

(a) Evaluate the part in parentheses for $\theta = 2°$, $30°$, and $60°$, and give each result correct to five decimals. (Use tables)

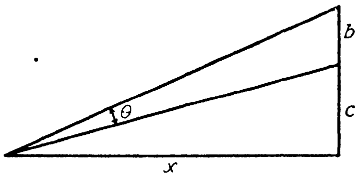
(b) For what positive range of values of θ is $\frac{1}{4} \sin^2 \theta/2 < 0.001$?

Note: In physics courses it is customary to use $T = 2\pi\sqrt{L/G}$ for small angles θ.

TRIGONOMETRY AND CALCULUS 16.01–16.05

16.01. Art.* A tapestry b feet in height is hung on a wall so that its lower edge is c feet above an observer's eye. The vertical angle

θ, subtended by the tapestry in the eye of the observer, is given by the formula noted below:

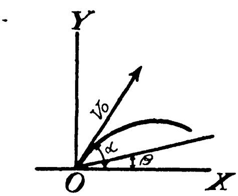


$$\theta = \arc \tan \frac{b + c}{x} - \arc \tan \frac{c}{x},$$

where x is the distance of the observer from the wall. If by calculus the value of x is found so that θ has a maximum value, then it is assumed that a most favorable view of the tapestry is obtained by the observer.

This formula is useful in determining how far the railing should be from paintings in a gallery. The solution requires the use of calculus.

16.02. PROJECTILES. The equation for the maximum range of a projectile when shot in a vacuum on an inclined plane is given by the equation:



$$R = \frac{V_0^2}{g} \cdot \frac{1}{1 + \sin \beta},$$

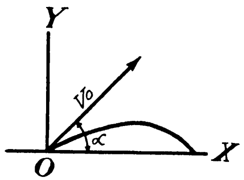
where V_0 is the velocity of the gun, β is the angle of the inclined plane, and g is equal to 32, when the units of measurement are in feet.

A revolver can give a bullet a velocity of 200 feet per second. Is it possible to hit the top of an object which is one-fourth mile away and 100 feet high?

Note: Actually, wind resistance must always be taken into account, but the formula needed requires higher mathematics along with the trigonometry.

* Adapted from W. A. Granville, P. F. Smith, W. R. Longley, *Elements of Differential and Integral Calculus* p. 114. Ginn and Company, Boston, 1934.

16.03. Projectiles. The equation a trajectory* takes when the projectile is shot in a vacuum is



$$y = x \tan \alpha - \frac{gx^2}{2V_0^2 \cos^2 \alpha},$$

where y is the vertical distance the projectile has traveled, x the horizontal distance, V_0 the initial velocity, α the angle the gun makes with the horizontal, and $g = 32$.

If $V_0 = 2000$ ft. per second, $\alpha = 30°$, and $y = 1200$ ft., how far on the horizontal has the projectile traveled from the gun?

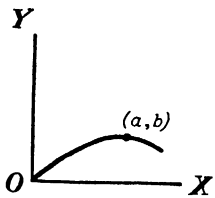
16.04. Projectiles. The equation for the velocity of a projectile at any point on a trajectory is given by the equation:†

$$V = \sqrt{V_0^2 - 2\,gy},$$

where V_0 is the initial velocity, y the height the projectile has reached, and $g = 32$.

If the initial velocity of a projectile is 2000 feet per second and if the projectile has attained a height of 1200 feet, what is the velocity of the projectile at that point?

16.05. Projectiles. To determine whether a projectile is able to strike a certain object, the following equation is used when the rojectile is shot in a vacuum:



$$V_0^2 = g\sqrt{a^2 + b^2} + b,$$

where a is the horizontal distance, b is the vertical distance, and $g = 32$.

If a gun has an initial velocity of 1500 feet per second, can a person hit an object which is on a hill at a point $\frac{1}{8}$ of a mile high? Its horizontal distance is $\frac{1}{4}$ mile.

* A trajectory is the path the projectile takes when projected into space.

† Assuming no wind resistance

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Abbreviations: Ar.—Arithmetic. Al.—Algebra. G.—Geometry. T.—Trigonometry