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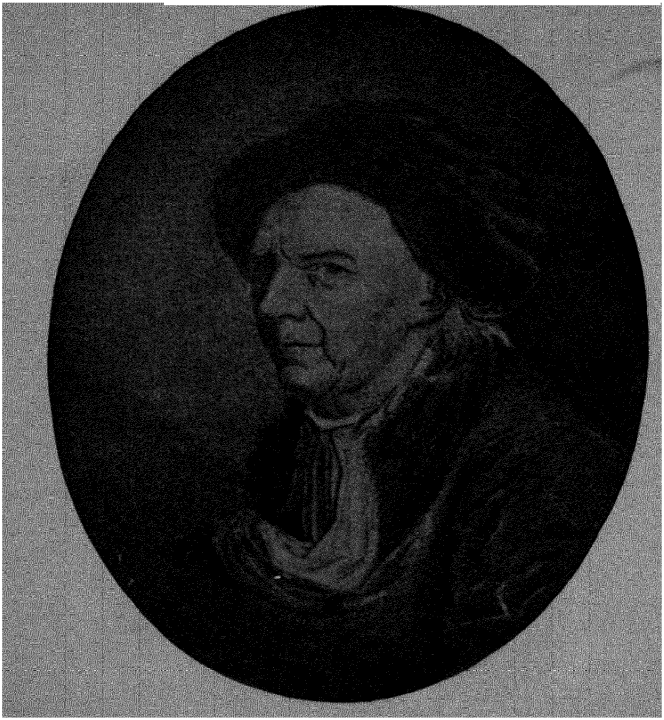

LEONHARD EULER

Leonhard Euler (1707–1783), the son of a Lutheran minister, was born at Basel, Switzerland, where he was the favorite pupil of the great mathematician John Bernoulli. At the age of twenty he went to Russia and three years later became a professor in the University of St. Petersburg (now Petrograd). In 1741 he was invited by Frederick the Great to Berlin, where he lived until 1766. He then returned to Russia and soon after this became blind; yet for seventeen years, until his death, he continued his remarkable production of mathematical literature. He revised almost all of the branches of pure mathematics and gave them a consistent form and their present arrangement. He separated trigonometry from astronomy and made it a distinct branch of mathematics. He introduced e as the symbol for the base of the natural system of logarithms, and π for the ratio of the circumference to the diameter of a circle. He was probably the most prolific writer of mathematical literature that the world has known. It is only in the present century that plans have matured for a complete edition of his works.

PLANE TRIGONOMETRY

BY

PAUL R. RIDER, PH.D.
ASSOCIATE PROFESSOR OF MATHEMATICS, WASHINGTON UNIVERSITY
SAINT LOUIS, MISSOURI

AND

ALFRED DAVIS, M. A.
HEAD ASSISTANT, DEPARTMENT OF MATHEMATICS, SOLDAN HIGH SCHOOL
SAINT LOUIS, MISSOURI

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PREFACE

This book has been developed as the result of a number of years of teaching experience, on the part of the authors, both in secondary and in university fields, and it is hoped that it will prove a suitable text-book for secondary and technical schools and for colleges.

It is believed that the explanations will be found especially clear, as the methods of presentation have been thoroughly tested in the classroom. Some of the other features of the book are its completeness, the scheme of arrangement of computations, and the abundant supply of selected problems, from which suitable choice can be made for any course in the subject.

The authors wish to acknowledge their indebtedness to Professor W. L. Crum of Harvard University and Professor F. W. Bubb of Washington University, who read the manuscript carefully and critically and gave many valuable suggestions for its improvement.

PAUL R. RIDER
ALFRED DAVIS

SAINT LOUIS, MISSOURI,
July, 1923

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PLANE TRIGONOMETRY

CHAPTER I

INTRODUCTION

1. **Trigonometry.** The word trigonometry is derived from two Greek words, **trigonon,** meaning a triangle, and **metria,** meaning measurement. Evidently one would expect the subject to treat of the measurement of triangles. It does include this as an important phase, but it is much more comprehensive. For example, it considers angles greater than 180°, it is used in simplifying certain algebraic processes, and is absolutely essential in many practical applications of mathematics, as well as in the development of its theoretical branches.

2. **The development of trigonometry.** Crude beginnings of the science may be traced back to a papyrus written by an Egyptian named **Ahmes,** who lived prior to 1700 B.C. Some of the information given in his writings may in turn be traced to other writings of about 3400 B.C., or long before the time of Moses. It was, however, **Hipparchus,** a Greek who lived about 150 B.C., who founded trigonometry. He was born in Bithynia, Asia Minor, and studied in Alexandria, Egypt. He established his home on the island of Rhodes, and there did his most important work. He is also considered to be the founder of observational astronomy, and it was to aid in carrying on work in this subject that he invented trigonometry. As a result

of this the latter science was for a long time studied as a part of astronomy. None of the works of Hipparchus has been preserved, but we know what he did through **Theon** of Alexandria. If it is somewhat surprising that the development of spherical trigonometry, which has for its primary purpose the study of triangles on a sphere, preceded that of plane trigonometry, we must remember that the science came into existence because of its need in the study of astronomy.

After Hipparchus, **Claudius Ptolemy,** who lived about A.D. 135, and who was a Greek though born in Egypt, was the next to improve trigonometry. His great work was the *Almagest.* This dealt mainly with astronomy, but it showed how to develop a table of chords, and contained a chapter devoted entirely to trigonometry.

The next notable advance was made by a German, Johann Mueller (1436–1476), who was called **Regiomontanus.** He wrote a text-book, *De triangulis,* in which he separated trigonometry from astronomy, and made it a part of geometry. This was a rhetorical work; that is, every step was expressed at length in words, no symbols being employed.

It was due, however, to the work of the Swiss, **Leonhard Euler** (1707–1783), that trigonometry was eventually established as an independent science.

During the eighteenth century **John Bernoulli** (1667–1748) and Euler developed the science as a means of analysis. **Abraham De Moivre** (1667–1754) applied it to the treatment of complex numbers. **Preston A. Lambert** (1728–1777) extended it to include hyperbolic functions. **Joseph Fourier** (1768–1830) and others introduced Fourier series, by which many phenomena of physical science, particularly radiation, can be represented in terms of trigonometric functions. **Niels Henrik Abel** (1802–1829) and other mathematicians of the nineteenth century developed elliptic functions and similar important outgrowths. Thus the subject continues to branch out and develop, and no one can predict to what results it will eventually lead.

3. The importance of trigonometry. Some of the important results to be gained from the study of other branches of mathematics may be realized in the study of trigonometry. It should develop the ability to think clearly and accurately. It may be made to develop accuracy, rapidity, and neatness; accuracy because this is essential in all mathematics; rapidity because it provides easy and short roads to conclusions and results; neatness—without which other advantages are frequently turned to naught—because of the orderly form required in the arrangement of the solutions of problems. But by far its greatest value consists in its practical application to problems that one meets on every hand. Indeed, it is one of the most practical branches of mathematics, and, with the proper background of algebra and geometry, it should be an unusually interesting subject to the student. It should serve as a fitting climax to a high school course in mathematics, and as a suitable beginning for a college course.

As has been stated, trigonometry had its origin in astronomy, and one could know little of that science were it not for the trigonometric functions. By means of them we can reach to other suns, and learn of their stupendous magnitudes and velocities; they assist us in tracing out the paths that these heavenly bodies have traversed through the centuries and in predicting their movements for ages to come.

Trigonometry has long since become a necessity in other sciences. The surveyor can measure a straight line in a convenient place and then by measuring angles he can determine the distances between points with greater accuracy and with less trouble than by direct measurement. For example, in the accompanying figure, the distance BC across the stream can be obtained to a great degree of accuracy by measuring the angle A and the line AC, which is perpendicular to BC. The navigator would be helpless without its aid. It has assisted science and engineering to reach their present high stage of development, and they could not exist without it. Its uses will be more specifically illustrated in the problems which follow throughout the text.

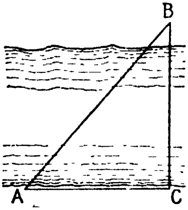


Fig. 1.

4. Directions for study. The student should be provided with compasses, a protractor, a ruler divided in accordance with both the English and the metric systems, and coordinate paper.

The arrangement of the subject matter in the text, the explanations, and the suggestions are made with a view to aiding the student in the mastery of the subject. However, the following additional suggestions may prove helpful:

1. Read carefully the assigned lesson. Be sure that every word is understood and that the principles are clearly grasped. It frequently happens that much of the difficulty of the student is due to careless reading or to guessing the meanings of words.

2. Connect the ideas involved with other facts already known. Try to justify mentally each point in the argument by calling up reasons.

3. Try to ascertain the main thought or the fundamental idea. This will aid in the untangling of complicated parts.

4. Express the newly acquired information in your own words. It is only as we are able to give expression to ideas that we are really master of them. Better still, endeavor to apply the new principle learned in some practical way. This will help to fix it in your mind.

5. If there is any point in the lesson which is not clear, try to formulate your difficulty and put it in writing. Try again to understand the matter, and if you fail present it to the class or to the teacher at the earliest opportunity.

6. In attempting to understand a difficult lesson use every mental resource you can command. For further help, consult other parts of the text through the table of contents and the index, refer to other texts, etc. If still in difficulty, discuss the matter with someone who can help you.

7. Be determined to master the lesson. A half-hearted attitude is almost certain to end in defeat. Do not discount, or allow anyone to discount for you, the importance of putting your best effort into the task you have undertaken.

8. Take time to study the lesson. The mental operations of real study require time as well as concentrated effort.

9. Remember in working problems that accuracy is more important than speed. Strive for accuracy; speed will come with practice.

10. Form the habit of self-reliance. Check your work to test its accuracy, and as a rule avoid answer books.

11. Above all, be sure to gain a thorough mastery of each point as it comes. Do not allow anything to pass not understood or poorly understood. Failure here is certain to result in discouragement and ultimately in defeat. Do

not depend on getting the matter later or on "cramming" before an examination.

5. Some important points in algebra. It is because of their importance in the understanding of trigonometry that we recall at this place several points in algebra. These will be convenient for ready reference.

Factoring. Some of the principal types of factoring are the following:

1. **The common monomial,** as

$$ax + bx = x(a + b).$$

2. **Trial,** as

$$2x^2 + 5xy + y^2 = (2x + y)(x + 2y).$$

This may also be considered to include such problems as

$$x^2 + 2xy + y^2 = (x + y)^2,$$
$$x^2 + x - 6 = (x + 3)(x - 2).$$

3. **Grouping,** as

$$ax + bx - ay - by = (a + b)(x - y).$$

4. $\boldsymbol{a^n \pm b^n}$, where n is a positive whole number. The purpose in this case is to determine when the expression may be divided by $a + b$ or by $a - b$ without a remainder. It is out of place here to give a complete discussion of this problem. We simply state the following rule: *If when a is replaced by b the expression becomes zero in value then $a - b$ is a factor, the other factor being*

$$a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1};$$

if the expression becomes zero when $-b$ replaces a then $a + b$ is a factor, the other factor being the same as before except that the signs are alternately plus and minus, the first being plus. In particular

$$a^2 - b^2 = (a + b)(a - b).$$

It should be noted that, when in the process of solving an equation, a factor which contains the unknown is divided out, this factor should be placed equal to zero and solved for the unknown.

The final test for the solution of an equation is whether or not the values obtained will satisfy the original equation. This is especially important in radical equations and in trigonometric equations.

The quadratic equation, $ax^2 + bx + c = 0$. We give the following three methods of solving a quadratic equation:

1. **Factoring.** For example,

$$\begin{aligned} 2x^2 + 5x + 2 &= 0, \\ (2x + 1)(x + 2) &= 0, \\ x &= -\tfrac{1}{2},\ -2. \end{aligned}$$

2. **Completing the square.**

$$\begin{aligned} 2x^2 + 5x + 2 &= 0, \\ x^2 + \frac{5}{2}x &= -1, \\ x^2 + \frac{5}{2}x + \frac{25}{16} &= \frac{25}{16} - 1 = \frac{9}{16}, \\ x + \frac{5}{4} &= \pm\frac{3}{4}, \\ x &= -\tfrac{1}{2},\ -2. \end{aligned}$$

3. **Formula.** The formula for the roots of a quadratic equation is obtained by solving the equation $ax^2 + bx + c = 0$ by completing the square. The result is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The **reciprocal** of a number is 1 divided by the number. Thus the reciprocal of x is $1/x$. Evidently, when the product of two numbers is unity, either is the reciprocal of the other. It is evident also that the reciprocal of a common fraction may be found by inverting the fraction.

Proportion. When two like quantities are written in the form of a fraction, or when we indicate that one of these quantities is to be divided by the other, we have a **ratio.** Two ratios written equal to each other form a **proportion.** A proportion is therefore an equation and may be handled exactly as any other equation. This is illustrated in the following:

1. $\frac{a}{b} = \frac{c}{d}$ (a given proportion).

2. Clearing of fractions, we get $ad = bc$.

3. Dividing the equation in 2 by ac, we get $\frac{b}{a} = \frac{d}{c}$ (**inversion**).

4. Dividing the equation in 2 by cd, we get $\frac{a}{c} = \frac{b}{d}$ (**alternation**).

5. Adding one to both members of the equations in 1 and 3, we get $\frac{a+b}{b} = \frac{c+d}{d}$ and $\frac{a+b}{a} = \frac{c+d}{c}$ respectively (**composition**).

6. Subtracting one from both members of the equation in 1, and subtracting both members of the equation in 3 from one, we get $\frac{a-b}{b} = \frac{c-d}{d}$ and $\frac{a-b}{a} = \frac{c-d}{c}$ respectively (**division**).

7. Dividing either equation in 6 by the corresponding equation in 5, we get $\frac{a-b}{a+b} = \frac{c-d}{c+d}$ (**composition and division**).

If $\frac{a}{b} = \frac{b}{c}$, by clearing of fractions and extracting the square root of each side of the resulting equation we get

$$b = \pm\sqrt{ac}.$$

The quantity b is called a **mean proportional** between, or **geometric mean** of, a and c.

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$, and call the value of each ratio r. Then

$$\frac{a}{b} = r, \qquad a = br; \qquad \frac{c}{d} = r, \qquad c = dr; \qquad \text{etc.}$$

Adding, and dividing by the coefficient of r, we find that

$$\frac{a + c + e + g}{b + d + f + h} = r = \frac{a}{b} = \frac{c}{d}, \qquad \text{etc.}$$

(In a series of equal ratios the sum of the numerators is to the sum of the denominators as any numerator is to its denominator.)

Radicals. Radicals may be simplified by

1. Removing from under the radical sign all factors having exponents identical with, or multiples of, the index; thus

$$\sqrt{xy^2z^3w^4} = yzw^2\sqrt{xz}.$$

2. Dividing index and exponent by a common factor; thus

$$\sqrt[18]{x^6y^{12}} = \sqrt[3]{xy^2}.$$

3. Removing fractions from under the radical sign, or removing radicals from the denominator of a fraction; thus

$$\sqrt{\frac{2}{xy^3}} = \sqrt{\frac{2xy}{x^2y^4}} = \frac{1}{xy^2}\sqrt{2xy},$$

$$\frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{3\sqrt{5}}{5} = \frac{3}{5}\sqrt{5}.$$

6. Some important points in geometry. We state in this section, for reference, a few of the important theorems, definitions, etc., of geometry which will be useful in trigonometry.

1. Triangles are **similar** (have the same shape) when they are mutually equiangular, or when their corresponding sides are proportional. It follows from this that *two right triangles are similar if an acute angle of one is equal to an acute angle of the other.* This may be considered the fundamental proposition, the chief corner stone of trigonometry.

2. *The sum of the angles of a triangle is 180°.*

3. *The square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides.* This is the **theorem of Pythagoras.** It is probably the most important as well as the most famous theorem in elementary geometry. There are many proofs for it. President James A. Garfield is credited with the discovery of a proof, and at least two other proofs have been found by high school students in recent years. One of the most convenient proofs is the following:

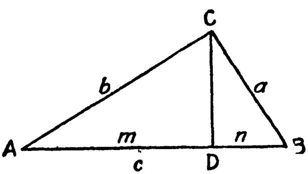


FIG. 1

The three right triangles in the figure are similar, and hence their corresponding sides are in proportion. Therefore

$$\frac{c}{a} = \frac{a}{n}, \qquad \frac{c}{b} = \frac{b}{m},$$

or

$$a^2 = cn, \qquad b^2 = cm.$$

Adding the last two equations, we get

$$a^2 + b^2 = c(m + n) = c^2.$$

4. Triangles are **congruent** (frequently called *equal,* although this term is less specific) when they can be made to coincide in all their parts.

5. Two angles are **complementary** when their sum is 90°.

6. Two angles are **supplementary** when their sum is 180°.

7. *The base angles of an isosceles triangle are equal, and conversely if two angles of a triangle are equal the sides opposite these angles are equal.*

8. *If two angles of a triangle are unequal the sides opposite them are unequal, and the greater side is opposite the greater angle; and the converse.*

9. *Two angles are equal if their sides are perpendicular, or parallel, right to right and left to left; they are supplementary if their sides are perpendicular, or parallel, right to left and left to right.*

10. *In any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides minus twice the product of one of these sides and the projection of the other side upon it.*

11. *In an obtuse triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides plus twice the product of one of these sides and the projection of the other side upon it.*

It should be noticed that 10, 11, and 3 are closely related to each other. The theorem of Pythagoras is the dividing case between the other two. These theorems may be grouped together into a single statement called the law of cosines, as may be seen when that law is introduced.

12. *The area of a triangle is equal to half the product of its base by its altitude.*

13. *The formula for the area of a triangle in terms of its sides is*

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where a, b, c are the sides of the triangle, and s is half their sum. This is known as **Hero's,** or the **Heronic, formula** in honor of Hero (or Heron), a Greek who lived in Alexandria about 120 B.C. This formula will be derived in trigonometry by a method different from that employed in geometry.

14. *A line perpendicular to a radius of a circle at its extremity is tangent to the circle, and the converse.*

15. *A line drawn from the center of a circle perpendicular to a chord bisects the chord and the arc subtended by it.*

16. *The ratio of two central angles in a circle is equal to the ratio of the corresponding intercepted arcs.* The arc of a circle may therefore be said to be measured by its central angle.

17. *The length of a circle is $2\pi r$, where r is the radius of the circle, and π (pi) is a trifle less than 3.1416. It is evident that π represents the value of the ratio* $\frac{\textit{circumference}}{\textit{diameter}}$.

18. *The area of a circle is πr^2.*

Other facts from algebra and geometry may be found when needed by consulting any good elementary text-books on these subjects.

CHAPTER II

ANGLES

7. Angle. An **angle** is formed if two straight lines extend from a point. In Fig. 2 an angle is formed at O by the lines OX and OP which extend from the point O.*

8. Generation of an angle. The angle at O may be thought of as generated by the rotation of the line OP about the point O as a pivot from coincidence with OX to its present position. (See Fig. 3.) The line OX is called the **initial side** of the angle, and the line OP is called the **terminal side.**

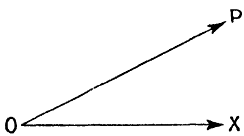


FIG. 2

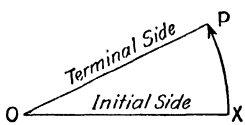


FIG. 3

If the rotating line (sometimes called the generating line) turns through less than one-fourth of a complete rotation the angle is **acute,** if through exactly one-fourth of a complete rotation it is a **right** angle, if through more than one-fourth but less than one-half of a rotation it is **obtuse,** if through one-half of a rotation it is a **straight** angle, if through more than one-half but less than a whole rotation it is a **reflex** angle, and if through exactly a complete rotation it is a **perigon.**

* Since a straight line extends indefinitely in either direction, and since each of the lines OX and OP may be considered as extending in only one direction from O, they are sometimes called **half-lines** or **rays.**

If an angle is designated by three letters, the letter on the initial side should be read first, then the letter at the vertex, and finally the letter on the terminal side. The angle in Fig. 3 should therefore be read XOP. If there is no danger of misunderstanding, the angle may be designated by the letter at the vertex. Small letters within the angles are frequently used for convenience. (See Fig. 5.)

9. Positive and negative angles. It is evident that there is a choice of directions for rotating the generating line from the position OX to the position OP. One of these directions corresponds to the motion of the hands of a clock and is called **clockwise**; the other direction is called **counter-clockwise.** If the rotation of the generating line is counter-clockwise the angle is **plus** (+) or **positive.** If the rotation is clockwise the angle is **minus** (−) or **negative.** (See Fig. 4.) A small curved arrow, starting at the initial side of an angle and ending with its tip on the terminal side, is often used to indicate the direction of rotation. (See Fig. 5.)

There is no intrinsic reason why a clockwise rotation should give a negative rather than a positive angle. The choice is purely arbitrary. However, we have followed the custom in our definition.

It is evident from our definition that angles may have any value whatever, for we may rotate the generating line as many times as we please in either direction.

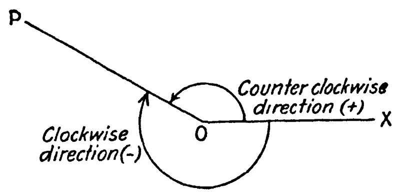


FIG. 4

Any given position of OP represents an indefinitely large number of positive and negative angles. On the other hand, to each angle, whether positive, negative, or zero, there corresponds one and only one position of OP.

Angles are equal if they are generated by the same amount of rotation in the same direction.

If the notion of angle given above is different from what the student learned in geometry, he should appreciate the superiority of the new idea, since it includes what he learned in geometry and is more general in its application. This new notion of an angle is very important in the thorough understanding of trigonometry, in the study of astronomy, and in the study of higher mathematics. Much of our work, however, will be with positive angles generated by less than one complete rotation.

10. Addition and subtraction of angles. Two angles may be added geometrically, or graphically, by placing them in the same plane with their vertices in coincidence and with the initial side of one on the terminal side of the other, as in Figs. 5 and 6.

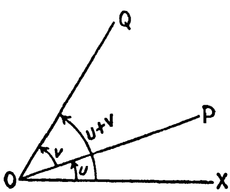


Fig. 5

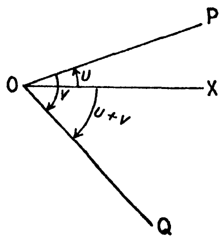


Fig. 6

Let u and v be two angles to be added, v being positive in Fig. 5 and negative in Fig. 6. The sum, $u + v$, is the angle XOQ, as indicated. For the particular angles illustrated, v is in both instances greater in absolute value (that is, greater in actual magnitude irrespective of whether positive or negative) than u. Consequently the sum $u + v$ is positive in Fig. 5 and negative in Fig. 6.

Subtraction may be performed by changing the sign of the angle to be subtracted and then adding as above.

11. Measurement of angles. The ordinary or sexagesimal system. Angles are measured by various units. The student is already familiar with the **right angle,** which is generated by one-fourth of a complete rotation, and with the **straight angle,** which is generated by one-half of a complete rotation. (See section 8.) He is also familiar with the ordinary or **sexagesimal** system of measurement (*sexagesimus* is the Latin word for *sixtieth*).

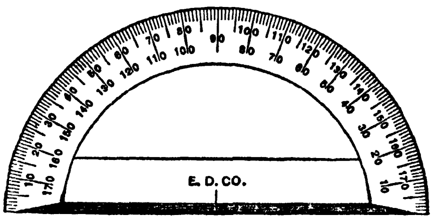
The main unit in this system is the **degree,** which is 1/360 of a rotation. Each degree is divided into sixty equal parts called **minutes,** and each minute is subdivided into sixty equal parts called **seconds.** An angle of 36 degrees, 17 minutes, 40 seconds is usually written $36^\circ\ 17$ $40''$. The units of the system may be summarized as follows:

$60''$ (seconds)	=	**$1'$ (minute)**
$60'$ (minutes)	=	**1° (degree)**
90°	=	**a right angle**
180°	=	**a straight angle**
360°	=	**a perigon, or 1 rotation**

It should be noted that, since the angles at the center of a circle are proved by plane geometry to have the same ratio as their intercepted arcs, the angles and the arcs of a circle are measured by the same angular units. This is true whatever the system used.

Historical Note. The sexagesimal system originated with the Chaldeans, who lived in Mesopotamia many centuries before the birth of Christ. There is no certainty as to how it came to be used. The Chaldeans represented fractions by it, that is, the denominators were always sixty or some power of sixty. The decimal system was not perfected until many centuries after the birth of Christ. The system of the Chaldeans was later developed by the Greeks into the form in which it is now used.

Angles are measured in actual practice by numerous instruments which usually use this system. One of these instruments is called a **protractor.** One type of protractor reads from 0° to 180° in either direction as shown in the accompanying illustration. This instrument is con-



A protractor

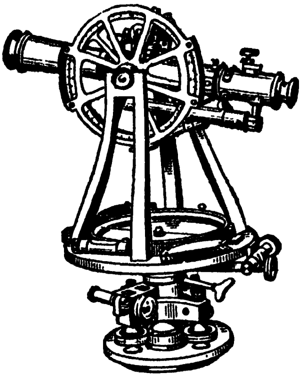
venient for class-room use. The student should obtain a protractor and practice measuring angles and constructing angles of given sizes.

Another instrument, used by surveyors for measuring angles in the field, is called a **transit,** or **theodolite.** This instrument is usually mounted on a tripod, and consists of the following principal parts:

1. A telescope for sighting straight lines.
2. Two graduated circles (that is, circles marked off

into divisions in the same way as a protractor) for reading the measures of angles in horizontal and vertical planes.

3. A compass for giving direction.


A transit or theodolite

4. A spirit level and a plumb-line for adjusting the instrument.

5. A vernier and a microscope for the accurate reading of angles.

Other instruments for measuring angles are used in astronomy, navigation, military observation, aiming of artillery, etc.

It is evident that angles can be measured only with approximate accuracy. In ordinary work no attempt is made to read them more accurately than to the nearest tenth of a minute. An object one inch long at a distance of three and a quarter miles will subtend an angle of about one second. This is too small to be considered in ordinary measurements.

However, in astronomy very great accuracy is needed. This is gained by the use of large graduated circles; micrometers; many successive readings of an angle, from which great accuracy is obtained by the application of the theory of least squares; etc.

It is not to be inferred that all instruments for measuring angles are graduated according to the sexagesimal system. There are other systems, some of which we shall consider in the following sections.

12. The centesimal system. The French use a system based on the number one hundred, and therefore called the **centesimal** system (*centesimus* is the Latin word for *hun-*

dredth). This system was perfected about A.D. 1600. The right angle is divided into one hundred equal parts, each of which is called a **grade** (pronounced grăd), the grade is divided into one hundred centesimal **minutes,** and the minute is subdivided into one hundred centesimal **seconds.** Ten centesimal minutes constitute a **decigrade,** and ten grades constitute a **dekagrade.** Thus

$$
\begin{aligned}
\mathbf{100''}\ \textbf{(seconds)} &= \mathbf{1'}\ \textbf{(minute)}\\
\mathbf{10'}\ \textbf{(minutes)} &= \textbf{1 decigrade}\\
\textbf{10 decigrades} &= \mathbf{1^g}\ \textbf{(grade)}\\
\mathbf{10^g}\ \textbf{(grades)} &= \textbf{1 dekagrade}\\
\mathbf{400^g}\ \textbf{(grades)} &= \textbf{1 rotation}
\end{aligned}
$$

The convenience of this system lies in the ease with which we change from certain units to others. For example,

$$50.6742^g = 50^g\ 67'\ 42''.$$

The system is coming into more general use. It was used to some extent in artillery work in the United States Army during the recent war. It is not likely to replace the present system for some time, however, since this would mean the changing of our tables for computation, and the graduations on surveying, astronomical, and other instruments.

We may note that, since $100^g = 90°$, *to change from grades to degrees we simply deduct one-tenth of the number of grades.* Thus, $135^g = 135° - 13.5° = 121.5°$. *For the reverse operation we take ten-ninths of the number of degrees, or move the decimal point one place to the right and divide by nine.*

Exercise 1

1. Using OX as an initial line, construct by means of a protractor the following angles: 65°, 152°, 215°, 305°.

2. Construct − 35°, − 95°, − 189°, − 280°.

3. Construct 400°, − 520°, 750°, 720°, 1000°.

4. Draw six angles at random and measure them as accurately as possible with the protractor.

5. Draw a triangle at random. Measure its angles and add them. What should the sum be? If different from this how do you account for the difference?

6. Combine graphically, using the protractor: $65° + 85°$, $45° - 73° + 115°$.

7. Subtract graphically $125°$ from $67°$, $-65°$ from $-238°$.

8. Change $53''$ to the decimal part of a degree.

9. Change $0.975°$ to minutes and seconds.

10. Change $57° 17' 45''$ to degrees, to minutes, to seconds.

11. Find an angle equal to $67° 54' 23'' + 34° 18' 52'' + 41.357° - 185.078°$.

12. Change 85.473^g to degrees, minutes and seconds.

13. Change $35° 27' 24.7''$ to grades, minutes and seconds.

13. The mil. A unit of angular measurement which found great favor during the recent war on account of its extremely practical value is the **mil**. It is defined as 1/6400 of a rotation, or 1/1600 of a right angle. It is, to a very close approximation, the thousandth part of a radian (to be defined later; see section 14). From this fact it receives its name (*mille* is the Latin word for *thousand*). Moreover it is practically the angle subtended by a length of one unit at a distance of one thousand units. That is, the base of an isosceles triangle represents one unit and its altitude one thousand units. Also it would make no essential difference if we used an isosceles triangle with its base equal to one and each leg equal to one thousand. Or we might use a right triangle with one leg equal to unity and the other leg, or the hypotenuse, equal to one thousand.

From this practical definition of the mil the following statements, though not all strictly true, are sufficiently accurate for all practical purposes if the angle concerned is not too large.

An object 1 yard long at a distance of 1000 yards will subtend an angle of 1 mil, and an object 3 yards long will subtend an angle of 3 mils at the same distance. An object that subtends an angle of 5 mils at a distance of 1000 yards is 5 yards long.

Suppose now that we have an object 8 yards long at a distance of 2000 yards, as shown in Fig. 7.

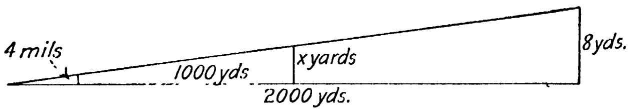


Fig. 7
(Not drawn to scale)

It is evident from the similarity of triangles that the unknown x at a distance of 1000 yards is 4 yards; hence the subtended angle is 4 mils.

In general, if we denote the range or distance to an object in any units by R, the length of the object in the same units by L, and the number of mils in the angle subtended by the object by M, as shown in Fig. 8, an approximate formula

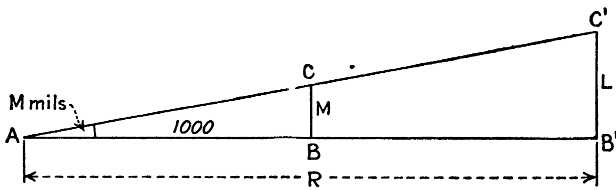


Fig. 8

can be developed connecting R, L, and M. The length CB at a distance of 1000 units from A has the same numerical measure as the angle at A measured in mils. Therefore, from the similar triangles ABC and $AB'C'$ we have

$$\frac{L}{M} = \frac{R}{1000} = (0.001)R.$$

Clearing of fractions, we get

$$L = (0.001)RM, \tag{1}$$

and solving for M,

$$M = \frac{L}{(0.001)R}. \tag{2}$$

These approximate formulas enable us to find any one of the three quantities R, L, M if the other two are known. For angles not greater than 330 mils (not quite 19°) the error will be less than 2 per cent. Thus, if by one of these formulas we have found an angle to be 300 mils, our error is less than 2 per cent. of the correct value. This is approximately the same as 2 per cent. of the 300 mils, which is 6 mils. Or, if from one of these formulas we have found the height of a hill to be 25 yards, the error will be less than half a yard.

Exercise 2

1. An object 20 feet long is 500 feet away. How many mils does it subtend?

2. A tree 250 yards distant subtends an angle of 30 mils. How tall is it?

3. A freight car which is known to be 33 feet long subtends an angle of 20 mils. If it is perpendicular to the line of vision, how far away is it?

4. A hill at a distance of 1560 meters subtends an angle of 40 mils. How high is it?

5. What angle does a pole 25 feet high subtend at a distance of 100 yards?

6. A balloon known to be 150 feet long is directly overhead and subtends an angle of 125 mils. How high is it?

7. A hill 50 meters high is 1500 meters away. At what angle with the horizontal must a gun be pointed in order for the projectile just to clear the top of the hill, if an allowance of 10 mils must be made for the fall of the projectile?

8. A tree 75 feet high is at a distance of 500 feet from a given point on the ground; 1500 feet farther on is a hill 350 feet high. How far from the top of the hill will a line from the point on the ground through the top of the tree strike?

9. A gun is 2500 yards from its target. A shot is fired and the projectile is observed to strike even with the target but 8 mils to the right. By how much did it miss the target?

10. A car 33 feet long directly northeast of an observer stands on a north and south track. If it subtends an angle of 10 mils, find the approximate distance to the south end.

11. Change into mils: $10°$, $15°$, $10'$, $10''$, 10^g.

12. Change into degrees, minutes and seconds: 10 mils, 50 mils, 100 mils.

13. Change into grades: 10 mils, 50 mils, 100 mils.

14. The radian. The unit in the so-called **circular** measure of angles is the **radian.** The radian is defined as the angle at the center of a circle subtended by an **arc** equal in length to the radius of the circle.

To illustrate, if we take a string equal in length to the radius of a wheel and wrap it around the rim, the angle formed at the center of the wheel by radii drawn to the ends of the string will be a radian. It may be written as 1^r, $1^{(r)}$, or 1 rad.

Our definition of the radian is based on the following geometric fact: Whatever the circles, for a given angle at their centers, the ratio of an intercepted arc to its radius is constant.

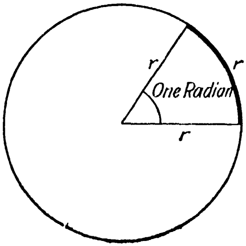


FIG. 9

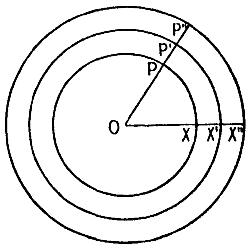


FIG. 10

Thus, in Fig. 10,

$$\frac{\text{arc } XP}{OX} = \frac{\text{arc } X'P'}{OX'} = \frac{\text{arc } X''P''}{OX''}.$$

It would seem reasonable that the central angle which makes the value of this ratio unity might be a convenient measure in some instances. In higher mathematics it proves so convenient that the radian is used almost exclusively as the unit of angular measurement.

The name radian was first used in print in 1873 in examination questions at Queen's College, Belfast, set by James Thomson, a brother of the famous scientist Lord Kelvin.

It is evident from our knowledge of plane geometry that the value of the radian in sexagesimal units must be less than 60°. We learned that the radius of a circle is equal in length to the side of a regular inscribed hexagon, and hence the radius as a chord subtends an angle of 60° at the center. Since the shortest distance between two points is the straight line segment connecting them, it is obvious that the arc subtending the 60° angle is greater than its chord and consequently greater than its radius. It is therefore greater than the arc subtending the radian. The radian must therefore be less than 60° as is shown in Fig. 11.

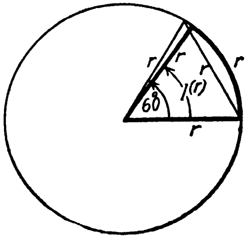


Fig. 11

A more exact value for the radian may be found as follows: We know from geometry that the circumference of a circle is 2π times the radius, where $\pi = 3.1416$ approximately. Therefore the radius must be applied to the circumference 2π (somewhat more than six) times to go around it once. Consequently the number of radians at the center of a circle is equal to

$$\frac{\text{circumference}}{\text{radius}} = \frac{2\pi r}{r} = 2\pi. \tag{3}$$

Therefore,

$$\begin{aligned}
360° &= 2\pi^{(r)} &&= 6.2832^{(r)}\\
180° &= \pi^{(r)} &&= 3.1416^{(r)}\\
90° &= \frac{\pi^{(r)}}{2} &&= 1.5708^{(r)}\\
60° &= \frac{\pi^{(r)}}{3} &&= 1.0472^{(r)}\\
45° &= \frac{\pi^{(r)}}{4} &&= 0.7854^{(r)}\\
30° &= \frac{\pi^{(r)}}{6} &&= 0.5236^{(r)}
\end{aligned}$$

$$1^\circ = \frac{\pi^{(r)}}{180} = 0.01745^{(r)}$$

$$1' = \frac{\pi^{(r)}}{10800} = 0.0002909^{(r)}$$

$$1'' = \frac{\pi^{(r)}}{648000} = 0.000004848^{(r)}$$

Thus to change degrees to radians, multiply by $\pi/180$, or 0.01745.

To convert radians to degrees, we have $2\pi^{(r)} = 360^\circ$, and consequently

$$\begin{aligned} 1^{(r)} &= \frac{180^\circ}{\pi} \\ &= 57.29578^\circ \\ &= 57^\circ\ 17'\ 45'' \\ &= 3437.75' \\ &= 206265''. \end{aligned} \tag{4}$$

Therefore to change radians to degrees, multiply by $180/\pi$, or for an approximate result by 57.3.

Example 1. How many radians are there in $37^\circ\ 43'\ 26''$?

Solution.
$$\begin{aligned} 37^\circ\ 43'\ 26'' &= 37.7239^\circ \\ &= (37.7239)(0.01745)^{(r)} \\ &= 0.65828^{(r)}. \end{aligned}$$

Example 2. Change 2.25 radians to degrees minutes and seconds.

Solution.
$$\begin{aligned} 2.25^{(r)} &= 2.25(57.2958)^\circ \\ &= 128.91555^\circ \\ &= 128^\circ\ 54'\ 56''. \end{aligned}$$

These examples might have been worked by using tables.

NOTE. Numerous editions of tables may be obtained, for example, *Logarithmic and Trigonometric Tables* (Revised Edition, 1920), prepared under the direction of E. R. Hedrick, New York, the Macmillan Company. This note covers all references to tables in this book.

From our definition of a radian we see that in a circle whose radius is 3 inches, an arc of 3 inches will subtend an angle of one radian, an arc of 6 inches will subtend an angle of 2 radians, an arc of 1 inch will subtend an angle of 1/3 radian, etc. And in the same circle 1 radian intercepts an arc of 3 inches, 1/2 radian intercepts an arc of 1.5 inches, etc. Again, if an arc of 5 inches subtends an angle of two radians at the center of a circle, 1 radian is subtended by an arc of 2.5 inches; that is, the radius of the circle is 2.5 inches.

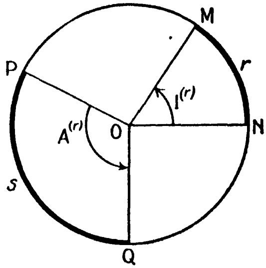


FIG. 12

Now suppose an arc PQ, s units in length as shown in Fig. 12, subtends an angle of A radians at the center of a circle whose radius is r units. Then the arc MN, r units in length, subtends an angle of 1 radian; an arc of one unit subtends an angle of $1/r$ radians; and an arc of s units subtends an angle of s/r radians. That is,

$$A = \frac{s}{r}.$$

Clearing of fractions, we get

$$(5) \qquad Ar = s.$$

This may be clearer if we consider it as follows: From plane geometry we know that in the same circle, or in equal circles, two central angles have the same ratio as their intercepted arcs. Hence,

$$\frac{A}{1} = \frac{s}{r}, \qquad A = \frac{s}{r}, \qquad Ar = s.$$

Stated in words this relation is: **The length of the arc is equal to the length of the radius multiplied by the central angle measured in radians,** or more briefly, **arc equals radius times angle.**

It is readily seen that for a unit circle, that is, a circle whose radius is 1, the angle expressed in radians is numerically equal to the arc expressed in linear units. For example, in a circle having a radius measuring 1 inch, an arc of 2.3 inches will subtend an angle of 2.3 radians at the center.

Example. What is the length of the arc subtended by a central angle of 95° in a circle whose radius is 12 feet?

Solution. Let A represent the number of radians in the angle, and s the number of feet in the arc. Then

$$A = 95 \times \frac{\pi}{180} = 1.66,$$
$$s = 12 \times 1.66 = 19.92.$$

Hence the length of the arc is 19.92 feet.

NOTE. The various systems of angular measurement are not essentially different. In any system the unit or comparison angle is an angle which is a definite part of one complete positive rotation. That is, the unit angle is one which bears a certain definite ratio to one rotation. For the degree this ratio is 1/360, for the right angle it is 1/4, for the mil it is 1/6400, for the radian it is the ratio of the radius to the circumference of a circle or $1/2\pi$.

Exercise 3

1. Reduce the following angles to radians (do not substitute the numerical value of π): 10°, 35°, − 45°, 70°, 150°, − 280°, 18°, 400°, 10° 30′, 24° 45′, 480° 45′.

2. Reduce the following angles to radians giving your result correct to two decimal places: 10°, 18° 24′ 16″, 370° 15′ 08″, 138°, 142° 25′ 30″.

3. Express the following angles in degrees, minutes and seconds. (When it is quite clear that radian measure is to be used, the symbol for radians is commonly omitted. Thus the angle π radians may be written $\pi^{(r)}$ or simply π.) $\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, \frac{3\pi}{5}, \frac{7\pi}{15}, \frac{3\pi}{10}, \frac{7\pi}{30}$.

4. Using a protractor, construct the following angles approximately: $2^{(r)}$, $5\frac{1}{2}^{(r)}$, $-3\frac{1}{2}^{(r)}$, $8^{(r)}$.

5. Construct the following angles: $\frac{\pi}{3}, \frac{\pi}{4}, \frac{4\pi}{5}, \frac{8\pi}{15}, \frac{3\pi}{2}, \frac{2\pi}{3}$.

6. When a radian is taken as the unit, what is the measure of each of the following angles: 170°, 245°, 375°, 67° 43′ 52″? Find the measure of each of these angles taking the right angle as the unit.

7. If the right angle is taken as the unit, find the measure of (*a*) the angles in problem 4, (*b*) the angles in problem 5.

8. A circle has a radius of 15 inches. Find the central angle which is subtended by an arc of 25 inches; by an arc of 1 inch.

9. An arc of 4 feet 3 inches subtends an angle of 1.2 radians in a circle whose radius is r. Find the numerical value of r.

10. One angle of a triangle is 25°, another angle is 1.3 radians. Find the third angle in degrees and also in radians.

11. A central angle in a circle of radius 10 subtends an arc of 14. How many radians are there in the angle? How many degrees?

12. Find in degrees and in radians the angle between the tangents to a circle at two points whose distance apart, measured on the arc of the circle, is 350 feet, the radius of the circle being 800 feet?

13. On a railroad line a reversed curve is composed of two circular arcs, one of $20° 30'$ and radius 2200 feet, and the other of $18° 15'$ and radius 2500 feet. What is the total length of the curve?

14. Considering the earth to be a sphere of radius 3960 miles, find the number of degrees of longitude in an equatorial distance of 100 miles.

15. Find the distance intercepted at the earth's equator by a central angle of 1 minute. (The length of one minute of arc on the equator is taken as the definition of a **knot** or **geographical mile.**)

16. The radius of the earth subtends an angle of $8.82''$ at the sun. What is the approximate distance from the earth to the sun? (The radius of the earth may be considered as approximately the same as the arc of a circle whose radius is the distance to the sun.)

17. The radius of the earth subtends an angle of $57'$ at the moon. What is the approximate distance from the earth to the moon?

18. The radius of the earth's orbit, which is approximately 93,000,000 miles, subtends an angle of about $0.76''$ at the star Alpha Centauri. What is the approximate distance of this star from the earth?

19. An explorer reaches latitude $87° 29' 30''$. How far is he from the pole?

20. Through what angle does the earth turn between 8:00 A.M. and 5:30 P.M.?

21. The **pitch** of a screw is the distance that it moves forward along its axis when the head makes one complete turn. What is the pitch of a screw that moves forward a distance of 0.125 inch in turning through $1200°$?

22. If the pitch of a screw is 0.1 inch, find the distance

that the screw moves through when the head turns through (*a*) 750°, (*b*) 3 radians.

15. Angular velocity. If a wheel turns completely round thirty times in a second, we say that it is rotating at the rate of thirty revolutions per second, or 30 RPS. (In like manner the expression revolutions per minute is abbreviated RPM.) A spoke on this wheel will turn through 360° in each rotation, or through $30 \times 360° = 10800°$ per second. Since the spoke turns through 2π radians in each rotation, in each second it turns through $30 \times 2\pi$ radians, or 60π radians per second. The wheel is said to have an angular velocity of 30 RPS, or 10800° per second, or $60\pi^{(r)}$ per second.

Suppose now that the wheel has a radius of 2 feet. When a spoke has traveled through an angle of one radian, a point on the rim will have traveled 2 feet. For any number of radians through which the spoke turns the point on the rim will travel twice that number of feet. But the wheel turns through 60π radians per second. Hence a point on the rim moves through $60\pi \times 2$ feet per second, or it has a velocity of 120π feet per second.

In general, if the radius of a wheel is r units, and the angular velocity of the wheel is w radians per second, then, designating by v the velocity in linear units per second of a point on the rim, we have

$$v = wr. \tag{6}$$

For, as the wheel turns through one radian a point on the rim moves through r units. Since the wheel turns through w radians in a second the point on the rim must travel wr units, that is, it has a linear velocity of wr units per second. Obviously, if the unit of time were the minute, the hour, or even the year, instead of the second, equation (6) would still hold.

Formula (6) enables us to find any one of these three quantities involved when the other two are known. If we know the linear velocity and the angular velocity we can find the radius. If we know the radius and the angular velocity we can find the linear velocity, etc.

Equation (6) may be obtained directly from equation (5). For if a point P on the rim of a wheel moves with the constant linear velocity v and describes the arc s in the time t, then $v = s/t$. If during the same time t the angle A is generated, A/t is the angular velocity of the point P (or of the wheel); we designate it by w. Now if A is measured in radians we may use

$$(5) \qquad Ar = s,$$

and dividing by t we get

$$\frac{A}{t}r = \frac{s}{t},$$

or

$$wr = v,$$

which is the same as (6). Since A is measured in radians, w is measured in radians per unit of time.

Example. If a point moves 10 yards in 3 seconds on the rim of a wheel whose radius is 5 feet, what is its angular velocity?

Solution. The linear velocity of the point P is 10/3 yd. per sec. = 30/3 or 10 ft. per sec. (It should be noted that like quantities must be reduced to the same unit.)

Substituting in equation (6), we get

$$10 = 5w,$$

or

$$w = 2^{(r)} \text{ per sec.}$$

Exercise 4

1. A wheel is rotating at the rate of 10π radians per second. Find the number of revolutions it makes per minute; the number per hour.

2. Express an angular velocity of 2.7 revolutions per second in degrees per second; in radians per second.

3. Find the angular velocity of the minute hand of a watch.

4. If the minute hand of a clock is 4 inches long, find the linear speed of its tip.

5. Find the angular rate at which the minute hand of a watch overtakes the hour hand.

6. A train is running at the rate of 15 miles an hour on a circular curve having a radius of 500 feet. Find the angular velocity of the train in radians per hour; in degrees per minute.

7. A fly-wheel 24 feet in diameter has an angular velocity of 8 radians per second. Find the linear velocity of a point on the rim. Find the number of revolutions per minute that the wheel is making.

8. What is the angular velocity of a point on the earth due to the rotation of the earth?

9. Find the linear velocity of a point on the equator, assuming the earth to be a sphere of radius 3960 miles.

10. A wheel of an automobile is 34 inches in diameter. The automobile is traveling at the rate of 30 miles an hour. What is the angular velocity of the wheel?

11. Assuming the path of the earth about the sun to be a circle with a radius of 93,000,000 miles, find the velocity of the earth in miles a second.

12. A carriage moves at the rate of 7 miles an hour. If a wheel has a radius of 2 feet, what is its angular velocity?

13. A belt travels around two pulleys whose diameters are 10 inches and 4 feet respectively. The large pulley makes 100 revolutions per minute. Find the angular velocity of the small pulley and the speed in feet per minute of a point on the belt.

CHAPTER III

THE TRIGONOMETRIC FUNCTIONS

16. Directed lines. We have defined a positive angle as one generated by the counter-clockwise rotation of a line, and a negative angle as one generated by clockwise rotation. In like manner we may consider motion in one direction along a straight line as positive, and motion in the opposite direction as negative. We are already familiar with this in connection with the thermometer. When the liquid rises in the tube we say it is moving in a positive direction, and when it is descending we say it is moving in a negative direction. In Fig. 13 from A to B is plus, indicated by the

FIG. 13

arrow head pointing in that direction. From B to A must consequently be minus. From B to A might have been called plus, in which case from A to B would have been minus. In the case of horizontal lines, however, an arbitrary custom has determined that plus shall be to the right. In like manner for vertical lines, as with the thermometer, the upward direction is plus, while the downward direction is minus. A line upon which positive and negative directions have been determined, such as the line AB in Fig. 13, is called a **directed** line. For conciseness we shall often say the direction AB instead of the direction from A towards B, and similarly the direction BA for the direction from B towards A.

The point A may be called the **initial point** and the point B the **terminal point** of the line segment AB.

Two line segments are added by a method similar to that of adding angles. The initial point of the second segment is placed on the terminal point of the first, and the sum is the segment from the initial point of the first segment to the terminal point of the second. The proper direction must be preserved for each segment. Obviously several segments can be added by a continuation of the process described.

Subtraction of line segments is accomplished by changing the direction of the segment to be subtracted and then proceeding as in addition. Thus, if A, B, and C are points

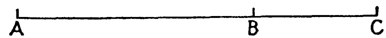


FIG. 14

arranged in order along a directed line (Fig. 14), then

$$AB + BC = AC,$$
$$AB + BC - AC = 0,$$

or we may write

$$AB + BC + CA = 0.$$

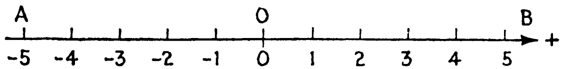


FIG. 15

On the directed line in Fig. 15 we select O as a zero point, or point from which to measure distances. We also choose a fixed length as a unit of measure, and apply it consecutively a number of times to the line AB in both directions from O. The points determined in this way may be thought of as corresponding to the numbers 1, 2, 3, $\cdots$, as we go from O towards B, and to $-1, -2, -3, \cdots$, as we go from O towards A. It is clear that other points on the line may be thought of as corresponding to other numbers; for example, a point half way between the point corresponding to 3 and the point corresponding to 4 would correspond to the number $3\frac{1}{2}$. In fact, to every point on the line there corresponds one and only one real number (positive, negative, or zero), and to every real number there corresponds one and only one point on the line. This holds for numbers such as $-10/3$, $\sqrt{2}$, π, etc. Instead of saying the point corresponding to 2/7, or the point corresponding to -5, we shall say for brevity the point 2/7 or the point -5.

17. Rectangular co-ordinates. Suppose now that we have two straight lines intersecting each other at right angles at the point O as in Fig. 16. We shall use these as lines of reference and as such they are called **axes.** The horizontal line extending from left to right is usually designated as the X-axis. Distances measured along it or parallel to it are called **abscissas,** and are denoted by the letter x. The line at right angles to the X-axis is designated as the Y-axis. Distances measured along it or parallel to it are called **ordinates,** and are denoted by y. Abscissas and ordinates together are called **co-ordinates.** It is convenient to consider the zero point of both

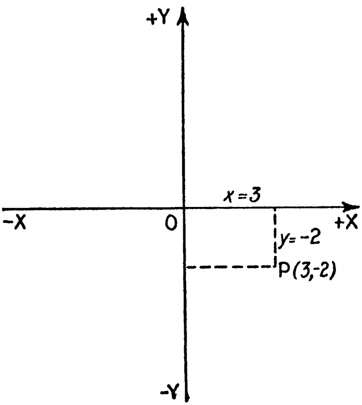


FIG. 16

axes to be O, and to measure all positive values of x to the right and all negative values to the left of the Y-axis, all positive values of y above and all negative values below the X-axis. This is consistent with the custom referred to above. The point O is called the **center** or **origin** of co-ordinates. We shall ordinarily use the same unit of measure on both axes. However, this is not necessary, and is sometimes inconvenient. To illustrate more clearly the meaning of abscissa and ordinate, a point 3 units to the right of the Y-axis and 2 units below the X-axis has the abscissa 3 and the ordinate -2; or, it has the co-ordinates 3 and -2. The abscissa, or x, is always placed first, and the point may be designated as $P(3, -2)$; or $x = 3$, $y = -2$; or simply $(3, -2)$.

It is evident that to every point in the plane of the axes there corresponds one and only one pair of real numbers, the abscissa and the ordinate; and that to every pair of real numbers, taken in a given order, there corresponds one point and only one in the plane of the axes. It should be noted that (4, 5) and (5, 4) give different points and are different pairs of numbers. The order of the numbers is therefore important.

It will be noted that the axes divide the plane into four parts called **quadrants,** numbered as indicated in Fig. 17. The order of numbering is in accordance with counter-clockwise rotation. That is, a line generating a positive angle turns first through quadrant I, then through quadrant II, etc. The signs of x and y in each quadrant are indicated in the figure and in the following table:

TABLE A

Quadrant................	I	II	III	IV
x (abscissa)...............	+	−	−	+
y (ordinate)...............	+	+	−	−

The process of locating points with reference to co-ordinate axes is called plotting. Plotting is most con-

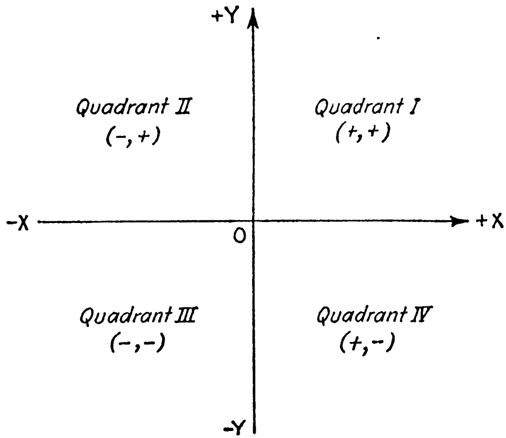


FIG. 17

veniently done on paper ruled with small squares, and called squared, graph, or co-ordinate paper.

Exercise 5

1. Draw two axes at right angles to each other and plot the following points: (4, 7), (− 5, − 3), (6, − 9), (0, − 7), (16, − 1), (5, 0).

2. What are the co-ordinates of the origin (intersection of axes)?

3. Plot ($\sqrt{3}$, 4), (π, − 6), ($\sqrt{2}$, $\sqrt{5}$). (The co-ordinates which are radicals may be found by the methods learned in plane geometry, or their decimal values may be obtained.)

4. (*a*) How far from the origin is each of the points in problem 1? (*b*) How far from the origin is each point in problem 3?

5. A point whose abscissa is 4 is 5 units from the origin. What are its co-ordinates?

6. A point whose ordinate is -2 is 3 units from the origin. What are its co-ordinates?

7. A point whose abscissa is -3 is 3 units from the origin. What are its co-ordinates?

8. The hour hand of a clock is 6 inches long. Taking horizontal and vertical axes through the point at which the hands are pivoted, find the co-ordinates of the tip of the hour hand at (*a*) one, (*b*) two, (*c*) three, (*d*) four, (*e*) six, (*f*) seven, (*g*) ten-thirty.

9. (*a*) In the equation $x^2 + y^2 = 25$ give values to one of the letters and solve for the other. Plot the various pairs of corresponding values and draw a smooth curve connecting the points. (*b*) Show that all points whose co-ordinates satisfy the equation must lie on the circle whose center is at the origin and whose radius is 5.

10. Plot the following points and connect them with a smooth curve: $(0, 0)$, $\left(\frac{\pi}{6}, .50\right)$, $\left(\frac{\pi}{4}, .71\right)$, $\left(\frac{\pi}{3}, .87\right)$, $\left(\frac{\pi}{2}, 1\right)$, $\left(\frac{2\pi}{3}, .87\right)$, $\left(\frac{3\pi}{4}, .71\right)$, $\left(\frac{5\pi}{6}, .5\right)$, $(\pi, 0)$, $\left(\frac{7\pi}{6}, -.5\right)$, $\left(\frac{3\pi}{2}, -1\right)$, $\left(\frac{7\pi}{4}, -.71\right)$, $(2\pi, 0)$.

11. Plot and connect with a smooth curve: $(0, 1)$, $\left(\frac{\pi}{6}, .87\right)$, $\left(\frac{\pi}{4}, .71\right)$, $\left(\frac{\pi}{3}, .50\right)$, $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{2\pi}{3}, .50\right)$, $\left(\frac{3\pi}{4}, -.71\right)$, $\left(\frac{5\pi}{6}, -.87\right)$, $(\pi, -1)$, $\left(\frac{7\pi}{6}, -.87\right)$, $\left(\frac{3\pi}{2}, 0\right)$, $\left(\frac{7\pi}{4}, .71\right)$, $(2\pi, 1)$.

12. Plot the following:

$$(a)\ y = x^2 + 3x - 4,$$
$$(b)\ y = x^2,$$
$$(c)\ y = x^3,$$
$$(d)\ x^2 + 4y^2 = 36,$$
$$(e)\ x^2 - y^2 = 49,$$
$$(f)\ xy = 36,$$
$$(g)\ y^2 = (x - 1)(x - 2)(x - 3),$$
$$(h)\ y^2 = (x - 1)^2(x - 4).$$

18. The trigonometric functions. It is convenient to think of a rotating line which generates an angle as rotating about the origin of co-ordinates as a pivot, as shown in Fig. 18. We shall call the rotating line OP the **radius,** and

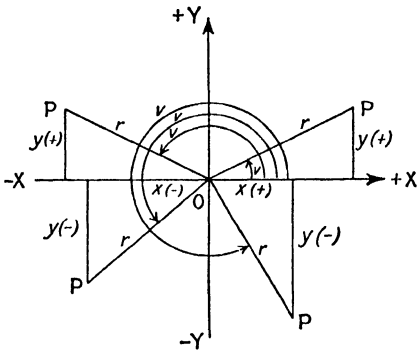


FIG. 18

shall denote it by r. Its initial position is OX. For an angle between 0° and 90° r is in the first quadrant; for an angle between 90° and 180° r is in the second quadrant; for an angle between 180° and 270° r is in the third quadrant; for an angle between 270° and 360° r is in the fourth quadrant; for an angle between 360° and 450° r is again in the first quadrant; etc.

In whatever quadrant we find r there are three linear quantities to be noted,—the radius OP, and the abscissa and the ordinate of the point P. These quantities may be used to write six ratios with names as indicated in Table B.

TABLE B

Ratio	Representation	Name	Abbreviation	Exception
$\frac{\text{ordinate}}{\text{radius}}$	$\frac{y}{r}$	sine of angle v	sin v	
$\frac{\text{abscissa}}{\text{radius}}$	$\frac{x}{r}$	cosine of angle v	cos v	
$\frac{\text{ordinate}}{\text{abscissa}}$	$\frac{y}{x}$	tangent of angle v	tan v	$x = 0$
$\frac{\text{abscissa}}{\text{ordinate}}$	$\frac{x}{y}$	cotangent of angle v	cot v	$y = 0$
$\frac{\text{radius}}{\text{abscissa}}$	$\frac{r}{x}$	secant of angle v	sec v	$x = 0$
$\frac{\text{radius}}{\text{ordinate}}$	$\frac{r}{y}$	cosecant of angle v	csc v	$y = 0$

The exceptions are due to the fact that we can not divide a number by zero. If the quantity in the denominator of a ratio becomes zero the value of the ratio does not exist. It is customary to say, however, that a ratio whose denominator is zero (and whose numerator is finite) is **infinite,** or that its value is **infinity** (infinity is represented by the symbol ∞). This simply means that as the denominator approaches zero, the ratio becomes and remains numerically larger than any number that we may assign.

The six quantities given in Table B are called **functions** of the angle v; that is, their values depend on that of the angle v. Five trigonometric functions which are less frequently used are listed in Table C.

TABLE C

Function	Name	Abbreviation
$1 - \cos v$	versed sine of angle v	vers v
$1 - \sin v$	coversed sine of angle v	covers v
$\sec v - 1$	external secant of angle v	exsec v
$\csc v - 1$	coexternal secant of angle v	coexsec v
$(1 - \cos v)/2$	haversine* of angle v	hav v

For any given angle the values of the functions are definitely fixed. This may be shown as follows:

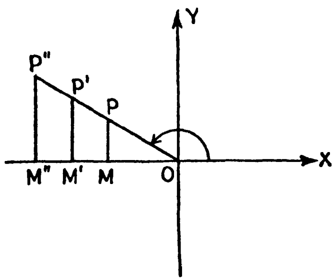


FIG. 19

Let XOP (Fig. 19) be any angle. By the definition of the sine of an angle,

$$\sin XOP = \frac{MP}{OP} = \frac{M'P'}{OP'} = \frac{M''P''}{OP''}.$$

These ratios are equal because the right triangles MOP, $M'OP'$, $M''OP''$, formed by the line segments involved, are similar, having an acute angle common. (The student should note carefully that the angle whose sine is being considered is the angle XOP whether or not it is an angle of the triangle formed by the line segments.) Hence, for a given angle, the sine has a fixed value independent of the length of the lines. The same can in like manner be shown

* The word *haversine* is in itself an abbreviation of *half versed sine*.

to be true for all the other functions. This is the reason for calling the trigonometric functions single-valued functions of the angle.

It should be noted that the abscissas in the second and the third quadrants are negative because they extend to the left from the Y-axis, and that the ordinates in the third and fourth quadrants are negative because they extend downward from the X-axis. For the present we shall consider r as positive only. The ratios will be positive when the signs of both terms involved are the same, and negative when the signs are different, in accordance with what the student has learned in algebra. Tables D and E and Fig. 20 show the signs of the six principal functions in all four quadrants.

TABLE D

Function	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
sin	$\frac{y(+)}{r}$, or +	$\frac{y(+)}{r}$, or +	$\frac{y(-)}{r}$, or −	$\frac{y(-)}{r}$, or −
cos	$\frac{x(+)}{r}$, or +	$\frac{x(-)}{r}$, or −	$\frac{x(-)}{r}$, or −	$\frac{x(+)}{r}$, or +
tan	$\frac{y(+)}{x(+)}$, or +	$\frac{y(+)}{x(-)}$, or −	$\frac{y(-)}{x(-)}$, or +	$\frac{y(-)}{x(+)}$, or −
cot	$\frac{y(+)}{x(+)}$, or +	$\frac{x(-)}{y(+)}$, or −	$\frac{x(-)}{y(-)}$, or +	$\frac{x(+)}{y(-)}$, or −
sec	$\frac{r}{x(+)}$, or +	$\frac{r}{x(-)}$, or −	$\frac{r}{x(-)}$, or −	$\frac{r}{x(+)}$, or +
csc	$\frac{r}{y(+)}$, or +	$\frac{r}{y(+)}$, or +	$\frac{r}{y(-)}$, or −	$\frac{r}{y(-)}$, or −

TABLE E

Quadrant	Function					
	sin	cos	tan	cot	sec	csc
I.	+	+	+	+	+	+
II.	+	−	−	−	−	+
III.	−	−	+	+	−	−
IV	−	+	−	−	+	−

HISTORICAL NOTE. The word *sine* is from the Latin word *sinus*, meaning *gulf*. The origin of the name is interesting. The Hindoos used the word *jiva*, meaning bowstring or chord, which the Arabs called *jiba*. This word was not so common in the Arabic as another word which it resembled, namely, *jaib*. This word means gulf. Sometime in the twelfth century **Plato Tiburtinus** translated the work of the Arab **Al-Battani** from Arabic into Latin. Out of this the literal translation of *jaib* gave us the word sine.

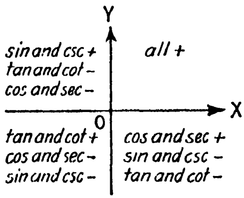


FIG. 20

Edmund Gunter (1581–1626), an Englishman, was the first to use the word *cosine*. He obtained this by contracting the Latin words *complementi sinus*, meaning the *sine of the complementary angle*. He also invented the name *cotangent*.

Thomas Finck, a physician and surgeon, and a native of Denmark, invented the words *tangent* and *secant* about 1583. The reason for the appropriateness of these names will appear later.

Albert Girard, a mathematician of Lorraine, early in the seventeenth century, was the first to use the abbreviated forms sin, tan, sec, etc. However, they were not generally adopted until about 1748 when they were used by **Thomas Simpson** in England and by **Leonhard Euler** on the continent.

19. Relations among the functions. It will be observed that the connecting bars in the left-hand column of Table D indicate certain reciprocal relations. That is, the cotangent, secant, and cosecant are the reciprocals of the tangent, cosine, and sine respectively. For this reason it will be found convenient to visualize and learn the functions in the order of Table D.

The reciprocal relations are stated in the following equations:

$$\frac{1}{\sin v} = \csc v, \qquad \frac{1}{\csc v} = \sin v,$$

$$\frac{1}{\cos v} = \sec v, \qquad \frac{1}{\sec v} = \cos v,$$

$$\frac{1}{\tan v} = \cot v, \qquad \frac{1}{\cot v} = \tan v.$$

Clearing of fractions, we get

$$\sin v \csc v = 1, \tag{7}$$

$$\cos v \sec v = 1, \tag{8}$$

$$\tan v \cot v = 1. \tag{9}$$

Starting with the definitions given in Table B, namely,

$$\sin v = \frac{y}{r}, \qquad \cot v = \frac{x}{y},$$

$$\cos v = \frac{x}{r}, \qquad \sec v = \frac{r}{x},$$

$$\tan v = \frac{y}{x}, \qquad \csc v = \frac{r}{y},$$

we have, upon clearing of fractions,

$$
\begin{aligned}
y &= r \sin v, & x &= y \cot v,\\
x &= r \cos v, & r &= x \sec v,\\
y &= x \tan v, & r &= y \csc v.
\end{aligned}
$$

Or, as is indicated in Fig. 21, we have a choice of two ways of expressing any one of the three linear quantities involved. It is evident that if we know the functions of the angle and one of these lines the other two lines may be found. The values of the functions for different angles may be found in tables. We shall refer to these later.

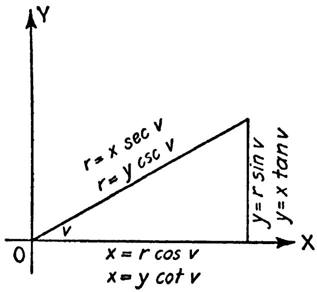


FIG. 21

From the definitions of sine and cosine it is evident that

$$\frac{\sin v}{\cos v} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x}.$$

But by definition

$$\frac{y}{x} = \tan v.$$

Hence

(10) $$\frac{\sin v}{\cos v} = \tan v,$$

and similarly

(11) $$\frac{\cos v}{\sin v} = \cot v.$$

Again, in any quadrant, since the square of a negative quantity is positive, we have, from the theorem of Pythagoras,

$$x^2 + y^2 = r^2.$$

Dividing this equation through by r^2, we get

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1.$$

Substituting the proper trigonometric functions for the two ratios in the left-hand member of the equation gives

$$\cos^2 v + \sin^2 v = 1. \tag{12}$$

If we divide (12) through by $\cos^2 v$ we get

$$1 + \frac{\sin^2 v}{\cos^2 v} = \frac{1}{\cos^2 v}.$$

Substituting from (10) and remembering that $1/\cos v = \sec v$, we get

$$1 + \tan^2 v = \sec^2 v. \tag{13}$$

In like manner we may divide (12) by $\sin^2 v$ and substitute from (11), getting

$$\cot^2 v + 1 = \csc^2 v. \tag{14}$$

Relations (13) and (14) may also be derived by the method used in deriving (12), dividing the equation $x^2 + y^2 = r^2$ by x^2 to obtain (13), and by y^2 to obtain (14), each time substituting the proper functions for the ratios.

Formula (12) may be solved for the sine or cosine, giving

$$\sin v = \pm \sqrt{1 - \cos^2 v}, \qquad \cos v = \pm \sqrt{1 - \sin^2 v}.$$

Equations (13) and (14) may be solved in like manner for the functions contained in them.

TABLE F

	$\sin v$	$\cos v$	$\tan v$	$\cot v$	$\sec v$	$\csc v$
$\sin v$	$\sin v$	$\pm\sqrt{1-\cos^2 v}$	$\dfrac{\tan v}{\pm\sqrt{1+\tan^2 v}}$	$\dfrac{1}{\pm\sqrt{1+\cot^2 v}}$	$\dfrac{\pm\sqrt{\sec^2 v-1}}{\sec v}$	$\dfrac{1}{\csc v}$
$\cos v$	$\pm\sqrt{1-\sin^2 v}$	$\cos v$	$\dfrac{1}{\pm\sqrt{1+\tan^2 v}}$	$\dfrac{\cot v}{\pm\sqrt{1+\cot^2 v}}$	$\dfrac{1}{\sec v}$	$\dfrac{\pm\sqrt{\csc^2 v-1}}{\csc v}$
$\tan v$	$\dfrac{\sin v}{\pm\sqrt{1-\sin^2 v}}$	$\dfrac{\pm\sqrt{1-\cos^2 v}}{\cos v}$	$\tan v$	$\dfrac{1}{\cot v}$	$\pm\sqrt{\sec^2 v-1}$	$\dfrac{1}{\pm\sqrt{\csc^2 v-1}}$
$\cot v$	$\dfrac{\pm\sqrt{1-\sin^2 v}}{\sin v}$	$\dfrac{\cos v}{\pm\sqrt{1-\cos^2 v}}$	$\dfrac{1}{\tan v}$	$\cot v$	$\dfrac{1}{\pm\sqrt{\sec^2 v-1}}$	$\pm\sqrt{\csc^2 v-1}$
$\sec v$	$\dfrac{1}{\pm\sqrt{1-\sin^2 v}}$	$\dfrac{1}{\cos v}$	$\pm\sqrt{1+\tan^2 v}$	$\dfrac{\pm\sqrt{1+\cot^2 v}}{\cot v}$	$\sec v$	$\dfrac{\csc v}{\pm\sqrt{\csc^2 v-1}}$
$\csc v$	$\dfrac{1}{\sin v}$	$\dfrac{1}{\pm\sqrt{1-\cos^2 v}}$	$\dfrac{\pm\sqrt{1+\tan^2 v}}{\tan v}$	$\pm\sqrt{1+\cot^2 v}$	$\dfrac{\sec v}{\pm\sqrt{\sec^2 v-1}}$	$\csc v$

In fact it is possible to express any function in terms of any other function. To illustrate how this may be done in a specific case, let us express the tangent in terms of the cosine. We have

$$\tan v = \frac{\sin v}{\cos v}.$$

Substituting for sin v its value in terms of cos v, we get

$$\tan v = \frac{\pm\sqrt{1 - \cos^2 v}}{\cos v}.$$

In Table F all of the functions are expressed in terms of each single function. The student should verify this table.

20. To find the values of the other functions when the value of one function is given. We shall now show, by means of examples, how the values of all the other functions can be obtained when the value of one function is given.

Example 1. Given sin $v = 3/5$; find the other functions of v.
Solution. Angle v may be in either quadrant I or quadrant II.

$$\cos v = \pm\sqrt{1 - \sin^2 v} = \pm\sqrt{1 - 9/25} = \pm\sqrt{16/25} = \pm 4/5.$$

We use the plus sign if v is in the first quadrant and the minus if it is in the second.

$$\tan v = \frac{\sin v}{\cos v} = \frac{3/5}{\pm 4/5} = \pm\frac{3}{4}.$$

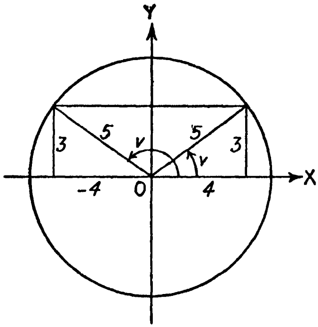
Using the reciprocal relations, we get

$$\cot v = \frac{1}{\tan v} = \pm\frac{4}{3},$$

$$\sec v = \frac{1}{\cos v} = \pm\frac{5}{4},$$

$$\csc v = \frac{1}{\sin v} = \frac{5}{3}.$$

In this problem, wherever there is a choice of signs for the answer, the plus is used when the angle is considered to be in quadrant I, and the minus is used when the angle is considered to be in quadrant II.



The problem may be solved graphically or geometrically as follows:

Since $\sin v = 3/5 = y/r$, we may consider that y and r are 3 and 5 (or any two numbers proportional to 3 and 5) respectively. Let us draw a circle of radius 5 units, and take a point on the Y-axis at a distance of 3 units above the X-axis. A line through this point parallel to the X-axis will cut the circle in two points, and consequently there will be two positions for the angle v, one in quadrant I and the other in quadrant II.

By the theorem of Pythagoras we have

$$x = \pm\sqrt{25 - 9} = \pm\sqrt{16} = \pm 4.$$

Thus, corresponding to the angle in quadrant I we have an abscissa 4, and corresponding to the angle in quadrant II we have an abscissa -4. We can now write all of the functions of both angles directly from the figure.

Quadrant I	Quadrant II
$\sin v = 3/5,$	$\sin v = \quad 3/5,$
$\cos v = 4/5,$	$\cos v = -4/5,$
$\tan v = 3/4,$	$\tan v = -3/4,$
$\cot v = 4/3,$	$\cot v = -4/3,$
$\sec v = 5/4,$	$\sec v = -5/4,$
$\csc v = 5/3.$	$\csc v = \quad 5/3.$

Example 2. Given $\tan v = 2$; find the other functions of v.

Solution. Since $\tan v$ is positive, angle v may be in either quadrant I or quadrant III. If v is in quadrant III both $\sin v$ and $\cos v$ are negative.

$$\sin v = \pm\frac{\tan v}{\sqrt{1 + \tan^2 v}} = \pm\frac{2}{\sqrt{5}} = \pm\frac{2\sqrt{5}}{5},$$

$$\cos v = \pm\sqrt{1 - \sin^2 v} = \pm\sqrt{1 - 4/5} = \pm\frac{1}{\sqrt{5}} = \pm\frac{\sqrt{5}}{5},$$

$$\tan v = 2,$$

$$\cot v = \frac{1}{\tan v} = \frac{1}{2},$$

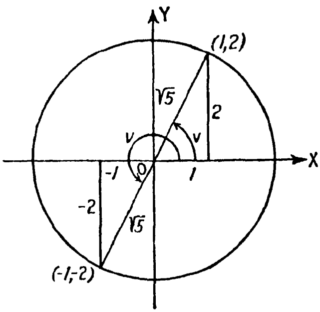
$$\sec v = \frac{1}{\cos v} = \pm\sqrt{5},$$

$$\csc v = \frac{1}{\sin v} = \pm\frac{\sqrt{5}}{2}.$$

It is seen that the radical form $2/\sqrt{5}$ has been changed to the form $2\sqrt{5}/5$. This change facilitates the computation when the decimal value of the quantity is desired; for $\sqrt{5} = 2.23607$, and multiplying this number by 2 and dividing by 5 is easier than dividing 2 by 2.23607.

To solve the problem geometrically we note that

$$\tan v = \frac{2}{1} \text{ or } \frac{-2}{-1}.$$



We may therefore make the abscissas 1 and -1, and the ordinates 2 and -2, and complete the figure as illustrated.

For either position of the terminal side of angle v, we have by the theorem of Pythagoras

$$r = \pm\sqrt{1+4} = \pm\sqrt{5}.$$

We use the plus sign, as agreed upon in section 18. We may now write all the functions of angle v in either quadrant directly from the figure.

Quadrant I	Quadrant II
$\sin v = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$,	$\sin v = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$,
$\cos v = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$,	$\cos v = \frac{-1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$,
$\tan v = 2$,	$\tan v = 2$,
$\cot v = \frac{1}{2}$,	$\cot v = \frac{1}{2}$,
$\sec v = \sqrt{5}$,	$\sec v = -\sqrt{5}$,
$\csc v = \frac{\sqrt{5}}{2}$.	$\csc v = -\frac{\sqrt{5}}{2}$.

In like manner we may find graphically the functions of an angle in any quadrant when the value of one function is given.

Exercise 6

1. Find the other functions of the angle v if

   (*a*) $\sec v = \sqrt{2}$, v being in quadrant I;
   (*b*) $\cos v = -4/5$, v being in quadrant III;
   (*c*) $\tan v = -2/3$, v being in quadrant IV;
   (*d*) $\cot v = 1/5$, v being in quadrant III;
   (*e*) $\cos v = -2/5$, v being in quadrant II;
   (*f*) $\csc v = -41/9$, v being in quadrant IV.

2. Find the other functions of v if

   (*a*) $\sin v = 1/2$, (*b*) $\cos v = 2/3$,
   (*c*) $\tan v = -2/5$, (*d*) $\csc v = 4/3$,
   (*e*) $\cot v = 5/2$, (*f*) $\sec v = 5/4$,
   (*g*) $\sec v = -2$, (*h*) $\cos v = -1/4$,
   (*i*) $\tan v = 0.5$, (*j*) $\sin v = -0.8$.

3. Find the other functions of v if

   (*a*) $\sin v = \dfrac{\sqrt{3}}{2}$, (*b*) $\cos v = \dfrac{\sqrt{2}}{2}$,
   (*c*) $\tan v = \dfrac{1}{\sqrt{3}}$, (*d*) $\sec v = \sqrt{2}$.

4. Find the other functions of v if

   (*a*) $\tan v = \dfrac{a}{b}$, (*b*) $\sin v = \dfrac{2mn}{m^2 + n^2}$.

5. Given $\tan v = 4/3$. Find

   (*a*) vers v, (*b*) covers v,
   (*c*) exsec v, (*d*) hav v.

6. Given covers $v = 4/3$. Find $\sin v$, $\cos v$, $\tan v$.

7. Given hav $v = 1/4$. Find $\sin v$, $\cos v$, $\tan v$.

8. Given exsec $v = 0.75$. Find $\sin v$, $\cos v$, $\tan v$.

9. How does formula (12) show that $\cos v$ and $\sin v$ cannot be greater than 1? In case either function is 1, what must be the value of the other?

10. Show that

(*a*) $\cos v \tan v = \sin v$,

(*b*) $\cot v \cos v = \csc v - \sin v$,

(*c*) $\dfrac{1 + \sin v}{\cos v} = \dfrac{\cos v}{1 - \sin v}$,

(*d*) $(\tan v - \sin v)^2 + (1 - \cos v)^2 = (1 - \sec v)^2$,

(*e*) $\dfrac{\cos^2 v}{1 - \sin v} = 1 + \sin v$,

(*f*) $\cot v + \tan v = \dfrac{\csc^2 v + \sec^2 v}{\csc v \sec v}$,

(*g*) $\dfrac{\sin v + \tan v}{\cot v + \csc v} = \sin v \tan v$,

(*h*) $\dfrac{1 - 2\cos^2 v}{\sin v \cos v} = \tan v - \cot v$,

(*i*) $(\sin v + \cos v)^2 + (\sin v - \cos v)^2 = 2$,

(*j*) $\sin^2 v - \cos^2 v = \sin^4 v - \cos^4 v$,

(*k*) $\tan^2 v - \sin^2 v = \tan^2 v \sin^2 v$,

(*l*) $\sin^6 v + \cos^6 v = 1 - 3 \sin^2 v \cos^2 v$,

(*m*) $\dfrac{\csc v}{\csc v - 1} + \dfrac{\csc v}{\csc v + 1} = 2 \sec^2 v$,

(*n*) $\dfrac{1 - \tan v}{1 + \tan v} = \dfrac{\cot v - 1}{\cot v + 1}$.

11. Express $\cot v + \dfrac{\sin v}{1 + \cos v}$ in terms of $\sin v$.

12. Express $(\csc v - \cot v)^2$ in terms of $\tan v$.

13. Express $\dfrac{\sec^2 v \csc^2 v + \sec^2 v - \csc^2 v - 1}{\tan^2 v - \csc^2 v + 1}$ in terms of cot v.

21. Functions of 45°, 135°, 225°, 315°, etc. If we take angle v to be 45° (Fig. 22), any point on its terminal side will have its abscissa and its ordinate equal, for they are the sides of an isosceles right triangle. Let us for convenience denote both abscissa and ordinate by s. Then $r = \sqrt{s^2 + s^2} = s\sqrt{2}$.

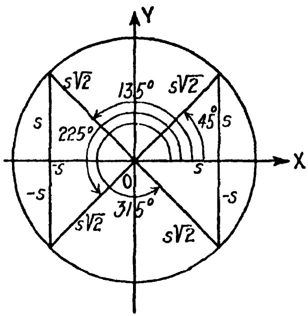


FIG. 22

If now we consider angles of 135°, 225°, 315°, we see that the terminal sides of these angles bisect quadrants II, III, IV, respectively. If we use the same absolute values for the ordinates and the abscissas, r will be equal to $s\sqrt{2}$ in every quadrant. Hence we have

$$\begin{aligned}
\sin\ 45° &= s/s\sqrt{2} = 1/\sqrt{2} = \sqrt{2}/2 = 0.707,\\
\cos\ 45° &= s/s\sqrt{2} = 1/\sqrt{2} = \sqrt{2}/2 = 0.707,\\
\tan\ 45° &= s/s = 1,\\
\cot\ 45° &= s/s = 1,\\
\sec\ 45° &= s\sqrt{2}/s = \sqrt{2} = 1.414,\\
\csc\ 45° &= s\sqrt{2}/s = \sqrt{2} = 1.414;\\
\sin 135° &= s/s\sqrt{2} = \sqrt{2}/2 = 0.707,\\
\cos 135° &= -\ s/s\sqrt{2} = -\ \sqrt{2}/2 = -\ 0.707,\\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

It is evident that the functions of 405° (360° + 45°) are exactly the same as the functions of 45°, since the terminal sides of 405° and 45° coincide. For a like reason the functions of 495° are the same as those of 135°; etc.

22. Functions of 30°, 150°, 210°, 330°, etc. If we draw the ordinate from a point on the terminal side of an angle of 30°, we have a right triangle in which the other acute angle is 60° (Fig. 23). We know from geometry that the triangle is half an equilateral triangle whose side is r, and the ordinate will therefore be $r/2$. To avoid fractions we shall call the ordinate s; then the radius will be $2s$. Using the theorem of Pythagoras, we find that the abscissa is $x = \sqrt{4s^2 - s^2} = s\sqrt{3}$.

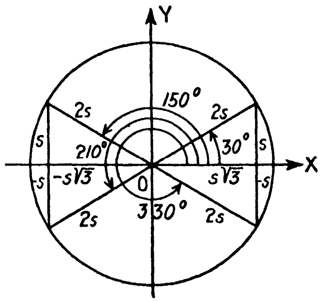


FIG. 23

If we now consider angles of 150°, 210°, 330°, as well as 30°, we find radii, abscissas, and ordinates to be as indicated in Fig. 23. Hence

$$
\begin{aligned}
\sin\ 30^\circ &= s/2s = 1/2 = 0.500,\\
\cos\ 30^\circ &= s\sqrt{3}/2s = \sqrt{3}/2 = 0.866,\\
\tan\ 30^\circ &= s/s\sqrt{3} = 1/\sqrt{3} = \sqrt{3}/3 = 0.577,\\
\cot\ 30^\circ &= s\sqrt{3}/s = \sqrt{3} = 1.732,\\
\sec\ 30^\circ &= 2s/s\sqrt{3} = 2/\sqrt{3} = 2\sqrt{3}/3 = 1.555,\\
\csc\ 30^\circ &= 2s/s = 2;\\
\sin 150^\circ &= s/2s = 1/2 = 0.500,\\
\cos 150^\circ &= -s\sqrt{3}/2s = -\sqrt{3}/2 = -0.866,
\end{aligned}
$$

. . . .

It is evident that the functions of 390° are the same as the functions of 30°, since the terminal sides of these angles coincide; the functions of 510° are the same as those of 150°; etc.

23. Functions of 60°, 120°, 240°, 300°, etc. For an angle of 60° we have a right triangle whose acute angles are 60° and 30° (Fig. 24). If we call the abscissa s, the radius will be $2s$, and we readily find the ordinate to be $s\sqrt{3}$.

Considering angles of 120°, 240°, 300°, as well as 60°, we find radii, abscissas, and ordinates to be as indicated. Therefore

$$\begin{aligned}
\sin 60^\circ &= s\sqrt{3}/2s = \sqrt{3}/2 = 0.866,\\
\cos 60^\circ &= s/2s = 1/2 = 0.500,\\
\tan 60^\circ &= s\sqrt{3}/s = \sqrt{3} = 1.732,\\
\cot 60^\circ &= s/s\sqrt{3} = \sqrt{3}/3 = 0.577,\\
\sec 60^\circ &= 2s/s = 2,\\
\csc 60^\circ &= 2s/s\sqrt{3} = 2\sqrt{3}/3 = 1.555;\\
&\ldots\ldots\ldots
\end{aligned}$$

The functions of 420° are the same as the functions of 60°; etc.

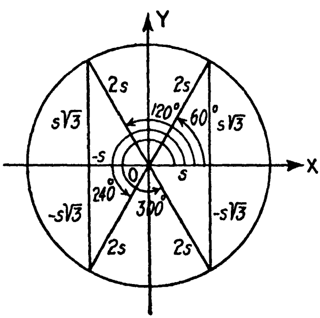


FIG. 24

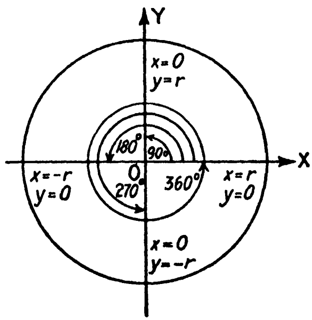


FIG. 25

24. Functions of 0°, 90°, 180°, 270°, 360°, etc. It is apparent (Fig. 25) that for the angles 0° and 360° the abscissa is equal to the radius (that is, $x = r$) and the ordinate is zero; for 90° the abscissa is zero and the ordinate is equal to the radius ($y = r$); for 180° the abscissa is the negative of the radius (that is, $x = -r$), and the ordinate is zero; for 270° the abscissa is zero and the ordinate is the negative of the radius ($y = -r$).

Hence, from the definitions of the functions, we have the results given in Table G.

TABLE G

	0°	90°	180°	270°
sin	$y/r = 0/r = 0$	$y/r = r/r = 1$	$y/r = 0/r = 0$	$y/r = -r/r = -1$
cos	$x/r = r/r = 1$	$x/r = 0/r = 0$	$x/r = -r/r = -1$	$x/r = 0/r = 0$
tan	$y/x = 0/x = 0$	$y/x = r/0 = \infty$	$y/x = 0/-r = 0$	$y/x = -r/0 = \infty$
cot	$x/y = x/0 = \infty$	$x/y = 0/r = 0$	$x/y = -r/0 = \infty$	$x/y = 0/-r = 0$
sec	$r/x = r/r = 1$	$r/x = r/0 = \infty$	$r/x = r/-r = -1$	$r/x = r/0 = \infty$
csc	$r/y = r/0 = \infty$	$r/y = r/r = 1$	$r/y = r/0 = \infty$	$r/y = r/-r = -1$

TABLE H

	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1
cos	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{3}\sqrt{3}$	0
sec	1	$\frac{2}{3}\sqrt{3}$	$\sqrt{2}$	2	∞
csc	∞	2	$\sqrt{2}$	$\frac{2}{3}\sqrt{3}$	1

$$\sqrt{2} = 1.4142$$
$$\sqrt{3} = 1.7321$$

It is clear that the functions of 360° are the same as those of 0°.

It should be noted that in cases where the denominator of the ratio is zero no value exists. We sometimes say that the value is indefinitely large or that it is infinity (∞). (See p. 41.)

The functions of 0°, 30°, 45°, 60°, and 90° are used so frequently in trigonometry that it will be convenient to memorize them. Table II will be useful for reference. However, the student should bear in mind the methods used in deriving these values.

25. The right triangle. Very many of the problems with which trigonometry is concerned are related to the right triangle. Now every right triangle may be placed with respect to the two rectangular co-ordinate axes in the position shown in Fig. 26. (It should be noted that in any triangle each side is usually designated by the small letter corresponding to the capital letter which designates the opposite vertex.) It is apparent from this figure, since a is the ordinate, b the abscissa, and c the radius of the point B, that

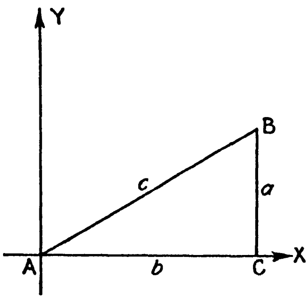


FIG. 26

$$\sin A = \frac{\text{side opposite } A}{\text{hypotenuse}} = \frac{a}{c},$$

$$\cos A = \frac{\text{side adjacent } A}{\text{hypotenuse}} = \frac{b}{c},$$

$$\tan A = \frac{\text{side opposite } A}{\text{side adjacent } A} = \frac{a}{b},$$

$$\cot A = \frac{\text{side adjacent } A}{\text{side opposite } A} = \frac{b}{a},$$

$$\sec A = \frac{\text{hypotenuse}}{\text{side adjacent } A} = \frac{c}{b},$$

$$\csc A = \frac{\text{hypotenuse}}{\text{side opposite } A} = \frac{c}{a}.$$

The foregoing equations may be taken as the definitions of the functions of the angle A. The student will readily see that they do not differ from our previous definitions, but that they can not be used for any angle except an acute angle of a right triangle.

By clearing of fractions we get

$$a = c \sin A,$$
$$b = c \cos A,$$
$$a = b \tan A,$$
$$b = a \cot A,$$
$$c = b \sec A,$$
$$c = a \csc A.$$

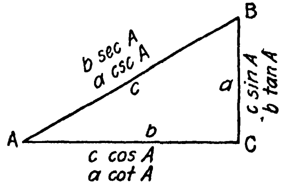


FIG. 27

These equations give a choice of two ways of expressing each side of a right triangle in terms of one other side and one acute angle. (Compare with p. 46.) Their use may be illustrated by means of an example. Suppose we wish to find the side a in a right triangle in which we know that $A = 30°$ and $b = 18$ inches. We have

$$a = b \tan A = 18 \tan 30°.$$

From Table H we see that $\tan 30° = \frac{1}{3}\sqrt{3}$. Therefore

$$a = 18 \times \tfrac{1}{3}\sqrt{3} = 6\sqrt{3} = 10.39.$$

Thus the side a is 10.39 inches.

However, in order to solve all problems of this type we should need to know the functions of all acute angles. How these may be obtained is told in the next section.

It is interesting to note that from the above definitions each function of A is also a function of B. Thus, if we observe that the side adjacent to A is the side opposite B, etc., and that $B = 90° - A$, we see that

$$(15)\qquad \begin{cases} \sin A = \cos B = \cos (90° - A), \\ \cos A = \sin B = \sin (90° - A), \\ \tan A = \cot B = \cot (90° - A), \\ \cot A = \tan B = \tan (90° - A), \\ \sec A = \csc B = \csc (90° - A), \\ \csc A = \sec B = \sec (90° - A). \end{cases}$$

Let us group the six functions into three pairs, sine and cosine, tangent and cotangent, secant and cosecant. In any pair let us call either function the **co-function** of the other. Then we may say that **the function of an acute angle is equal to the corresponding co-function of its complement.** In fact, cosine means the sine of the complementary angle, etc. A good illustration of this is afforded by the functions of 30° and 60°. (See Table H.)

26. Tables of the trigonometric functions. We have already found the values of the functions of 30°, 45°, 60°, 90°, etc. The methods of finding the values of the functions of other angles are more advanced. (See Chapter X.) However, these values have been computed and tabulated, and we shall resort to tables for obtaining them.

The table on pp. 61–62 gives the values of the functions of angles for every degree from 0° to 90°. For angles from 0° to 45° we find the angle in the left-hand column and follow the line across the page to the column headed by the name of the function we desire.

For example, to find cot 23° we follow down the left

VALUES OF THE TRIGONOMETRIC FUNCTIONS

Angle	sin	cos	tan	cot	sec	csc	
0°	.0000	1.0000	.0000	∞	1.0000	∞	**90°**
1°	.0175	.9998	.0175	57.2900	1.0002	57.2987	89°
2°	.0349	.9994	.0349	28.6363	1.0006	28.6537	88°
3°	.0523	.9986	.0524	19.0811	1.0014	19.1073	87°
4°	.0698	.9976	.0699	14.3007	1.0024	14.3356	86°
5°	.0872	.9962	.0875	11.4301	1.0038	11.4737	85°
6°	.1045	.9945	.1051	9.5144	1.0055	9.5668	84°
7°	.1219	.9925	.1228	8.1443	1.0075	8.2055	83°
8°	.1392	.9903	.1405	7.1154	1.0098	7.1853	82°
9°	.1564	.9877	.1584	6.3138	1.0125	6.3925	81°
10°	.1736	.9848	.1763	5.6713	1.0154	5.7588	**80°**
11°	.1908	.9816	.1944	5.1446	1.0187	5.2408	79°
12°	.2079	.9781	.2126	4.7046	1.0223	4.8097	78°
13°	.2250	.9744	.2309	4.3315	1.0263	4.4454	77°
14°	.2419	.9703	.2493	4.0108	1.0306	4.1336	76°
15°	.2588	.9659	.2679	3.7321	1.0353	3.8637	75°
16°	.2756	.9613	.2867	3.4874	1.0403	3.6280	74°
17°	.2924	.9563	.3057	3.2709	1.0457	3.4203	73°
18°	.3090	.9511	.3249	3.0777	1.0515	3.2361	72°
19°	.3256	.9455	.3443	2.9042	1.0576	3.0716	71°
20°	.3420	.9397	.3640	2.7475	1.0642	2.9238	**70°**
21°	.3584	.9336	.3839	2.6051	1.0711	2.7904	69°
22°	.3746	.9272	.4040	2.4751	1.0785	2.6695	68°
23°	.3907	.9205	.4245	2.3559	1.0864	2.5593	67°
24°	.4067	.9135	.4452	2.2460	1.0946	2.4586	66°
25°	.4226	.9063	.4663	2.1445	1.1034	2.3662	65°
26°	.4384	.8988	.4877	2.0503	1.1126	2.2812	64°
27°	.4540	.8910	.5095	1.9626	1.1223	2.2027	63°
28°	.4695	.8829	.5317	1.8807	1.1326	2.1301	62°
29°	.4848	.8746	.5543	1.8040	1.1434	2.0627	61°
	cos	sin	cot	tan	csc	sec	Angle

VALUES OF THE TRIGONOMETRIC FUNCTIONS—*Continued*

Angle	sin	cos	tan	cot	sec	csc	
30°	.5000	.8660	.5774	1.7321	1.1547	2.0000	**60°**
31°	.5150	.8572	.6009	1.6643	1.1666	1.9416	59°
32°	.5299	.8480	.6249	1.6003	1.1792	1.8871	58°
33°	.5446	.8387	.6494	1.5399	1.1924	1.8361	57°
34°	.5592	.8290	.6745	1.4826	1.2062	1.7883	56°
35°	.5736	.8192	.7002	1.4281	1.2208	1.7434	55°
36°	.5878	.8090	.7265	1.3764	1.2361	1.7013	54°
37°	.6018	.7986	.7536	1.3270	1.2521	1.6616	53°
38°	.6157	.7880	.7813	1.2799	1.2690	1.6243	52°
39°	.6293	.7771	.8098	1.2349	1.2868	1.5890	51°
40°	.6428	.7660	.8391	1.1918	1.3054	1.5557	**50°**
41°	.6561	.7547	.8693	1.1504	1.3250	1.5243	49°
42°	.6691	.7431	.9004	1.1106	1.3456	1.4945	48°
43°	.6820	.7314	.9325	1.0724	1.3673	1.4663	47°
44°	.6947	.7193	.9657	1.0355	1.3902	1.4396	46°
45°	.7071	.7071	1.0000	1.0000	1.4142	1.4142	45°
	cos	sin	cot	tan	csc	sec	Angle

hand column to 23°; then following a line across the page to the column headed cot, we find 2.3559, which is the value sought.

From 45° to 90° the angles are given at the right-hand side of the page, and the names of the functions are at the bottoms of the columns. That the table may thus be made to render double service depends on the fact noted in the preceding section—that the functions of an acute angle are the co-functions of its complement.

To find sin 66° we find the angle in the column to the right and follow a line across the page to the column which has sin at the bottom, and read .9135, which is the value desired.

27. Interpolation. The table on pp. 61–62 gives the values of functions of angles which involve only a whole number of degrees. We may use it to find the functions of angles in which fractional parts of a degree occur, by a process called **interpolation.** This means a process of inserting or placing between. It is based on the assumption that a change in the angle and the corresponding change in the value of the function are proportional. This is not strictly true, as will be noted later. For cases, however, in which the change in the function is small for a change in the angle, the result obtained by this method of interpolation is sufficiently accurate for use. For example, interpolating for the tangent between 1° and 2° would give a satisfactory result, but for the cotangent it would not, since in this interval the change in the cotangent is great.

If we wish to find sin 20° 30′, by employing the table we find

$$\begin{aligned} \sin 21^\circ &= .3584 \\ \sin 20^\circ &= \underline{.3420} \\ \text{difference} &= .0164 \end{aligned}$$

That is, at 20° a change of 1° in the angle gives a change of .0164 in the sine. Since 30′ is half a degree we must take half the difference for 1°, that is, half .0164, or .0082. Since sin 21° is greater than sin 20° this difference must be added to sin 20°, giving

$$\sin 20^\circ\ 30' = .3502.$$

It is evident from this illustration that in interpolating to get the functions of an angle between those given in the table we

1. *Find the difference for* 1°.

2. *Change the difference between the angle under consideration and the next preceding to the fractional part of a degree.*

3. *Multiply the results obtained in* 1 *and* 2.

4. *Add to or subtract from, as the case requires, the function of the angle next preceding.*

Example 1. Find cos 31.7°.

Solution. The required value must lie between cos 31° and cos 32°.

$$
\begin{aligned}
\cos 31^\circ &= .8572 \\
\cos 32^\circ &= .8480 \\
\text{difference} &= .0092 \\
(.7) \times (.0092) &= .0064\cancel{4}
\end{aligned}
$$

The last figure is stricken out,* as we use only four places.

$$\cos 31.7^\circ = .8572 - .0064 = .8508.$$

Note that in the first quadrant the cosine decreases as the angle increases; this is why the difference must be subtracted.

Example 2. Find tan 25° 21′ 45″.

Solution. The value lies between tan 25° and tan 26°.

$$
\begin{aligned}
\tan 26^\circ &= .4877 \\
\tan 25^\circ &= .4663 \\
\text{difference} &= .0214 \\
21'\,45'' &= .3625^\circ \\
(.3625) \times (.0214) &= .0078 \\
\tan 25^\circ\,21'\,45'' &= .4663 + .0078 = .4741
\end{aligned}
$$

It may also be necessary to find angles from given values of functions when the value of the function lies between those given in the table. This requires interpolation of reverse order as compared with that above.

* If it becomes necessary to discard a 5 that terminates a number, increase by 1 the digit in the last place to be retained if that digit is odd, but leave it unchanged if it is even; that is, *keep the last digit retained even.*

Of course if the discarded part is greater than 5 the digit in the last place retained is increased by 1, but if the discarded part is less than 5 the digit in the last place retained is unchanged.

Example 3. Given $\tan A = .5821$; find A.

Solution. Looking in the tangent column, we find that the angle must lie between 30° and 31°.

$\tan 31° = .6009$	$\tan A = .5821$
$\tan 30° = \underline{.5774}$	$\tan 30° = \underline{.5774}$
difference $= .0235$	difference $= .0047$

Now a difference of .0235 in the tangent at 30° gives a difference of 1° in the angle. Evidently a difference of .0047 in the tangent will give a difference in the angle of .0047/.0235 of 1°, or $47/235 \times 60' = 12.0'$.

Hence the acute angle A whose tangent is .5821 is 30° 12.0′.

In general, to find an angle from the value of a function when this value does not appear in the table:

1. *Find the difference between the values of the function on either side of the given value.*

2. *Find the difference between the given value and the value of the function for the angle next preceding the one required,*

3. *Divide the result obtained in 2 by that obtained in 1. This will be the fractional part of a degree required. It may be given as a decimal or converted to minutes, etc.*

4. *Add the fractional part of a degree to the smaller angle.*

More extensive tables are used in like manner.

Exercise 7

1. Using the table on pp. 61–62, find the values of the following functions:

(*a*) sin 13°,	(*h*) cos 89°,	(*o*) tan 25°,
(*b*) sin 7°,	(*i*) cos 88°,	(*p*) cot 35°,
(*c*) sin 22°,	(*j*) cos 1°,	(*q*) cot 65°,
(*d*) sin 34°,	(*k*) cos 17°,	(*r*) tan 55°,
(*e*) sin 63°,	(*l*) cos 40°,	(*s*) sec 14°,
(*f*) sin 84°,	(*m*) cos 24°,	(*t*) csc 5°,
(*g*) sin 73°,	(*n*) cos 0°,	(*u*) sec 71°.

2. Using the table on pp. 61–62, find the angle A, if

(*a*) $\sin A = 0.3746$,
(*b*) $\sin A = 0.9659$,
(*c*) $\cos A = 0.7431$,
(*d*) $\cos A = 0.2078$,
(*e*) $\tan A = 1.1504$,
(*f*) $\tan A = 0.8480$,
(*g*) $\cot A = 2.9042$,
(*h*) $\cot A = 0.3839$,
(*i*) $\sec A = 1.0785$,
(*j*) $\csc A = 1.3673$,
(*k*) $\sin A = 0.2462$,
(*l*) $\sin A = 0.7323$,
(*m*) $\cos A = 0.8361$,
(*n*) $\cos A = 0.3714$,
(*o*) $\tan A = 0.4470$,
(*p*) $\tan A = 2.5002$,
(*q*) $\cot A = 1.8781$,
(*r*) $\cot A = 0.1745$.

3. Find the functions of angle A of a right triangle ABC if $a = 7$ and $b = 14$.

4. If $\sin A = 4/5$ and $c = 100.5$, find a and b.

5. If $B = 60°$ and $c = 54$, find a and b.

6. Express as functions of the complementary angle,

(*a*) $\sin 45°$,
(*b*) $\cos 30°$,
(*c*) $\tan 15°$,
(*d*) $\sec 52°$,
(*e*) $\csc 67°$,
(*f*) $\cot 81°$.

7. Express as functions of an angle less than 45°,

(*a*) $\sin 68°$,
(*b*) $\cos 70°$,
(*c*) $\tan 51°$,
(*d*) $\cot 55°$,
(*e*) $\sec 88°$,
(*f*) $\csc 64°$.

8. Find the acute angle A if

(*a*) $\cos A = \sin A$,
(*b*) $\cos A = \sin 2A$,
(*c*) $\cot (45° + A) = \tan A$,
(*d*) $\sec (A - 30°) = \csc 2A$.

Suggestion. See end of section 25.

9. Find the following functions:

(*a*) $\sin 21.3°$,
(*b*) $\cos 21.3°$,
(*g*) $\cot 35° 35'$,
(*h*) $\cot 70° 22'$,

(*c*) $\sin 18° 15'$,
(*d*) $\cos 42° 15'$,
(*e*) $\tan 80° 35'$,
(*f*) $\sin 64° 20'$,
(*i*) $\sin 55° 21' 30''$,
(*j*) $\cos 70° 18' 25''$,
(*k*) $\tan 15° 25' 30''$,
(*l*) $\cot 65° 14' 32''$.

10. Find the angle A if

(*a*) $\sin A = 0.6587$,
(*b*) $\cos A = 0.9640$,
(*c*) $\cos A = 0.8100$,
(*d*) $\tan A = 0.7715$,
(*e*) $\sin A = 0.7329$,
(*f*) $\cot A = 0.4913$,
(*g*) $\tan A = 0.2235$,
(*h*) $\cot A = 0.2235$,
(*i*) $\sec A = 1.3584$,
(*j*) $\csc A = 2.3500$,
(*k*) $\tan A = 3.0000$,
(*l*) $\cot A = 2.1745$.

11. Find

(*a*) $\sin 24° 15'$,
(*b*) $\cos 64° 27'$,
(*c*) $\sin 20° 05' 15''$,
(*d*) $\tan 30° 42' 48''$,
(*e*) $\cos 40° 21' 45''$,
(*f*) $\cot 17° 40' 12''$,
(*g*) $\tan 57° 19' 20''$,
(*h*) $\cot 72° 42' 25''$,
(*i*) $\sin 48.35°$,
(*j*) $\cos 23.15°$.

12. Find A if

(*a*) $\tan A = 0.6048$,
(*b*) $\sin A = 0.9085$,
(*c*) $\sin A = 0.5938$,
(*d*) $\tan A = 1.6205$,
(*e*) $\cos A = 0.5172$,
(*f*) $\cot A = 4.1900$,
(*g*) $\tan A = 15.0000$,
(*h*) $\sin A = 0.6156$,
(*i*) $\tan A = 1.0186$,
(*j*) $\cot A = 2.0342$.

13. Take a sheet of paper (ordinary graph paper is desirable), draw axes at right angles to each other and construct a circle with the center at the origin. Measure the angles 5°, 10°, 15°, 20°, 25°, etc., to 90°. Using this diagram determine by measurement and computation the functions of these angles. What is your conclusion regarding this as a method for finding the values of the functions?

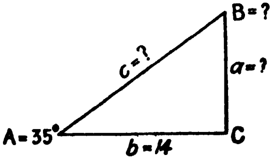
In the following problems select from the definitions of the trigonometric functions, or from Fig. 27, a formula which will involve one unknown with two knowns.

Always draw the diagram which will, as nearly as possible, represent the conditions of the problem.

14. In the right triangle ABC, $b = 14$, $A = 35°$. Find a, c, and B.

Solution.

$$\begin{array}{ll} a = b \times \tan A & \cos A = b/c \\ a = b \times \tan 35° & c = b/\cos A \\ a = 14 \times 0.7002 & c = 14/0.8192 \\ a = 9.803 & c = 17.09 \end{array}$$



Check*

$$a^2 = c^2 - b^2 = (c + b)(c - b)$$
$$a^2 = (17.090 + 14)(17.090 - 14)$$
$$= 31.090 \times 3.090 = 96.0681.$$
$$a = 9.80 +.$$

Note. The area of a right triangle may be found by taking half the product of the two legs.

15. Solve the following right triangles for the unknown parts and for the area:

$$\begin{array}{lll} (a) & a = 6, & c = 14; \\ (b) & a = 3, & b = 5; \\ (c) & A = 30°, & b = 6. \end{array}$$

16. A tree broken over by the wind forms a right triangle with the ground. If the angle which the broken part makes with the ground is 50° and the distance from the top of the tree to the foot of the stump is 55 feet, find the length of the tree.

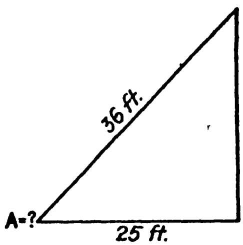
* The importance of checking will be referred to on pp. 98–99, which see.

17. Find the angle of ascent of a railway with a $2\frac{1}{2}$ per cent. grade; that is, there is a rise of $2\frac{1}{2}$ feet vertically in a horizontal distance of 100 feet.

18. A person wishes to find the width of a river without crossing it. He measures a line AB along the bank, A being directly opposite a tree located at a point C on the other side. The angle ABC is measured 55°, and AB is found to be 110 feet. Find the width of the river.

19. A vessel whose masts are 75 feet high is observed from the edge of the water to subtend an angle of 6°. How far is the vessel from the shore?

20. A ladder 36 feet long rests against a wall, and its foot is 25 feet distant from the wall. Find the angle it makes with the ground.



21. Find the height of a tree which casts a shadow 60 feet long on the level ground when the altitude of the sun is 50° (that is, the rays of the sun make an angle of 50° with the horizontal).

A problem such as 20 above may be solved graphically as follows:

Draw a right triangle to scale (1 foot = 1/16 inch, for example) as in the figure and measure the angle, or any other part, required.

What is your conclusion regarding this as a method of solving problems?

22. Solve problems 15, 17, 18, and 20 graphically. Compare your results with those already obtained.

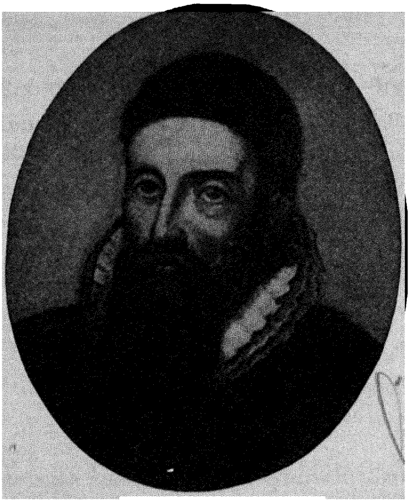
23. From a window 30 feet above the ground a tree 100 feet high at a distance of 200 feet is observed. Find the angle of elevation of the top of the tree and the angle of depression of the bottom.

The **angle of elevation** of an object which is above the observer is the angle which the line of sight to the object makes with the horizontal. If the object is below the observer, the angle which the line of sight makes with the horizontal is called the **angle of depression.**

24. From the top of a cliff 250 feet high the angle of depression of a boat is $10°$. How far out is the boat?

25. From the top of a tower 63 feet high the angles of depression of two objects situated in the same horizontal line with the base of the tower, and on the same side, are $31° 15'$ and $45° 51'$. Find the distance between these two objects.

26. A balloon which is passing over a level road is observed from two points in the road 700 feet apart. The angles of elevation are $35° 15'$ and $41° 14'$. Find the height of the balloon.



Sir John Napier (1550–1617), Baron of Merchiston, Scotland, was the nventor of logarithms. His rule of circular parts for the solution of right spherical triangles is probably "the happiest example of artificial memory known." He also simplified the solution of oblique spherical triangles by his discovery of the formulas known as Napier's Analogies. In 1617 he introduced his "rods" or "bones," an instrument by which one number can be multiplied or divided by another number.

His discovery of logarithms is most remarkable when one considers the times in which he lived. He received no hint from previous writings, neither was it a lucky accident. It was the result of a deliberate purpose and effort to reduce the labor of computation. That a discovery of such far-reaching importance in astronomy and other sciences should be the product of a single mind is most remarkable. His work places him in the same rank with Copernicus, Kepler, and Galileo.

CHAPTER IV

LOGARITHMS

28. Exponents. The student should remember that the notation a^4 means $a \cdot a \cdot a \cdot a$, and that in general a^n means $a \cdot a \cdot a \cdots$ to n factors. In the expressions a^4 and a^n the 4 and the n are called *exponents*. Let us recall here the laws of exponents learned in algebra.

Law	Illustration
$b^m \times b^n = b^{m+n}$,	$b^6 \times b^3 = b^{6+3} = b^9$,
$b^m \div b^n = b^{m-n}$,	$b^6 \div b^3 = b^{6-3} = b^3$,
$(b^m)^n = b^{mn}$,	$(b^6)^3 = b^{6 \times 3} = b^{18}$,
$\sqrt[n]{b^m} = b^{m/n}$,	$\sqrt[3]{b^6} = b^{6 \div 3} = b^2$.

Thus, in multiplying two powers of the same base b, we add the exponents, in dividing we subtract exponents, in raising to a power we multiply the exponent by the power to which the quantity is to be raised, in extracting a root we divide the exponent by the index of the root.

These facts suggest the possibility of replacing multiplication and division by addition and subtraction respectively, and of replacing the operations of raising to powers and extracting roots by multiplication and division respectively.

This possibility is more definitely shown if we consider the base b to be 2, and let the successive exponents be 1, 2, 3, etc. If the exponent is 1, the number is $2^1 = 2$; if the exponent is 2, the number is $2^2 = 4$; if the exponent is 3, the number is $2^3 = 8$; etc. This gives us the following table:

Exponent........	1	2	3	4	5	6	7	8	9
Number.........	2	4	8	16	32	64	128	256	512

By means of this table we can multiply two numbers by adding the corresponding exponents. For example, if we wish to multiply 4 and 32 we add the corresponding exponents 2 and 5, getting the new exponent 7. Beneath this new exponent 7 we find the number 128, which is the desired product. Division can be performed by subtracting exponents. In similar manner the other operations can be effected.

The main difficulty with the table is that it is only of use in operations involving the few numbers given. This difficulty may be obviated by extending the table over a greater range and inserting other numbers. A method of doing this will now be explained.

It is observed that the exponents form an arithmetical progression whose common difference is 1. That is, each is found by adding 1 to the next preceding. The numbers form a geometric progression whose ratio is 2. That is, each is found by multiplying the next preceding by 2.

The table may therefore be extended as far as we please to the right by multiplying a number by 2 for the next following number, and by adding 1 to the exponent for the corresponding exponent, and to the left by dividing a number by 2 for the new number, and by subtracting 1 from the exponent for the new exponent.

Moreover, numbers with their corresponding exponents may be inserted in the table by taking the geometric mean of the two adjacent numbers for the new number, and the arithmetical mean of the two adjacent exponents for the new exponent. Thus, if we wish to insert a number with its exponent between 16 and 32, we find that the geometric mean of 16 and 32 is

$$\sqrt{16 \times 32} = 16\sqrt{2} = 22.624,$$

and that the arithmetical mean between 4 and 5 is

$$\frac{4+5}{2} = 4.5.$$

Hence we have at this point in the two series

Exponent............	4	4.5	5
Number.............	16	22.624	32

However, this method of inserting exponents and numbers is not practicable. The methods of advanced mathematics for finding the exponent corresponding to any given number are more convenient. Exponents have been computed by these methods, and arranged in tables for use. We shall consider these tables shortly.

Historical. It is interesting to note that a scheme such as that suggested above was invented long before much was known about exponents. In fact, the series of numbers which we have called exponents was called by another name—**logarithms**—which name they still retain. The word logarithm is derived from two Greek words, **logos** meaning *ratio,* and **arithmos** meaning *number.* This name was chosen by Napier, the inventor of logarithms, because as was shown in the preceding section, the integers 1, 2, 3, $\cdots$, considered as exponents, correspond to numbers of which any successive two are in the same ratio.

Sir John Napier (1550–1617), who was Baron of Merchiston, Scotland, was led to the invention of a system of logarithms by the progress of science and the increased need for extended computations. He first published the results of his work in 1614. Logarithms had also been invented independently by **Joost Bürgi** (1552–1632), a Swiss; he, however, failed to publish his work until 1620, three years after Napier's death.

Upon reading Napier's book on logarithms, **Henry Briggs** (1556–1631), at that time professor of geometry at Gresham College, London, admired it so much that he traveled to Scotland to visit its author. The meeting, as narrated by Florian Cajori in his *History of Mathematics*, (1919), p. 150, was a most remarkable one. "Briggs was delayed in his journey, and Napier complained to a common friend, 'Ah, John, Mr. Briggs will not come.' At that very moment knocks were heard at the gate, and Briggs was brought into the lord's chamber. Almost one-quarter of an hour was spent, each beholding the other without speaking a word. At last Briggs began: 'My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help in astronomy, viz., the logarithms; but, my lord, being by you found out, I wonder nobody found it out before, when now known it is so easy.'" The conference led to numerous changes and improvements. Briggs returned to London and began the construction of logarithmic tables with the base 10. The **natural** logarithms (sometimes called **Napierian** logarithms, although not exactly the same as the logarithms first used by Napier) have the base $e = 2.71828. . . .$ They are used in many theoretical investigations. Briggs's table was computed with the aid of Adrian Vlacq (born in Holland about 1600 and died 1667). It gave the logarithms to fourteen places of decimals, and was published in 1624. Logarithms employing the number 10 as a base are called **Briggsian** or **common** logarithms. They are the logarithms in common use.

29. Common logarithms. In section 28 we have shown that a system of logarithms might be constructed having the base 2. In fact, a system might be constructed having any positive number except 1 for a base. However, 10 is the base that is found most convenient for practical use.

If b is the base of a system of logarithms, and the number n is such that

$$b^x = n,$$

we write

$$\log_b n = x.$$

This is read *the logarithm of n to the base b equals x.* Thus, for example, in the common system

$$10^3 = 1000, \quad \text{or} \quad \log_{10} 1000 = 3.$$

It is evident, then, that*

$$\begin{array}{lll}
10^{-3} = 0.001, & \text{or} & \log_{10} 0.001 = -3, \\
10^{-2} = 0.01, & \text{or} & \log_{10} 0.01 = -2, \\
10^{-1} = 0.1, & \text{or} & \log_{10} 0.1 = -1, \\
10^{0} = 1, & \text{or} & \log_{10} 1 = 0, \\
10^{1} = 10, & \text{or} & \log_{10} 10 = 1, \\
10^{2} = 100, & \text{or} & \log_{10} 100 = 2, \\
10^{3} = 1000, & \text{or} & \log_{10} 1000 = 3.
\end{array}$$

Every integral power of 10 of course has a whole number for its logarithm. A number between two consecutive integral powers of 10 has for its logarithm a whole number plus a fraction. For example, $\log_{10} 32 = 1.5 +$.

The whole number part of the logarithm is called the **characteristic**; the fraction, usually expressed as a decimal, is called the **mantissa.**

By reference to the above table it is seen that numbers between 1 and 10 (that is, numbers consisting of one place

* The student should remember the meaning of negative and zero exponents. For instance, $10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = 0.001$, $10^0 = 1$.

to the left of the decimal point) have the characteristic 0, numbers between 10 and 100 (having two places to the left of the decimal point) have the characteristic 1, etc. In general,

For all numbers greater than 1 the logarithm is positive, and the characteristic is one less than the number of places to the left of the decimal point.

The above table also shows, for example, that the logarithm of any number between 0.001 and 0.01 must lie between -3 and -2. It is considered as being -3 plus a decimal quantity. Hence the mantissa is positive. The characteristic for any positive number less than 1 is negative. For example,

$$\log_{10} 0.00552 = \bar{3}.74194,$$

the negative sign being written above the characteristic, instead of in front of the logarithm, to show that it belongs merely to the characteristic and not to the entire logarithm. The foregoing, however, is usually written

$$\log_{10} 0.00552 = 7.74194 - 10,$$

where we have added 10 to and subtracted 10 from the characteristic, leaving the value the same but changing the form. It is to be noted that the mantissa remains unchanged.

Since numbers between 1 and 0.1 have the characteristic -1, or $9 - 10$; numbers between 0.1 and 0.01 the characteristic -2, or $8 - 10$; etc.; we have, in general,

For numbers less than 1 and greater than 0 the characteristic is negative and numerically one greater than the number of zeros between the decimal point and the first significant figure (the first figure different from zero).

If we wish to express the logarithm with a -10 appended, we have the following rule:

For numbers less than 1 and greater than 0 the characteristic is found by subtracting from 9 the number of zeros between the decimal point and the first significant figure, and writing -10 after the result.

Changing the position of the decimal point in a number does not change the mantissa of the logarithm. It changes the characteristic only. This may be illustrated as follows: Suppose that

$$10^{3.41162} = 2580, \qquad \text{or} \qquad \log_{10} 2580 = 3.41162.$$

Multiplying both sides by 10, we obtain

$$10^{4.41162} = 25800, \qquad \text{or} \qquad \log_{10} 25800 = 4.41162.$$

Dividing both sides of this by 1,000,000 (that is 10^6), we get

$$10^{\bar{2}.41162} = 0.0258, \qquad \text{or} \qquad \log_{10} 0.0258 = \bar{2}.41162 = 8.41162 - 10.$$

The foregoing property is peculiar to the system of logarithms having 10 as a base. It shows one of the advantages of the common system.

Although negative numbers have no real logarithms, we deal with negative numbers just as we do with positive numbers. Since the different signs may affect the result, we must keep account of them. This is usually done as in the following illustration:

$$\log_{10} (-321) = 2.50651_n.$$

An even number of n's in a product, or quotient, will give a positive result, while an odd number will give a negative result.

NOTE. Hereafter we shall not write the base 10 when using common logarithms. Thus, log 45 will mean $\log_{10} 45$.

Also, when there is no danger of misunderstanding, we shall sometimes omit the -10 after a logarithm. That is, we shall write $\log 0.2 = 9.30103$ instead of $\log 0.2 = 9.30103 - 10$.

Exercise 8

1. Determine the characteristic for the logarithm of each of the following numbers:

(*a*) 436,	(*g*) 0.43,
(*b*) 25,	(*h*) 0.04,
(*c*) 3280,	(*i*) 0.0075,
(*d*) 4,	(*j*) 1.0075,
(*e*) 0.136,	(*k*) 0.1075,
(*f*) 0.2,	(*l*) 52.684.

2. Given $\log 2 = 0.30103$; find

(*a*) $\log 20$,	(*e*) $\log 0.002$,
(*b*) $\log 0.02$,	(*f*) $\log 0.000002$,
(*c*) $\log 2000$,	(*g*) $\log 0.2$,
(*d*) $\log 0.20$,	(*h*) $\log 2000000$.

3. Given that the mantissa of the logarithm of 35 is 54407; find

(*a*) $\log 35$,	(*e*) $\log 35000$,
(*b*) $\log 0.35$,	(*f*) $\log 350$,
(*c*) $\log 3.5$,	(*g*) $\log 0.035$,
(*d*) $\log 0.000035$,	(*h*) $\log 0.0035$.

4. Given $\log 2 = 0.30103$, $\log 3 = 0.47712$, $\log 5 = 0.69897$, $\log 7 = 0.84510$; find

(*a*) $\log 4$,	(*k*) $\log 20$,	(*u*) $\log 40$,	(5) $\log 72$,
(*b*) $\log 6$,	(*l*) $\log 21$,	(*v*) $\log 42$,	(6) $\log 75$,
(*c*) $\log 8$,	(*m*) $\log 24$,	(*w*) $\log 45$,	(7) $\log 80$,
(*d*) $\log 9$,	(*n*) $\log 25$,	(*x*) $\log 48$,	(8) $\log 81$,

(*e*) log 10,	(*o*) log 27,	(*y*) log 49,	(9) log 125,
(*f*) log 12,	(*p*) log 28,	(*z*) log 50,	(10) log 128,
(*g*) log 14,	(*q*) log 30,	(1) log 56,	(11) log 140,
(*h*) log 15,	(*r*) log 32,	(2) log 60,	(12) log 210,
(*i*) log 16,	(*s*) log 36,	(3) log 64,	(13) log 243,
(*j*) log 18,	(*t*) log 35,	(4) log 70,	(14) log 245.

30. Laws of logarithms. Since logarithms are exponents, they are governed by the same laws that govern exponents (see section 28). These laws are

1. *The logarithm of the product of two or more numbers is equal to the sum of the logarithms of the numbers.*

2. *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.*

3. *The logarithm of a number raised to a power is equal to the logarithm of the number multiplied by the exponent of the power.*

4. *The logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.*

31. Use of tables. Tables of common logarithms of numbers (see Note, p. 26) usually give the mantissas only.

As an illustration of the process of finding the logarithm of a number let us find the logarithm of 127. We find the number 127 in the left-hand column headed N and read the the mantissa opposite in the column headed 0. The characteristic must be 2, since the number is in the hundreds. Therefore

$$\log 127 = 2.10380.$$

To find the logarithm of 1276, follow along the same line as before to the column headed 6, obtaining

$$\log 1276 = 3.10585.$$

To find the logarithm of 12764 requires the use of interpolation. (See the explanation of this process which has already been given in section 27.)

Our number lies between 12760 and 12770. From the table,

$$\begin{aligned} \log 12770 &= 4.10619 \\ \log 12760 &= \underline{4.10585} \\ \text{difference} &= 0.00034 \end{aligned}$$

This difference in logarithms corresponds to a difference of 10 in the number. A difference of 4 in the number will give 0.4 of the difference in logarithms. Thus

$$\begin{aligned} 0.4 \times 0.00034 &= 0.000136 \\ \log 12760 &= \underline{4.10585} \\ \log 12764 &= 4.10599 \end{aligned}$$

(We retain only five decimal places in the logarithm.) For numbers having more than five significant places such a table can not be used to advantage. After the fifth place, then, we may consider the number to consist of zeros. For example, if we desire the logarithm of 5763243 we merely find the logarithm of 5763200.

The reverse process consists in finding the number when the logarithm is given. The number is usually called the **antilogarithm.** If the mantissa is found exactly in the tables, the number or antilogarithm is obtained by taking the first three figures from the left-hand column and the fourth from the top or bottom of the page. Subsequent digits are zeros.

If the mantissa can not be found exactly in the tables interpolation will be needed to get the fifth figure. For illustration, let us find the antilogarithm of 2.14460.

The given mantissa lies between the mantissas 14457 and 14489.

$$\begin{aligned} \log 1396 &= .14489 \text{ (mantissa only)} \\ \log 1395 &= .14457 \quad \text{“} \quad \text{“} \\ \text{difference} &= .00032 \end{aligned}$$

Representing the antilogarithm by x, we have

$$\begin{aligned} \log x &= .14460 \text{ (mantissa only)} \\ \log 1395 &= .14457 \quad \text{“} \quad \text{“} \\ \text{difference} &= .00003 \end{aligned}$$

Now a difference of .00032 in the logarithm accounts for a difference of 1 in the antilogarithm if we consider the latter to consist of four places, or a difference of 10 if we consider it to consist of five places. A difference of .00003 will account for

$$\frac{.00003}{.00032} \times 10 = \frac{3}{32} \times 10 = 1 -.$$

Thus, the sequence of figures in our antilogarithm is 13951. Making use of the characteristic 2 to locate the position of the decimal point, we get

$$\text{antilog } 2.14460 = 139.51.$$

To summarize: *Take the difference between the given mantissa and the mantissa next smaller, and divide by the difference between the mantissas adjacent to the given mantissa. This fraction times* 10 *will give the fifth figure in the antilogarithm.*

Note. Many of the results of trigonometry are only approximate. Their use, however, is justified because much of the practical work involved is based on measurements. Measurements can not be accurate beyond the sixth or seventh figure. The inaccuracy is due to human limitations. For example, the eye is limited in its ability to distinguish small quantities; the senses of touch and of hearing are imperfect; and instruments of precision, because made by man, can not be perfect, neither can they be manipulated with sufficient skill to realize all their possibilities. However, a mile can be measured to one-tenth of an inch of its true length, and by applying the greatest possible skill, an inch may be measured to within about a millionth part of itself. Ordinary measures, such as the carpenter and the surveyor make, are sufficiently accurate when made to the hundredth of a foot or to the tenth of an inch. The machinist, however, needs to measure to greater accuracy. It is found that four-place tables are usually sufficiently accurate for ordinary work. However, five-place tables will give all the results that can be obtained from four-place tables, and will provide all the necessary accuracy for computations except in rare cases. For extremely exact scientific work six-place and even seven-place tables are sometimes used, and in rare cases as many as fifteen places are used.

Since no measurement can be absolutely perfect, a number representing a measurement can not be exactly correct. It is evident, then, that the degree of accuracy represented by a number is of prime importance. Let us consider, for example, a number which represents approximately the square root of 2. If we write $\sqrt{2} = 1.414$, the value is given to thousandths, everything beyond the third place of decimals being discarded. Ordinarily we do not know the value of the part of a number discarded. If this were true of the above, we should simply know that the value of $\sqrt{2}$ is something between 1.4135 and 1.4145. That is, so far as we could know, $\sqrt{2}$ to four places of decimals might, for instance, be 1.4142 or 1.4138, for in the latter case the value is nearer 1.414 than 1.413.

From this it should be clear that a final digit ought not to be discarded because it happens to be a zero. If a measurement to two places of decimals is 4.60, the length is between 4.595 and 4.605, and to discard the zero would be to indicate that the length is between 4.55 and 4.65, which is not so accurate. Neither should zeros be annexed, since this would be claiming a degree of accuracy not justified.

Exercise 9

1. Find the logarithm of

(*a*) 68,	(*k*) 6784,
(*b*) 68.3,	(*l*) 678.4,
(*c*) 359,	(*m*) 67.84,
(*d*) 360.4,	(*n*) 6.784,
(*e*) 5943,	(*o*) 0.6784,
(*f*) 4006,	(*p*) 54326,
(*g*) 0.7,	(*q*) 38794,
(*h*) 0.094,	(*r*) 6312.90,
(*i*) 0.000345,	(*s*) 0.34732,
(*j*) 235700,	(*t*) 0.00000087695.

2. Find the antilogarithms of

(*a*) 0.69897,	(*j*) 0.63994,
(*b*) 1.76042,	(*k*) $\bar{2}.69085$,
(*c*) 2.93601,	(*l*) 1.51416,
(*d*) 4.26174,	(*m*) 7.19767 − 10,
(*e*) $\bar{1}.81278$,	(*n*) 1.48762,
(*f*) 9.96741 − 10,	(*o*) 8.82326 − 10,
(*g*) $\bar{7}.76253$,	(*p*) 5.18752,
(*h*) 3.63337,	(*q*) 6.15465,
(*i*) 8.84442 − 10,	(*r*) $\bar{1}.79029$.

3. Find by logarithms the values of

(*a*) 35.8×41.6,
(*b*) 5.25×4.84,
(*c*) 14.26×3.86,
(*d*) 529.6×29.64,
(*e*) 5028×46.09,
(*f*) $43.8 \times 13 \times 32.8$,
(*g*) $7.283 \times 283.4 \times 5.437$,
(*h*) 2.56×0.4875,
(*i*) 0.725×0.02755,
(*j*) 127400×0.1283,
(*k*) 0.256×0.3465,
(*l*) $0.0532 \times 0.00567 \times 0.0012$.

4. Find by logarithms the values of

(*a*) $63 \div 12$,
(*b*) $24284 \div 631$,
(*c*) $47248 \div 920$,
(*d*) $45990 \div 623$,
(*e*) $0.41831 \div 0.057$,
(*f*) $59.29 \div 0.66$,
(*g*) $2.451 \div 190$,
(*h*) $0.98902 \div 5.9$,
(*i*) $0.8772 \div 99$,
(*j*) $0.02736 \div 0.35$.

32. Cologarithms. When one number is to be divided by another we may change the problem to one of multiplication by using the reciprocal of the divisor. For example, $3 \div 2 = 3 \times \frac{1}{2}$.

The logarithm of the reciprocal of a number is called the **cologarithm** of the number, and is abbreviated **colog.** Thus

$$\text{colog } 2 = \log \tfrac{1}{2}.$$

Treating $\frac{1}{2}$ as a problem in division, we have

$$\begin{aligned} \log 1 &= 10.00000 - 10 = 0 \\ \log 2 &= 0.30103 \\ \log \tfrac{1}{2} &= 9.69897 - 10 \end{aligned}$$

(We have expressed log 1 in the form $10 - 10$ so as to be able to subtract log 2 more conveniently.) Hence

$$\text{colog } 2 = 9.69897 - 10.$$

It is readily seen that in solving a problem in division by means of logarithms we may either subtract the logarithm or add the cologarithm of the divisor. There is no advantage, but rather a disadvantage, in using the cologarithm when only two numbers are involved in a division problem. There is, however, considerable advantage when more than two numbers are involved.

We see from the above illustration that the cologarithm of a number is obtained by finding the logarithm in the usual manner, and then subtracting it from log 1, that is, from 10 − 10, which is of course 0. This may be done mentally after a little practice, and the result may be written without using scratch paper.

It may be remarked that a convenient way of finding the cologarithm is the following: *Subtract from* 9 *each digit of the logarithm except the last digit different from zero, which is subtracted from* 10; *then affix* − 10.

Indeed, since it is more convenient to write numbers from left to right rather than from right to left, all subtractions are more conveniently done beginning at the left, and with a little practice this is just as rapid as the other way. One has merely to glance ahead to see whether the figure in the minuend should be decreased by 1 or remain as it is. This is not more difficult than the glancing ahead which is necessary in ordinary reading. For example, in the following subtraction:

$$\begin{array}{r} 1063214563 \\ \underline{783402689} \end{array}$$

one can instantly see that it will not be 10 from which we take 7, because 8 is greater than 6 and it would be necessary to borrow from the 10. Again we notice that 6 must be decreased by 1 because, while we next have 3 from 3, beyond this the 4 of the subtrahend is greater than the 2 of the minuend, etc. Hence we can write the result rapidly from left to right,

$$279811874.$$

Exercise 10

1. Find the value of

$$\frac{538 \times 1.78}{256 \times 0.4075 \times 836.76} = A$$

Solution.

log 538	2.73078
log 1.78	0.25042
colog 256	7.59176 − 10
colog 0.4075	0.38987
colog 836.76	7.06361 − 10
log A	18.02644 − 20
log A	8.02644 − 10
A	0.010063

2. Find the cube root of 0.3475.

Solution.

$$\begin{aligned} \log 0.3475 &= 9.54095 - 10 \\ &= 29.54095 - 30 \\ \tfrac{1}{3}\log 0.3475 &= 9.84698 - 10 \\ \sqrt[3]{0.3475} &= 0.70303 \end{aligned}$$

Since we can not divide − 10 by 3 and get a multiple of 10, we add and subtract 20, obtaining 29.54095 − 30. This is perhaps the most convenient procedure in a case of this kind.

3. Find the value of

$$(a)\ \frac{97.3 \times 71.48}{436.5 \times 96.715},$$

(b) $\dfrac{1.1 \times 0.7832}{7.03 \times 0.05 \times 0.00062}$,

(c) $\dfrac{298.7 \times \sqrt[3]{0.462} \times \sqrt{11}}{4.576 \times 5.678 \times 0.06785}$,

(d) $\sqrt[5]{\left(\dfrac{5.75 \times 42.4 \times 0.097}{3.426 \times 0.0065853}\right)^2}$,

(e) $\dfrac{\sqrt{3.8456} \times 8}{60054 \times 0.00912}$.

4. Find the value to five significant figures of the following:

(a) 57^3,	(b) 648^4,	(c) 0.05^7,
(d) 6.58^5,	(e) 0.0006745^8,	(f) 0.050408^5,
(g) $\sqrt[3]{7}$,	(h) $11^{4/7}$,	(i) $13\sqrt{35}$,
(j) $\sqrt[25]{100}$,	(k) $\sqrt[9]{0.01534}$,	(l) $15467^{5/9}$.

5. Light travels at the rate of 186,330 miles a second. It takes light about $4\frac{1}{2}$ years to travel from the nearest star to us. (The star is said to be $4\frac{1}{2}$ light years distant.) What is the approximate distance in miles from the star to the earth?

6. The largest number that can be represented by three figures is 9^{9^9}, that is, 9 with an exponent 9^9. How many digits should there be in the number?

33. Computation of logarithms. The topic of the computation of logarithms properly belongs to advanced algebra or calculus. Consequently we simply give at this point a brief statement concerning the method employed in obtaining logarithms. A fuller discussion of the matter will be found in Chapter X.

Logarithms are computed by using the series

$$\log_e m = \log_e n + 2\left[\left(\frac{m-n}{m+n}\right) + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \cdots\right].$$

It is noted that the base e (see section 28) is used.

The logarithm of 2 to the base e is found by setting $m = 2$ and $n = 1$, the logarithm of 3 by setting $m = 3$ and $n = 2$, etc.

We can change logarithms having the base e to logarithms having the base 10, and *vice versa* by the following formulas:

$$\log_{10} N = M \log_e N, \qquad \log_e N = \frac{\log_{10} N}{M},$$

where $M = \log_{10} e = 0.43429448\cdots$. (For a proof of the foregoing see p. 272.) The number M is called the **modulus** of the common system (base 10) with respect to the natural system (base e).

The reciprocal of this modulus, $1/M = 2.30258509\cdots$, is called the modulus of the natural system with respect to the common system. It can easily be proved that $1/M = \log_e 10$ (see reference just cited).

Thus we have

$$\log_{10} N = (0.43429448) \log_e N,$$
$$\log_e N = (2.30258509) \log_{10} N.$$

Exercise 11

1. (*a*) Find $\log_e 2$ from the series of section 33.
   (*b*) Find $\log_{10} e$.
2. (*a*) Find $\log_e 3$.
   (*b*) Find $\log_{10} 3$.

3. The formula for the work W done by a gas expanding at a constant temperature from a volume V_0 to a volume V_1 is

$$W = P_0 V_0 \log_e \frac{V_1}{V_0},$$

where P_0 is the pressure when the volume is V_0. Find W if $P_0 = 90$, $V_0 = 250$, $V_1 = 450$.

4. The formula for the time t that it takes a body to cool from a temperature of T_0 to a temperature T_1 is

$$t = \frac{1}{k} \log_e \frac{T_0}{T_1}.$$

(*a*) If $k = 0.08514$, and the time is measured in minutes, find the time it takes a body to fall from a temperature of 30° to a temperature of 18°.

(*b*) If $k = 0.08514$, find the temperature after 5 minutes, if the original temperature was 25°.

(*c*) What value of k must be used in (*a*) and (*b*) if we wish to employ common instead of natural logarithms?

34. Exponential equations. An exponential equation is one which contains an unknown quantity in an exponent. Thus,

$$2^x = 12$$

is an exponential equation. Heretofore in our discussion the use of logarithms has been a great convenience, but not a necessity. In solving an equation such as the above we have practically no method of attack except the use of logarithms. To solve the equation we take the logarithm of each side, getting

$$x \log 2 = \log 12,$$

from which

$$x = \frac{\log 12}{\log 2} = \frac{1.07918}{0.30103}.$$

This fraction may be divided out or solved by logarithms, giving

$$x = 3.585.$$

Exercise 12

1. Solve the following for x:

$$(a)\quad 5^x = 125.$$
$$(b)\quad 11^x = 2560.$$

2. Solve for x and y:

$$\begin{cases} 2^x \times 3^y = 200, \\ 2^x \times 8^y = 256. \end{cases}$$

3. Solve for n: $l = ar^{n-1}$. This is the formula for the nth term of a geometric progression.

4. Solve for n: $A = P(1 + r)^n$. This is the formula for compound interest, A being the amount, P the principal, r the rate, and n the number of years.

5. In how many years will \$5500 amount to \$9000 at $4\frac{1}{2}\%$ compound interest?

6. How long will it take a sum of money to double itself at 4% compound interest?

7. Solve for n:

$$(a)\ S = R\frac{(1 + r)^n - 1}{r}, \qquad (b)\ V = \frac{R}{r}\left[1 - \frac{1}{(1 + r)^n}\right].$$

35. Logarithms of the trigonometric functions. A table giving the values of the functions of angles is called a table of the **natural** functions to distinguish it from tables which give the logarithms of these same values; the latter are called tables of **logarithmic** functions. We might in all cases find the natural function and then the logarithm of that function from the table of logarithms of numbers.

However, we have tables which omit one step in this process by giving the logarithm directly from the known value of the angle.

For example, suppose we wish to find

$$x = \sin 50° \tan 15°.$$

The long way would be as follows:

$$\begin{aligned} \sin 50° = 0.7660,\ \log 0.7660 &= 9.88423 - 10 \\ \tan 15° = 0.2680,\ \log 0.2680 &= \underline{9.42813 - 10} \\ \log x &= 9.31236 - 10 \\ x &= 0.20529 \end{aligned}$$

This process uses tables that we have already explained. The short method would be to use a table of logarithmic functions.

For angles from 0° to 45° this table is read downward and for angles from 45° to 90° it is read upward. The number of degrees usually appears at the top or at the bottom of the page. The number of minutes is in the extreme left-hand column for downward reading, and at the right for upward reading. Interpolation must be used in finding the functions of angles involving seconds.

To find log sin 18° 35′ turn to the page with 18° at the top. Read down the left-hand column to 35 and read the logarithm opposite this in the column headed log sin. The logarithm is found to be 9.50336 − 10 (the − 10 is not written, it is understood).

To find log cot 18° 35′ 45″, continue along the same line as before to the column headed log cot, and read 0.47339. The required logarithm must be between this logarithm and that next preceding, namely, 0.47297. The decrease in logarithms for 1′ is 0.00042. The decrease for 45″ is

45/60 or 3/4 of this, or 0.00032. Hence log cot 18° 35′ must be decreased by 0.00032, giving log cot 18° 35′ 45″ = 0.47307.

Exercise 13

1. Find the following by using the tables:

(*a*) log sin 29°,
(*b*) log cos 31°,
(*c*) log sin 78°,
(*d*) log tan 74°,
(*e*) log cot 17°,
(*f*) log cot 80°,
(*g*) log tan 12°,
(*h*) log sin 31°,
(*i*) log cos 49°,
(*j*) log sin 6° 31′,
(*k*) log sin 7° 46′,
(*l*) log cos 53° 21′,
(*m*) log cot 30° 26′,
(*n*) log sin 26° 45′,
(*o*) log tan 35° 15′ 18″,
(*p*) log sin 12° 13′ 14″,
(*q*) log cos 58° 37′ 50″,
(*r*) log cot 81° 25′ 7″,
(*s*) log sin 9.69°,
(*t*) log tan 54.37°.

2. Find the angle x if

(*a*) log sin x = 9.53871,*
(*b*) log cos x = 9.99483,
(*c*) log tan x = 0.30587,
(*d*) log cot x = 1.54623,
(*e*) log tan x = 0.18750,
(*f*) log cos x = 9.71705,
(*g*) log tan x = 9.28865,
(*h*) log cos x = 9.53871,
(*i*) log cos x = 9.72868,
(*j*) log cos x = 9.88150,
(*k*) log cos x = 9.89530,
(*l*) log sin x = 9.90151,
(*m*) log sin x = 9.80070,
(*n*) log sin x = 9.99483,
(*o*) log cot x = 9.18854,
(*p*) log cot x = 0.18750,
(*q*) log tan x = 0.06735,
(*r*) log tan x = 0.10235,
(*s*) log tan x = 1.55553,
(*t*) log cot x = 7.49859.

36. The slide-rule. Napier said of logarithms that "by shortening the labors they doubled the life of the astronomer." This is no exaggeration, for the time and energy

* The − 10 is understood and not written. The student will have no difficulty in knowing when it is understood.

conserved by labor-saving devices can be employed in other useful effort. The application of logarithms for ordinary purposes has been greatly facilitated by the slide-rule. This instrument was invented by William Oughtred (1574–1660), an Episcopal minister who lived near London. It is now widely used, not only by engineers and computers, but also by merchants, chemists, managers, and others, for performing such operations as permit the use of logarithms. We shall give only a brief description of the instrument here since complete instructions are furnished by the dealers who sell these instruments.

As indicated in Fig. 28, the ordinary slide-rule consists of

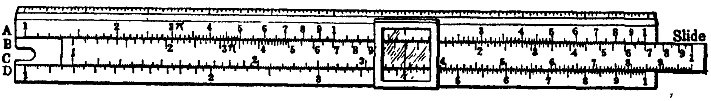


FIG. 28

a ruler in which is found a freely moving slide. Each of these has two logarithmic scales, A and D on the rule, B and C on the slide. The distance of any division on either of these scales from the left-hand graduation of that scale represents the mantissa of the logarithm of a number. The figure printed at this division is the antilogarithm, or the number itself. Evidently then, if we can add the mantissas by moving the slide, the numbers read from the rule will be the products desired.

Let us consider the scales A and B, which are identical. Fig. 29 shows the left half of these scales. This part of the

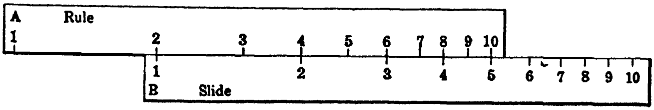


FIG. 29

scale extends from the division marked 1 (the antilogarithm of 0) to the division marked 10 (the antilogarithm of 1). The second half of the scale continues to the division marked 100. The markings 10 and 100 are usually both replaced by 1, and the decimal point must be obtained from an independent consideration. If now we move the slide B so that its index or division marked 1 coincides with the division marked 2 on A, then above 2 on the slide we find 4, the product of 2 and 2; above 3 we find 6; above 4 we find 8; etc. In other words we have multiplied by 2 each number on the slide, by a simple mechanical method of adding logarithms. By reversing the process we may divide each number by the adjacent number on B, the quotient being read over the index 1 of B. These processes will perhaps be clearer if the student constructs a scale like B on the edge of a separate strip of paper and actually slides this along the scale A.

The scales C and D are also identical, and operate in the same way; the entire scale, however, is the equivalent of half the scale on A and B. The scales C and D are therefore more accurate in the results obtained.

If the slide is turned over, scales marked S and T (sine and tangent) will be found. These scales enable us to compute results when the trigonometric functions are involved.

The numbers on the D scale are the square roots of those immediately above on the A scale. (Why?)

The ordinary rule, about ten inches long, will give results correct to three significant figures. However, there are larger and more complicated rules that give a higher degree of accuracy. Special rules are made for merchants, chemists, and others.

The rule may be used as a valuable aid in checking, and in arriving at approximate results.

37. Errors in interpolating. The process of interpolation assumes that the change in logarithms is directly proportional to the change in numbers for the interval under consideration. This would mean that the curve plotted to represent the logarithms of numbers is a straight line for that interval. While this is not absolutely true, it is sufficiently near the truth for practical purposes.

The logarithmic curve is represented in Fig. 30. It

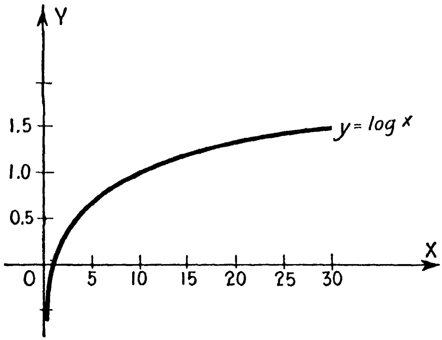


FIG. 30

crosses the X-axis at the point (1, 0), and approaches the Y-axis when prolonged downward, but never reaches it. The curvature, which is never very great (it is even exaggerated in Fig. 30 because we have taken a larger unit on the Y-axis than on the X-axis), is greatest at the point where $x = \frac{1}{2}\sqrt{2} = 0.7071$, as can be shown by methods of the calculus. When we take a small portion of the curve, say from $x = 0.7071$ to $x = 0.7072$, the part of the curve considered is so minute that it does not differ perceptibly from a straight line, hence the assumption that it is a straight line is sufficiently accurate for our purposes. This is shown in Fig. 31, which was constructed by means of seven-place tables. From these tables we find

$$\begin{aligned} \log 0.7072 &= 9.8495423 - 10 \\ \log 0.7071 &= 9.8494808 - 10 \\ \text{difference} &= 0.0000615 \end{aligned}$$

Interpolating for 0.70716, we get

$$\log 0.70716 = 9.8495115 - 10,$$

which is precisely the value obtained from the tables themselves.

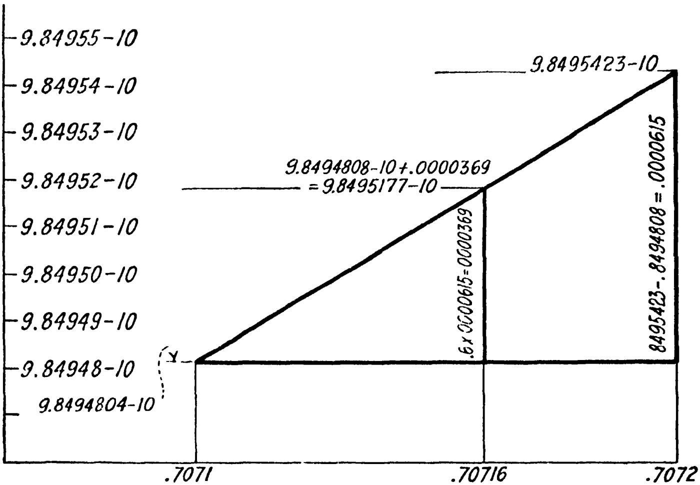


Fig. 31

However, the curves of the trigonometric functions, which we shall construct later (section 75), show that for certain intervals, interpolation for the functions, and consequently for the logarithms of the functions, is of little value. The value of a small angle (2° and less) can not be determined with any great accuracy from the cosine, because for small angles this function changes so slowly; neither can we determine the value of the sine, the tangent, or the cotangent from a small angle, because for small angles these functions change so rapidly. Similarly, for angles near 90° the sine changes slowly and the cosine, tangent, and cotangent change rapidly, so that the use of this portion of the table is inaccurate for certain purposes. To overcome these difficulties various devices are used, the most satisfactory of which is probably a special table giving the logarithms for every second or for every ten seconds. Another method is explained in section 90.

Exercise 14

1. Find

(*a*) log sin $1° 2' 52''$, (*d*) log sin $4' 18''$,
(*b*) log tan $32' 4''$, (*e*) log cot $88° 53' 4''$,
(*c*) log cos $89° 34' 36''$, (*f*) log sin $3' 42''$.

2. Find A if

(*a*) $\log \sin A = 7.85387$, (*d*) $\log \sin A = 6.72306$,
(*b*) $\log \tan A = 7.86432$, (*e*) $\log \cot A = 1.55407$,
(*c*) $\log \cos A = 7.67604$, (*f*) $\log \tan A = 7.23561$.

3. Find log tan $89° 55' 17''$.

4. Find

(*a*) log tan $89° 15' 32''$, (*c*) log sin $88° 47' 28''$,
(*b*) log cos $58' 22''$, (*d*) log cot $1° 21' 37''$.

5. Find A if $\log \tan A = 1.54602$.

6. Find A if

(*a*) $\log \cos A = 8.22925$, (*c*) $\log \tan A = 2.80571$,
(*b*) $\log \cot A = 8.18633$, (*d*) $\log \sin A = 7.45702$.

7. Find

(*a*) log sin $25' 25.37''$, (*d*) log cot $89° 59' 18.42''$,
(*b*) log cos $89° 28' 32.84''$, (*e*) log tan $24.75''$,
(*c*) log tan $19' 47.65''$, (*f*) log sin $57.38''$.

CHAPTER V

RIGHT TRIANGLES

38. Right triangles. We are now ready to employ logarithms in solving right triangles, that is, in finding the remaining parts when certain parts are given. (We say that a triangle has six parts—three sides and three angles.) We can always find the remaining parts of a triangle if we have three parts given, provided one of the known parts is a side. In a right triangle the right angle is one known part. If, then, in addition, we know two sides or one side and an acute angle we can solve the triangle.

In solving the problems in Exercise 15,

1. Draw the right triangle to scale as accurately as possible, letter the figure, and underline the known parts. Often it is advantageous to write in the dimensions of known sides and angles.

2. Obtain approximate values for the unknown parts desired by measuring the sides with a ruler and the angles with a protractor. This frequently prevents absurd results.

3. Select from the definitions of the trigonometric functions, or from the relations

$$A + B = 90^\circ, \qquad a^2 + b^2 = c^2,$$

a formula which will involve two known parts and an unknown part which is to be found.

4. Select a formula which will involve the computed part in a new relation for final checking. Checking is one of the most important parts of trigonometry, and in fact of all mathematical and scientific work. The skillful

mathematician and computer is the one who can readily discover and use checks for his work. He will not rely on his results until he has verified them. The habit of checking acquired in the study of mathematics should make one confirm all statements before making them positively. All possible assurance must first be found and possible sources of error must be investigated; even then the thoughtful person will frequently speak with reservations.

It is usually desirable that the student have no answers available, so that he may form the habit of finding out for himself whether his work is right or wrong.

A convenient check for right triangles is obtained by writing the theorem of Pythagoras in the following form, which is convenient for the use of logarithms:

$$a^2 = c^2 - b^2 = (c + b)(c - b).$$

5. Arrange the outline for the solution to the left of a vertical line, as indicated in the model solution below. Begin with the known parts and conclude with the checks. The outline should be complete before any numerical quantities are written in. The advantages of this arrangement are apparent. The student will understand the problem before he does any computing. In filling in the numerical quantities he can frequently find more than one quantity at the same time from the tables; this will be especially true in extended problems. If the check shows need for correction the error is more easily found than when the work is less systematically arranged. If anyone else wishes to understand or check the work he can follow the outline readily without loss of time or energy. This outline will show all of the work that the student does not do mentally, no scratch paper will be needed, hence the greater convenience for checking. It is needless to add that this form is valued by expert computers.

6. Carry out the computations and the check by filling in the numerical quantities to the right of the vertical line.

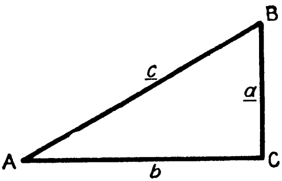
The following general rules will be of use in determining the degree of accuracy to be expected, and in avoiding useless labor, not only in connection with right triangles, but for all trigonometric work:

1. Distances expressed to two significant figures call for angles expressed to the nearest $30'$ and *vice versa.*

2. Distances expressed to three significant figures call for angles expressed to the nearest $5'$, and *vice versa.*

3. Distances expressed to four significant figures call for angles expressed to the nearest minute, and *vice versa.*

4. Distances expressed to five significant figures call for angles expressed to the nearest tenth of a minute, and *vice versa.*



Example. In a right triangle, $c = 20$, $a = 6.84$; find the other parts of the triangle.

Solution.

	Outline		Outline filled in	
$\sin A = a/c$	a		a	6.84
	c		c	20
	$\log a$		$\log a$	0.83506
	$\log c$		$\log c$	1.30103
	$\log \sin A$		$\log \sin A$	9.53403
	A		A	19° 59′ 57″
$B = 90° - A$	B		B	70° 00′ 03″
$b^2 = c^2 - a^2$	$c + a$		$c + a$	26.84
$= (c + a)(c - a)$	$c - a$		$c - a$	13.16
	$\log (c + a)$		$\log (c + a)$	1.42878
	$\log (c - a)$		$\log (c - a)$	1.11926
	$\log b^2$		$\log b^2$	2.54804
	$\log b$		$\log b$	1.27402
	b		b	18.794
Check	$\log c$		$\log c$	1.30103
$b = c \cos A$	$\log \cos A$		$\log \cos A$	9.97299
	$\log b$		$\log b$	1.27402

Exercise 15

1. In the following triangles, in which C is a right angle, find the remaining parts.

(*a*) $A = 37°$, $b = 53$; (*e*) $B = 17° 26' 30''$, $b = 92.37$;
(*b*) $B = 56°$, $c = 84$; (*f*) $A = 57° 10' 18''$, $c = 0.0286$;
(*c*) $a = 23$, $b = 17$; (*g*) $a = 0.2571$, $b = 0.8564$;
(*d*) $a = 18.5$, $c = 37.2$; (*h*) $b = 1892$, $A = 13° 52' 45''$.

2. A rectangle is 87 feet by 136 feet. Find the length of the diagonal, and the angles that it makes with the sides.

3. Find the radius of a circle in which a 59-foot chord subtends an angle of 12° at the center.

4. The length of a kite string is 150 yards. Find the height of the kite if the string makes an angle of 35° with the horizontal, assuming that there is no sag.

5. From the top of a cliff rising vertically 150 feet out of the water, the angle of depression of a boat was found to be 25° 20′. Find the distance of the boat from the foot of the cliff.

6. If a ladder 40 feet long is placed so as to reach a window 30 feet high, what angle does the ladder make with the ground, and how far is its foot from the base of the building?

7. A ladder 42 feet long is placed so that it will reach a window 30 feet high on one side of a street; by turning it over without moving its foot it will reach a window 25 feet high on the other side. How wide is the street?

8. At a point 160 feet from a building the angle of elevation of the top of the building is 37°. How high is the building?

9. A flagstaff 39 feet high is on the top of a building which stands on level ground. From a certain point on the ground the angles of elevation of the top and bottom of the flagstaff are 60° and 45° respectively. Find the height of the building.

10. Find the length of a belt passing around two pulleys whose radii are 14 inches and 5 inches respectively, and whose distance apart between centers is 10 feet.

11. A person on a ship sailing due south at the rate of 15 miles an hour observes a lighthouse due west at 3 P.M. At 5 P.M. the lighthouse is 52° west of north. How far from the lighthouse was the ship at (*a*) 3 P.M.? (*b*) 5 P.M.? (*c*) 4 P.M.?

12. The length of a degree of longitude at the equator is about 69.1 miles. Find the length of a degree of longitude at St. Louis, approximately 38° 38′ north latitude.

13. An aeroplane casts a shadow which is 275 feet from a point directly beneath. If the altitude (or angle of elevation) of the sun is 70°, how high is the aeroplane?

14. Two pulleys, 6 inches and 10 inches in diameter respectively, are running on shafts 80 inches apart. Find the length of a crossed belt connecting them.

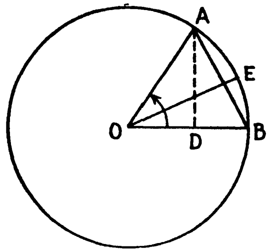
15. At a distance of d feet from the foot of a tree, the angle of elevation of the top of the tree is A. Find an expression that will represent the height of the tree, and one that will represent the length of a line from the point of observation to the top of the tree.

16. A roof 20 feet by 30 feet is inclined at an angle of 30° to the horizontal. Find the angle that a diagonal of the roof makes with the horizontal.

17. To measure the width of a river, a point R is selected on the near side directly opposite a point T on the far side. A base line $RS = 100$ feet is laid off along the bank perpendicular to RT. Then the angle RST is measured and found to be 75° 40′. How wide is the river?

18. A wall extending east and west is 6 feet high. The sun has an altitude of 49° 30′ and is 47° 15′ east of south. Find the width of the shadow of the wall.

19. Find the area of a 25° segment which subtends a 9 foot arc.



NOTE. When the angle at the center of a circle is measured in radians the area of the sector is one-half the length of the subtended arc times the radius. Hence

$$\text{sector } AOB = \tfrac{1}{2} \text{ arc } AB \times OB$$
$$= \tfrac{1}{2}R^2 \times O,$$

where O is the angle at the center of the circle.

$$\begin{aligned} \text{Area of triangle } AOB &= \tfrac{1}{2}AB \times OE \\ &= \tfrac{1}{2}DA \times OB \\ &= \tfrac{1}{2}R^2 \sin O. \\ \text{Area of segment } AEB &= \tfrac{1}{2}R^2O - \tfrac{1}{2}R^2 \sin O \\ &= \tfrac{1}{2}R^2(O - \sin O). \end{aligned}$$

Angle O must be measured in radians.

20. In a circle whose radius is 12 feet a chord is 7 feet from the center. Find the area of both segments.

21. A sector of a circle of radius 9.5 inches has an angle of 32° 41′. A chord joins the extremities of the radii forming the sector. What is the area of the small segment?

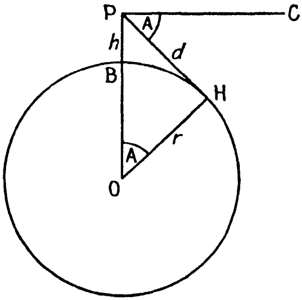
22. A five-pointed star is inscribed in a circle of radius 11 inches. What is the area of the star?

23. In surveying a circular railroad curve successive chords of 110 feet each are laid off. What is the radius of the curve if an angle between two successive chords is 175°?

24. **Distance and dip of the horizon.** In the accompanying figure let O represent the center of the earth, P the position of an observer and h his height above surface of the earth, r the radius of the earth, H a point on the horizon,

and angle A the **dip** of the horizon. Then, since the angle at H is a right angle,

$$d^2 = (r + h)^2 - r^2 = 2rh + h^2, \quad \text{and} \quad d = \sqrt{2rh + h^2}.$$



Now h^2 is of necessity small, since we can not rise to a great distance above the earth as compared with its radius. Hence we have approximately,

$$d = \sqrt{2rh}. \qquad (a)$$

Evidently,

$$\tan A = d/r, \quad \text{or} \quad d = r \tan A.$$

However, from (a) we may obtain a sufficiently accurate result for ordinary purposes if h and r be expressed in miles. Of course d will then be obtained in miles. Take $r = 3960$ miles. Then

$$d = \sqrt{2 \times 3960 \times h/5280}, \qquad \text{or} \qquad d = \sqrt{\tfrac{3}{2}h} \text{ miles.}$$

Conversely, $h = \frac{2}{3}d^2$ feet. That is, *the distance of the horizon in miles is approximately equal to the square root of* $\frac{3}{2}$ *the height in feet of the point of observation. The height in feet of the point of observation is* $\frac{2}{3}$ *the square of the distance of the horizon in miles.*

25. A person is standing on a mountain 5000 feet high. How far away is the horizon?

26. What is the distance to the horizon if a person is 6000 feet above the earth? 10,000 feet?

27. How far above the earth must a person be to see 8 miles? 15 miles? 25 miles? 60 miles? 120 miles?

28. If the great pyramid of Egypt is 486 feet high, how far may it be seen across the desert?

29. A vessel carries a mast whose top is 100 feet above the water. How far out at sea is the ship when the top of the mast is just visible?

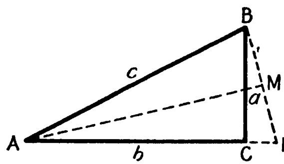
30. How far above the earth would one have to be in order to see one-fifth of its surface?

31. Prove that in any right triangle, $c - b = a \tan \frac{1}{2}A$. This formula may be used to solve problems such as 27 above by first finding angle A.

Suggestion. Bisect angle A. Draw BM perpendicular to this bisector to meet b produced at N. Then $CN = c - b$, and angle $CBN = \frac{1}{2}A$.

32. Prove that in any right triangle, $\tan \frac{1}{2}A = \sqrt{\dfrac{c-b}{c+b}}$.

This formula can be used in finding the angle A in a right triangle when the side b is very nearly equal to the hypotenuse c. It is better in such a case than the relation $\cos A = b/c$, since the angle A is small, and consequently, as was stated in section 37, can not be accurately determined from the cosine.



Suggestion. Use the relation $a = \sqrt{c^2 - b^2} = \sqrt{(c+b)(c-b)}$ in problem 31.

33. Prove the following relations for the parts of a right triangle:

$$(a)\ \sin 2A = \sin 2B,$$

$$(b)\ \cos 2A = \sin (B - A),$$

$$(c)\ \tan (B - A) = \frac{(b+a)(b-a)}{2ab},$$

$$(d)\ \cos (B - A) = \frac{2ab}{c^2}.$$

NOTE. For small angles and very great distances it is obvious that the hypotenuse of a right triangle may be considered as approximately equal to the longer leg. This assumes the shorter leg to be equal to the chord of a circle instead of perpendicular to the longer leg. The results are sufficiently accurate with this assumption, and the work is frequently very much simplified.

34. Find the angle subtended by a circular target 5 feet 6 inches in diameter at a distance of one-third of a mile.

35. If a railroad rises 15 inches in every 1200 feet, what angle does the track make with the horizontal?

36. Find the angle subtended by a man 5 feet 6 inches in height at a distance of half a mile.

37. What must be the height of a tower if it subtends an angle of $50'$ at a distance of 4500 feet?

38. The moon's angular diameter is observed to be $31' 5''$. Its diameter in miles is 2160. What is the distance of the moon? How many moons touching each other would fill the $180°$?

39. Assuming that the diameter of the earth is 3960 miles, what would be the angular diameter of the earth observed from the moon?

40. Find the approximate distance at which a coin one-half inch in diameter would hide the disc of the moon.

41. The inclination of a railway is $1°$. How much does it rise in a mile of its length?

42. According to recent estimates the Great Nebula in Andromeda is 600,000 light years distant and has a diameter of 20,000 light years. (A light year is the distance light travels in a year at approximately 186,330 miles per second.) (*a*) What angle should it subtend to an observer on the earth? (*b*) What angle would the distance from the earth to the sun (93,000,000 miles) subtend at the nebula?

43. The angle subtended by the radius of the sun is 16′ 2″ at the center of the earth. If the distance of the sun is 93,000,000 miles, what is the radius of the sun?

44. When the planet Venus is most brilliant, its diameter subtends an angle of 40″ as seen from the earth. The diameter of the planet is 7700 miles. What is its distance from the earth at such time?

45. The distance from the earth to the sun, 93,000,000 miles, observed at the nearest fixed star, Alpha Centauri, is 0.76″. At what distance would a disc one inch in diameter subtend the same angle?

46. Neptune, the most remote known planet of the solar system, is 34,800 miles in diameter, and is 30 times as far from the sun as the earth is. What angle does it subtend at the sun?

39. Isosceles triangles and regular polygons. The perpendicular from the vertex of an isosceles triangle divides it into two equal right triangles. Hence we can solve the isosceles triangle by solving one of these right triangles. It is sometimes convenient to place the isosceles triangle as indicated in Fig. 32.

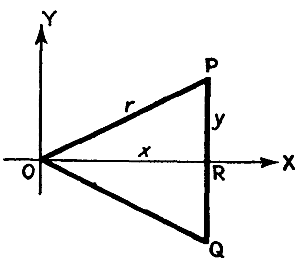


FIG. 32

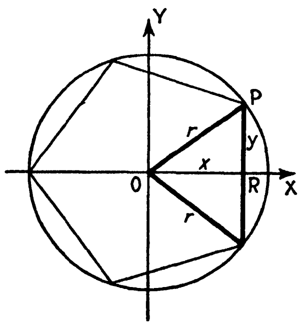


FIG. 33

To solve a regular polygon of n sides, we may divide it into n equal isosceles triangles. The solution of one of these triangles will enable us to solve the polygon. (See Fig. 33.)

Exercise 16

1. The equal angles of an isosceles triangle are each 40° 15′, and the base is 15 inches. Find the remaining parts and the area.

2. The equal sides of an isosceles triangle are each 11.52 inches, and the vertex angle is 32° 15′. Find the base.

3. The equal sides of a wedge are 4.2 inches, and the base is 1.6 inches. Find the angles.

4. A barn has a gable roof whose rafters are 40 feet long. The width of the barn is 60 feet. What is the pitch of the roof (the angle that the rafters make with the horizontal)? What is the area of one of the gable ends?

5. A barn is 30 feet by 60 feet and the pitch of the roof is 45°. Find the area of the gable ends, the length of the rafters, and the area of the roof.

6. The radius of a circle is 40 inches, and the length of a chord is 70 inches. What is the angle subtended at the center?

7. Find the radius of a circle in which a chord of 7 inches subtends an angle of 142° at the center.

8. Find the area of a circular sector if the radius of the circle is 14 inches and the angle at the center is 35°.

9. Find the radius, the apothem, and the area of the following regular polygons:

(*a*) A decagon whose side is 10 inches,

(*b*) A nine-sided polygon whose side is 15 inches,

(*c*) A twenty-sided polygon whose side is 6.758 inches.

10. A metal nut $\frac{3}{4}$ inch thick is in the shape of a regular hexagon, the distance between the parallel sides being $1\frac{3}{4}$

inches. The circular hole through the center is $\frac{3}{4}$ inch in diameter. Find the amount of metal in the nut.

11. The radius of a circle is 100 feet. Find the perimeter and the area of (*a*) a regular inscribed pentagon, (*b*) a regular inscribed decagon, (*c*) a regular circumscribed pentagon, (*d*) a regular circumscribed decagon.

12. What is the radius of a circle inscribed in an equilateral triangle whose perimeter is 66 feet?

13. The area of a regular pentagon is 560 square feet. Find the radii of the circumscribed and inscribed circles.

14. Show that the area of a regular polygon of n sides circumscribed about a circle of radius r is

$$nr^2 \tan \frac{180^\circ}{n}.$$

15. Show that the perimeter of a regular polygon of n sides inscribed in a circle of radius r is

$$2nr \sin \frac{180^\circ}{n}.$$

40. Oblique triangles. Oblique triangles can always be solved by splitting them up into right triangles. The following examples illustrate the methods used in the four typical cases which arise. Usually, however, it will be found convenient to have other methods and formulas for solving oblique triangles. These methods and formulas will be developed later.

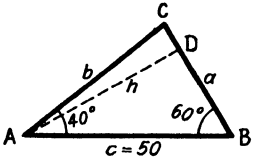
Case I. Two angles and a side given.

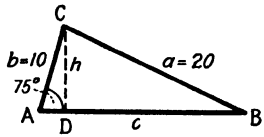
Example. In the triangle ABC, $A = 40^\circ$, $B = 60^\circ$, $c = 50$. Find the remaining parts.

Solution. Obviously, $C = 180^\circ - (A + B) = 180^\circ - (40^\circ + 60^\circ) = 80^\circ$.

To find the other parts, draw the altitude from one end of the known side. Suppose that this altitude is $AD = h$. Then in the right triangle ABD, $h = 50 \sin 60° = 43.30$.

Now in the right triangle ADC, $b = h/\sin C = 43.30/\sin 80° = 43.97$. Side a may be found in a similar manner.





Case II. Two sides and an angle opposite one of them given. (See discussion.)

Example. In a triangle ABC, $A = 75°$, $a = 20$, $b = 10$. Find B, C, and c.

Solution. Draw the altitude $CD = h$. In the right triangle ADC, $h = b \sin A = 10 \sin 75° = 9.744$. Then $\sin B = h/a = 9.744/20 = .4872$, $B = 29° 09'$. $C = 180° - (A + B) = 73° 51'$.

Side c may be obtained by computing the segments AD and DB and adding them.

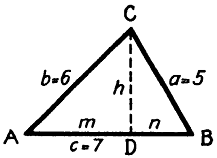
Case III. Two sides and the included angle given.

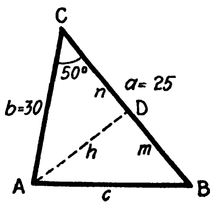
Example. Two sides of a triangle are 25 and 30, and the included angle is 50°. Find the other parts.

Solution. Let $a = 25$, $b = 30$, $C = 50°$. Draw an altitude to one of the known sides. Suppose that this altitude is $AD = h$, and that it divides the side BC into the segments $BD = m$ and $DC = n$. Then $h = b \sin C = 30 \sin 50° = 22.98$, $n = b \cos C = 30 \cos 50° = 19.28$, $m = a - n = 25 - 19.28 = 5.72$,

$$c = \sqrt{h^2 + m^2} = \sqrt{(22.98)^2 + (5.72)^2} = 23.68.$$

Angles A and B can now be found quite easily.





Case IV. Three sides given.

Example. The three sides of a triangle are $a = 5$, $b = 6$, $c = 7$. Find the angles.

Solution. Draw one of the altitudes, say $CD = h$, dividing the side c into the segments $AD = m$ and $DB = n$. Then

$$h^2 = 36 - m^2 = 25 - n^2,$$

and therefore

$$m^2 - n^2 = 11,$$

or

$$(m + n)(m - n) = 11.$$

But

$$m + n = 7.$$

Dividing, we get

$$m - n = 11/7 = 1.57.$$

Thus

$$m = 4.28, \qquad n = 2.72.$$

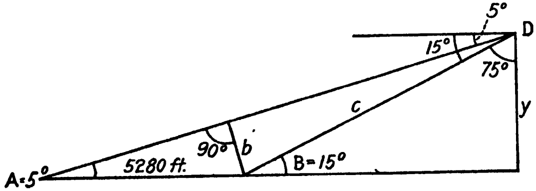
Angles A and B can now be found very readily.

Exercise 17

1. Find the remaining parts of the following triangles:

(*a*) $A = 30°,\ B = 80°,\ a = 15,$
(*b*) $A = 35°,\ b = 17,\ c = 32,$
(*c*) $A = 70°,\ a = 8,\ c = 5,$
(*d*) $B = 100°,\ C = 30°,\ b = 75,$
(*e*) $a = 2.3,\ b = 1.5,\ c = 1.6,$
(*f*) $a = 26,\ c = 40,\ B = 62°.$

2. From the top of a hill the angles of depression of two successive mile stones on a straight level road are 5° and 15° respectively. What is the height of the hill?



Solution. Drop a line from B perpendicular to AD.

$b = 5280 \sin 5°$	$\log 5280$	3.72263
	$\log \sin 5°$	8.94030
	$\log b$	2.66293
$c = b/\sin 10°$	$\log \sin 10°$	9.23967
	$\log c$	3.42326
$y = c \cos 75°$	$\log \cos 75°$	9.41300
	$\log y$	2.83626
	y	685.9 feet

3. Show that in problem 2,

$$y = \frac{5280 \tan A \tan B}{\tan B - \tan A}.$$

4. Verify the formula

$$y = \frac{5280 \sin A \sin B}{\sin (B - A)}$$

by using it to solve problem 2 and comparing the result with that already obtained.

5. An observer in a balloon notes that the angles of depression of two tents, which are in the same vertical plane with the balloon, are 6° and 12°. If the tents are 3000 feet apart, how high is the balloon?

6. In order to find the height of a hill a line 150 feet long is measured in the same horizontal plane with the base of the hill and in the same vertical plane with its top. The angles of elevation of the top of the hill from the ends of this line are 30° and 45° respectively. Find the height of the hill.

7. At a certain distance from the foot of a vertical cliff the angle of elevation of the top of a flag pole 50 feet tall standing on the edge of the cliff is 40°. From the same position the angle of elevation of the foot of the pole is 35°. How high is the cliff?

8. At a point in the same horizontal plane with the foot of a vertical cliff 150 feet high the angles of elevation of the top and the bottom of a flag pole standing on the edge of the cliff are 20° and 16° respectively. How tall is the pole?

9. From a window 30 feet above the street the angle of depression of the near side of the street is 50°, of the far side is 13°. How wide is the street?

10. The length of a lake subtends an angle of 44° 35′ at a certain point which is distant from the ends of the lake 456 feet and 580 feet. How long is the lake?

11. The sides of a triangle are 18, 24, and 30. Find the altitude to the longest side.

12. A rod 3 feet long is suspended by a string which is fastened to its two ends and passed over a smooth peg. The rod is inclined at an angle of 20° to the horizontal, and each part of the string is at an angle of 55° to the horizontal. Find the total length of the string.

41. Parallel sailing. In **parallel sailing** a ship travels due east or due west, that is, along the same parallel of latitude. The distance sailed is called the **departure**; it is expressed in knots. (A **knot**—sometimes called a **nautical** or **geographic mile**—is the length of an arc of one minute on the equator. It is approximately 1.15 ordinary miles.) The departure is evidently measured along a parallel from a meridian passing through the original position of the vessel to a meridian passing through its final position. In Fig. 34 the length of the arc AB is the departure between A and B. The angular measure of the corresponding equatorial arc EQ is called

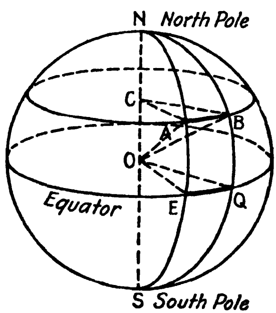


FIG. 34

the **difference in longitude** between A and B. But since EQ is an arc of a great circle, its angular measure in minutes is the same as its linear measure in knots.

Now if we measure arc AB and arc EQ in knots, we have

$$\frac{\text{arc } AB}{\text{arc } EQ} = \frac{CA}{OE} = \frac{CA}{OA} = \sin AOC$$
$$= \cos EOA = \cos \text{latitude}.$$

Clearing of fractions, we get

$$\text{arc } AB = \text{arc } EQ \times \cos \text{latitude}.$$

Recalling that arc EQ expressed in knots is equal to the difference in longitude expressed in minutes we see that

(16) **depart. = diff. long. × cos lat.,**

where it is understood that the departure is measured in knots and the difference in longitude is measured in minutes.

Example. A ship whose position was lat. 39° 45′ N., long. 68° 12′ W., sailed due east 150 knots. Find the longitude of the place reached.

Solution. $$\text{diff. long.} = \frac{\text{depart.}}{\text{cos lat.}}$$

lat.	39° 45′ N.
long.	68° 12′ W.
depart.	150
log depart.	2.17609
log cos lat.	9.88584
log diff. long.	2.29025
diff. long.	195.1′ = 3° 15.1′
new long.	64° 56.9′ W.

Exercise 18

1. A ship whose position is latitude 25° 20′ N., longitude 36° 40′ W., sails due west 150 knots. Find the longitude of the place reached.

2. A ship in latitude 50° 15′ N., longitude 62° 18′ W., sails due west a distance of 70 knots. What is the new position?

3. A ship is in latitude 43° 17′ N. and longitude 75° 16′ W. It sails due east 150 knots. What is the position reached?

4. A ship in latitude 26° 49′ N. and longitude 56° 18′ W. sails due east 226 knots. What is the new longitude?

42. Plane sailing. The angle which the path of a ship makes with a meridian* is called the **course** of the ship. If the course is constant, that is, if the ship crosses all meridians at the same angle, the path of the ship is called a **rhumb-line.** The **distance** between two positions of the ship is the length of that portion of the rhumb-line between the two positions. The **difference in latitude** between the two positions is the angular measure of the arc of a meridian intercepted between parallels of latitude through the two positions. Since a meridian is a great circle, the length of the meridional arc expressed in knots is the same as its angular measure in minutes.

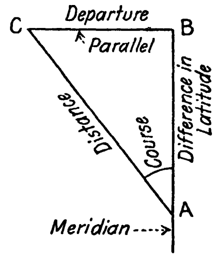


FIG. 35

If we are considering a small portion of the earth's surface we may without serious error regard it as a plane. That part of navigation which treats the surface of the earth as a plane is called **plane sailing.** In plane sailing meridians are considered parallel straight lines, a rhumb-line is considered the hypotenuse of a right triangle of which the departure and the difference in latitude are sides. (See Fig. 35.) Thus we have

$$\text{(17)} \qquad \textbf{sin course} = \frac{\textbf{depart.}}{\textbf{dist.}},$$

$$\text{(18)} \qquad \textbf{cos course} = \frac{\textbf{diff. lat.}}{\textbf{dist.}},$$

* The angle between two curves that intersect at a point is measured by the angle between the lines tangent to the two curves at that point.

where departure and distance are measured in knots and difference in latitude is measured in minutes.

Example. A ship sails from lat. 32° 18′ N., long. 69° 20′ W. on a course N. 17° 25′ W. a distance of 344 knots. Find its departure and its new position.

Solution.

	old lat.	32° 18′ N.
	old long.	69° 20′ W.
	course	N. 17° 25′ W.
	dist.	344
diff. lat. = dist. × cos course	log cos course	9.97962
depart. = dist. × sin course	log dist.	2.53656
	log sin course	9.47613
	log diff. lat.	2.51618
diff. long. = $\dfrac{\text{depart.}}{\text{cos lat.}}$	log depart.	2.01269
	log cos lat.	9.92699
	log diff. long.	2.08570
	depart.	103.0
	diff. lat.	328.2′ = 5° 28.2′
	diff. long.	121.8′ = 2° 01.8′
	new lat.	37° 46.2′ N.
	new long.	71° 21.8′ W.

43. Middle latitude sailing. It is readily seen that the departure between two meridians decreases as we go from the equator toward either pole. Although it is immaterial whether we compute the departure on the parallel through the original position of a vessel or on the one through its final position, a better value is obtained if we compute it on a parallel at the **middle latitude** (that is, a latitude half way between the original latitude and the final latitude). By this method, called **middle latitude sailing,** we have

(19) **depart. = diff. long. × cos mid lat.**

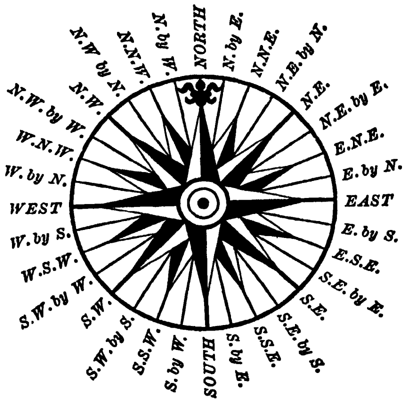
Exercise 19

The accompanying diagram of the mariner's compass will be found of use in some of the following problems.

1. A ship sails from latitude 8° 45′ S. on a course N. 30° E.

(30° east of north) 300 geographic miles. Find the new latitude and the departure.

2. A ship sails from latitude 45° N. on a course N.E. 260 knots. Find the new latitude and the departure.



Mariner's compass
FIG. 36.

3. A ship sails from a point on the equator on a course N.E. by N. 185 knots. Find the difference in latitude and the departure.

4. A ship sails from latitude 32° 20′ N., on a course between N. and W., a distance of 356 knots, and a departure of 110 knots. Find the new latitude and the course.

5. A ship sails on a course between S. and E. 245 knots, from latitude 3° 52′ S. to latitude 6° 8′ S. Find the course and the departure.

6. A ship sails on a course between S. and E., making a difference of latitude of 5° 45′ and a departure of 210 knots. Find the distance and the course.

7. A ship in latitude 42° 45′ N., longitude 58° 30′ W., sails S. 33° 45′ E. 300 knots. Find the new latitude and longitude, using the method of middle latitude sailing.

8. A ship sails E.N.E. 412 knots from latitude 31° 14′ N., longitude 42° 19′ W. Find the new position, using middle latitude.

9. Find the bearing (direction) and distance of Havana, 23° 09′ N., 82° 22′ W., from Cape Cod, 42° 02′ N., 70° 03′ W., using middle latitude.

10. Leaving latitude 17° 04′ N., longitude 118° 51′ E., a ship sails 221 knots, making a departure of 165 knots. Find the possible positions it may reach.

44. Components. The trigonometric functions have direct application in physics and mechanics. A displacement (change of position), velocity, force, or similar quantity is represented by a line having a certain length and a certain direction. For example, suppose an automobile is traveling at the rate of 30 miles an hour along a straight level road which makes an angle of 30° with north toward the east. Its velocity or speed may be represented by a line OP 30 units long extending from the origin and making an angle of 60° with the X-axis (see Fig. 37), the X-axis extending eastward, and the Y-axis northward. We speak of the line OP as the vector OP. If we project this vector on the X- and Y-axes and call the projections OP_x and OP_y respectively, we have

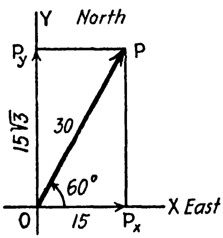


FIG. 37

$$OP_x = 30 \cos 60° = 30 \times \tfrac{1}{2} = 15,$$

$$OP_y = 30 \sin 60° = 30 \times \frac{\sqrt{3}}{2} = 25.98.$$

That is, at the end of an hour the automobile will be 15 miles east and 25.98 miles north of its position at the beginning of the hour. Thus we may think of the velocity of the automobile as composed of an easterly velocity of 15 miles an hour and a northerly velocity of 25.98 miles an hour. We call the projections OP_x and OP_y the **components** of the velocity represented by OP. We say that OP is **resolved** into its components along, or parallel to, OX and OY. Conversely, we say that OP is the **resultant** of OP_x and OP_y. It is readily seen that if the initial point of the line representing the velocity is at the origin, then the components are the co-ordinates of its terminal point.

In general, if a directed line segment V makes an angle A with the X-axis, and if V_x and V_y are its components parallel to the X- and Y-axes respectively, then

$$(20) \qquad \mathbf{V_x = V \cos A, \qquad V_y = V \sin A.}$$

The student can easily verify this by means of a figure.

A force F may similarly be resolved into its components. Suppose we have a force of F pounds pulling in a direction which makes an angle A with the line OX. If its components, or projections, are F_x and F_y, then

$$(21) \qquad F_x = F \cos A, \qquad F_y = F \sin A.$$

The force F has the same effect on an object at O as forces F_x pulling along OX and F_y pulling along OY. In the figure F_x is negative. This is taken care of in the first equation of (21), because $\cos A$ is there negative.

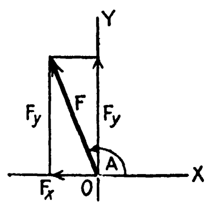


FIG. 38

If we know the magnitude and direction of a force or velocity, we can find its components, and conversely, if we know its components, we can find its magnitude and direction.

Example 1. Suppose for instance that a boat which can travel at the rate of 4 miles an hour in still water is pointed directly across a stream having a current of 3 miles an hour. What will be the actual speed of the boat, and in what direction will it go?

Solution. In still water the boat would go out at right angles to the bank at the rate of 4 miles an hour. But the current carries it down stream 3 units for every 4 units that it goes across. If then we take an X-axis parallel to the bank and a Y-axis perpendicular to it, the 3 and 4 may be considered as the X and Y components respectively of the actual velocity of the boat. The actual magnitude or amount of this velocity will of course be $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$. If A is the angle that it makes with the X-axis, then $\tan A = 4/3$, as can easily be seen in the figure, and $A = 53° +$.

In the notation used above, we have, in general,

$$V = \sqrt{V_x^2 + V_y^2}, \qquad \tan A = \frac{V_y}{V_x}. \tag{22}$$

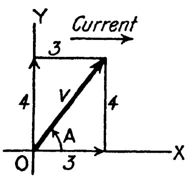


FIG. 39

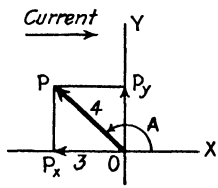


FIG. 40

It should be noticed that V is the diagonal of the rectangle whose sides are V_x and V_y.

Example 2. How must the boat of the above problem be pointed in order to go straight across the stream?

Solution. It is readily seen that the boat must be pointed so that its velocity of 4 miles an hour will have a component parallel to the bank which will exactly counterbalance the effect of the current. Taking the axes as before, we have

$$V = 4, \qquad V_x = -3,$$

and consequently

$$\cos A = -\tfrac{3}{4} = -0.75, \qquad A = 138° 35'.$$
$$V_y = 4 \sin A = 2.65.$$

(We may also obtain V_y by means of the theorem of Pythagoras.)

Thus, to go straight across the river, the boat should be pointed at an angle of 138° 35′ with the down-stream direction. The boat will then cross the river at the rate of 2.65 miles an hour.

Example 3. Suppose we have two forces of 100 pounds and 80 pounds acting on a weight as shown in Fig. 41. What will be their horizontal effect and what will be their vertical or lifting effect?

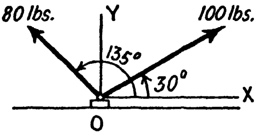


FIG. 41

Solution. The horizontal component of the 100-pound force is 100 cos 30° = 86.60. The horizontal component of the 80-pound force is 80 cos 135° = − 56.57. Thus the total horizontal force tending to move the weight to the right is 86.60 − 56.57 = 30.03 pounds. The total lifting force is 100 sin 30° + 80 sin 135° = 50 + 56.57 = 106.57 pounds.

The components of the resultant force (the single force that is equivalent to the two given forces) are

$$R_x = 30.03, \qquad R_y = 106.57.$$

The resultant force is

$$R = \sqrt{R_x^2 + R_y^2} = 110.72,$$

and if A is the angle that it makes with the X-axis,

$$\tan A = \frac{R_y}{R_x} = \frac{106.57}{30.03} = 3.5488, \qquad A = 74^\circ\ 16'.$$

That is, a single force of 110.72 pounds acting at an angle of 74° 16′ with the X-axis will have the same effect as the two given forces.

Exercise 20

1. The westward and southward components of the velocity of a ship are 6.7 and 12.5 knots an hour respectively. Find the direction and the rate of the motion of the ship.

2. A canal boat is drawn by means of a cable. If a force of 4,000 pounds is required to pull the boat, what force will be required when the cable makes an angle of 12° with the bank of the canal?

3. A stone is supported by two ropes which make angles of 30° and 20° with the vertical. The pull on the first

rope is 75 pounds. Find the weight of the stone and the pull on the second rope.

4. A boat is being rowed north at the rate of 5 miles an hour, and the tide carries it west at the rate of 3 miles an hour. Find the actual speed and the actual direction of the boat.

5. A river flows at the rate of 1.5 miles an hour. (*a*) In what direction must a man swim in order to go straight across, if his rate of swimming in still water is 2.5 miles an hour? (*b*) How long will it take him to cross if the river is $\frac{1}{2}$ mile wide?

6. An aeroplane has a rate of 70 miles an hour in still air. A wind having a velocity of 15 miles an hour is blowing from the southwest. (*a*) If the aeroplane is pointed southeast, what will be the magnitude and the direction of its velocity with reference to the ground? (*b*) How must the aeroplane be pointed in order to fly southeast, and what will be its actual velocity?

7. A balloon is rising at the rate of 10 feet a second and is being carried horizontally by a wind which has a velocity of 15 miles an hour. Find its actual velocity and the angle which its path makes with the horizontal.

8. (*a*) A barge is being towed due north at the rate of 15 miles an hour. A man walks across the deck, at right angles to the direction of motion of the barge, at the rate of 6 feet a second. Find the direction and the magnitude of his actual velocity. (*b*) If the barge is 50 feet long and 20 feet wide, and the man walks along the diagonal of the deck, find the direction and the magnitude of his actual velocity.

9. A ship is traveling at the rate of 20 knots an hour. (A knot is approximately 1.15 miles.) When directly

opposite a target it fires a gun whose projectile has a velocity of 2000 feet a second. (*a*) At what angle with the path of the ship must the gun be pointed in order to hit the target? (*b*) If the ship is 3000 feet from the target, at the time the gun is fired, how long will it take the projectile to reach the target?

10. To a person on a train traveling at the rate of 40 miles an hour, raindrops, which are actually falling vertically, appear to fall at an angle of 15° with the horizontal. Find the actual velocity with which the rain is falling.

11. A force of 150 pounds makes an angle of 125° with the positive end of the X-axis. Find the components of the force along the X- and Y-axes.

12. A force has components of 16 and -24 along the X- and Y-axes respectively. Find its magnitude and direction.

13. A force of 72 pounds acts towards the northwest; another force of 56 pounds acts in a direction 15° east of north. Find the magnitude and the direction of their resultant.

14. A weight of 50 pounds is supported by two strings, each of which is inclined to the vertical at an angle of 30°. Find the pull on each string.

15. A force of 200 pounds, when applied at the origin, makes an angle of 67° 35′ with the positive X-axis. Another force of 358 pounds makes an angle of 12° 40′ with the positive X-axis. Find the magnitude and the direction of the resultant of the two forces.

16. Show that the projection of a plane area upon a fixed plane is equal to the given area times the cosine of the angle between the planes.

17. A hillside has a slope of 10°, and contains 20 acres. How much more is this than the projection of the hillside on the horizontal plane?

18. A mine shaft is 1600 feet in length. It slopes downward at an angle of 45° to the horizontal for 1000 feet, and at an angle of 30° for the remainder of its length. (*a*) What is the total depth reached? (*b*) If the first part of the shaft runs 20° west of north, and the second part northwest, find the co-ordinates of the point directly above the lower end of the shaft. Use an east and west line as the *X*-axis, a north and south line as the *Y*-axis, and the opening of the mine as the origin.

19. A man playing five holes of golf walks 260 yards east, then 140 yards 20° south of east, then 300 yards south, then 200 yards 40° west of north, and finally 220 yards 30° west of south, arriving at the fifth hole. How far is the fifth hole from the first tee?

45. Simple harmonic motion. Let us consider a point P moving on the circumference of a circle of radius r. Suppose that the center of the circle is at the origin of the system of co-ordinates. Let P_x and P_y be the projections of the point P on the X- and Y-axes respectively. Then as P moves around the circumference, P_x will move back and forth along the diameter which lies upon the X-axis, and P_y will move up and down the diameter which lies upon the Y-axis. When the speed of P around the circumference is constant, the motion of P_x, or of P_y, is called **simple harmonic motion** (sometimes abbreviated S.H.M.).

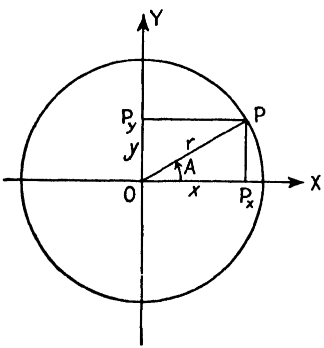


FIG. 42

This is practically the manner in which a point on a vibrating string or other vibrating object moves. The study of simple harmonic motion is therefore very important in physics.

Let us now consider some of the characteristics of this sort of motion. Obviously, if we designate by A the angle P_xOP, we have

$$(23) \qquad OP_x = x = r \cos A, \qquad OP_y = y = r \sin A.$$

Since P is moving with a constant speed, the line OP moves with a constant angular speed (in radians a second, degrees a second, or other units) which we may designate by c, c being a constant. Now let A be the angle turned through in t units of time. Then $A = ct$, provided, of course, that A is measured in terms of the angular unit that is used in expressing c. Thus equations (23) reduce to

$$(24) \qquad x = r \cos ct, \qquad y = r \sin ct.$$

Each of these is called the equation of the respective simple harmonic motion.

To illustrate this by means of a numerical example, let us take a circle of radius 10 inches, and assume that we have a point moving around its circumference at the rate of 2 radians per second.

For this particular case we have

$$x = 10 \cos 2t, \qquad y = 10 \sin 2t.$$

At the end of 0.1 second, $2t = 0.2$, and

$$x = 10 \cos 0.2^{(r)} = 9.8007,$$
$$y = 10 \sin 0.2^{(r)} = 1.9896.$$

That is, the line OP has turned through an angle of 0.2 radians, the point P_x has moved over until it is 9.80 inches from O, the point P_y has moved up a distance of 1.99 inches from O.

At the end of 1 second, $2t = 2$, and

$$x = 10 \cos 2^{(r)} = -4.1615,$$
$$y = 10 \sin 2^{(r)} = 9.0930.$$

That is, the line OP has turned through an angle of 2 radians, so that OP is in the second quadrant. The point P_x has moved so that it is at a distance -4.16 inches from the point O, the negative sign indicating that it is to the left of O. The point P_y has moved up to the extremity of the diameter and has returned to a position 9.09 inches above O.

For P_x to get across the diameter and back again, P obviously must go completely around the circumference. Since P moves at the angular rate of 2 radians a second and there are 2π radians in the entire circumference, it will take P $2\pi/2$ or π seconds to make a complete circuit; consequently, it will take P_x π seconds to go across the diameter and back again.

For the general case of harmonic motion whose equations are (24), the point P will make a complete revolution in the time $2\pi/c$, and of course P_x will move across the diameter and back again in this time. The time $2\pi/c$ is called the **period** of the simple harmonic motion. The reciprocal of the period, or $c/2\pi$, is called the **frequency.** The quantity r is called the **amplitude** of the simple harmonic motion.

It can readily be seen that the time required for P_x to move from any given position to one extremity of the diameter, then to the other extremity of the diameter, and back to its original position is exactly the period, since during this time the point P, of which P_x is the projection, will move entirely around the circumference. The frequency is the number of times a second (or other given unit of time) that the point P_x makes a complete trip across the diameter and back; that is, it is the frequency with which such a complete trip is made.

Exercise 21

1. The equation of a simple harmonic motion is $x = 5 \cos t$. (*a*) Find its amplitude, period, and frequency. (*b*) Find the value of x when $t = 0.1, 0.2, 0.3, 0.4, 0.5, 1, 2, \pi, 3\pi/2$.

2. The equation of a simple harmonic motion is $x = 8 \cos 0.2t$. (*a*) Find the value of x corresponding to $t = 1, 2, 3, 4, 5, 10, 20, 50, 100, \pi, 2\pi, 3\pi, 4\pi, 5\pi$. (*b*) Find the period of the simple harmonic motion.

3. The amplitude of a simple harmonic motion is 4 and its period is $2\pi/3$. Write an equation for it.

4. The equation of a simple harmonic motion is $y = 0.6 \sin \pi t/4$. (*a*) What is its period? (*b*) Find the value of y when $t = 1, 2, 3, 10, 25, 0.1, 2.5$.

5. If the amplitude of a simple harmonic motion is 10 and its period is 2, write an equation for it.

6. A tuning fork for the note middle C makes 256 complete vibrations a second. That is, its frequency is 256. Its period is therefore 1/256. Show that the motion of a point on the fork is described by the equation $x = a \cos 512\pi t$, where a is half the distance between the extreme positions of the fork.

7. Write an equation representing the motion of a point on a tuning fork which sounds the note E of the middle octave. (Such a fork makes 320 complete vibrations a second.)

8. The length of the simple pendulum in the accompanying figure is l; arc $OP = s$, and arc $OP_0 = s_0$. If the

pendulum is released from the position CP_0, where the angle $v_0 = OCP_0$ is small, then it can be shown that an equation representing the motion of the bob of the pendulum is $s = s_0 \cos(\sqrt{g/l})t$, in which g is the acceleration due to gravity, and t is the time. Find the period and the frequency of the pendulum.

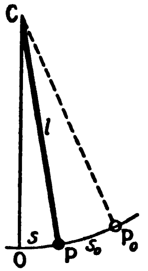


FIG. 43.

9. If, in problem 8, $l = 3$ feet, $v_0 = 10°$, and $g = 32.2$ feet per second per second, (*a*) write the equation of motion for the pendulum bob and find the amplitude, period, and frequency. (*b*) Locate the bob at the end of 1 sec., 2 sec., 3 sec., 10 sec.

CHAPTER VI

FORMULAS AND IDENTITIES

46. Functions of $-v$. Let us consider the functions of $-v$, where v is any angle whatever.

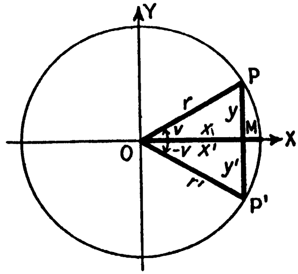


FIG. 44

In Fig. 44 the angle v is for definiteness shown in the first quadrant, but in the following considerations v is not restricted to the first or to any other quadrant. It is readily seen that in the equal right triangles OMP' and OMP, $x' = x$, $y' = -y$ (since y' and y extend in opposite directions), and $r' = r$ (since we have agreed to consider the radius as positive). Consequently

$$
(25) \quad \left\{
\begin{aligned}
\sin(-v) &= \frac{y'}{r'} = \frac{-y}{r} = -\sin v,\\
\cos(-v) &= \frac{x'}{r'} = \frac{x}{r} = \cos v,\\
\tan(-v) &= \frac{y'}{x'} = \frac{-y}{x} = -\tan v,\\
\cot(-v) &= \frac{x'}{y'} = \frac{x}{-y} = -\cot v,\\
\sec(-v) &= \frac{r'}{x'} = \frac{r}{x} = \sec v,\\
\csc(-v) &= \frac{r'}{y'} = \frac{r}{-y} = -\csc v.
\end{aligned}
\right.
$$

The student should satisfy himself that these formulas hold for the other cases by drawing figures with v in the second, third, and fourth quadrants. The angle v might also be taken as greater than one rotation, or even as a negative angle.

It might be noted that the formula for tan $(-v)$ could be proved as follows:

$$\tan(-v) = \frac{\sin(-v)}{\cos(-v)} = \frac{-\sin v}{\cos v} = -\tan v,$$

and that cot $(-v)$, sec $(-v)$, and csc $(-v)$ could be obtained from reciprocal relations.

47. Functions of 180° − *v*. Let us now consider the functions of $180° - v$, where again v may be any angle whatever. Reference to Fig. 45 shows that

$$(26)\quad \begin{cases} \sin(180° - v) = \dfrac{y'}{r'} = \dfrac{y}{r} = \sin v, \\[2ex] \cos(180° - v) = \dfrac{x'}{r'} = \dfrac{-x}{r} = -\cos v, \\[2ex] \tan(180° - v) = \dfrac{y'}{x'} = \dfrac{y}{-x} = -\tan v, \\[2ex] \cot(180° - v) = \dfrac{x'}{y'} = \dfrac{-x}{y} = -\cot v, \\[2ex] \sec(180° - v) = \dfrac{r'}{x'} = \dfrac{r}{-x} = -\sec v, \\[2ex] \csc(180° - v) = \dfrac{r'}{y'} = \dfrac{r}{y} = \csc v. \end{cases}$$

48. Functions of 180° + *v*. From Fig. 46 it is evident that

$$(27)\quad \begin{cases} \sin\ (180^\circ + v) = \dfrac{y'}{r'} = \dfrac{-y}{r} = -\sin\ v, \\ \cos\ (180^\circ + v) = \dfrac{x'}{r'} = \dfrac{-x}{r} = -\cos\ v, \\ \tan\ (180^\circ + v) = \dfrac{y'}{x'} = \dfrac{-y}{-x} = \tan\ v, \\ \cot\ (180^\circ + v) = \dfrac{x'}{y'} = \dfrac{-x}{-y} = \cot\ v, \\ \sec\ (180^\circ + v) = \dfrac{r'}{x'} = \dfrac{r}{-x} = -\sec\ v, \\ \csc\ (180^\circ + v) = \dfrac{r'}{y'} = \dfrac{r}{-y} = -\csc\ v. \end{cases}$$

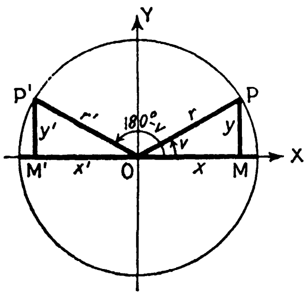


FIG. 45

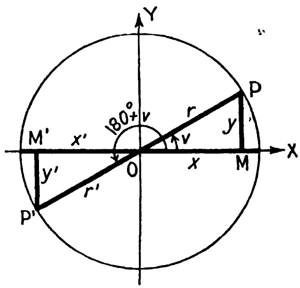


FIG. 46

49. Functions of 360° − *v*. Fig. 44 shows that the functions of 360° − v are the same as the functions of $-v$. Thus

$$(28)\quad \begin{cases} \sin\ (360^\circ - v) = -\sin\ v, \\ \cos\ (360^\circ - v) = \cos\ v, \\ \tan\ (360^\circ - v) = -\tan\ v, \\ \cot\ (360^\circ - v) = -\cot\ v, \\ \sec\ (360^\circ - v) = \sec\ v, \\ \csc\ (360^\circ - v) = -\csc\ v. \end{cases}$$

50. Functions of $360° + v$. It should be quite clear that the functions of $360° + v$ are the same as the corresponding functions of v.

51. Functions of $90° - v$. It was shown in section 25 that for any acute angle A, $\sin (90° - A) = \cos A$, etc. That is, any function of an acute angle is equal to the co-function of the complementary angle. That formulas (15) of section 25 are true for any angle may be shown by Fig. 47 as follows: Right triangles OMP and $OM'P'$ are equal, and consequently $x' = y$, $y' = x$, $r' = r$. Therefore

$$(29)\quad \begin{cases} \sin (90° - v) = \dfrac{y'}{r'} = \dfrac{x}{r} = \cos v, \\[2ex] \cos (90° - v) = \dfrac{x'}{r'} = \dfrac{y}{r} = \sin v, \\[2ex] \tan (90° - v) = \dfrac{y'}{x'} = \dfrac{x}{y} = \cot v, \\[2ex] \cot (90° - v) = \dfrac{x'}{y'} = \dfrac{y}{x} = \tan v, \\[2ex] \sec (90° - v) = \dfrac{r'}{x'} = \dfrac{r}{y} = \csc v, \\[2ex] \csc (90° - v) = \dfrac{r'}{y'} = \dfrac{r}{x} = \sec v. \end{cases}$$

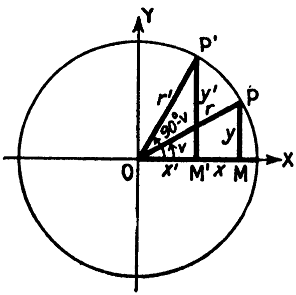


FIG. 47

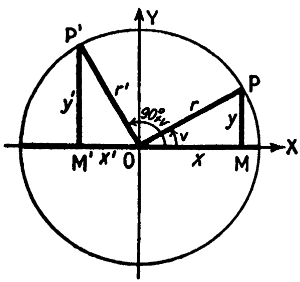


FIG. 48

52. Functions of $90° + v$. It is seen that in Fig. 48, x' and y are equal numerically but are of opposite signs; that is, $x' = -y$. Similarly, y' and x are numerically equal and are of the same sign; that is, $y' = x$. Also $r' = r$. Therefore

$$(30)\quad \begin{cases} \sin\ (90° + v) = \dfrac{y'}{r'} = \dfrac{x}{r} = \cos v, \\ \cos\ (90° + v) = \dfrac{x'}{r'} = \dfrac{-y}{r} = -\sin v, \\ \tan\ (90° + v) = \dfrac{y'}{x'} = \dfrac{x}{-y} = -\cot v, \\ \cot\ (90° + v) = \dfrac{x'}{y'} = \dfrac{-y}{x} = -\tan v, \\ \sec\ (90° + v) = \dfrac{r'}{x'} = \dfrac{r}{-y} = -\csc v, \\ \csc\ (90° + v) = \dfrac{r'}{y'} = \dfrac{r}{x} = \sec v. \end{cases}$$

53. Functions of $270° - v$. In Fig. 49, $x' = -y$, $y' = -x$, $r' = r$, and

$$(31)\quad \begin{cases} \sin\ (270° - v) = \dfrac{y'}{r'} = \dfrac{-x}{r} = -\cos v, \\ \cos\ (270° - v) = \dfrac{x'}{r'} = \dfrac{-y}{r} = -\sin v, \\ \tan\ (270° - v) = \dfrac{y'}{x'} = \dfrac{-x}{-y} = \cot v, \\ \cot\ (270° - v) = \dfrac{x'}{y'} = \dfrac{-y}{-x} = \tan v, \\ \sec\ (270° - v) = \dfrac{r'}{x'} = \dfrac{r}{-y} = -\csc v, \\ \csc\ (270° - v) = \dfrac{r'}{y'} = \dfrac{r}{-x} = -\sec v. \end{cases}$$

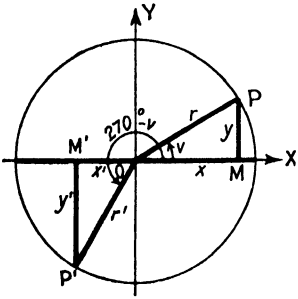


FIG. 49

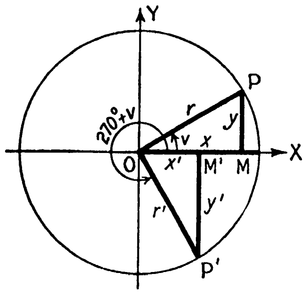


FIG. 50

54. Functions of $270° + v$. In Fig. 50, $x' = y$, $y' = -x$, $r' = r$, and

$$(32)\quad \left\{\begin{aligned}
\sin\,(270° + v) &= \frac{y'}{r'} = \frac{-x}{r} = -\cos v,\\
\cos\,(270° + v) &= \frac{x'}{r'} = \frac{y}{r} = \sin v,\\
\tan\,(270° + v) &= \frac{y'}{x'} = \frac{-x}{y} = -\cot v,\\
\cot\,(270° + v) &= \frac{x'}{y'} = \frac{y}{-x} = -\tan v,\\
\sec\,(270° + v) &= \frac{r'}{x'} = \frac{r}{y} = \csc v,\\
\csc\,(270° + v) &= \frac{r'}{y'} = \frac{r}{-x} = -\sec v.
\end{aligned}\right.$$

55. Summary. Formulas (25) to (32) may be summarized for reference as in Table J. The upper sign preceding a function in the body of the table corresponds to the upper sign in the heading of the same column, and similarly for the lower sign.

TABLE J

	$-v$	$90° \pm v$	$180° \pm v$	$270° \pm v$	$360° \pm v$
sin	$-\sin v$	$\cos v$	$\mp \sin v$	$-\cos v$	$\pm \sin v$
cos	$\cos v$	$\mp \sin v$	$-\cos v$	$\pm \sin v$	$\cos v$
tan	$-\tan v$	$\mp \cot v$	$\pm \tan v$	$\mp \cot v$	$\pm \tan v$
cot	$-\cot v$	$\mp \tan v$	$\pm \cot v$	$\mp \tan v$	$\pm \cot v$
sec	$\sec v$	$\mp \csc v$	$-\sec v$	$\pm \csc v$	$\sec v$
csc	$-\csc v$	$\sec v$	$\mp \csc v$	$-\sec v$	$\pm \csc v$

Note that on each line we have the same functions except for those columns headed $90° \pm v$, and $270° \pm v$, in which columns we find the co-functions.

56. Reduction of functions of angles to functions of acute angles. It will be noted that the trigonometric tables use angles from 0° to 45°. We have already seen however that they can be used to find the functions of angles between 45° and 90°. We are now in a position to find the functions of any angle whatever, by using formulas (25) to (32).

Example 1. Find sine, cosine, and tangent of 110°.

Solution. Since $110° = 180° - 70°$ we have

$$\sin 110° = \sin (180° - 70°) = \sin 70° = 0.9397,$$
$$\cos 110° = \cos (180° - 70°) = -\cos 70° = -0.3420,$$
$$\tan 110° = \tan (180° - 70°) = -\tan 70° = -2.7475.$$

Or since $110° = 90° + 20°$,

$$\sin 110° = \sin (90° + 20°) = \cos 20° = 0.9397,$$
$$\cos 110° = \cos (90° + 20°) = -\sin 20° = -0.3420,$$
$$\tan 110° = \tan (90° + 20°) = -\cot 20° = -2.7475.$$

Example 2. Find sine, cosine, and tangent of 615°.

Solution. Since $615° = 360° + 255°$, the functions of 615° are precisely the same as those of 255°. But $255° = 180° + 75°$, and therefore

$$\sin 615° = \sin 255° = \sin (180° + 75°) = -\sin 75° = -0.9659,$$
$$\cos 615° = \cos 255° = \cos (180° + 75°) = -\cos 75° = -0.2588,$$

$\tan 615° = \tan 255° = \tan(180° + 75°) = \tan 75° = 3.7321.$

Or, we could express 255° as 270° − 15°.

It is hardly worth while to attempt to remember the results given in Table J. It is better to draw a figure for each separate problem as needed.

Exercise 22

1. Show that

(*a*) $\sin 70° = \sin 110°$,
(*b*) $\cos 30° = -\sin(-60°)$,
(*c*) $\cos 300° = \sin 30°$,
(*d*) $\cot 165° = -\tan 75°$.

2. What other angle less than 360°

(*a*) has the same sine as 88°?
(*b*) has the same cosine as 145°?
(*c*) has the same tangent as 310°?
(*d*) has the same cosecant as 150°?
(*e*) has the same secant as 340°?
(*f*) has the same cotangent as 245°?

3. Reduce to functions of an angle less than 45°:

(*a*) $\sin 175°$,
(*b*) $\sin(-167°)$,
(*c*) $\sin 520°$,
(*d*) $\sin 125° 26'$,
(*e*) $\sec 267° 28'$,
(*f*) $\csc 325° 41' 51''$.

4. Express as functions of angle v:

(*a*) $\sin(810° - v)$,
(*b*) $\tan(990° - v)$,
(*c*) $\cot(360° - v)$,
(*d*) $\sec(v - 90°)$,
(*e*) $\sec(-180° - v)$,
(*f*) $\csc(630° + v)$.

5. Find the numerical values of

(*a*) $\sin 145°$,
(*b*) $\cos 246°$,
(*c*) $\tan 285°$,
(*d*) $\cot 572° 38'$,
(*e*) $\sec 321°$,
(*f*) $\csc 642° 50' 30''$,
(*g*) $\cot 21° 13' 38''$,
(*h*) $\sin 462° 31' 06''$.

6. If angle v is in the third or fourth quadrants, in what quadrant is $\frac{1}{2}v$, $2v$?

7. Construct the angle for each of the following and compute the values of all the functions:

(*a*) $\sin v = 1/2$,
(*b*) $\tan v = -\sqrt{3}$,
(*c*) $\cos v = -1/3$,
(*d*) $\csc v = 3$,
(*e*) $\sec v = 5/3$,
(*f*) $\tan v = -5$,
(*g*) $\cos v = 0.2$,
(*h*) $\sin v = -5/6$.

8. If angle v is an angle of a triangle and $\sin v = 0.2564$, find v.

9. Prove

(*a*) $\cos 240° \cos 120° - \sin 120° \cos 150° = 1$,
(*b*) $\tan 315° \sec 900° + \cot 495° \csc 450° = 0$.

10. Simplify

$$\sin (90° + v) \sin (180° + v) + \cos (90° + v) \cos (180° - v).$$

57. Functions of the sum and difference of two angles. The formulas for the functions of the sum and of the difference of two angles are of such importance that their derivation should be thoroughly mastered.

We consider first $\cos (u + v)$.

Place the angles u and v with reference to the co-ordinate axes as shown in Fig. 51.

Take a point P on the terminal side of $u + v$, and drop a perpendicular PQ to the terminal side of u.

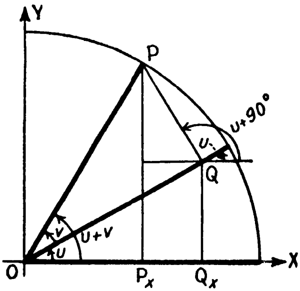


FIG. 51

Project points P and Q on the X-axis, calling the projections P_x and Q_x respectively.

Now if we take into consideration the signs of the line segments involved, we have (see section 16)

$$(33) \qquad OP_x = OQ_x + Q_xP_x.$$

But

$$OP_x = OP \cos (u + v),$$
$$OQ_x = OQ \cos u,$$
$$Q_xP_x = QP \cos (u + 90°) = - QP \sin u.$$

Therefore, substituting in (33), we get

$$OP \cos (u + v) = OQ \cos u - QP \sin u.$$

Dividing by OP gives

$$\cos (u + v) = \frac{OQ}{OP} \cos u - \frac{QP}{OP} \sin u.$$

But

$$\frac{OQ}{OP} = \cos v, \qquad \frac{QP}{OP} = \sin v,$$

and consequently

$$(34) \qquad \mathbf{\cos (u + v) = \cos u \cos v - \sin u \sin v.}$$

The foregoing proof will hold for all values of u and v if we are careful to take into consideration the proper sign of each function and of each line segment involved. It will be necessary, however, to consider as negative a segment measured backwards along the terminal side of an angle. Such a segment would be OP' in Fig. 52.

If in (34) we replace v by $-v$ we get
$\cos(u - v) = \cos u \cos(-v) - \sin u \sin(-v)$, or

$$(35) \qquad \cos(u - v) = \cos u \cos v + \sin u \sin v.$$

To develop the formula for $\sin(u + v)$ we use (34) and replace u by $90° - u$, and v by $-v$. We get

$$\cos(90° - u - v) = \cos(90° - u)\cos(-v) - \sin(90° - u)\sin(-v),$$

which becomes

$$(36) \qquad \sin(u + v) = \sin u \cos v + \cos u \sin v.$$

The formula for $\sin(u + v)$ might also have been obtained by projecting the points P and Q on the Y-axis.

If in (36) we replace v by $-v$ we get

$$\sin(u - v) = \sin u \cos(-v) + \cos u \sin(-v),$$

or

$$(37) \qquad \sin(u - v) = \sin u \cos v - \cos u \sin v.$$

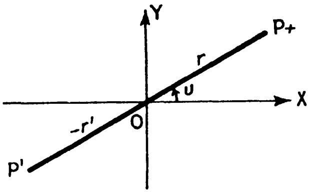


FIG. 52

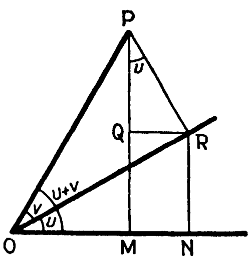


FIG. 53

The following proofs of the foregoing formulas are frequently used:

In Fig. 53 let angles u and v and also $u + v$ be acute.

Line PR is dropped from a point P on the terminal side of $u + v$ perpendicular to the terminal side of u, which is also the initial side of v.

The angle QPR is equal to the angle u because its sides are perpendicular to the sides of angle u right to right and left to left (plane geometry).

It is evident that

$$\text{(38)} \quad \sin(u+v) = \frac{MP}{OP} = \frac{MQ}{OP} + \frac{QP}{OP} = \frac{NR}{OP} + \frac{QP}{OP}.$$

But

$$\frac{NR}{OR} = \sin u, \qquad NR = OR \sin u,$$

and

$$\frac{QP}{RP} = \cos u, \qquad QP = RP \cos u.$$

Substituting these values for NR and QP in (38) we get

$$\sin(u+v) = \frac{OR}{OP}\sin u + \frac{RP}{OP}\cos u.$$

But it is further evident that

$$\text{(39)} \qquad \frac{OR}{OP} = \cos v, \qquad \frac{RP}{OP} = \sin v.$$

Therefore

$$\text{(36)} \qquad \sin(u+v) = \sin u \cos v + \cos u \sin v.$$

Also

$$\text{(40)} \quad \cos(u+v) = \frac{OM}{OP} = \frac{ON}{OP} - \frac{MN}{OP} = \frac{ON}{OP} - \frac{QR}{OP}.$$

But

$$\frac{ON}{OR} = \cos u, \qquad ON = OR \cos u,$$

$$\frac{QR}{PR} = \sin u, \qquad QR = PR \sin u.$$

Substituting in (40) we get

$$\cos(u+v) = \frac{OR}{OP}\cos u - \frac{PR}{OP}\sin u,$$

and using (39) we reduce this to

$$(34) \qquad \cos(u+v) = \cos u \cos v - \sin u \sin v.$$

If the angles u and v are both acute, but their sum obtuse, as shown in Fig. 54, with the lettering as indicated, the proof is the same as the above, word for word. It should be noted, however, that OM is negative, hence the cosine of $u+v$ is negative. This is taken care of in the fact that $ON - QR$ is a negative quantity, since we are subtracting a quantity QR from a smaller quantity ON.

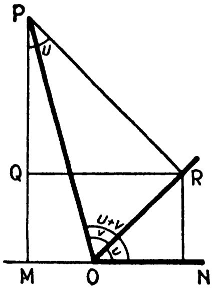


FIG. 54

In the second method of proof we have assumed that u and v are acute angles. If our formulas are derived by this method it will be necessary, if we wish to consider them perfectly general, to show that they are true whatever the values of u and v. This can be done without difficulty, but we shall omit the proof. An illustration of how the formulas may be extended is afforded by the following examples:

Example 1. Show that the formula for $\sin(u+v)$ holds true when one of the angles, say u, is obtuse.

Solution. Let $u = 90° + u'$. Then

$$\sin(u+v) = \sin(90° + u' + v) = \cos(u'+v).$$

But we know that since u' and v are both acute,

$$\begin{aligned}\cos(u'+v) &= \cos u' \cos v - \sin u' \sin v \\ &= \cos(u - 90°)\cos v - \sin(u-90°)\sin v \\ &= \sin u \cos v + \cos u \sin v.\end{aligned}$$

Therefore

$$\sin(u+v) = \sin u \cos v + \cos u \sin v.$$

Example 2. Show that the formula for $\cos(u+v)$ holds if u is in the third quadrant and v in the second.

Solution. Let $u = 180° + u'$, $v = 90° + v'$. Then

$$\cos (u + v) = \cos (180° + u' + 90° + v')$$
$$= \cos (270° + u' + v') = \sin (u' + v').$$

But u' and v' are both acute, and therefore

$$\sin (u' + v') = \sin u' \cos v' + \cos u' \sin v'$$
$$= \sin (u - 180°) \cos (v - 90°) + \cos (u - 180°) \sin (v - 90°)$$
$$= - \sin (180° - u) \cos (90° - v) + \cos (180° - u)[- \sin (90° - v)]$$
$$= - \sin u \sin v - \cos u(- \cos v)$$
$$= \cos u \cos v - \sin u \sin v.$$

Thus $\cos (u + v) = \cos u \cos v - \sin u \sin v$.

Assuming that the formulas for $\sin (u+v)$ and $\cos (u+v)$ are true for all angles, we may replace v by $- v$ and obtain formulas (35) and (37).

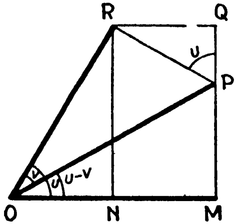


FIG. 55

Formulas (35) and (37) are also sometimes obtained by the following geometric proof:

Referring to Fig. 55, in which PR is perpendicular to OR, we see that

$$\sin (u - v) = \frac{MP}{OP} = \frac{MQ}{OP} - \frac{PQ}{OP} = \frac{NR}{OP} - \frac{PQ}{OP}.$$

But

$$\frac{NR}{OR} = \sin u, \qquad NR = OR \sin u,$$
$$\frac{PQ}{PR} = \cos u, \qquad PQ = PR \cos u.$$

Substituting we get

$$\sin (u - v) = \frac{OR}{OP} \sin u - \frac{PR}{OP} \cos u.$$

But

(41) $$\frac{OR}{OP} = \cos v, \qquad \frac{PR}{OP} \sin v =,$$

and consequently

(37) $\sin (u - v) = \sin u \cos v - \cos u \sin v.$

Moreover,

$$\cos (u - v) = \frac{OM}{OP} = \frac{ON}{OP} + \frac{NM}{OP} = \frac{ON}{OP} + \frac{RQ}{OP}.$$

But

$$\frac{ON}{OR} = \cos u, \qquad ON = OR \cos u,$$

$$\frac{RQ}{RP} = \sin u, \qquad RQ = RP \sin u.$$

Therefore

$$\cos (u - v) = \frac{OR}{OP} \cos u + \frac{RP}{OP} \sin u,$$

which by the use of (41) becomes

(35) $\cos (u - v) = \cos u \cos v + \sin u \sin v.$

If this method is used it will be necessary to extend the formulas to include all angles.

These four formulas may be arranged as follows for visualizing as an aid to the memory:

(36) $\boldsymbol{\sin (u + v) = \sin u \cos v + \cos u \sin v,}$
(37) $\boldsymbol{\sin (u - v) = \sin u \cos v - \cos u \sin v,}$
(34) $\boldsymbol{\cos (u + v) = \cos u \cos v - \sin u \sin v,}$
(35) $\boldsymbol{\cos (u - v) = \cos u \cos v + \sin u \sin v.}$

They are sometimes called the **addition** and **subtraction** formulas of the sine and cosine.

To find the tangent of $u + v$, and of $u - v$, we proceed as follows:

$$\tan (u + v) = \frac{\sin (u + v)}{\cos (u + v)} = \frac{\sin u \cos v + \cos u \sin v}{\cos u \cos v - \sin u \sin v}.$$

Since it is desirable to express the tangent of $u + v$ in terms of the tangents we divide numerator and denominator of the last fraction by $\cos u \cos v$, getting

$$\tan(u+v) = \frac{\dfrac{\sin u \cos v}{\cos u \cos v} + \dfrac{\cos u \sin v}{\cos u \cos v}}{\dfrac{\cos u \cos v}{\cos u \cos v} - \dfrac{\sin u \sin v}{\cos u \cos v}}.$$

Hence

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}. \tag{42}$$

In like manner

$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}. \tag{43}$$

For the cotangent of $u+v$ and of $u-v$ we obtain, by a method like that used in deriving the tangent formulas,

$$\cot(u+v) = \frac{\cot u \cot v - 1}{\cot v + \cot u}, \tag{44}$$

$$\cot(u-v) = \frac{\cot u \cot v + 1}{\cot v - \cot u}. \tag{45}$$

58. Functions of twice an angle. If in formulas (36), (34), (42), and (44) we substitute v for u we get

$$\sin(v+v) = \sin v \cos v + \cos v \sin v,$$

or

$$\sin 2v = 2 \sin v \cos v; \tag{46}$$

$$\cos(v+v) = \cos v \cos v - \sin v \sin v,$$

or

$$\cos 2v = \cos^2 v - \sin^2 v; \tag{47}$$

$$\tan(v+v) = \frac{\tan v + \tan v}{1 - \tan v \tan v},$$

or

(48) $$\tan 2v = \frac{2 \tan v}{1 - \tan^2 v};$$

$$\cot(v + v) = \frac{\cot v \cot v - 1}{\cot v + \cot v},$$

or

(49) $$\cot 2v = \frac{\cot^2 v - 1}{2 \cot v}.$$

Two other formulas for $\cos 2v$ may be derived as follows: Remembering that

$$\sin^2 v = 1 - \cos^2 v, \qquad \cos^2 v = 1 - \sin^2 v,$$

and substituting these equations separately in (47) we get, respectively,

(50) $$\cos 2v = 2 \cos^2 v - 1,$$

(51) $$\cos 2v = 1 - 2 \sin^2 v.$$

59. Functions of half an angle. Using the relation connecting sine and cosine, and the formula for $\cos 2v$, namely,

(12) $$\cos^2 v + \sin^2 v = 1,$$

(47) $$\cos^2 v - \sin^2 v = \cos 2v,$$

we get, by adding,

$$2 \cos^2 v = 1 + \cos 2v.$$

From this we get

(52) $$\cos^2 v = \frac{1 + \cos 2v}{2},$$

or

$$\cos v = \pm \sqrt{\frac{1 + \cos 2v}{2}}.$$

This formula gives the cosine of an angle in terms of double that angle. It is obviously equivalent to

(53) $$\cos \tfrac{1}{2}v = \pm \sqrt{\frac{1 + \cos v}{2}}.$$

By subtracting (47) from (17) we get

$$2\sin^2 v = 1 - \cos 2v,$$

(54) $$\sin^2 v = \frac{1 - \cos 2v}{2},$$

$$\sin v = \pm\sqrt{\frac{1 - \cos 2v}{2}},$$

or

(55) $$\sin \tfrac{1}{2}v = \pm\sqrt{\frac{1 - \cos v}{2}}.$$

The sign to be used before the square root symbols in the foregoing formulas depends on the quadrant in which the half angle lies.

Formulas (53) and (55) can also be obtained very easily from (50) and (51).

Dividing (55) by (53), we get

(56) $$\tan \tfrac{1}{2}v = \pm\sqrt{\frac{1 - \cos v}{1 + \cos v}}.$$

Multiplying numerator and denominator of the right-hand side by $\sqrt{1 - \cos v}$, we get

$$\tan \tfrac{1}{2}v = \frac{1 - \cos v}{\sqrt{1 - \cos^2 v}},$$

or

(57) $$\tan \tfrac{1}{2}v = \frac{1 - \cos v}{\sin v}.$$

If we multiply numerator and denominator of the right-hand side of (56) by $\sqrt{1 + \cos v}$, we get

(58) $$\tan \tfrac{1}{2}v = \frac{\sin v}{1 + \cos v}.$$

Similarly, we obtain the formulas

$$\cot \tfrac{1}{2}v = \pm\sqrt{\frac{1+\cos v}{1-\cos v}}, \tag{59}$$

$$\cot \tfrac{1}{2}v = \frac{\sin v}{1-\cos v}, \tag{60}$$

$$\cot \tfrac{1}{2}v = \frac{1+\cos v}{\sin v}. \tag{61}$$

60. Sums, differences, and products of two functions. Repeating, for convenience, formulas (34) to (37), we have

$$\sin(u+v) = \sin u \cos v + \cos u \sin v, \tag{36}$$
$$\sin(u-v) = \sin u \cos v - \cos u \sin v, \tag{37}$$
$$\cos(u+v) = \cos u \cos v - \sin u \sin v, \tag{34}$$
$$\cos(u-v) = \cos u \cos v + \sin u \sin v. \tag{35}$$

Adding (36) and (37) gives

$$\sin(u+v) + \sin(u-v) = 2\sin u \cos v, \tag{62}$$

from which we get, by interchanging the right and left sides of the equation and dividing by 2,

$$\sin u \cos v = \tfrac{1}{2}[\sin(u+v) + \sin(u-v)]. \tag{63}$$

By subtracting (37) from (36), we get

$$\sin(u+v) - \sin(u-v) = 2\cos u \sin v, \tag{64}$$

or

$$\cos u \sin v = \tfrac{1}{2}[\sin(u+v) - \sin(u-v)]. \tag{65}$$

Combining (34) and (35) by addition and subtraction, we get, respectively,

$$\cos(u+v) + \cos(u-v) = 2\cos u \cos v, \tag{66}$$
$$\cos(u+v) - \cos(u-v) = -2\sin u \sin v, \tag{67}$$

from which we obtain

$$\cos u \cos v = \tfrac{1}{2}[\cos(u+v) + \cos(u-v)], \tag{68}$$
$$\sin u \sin v = -\tfrac{1}{2}[\cos(u+v) - \cos(u-v)]. \tag{69}$$

Now if we let

$$u + v = A,$$
$$u - v = B,$$

we get by addition,

$$2u = A + B, \qquad u = \tfrac{1}{2}(A + B),$$

and by subtraction,

$$2v = A - B, \qquad v = \tfrac{1}{2}(A - B).$$

Substituting in (62), (64), (66), and (67), we obtain, respectively,

(70) $\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B),$

(71) $\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B),$

(72) $\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B),$

(73) $\cos A - \cos B = -2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B).$

Dividing (71) by (70), we get

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)}$$
$$= \cot \tfrac{1}{2}(A + B) \tan \tfrac{1}{2}(A - B),$$

or

(74) $$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}.$$

Exercise 23

1. Find the values of sin 15°, cos 15°, tan 15°, and cot 15°.

Solution. Since $15° = 45° - 30°$, and we know the values of the functions of 45° and 30°, we may proceed as follows:

$$\sin 15° = \sin (45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$$
$$= \tfrac{1}{2}\sqrt{2} \times \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{2} \times \tfrac{1}{2}$$
$$= \tfrac{1}{4}\sqrt{6} - \tfrac{1}{4}\sqrt{2}$$
$$= \tfrac{1}{4}(\sqrt{6} - \sqrt{2}).$$

In like manner find the other values.

2. Find sine, cosine, tangent, and cotangent of 75°. Compare with problem 1 above.

3. Find the functions of 60° from the functions of 30°.

4. Find the functions of 150° from the functions of 75°.

5. Given the cosine of 45°. Find the functions of 22° 30′.

6. If $\cos v = \frac{1}{12}$, find the functions of $\frac{1}{2}v$.

7. In a right triangle $\cos A = b/c$ and $\cos B = a/c$. Show that

$$\sin \tfrac{1}{2}A = \sqrt{\frac{c-b}{2c}}, \qquad \text{and} \qquad \sin \tfrac{1}{2}B = \sqrt{\frac{c-a}{2c}}.$$

8. Reduce to a product:

(*a*) $\sin 4v - \sin 2v$,
(*b*) $\sin 3v + \sin v$,
(*c*) $\cos 5v + \cos 9v$,
(*d*) $\sin \frac{1}{2}v + \sin \frac{3}{2}v$,
(*e*) $\cos 17° - \cos 23°$,
(*f*) $\sin (25° + v) - \sin (25° - v)$,
(*g*) $\cos (40° - v) - \cos (40° + v)$,
(*h*) $\sin v + \cos v$,
(*i*) $\dfrac{\sin v + \sin u}{\cos v - \cos u}$.

9. Prove

(*a*) $\sin 40° + \sin 20° = \cos 10°$,
(*b*) $\cos 20° - \cos 80° = \sin 50°$,
(*c*) $\cos 20° + \cos 100° + \cos 140° = 0$,
(*d*) $\dfrac{\sin 3v + \sin 5v}{\cos 3v - \cos 5v} = \cot v$,
(*e*) $\dfrac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°} = \frac{1}{3}\sqrt{3}$,
(*f*) $\sin 3v + 2 \sin 5v + \sin 7v = 4 \sin 5v \cos^2 v$,
(*g*) $\sin v = \cos (30° - v) - \cos (30° + v)$,

(*h*) $\tan v \pm \tan u = \dfrac{\sin (v \pm u)}{\cos v \cos u}$,

(*i*) $\sin^2 v - \sin^2 u = \sin (v + u) \sin (v - u)$,

(*j*) $\cos^2 v - \cos^2 u = -\sin (v + u) \sin (v - u)$
$= \cos (v + u) \cos (v - u)$.

10. Prove $\sin 3v = 3 \sin v - 4 \sin^3 v$.

Solution.

$$\sin 3v = \sin (2v + v) = \sin 2v \cos v + \cos 2v \sin v.$$

Substituting

$$\sin 2v = 2 \sin v \cos v, \quad \text{and} \quad \cos 2v = 1 - 2 \sin^2 v,$$

we get

$$\sin 3v = 2 \sin v \cos^2 v + \sin v - 2 \sin^3 v.$$

Substituting $1 - \sin^2 v$ for $\cos^2 v$, we get

$$\begin{aligned} \sin 3v &= 2 \sin v - 2 \sin^3 v + \sin v - 2 \sin^3 v \\ &= 3 \sin v - 4 \sin^3 v. \end{aligned}$$

11. Prove

(*a*) $\cos 3v = 4 \cos^3 v - 3 \cos v$,

(*b*) $\tan 3v = \dfrac{3 \tan v - \tan^3 v}{1 - 3 \tan^2 v}$,

(*c*) $\cot 3v = \dfrac{\cot^3 v - 3 \cot v}{3 \cot^2 v - 1}$,

(*d*) $\sin 4v = 4 \sin v \cos^3 v - 4 \sin^3 v \cos v$,

(*e*) $\cos 4v = 8 \cos^4 v - 8 \cos^2 v + 1$.

12. Prove

$$\frac{\tan (45^\circ + v) - \tan (45^\circ - v)}{\tan (45^\circ + v) + \tan (45^\circ - v)} = \sin 2v.$$

13. If A, B, and C are the angles of a triangle, prove that

(*a*) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$,

(*b*) $\sin 6A + \sin 6B + \sin 6C = 4 \sin 3A \sin 3B \sin 3C$,

(*c*) $\sin 6A + \sin 6B - \sin 6C = 4 \cos 3A \cos 3B \sin 3C$,

(*d*) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C,$
(*e*) $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C.$

14. Prove that

(*a*) $\cos (60° + v) + \sin (30° + v) = \cos v,$
(*b*) $\sin (60° + v) - \sin (60° - v) = \sin v,$
(*c*) $\sin (v + 60°) + \sin (v - 60°) = \sin v,$
(*d*) $\cos (v + 30°) + \cos (v - 30°) = \sqrt{3} \cos v,$

(*e*) $\tan (45° + v) = \dfrac{1 + \tan v}{1 - \tan v},$

(*f*) $\tan (45° - v) = \dfrac{1 - \tan v}{1 + \tan v}.$

15. If $\tan v = \frac{2}{3}$ and $\sin u = \frac{3}{13}$, find the functions of $(u + v)$.

16. Prove

(*a*) $\dfrac{\sin (v + u)}{\sin (v - u)} = \dfrac{\tan v + \tan u}{\tan v - \tan u} = \dfrac{\cot u + \cot v}{\cot u - \cot v},$

(*b*) $\dfrac{\cos (v + u)}{\cos (v - u)} = \dfrac{1 - \tan v \tan u}{1 + \tan v \tan u} = \dfrac{\cot v \cot u - 1}{\cot v \cot u + 1},$

(*c*) $\sin (v + u) \sin (v - u)$
$= \sin^2 v - \sin^2 u = \cos^2 u - \cos^2 v,$

(*d*) $\cos (v + u) \sin (v - u)$
$= \cos^2 v - \sin^2 u = \cos^2 u - \sin^2 v.$

17. Expand the following:

(*a*) $\sin (v + u + w)$.

Solution.

$$\begin{aligned} \sin (v + u + w) &= \sin (\overline{v + u} + w) \\ &= \sin (v + u) \cos w + \cos (v + u) \sin w \\ &= \sin v \cos u \cos w + \cos v \sin u \cos w \\ &\quad + \cos v \cos u \sin w - \sin v \sin u \sin w. \end{aligned}$$

(*b*) $\cos(v + u + w)$,
(*c*) $\tan(v + u + w)$,
(*d*) $\cot(v + u + w)$,
(*e*) $\sin(v + u - w)$,
(*f*) $\sin(v - u - w)$,
(*g*) $\cos(v - u - w)$.

18. Prove that

(*a*) $\dfrac{3\cos v + \cos 3v}{3\sin v - \sin 3v} = \cot^3 v$,

(*b*) $\sin v + \sin 3v + \sin 5v = \dfrac{\sin^2 3v}{\sin v}$,

(*c*) $\dfrac{\sin 3v + \sin 5v}{\cos 3v - \cos 5v} = \cot v$,

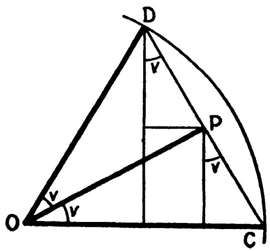
(*d*) $\sin 3v = 4 \sin v \sin(60° + v) \sin(60° - v)$,

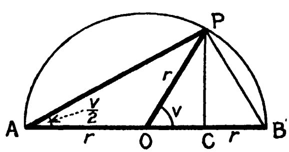
(*e*) $\cos(v + u) \sin u - \cos(v + w) \sin w$
$$= \sin(v + u)\cos u - \sin(v + w)\cos w,$$

(*f*) $\cos(v + u + w) + \cos(v + u - w)$
$$+ \cos(v - u + w) + \cos(u + w - v)$$
$$= 4\cos v \cos u \cos w.$$

19. In the accompanying figure the line *OP* is drawn perpendicular to the chord *CD*. Prove that

$$\sin 2v = 2 \sin v \cos v,$$
$$\cos 2v = \cos^2 v - \sin^2 v.$$





20. Use the accompanying figure to prove that

$$\sin \tfrac{1}{2}v = \sqrt{(1 - \cos v)/2}, \qquad \cos \tfrac{1}{2}v = \sqrt{(1 + \cos v)/2}.$$

Suggestion.

$AC = r(1 + \cos v),\quad AP = \sqrt{\overline{AC}^2 + \overline{CP}^2} = r\sqrt{2(1 + \cos v)}.$

21. Show that the area of a regular polygon of n sides inscribed in a circle of radius r is $\dfrac{nr^2}{2} \sin \dfrac{360°}{n}$.

61. $a \cos u \pm b \sin u$. It is always possible, and often very convenient, to change an expression of the form $a \cos u \pm b \sin u$ to the form $r \sin (u \pm v)$, or to the form $r \cos (u \pm v)$, where $r = \sqrt{a^2 + b^2}$ and v is an angle of constant value (provided a and b are constant). These transformations adapt the expressions to logarithmic computations, and are often of advantage in solving trigonometric equations. It will be shown below that we can thus combine two simple harmonic motions of the same period into a single simple harmonic motion of the same period.

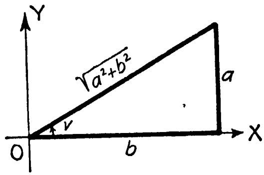


FIG. 56

The above transformations may be made in the following manner: Multiplying and dividing the expression $a \cos u + b \sin u$ by $\sqrt{a^2 + b^2}$ we get

$a \cos u + b \sin u$

$$= \sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}} \cos u + \frac{b}{\sqrt{a^2 + b^2}} \sin u\right).$$

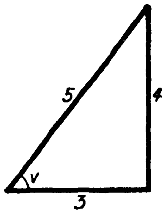
From Fig. 56 it is obvious that

$$\frac{a}{\sqrt{a^2 + b^2}} = \sin v, \qquad \frac{b}{\sqrt{a^2 + b^2}} = \cos v.$$

Hence

$$a \cos u + b \sin u = \sqrt{a^2 + b^2}\,(\sin v \cos u + \cos v \sin u)$$
$$= \sqrt{a^2 + b^2}\,\sin (u + v).$$

Example 1. Reduce $3 \cos u - 4 \sin u$ to the form $r \cos (u + v)$.
Solution. Multiply and divide by $\sqrt{3^2 + 4^2} = 5$. We get



$$3 \cos u - 4 \sin u = 5\left(\frac{3}{5}\cos u - \frac{4}{5}\sin u\right).$$

From the figure it is evident that

$$\frac{3}{5} = \cos v, \qquad \frac{4}{5} = \sin v.$$

Therefore

$$\begin{aligned} 3 \cos u - 4 \sin u &= 5(\cos v \cos u - \sin v \sin u) \\ &= 5 \cos (u + v) \\ &= 5 \cos (u + 53^\circ\, 08'). \end{aligned}$$

Example 2. Express the combination of two simple harmonic motions $x_1 = a \cos ct$ and $x_2 = b \sin ct$ as a single simple harmonic motion of the form $x = r \cos (ct - v)$.

Solution.

$$\begin{aligned} x = x_1 + x_2 &= a \cos ct + b \sin ct \\ &= \sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}}\cos ct + \frac{b}{\sqrt{a^2 + b^2}}\sin ct\right). \end{aligned}$$

Now if

$$\cos v = \frac{a}{\sqrt{a^2 + b^2}}, \qquad \sin v = \frac{b}{\sqrt{a^2 + b^2}},$$

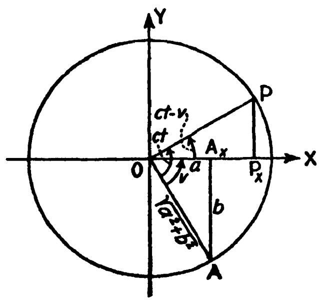
we have

$$x = \sqrt{a^2 + b^2}\,(\cos v \cos ct + \sin v \sin ct) = \sqrt{a^2 + b^2}\cos (ct - v).$$

A reference to the accompanying figure will show that $r \cos (ct - v)$ is represented by OP_x. Thus the simple harmonic motion

$$x = r \cos (ct - v) \qquad (a)$$

is a motion of the point P_x back and forth along the X-axis as the point P, of which P_x is the projection, moves with uniform velocity around the circumference of the circle.



It should be clear that when

$t = 0$, P is at the point A, and the position of P_x is given by

$$x = r \cos(-v) = r \cos v = OA_x.$$

That is, P_x is at the point A_x. This is evident from the figure, or it may be obtained by substituting $t = 0$ in (a).

The angle v is called the **phase angle.** The angle $ct - v$ is called the **epoch angle** of the single simple harmonic motion.

Exercise 24

1. Reduce $\sin u - \cos u$ to the form $r \sin(u - v)$, and find the angle v.

2. Reduce $\sin u + 2 \cos u$ to the form $r \sin(u - v)$, and find v.

3. Reduce each of the following expressions to the product of a number and a sine or cosine:

(a) $12 \cos u - 5 \sin u$,

(b) $\cos u + \sqrt{3} \sin u$,

(c) $\cos u + \sin u$,

(d) $3 \sin u - 2 \cos u$,

(e) $\frac{1}{2} \sin u + \frac{\sqrt{3}}{2} \cos u$,

(f) $0.4 \cos u + 1.5 \sin u$.

4. Reduce $0.3642 \cos u - 1.2476 \sin u$ to the product of a number and a sine or cosine. (*Suggestion.* Use logarithms.)

5. (a) Express the combination of the two simple harmonic motions $x_1 = 4 \cos 2t$, $x_2 = 3 \sin 2t$ as a single simple harmonic motion. (b) Find the amplitude and the phase angle of the single simple harmonic motion.

6. (a) Express the sum of the two simple harmonic motions $x_1 = 2 \sin(\pi t/4)$, $x_2 = 7 \cos(\pi t/4)$ as a single simple harmonic motion. (b) Find the amplitude and the phase angle of the single simple harmonic motion.

7. Reduce $m \cos u + n \sin u$ to the form $r \cos(u - v)$.

CHAPTER VII

OBLIQUE TRIANGLES

62. Introduction. One of the most important uses of trigonometry is its application to the solution of triangles. We have already considered the solution of the right triangle. We shall now take up the solution of triangles in general by methods that will not require breaking them up into right triangles.

The student has already learned from geometry how to construct triangles when three parts, one of which is a side, are given. He might use that information here to solve triangles graphically, but he knows that the method is not sufficiently accurate. The trigonometric method uses the same given conditions, but computes the unknown parts to a higher degree of accuracy.

Problems in the solution of oblique triangles may be divided into the following four cases, as was stated in section 40:

Case I. Two angles and one side given.

Case II. Two sides and an angle opposite one of them given.

Case III. Two sides and the included angle given.

Case IV. Three sides given.

Before proceeding to a systematic treatment of the solution of oblique triangles we shall develop certain laws which are used in handling the various cases.

63. The law of sines. *The sides of a triangle are proportional to the sines of the angles opposite them.*

Francis Vieta (1540–1603) was born in Poitou and died in Paris. As a lawyer he was constantly employed in the service of the state under Henry III and Henry IV. He studied mathematics as a diversion. So great was his devotion to the science that he remained in his room studying for several days at a time without eating or sleeping more than was necessary to sustain himself. His fame rests entirely on his achievements in mathematics. He is known as the father of modern algebra. He systematized and elaborated the methods of computing plane and spherical triangles with the aid of the six trigonometric functions. He was the first to apply algebraic transformations to trigonometry and gave the general formulas for sin nA and cos nA in terms of sin A and cos A. An equation of the 45th degree,

$$45y - 3795y^3 + 95634y^5 - \cdots + 945y^{41} - 45y^{43} + y^{45} = C,$$

was proposed to the mathematicians of the world by Adrianus Romanus of Holland. In a few minutes Vieta, by the substitution of trigonometric functions, solved the equation for twenty-three roots including the one root previously known. He failed to obtain all forty-five roots because those remaining involved negative sine functions, which had no meaning for him. He solved the cubic equation by first reducing it to the quadratic form.

Proof. Fig. 57 represents an acute triangle, Fig. 58 an obtuse triangle.

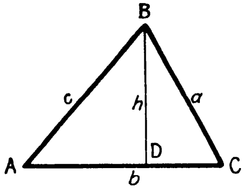


FIG. 57

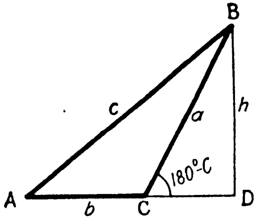


FIG. 58

Draw BD perpendicular to AC or AC prolonged. In either figure, right triangle BDC gives

$$\sin C = \frac{h}{a}, \tag{75}$$

since in Fig. 58 $\sin(180° - C) = \sin C$.

From right triangle ADB we have

$$\sin A = \frac{h}{c}. \tag{76}$$

Dividing (76) by (75) gives

$$\frac{\sin A}{\sin C} = \frac{a}{c},$$

or by alternation (see p. 8)

$$\frac{a}{\sin A} = \frac{c}{\sin C}. \tag{78}$$

Similarly, by drawing perpendiculars from A and C we get

$$\frac{b}{\sin B} = \frac{c}{\sin C}, \tag{79}$$

and

(80) $$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

Hence we may write the law of sines as follows:

(81) $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The student will observe that angle B was any angle in Fig. 57 and Fig. 58. Therefore we might have obtained (79) and (80) by an interchange of letters. This obviously gives the results that would have been obtained if each of the other angles had been selected in turn.

The only possible case not considered is the right triangle. The law is true for this as a special case.

For if C is the right angle, we have, by definition,

$$\frac{a}{c} = \sin A, \qquad \frac{b}{c} = \sin B,$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B} = c.$$

Now $\sin C = \sin 90° = 1$, and c may therefore be written $\frac{c}{1} = \frac{c}{\sin C}$. Consequently

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

That is, the law of sines holds for right triangles.

The law of sines may also be proved as follows:

Draw the perpendicular bisectors of the sides of the triangle ABC (Fig. 60). By plane geometry we know that these bisectors will meet in a point O, which is the center of the circumscribed circle. Draw this circle and connect its center with the vertices of the triangle. Let R be the radius of the circle, and let A, B, C represent the angles of the triangle.

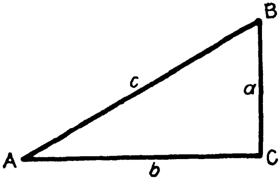


FIG. 59

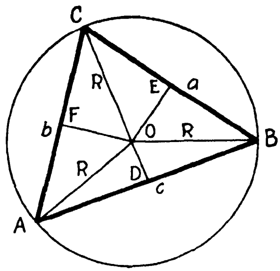


FIG. 60

Then angle $AOB = 2C$. (Why?)
Hence, angle $AOD = C$.
Now

$$\sin C = \sin AOD = \frac{AD}{R} = \frac{\frac{1}{2}c}{R} = \frac{c}{2R},$$

and consequently

$$2R = \frac{c}{\sin C}.$$

In like manner we can show that

$$2R = \frac{a}{\sin A},$$

$$2R = \frac{b}{\sin B}.$$

Therefore

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R = D, \tag{82}$$

where D is the diameter of the circumscribed circle.

If one of the angles of the triangle ABC is obtuse, the proof will require a slight modification.

The student will note that this proof shows how we may obtain the diameter of the circumscribed circle for any triangle.

HISTORICAL NOTE. The first clear statement and proof of the law of sines seem to have been given in a treatise on trigonometry by a Persian, **Nasir-Eddin** (1201–1274), who was also the first to treat trigonometry as a subject separate from astronomy.

64. The law of cosines. *The square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of these two sides by the cosine of their included angle.*

Proof. Fig. 61 represents the side c as greater than the projection of b upon it, while Fig. 62 represents the side c as shorter than the projection of b upon it.

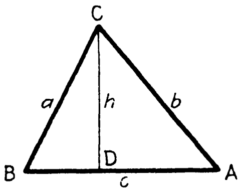


FIG. 61

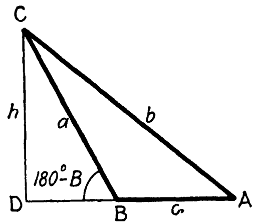


FIG. 62

In either figure,

(83) $$a^2 = h^2 + \overline{BD}^2.$$

In Fig. 61,

$$\overline{BD} = c - \overline{DA}.$$

In Fig. 62,

$$\overline{BD} = \overline{DA} - c.$$

Thus, in either case,

$$\overline{BD}^2 = \overline{DA}^2 - 2c \times \overline{DA} + c^2.$$

Substituting in (83), we get

$$a^2 = h^2 + \overline{DA}^2 - 2c \times \overline{DA} + c^2. \tag{84}$$

Now

$$h^2 + \overline{DA}^2 = b^2,$$
$$\overline{DA} = b \cos A.$$

Substituting these in (84) gives

$$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A.} \tag{85}$$

Similarly,

$$\boldsymbol{b^2 = c^2 + a^2 - 2ca \cos B,} \tag{86}$$
$$\boldsymbol{c^2 = a^2 + b^2 - 2ab \cos C.} \tag{87}$$

It will be noticed that if the angle C, for example, is a right angle, then $\cos C = 0$, and the last term in formula (87) drops out, leaving $c^2 = a^2 + b^2$, the theorem of Pythagoras. Thus the law of cosines holds for right triangles as well as for oblique triangles.

Formulas (86) and (87) are obtained from (85) by what is called a **cyclic** change of letters. This may be effected in the following way:

Arrange the letters in order around the circumference of a circle, as shown in Fig. 63. Then replace each letter in the given formula by the letter next in order. Thus a new formula may be obtained if

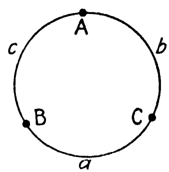


FIG. 63

b takes the place of a,
c takes the place of b,
a takes the place of c,

and similarly for the capital letters.

In this manner (85) is changed into (86), which in turn may be changed into (87).

Solving (85) for $\cos A$, (86) for $\cos B$, and (87) for $\cos C$, we get

$$(88)\qquad \begin{cases} \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}, \\ \cos B = \dfrac{c^2 + a^2 - b^2}{2ca}, \\ \cos C = \dfrac{a^2 + b^2 - c^2}{2ab}. \end{cases}$$

It is evident that the law of cosines in the form (85), (86), and (87) may be used to find the third side of a triangle when the other two sides and their included angle are known, while in the form (88) it may be used to find the angles when the three sides are known.

NOTE. The discovery of the law of cosines seems to be due to the French mathematician **Vieta** (1540–1603).

This law is becoming increasingly important in practical engineering, where its use is facilitated by calculating machines and by tables of squares and of products.

65. The law of tangents. *In any triangle the tangent of half the difference of any two angles is to the tangent of half their sum as the difference of the sides opposite the given angles is to the sum of these sides.*

Proof. From the law of sines we have

$$\frac{\sin A}{\sin B} = \frac{a}{b}.$$

By composition and division (see p. 8), we get

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{a - b}{a + b}.$$

But by (74), p. 148,

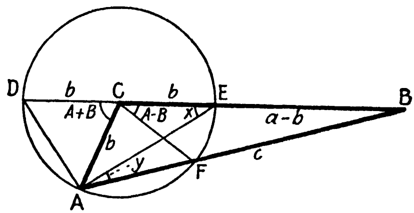
$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}.$$

Therefore,

$$\tan \tfrac{1}{2}(A - B) \over \tan \tfrac{1}{2}(A + B) = \frac{a - b}{a + b}. \tag{89}$$

There are a number of geometric proofs of the law of tangents, one of which follows.

In triangle ABC let a be greater than b. With C as center and b as radius draw a circle cutting a and c at E and F respectively. Prolong BC to meet the circle at D. Draw DA, AE, and CF.



Then $EB = a - b$, and $DB = a + b$.

Angle $DCA = A + B$ (exterior angle of a triangle).

Angle $x = \frac{1}{2}(A + B)$ (a central angle and an inscribed angle intercepting the same arc).

Angle $AFC = A$ (base angles of an isosceles triangle).

Angle $AFC =$ angle $FCB + B$ (exterior angle).

Hence angle $FCB = A - B$.

Moreover, angle $y = \frac{1}{2}(A - B)$ (a central angle and an inscribed angle intercepting the same arc).

Applying the law of sines to triangle ABE, we get

$$\frac{a - b}{c} = \frac{\sin y}{\sin AEB} = \frac{\sin \frac{1}{2}(A - B)}{\sin (180° - x)} = \frac{\sin \frac{1}{2}(A - B)}{\sin x},$$

or

$$\frac{a - b}{c} = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)}. \tag{90}$$

In like manner, we get from triangle ABD

$$\frac{a + b}{c} = \frac{\sin DAB}{\sin ADB}.$$

But angle DAB = angle $DAE + y = 90° + \frac{1}{2}(A - B)$, and angle $ADB = 90° - x = 90° - \frac{1}{2}(A + B)$.

Therefore

$$(91) \qquad \frac{a+b}{c} = \frac{\sin\,[90° + \frac{1}{2}(A - B)]}{\sin\,[90° - \frac{1}{2}(A + B)]} = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)}.$$

Dividing (90) by (91) gives

$$\frac{a-b}{a+b} = \frac{\sin \frac{1}{2}(A - B) \cos \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}.$$

Incidentally we are enabled to obtain two equations which are very serviceable in checking solutions of triangles.

Since $A + B = 180° - C$, we have $\frac{1}{2}(A + B) = 90° - \frac{1}{2}C$, and consequently

$$\sin \tfrac{1}{2}(A + B) = \sin\,(90° - \tfrac{1}{2}C) = \cos \tfrac{1}{2}C,$$
$$\cos \tfrac{1}{2}(A + B) = \cos\,(90° - \tfrac{1}{2}C) = \sin \tfrac{1}{2}C.$$

Using these results in (90) and (91) respectively, we get

$$(92) \qquad \begin{cases} \dfrac{a-b}{c} = \dfrac{\sin \frac{1}{2}(A - B)}{\cos \frac{1}{2}C}, \\[2ex] \dfrac{a+b}{c} = \dfrac{\cos \frac{1}{2}(A - B)}{\sin \frac{1}{2}C}. \end{cases}$$

These equations are known as **Mollweide's equations,** after the German astronomer **Karl Brandan Mollweide** (1774–1825), although they were published by others before his time.

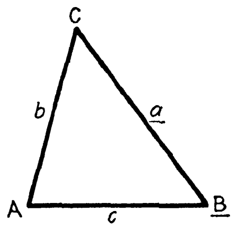
Each contains all six parts of the triangle, and hence an error is likely to be detected by a lack of agreement between the members of one of these equations.

Other formulas may of course be obtained from (92) by a cyclic changing of the letters.

Historical Note. The law of tangents seems first to have been expressed in the form in which we have given it by Vieta. It was however discovered about a decade earlier, but expressed in a more complicated form by a Danish physician and mathematician named **Thomas Finck** (or **Fink**) (1561–1656). Finck also pointed out the principal use of the law of tangents, namely, solving oblique triangles when two sides and the included angle are given. To him we also owe the names tangent and secant as applied to the functions which bear these names.

We are now ready to take up the solutions of the four cases mentioned in section 62.

66. Case I. Two angles and a side given. Problems of this type may be solved by the application of the law of sines.



Example. Given $a = 804$, $A = 79°\ 59'$, $B = 46°\ 36'$; find the other parts.

Solution.

$$C = 180° - (A + B)$$

$$b = \frac{a \sin B}{\sin A}$$

$$c = \frac{a \sin C}{\sin A}$$

Check

$$\frac{a-b}{c} = \frac{\sin \frac{1}{2}(A - B)}{\cos \frac{1}{2}C} = x$$

A	79° 59′
B	46° 36′
$A + B$	126° 35′
C	53° 25′
a	804
$\log \sin B$	9.86128
$\log a$	2.90526
$\text{colog} \sin A$	0.00667
$\log \sin C$	9.90471
$\log b$	2.77321
$\log c$	2.81664
b	593.21
c	655.60
$a - b$	210.79
$\log (a - b)$	2.32385
$\log c$	2.81664
$\log x$	9.50721
$A - B$	33° 23′
$\frac{1}{2}(A - B)$	16° 41′ 30″
$\frac{1}{2}C$	26° 42′ 30″
$\log \sin \frac{1}{2}(A - B)$	9.45822
$\log \cos \frac{1}{2}C$	9.95100
$\log x$	9.50722

The student may make a rough estimate of the results required, either mentally or by drawing a triangle to scale from the given parts and measuring the unknown parts.

The arrangement of the work in this problem is such as to place adjacent to each other the numbers to be added and subtracted, and at the same time minimize the amount of writing. This is one of the advantages of the form of solution used.

As a check we might have employed any formula or formulas which would involve the unknown parts with the known. For example, we might have used the law of cosines, $c^2 = a^2 + b^2 - 2ab \cos C$. This formula contains all of the unknown parts, with a, which is known. However, it is not suited to the use of logarithms, and is therefore not convenient in most cases.

A check to within 1 or 2 in the last place of decimals is sufficiently accurate, since the tabular values of logarithms are only approximate.

It should be noticed that in checking we do not need to find the quantities $(a - b)/c$ and $\sin \frac{1}{2}(A - B)/\cos \frac{1}{2}C$. It is sufficient if the logarithms of these quantities agree.

Exercise 25

1. Solve the following triangles:

(a) $B = 65° 25' 30''$, $C = 81° 24' 36''$, $b = 724.32$;
(b) $B = 38° 37' 24''$, $C = 75° 32' 48''$, $c = 129.63$;
(c) $A = 48° 29' 12''$, $C = 115° 33' 48''$, $a = 14.829$;
(d) $A = 68° 41' 30''$, $C = 110° 16' 30''$, $c = 9.4326$;
(e) $A = 60° 25' 31''$, $B = 69° 26'$, $c = 173$;
(f) $B = 43° 41' 18''$, $C = 75° 2' 42''$, $b = 81.5$;
(g) $A = 11° 11' 18''$, $C = 57° 37' 24''$, $c = 444.79$;
(h) $B = 20° 20' 12''$, $C = 12° 28' 30''$, $a = 673.75$;
(i) $A = 28° 14' 44''$, $C = 109° 32' 30''$, $b = 730.8$;
(j) $B = 102° 38' 16''$, $C = 20° 3' 9''$, $b = 479.36$;
(k) $B = 30° 36' 48''$, $C = 107° 15' 30''$, $b = 14.379$.

2. Two observers 2 miles apart on a horizontal plane observe a balloon in the same vertical plane with themselves. The angles of elevation are 50° and 55°. Find the distance from the balloon to each observer, and its height above the ground.

3. The bases of a trapezoid are a and b. The angles at the extremities of one base are A and B. Find the legs. Compute the results when $a = 20$, $b = 10$, $A = 65°$, $B = 40°$.

4. In a parallelogram we have given the diagonal d and the angles A and B which this diagonal makes with two adjacent sides. Find the sides. Solve when $d = 16.583$, $A = 36° 14'$, $B = 14° 27'$.

5. A man traveling due west along a straight road notices that when he is due south of a certain house the straight line drawn to a distant standpipe makes an angle of 30° with the road. A mile farther on the house and the pipe are northeast and northwest respectively. Find the distances from the house to the tower and from the house to the nearest point on the road.

6. A hill is inclined 35° with the horizon. An observer moves 120 yards farther away from the foot of the hill and finds the elevation of a point one-third of the way up the hill to be 10°. Find the height of the hill.

7. A line 1250 feet long is measured along the straight bank of a river. A point P is on the opposite bank. Angles BAP and PBA are found to be 65° 38′ and 54° 31′ respectively. Find the breadth of the river.

8. From a given position an observer notes that the angle of elevation of a rock is 47°. After walking 1000 feet towards the rock up a slope of 32° he finds the altitude to be 75°. Find the vertical height of the rock above each point of observation.

9. A flagstaff a feet high stands on the top of a building. From a point on the plain the angles of elevation of the top and of the bottom of the staff are observed to be v and u respectively. Show that the height of the tower is

$$\frac{a \sin u \cos v}{\sin (v - u)}.$$

10. One side of a parallelogram is 45.6 yards, and the angles which the diagonals make with this side are 36° 30′ and 54° 35′. Find the lengths of the diagonals.

11. Two observers, A and B, 1600 feet apart on a plain, observe an aeroplane at the same instant. The angle of elevation for A is 68° 25′. The angles which the lines of sight make with the lines joining A and B are 43° 27′ and 23° 45′ respectively. Find the height of the aeroplane.

12. Find the radius of the circle circumscribed about the triangle $A = 50°$, $B = 20°$, $a = 35$.

13. The sides of a triangular grass plot are 27, 32, and 35 feet. Find the radius of action of an automatic lawn sprinkler which will water all parts of the plot simultaneously.

67. Case II. Two sides and an angle opposite one of them given. The problem of solving a triangle when two sides and an angle opposite one of them are given may present difficulties that are not found in the other cases. This case is called the *ambiguous case*, for a reason that appears below. The student will recall a similar problem in plane geometry in which he was to construct a triangle having given two sides and an angle opposite one of them. The discussion here will be similar to the one needed there.

Let us suppose that the given angle is A, and that the given sides are a and b.

We first consider the case in which A is acute, as shown in Fig. 64.

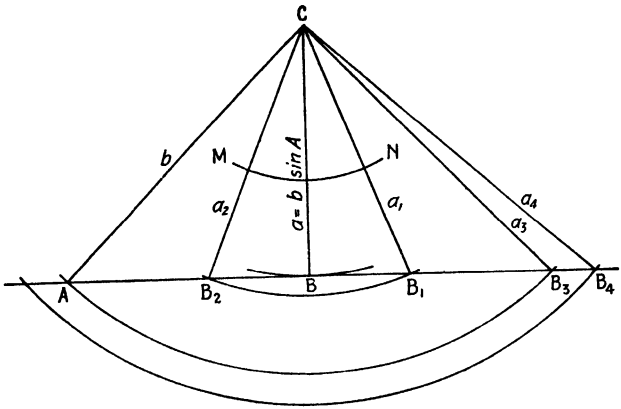


FIG. 64

It is evident that various cases may occur depending on the length of a with respect to b and also with respect to $b \sin A$.

Let us take a pair of compasses and with C as center and a as radius test the various cases by constructing arcs.

If a is less than $b \sin A$, the arc will be such as MN, and there will be no triangle.

If $a = b \sin A$, the arc will be tangent to the base line at the point B, and there will be but one triangle, the right triangle ABC.

If a is greater than $b \sin A$ but less than b, the arc will cut the base line in two points such as B_1 and B_2. Consequently we get two triangles, AB_1C and AB_2C. This is the condition which makes Case II ambiguous and gives it its name. Since either triangle satisfies the requirements of the problem, we must solve both triangles. It is to be noted that angle B_1 is acute, that angle B_2 is obtuse, and that they are supplementary.

If $a = b$, the arc passes through A, and we get but one solution, the isosceles triangle AB_3C.

If a is greater than b, there is but one triangle, such as AB_4C.

There are no other possible conditions when A is acute.

If A is a right angle, as shown in Fig. 65, it is evident

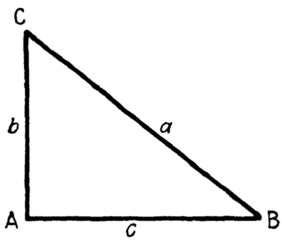


Fig. 65

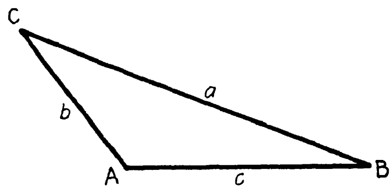


Fig. 66

that we can not have a triangle unless a is greater than b, in which case we have only one construction.

If A is obtuse, as shown in Fig. 66, the arc having a as radius cannot cut the base line on the proper side of the point A unless a is greater than b. Thus we have no triangle unless a is greater than b, and then we have only one.

Our conclusions may be summarized as follows:

$A < 90°$	
$a < b \sin A$	no solution
$a = b \sin A$	one solution (right triangle)
$b \sin A < a < b$	two solutions
$a = b$	one solution (isosceles triangle)
$a > b$	one solution

$A \gtreqless 90°$	
$a \leqq b$	no solution
$a > b$	one solution

Most of these points can be decided by inspection of the data. The only difficulty arises when two solutions are possible. The number of solutions can frequently be determined by drawing a rough figure to an approximate scale, or by estimating the value of $\sin A$. However, the question is finally settled when we know the value of $b \sin A$.

Case II is solved by the application of the law of sines. Using the formula

$$\sin B = \frac{b \sin A}{a},$$

we compute $\log \sin B$.

If $\log \sin B = 0$, then $\sin B = 1$, $B = 90°$; hence there is a right triangle.

If $\log \sin B > 0$, then $\sin B > 1$; hence there is no solution.

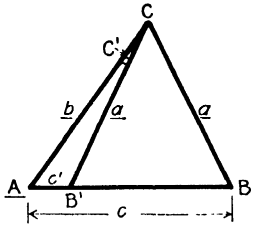
If $\log \sin B < 0$ and $a > b$, there is one solution.

If $\log \sin B < 0$ and $a < b$, there are two solutions.

Whether or not there are two solutions or only one will also be apparent when we attempt to find the angles C and C' by adding A and B, A' and B', and subtracting from 180°. If $A + B' \geqq 180°$, a second solution can obviously not exist.

In case there are two solutions it will be noted, as stated above, that angles B and B' are supplementary.

Example. Given $A = 38° \; 14' \; 12''$, $a = 8.716$, $b = 9.787$; find the other parts.



Solution.

$$\sin B = \frac{b \sin A}{a}$$

2 solutions
$\log \sin B < 0$
$a < b$
A acute

$$c = \frac{a \sin C}{\sin A}$$

$$c' = \frac{a \sin C'}{\sin A}$$

Check 1st solution

$$\tan \frac{1}{2}(C - A) = \frac{c - a}{c + a} \tan \frac{1}{2}(C + A)$$

a	8.716
b	9.787
A	38° 14′ 12″
$\log b$	0.99065
$\log \sin A$	9.79163
colog a	9.05968
$\log \sin B$	9.84196
B	44° 01′ 28″
B'	135° 58′ 32″
C	97° 44′ 20″
C'	5° 47′ 16″
$\log \sin C$	9.99602
$\log a$	0.94032
colog $\sin A$	0.20837
$\log \sin C'$	9.00365
$\log c$	1.14471
$\log c'$	0.15234
c	13.954
c'	1.4202
$c - a$	5.238
$c + a$	22.670
$C + A$	135° 58′ 32″
$\frac{1}{2}(C + A)$	67° 59′ 16″
$\log (c - a)$	0.71917
colog $(c + a)$	8.64455
$\log \tan \frac{1}{2}(C + A)$	0.39333
$\log \tan \frac{1}{2}(C - A)$	9.75705
$\frac{1}{2}(C - A)$	29° 45′ 00″
$\frac{1}{2}(C + A)$	67° 59′ 16″
C	97° 44′ 16″
A	38° 14′ 16″

The second solution may be checked in like manner. Small differences in seconds are to be expected, especially when using angles near 0° or 90°.

The law of tangents may be conveniently adapted as a check for such problems. This does not bring in the angle B, but since C is obtained by using B this is immaterial.

Exercise 26

1. Solve all possible triangles in the following set:

(*a*) $a = 62.518,\quad b = 72.932,\quad B = 98° 23' 30''$;
(*b*) $a = 429.15,\quad c = 328.12,\quad A = 130° 33' 42''$;
(*c*) $b = 3912.7,\quad c = 3526.5,\quad C = 35° 25' 48''$;
(*d*) $b = 12968,\quad c = 1529.6,\quad B = 38° 28' 38''$;
(*e*) $a = 86.425,\quad c = 73.463,\quad C = 49° 18' 56''$;
(*f*) $b = 223.46,\quad c = 327.92,\quad C = 116° 19' 36''$;
(*g*) $b = 32.492,\quad c = 52.392,\quad B = 27° 49' 16''$;
(*h*) $a = 67.291,\quad c = 110.97,\quad A = 37° 19' 50''$;
(*i*) $b = 45.872,\quad c = 56.321,\quad B = 20° 14' 28''$;
(*j*) $a = 57.147,\quad b = 46.703,\quad B = 19° 17' 45''$;
(*k*) $a = 15.8,\quad b = 17.9,\quad A = 21° 17' 32''$;
(*l*) $a = 23.62,\quad c = 45.63,\quad A = 36° 42' 50''$.

2. In a parallelogram one side is a, one diagonal is d, and the diagonals intersect at an angle A. Find the other diagonal. Compute when $a = 36$, $d = 64$, $A = 25° 45'$.

3. In a parallelogram $ABCD$, $AD = 4$ inches, $BD = 3.5$ inches, and $A = 48° 50'$. Find AB.

4. Two points A and B are on opposite sides of a bay. From C, a point at the head of the bay, A is distant 12,390 feet, and B is distant 15,800 feet. From A the lines of sight to C and B make an angle of $48° 35'$. Find the distance from A to B.

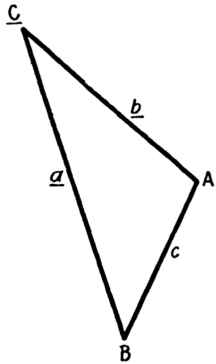
5. The two diagonals of a parallelogram make an angle of $22° 10' 12''$ with each other. One diagonal is 3325 feet and one side is 1732 feet. Find the length of the other diagonal.

6. A railroad embankment stands on a horizontal plane. It is required to find the distance from a point A in the plane to a point B at the top of the embankment. A point C is selected at the foot of the embankment and in the same

vertical plane with A and B. The distances AC and CB are measured and found to be 290 feet and 350 feet respectively; angle CAB measures $37° 45'$. Solve for the required length AB.

68. Case III. Two sides and the included angle given. This case can be conveniently solved by the law of tangents.

Example. Given $a = 55.14$, $b = 33.09$, $C = 30° 24'$; find the remaining parts.



Solution.

	a	55.14
	b	33.09
	C	$30° 24'$
	$a - b$	22.05
	$a + b$	88.23
$A + B = (180° - C)$	$A + B$	$149° 36'$
$\tan \frac{1}{2}(A - B) =$	$\frac{1}{2}(A + B)$	$74° 48'$
$\frac{a - b}{a + b} \tan \frac{1}{2}(A + B)$	$\log (a - b)$	1.34341
	$\text{colog } (a + b)$	8.05438
	$\log \tan \frac{1}{2}(A + B)$	0.56592
	$\log \tan \frac{1}{2}(A - B)$	9.96371
	$\frac{1}{2}(A - B)$	$42° 36' 32''$
	$\frac{1}{2}(A + B)$	$74° 48'$
	A	$117° 24' 32''$
$c = \frac{b \sin C}{\sin B}$	B	$32° 11' 28''$
	$\log b$	1.51970
	$\log \sin C$	9.70418
	$\text{colog} \sin B$	0.27348
	$\log c$	1.49736
Check	c	31.431
$c = \frac{a \sin C}{\sin A}$	$\log a$	1.74147
	$\log \sin C$	9.70418
	$\text{colog} \sin A$	0.05171
	$\log c$	1.49736

If the third side only is required and if the numbers are easy to handle, we may use the law of cosines to advantage.

For example, suppose $a = 4$, $b = 5$, and $C = 60°$.

From the law of cosines we have

$$\begin{aligned} c &= \sqrt{a^2 + b^2 - 2ab \cos C} \\ &= \sqrt{16 + 25 - 2 \times 20 \times \tfrac{1}{2}} \\ &= \sqrt{21} = 4.58 +. \end{aligned}$$

The angles may now be obtained by applying the law of sines.

If one angle is required we may proceed as follows:

In Fig. 67

$$\tan B = \frac{b \sin A}{c - m}.$$

Now $m = b \cos A$.

Hence, by substitution, we get

$$(93) \quad \tan B = \frac{b \sin A}{c - b \cos A}.$$

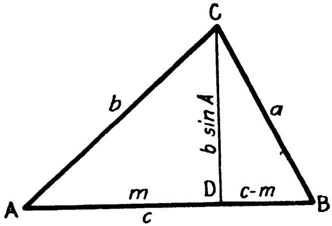


Fig. 67

The third angle may be obtained from a similar formula, or from the relation

$$C = 180° - (A + B).$$

The third side may now be found from the law of sines.

Neither of the last two methods is adapted to the use of logarithms.

Exercise 27

1. Solve the triangles in the following set:

(a) $a = 49.366$, $b = 26.437$, $C = 47° 16' 36''$;

(b) $a = 284.3$, $b = 286.5$, $C = 63° 38' 11''$;

(c) $a = 36.508$, $b = 8.9156$, $C = 132° 18' 20''$;

(*d*) $b = 247.81$, $c = 513.58$, $A = 147° \; 8' \, 49''$;
(*e*) $a = 67.375$, $c = 36.858$, $B = 12° \, 28' \, 30''$;
(*f*) $b = 284.12$, $c = 362.16$, $A = 126° \, 32' \, 10''$;
(*g*) $c = 395.71$, $a = 482.33$, $B = 137° \, 31' \, 12''$;
(*h*) $a = 0.0635$, $c = 0.10391$, $B = 83° \, 29' \, 22''$;
(*i*) $c = 0.00387$, $b = 0.00563$, $A = 68° \, 21' \, 40''$.

2. Two straight roads diverge from each other at an angle of 60°. One leads to a town 35 miles, the other to a town 20 miles distant from the intersection of the roads. Find the distance between the two towns.

3. Two ships leave a harbor at the same time. One sails northeast at the rate of 9 miles an hour, the other sails due north at 10.5 miles an hour. Find the distance between the ships at the end of 2 hours.

4. The points A and B are on the opposite sides of a mountain, while C is a point visible from each of them. From A to C and from B to C are 10 and 12 miles respectively, and the angle ACB is 70°. Find the distance between A and B.

5. The length of a certain lake subtends an angle of 46° 25′ from a point of observation. The distances from this point to the ends of the lake are 347 and 295 feet. Find the length of the lake.

6. A given parallelogram has a side a and a diagonal d. The angle that the given diagonal makes with the given side is v. Find the other diagonal and the other side.

7. Two circles whose radii are 12 and 16 inches intersect. The angle between the tangents drawn from the point of intersection is 29° 30′. Find the length of the line connecting the centers of the circles.

8. A military observer notes two enemy batteries which subtend at his observation post an angle of 40°. The

interval between the flash and the report of a gun is 5 seconds for one battery and 4 seconds for the other. If the velocity of sound is 1140 feet a second, how far apart are the two batteries?

69. Case IV. Three sides given. This problem may be solved by formulas (88). However these formulas are not adapted to the use of logarithms, and we shall develop them into other formulas that are so adapted.

It is convenient to obtain the angles from functions of half the angles.

Starting with the first of equations (88), namely

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \tag{94}$$

and subtracting both sides from 1, we get

$$\begin{aligned} 1 - \cos A &= 1 - \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{2bc - b^2 - c^2 + a^2}{2bc} \\ &= \frac{a^2 - (b - c)^2}{2bc} \\ &= \frac{(a + b - c)(a - b + c)}{2bc}. \end{aligned} \tag{95}$$

Let

$$2s = a + b + c. \tag{96}$$

Then

$$\frac{a + b + c}{2} = s. \tag{97}$$

Subtracting $2a$ from both sides of (96) gives

$$2(s - a) = b + c - a,$$

or

$$\frac{b + c - a}{2} = s - a.$$

Similarly,

$$\frac{c + a - b}{2} = s - b,$$

$$\frac{a + b - c}{2} = s - c.$$

Substituting these results in (95), we get

$$1 - \cos A = \frac{2(s - b)(s - c)}{bc}. \tag{98}$$

Adding 1 to both sides of (94), we get

$$\begin{aligned} 1 + \cos A &= 1 + \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{2bc + b^2 + c^2 - a^2}{2bc} \\ &= \frac{(b + c)^2 - a^2}{2bc} \\ &= \frac{(b + c + a)(b + c - a)}{2bc} \\ &= \frac{2s(s - a)}{bc}. \end{aligned} \tag{99}$$

Now from formulas (55) and (54) we get, respectively,

$$1 - \cos A = 2 \sin^2 \tfrac{1}{2}A,$$
$$1 + \cos A = 2 \cos^2 \tfrac{1}{2}A.$$

Substituting in (98) and (99), dividing out 2, and extracting the square root, we obtain

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(s - b)(s - c)}{bc}}, \tag{100}$$

$$\cos \tfrac{1}{2}A = \sqrt{\frac{s(s - a)}{bc}}. \tag{101}$$

Dividing (100) by (101), we get

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \tag{102}$$

By a cyclic change of the letters we get similar formulas for the other two angles. It is of advantage to solve for all three angles so that an easy check may be had in adding them.

There is a choice of three formulas in getting any angle. However, for angles near 0° the cosine is changing slowly and so will not give an accurate result, and for angles near 90° the same may be said of the sine. In the former case the sine, and in the latter the cosine, may be used to advantage. In general the tangent is to be preferred.

In case all the angles are computed from these formulas, (102) may be put in a more convenient form.

If we multiply both numerator and denominator of the fraction under the radical sign by $(s-a)$ we get

$$\begin{aligned}\tan \tfrac{1}{2}A &= \sqrt{\frac{(s-a)(s-b)(s-c)}{s(s-a)^2}}\\ &= \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\end{aligned}$$

If we set

$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = r, \tag{103}$$

then

$$\tan \tfrac{1}{2}A = \frac{r}{s-a}. \tag{104}$$

In like manner,

$$\tan \tfrac{1}{2}B = \frac{r}{s-b}, \tag{105}$$

$$\tan \tfrac{1}{2}C = \frac{r}{s-c}. \tag{106}$$

We can now compute $\log r$ once for all three angles and having obtained it, there is little work to be added to solve the problem completely.

The half angle formulas may be derived geometrically as follows:

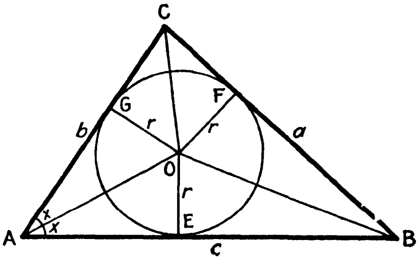


FIG. 68

As shown in Fig. 68, a circle with center O and radius r is inscribed in the triangle ABC.

Let

$$2s = a + b + c.$$

Now

$$AE = AG,$$
$$BE = BF,$$
$$CG = CF.$$

Therefore

$$s = AE + BF + CF = AE + a,$$

and

$$AE = s - a.$$

Consequently,

$$\tan \tfrac{1}{2}A = \frac{r}{s - a}.$$

In like manner, or by changing the letters,

$$\tan \tfrac{1}{2}B = \frac{r}{s - b},$$

$$\tan \tfrac{1}{2}C = \frac{r}{s - c}.$$

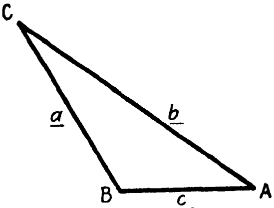
If we compare these results with (104), (105), (106), it is evident that we have incidentally given a new meaning to the expression

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\,.$$

It is the value of the radius of the inscribed circle.

Example. Given $a = 51$, $b = 65$, $c = 20$; solve for the angles.

Solution.



$$\tan \tfrac{1}{2}A = \frac{r}{s-a}$$

$$\tan \tfrac{1}{2}B = \frac{r}{s-b}$$

$$\tan \tfrac{1}{2}C = \frac{r}{s-c}$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

a	51
b	65
c	20
$2s$	136
s	68
$s-a$	17
$s-b$	3
$s-c$	48
$\log(s-a)$	1.23045
$\log(s-b)$	0.47712
$\log(s-c)$	1.68124
colog s	8.16749
$\log r^2$	1.55630
$\log r$	0.77815
$\log \tan \frac{1}{2}A$	9.54770
$\log \tan \frac{1}{2}B$	0.30103
$\log \tan \frac{1}{2}C$	9.09691
$\frac{1}{2}A$	19° 26′ 24″
$\frac{1}{2}B$	63° 26′ 6″
$\frac{1}{2}C$	7° 07′ 30″
A	38° 52′ 48″
B	126° 52′ 12″
C	14° 15′
Check $A + B + C$	180°

NOTE. It is an easy and valuable check to add the values of $s - a$, $s - b$, and $s - c$ as soon as these have been found. Since this gives $3s - a - b - c = 3s - 2s = s$, the sum should be equal to the value of s. This simple check often prevents working the entire problem with an incorrect value for one of the expressions $s - a$, $s - b$, $s - c$.

For convenience log r may be written at the bottom of a slip of paper and placed in turn above log $(s - a)$, log $(s - b)$, log $(s - c)$. This is a device used by computers to facilitate speed and accuracy in this and similar situations.

Exercise 28

1. Solve the following triangles:

(*a*)	$a = 125.36$,	$b = 176.43$,	$c = 101.23$;
(*b*)	$a = 23.586$,	$b = 25.743$,	$c = 10.047$;
(*c*)	$a = 10.057$,	$b = 19.436$,	$c = 15.067$;
(*d*)	$a = 22.498$,	$b = 24.672$,	$c = 31.526$;
(*e*)	$a = 7225$,	$b = 5846$,	$c = 11912$;
(*f*)	$a = 0.563$,	$b = 0.7704$,	$c = 0.35892$;
(*g*)	$a = 0.0065$,	$b = 0.009675$,	$c = 0.003476$;
(*h*)	$a = 50014$,	$b = 70023$,	$c = 90054$;
(*i*)	$a = 121.62$,	$b = 9.821$,	$c = 113.94$;
(*j*)	$a = 42.391$,	$b = 23.168$,	$c = 51.833$;
(*k*)	$a = 0.0587$,	$b = 11.675$,	$c = 0.09765$.

2. Three towns A, B, and C are situated so that $AB = 300$ miles, $AC = 194$ miles, and $BC = 160$ miles, B being due north of C. How many miles is A north of C, and how many miles west of C?

3. Three towns are so situated that the first is 205 miles from the second, and 185 miles from the third. The second is 151 miles due north from the third. Find the directions from the first town to the second and the third.

4. The sides of a triangle are 14, 16, and 18 inches respectively. Find the length of the perpendicular from the vertex of the largest angle to the side opposite.

5. What is the visual angle of an object 8 feet long when the ends of the object are 6 feet and 9 feet from the eye of the observer?

6. If the sides of a triangle are 16, 20, and 27, find the length of the line bisecting the largest angle and drawn to the opposite side.

7. The sum of the sides of a triangle is 200. The angles are in the ratio of 1 : 2 : 3. Find the sides.

8. (*a*) Find the radius of the largest circle that can be cut from a triangular piece of tin whose sides are 12, 17, and 22 inches. (*b*) Locate the center of the circle.

Exercise 29

1. In measuring across a barrier from A to B, a point C is selected so that CA is 586 feet, CB 597 feet, and angle ACB 87° 45′. Find the distance from A to B.

2. If the sides of a triangle are to each other as 5 : 7 : 9, what are the angles?

3. Find the area of a circle circumscribed about the triangle whose sides are 12, 24, 30.

4. A grass plot is a triangle whose sides are 47.5 feet, 64.5 feet, and 85 feet respectively. What is the radius of the largest circular flower bed that can be made in the plot?

5. A tower 155 feet high is on the top of a hill. At a point 1111 feet down the hill the tower subtends an angle of 6° 47′ 10″. What is the angle of inclination of the hill?

6. A ladder 45 feet long is set with one end 17 feet from the base of a buttress, the other reaches a point 38 feet up the face of the buttress. Find the angle of inclination of the face of the buttress with the vertical.

7. What is the length of a diagonal cutting off one vertex of a regular polygon of 10 sides, if the radius is 32 feet?

8. Find the number of square yards of canvas in a conical tent with a circular base, the semi-vertical angle (the angle between the axis of the cone and an element) being 30°, and the central pole 14 feet high.

9. Find the projection of the median of an equilateral triangle of side a upon a side.

10. Find the sine of the angle formed at the intersection of the diagonals of a cube.

11. A rectangular solid is 5 inches long, 4 inches wide, and 3 inches high. What is the tangent of the angle formed by the intersection of the diagonals?

12. When the moon is setting the angle at the moon subtended by the radius of the earth is $57' \, 3''$. If the radius of the earth is 3960 miles, what is the distance of the moon from the center of the earth?

13. The angle at the center of the earth subtended by the radius of the sun is $16' \, 2''$. If the sun is 93,000,000 miles from the earth, what is its diameter in miles?

14. A man in a balloon observes that an object on the ground has an angle of depression of $36° \, 30'$. The balloon moves east 2.5 miles at the same height and the angle of depression of the same object is $22° \, 15'$. What is the height of the balloon?

15. A football player stands c feet behind and opposite the middle of the goal. He sees the angle of elevation of the nearer crossbar to be v, and that of the farther one u. Show that the length of the field is $c(\tan v \cot u - 1)$.

16. Solve the following triangles:

(*a*) $A = 1° \, 15' \, 20''$, $b = 127$, $c = 254$;
(*b*) $a = 156$, $b = 98$, $c = 58$;
(*c*) $B = 91° \, 40'$, $C = 1° \, 20'$, $c = 24$;
(*d*) $B = 50' \, 15''$, $a = 15$, $b = 5$.

70. Areas of oblique triangles.

1. *When two sides and the included angle are given.* In Fig. 69 let A, b, and c be given, and let S denote the area of the triangle. Since the area is equal to half the product of the base by the altitude, we have

$$S = \tfrac{1}{2}bh,$$

or

(107) $$\boldsymbol{S = \tfrac{1}{2}bc \sin A.}$$

Similarly,

$$\boldsymbol{S = \tfrac{1}{2}ca \sin B,}$$
$$\boldsymbol{S = \tfrac{1}{2}ab \sin C.}$$

2. *When a side and two angles are given.* We can always reduce this problem to one in which the two given angles are adjacent to the given side. For if two angles are given we can always find the third. Let us therefore assume that c, A, and B are given.

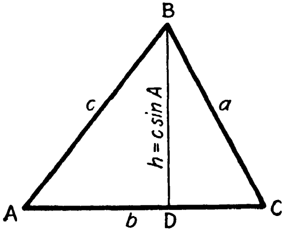


FIG. 69

We have already shown that

(107) $$S = \tfrac{1}{2}bc \sin A.$$

From the law of sines we have

$$b = \frac{c \sin B}{\sin C}.$$

Substituting this value of b in (107), we get

(108) $$S = \frac{c^2 \sin A \sin B}{2 \sin C}.$$

But

$$\sin C = \sin (180^\circ - \overline{A + B}) = \sin (A + B),$$

and therefore (108) reduces to

(109) $$\boldsymbol{S = \frac{c^2 \sin A \sin B}{2 \sin (A + B)}.}$$

Of course we also have

$$\boldsymbol{S = \frac{a^2 \sin B \sin C}{2 \sin (B + C)},}$$
$$\boldsymbol{S = \frac{b^2 \sin C \sin A}{2 \sin (C + A)}.}$$

3. *When three sides are given.*

From (46) we may write

$$\sin A = 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A. \tag{110}$$

Repeating (100) and (101), we have

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}, \tag{100}$$

$$\cos \tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}. \tag{101}$$

Substituting these expressions for $\sin \frac{1}{2}A$ and $\cos \frac{1}{2}A$ in (110) gives

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}. \tag{111}$$

But

$$S = \tfrac{1}{2}bc \sin A. \tag{107}$$

Substituting in (107) the expression for $\sin A$ given in (111), we get

$$\boldsymbol{S = \sqrt{s(s-a)(s-b)(s-c)}}. \tag{112}$$

The student will recall that this is the formula derived in geometry for obtaining the area of a triangle when three sides are given. It is known as **Hero's** (or **Heron's**) formula. (See p. 12.)

4. *When two sides and an angle opposite one of them are given,* we can find the other two angles by the law of sines, and then find the area by the method of 1 or by that of 2.

Exercise 30

1. Solve for the areas of the triangles in the first problem of each of the exercises 25 to 28 inclusive.

2. Find the area of an equilateral triangle in terms of a side.

3. Two sides of a triangle are 14.28 chains and 7.64 chains. The included angle is 45° 33′. What is the area? (A **surveyor's** or **Gunter's chain** is 66 feet long.)

4. The sides of a triangular field which contains 15 acres are as 3 : 5 : 7. Find the sides.

5. Two adjacent sides of a triangle are 55.6 rods and 35.7 rods, and the area is 375 square rods. Find the third side and the angles.

6. Two adjacent sides of a parallelogram are 15 yards and 12 yards, and the angle between them is 75°. Find the area of the parallelogram.

7. Show that the area of a quadrilateral is one-half the product of its diagonals into the sine of the included angle.

8. Show that the area of a regular polygon having n sides, of which one side is s, is $\frac{ns^2}{4}\cot\frac{180°}{n}$.

9. One side of a regular decagon is 45. Find the area.

71. Solution of triangles by the use of the haversine. It seems that it is not generally known that plane trigonometry can be simplified both in teaching and in computation by the use of the haversine, a function of especial importance in navigation and in nautical problems. It may be used to advantage in the solution of oblique triangles in Cases III and IV:

Case III. *When two sides and the included angle are given.*
Case IV. *When the three sides are given.*

The proofs of the formulas which are used in solving these two cases are simple and direct, and in Case IV the radical symbols disappear. If the solution of any case is carried out in the usual way, haversines may be employed for checking results.*

By definition,

$$\text{hav } A = \tfrac{1}{2}(1 - \cos A).$$

By employing the law of cosines, we get

$$2 \text{ hav } A = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{a^2 - (b - c)^2}{2bc}.$$

Clearing of fractions, we get

$$4\,bc \text{ hav } A = a^2 - (b - c)^2,$$

and hence

(113) $$\boldsymbol{a^2 = (b - c)^2 + 4bc \text{ hav } A.}$$

In like manner,

$$\boldsymbol{b^2 = (c - a)^2 + 4ca \text{ hav } B,}$$

$$\boldsymbol{c^2 = (a - b)^2 + 4ab \text{ hav } C.}$$

If a, b, and c are given and we wish to find A, we solve for hav A, obtaining

$$\text{hav } A = \frac{a^2 - (b - c)^2}{4bc} = \frac{(a + b - c)(a - b + c)}{4bc},$$

or

(114) $$\boldsymbol{\text{hav } A = \frac{(s - b)(s - c)}{bc}.}$$

* Further information with regard to the use of the haversine may be found in an article entitled "Concerning Haversines in Plane Trigonometry," by G. W. Evans in the *American Mathematical Monthly*, vol. XXVI, no. 4 (Feb., 1919), p. 69. Any good treatise on navigation will show the nautical uses of haversines.

Also

$$\text{hav } B = \frac{(s-c)(s-a)}{ca},$$

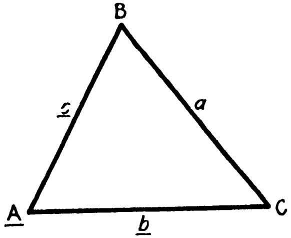
$$\text{hav } C = \frac{(s-a)(s-b)}{ab}.$$

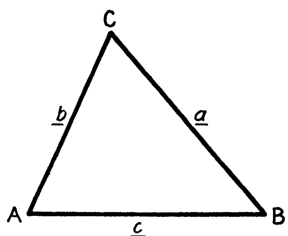
Special tables of haversines or logarithms of haversines are needed to apply these formulas. (See Note, p. 26.)

Example 1. Given $A = 50°$, $b = 40$, $c = 30$; find a.
Solution. The solution of Case III is not adapted to the use of logarithms, hence we use natural values.

$$a = \sqrt{(b-c)^2 + 4bc \text{ hav } A}$$

A	50°
b	40
c	30
bc	1200
hav A	0.17861
bc hav A	214.332
$b - c$	10
$(b-c)^2$	100
$4bc$ hav A	857.328
a^2	957.328
a	30.94





Example 2. Given $a = 60$, $b = 50$, $c = 40$; find the angles.
Solution. Case IV can be conveniently solved by logarithms.

$$\text{hav } A = \frac{(s-b)(s-c)}{bc}$$
$$\text{hav } B = \frac{(s-c)(s-a)}{ac}$$
$$\text{hav } C = \frac{(s-a)(s-b)}{ab}$$

a	60
b	50
c	40
$2s$	150
s	75
$s-a$	15
$s-b$	25
$s-c$	35
$\log(s-a)$	1.17609
colog a	8.22185
$\log(s-b)$	1.39794
colog b	8.30103
log hav C	9.09691
$\log(s-b)$	1.39794
colog b	8.30103
$\log(s-c)$	1.54407
colog c	8.39794
$\log(s-a)$	1.17609
colog a	8.22185
log hav A	9.64098
log hav B	9.33995
A	82° 49′ 09″
B	55° 46′ 18″
C	41° 24′ 34″
Check $A+B+C$	180° 00′ 01″

Exercise 31

1. Solve the following triangles by the use of haversines:

(a) $a = 250,\quad b = 650,\quad c = 501;$
(b) $a = 630,\quad b = 106,\quad c = 700;$
(c) $a = 53,\quad b = 67,\quad c = 75;$
(d) $a = 6.5,\quad b = 7.35,\quad c = 9.15;$
(e) $a = 24,\quad b = 16,\quad C = 35° 30';$
(f) $A = 65° 40',\quad b = 65,\quad c = 109.$

2. Find the value of z in the equation $z = \text{hav}(L-d) + \cos d \cos L \text{ hav } t$, letting $t = 13° 21' 38''$, $d = 12° 11' 40''$, $L = 41° 18' 19''$.

72. Vectors. If an object is at the point A in Fig. 70 and is displaced or moved to the point B, the displacement may be represented by the directed line-segment AB. It will be noted that this line-segment represents both the amount and the direction of the displacement. Now let AD represent another displacement. If an object originally at A is given both of these displacements, it will arrive at the point C, the figure $ABCD$ being a parallelogram. The order in which the displacements are given is immaterial; that is, the object may be moved from A to B and then from B to C, or it may be moved from A to D and then from D to C. The displacement AC is called the **resultant** of the displacements AB and AD. (Compare section 44.) Obviously the resultant is a diagonal of the parallelogram of which AB and AD are sides.

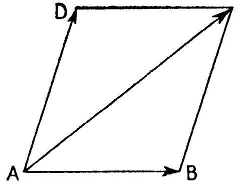


Fig. 70

It can be proved experimentally that two forces acting on the same object combine into a resultant force according to this same parallelogram law. Thus, if in Fig. 70 AB and AD represent in magnitude and direction two forces acting on an object at A, then the diagonal AC will represent in magnitude and direction the resultant of the two given forces. That is, the single force represented by AC will have the same effect on the object as the two forces represented by AB and AD.

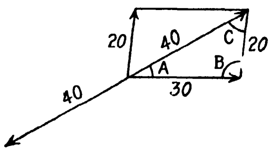
Velocities and many other directed quantities (quantities which have both magnitude and direction) also combine according to the parallelogram law. Such quantities are called **vector** quantities. The directed line-segment representing the vector quantity is called a vector.

The resultant of any two vectors may of course be found graphically by completing the parallelogram of which they form adjacent sides and drawing the proper diagonal. This is called the geometric addition of the vectors. Two vectors may be added geometrically by placing the initial point of one on the terminal point of the other, preserving the proper direction of each vector, and then drawing a third vector from the initial point of the first to the terminal point of the second. This can readily be seen by reference to Fig. 70.

The addition of line-segments described on p. 35 is a special case of addition of vectors in which both vectors have the same direction or opposite directions.

Trigonometry is essential in dealing with vectors. Its application may be illustrated by the following examples:

Example 1. Three forces of 20, 30, and 40 pounds, respectively, are in equilibrium. Find the angles that they make with each other.



Solution. Since the forces are in equilibrium, any one of them must be equal in magnitude and opposite in direction to the resultant of the other two. That is, we have a parallelogram in which the diagonal is, for example, 40, and in which the two sides are 20 and 30. (See figure.) Our problem is thus reduced to that of finding the angles of a triangle whose sides are 20, 30, and 40. This may of course be done by employing the law of tangents, the law of cosines, or haversines. Since the numbers are simple, let us use the law of cosines. The numerical part of the solution follows.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

a	20
b	40
c	30
a^2	400
b^2	1600
c^2	900
$b^2 + c^2 - a^2$	2100
$2bc$	2400
$c^2 + a^2 - b^2$	-300
$2ca$	1200
$a^2 + b^2 - c^2$	1100
$2ab$	1600
$\cos A$	0.8750
$\cos B$	-0.2500
$\cos C$	0.6875
A	28° 57′ 00″
B	104° 28′ 40″
C	46° 34′ 00″
Check $A + B + C$	179° 59′ 40″

Therefore,

angle between 40 lb. and 30 lb. forces = $180° - A$ = $151° 03' 00''$
angle between 20 lb. and 30 lb. forces = $180° - B$ = $75° 31' 20''$
angle between 20 lb. and 40 lb. forces = $180° - C$ = $133° 26' 00''$

Check $360° 00' 20''$

It may be noted that since the three forces are represented by the sides of the triangle ABC, the forces are proportional to the sines of the opposite angles.

Example 2. An aeroplane having a speed of 70 miles an hour in still air is pointed in a direction 30° east of north. If a wind having a velocity of 15 miles an hour is blowing from the northwest, what will be the actual speed and the actual direction of the aeroplane relative to the ground?

Solution. Referring to Fig. 71, we see that the vector AB represents the velocity of the aeroplane due to its own power, and that the vector AD represents the velocity of the wind. We complete the parallelogram of velocities $ABCD$ as shown, and draw the diagonal AC, which represents the speed of the aeroplane relative to the ground.

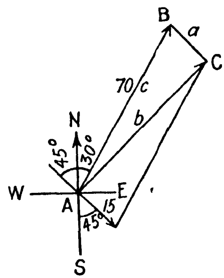


FIG. 71

It is readily seen that angle $DAB = 180° - (30° + 45°) = 105°$, and that angle B, which is supplementary to it, is therefore 75°. Thus in the triangle ABC we have $a = 15$, $c = 70$, $B = 75°$. Since the numbers are simple, let us employ the law of cosines to find b. We have

$$\begin{aligned} b^2 &= c^2 + a^2 - 2ca \cos B \\ &= 4900 + 225 - 2100 \cos 75° \\ &= 4581.1, \\ b &= 67.7. \end{aligned}$$

Also

$$\sin A = \frac{a \sin B}{b} = \frac{15 \sin 75°}{67.7} = 0.214,$$

$$A\ (= BAC) = 12° 20',$$

$$DAC = ACB = 12° 20',$$

$$NAC = NAB + BAC = 30° + 12° 20' = 42° 20'.$$

Thus the aeroplane actually travels in a direction 42° 20′ east of north at a speed of 67.7 miles an hour relative to the ground.

Example 3. An aeroplane which has a speed of 60 miles an hour in still air is ordered to fly in a direction northeast of the flying field at which it is stationed and return in 3 hours. A 15-mile wind is blowing from a direction 30° west of north. (*a*) How must the aeroplane be pointed on its outward trip? (*b*) How must it be pointed on its return trip? (*c*) When must it turn in order to get back at the specified time, and how far out does it go?

Solution. (*a*) In Fig. 72 the vector AD represents the velocity of the wind, the vector AB represents the velocity of the aeroplane due to its own power. The resultant of these two velocities must be along

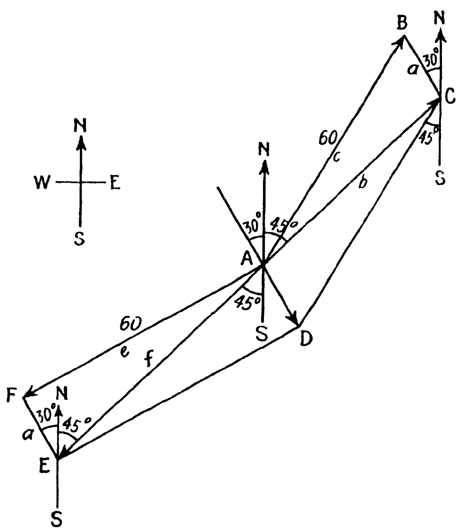


FIG. 72

the northeast direction AC. That is, AC is the diagonal of the parallelogram of velocities. In the triangle ABC we have $a = 15$, $c = 60$, $C = 180° - (30° + 45°) = 105°$. Therefore

$$\sin A = \frac{a \sin C}{c} = \frac{15 \sin 105°}{60} = 0.241,$$

$$A\ (= BAC) = 14°,$$

$$NAB = NAC - BAC = 45° - 14° = 31°,$$

$$B = 180° - (A + C) = 61°,$$

$$b = \frac{c \sin B}{\sin C} = \frac{60 \sin 61°}{\sin 105°} = 54.3.$$

Thus the direction in which the aeroplane must be pointed is 31° east of north, and the actual speed on the outward trip is 54.3 miles an hour.

(*b*) On the return trip, the velocity of the wind is again represented by AD, the velocity of the aeroplane due to its own power is represented by AF. The resultant of these two velocities is AE, which extends southwest. In the triangle AEF, $a = 15$, $e = 60$, $E = 75°$. Thus

$$\sin A = \frac{a \sin E}{e} = \frac{15 \sin 75°}{60} = 0.241,$$

$$A \ (= FAE) = 14°,$$
$$NAF = NAE - FAE = 135° - 14° = 121°,$$
$$F = 180° - (A + E) = 91°,$$
$$f = \frac{e \sin F}{\sin E} = \frac{60 \sin 91°}{\sin 75°} = 62.1.$$

Therefore, to return to the starting point, the aeroplane must be pointed in a direction 121° west of north (or 59° west of south), and its speed with respect to the ground on the homeward trip is 62.1 miles an hour.

(*c*) Let T represent the time, in hours, from the time of starting until the time of turning, that is, the time of the outward trip. Then $3 - T$ will be the time of the return trip. Since 54.3 and 62.1 are the velocities of the outward and return trips, respectively, $54.3T$ will be the number of miles in the outward trip and $62.1(3 - T)$ will be the number of miles on the homeward trip. Therefore

$$54.3T = 62.1(3 - T),$$
$$T = 1.6.$$

The turn should therefore be made about 1.6 hours after starting.

If R represents the number of miles from the starting point to the turning point, sometimes called the **radius of action,** then

$$R = 54.3T = 54.3 \times 1.6 = 86.9.$$

Example 4. An aeroplane having a speed of 60 miles an hour in still air is aboard a ship equipped with a landing deck. It is ordered to scout in a northeast direction. When it starts the ship sets out due north at the rate of 20 miles an hour. If a 15-mile wind is blowing from a direction 30° west of north, find the directions in which the aeroplane must be pointed going and returning.

Solution. Obviously, the direction in which the aeroplane must be pointed on the outward trip is the same as in the preceding example, namely, 31° east of north.

In Fig. 73 let A represent the starting point, and AC the distance that the aeroplane has gone in one hour. From the preceding example, $AC = 54.3$.

In 1 hour, however, the ship is at the point B, which is 20 miles north of A. But the ship continues to steam north at the rate of 20 miles an hour, and the pilot of the aeroplane must make allowance for this exactly as he does for the wind. It is clear that the position of the aeroplane relative to the ship would be affected in the same way if the latter were stationary and the former were moving south at the rate of 20 miles an hour. It is more convenient to consider the problem in this way, that is, to consider the velocity of the aeroplane relative to the ship. This velocity, which must be along the line CB, is, then, the resultant of the 20 miles an hour motion south, represented by CD; the velocity of the wind, represented by CE; and the velocity of the aeroplane due to its own power, represented by CF.

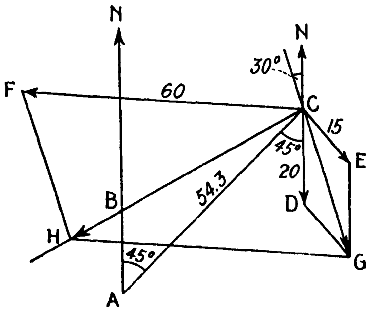


Fig. 73

We first find the resultant CG of the vectors CD and CE. This can readily be done, since in the triangle CEG we know $c = 20$, $g = 15$, $E = 180° - 30° = 150°$.

Having found CG, we find CF from the triangle CFH. The vector CF gives us the direction in which the aeroplane must be pointed on the return trip.

If the actual path of the aeroplane relative to the earth is desired, it can be obtained by finding the resultant of CE and CF. The vector CH gives the path relative to the ship.

The foregoing problems in aerial navigation can be solved graphically very quickly and with sufficient accuracy for practical purposes. A graduated ruler, a protractor, and a pair of compasses are the only mathematical instruments required.

Exercise 32

1. Complete the solution of example 4, section 72.

2. Solve examples 2, 3, and 4, of section 72, graphically.

3. Two forces of 8 and 11 pounds act at an angle of 75°. Find the magnitude of their resultant, and the angle that it makes with the 8 pound force.

4. Three forces of 7, 9, and 13 pounds are in equilibrium. Find the angles that they make with each other.

5. A train is traveling at the rate of 30 miles an hour, and rain falls with a velocity of 22 feet a second at an angle of 30° with the vertical and in the same direction as the motion of the train. Find the direction of the splashes made on the windows by the raindrops.

6. A motor boat which has a speed of 15 miles an hour sets out to cross a stream which has a current of 5 miles an hour. The boat points up stream at an angle of 30° with the bank. Find its actual speed and the actual direction that it takes.

7. A man wishes to row across a river which is a mile wide and to reach a landing which is one-half mile up stream from a point directly opposite the starting point. If the current is 3 miles an hour and the man can row 4 miles an hour, (*a*) how must he point his boat, and (*b*) how long will it take him to reach the landing?

8. Resolve a force of 100 pounds into components of 60 and 50 pounds, and find the angle between the components.

9. An aeroplane has a rate of 75 miles an hour. The pilot wishes to fly in a direction 65° east of north. A 15-mile wind is blowing from the southeast. In what direction must the aeroplane be pointed?

10. An aeroplane has a speed of 60 miles an hour, and can carry a supply of gasoline sufficient to last for 5 hours. A 12-mile wind is blowing from the north. (*a*) How far can the aeroplane fly north and return to its starting point, and when must it turn? (*b*) How far west can it fly and return, and when must it turn?

11. An aviator whose aeroplane flies 65 miles an hour wishes to go due north and return to his hangar at the end of two hours. A 13-mile wind is blowing from E.N.E. Find (*a*) the directions in which he must steer on the outward and homeward trips, and (*b*) when he must turn.

12. A pilot is ordered to sail southwest from a given flying field and to report in four hours at another field 20 miles south of the first. His machine has a speed of 60 miles an hour, and a wind of 10 miles an hour is blowing from a direction 25° west of north. (*a*) In what direction must he point his aeroplane on the outward trip? (*b*) How must he point it on the return trip? (*c*) How far out does he go from the first field, and when does he turn?

Exercise 33

Miscellaneous Problems

1. The altitude of the sun is 35°. What will be the length of the shadow of a tree which is 75 feet high?

2. How many degrees are there in the angle at the center of a circle if the chord of the intercepted arc is 2/3 the radius?

3. Two straight railroads intersect at an angle of 65°. (*a*) If two trains leave the crossing at the same time, one traveling 30 miles an hour and the other 40 miles an hour, how far apart will they be at the end of 30 minutes? (*b*) If the first leaves the crossing 15 minutes ahead of the second,

how far apart will they be 30 minutes after the first leaves the crossing?

4. Three circles whose diameters are 25, 30, and 50 inches are tangent to each other. Find the angles between their lines of centers.

5. Find the area of a triangular field whose sides are 52, 63, and 76 rods.

6. The angle subtended by a tower on an inclined plane is 57° at a certain point on the plane; 200 feet farther down it is 27°. The plane is inclined 10°. What is the height of the tower?

7. An automobile is traveling north at the rate of 25 miles an hour. The wind is blowing from the direction W. 40° N. at the rate of 15 miles an hour. With what velocity and from what direction does it seem to strike the chauffeur?

8. Two forces of 475 and 530 pounds, making an angle with each other of 36° 35′, act at the same point. What is the resultant force?

9. The elevation of a balloon from a point due south is 45°; from a point due east it is 15°. If the distance between the two places is a, find the height of the balloon.

10. Each of two ships has a wireless apparatus with a range of 200 miles. One is 165 miles N. 23° 50′ E. and the other 157 miles N. 48° 46′ W. from a shore station. Can they communicate with each other directly?

11. Three forces of 255, 320, and 195 pounds are in equilibrium. What angles do they make with each other?

12. To measure across a pond from A to B, a point C is selected so that $AC = 489$ feet, $BC = 674$ feet, and angle $C = 78° 45'$. Find the distance AB.

13. In an isosceles triangle each of the equal sides is

179 feet and their included angle is 102°. Find the third side and the area of the triangle.

14. From one corner of a cube lines are drawn in two of its faces making angles of 30° and 40° respectively with its common edge. Find the angle between these two lines.

15. A point A is 5000 feet S.E. of B. From B the angle of elevation of a cloud is 40° 15′, the cloud bearing N. 75° E. At A the cloud bears N. 49° E. How high is the cloud above the surface of the earth?

16. From the window of a building another building is observed. The base has an angle of depression of 13° 52′, and the top has an angle of elevation of 23° 47′. A horizontal line from the eye of the observer meets the building 20 feet from the ground. Find the height of the building.

17. A tower 155 feet high stands on the top of a hill. At a point 250 feet down the slope of the hill the tower subtends an angle of 8° 43′ 21″. Find the angle of inclination of the hill.

18. Taking the radius of the earth as 3960 miles, find the difference in latitude between two points 600 miles apart on the same meridian.

19. Two streets meet at an angle of 78° 35′. How much land is there in the triangular corner lot which fronts 436.2 feet on one street and 328.3 feet on the other?

20. Two billiard balls are at distances of 6 inches and 2 feet from the same cushion, and are 2 feet 6 inches apart. At what angle must the first ball strike the cushion in order to rebound and hit the other ball squarely? (A billiard ball without "English" rebounds according to the law of reflected light.)

21. The distance of Venus from the sun is 36,000,000 miles and of the earth from the sun is 93,000,000 miles.

How far is Venus from the earth when its angular distance from the sun is $12° 35'$?

22. Find the length of the diagonals of a regular pentagon whose side is 12.

23. Find the area of the segment of a circle whose radius is 16.7 feet, the chord of the segment being 22.2 feet.

24. Two lighthouses are 90 and 80 feet high. From the top of either the other is just visible over the water. How far are they apart?

25. From a point 5 miles from one end of an island and 8 miles from the other, the island subtends at the eye an angle of $35° 34' 25''$. What is the length of the island?

26. The sides of a triangle are 12, 17, and 21. Find the cosine of the largest angle.

27. The diagonals of a quadrilateral are 56.5 and 78.4 yards. They intersect at an angle of $51° 35'$. Find the area of the quadrilateral.

28. Two straight paths cross each other at an angle of 71°. A fence is built to enclose with the two paths an acre of ground. The fence is built from a point on one path 5 rods from the intersection of the paths, and follows a straight line to the other path. Find the length of the fence and the angles it makes with the paths.

29. A meteor was observed by two persons 300 miles apart, the meteor and the persons being in the same vertical plane with the center of the earth. The angle of elevation observed by one person was 26°, and by the other was 18°. How high was the meteor?

30. At what latitude is the radius of the circle of latitude one-half the radius of the earth?

31. A line 600 feet long is measured along the bank of a river. From each end of the line an object is sighted on the

opposite bank. The angles which the line makes with these lines of sight are 62° 05′ 18″ and 81° 34′ 42″. Find the width of the river.

32. From the summit of a hill 500 feet high the angles of depression of the top and bottom of a tower are 46° and 57° respectively. What is the height of the tower?

33. From the top of a hill which stands on a level plain the angles of depression of two objects on the plain are 30° and 45°. The horizontal angle between them is 60°. If the distance between the two objects is c, find the height of the hill in terms of c.

34. Let A, B, and C represent three consecutive mile posts on a straight road. From each of these a distant spire is observed. At A it is N.E., at B it is E., and at C it is E. 30° S. Find the distance of the spire from B, and the shortest distance from the road to the spire.

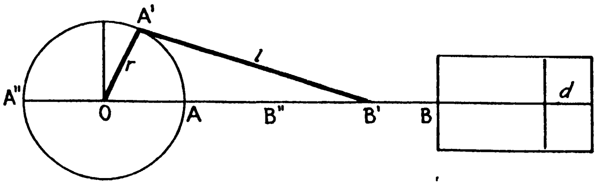
35. Find the radius of the largest cylindrical gas tank that can be built on a triangular lot whose sides are 78, 92, and 83 feet.

36. A bridge is to be constructed across a valley. If the length of the bridge is to be l, and the slopes of the two sides of the valley are A and B, how high will a pier need to be when built from the lowest point of the valley to support the bridge?

37. The altitude of a certain rock is observed to be 57°. At a distance of 1200 feet towards the rock up a slope of 25°, the angle of elevation is 78° 35′. Find the height of the rock above the first point of observation.

38. In a steam engine the crank (r) is 12 inches long, and the connecting rod (l) is 50 inches long. (See figure.) Find the distance (d) that the piston has moved from the end of the cylinder when the angle $B'OA'$, as indicated in the diagram, is 10°, 20°, 30°, and so on up to 180°.

39. *To find the horizontal distance AC and the vertical distance CD from a point A to an inaccessible object D, when*



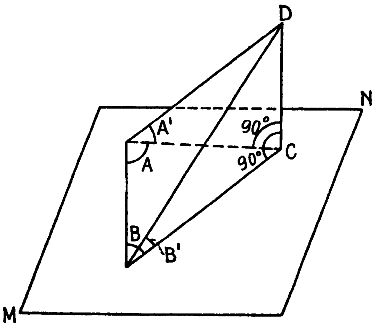
it is not convenient to measure a base line in the same vertical plane with D. (See figure.)

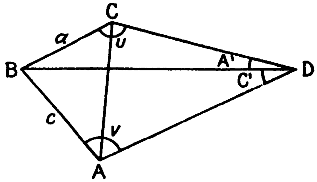
Draw AB, c feet long, in the plane MN in any convenient direction. Let C be the foot of the perpendicular from D. Measure angles A, A', B, and B'. Show that

$$BC = \frac{c \sin A}{\sin (A + B)}, \qquad AC = \frac{c \sin B}{\sin (A + B)},$$

$$CD = \frac{c \sin B \tan A'}{\sin (A + B)} = \frac{c \sin A \tan B'}{\sin (A + B)}.$$

The height CD can be found from both formulas in order to check.





40. **Pothenot's problem.** The following problem is usually called *Pothenot's problem,* but this name is not

historically correct. It was first solved by **Snellius**, in 1617, and again by Pothenot in 1730. The solution of Snellius was forgotten, so it received the name of Pothenot.

The distances between three places A, B, and C are known. It is required to find, without actual measurement, the distances of A, B, and C from a fourth point D, the angles ADB and CDB being assumed known. A problem of this sort is a practical problem in surveying and in military reconnaissance or orientation.

The angles of the triangle ABC may easily be found, since a, AC, and c are known. If the angles u and v (see accompanying figure) can be found the problem may be considered solved, since it will then resolve into a solution of oblique triangles when two angles and a side are given.

By the law of sines,

$$BD = \frac{a \sin u}{\sin A'} = \frac{c \sin v}{\sin C'}.$$

Hence

$$\frac{\sin u}{\sin v} = \frac{c \sin A'}{a \sin C'}.$$

By composition and division in the theory of proportion,

$$\frac{\sin u - \sin v}{\sin u + \sin v} = \frac{c \sin A' - a \sin C'}{c \sin A' + a \sin C'}.$$

Using (74), p. 148, we get

$$(a) \quad \frac{\tan \frac{1}{2}(u - v)}{\tan \frac{1}{2}(u + v)} = \frac{c \sin A' - a \sin C'}{c \sin A' + a \sin C'} = \frac{1 - \dfrac{a \sin C'}{c \sin A'}}{1 + \dfrac{a \sin C'}{c \sin A'}}.$$

But $u + v + ABC + ADC = 360°$, since the sum of the angles of a quadrilateral is equal to 360°. By transposing and dividing by 2 we get

$$\tfrac{1}{2}(u + v) = 180° - \tfrac{1}{2}(ABC + ADC).$$

Since we know ABC and ADC, we now have $\frac{1}{2}(u+v)$. Substituting in (a) we get the value of $\frac{1}{2}(u-v)$, and can then easily find u and v.

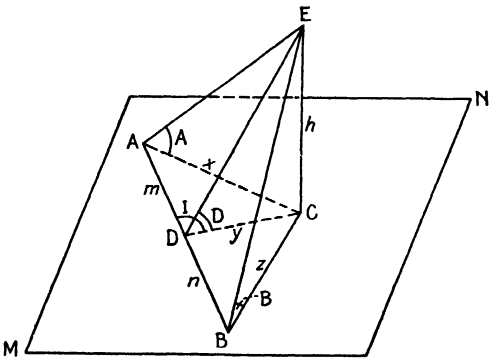
It will be noted that if D should lie on a circle passing through A, B, and C, the position of D can not be determined. For as D moves on the circumference, the angles A' and C' do not change; hence the position of D is not determined by the values of these angles.

41. In the accompanying figure let A, D, and B be three points in a horizontal line, and let m and n be known. Point E is inaccessible. Find x, y, z, and h.

Solution.

$$y = h \cot D, \qquad x = h \cot A, \qquad z = h \cot B. \qquad (a)$$

$$x^2 = m^2 + y^2 - 2my \cos I, \qquad z^2 = n^2 + y^2 + 2ny \cos I.$$



Whence

$$\frac{m^2 + y^2 - x^2}{m} = \frac{z^2 - n^2 - y^2}{n}.$$

Substituting from (a) for x, y, and z, and solving for h^2, we find that

$$h^2 = \frac{mn(m + n)}{m(\cot^2 B - \cot^2 D) + n(\cot^2 A - \cot^2 D)}.$$

Having found h, we can compute x, y, and z from (a).

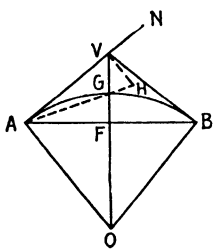
42. Three spherical shot, each 5 inches in diameter, lie on a floor in contact with each other, and a fourth shot of the same size is placed on them. Find the height of the center of the fourth ball above the floor.

43. The curved surface of a right circular cone whose semi-vertical angle is 45° is made by cutting out a sector from a circular sheet of metal, the diameter of the sheet being 56 inches. Find the angle of the required sector.

44. Prove that the area of a regular polygon inscribed in a circle is a geometric mean between the areas of an inscribed and of a circumscribed polygon of half the number of sides.

45. A person traveling eastward at the rate of 3 miles an hour finds that the wind seems to blow directly from the north; on doubling his speed he finds that it appears to come from the northeast. Determine the direction and velocity of the wind.

46. In railroad engineering a **simple curve** is the arc of a circle. In the accompanying figure AV and BV are tangent



to the simple curve AGB whose radius is R. We have the following definitions and designations: AV or BV is the **tangent distance,** denoted by T, AB is the **chord** or **long chord,** denoted by C, FG is the **middle ordinate,** denoted by M, GV is the **external distance,** denoted by E, BVN $(= AOB)$ is the **intersection angle,** denoted by I.

If we draw VH perpendicular to AV and intersecting G produced in H, then triangles VGH and AOG are similar, and therefore $HV = GV = E$. Prove that

(a) $T = R \tan \frac{1}{2}I$, (b) $M = R \operatorname{vers} \frac{1}{2}I$,
(c) $C = 2R \sin \frac{1}{2}I$, (d) $C = 2T \cos \frac{1}{2}I$,
(e) $C = 2M \cot \frac{1}{4}I$, (f) $C = 2E \cos \frac{1}{2}I \cot \frac{1}{4}I$,
(g) $M = E \cos \frac{1}{2}I$, (h) $M = \frac{1}{2}C \tan \frac{1}{4}I$,
(i) $M = T \cos \frac{1}{2}I \tan \frac{1}{4}I$, (j) $E = T \tan \frac{1}{4}I$,
(k) $E = R \operatorname{exsec} \frac{1}{2}I$, (l) $E = R \operatorname{vers} \frac{1}{2}I \sec \frac{1}{2}I$.

CHAPTER VIII

GRAPHIC REPRESENTATIONS OF THE FUNCTIONS

73. Line representations of the trigonometric functions. Using the definitions of the trigonometric functions as ratios of two quantities, we shall now show how to represent the functions of any angle by means of lines. This method is as indicated in Fig. 74.

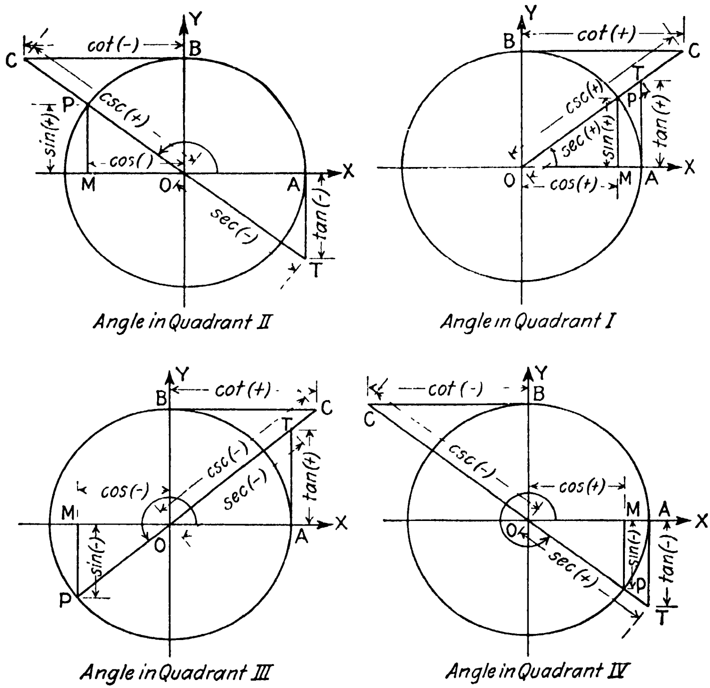


FIG. 74

René Descartes (1596–1650) was born in Touraine, France, and died in Stockholm, Sweden. In 1629 he moved to Holland where he pursued his studies in seclusion for twenty years. The last year of his life was spent in Sweden, whither he had gone at the invitation of Queen Christina to be her tutor. Another of his well-known pupils was Princess Elizabeth of Bohemia. He did important work in physics, anatomy, and philosophy, but his greatest work was in mathematics. He is regarded as the first of the modern mathematicians. He invented analytic geometry, a means of relating algebra and geometry in such manner that algebraic equations may be represented graphically with reference to two intersecting straight lines. This device has become a powerful means of research and without it Sir Isaac Newton could not have made his greatest discoveries. Descartes founded what is known as the Cartesian school of thought, and is the father of the modern mathematical method of investigating science.

In representing the functions by means of lines, we employ the radius as the unit of measurement. (That is, we take the radius to be equal to 1.) Hence we call the circle the **unit circle.**

If we denote by $P(x, y)$ the point at which the terminal line cuts the circle, then

$$\sin v = \text{ordinate of } P,$$
$$\cos v = \text{abscissa of } P.$$

For, from the definitions of the functions as ratios, we have

$$\sin v = \frac{y}{r} = \frac{y}{1} = y,$$

$$\cos v = \frac{x}{r} = \frac{x}{1} = x.$$

It is evident that in order to complete this scheme of representing the functions we must write them as ratios of which the denominator will be r, or 1. This is accomplished by the selection of similar right triangles. Moreover, we wish to select these lines so that they will have the proper signs.

The tangent will in all cases be drawn from A, the point where the circle cuts the X-axis on the right. The cotangent will in all cases be drawn from B, the point where the circle cuts the Y-axis above.

Since the right triangles MOP and AOT are similar, having the angle v common, we see that

$$\tan v = \frac{y}{x} = \frac{AT}{r} = \frac{AT}{1} = AT.$$

The right triangles MOP and BOC are similar, having angles v and C equal (alternate interior angles). Therefore

$$\cot v = \frac{x}{y} = \frac{BC}{r} = \frac{BC}{1} = BC.$$

The secant and the cosecant are measured along the terminal line of the angle. We shall assume that when measured in the same direction as the terminal line* they are positive, and when measured in the reverse direction they are negative. Then from similar triangles,

$$\sec v = \frac{r}{x} = \frac{OT}{OA} = OT,$$

$$\csc v = \frac{r}{y} = \frac{OC}{OB} = OC.$$

The following representations might also be mentioned:

$$\begin{aligned} \text{vers } v &= 1 - \cos v = MA, \\ \text{exsec } v &= \sec v - 1 = PT, \\ v &= \text{arc } AP \text{ (if } v \text{ is measured in radians).} \end{aligned}$$

How could we represent covers $v = 1 - \sin v$, coexsec $v = \csc v - 1$, and hav $v = \frac{1}{2}(1 - \cos v)$?

It should be carefully noted by the student that the functions are not lines. They are ratios and therefore abstract numbers. The values of the functions are given by the measures of the lines in terms of the radius. The use of the circle explains why the trigonometric functions are sometimes called **circular** functions. It also explains the origin of the terms tangent and secant. The tangent is a line tangent to the circle, while the cotangent bears a similar relation to the complementary angle, etc.

74. Variations of the functions when the angle varies. By observing Fig. 75 we see that for an angle of $0°$ the line representing the sine disappears; that is, $\sin 0° = 0$. As the angle increases, the sine increases, until at $90°$ it reaches its maximum value of 1; as the angle increases

* The terminal line is considered as directed from the origin out.

further, the value of the sine decreases to 0 at 180°, and to − 1 at 270°. It has now reached its minimum value, and as the angle increases beyond this the sine increases from − 1 to 0, which value it reaches when the angle reaches 360°.

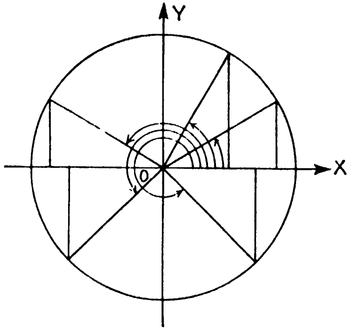


FIG. 75

In like manner, the cosine, starting with a maximum value of 1 when the angle is 0°, decreases to 0 at 90°, to − 1 at 180°, and then increases from this minimum value to 1 at 360°.

Thus we see that the values of both sine and cosine vary between 1 and − 1, twice taking on all possible values between these limits as the angle varies from 0° to 360°. While this can readily be visualized from the line representations of the functions, it can be deduced from the ratio definitions. In section 24 we gave the functions of 0°, 90°, 180°, etc., and the manner of variation of the functions between these angles is evident. The student should verify this.

We shall now see how the other functions vary when the angle varies, by considering their line representations. As in the case of the sine and cosine, however, we might study how the other functions vary by considering their ratio definitions.

By studying Fig. 76 we see that, since the length of the tangent line is determined by the point of intersection of this line with the terminal line, at 0° the tangent is 0. The tangent increases as the angle increases, until at 90° the terminal line is parallel to the tangent line, and there can be no point of intersection. That is, there is no value of the tangent for this angle. However, since the tangent has been increasing rapidly just before the angle became 90°,

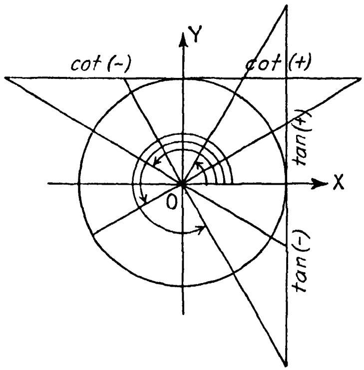


FIG. 76

we sometimes say that the tangent is indefinitely great when the angle is 90°, or that it is infinite or has the value infinity (∞). In the second quadrant the terminal line must be prolonged backward to intersect the tangent line. This measures the tangent below the X-axis, and hence it is negative. As the angle increases beyond 90°, the tangent, which has just extended indefinitely far in a positive direction, now begins at an indefinite distance in the negative direction, or, as we sometimes say, the tangent jumps instantly from $+\infty$ to $-\infty$.*

Thus the tangent does not have a continuous change in value; there is a break at 90°. The tangent increases from

* When v approaches 90° from the first quadrant the limit of $\tan v$ is $+\infty$. When v approaches 90° from the second quadrant the limit of $\tan v$ is $-\infty$.

$-\infty$ to 0 as the angle increases from 90° to 180°. As the angle increases through the third quadrant the terminal line must be prolonged backward and the values of the tangent are the same as in the first quadrant. As the angle increases to 360° the tangent repeats the values of the second quadrant. The tangent therefore passes through all possible values twice in one complete rotation of the terminal line.

In like manner, since the length of the cotangent line is determined by the intersection of the tangent line at B with the terminal line, the cotangent starts with the value $+\infty$ at 0°, decreases to 0 at 90°; continues to decrease to $-\infty$ at 180°, and instantly passes to $+\infty$; then decreases through the value 0 at 270° to $-\infty$ at 360°, where it changes to $+\infty$ as the angle passes through 360°. Hence the cotangent also passes through all possible values twice in one complete rotation of the terminal line.

Since the secant is measured by the terminal line drawn to the tangent, the secant is 1 at 0°; it increases to $+\infty$ at 90°, where it changes to $-\infty$; increases to -1 at 180°; and then decreases back to $-\infty$ at 270°, where it changes to $+\infty$; and comes back to 1 again at 360°. It should be noted that the secant does not have any of the values of the cosine except 1 and -1. That is, it does not have values between $+1$ and -1. It has all other possible values.

The cosecant is measured on the terminal line, and is limited by its intersection with the cotangent line at B. Starting with the value $+\infty$ at 0°, it decreases to 1 at 90°; increases back to $+\infty$ at 180°, where it changes to $-\infty$; increases to -1 at 270°; then decreases to $-\infty$ at 360°, and changes to $+\infty$ as the angle passes 360°.

The preceding facts are summarized in Table K.

TABLE K

Quadrant	I	II	III	IV
Angle	$0°$ to $90°$ $0^{(r)}$ to $\frac{\pi^{(r)}}{2}$	$90°$ to $180°$ $\frac{\pi^{(r)}}{2}$ to $\pi^{(r)}$	$180°$ to $270°$ $\pi^{(r)}$ to $\frac{3\pi^{(r)}}{2}$	$270°$ to $360°$ $\frac{3\pi^{(r)}}{2}$ to $2\pi^{(r)}$
sin	0 to 1	1 to 0	0 to -1	-1 to 0
cos	1 to 0	0 to -1	-1 to 0	0 to 1
tan	0 to ∞	$-\infty$ to 0	0 to ∞	$-\infty$ to 0
cot	∞ to 0	0 to $-\infty$	∞ to 0	0 to $-\infty$
sec	1 to ∞	$-\infty$ to -1	-1 to $-\infty$	∞ to 1
csc	∞ to 1	1 to ∞	$-\infty$ to -1	-1 to $-\infty$

From the representation of the functions by lines some of the formulas we have already derived become very apparent. For instance, by applying the theorem of Pythagoras to the right triangle concerned, we find

$$\begin{aligned}\cos^2 v + \sin^2 v &= 1,\\ 1 + \tan^2 v &= \sec^2 v,\\ \cot^2 v + 1 &= \csc^2 v.\end{aligned}$$

Also the formulas of sections 46–54 can easily be proved by using the line representations.

75. Graphs of the functions. We are now ready to construct graphs of the functions. Let us first consider the equation

$$y = \sin x.$$

In the following discussion we shall assume that the angle x is measured in radians, unless otherwise specified. The foregoing equation then says that *y is equal to the sine of x radians.* For instance, if $x = .5$, then $y = \sin .5^{(r)} = \sin 28° 38' = .48$. However, it is probably more convenient to give x values involving π. Some corresponding pairs of values are shown in the following table:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
	0.00	0.52	0.79	1.05	1.57	2.09	2.36	2.62
y	0.00	0.50	0.71	0.87	1.00	0.87	0.71	0.50

x	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$-\frac{\pi}{4}$
	3.14	3.67	4.71	5.50	6.28	7.07	−0.79
y	0.00	−0.50	−1.00	−0.71	0.00	0.71	−0.71

If we plot these points on a set of rectangular axes and connect them with a smooth curve, we get a wave curve such as that in Fig. 77.

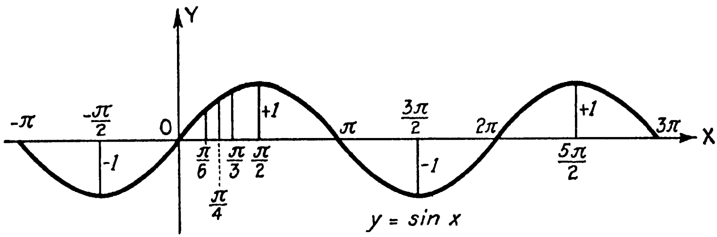


FIG. 77

The values of the function for given angles could have been obtained by measuring the lines of the unit circle.

Also by using the unit circle we may construct the sine curve as indicated in Fig. 78. The distance from the origin O to the point on the X-axis marked $\pi/6$ is exactly the same as the length of the arc of the circle from the point marked 0 to the point marked $\pi/6$, etc. The method by which we obtain the ordinate corresponding to a given abscissa is evident from the figure.

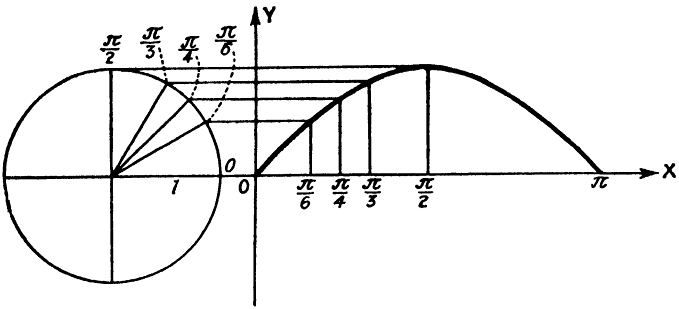


FIG. 78

It may be noted that we may easily construct the entire graph if we plot it carefully for the first quadrant of the unit circle. The same values are repeated in reverse order for the second quadrant, and the portion of the curve below the X-axis is the same as the part above the X-axis inverted and placed in proper position.

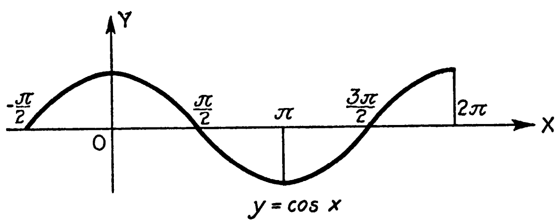


FIG. 79

The student should verify the curves of Figs. 79–83 by plotting them on graph paper.

76. Periods of the functions. The trigonometric functions are said to be **periodic** because their values, and consequently the curves that represent the functions, repeat.

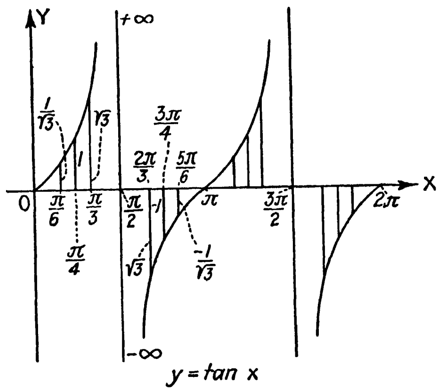


FIG. 80

The sine curve is complete between 0 and 2π radians; it is repeated between 2π and 4π. In fact, if we take a length of 2π units on the X-axis, we get a complete pattern of the

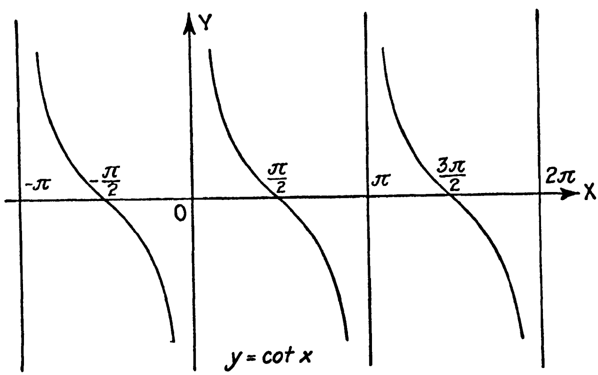


FIG. 81

curve. Consequently the sine is said to have a **period** of 2π radians. The cosine also has the period 2π radians. The tangent and cotangent complete a period in π radians. What period does the secant have? The cosecant?

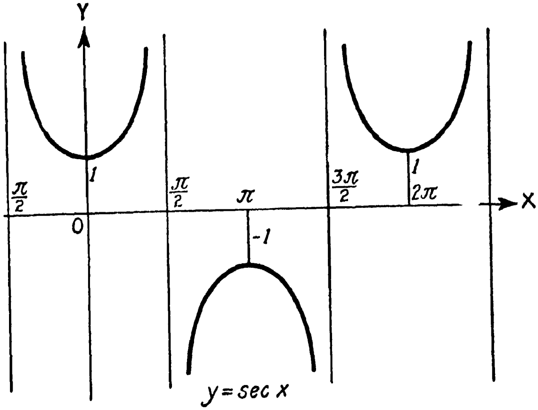


FIG. 82

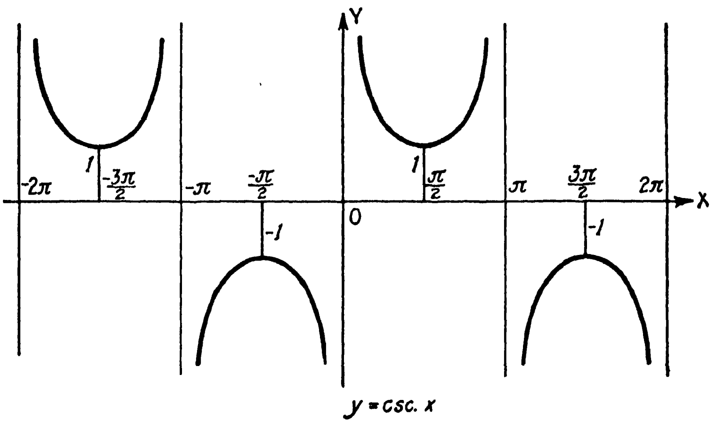


FIG. 83

Exercise 34

1. Plot the following curves:

(*a*) $y = 2 \sin x$, (*b*) $y = 2 \cos x$, (*c*) $y = \frac{1}{2} \sin x$.

2. Draw the graphs of the following equations:

(*a*) $y = \sin 2x$, (*b*) $y = \sin \frac{1}{2}x$, (*c*) $y = \cos \frac{1}{2}x$,
(*d*) $y = \cot \frac{1}{2}x$, (*e*) $y = \sin 3x$, (*f*) $y = \tan 2x$,
(*g*) $y = \sin \pi x$, (*h*) $y = \cos \dfrac{\pi x}{2}$, (*i*) $y = \sin \dfrac{2\pi x}{3}$.

3. Plot $y = \sin x \cos x$ and compare with the graph obtained in problem 1 (*a*).

4. Plot

(*a*) $y = \text{vers } x$, (*b*) $y = \text{covers } x$,
(*c*) $y = \text{exsec } x$, (*d*) $y = \text{hav } x$.

5. In what points will a line one unit above the X-axis intersect the curve $y = \tan x$? Explain.

6. If the graphs $y = \sin x$ and $y = \cos x$ are plotted on the same axes, where will they intersect? Why?

7. Plot $y = \sin (x + \pi/2)$ and compare with $y = \cos x$.

8. Plot $y = \cos (\pi/2 - x)$ and compare with $y = \sin x$.

9. Plot $y = \cos (x - \pi/4)$.

10. Plot $y = \sin (x - \frac{1}{2})$.

11. Given the equation $y = \sin x + \cos x$.

(*a*) Plot the curve by plotting the sine curve and cosine curve separately and adding their ordinates by using dividers.

(*b*) Plot the curve, first reducing the equation to the form $y = \sqrt{2} \sin (x + \pi/4)$.

12. Plot

(*a*) $y = \cos x - \sin x$, (*b*) $y = x + \sin x$.

13. Find the periods of the curves in problem 2.

CHAPTER IX

INVERSE FUNCTIONS. EQUATIONS

77. The inverse trigonometric functions. The algebraic equation

$$y = x^2,$$

in which y is expressed in terms of x, can be solved for x in terms of y; thus

$$x = \pm \sqrt{y}.$$

This suggests the possibility of solving the equation

$$(115) \qquad y = \sin v,$$

in which y is expressed in terms of v, and obtaining v in terms of y. Thus v is an angle whose sine is y. For abbreviation we use the notation

$$(116) \qquad \boldsymbol{v = \sin^{-1} y},$$

in which the symbol $\boldsymbol{\sin^{-1} y}$ is read an **angle whose sine is $\boldsymbol{y}$**, the **inverse sine of $\boldsymbol{y}$**, or the **antisine of $\boldsymbol{y}$**. The notation

$$(117) \qquad \boldsymbol{v = \arcsin y}$$

is also frequently used, with identically the same meaning as (116). The symbol **arcsin $\boldsymbol{y}$** is read an **arc** (or **angle**) **whose sine is $\boldsymbol{y}$**. We shall use the form (116) and say that v is an angle whose sine is y.

It should be carefully noted by the student that the -1 used here is not an exponent, although placed in the usual position for an exponent. If it is desired to have -1 as the exponent of a trigonometric function such as $\sin v$, it must be written $(\sin v)^{-1}$.

The quantity $\sin^{-1} y$ is called an **inverse** trigonometric function of y. Other inverse trigonometric functions are

$\cos^{-1} y$, meaning an **angle whose cosine is y,**
$\tan^{-1} y$, meaning an **angle whose tangent is y,**
$\cot^{-1} y$, meaning an **angle whose cotangent is y,**
$\sec^{-1} y$, meaning an **angle whose secant is y,**
$\csc^{-1} y$, meaning an **angle whose cosecant is y.**

We see that the two equations

$$y = \sin v, \qquad v = \sin^{-1} y$$

are equivalent. That is, they are merely different ways of expressing the same thing. However, there is an important difference between the trigonometric function and the inverse trigonometric function. If in the first of the above equations we assume the angle v to be known, there is but one corresponding value for its sine, or we say that y is a single-valued function of v. On the contrary, if in the second of the foregoing equations y is assumed to be known, v can have any number of values. In the algebraic equation cited, namely $y = x^2$, which written inversely is $x = \pm \sqrt{y}$, we have the same idea. To every value of x there corresponds one and only one value of y, but to every value of y (except 0) there correspond two values of x, and x is said to be a two-valued or double-valued function of y.

That the function $\sin^{-1} y$ has an infinite number of values corresponding to each value of y will be clearer from a consideration of Fig. 84, in which the radius of the circle is 1. Let $v = \sin^{-1} y$, or $v = \arcsin y$. That is, v is an angle or arc whose sine is y.

Then it is evident that whether y is positive or negative there will be two positions for the terminal side of the angle (unless of course $y = \pm 1$). If y is positive the terminal side will be in the first or the second quadrant; if y is negative the terminal side will be in the third or the fourth quadrant. If we remember that corresponding to any position of the terminal line there is an infinite number of angles, both positive and negative, we see very clearly that for any given value of y there is an infinite number of angles, that is, an infinite number of values of v. The value between $-\pi/2$ and $\pi/2$ is called the **principal value** of the inverse sine function. (In this connection see the last paragraph of this section.) Let us designate it by v'.

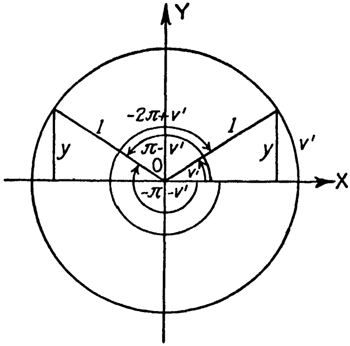


Fig. 84

Then we see that the various values of v are

$$v',\quad \pi - v',\quad 2\pi + v',\quad 3\pi - v',\quad 4\pi + v',\quad \cdots,$$
$$-\pi - v',\quad -2\pi + v',\quad -3\pi - v',\quad -4\pi + v',\quad \cdots.$$

From inspection of these values it is seen that the general expression for the angle v may be written

(118) $$v = \sin^{-1} y = n\pi + (-1)^n v',$$

where n is any positive or negative integer or zero.

Since the cosecant is the reciprocal of the sine, we may write

(119) $$v = \csc^{-1} \frac{1}{y}.$$

The principal value of the inverse cosecant, corresponding to a given value of $1/y$, is that value between $-\pi/2$ and

$\pi/2$. If we denote this value by v', we have, as before,

$$v = n\pi + (-1)^n v'.$$

Let us now consider the inverse function

$$v = \cos^{-1} x.$$

The principal value of the function corresponding to a given value of x is defined to be that value between 0 and π. Referring to Fig. 85, we see that if v' is the principal value of v, all of the values are

$$\begin{array}{llll} v', & 2\pi - v', & 2\pi + v', & \cdots, \\ -v', & -2\pi + v', & -2\pi - v', & \cdots. \end{array}$$

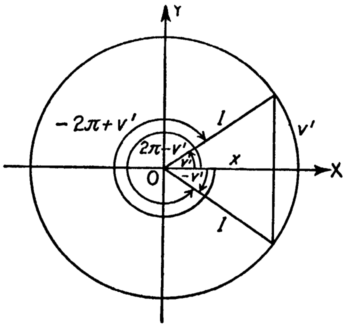


FIG. 85

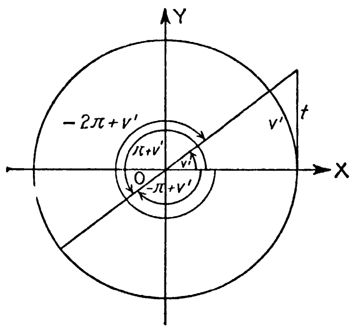


FIG. 86

Hence, from inspection, we may write the general formula

(120) $$v = \cos^{-1} x = 2n\pi \pm v',$$

where n is any positive or negative integer or zero.

The principal value of the inverse secant being defined as that between 0 and π, we may write, since the secant is the reciprocal of the cosine,

$$v = \sec^{-1}\frac{1}{x} = 2n\pi \pm v'.$$

For the inverse tangent and the inverse cotangent the principal value is that between $-\pi/2$ and $\pi/2$. If v' is the principal value of

$$v = \tan^{-1} t$$

corresponding to a given value of t, we see by referring to Fig. 86 that the various values of v are

$$\begin{gathered} v', \quad \pi + v', \quad 2\pi + v', \quad 3\pi + v', \quad \cdots, \\ -\pi + v', \quad -2\pi + v', \quad -3\pi + v', \quad \cdots. \end{gathered}$$

Hence we may write the general expression

$$v = \tan^{-1} t = n\pi + v', \tag{121}$$

where n is a positive or negative integer or zero.

Also

$$v = \cot^{-1} \frac{1}{t} = n\pi + v'. \tag{122}$$

We shall denote the principal values of the inverse trigonometric functions by $\mathbf{Sin}^{-1}\, x$, $\mathbf{Cos}^{-1}\, x$, $\mathbf{Tan}^{-1}\, x$, etc. If the notation arcsin x is used the principal value may be denoted by **Arcsin** x.

The graph of the equation

$$y = \sin^{-1} x, \tag{123}$$

in which y is expressed in radians, is given in Fig. 87. It will be noted that the curve extends indefinitely along the Y-axis in either direction, the value of x remaining between 1 and -1. For a given value of x we may have any number of values of y. The principal values of the function are indicated by the heavier part of the curve, which constitutes the **principal branch** of the function.

It is clear that this curve represents the equation x

$= \sin y$, which is equivalent to (123), and that it bears a close relation to the curve representing the equation

(124) $$y = \sin x.$$

Evidently (123) can be obtained from (124) by simply interchanging x and y. We can obtain the curve representing (123) from the sine curve representing (124) by

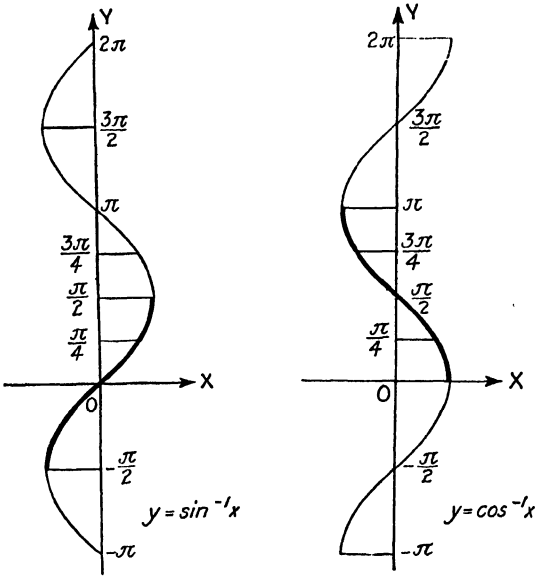


FIG. 87 FIG. 88

rotating the axes (without the curve) about O in a positive direction through 90°, and then rotating the sine curve out of the plane of the paper through 180°, using either axis as an axis of rotation. In like manner we can obtain the graphs of the other inverse circular functions from the graphs of the circular functions themselves.

The graphs of the inverse cosine and the inverse tangent are shown in Figs. 88 and 89 respectively. The principal branches are shown by the heavier parts of the curves. It might be remarked that the principal branches of the in-

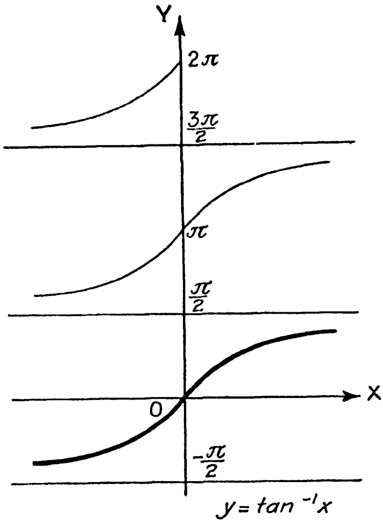


FIG. 89

verse cosine and inverse secant are composed of those values between 0 and π, and that the principal branches of the other inverse functions are composed of those values between $-\pi/2$ and $\pi/2$.

Exercise 35

1. Find $\sin^{-1} \sqrt{3}/2$.

Solution. Let $v = \sin^{-1} \sqrt{3}/2$. Then $\sin v = \sqrt{3}/2$, and the principal value of v is $60°$ or $\frac{\pi^{(r)}}{3}$. Therefore

$$v = \sin^{-1} \sqrt{3}/2 = n\pi + (-1)^n \frac{\pi}{3}.$$

2. Find the principal values and also the general values of

(a) $\sin^{-1}\frac{1}{2}$, (b) $\cos^{-1}(-1/\sqrt{2})$,
(c) $\sin^{-1}0$, (d) $\cos^{-1}0$,
(e) $\cot^{-1}\dfrac{1}{\sqrt{3}}$, (f) $\tan^{-1}1$,
(g) $\csc^{-1}\sqrt{3}$, (h) $\sin^{-1}0.37425$.

3. Find $\cos(\tan^{-1}\frac{5}{3})$.

Solution. Let $v = \tan^{-1}\frac{5}{3}$. (A figure would be of advantage.) Then $\tan v = \frac{5}{3}$, and $\cos v = \pm 3/\sqrt{34}$ $= \pm 3\sqrt{34}/34$.

4. Find

(a) $\tan(\cos^{-1}\frac{1}{3})$, (b) $\sec(\cot^{-1}2)$, (c) $\sin(\sec^{-1}\sqrt{3})$.

5. Show that

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$$

for the principal values.

Solution. Let $u = \tan^{-1}\frac{1}{2}$, $v = \tan^{-1}\frac{1}{3}$; that is, $\tan u = \frac{1}{2}$, $\tan v = \frac{1}{3}$. Then $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = u + v$, and

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = 1 = \tan\frac{\pi}{4},$$

Thus

$$u + v = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4} + n\pi,$$

or $u + v = \pi/4$, since we are using principal values.

6. Using principal values, prove that

(a) $\tan^{-1}\dfrac{1}{6} + \tan^{-1}\dfrac{5}{7} = \dfrac{\pi}{4}$,

(b) $\tan^{-1}\dfrac{3}{5} + \sin^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{27}{11}$,

(c) $\sin^{-1}\frac{3}{5} - \tan^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{29}$,

(d) $\tan^{-1}\frac{1}{3} - \tan^{-1}\frac{1}{4} = \tan^{-1}\frac{1}{13}$,

(e) $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \sin^{-1}\frac{77}{85}$,

(f) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$.

7. Prove that $\text{Tan}^{-1} m + \text{Tan}^{-1} n = \text{Tan}^{-1} \frac{m+n}{1-mn}$, provided $\text{Tan}^{-1} m + \text{Tan}^{-1} n$ is between $-\pi/2$ and $\pi/2$.

NOTE. In general, $\tan^{-1} m + \tan^{-1} n = \tan^{-1} \frac{m+n}{1-mn}$, but it should be observed that the particular values assigned to all but one of the three inverse functions are arbitrary; the particular value of that one is determined when the values of the others are assigned.

8. Prove that $\text{Tan}^{-1} x + \text{Cot}^{-1} x = \pi/2$ for $x > 0$.

9. Prove that $\text{Sin}^{-1} x + \text{Cos}^{-1} x = \pi/2$ for $-1 \leqq x \leqq 1$.

10. (a) Prove that $2\,\text{Sin}^{-1} x = \text{Cos}^{-1}(1 - 2x^2)$ for $0 \leqq x \leqq 1$. (b) Plot $y = 2\,\text{Sin}^{-1} x$ and $y = \text{Cos}^{-1}(1 - 2x^2)$.

11. Find the values of

(a) $\tan\left(\sin^{-1}\frac{1}{3} + \cos^{-1}\frac{1}{3}\right)$,

(b) $\cos\left(\sec^{-1}\frac{7}{3} + \sin^{-1}\frac{1}{3}\right)$,

(c) $\cos\left(2\sin^{-1}\frac{\sqrt{5}}{3}\right)$,

(d) $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$,

(e) $\sin\left(\sin^{-1}\frac{1}{2} + \sin^{-1}\frac{\sqrt{2}}{3}\right)$,

(f) $\sec\left(\frac{1}{2}\cos^{-1}\frac{m-n}{m+n}\right)$.

12. Prove that

$$(a)\ 2\ \mathrm{Tan}^{-1}\frac{1}{3}+\mathrm{Tan}^{-1}\frac{1}{7}=\frac{\pi}{4},$$

$$(b)\ \mathrm{Tan}^{-1}\frac{120}{119}=2\ \mathrm{Sin}^{-1}\frac{5}{13}.$$

78. Trigonometric equations. An equation which is satisfied by certain values only of the variable or variables that it contains is called a **conditional equation.** Examples of conditional equations are $x + 5 = 8$, which is satisfied by the value $x = 3$ only; $x^2 + y^2 = 25$, which is satisfied by a great many pairs of values of x and y, but certainly not by all pairs of values; $\sin v = \frac{1}{2}$, which is satisfied by $v = 30°, 150°, 390°, 510°$, etc.

An **identical equation,** or an **identity,** is an equation which is satisfied by all values (with perhaps some exceptions*) of the variable or variables that it contains. Examples of identities are $(x + 1)^2 = x^2 + 2x + 1$, $\sin 2v = 2 \sin v \cos v$, $\cos (u + v) = \cos u \cos v - \sin u \sin v$.

The equations which we are to consider here are conditional equations, identical equations having already been considered in various places throughout the book.

Trigonometric equations when solved for the value of an angle require the general expressions of formulas (118) to (122) for a complete solution. The problem is frequently considered sufficiently solved when all positive values of the angle less than 360° are obtained, since the terminal sides for additional values of the angle must coincide with the terminal sides of these. In fact the problem is often considered solved when the principal value of the angle is

* For example, the identity $\tan v = \dfrac{\sin v}{\cos v}$ is not defined for such values of v as $\pi/2$.

found, since, as has been shown, all other values can be found when this is known. Some of the methods of solving trigonometric equations are illustrated by the following examples.

Example 1. Solve the equation $\sin v = \cos v$.
Solution. Divide both sides by $\cos v$, getting

$$\tan v = 1.$$

Hence the principal value of v must be $\pi/4$ radians or $45°$, and the values of v less than $360°$ are $45°$ and $225°$. The general solution is

$$v = \tan^{-1} 1 = n\,180° + 45° = n\pi + \frac{\pi}{4},$$

where $n = 0, \pm 1, \pm 2, \cdots$.

A trigonometric equation can often be solved graphically to a sufficient degree of accuracy. Thus in Example 1 we could plot the curves $y = \sin v$ and $y = \cos v$ and note the values of v for which the two curves intersect.

Example 2. Solve the equation $\sin^4 v = \cos^4 v$.
Solution. Transposing, we get

$$\sin^4 v - \cos^4 v = 0.$$

Factoring gives

$$(\sin^2 v + \cos^2 v)(\sin^2 v - \cos^2 v) = 0.$$

Now $\sin^2 v + \cos^2 v = 1$ and can not be placed equal to zero. Hence, putting the second factor equal to zero and transposing, we get

$$\sin^2 v = \cos^2 v,$$
$$\sin v = \pm \cos v,$$
$$v = 45°, 135°, 225°, 315°, \cdots.$$

In general,

$$v = (2n + 1)45° = (2n + 1)\frac{\pi}{4},$$

where $n = 0, \pm 1, \pm 2, \cdots$.

In some cases the equation should be transformed by means of known relations which are applicable so that it will contain but one function, when it may be solved as an algebraic equation.

Example 3. Solve $2 \cos^2 v - 3 \sin v = 0$.

Solution. Substitute $1 - \sin^2 v$ for $\cos^2 v$, and then solve as a quadratic equation.

$$2 - 2 \sin^2 v - 3 \sin v = 0,$$
$$2 \sin^2 v + 3 \sin v - 2 = 0.$$

Factoring, we obtain

$$(2 \sin v - 1)(\sin v + 2) = 0.$$

The first factor gives

$$\sin v = \tfrac{1}{2},$$
$$v = 30^\circ,\ 150^\circ,\ \cdots.$$

The second factor gives $\sin v = -2$, which is impossible, since the sine of an angle can not be less than -1.

As in algebra, the safe and final test for the values of the angle obtained is to prove each one by substituting it in the original equation to determine whether or not it satisfies the equation. Thus, in the above example,

$$\sin 30^\circ = \sin 150^\circ = \tfrac{1}{2},$$
$$\cos 30^\circ = \frac{\sqrt{3}}{2}, \qquad \cos 150^\circ = -\frac{\sqrt{3}}{2}.$$

Substituting in the equation, we get

$$2\left(\pm\frac{\sqrt{3}}{2}\right)^2 - 3\left(\frac{1}{2}\right) = 2\left(\frac{3}{4}\right) - 3\left(\frac{1}{2}\right) = 0.$$

Hence the equation is satisfied by these values for the angle.

When an equation involves functions of different angles, such as v, $2v$, $\frac{1}{2} v$, $v + \frac{\pi}{4}$, etc., it is usually best to reduce everything to a single function of a single angle.

Example 4. Solve $\cos 2v = 2 \sin^2 v$.

Solution. Employing the formula for $\cos 2v$, we get

$$\cos^2 v - \sin^2 v = 2 \sin^2 v.$$

If we replace $\cos^2 v$ by $1 - \sin^2 v$, this reduces to

$$\sin^2 v = \tfrac{1}{4},$$

from which we get

$$\sin v = \pm \tfrac{1}{2},$$

$$v = 30°,\ 150°,\ 210°,\ 330°,\ \cdots,$$

or

$$v = n\ 180° \pm 30° = n\pi \pm \frac{\pi}{6}\cdot$$

Equations of the form $a \sin v + b \cos v = c$ can conveniently be solved by reducing the left-hand side to one of the forms $r \sin (v \pm w)$, $r \cos (v \pm w)$. (See section 61.)

Example 5. Solve the equation $3 \sin v - 4 \cos v = 1$.

Solution. Let

$$3 \sin v - 4 \cos v = r \sin (v - w) = r \sin v \cos w - r \cos v \sin w.$$

We see that

$$r \cos w = 3, \qquad r \sin w = 4,$$

from which we get, by division,

$$\tan w = \tfrac{4}{3} = 1.33333, \qquad w = 53°\ 07'\ 45''.$$

Also

$$r^2 \cos^2 w + r^2 \sin^2 w = 9 + 16 = 25,$$
$$r^2(\cos^2 w + \sin^2 w) = 25,$$
$$r^2 = 25, \qquad r = 5.$$

Therefore the original equation reduces to

$$5 \sin (v - 53°\ 07'\ 45'') = 1,$$

or

$$\sin (v - 53°\ 07'\ 45'') = \tfrac{1}{5} = 0.2.$$

From the tables,

$$v - 53°\ 07'\ 45'' = 11°\ 32'\ 13'',\ 168°\ 27'\ 47'',\ \cdots,$$
$$v = 64°\ 39'\ 58'',\ 221°\ 35'\ 32'',\ \cdots.$$

This method of solution is particularly valuable if the numbers involved are large, in which case logarithms should be used. The following arrangement for the logarithmic work is suggested, the foregoing problem, for convenience in checking, being used as an illustration:

$$\frac{r \sin w}{r \cos w} = \tan w$$

$$\sin (v - 53^\circ\ 07'\ 48'') = \frac{1}{r}$$

$r \sin w$	4
$r \cos w$	3
$\log r \sin w$	0.60206
$\log r \cos w$	0.47712
$\log \tan w$	0.12494
w	$53^\circ\ 07'\ 48''$
$\log r \cos w$	0.47712
$\log \cos w$	9.77815
$\log r$	0.69897
$\log 1$	0
$\log \sin (v - 53^\circ\ 07'\ 48'')$	9.30103
$v - 53^\circ\ 07'\ 48''$	$11^\circ\ 32'\ 13''$, $168^\circ\ 27'\ 47''$
v	$64^\circ\ 40'\ 01''$, $221^\circ\ 35'\ 35''$

This example might have been solved as follows:

Replace $\cos v$ by $\pm \sqrt{1 - \sin^2 v}$. Then

$$3 \sin v \mp 4\sqrt{1 - \sin^2 v} = 1.$$

Transposing, we get

$$3 \sin v - 1 = \pm 4\sqrt{1 - \sin^2 v}.$$

Squaring gives

$$9 \sin^2 v - 6 \sin v + 1 = 16 - 16 \sin^2 v,$$

or

$$25 \sin^2 v - 6 \sin v - 15 = 0.$$

By means of the formula for the roots of a quadratic equation, we obtain

$$\sin v = \frac{6 \pm \sqrt{36 + 1500}}{50} = 0.90384, \qquad \text{or} \qquad -0.66384.$$

From the first value we get

$$v = 64^\circ\ 40'\ 05'',\ 115^\circ\ 19'\ 55'',\ \cdots.$$

The negative value gives

$$v = 221^\circ\ 35'\ 35'',\ 318^\circ\ 24'\ 25'',\ \cdots.$$

If this method is used, all the values obtained must be tested. Thus

$$3 \sin 64^\circ\ 40'\ 05'' - 4 \cos 64^\circ\ 40'\ 05''$$
$$= 3(0.90384) - 4(0.42786) = 1.00008,$$
$$3 \sin 115^\circ\ 19'\ 55'' - 4 \cos 115^\circ\ 19'\ 55''$$
$$= 3(0.90384) - 4(-0.42786) = 4.42296 \neq 1.$$

We see that the value $v = 64^\circ\ 40'\ 05''$ satisfies the given equation to a reasonable degree of accuracy, but that the value $v = 115^\circ\ 19'\ 55''$

does not satisfy it. It can similarly be shown that $v = 221° 35' 35''$ satisfies the equation, but that $v = 318° 24' 25''$ does not.

Exercise 36

1. Solve the following equations:

(a) $2 \cos^2 v - \sin^2 v = 2$,
(b) $2 \cos^2 v + 3 \sin v = 0$,
(c) $\tan v + \cot v = 2$,
(d) $\sin v = 2 \cos v$,
(e) $\sec v = 4 \csc v$,
(f) $\cos 2v + \sin v = 0$,
(g) $\sin v + \cos v = 1$,
(h) $\sin 2v = 3 \sin^2 v - \cos^2 v$,
(i) $\sin^2 v = 1 - \sin 2v$,
(j) $\tan^2 v = \sin 2v$,
(k) $\sin 2v + 2 \cos 2v = 1$,
(l) $4 \sec^2 2v + \tan 2v = 7$,

2. Solve

(a) $\cos v + \cos 2v + \cos 3v = 0$,
(b) $\csc 2v + \cot 2v = 2$,
(c) $\sin 2v \cos 2v = -2 \sin v$,
(d) $\sin 2v + \cos v = 0$,
(e) $5 \cos v + 12 \sin v = 4$,
(f) $3264 \sin v - 5728 \cos v = 6018$,
(g) $0.1723 \cos v + 1.3284 \sin v = 0.8492$,
(h) $\sqrt{3} \cos v - \sin v = \sqrt{2}$,
(i) $\csc v = \cot v + \sqrt{3}$,
(j) $2 \sin^2 v + \sin^2 2v = 2$,
(k) $\tan^2 v + \cot^2 v = 10/3$,
(l) $\cos 3v - 2 \cos 2v + \cos v = 0$.

3. Solve

(a) $\sin (v + 12°) + \sin (v - 8°) = \sin 20°$,
(b) $\sin^4 v - \cos^4 v = 7/36$,
(c) $\sin^4 v + \cos^4 v = 5/8$,
(d) $\sin 3v = \cos 2v - 1$,
(e) $3 - 4 \cos^2 v = \cos 3v$,
(f) $\sin (60° - v) - \sin (60° + v) = \sqrt{3}/2$,

(*g*) $\sin (v - 30°) = \sin 2v$,

(*h*) $\tan (v + 15°) = 3 \tan (v - 15°)$.

4. Solve the following simultaneous equations for r and v in terms of x and y:

$$\begin{cases} x = r \cos v, \\ y = r \sin v. \end{cases}$$

5. Solve the following simultaneous equations for r, u, and v in terms of x, y, and z:

$$\begin{cases} x = r \sin u \cos v, \\ y = r \sin u \sin v, \\ z = r \cos u. \end{cases}$$

6. Solve for u and v

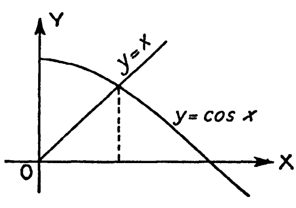
$$\begin{cases} \sin u - \sin v = 0.7038, \\ \cos u - \cos v = -0.7245. \end{cases}$$

7. Solve for u and v in terms of a and b

$$\begin{cases} \sin u + \sin v = a, \\ \cos u + \cos v = b. \end{cases}$$

8. Solve the equation $\cos x = x$.

Solution. Draw the graphs of $y = \cos x$ and $y = x$. (See figure.) The value of x for which these curves intersect is the solution of the equation. According to the graph this value is approximately $x = 0.74$.



A more accurate value may be obtained by writing the equation in the form $\cos x - x = 0$ and employing interpolation. We find that $0.74^{(r)} = 42° \; 23' \; 56''$. Tabulating for several values of x, we get the following results:

	x	$\cos x$	$\cos x - x$
42° 20′	.73886	.73924	.00038
42° 21′	.73915	.73904	−.00011
42° 22′	.73944	.73885	−.00059
42° 23′	.73973	.73865	−.00108
42° 24′	.74002	.73846	−.00156

Since we want the value of $\cos x - x$ to be zero, the required value of x is between 0.73886 and 0.73915. Using the ordinary methods of interpolation, we have

$$\frac{0 - .00038}{-.00011 - .00038} = \frac{-.00038}{-.00049} = \frac{38}{49} = .776,$$

$$x = .73886 + .776(.73915 - .73886)$$

$$= .73886 + .776(.00029) = 0.73908.$$

9. Solve the following equations:

(*a*) $\cos x = 2x$, (*b*) $\sin x = x - 1$,
(*c*) $\sin x = 1/x$, (*d*) $\tan x = 1 - x$,
(*e*) $\sin x = \log x$, (*f*) $\sin x = e^x$,
(*g*) $\cos x + \sin x = e^x$, (*h*) $\cos x = x^2$,
(*i*) $\log x = 1 - x$, (*j*) $x = 2 + \pi \sin x$,
(*k*) $x = 1 + \frac{\pi}{6} \sin x$, (*l*) $x = \sin 2x$.

10. Solve the following equations:

(*a*) $3^x = 2 \cos x$, (*b*) $\sin x = 10^x$.

CHAPTER X

ANALYTIC TRIGONOMETRY

79. Introduction. We wish now to get an insight into some of the more advanced, and in some respects more interesting, phases of trigonometry. Since most of the topics considered in this chapter can be treated more scientifically from the standpoint of higher branches of mathematics, such as the calculus, and as it is our main purpose merely to give an introduction to these topics, we shall, in some instances, only indicate the proofs of certain points, without entering into all the logical refinements necessary for strictly rigorous demonstrations.

80. Solution of algebraic equations. As an illustration of the application of trigonometry to algebra, we shall show how a quadratic equation of the type

$$x^2 + px - q = 0,$$

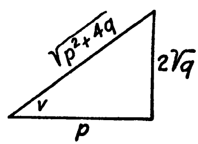
where q is positive, may be solved by the use of trigonometric functions.

We know that the roots of this equation are

$$x_1 = -\frac{p}{2} + \frac{1}{2}\sqrt{p^2 + 4q}, \qquad x_2 = -\frac{p}{2} - \frac{1}{2}\sqrt{p^2 + 4q}.$$

If we let $2\sqrt{q}/p = \tan v$ (see figure), then

$$p = 2\sqrt{q}\cot v, \qquad \sqrt{p^2 + 4q} = \frac{2\sqrt{q}}{\sin v},$$



and therefore

$$x_1 = -\sqrt{q}\left(\cot v - \frac{1}{\sin v}\right)$$

$$= \sqrt{q}\,\frac{1 - \cos v}{\sin v} = \sqrt{q}\tan \tfrac{1}{2}v.$$

Similarly,

$$x_2 = -\sqrt{q}\cot\tfrac{1}{2}v.$$

Example. Solve the equation $3.254x^2 + 17.83x - 4.18 = 0$.

Solution. Let us first reduce to the proper form by dividing through by 3.254, the coefficient of x^2. Then

$$x^2 + \frac{17.83}{3.254}x - \frac{4.18}{3.254} = 0.$$

The logarithmic work of the solution follows:

$p = \dfrac{17.83}{3.254}$	log 17.83	1.25115
	colog 3.254	9.48758
	log 4.18	0.62118
$q = \dfrac{4.18}{3.254}$	log p	0.73873
	log q	0.10876
	log $\sqrt{q}$	0.05438
	log 2	0.30103
	colog p	9.26127
$\tan v = \dfrac{2\sqrt{q}}{p}$	log tan v	9.61668
	v	22° 28′ 28″
	$\frac{1}{2}v$	11° 14′ 14″
	log tan $\frac{1}{2}v$	9.29815
	log $\sqrt{q}$	0.05483
	log cot $\frac{1}{2}v$	0.70185
$x_1 = \sqrt{q}\tan\frac{1}{2}v$	log x_1	9.35253
$x_2 = -\sqrt{q}\cot\frac{1}{2}v$	log $(-x_2)$	9.75623
	x_1	0.22518
	x_2	$-$ 5.7046

Quadratic equations of other types than the one considered may also be solved by trigonometry. In solving certain other algebraic equations and in treating various topics in algebra, trigonometry not only simplifies the work, but in some cases is quite necessary. For the trigonometric solution of certain cubic equations see any good work on college algebra. The solution of equations of the type $x^n = a$ is considered later in this chapter.

81. Imaginary numbers. We recall from algebra that an **imaginary number** is a number which is the product of a real number and the square root of minus one, as $3\sqrt{-1}$,

$-\frac{1}{8}\sqrt{-1}$, $\sqrt{-3}$ (the last of which is equal to $\sqrt{3}\,\sqrt{-1}$). We usually represent $\sqrt{-1}$ by the letter i. From the definition of i, we have

$$i^2 = -1, \qquad i^3 = i^2 \cdot i = -i, \qquad i^4 = 1, \qquad i^5 = i, \qquad \cdots.$$

82. Geometric representation of imaginary numbers. If we have an axis upon which we plot real numbers, and locate on it the number a, then the number $-a$ is at the same distance from the origin, but in the opposite direction. Thus, the multiplication of a by -1 may be represented geometrically by rotating the line Oa (the number a may be thought of as represented by the line Oa or by the point a) through an angle of 180°.

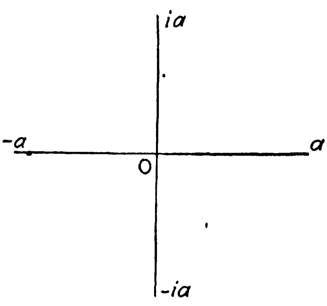


FIG. 90

Now the multiplication of a by i is a process which if performed twice will give $-a$. Therefore, it is quite natural to represent multiplication by i by means of a rotation through 90° (in a positive direction).

Pursuing this idea further, we see that multiplication by i^3 means a rotation through 270°. But $i^3 = -i$, and therefore the number $-ia$ is the same distance from O as the number ia, but in the opposite direction. Multiplying the number a by i^4 is equivalent to multiplying it by 1, since $i^4 = 1$. Geometrically this would be represented by rotation through 360°, which would bring the point back to its original position. Thus our geometric representation of multiplication by i is seen to be entirely consistent with the results of algebra.

83. Complex numbers. A number of the form $a + ib$, in which a and b are real numbers (but either may be plus or minus), is called a **complex number.** If a is zero the complex number reduces to the imaginary number ib (sometimes called a **pure imaginary**), and if b is zero the complex number reduces to the real number a. Thus it is seen that real numbers and pure imaginaries are special cases of complex number.

84. Geometric representation of complex numbers. Let us take two mutually perpendicular axes and mark off on them, according to the conventions of section 82, the numbers 1, -1, i, $-i$. The axis upon which the real numbers 1 and -1 are plotted will be called the **axis of reals, or real axis,** and the axis upon which the numbers i and $-i$ are plotted will be called the **axis of imaginaries,**

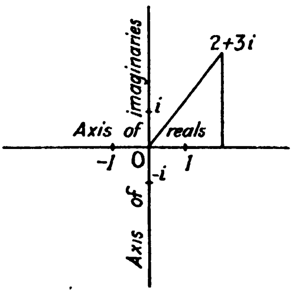


FIG. 91

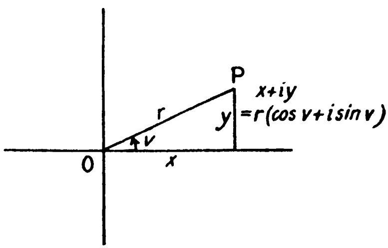


FIG. 92

or the **imaginary axis.** The complex number $2 + 3i$ will be plotted by a point two units to the right of the axis of imaginaries, and three units above the axis of reals. In general, the complex number $x + iy$ will be plotted by a point x units from the imaginary axis (to the right if x is positive, to the left if x is negative), and y units from the real axis (above if y is positive, below if y is negative).

(See Fig. 92.) It is obvious that according to this scheme of plotting, all real numbers will lie upon the axis of reals, and all pure imaginaries upon the axis of imaginaries.

The complex number $x + iy$ is represented by the point P in Fig. 92, or by the line OP in that figure. If we denote by r the length of the line OP, and by v the angle that OP makes with the real axis, we have

$$x = r \cos v, \qquad y = r \sin v.$$

Therefore the complex number $x + iy$ is equal to $r \cos v + ir \sin v$ or $r(\cos v + i \sin v)$. The form $x + iy$ is called the **rectangular form,** while $r(\cos v + i \sin v)$ is called the **trigonometric,** or **polar form** of the complex number.*

In the trigonometric form of the complex number, r is called the **radius,** the **modulus,** or the **absolute value,** and v is called the **angle,** the **argument,** or the **amplitude.** We have

$$r = \sqrt{x^2 + y^2}, \qquad v = \tan^{-1}\frac{y}{x} = \cos^{-1}\frac{x}{r} = \sin^{-1}\frac{y}{r}.$$

The absolute value of a number is expressed by enclosing the number with vertical straight lines. Thus, the absolute value of $x + iy$ is expressed by $|x + iy|$. It should be clear that $|3 + 4i| = \sqrt{3^2 + 4^2} = 5$, $|-2| = 2$, $|3| = 3$. Modulus is sometimes abbreviated **mod,** and amplitude **amp.**

Since the line OP (see Fig. 92) may be considered as a vector, it is obvious that complex numbers can be represented by vectors. On account of this fact they are important not only in theoretical mathematics,—where they are indispensable,—but also in various fields of applied mathematics where vectors are used. A particular example

* The expression $\cos v + i \sin v$ is sometimes abbreviated cis v.

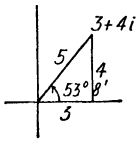
of a field in which they are very useful is the theory of alternating currents.

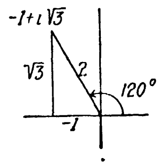
The complex number $3 + 4i$ can be reduced to the trigonometric form as follows:

$$r = \sqrt{3^2 + 4^2} = 5, \qquad \tan v = \tfrac{4}{3} = 1.33333,$$
$$v = 53^\circ\ 08' \text{ (from tables).}$$

Therefore

$$3 + 4i = 5(\cos 53^\circ\ 08' + i \sin 53^\circ\ 08').$$





Similarly, for the number $-1 + i\sqrt{3}$,

$$r = \sqrt{1 + 3} = 2, \qquad \tan v = \frac{\sqrt{3}}{-1} = -\sqrt{3}, \qquad v = 120^\circ.$$

Hence

$$-1 + i\sqrt{3} = 2(\cos 120^\circ + i \sin 120^\circ).$$

85. Addition and subtraction of complex numbers. As should be recalled from algebra, complex numbers are added by adding their real parts to find the real part of their sum, and their imaginary parts to find the imaginary part of their sum. This can of course be done geometrically by simply adding the vectors that represent the complex numbers (see section 72). To subtract a complex number we merely add its negative; that is, to subtract $a + ib$ we add $-a - ib$.

Exercise 37

1. Represent geometrically the following numbers, finding the modulus and the argument in each case:

(*a*) $2 - 2i$, (*b*) $-2 - 2i$, (*c*) $-2 + 2i$,
(*d*) $-2 - 4i$, (*e*) $3i$, (*f*) $3 + 0i$,
(*g*) $0.6 + 0.5i$, (*h*) 7, (*i*) -5,
(*j*) i, (*k*) $-i$, (*l*) $\frac{5}{4}i$,
(*m*) $\frac{1}{2} + \frac{i}{3}$, (*n*) $-8 + 2i$, (*o*) $7 - i$,
(*p*) $\sqrt{3} - i$, (*q*) $2 + i\sqrt{5}$.

2. Represent the following numbers geometrically, and write them in the form $x + iy$:

(*a*) $2(\cos 45° + i \sin 45°)$,
(*b*) $3(\cos 30° + i \sin 30°)$,
(*c*) $5(\cos 150° + i \sin 150°)$,
(*d*) $4(\cos 225° + i \sin 225°)$,
(*e*) $\frac{3}{4}(\cos 300° + i \sin 300°)$,
(*f*) $6(\cos 60° + i \sin 60°)$,
(*g*) $3(\cos 360° + i \sin 360°)$,
(*h*) $3(\cos 180° + i \sin 180°)$,
(*i*) $\sqrt{2}(\cos 315° + i \sin 315°)$,
(*j*) $4\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$,
(*k*) $2\left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}\right)$,
(*l*) $\left(\cos\frac{11\pi}{6} + i \sin\frac{11\pi}{6}\right)$,
(*m*) $5\left(\cos\frac{5\pi}{4} + i \sin\frac{5\pi}{4}\right)$.

3. Perform the operations indicated in the following problems, and illustrate the work graphically:

(*a*) $(3 - 5i) + (4 - 6i)$,
(*b*) $(5 + 3i) - (2 - i)$,
(*c*) $(1 + i) + (3 - i) - (6 + 2i)$,

(*d*) $(2.7 - 3.5i) - (0.5 - 5.2i)$,
(*e*) $(-4 + 2i) + (-3 - 5i)$,
(*f*) $(1 - 3i) - (-5 + i)$,
(*g*) $2(\cos 60° + i \sin 60°) - \sqrt{2}(\cos 45° + i \sin 45°)$,
(*h*) $2(\cos 300° + i \sin 300°) + (\cos 90° + i \sin 90°)$,
(*i*) $5(\cos 125° + i \sin 125°) - 3(\cos 280° + i \sin 280°)$.

86. Multiplication and division of complex numbers. A very interesting result is obtained if two complex numbers expressed in trigonometric form are multiplied together. Thus,

$$\begin{aligned} &r_1(\cos v_1 + i \sin v_1) \times r_2(\cos v_2 + i \sin v_2) \\ &\quad = r_1r_2(\cos v_1 \cos v_2 + i \cos v_1 \sin v_2 \\ &\qquad\qquad + i \sin v_1 \cos v_2 + i^2 \sin v_1 \sin v_2) \\ &\quad = r_1r_2[(\cos v_1 \cos v_2 - \sin v_1 \sin v_2) \\ &\qquad\qquad + i(\sin v_1 \cos v_2 + \cos v_1 \sin v_2)] \\ &\quad = r_1r_2[\cos (v_1 + v_2) + i \sin (v_1 + v_2)] \end{aligned}$$

(using the addition formulas for the cosine and sine). We see, therefore, that **the product of two complex numbers is a complex number whose radius is the product of the radii of the given numbers and whose angle is the sum of their angles.** It can readily be seen that this holds true if any number of complex numbers are multiplied together.

If one complex number is to be divided by another we express the quotient as a fraction and rationalize the denominator. Thus,

$$\begin{aligned} &\frac{r_1(\cos v_1 + i \sin v_1)}{r_2(\cos v_2 + i \sin v_2)} \\ &\quad = \frac{r_1(\cos v_1 + i \sin v_1)}{r_2(\cos v_2 + i \sin v_2)} \times \frac{\cos v_2 - i \sin v_2}{\cos v_2 - i \sin v_2} \\ &\quad = \frac{r_1[(\cos v_1 \cos v_2 + \sin v_1 \sin v_2) + i(\sin v_1 \cos v_2 - \cos v_1 \sin v_2)]}{r_2(\cos^2 v_2 + \sin^2 v_2)} \end{aligned}$$

$$= \frac{r_1}{r_2}[\cos(v_1 - v_2) + i \sin(v_1 - v_2)].$$

In words, **if we divide one complex number by another, the quotient is a complex number whose radius is the radius of the first complex number divided by that of the second, and whose angle is the angle of the first minus that of the second.** This was to have been expected since division is the reverse of multiplication.

Exercise 38

1. Perform the indicated operations in the following problems, illustrating the work graphically:

(*a*) $2(\cos 30° + i \sin 30°) \times 3(\cos 60° + i \sin 60°)$,
(*b*) $5(\cos 45° + i \sin 45°) \times 2(\cos 150° + i \sin 150°)$,
(*c*) $3(\cos 32° + i \sin 32°) \times 4(\cos 18° + i \sin 18°)$,
(*d*) $8(\cos 75° + i \sin 75°) \div 2(\cos 15° + i \sin 15°)$,
(*e*) $7(\cos 15° + i \sin 15°) \div 4(\cos 60° + i \sin 60°)$,
(*f*) $3(\cos 180° + i \sin 180°) \div 2(\cos 90° + i \sin 90°)$,
(*g*) $2(\cos 52° + i \sin 52°) \div 3(\cos 130° + i \sin 130°)$,
(*h*) $2(\cos 30° + i \sin 30°) \times (\cos 60° + i \sin 60°) \times 3(\cos 75° + i \sin 75°)$.
(*i*) $[3(\cos 30° + i \sin 30°)]^2$.

2. Perform the indicated operations and graph the results:

(*a*) $(2 + 2i)(\sqrt{3} + i)$,
(*b*) $(\frac{2}{3} - \frac{4}{5}i)(\frac{2}{3} + \frac{4}{5}i)$,
(*c*) $(3 - \sqrt{-5})(2 + \sqrt{-1})$,
(*d*) $7i(4 - 3i)(4 + 3i)$,
(*e*) $(-1 + i)(1 - i\sqrt{3})$,
(*f*) $(1 + i)^2$,
(*g*) $(1 + i)^3$,

(h) $(2 - 3i)(4 + 2i)(6 - 5i)$,
(i) $-3i \times 2i \times (0.1 - 0.5i)$,
(j) $(1 + i\sqrt{3}) \div (2\sqrt{3} + 2i)$,
(k) $(-3\sqrt{3} + 3i) \div (2 + 2i)$,
(l) $4i \div (2 + 2i\sqrt{3})$,
(m) $(6 \div i) \div (1 - 3i)$,
(n) $(-2 + 3i) \div 2i$,
(o) $25 \div (1 - 5i)$,
(p) $12i \div (1 - i)$,
(q) $1 \div i$,
(r) $(2 + 3i) \div (-1 - 4i)$,
(s) $\dfrac{2 - i\sqrt{5}}{3 - 4i}$,
(t) $\dfrac{(1 + i)(-2\sqrt{3} + 2i)}{1 + i\sqrt{3}}$,
(u) $4(\cos 60° + i \sin 60°)(1 - i)$.

87. De Moivre's theorem. It follows immediately from the method of multiplication given in the preceding section that if n is a positive integer

$$[r(\cos v + i \sin v)]^n = r^n(\cos nv + i \sin nv).$$

That is, **the *n*th power (*n* being a positive integer) of a complex number is a complex number whose radius is the *n*th power of the radius of the given number, and whose angle is *n* times the angle of the given number.** This is **De Moivre's theorem.**

HISTORICAL NOTE. **Abraham De Moivre** (1667–1754) was an English mathematician, but a native of France. He belonged to a French Protestant family and at the revocation of the Edict of Nantes, when he was eighteen, he took refuge in England. He was an intimate personal friend of Sir Isaac Newton and the *Principia* gave him an incentive to mathematical studies. He became so great a mathematician that Newton in his later years answered those who inquired of him regarding mathematics or of his *Principia*, "Go to Mr. De Moivre; he knows these things better than I do." The Royal Society called upon him to decide the contest between Newton and Leibnitz as to priority in the invention of the calculus. He was the first writer to develop the part of trigonometry dealing with complex numbers. He revolutionized higher trigonometry by the discovery of the theorem which bears his name. He also wrote on recurring series and on partial fractions. His work on probability surpasses all other early works on this topic except that of Laplace. He survived most of his intimate friends and his old age was spent in obscure poverty.

88. Extension of De Moivre's theorem. Suppose that n is a negative integer, say $-m$ $(m > 0)$. Then

$$\begin{aligned}[r(\cos v + i \sin v)]^{-m} &= r^{-m}\frac{1}{(\cos v + i \sin v)^m}\\ &= r^{-m}\frac{\cos 0 + i \sin 0}{\cos mv + i \sin mv}\\ &= r^{-m}[\cos(-mv) + i \sin(-mv)]\end{aligned}$$

(by the method of division given in the preceding section).

If $n = 1/p$, where p is an integer, we have

$$\left[r^{1/p}\left(\cos\frac{v}{p} + i \sin\frac{v}{p}\right)\right]^p = r(\cos v + i \sin v),$$

or, reversing the right- and left-hand sides of the equation and taking the pth root of each side,

$$[r(\cos v + i \sin v)]^{1/p} = r^{1/p}\left(\cos\frac{v}{p} + i \sin\frac{v}{p}\right).$$

Similarly, if p and q are integers,

$$[r(\cos v + i \sin v)]^{q/p} = r^{q/p}\left(\cos \frac{q}{p} v + i \sin \frac{q}{p} v\right).$$

89. Extracting roots of numbers. The preceding section shows, as might have been anticipated from the method of raising a complex number to a power, that **the pth root of a complex number is a complex number whose radius is the positive numerical pth root of the radius of the given number, and whose angle is that of the given number divided by p.**

Suppose, for example, that we want to find the cube root of -1. Since the radius of -1 is 1, and its angle is 180°, its trigonometric form is

$$1(\cos 180° + i \sin 180°). \tag{125}$$

The most general form for this, however, is

$$1[\cos (180° + k \times 360°) + i \sin (180° + k \times 360°)],$$

where k is an integer or zero, since this includes all angles whose terminal sides coincide with the line drawn from the origin to the point -1. (The reason for writing the number in its most general form will be apparent presently.)

Now the positive cube root of 1 is 1, and consequently

$$\sqrt[3]{-1} = 1\left(\cos \frac{180° + k \times 360°}{3} + i \sin \frac{180° + k \times 360°}{3}\right)$$

$$= 1[\cos (60° + k \times 120°) + i \sin (60° + k \times 120°)].$$

Let $k = 0$; then we get as one cube root

$$1(\cos 60° + i \sin 60°) = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Let $k = 1$; we get

$$1(\cos 180° + i \sin 180°) = -1.$$

Let $k = 2$; we get

$$1(\cos 300° + i \sin 300°) = \frac{1}{2} - i\,\frac{\sqrt{3}}{2}.$$

Thus we have all three of the cube roots of -1 expressed in the trigonometric and in the rectangular form. For if $k = 3$, we get $1(\cos 420° + i \sin 420°)$, which is the same as the first root that we obtained, since its angle differs from that of the first root by 360°, and any other integral value of k will give one of the three roots already found. If we had merely written the number -1 in the form (125) we should not have obtained all of its cube roots.

The results of the foregoing work are illustrated graphically in Fig. 93.

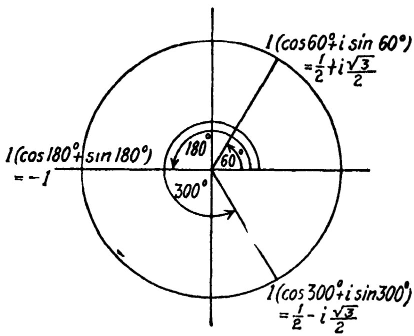


FIG. 93

Exercise. Verify the three numbers found to be the cube roots of -1.

Let us, more generally, find the pth roots of any complex number. We shall suppose the number written in its most general trigonometric form, namely

$$r[\cos (v + k\cdot 360°) + i \sin (v + k\cdot 360°)],$$

where k is any integer or zero. Then for the pth roots we have

$$\sqrt[p]{r}\left[\cos\left(\frac{v}{p}+k\cdot\frac{360^\circ}{p}\right)+i\sin\left(\frac{v}{p}+k\cdot\frac{360^\circ}{p}\right)\right],$$

in which $\sqrt[p]{r}$ is the positive numerical pth root of r. Designating the various roots by R_0, R_1, etc., we have, for $k = 0$,

$$R_0 = \sqrt[p]{r}\left(\cos\frac{v}{p}+i\sin\frac{v}{p}\right);$$

for $k = 1$,

$$R_1 = \sqrt[p]{r}\left[\cos\left(\frac{v}{p}+\frac{360^\circ}{p}\right)+i\sin\left(\frac{v}{p}+\frac{360^\circ}{p}\right)\right];$$

for $k = 2$,

$$R_2 = \sqrt[p]{r}\left[\cos\left(\frac{v}{p}+2\cdot\frac{360^\circ}{p}\right)+i\sin\left(\frac{v}{p}+2\cdot\frac{360^\circ}{p}\right)\right];$$

. . . .

for $k = p - 1$,

$$R_{p-1} = \sqrt[p]{r}\left[\cos\left(\frac{v}{p}+\overline{p-1}\cdot\frac{360^\circ}{p}\right)\right.$$
$$\left.+ i\sin\left(\frac{v}{p}+\overline{p-1}\cdot\frac{360^\circ}{p}\right)\right].$$

For $k = p$, we get

$$R_p = \sqrt[p]{r}\left[\cos\left(\frac{v}{p}+p\cdot\frac{360^\circ}{p}\right)+i\sin\left(\frac{v}{p}+p\cdot\frac{360^\circ}{p}\right)\right]$$
$$= \sqrt[p]{r}\left[\cos\left(\frac{v}{p}+360^\circ\right)+i\sin\left(\frac{v}{p}+360^\circ\right)\right],$$

which is obviously the same as R_0. Similarly, for any other possible value of k we get a root which is the same as one of the p roots $R_0, R_1, \cdots, R_{p-1}$ already found.

Exercise. To what do the above roots reduce if the given number is 1? In other words, find all of the pth roots of 1.

Example. Find the five fifth roots of $-3 + 4i$ and illustrate the work graphically.

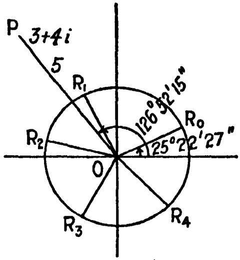
Solution. To reduce the number $-3 + 4i$ to the trigonometric form we find that

$$r = \sqrt{(-3)^2 + 4^2} = 5 \qquad \tan v = -\frac{4}{3}, \quad = -1.33333,$$

$$v = 126° \, 52' \, 15''.$$

Therefore

$$\begin{aligned} -3 + 4i \\ &= 5[\cos(126° \, 52' \, 15'' + k \cdot 360°) \\ &\quad + i \sin(126° \, 52' \, 15'' + k \cdot 360°)]. \end{aligned}$$



The five fifth roots are given by

$$\sqrt[5]{5}\left[\cos\left(\frac{126° \, 52' \, 15''}{5} + k \cdot \frac{360°}{5}\right) + i \sin\left(\frac{126° \, 52' \, 15''}{5} + k \cdot \frac{360°}{5}\right)\right]$$

$$= \sqrt[5]{5}[\cos(25° \, 22' \, 27'' + k \cdot 72°) + i \sin(25° \, 22' \, 27'' + k \cdot 72°)].$$

Giving k the values 0, 1, 2, 3, 4, we get as the roots

$$\begin{aligned} R_0 &= \sqrt[5]{5}\,(\cos \; 25° \, 22' \, 27'' + i \sin \; 25° \, 22' \, 27''), \\ R_1 &= \sqrt[5]{5}\,(\cos \; 97° \, 22' \, 27'' + i \sin \; 97° \, 22' \, 27''), \\ R_2 &= \sqrt[5]{5}\,(\cos 169° \, 22' \, 27'' + i \sin 169° \, 22' \, 27''), \\ R_3 &= \sqrt[5]{5}\,(\cos 241° \, 22' \, 27'' + i \sin 241° \, 22' \, 27''), \\ R_4 &= \sqrt[5]{5}\,(\cos 313° \, 22' \, 27'' + i \sin 313° \, 22' \, 27''). \end{aligned}$$

In the accompanying figure the vector OP represents the original number $-3 + 4i$. Since the fifth roots of this number all have the radius $\sqrt[5]{5} = 1.38$ (the value may be obtained by using logarithms), we draw a circle with this radius and they will all lie on its circumference. The first fifth root is represented by the point R_0 or by the vector OR_0 which has the angle $25° \, 22' \, 27''$. The other roots are spaced at equal intervals of 72°.

Exercise. Reduce one of these roots to the rectangular form and verify that it is a root of $-3 + 4i$ by raising it to the fifth power.

It is seen that the foregoing process enables us to solve equations of the type $x^p = a$. For we simply have to find the p pth roots of a.

Exercise 39

In the following problems illustrate your work graphically.

1. Raise to the powers indicated:

(a) $[3(\cos 25° + i \sin 25°)]^3$,
(b) $[1/2(\cos 45° + i \sin 45°)]^4$,
(c) $[4/7(\cos 300° + i \sin 300°)]^5$,
(d) $(2 + 2i)^3$,
(e) $(-\sqrt{3} + i)^2$,
(f) $(1 - 2i)^2$,
(g) $(5 + 6i)^3$,
(h) $(-i)^7$,
(i) $(1 + i)^9$,
(j) 2^5,
(k) $(-2)^3$.

2. Find the cube roots of $8(\cos 45° + i \sin 45°)$.
3. Find the square roots of $2(\cos 300° + i \sin 300°)$.
4. Find the cube roots of 1.
5. Find the fourth roots of -1.
6. Find the fourth roots of $81i$.
7. Find the fifth roots of $-32i$.
8. Find the cube roots of $\sqrt{3} + i$.
9. Find the square roots of $\dfrac{-1 + i\sqrt{3}}{2}$.
10. Find the square roots of $1 - i\sqrt{3}$.
11. Find the cube roots of $-1 + i$.
12. Find the cube roots of $2 + 3i$.
13. Find the sixth roots of $-5 - 12i$.
14. Solve the following equations:

(a) $x^5 - 1 = 0$,
(b) $x^6 + 64 = 0$,
(c) $x^5 + 10 = 0$,
(d) $x^5 - 235 = 0$,
(e) $x^{7/2} = 3$,
(f) $x^{5/3} + 2 = 0$.

15. Solve the equation $x^4 + x^3 + x^2 + x + 1 = 0$.

Suggestion. Multiply by $x - 1$, solve the resulting equation, and discard the root 1, which was introduced by multiplying by $x - 1$.

16. Solve $x^4 - x^3 + x^2 - x + 1 = 0$.

17. Solve $x^5 = a$.

90. Angles near 0° or 90°. We have shown that for those functions whose values change very rapidly for angles near 0° and 90°, the ordinary tables do not give sufficiently accurate results for many purposes. This difficulty may be remedied, as was suggested, by providing tables which give values of the functions for very small intervals of the angle. However, the difficulty may be met in another manner.

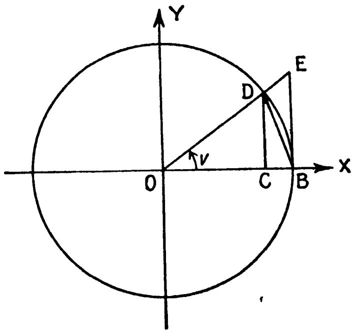


FIG. 94

It is evident from Fig. 94, in which v is an acute angle, that

$$\text{triangle } BOD < \text{sector } BOD < \text{triangle } BOE.$$

Moreover,

$$\text{area of triangle } BOD = \tfrac{1}{2}(OB \times CD),$$
$$\text{area of sector } BOD = \tfrac{1}{2}(OB \times \text{arc } BD),$$
$$\text{area of triangle } BOE = \tfrac{1}{2}(OB \times BE).$$

Now

$$CD = OD \sin v,$$
$$\text{arc } BD = OD \times v \text{ (in radians)},$$
$$BE = OB \tan v = OD \tan v.$$

Hence

$$\tfrac{1}{2}OB \times OD \sin v < \tfrac{1}{2}OB \times OD \times v < \tfrac{1}{2}OB \times OD \tan v.$$

Dividing out the common factor $\frac{1}{2}OB \times OD$, we get

$$\sin v < v < \tan v. \tag{126}$$

That is, if v is an acute angle measured in radians, it will always be greater than its sine and less than its tangent.

If we divide (126) by $\sin v$, and remember that $\tan v = \sin v/\cos v$ and that $1/\cos v = \sec v$, we find that

$$1 < \frac{v}{\sin v} < \sec v.$$

Now $\sec 0 = 1$. Hence we see that the ratio $v/\sin v$ must always be greater than 1, and yet as the angle decreases, approaching 0, the ratio must be less than the positive quantity $\sec v$, which we can make as near the value 1 as we please by decreasing the angle. It is evident, therefore, that as v approaches 0, $v/\sin v$ must approach 1 as its value. In other words, the limit, as v approaches 0, of $v/\sin v$ is 1. This may be written

$$\lim_{v \to 0} \frac{v}{\sin v} = 1.$$

It is clear also that

$$\lim_{v \to 0} \frac{\sin v}{v} = 1. \tag{127}$$

In like manner we may divide (126) by $\tan v$ and get

$$\cos v < \frac{v}{\tan v} < 1.$$

Now since $\cos 0 = 1$, we have

$$\lim_{v \to 0} \frac{v}{\tan v} = 1,$$

and

$$\lim_{v \to 0} \frac{\tan v}{v} = 1. \tag{128}$$

Thus, if v is a small angle, the value of $\sin v$ and of $\tan v$ is approximately the value of v in radians.

The truth of the above may be verified by reference to tables. To illustrate,

$$\sin 2° = 0.03490, \qquad \tan 2° = 0.03492, \qquad 2° = 0.03492^{(r)}.$$

Hence we may summarize as follows:

If the angle v is near 0,

$$\begin{aligned}&\sin v = \tan v = v^{(r)} \text{ approximately,}\\ &\cot v = \csc v = 1/v^{(r)} \text{ approximately,}\\ &\cos v \text{ and } \sec v \text{ are to be found as usual.}\end{aligned}$$

If the angle v is near 90°,

$$\begin{aligned}&\cos v = \cot v = (\pi/2 - v)^{(r)} \text{ approximately,}\\ &\tan v = \sec v = 1/(\pi/2 - v)^{(r)} \text{ approximately,}\\ &\sin v \text{ and } \csc v \text{ are to be found as usual.}\end{aligned}$$

In applying the above to problems we must remember to use the values of the angle in radians.

Example. Find log tan 2′ 54″.

Solution.

$$2' \, 54'' = 0.048333° = (0.048333 \times 0.017453)^{(r)}$$

$$\begin{array}{ll} \log 0.048333 & = 8.68425 \\ \log 0.017453 & = 8.24187 \\ \hline \log \tan 2' \, 54'' & = 6.92612 \end{array}$$

Let us compare this with other methods.

By interpolation in tables giving the functions for every minute we get

$$\log \tan 2' \, 54'' = 6.92324.$$

By interpolating in tables giving the functions for every 10″ we get

$$\log \tan 2' \, 54'' = 6.92595.$$

From tables with differences for every second we get

$$\log \tan 2' \, 54'' = 6.92612.$$

Using the S and T method explanation of which is given in tables (see Note, p. —), we have $2'\ 54'' = 174''$.

$$\begin{array}{ll} \log 174 & = 2.24055 \\ \log T & = 4.68557 \\ \hline \log \tan 2'\ 54'' & = 6.92612 \end{array}$$

It is thus seen that our new method gives log tan $2'\ 54''$ correct to five places of decimals.

Exercise 40

1. Find the functions of $1'$.

2. The inclination of a railroad track is $50'$. How many feet does it rise in 2 miles?

3. The moon is 238,862 miles from the earth and it subtends an angle of $31'\ 05''$ at the earth. What is the diameter of the moon in miles?

4. The sun is 92,897,416 miles from the earth and it subtends an angle of $31'\ 59''$ at the earth. What is the diameter of the sun in miles?

5. At Alpha Centauri, the nearest star to our sun, the distance from the earth to the sun subtends an angle of $0.75''$. (Note. $0.75''$ is called the **heliocentric parallax** of the star.) What is the approximate distance from the sun to the star?

6. A railroad track rises 70 feet in a mile. What is the angle of inclination?

7. The mean radius of the earth is approximately 3,957 miles. It subtends an angle of $8.8''$ at the sun. Find the distance from the earth to the sun. (Note. $8.8''$ is called the **geocentric parallax** of the sun.)

8. If the parallax of the moon is $57'\ 02.6''$, what is the distance from the earth to the moon?

9. Assuming the data of the above problems, and measuring the diameter of a dime, find how far from the eye a person should hold a dime to cut off the entire disc

of the sun. How far from the earth should the moon be to cut off the disc of the sun?

Better approximations for the trigonometric functions than those given in the preceding section may be obtained as follows:

91. Series representing the trigonometric functions. By De Moivre's theorem we have

$$\cos nv + i \sin nv = (\cos v + i \sin v)^n.$$

If we expand the second member by the binomial theorem, we obtain

$$(\cos v + i \sin v)^n = \cos^n v + n \cos^{n-1} v \cdot i \sin v + \frac{n(n-1)}{1 \cdot 2} \cos^{n-2} v \cdot i^2 \sin^2 v + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \cos^{n-3} v \cdot i^3 \sin^3 v + \cdots.$$

Therefore, since $i^2 = -1$, $i^3 = -i$, etc.,

$$\begin{aligned} \cos nv + i \sin nv &= \cos^n v + in \cos^{n-1} v \sin v \\ &- \frac{n(n-1)}{1 \cdot 2} \cos^{n-2} v \sin^2 v \\ &- i \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \cos^{n-3} v \sin^3 v + \cdots. \end{aligned} \tag{129}$$

Now in algebra it is shown that if a, b, c, d are real numbers and $a + ib = c + id$, then $a = c$ and $b = d$; that is, if two complex numbers are equal their real parts are equal and their imaginary parts are equal. Equating real parts and imaginary parts respectively in (129) we get

$$\cos nv = \cos^n v - \frac{n(n-1)}{1 \cdot 2} \cos^{n-2} v \sin^2 v + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cos^{n-4} v \sin^4 v + \cdots, \tag{130}$$

$$\text{(131)} \quad \sin nv = n \cos^{n-1} v \sin v$$
$$- \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} \cos^{n-3} v \sin^3 v$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5} \cos^{n-5} v \sin^5 v + \cdots.$$

In the foregoing formulas let us set $nv = x$, that is, $n = x/v$. We get

$$\cos x = \cos^n v - \frac{\frac{x}{v}\left(\frac{x}{v} - 1\right)}{1\cdot 2} \cos^{n-2} v \sin^2 v + \cdots$$
$$= \cos^n v - \frac{x(x-v)}{1\cdot 2} \cos^{n-2} v \left(\frac{\sin v}{v}\right)^2 + \cdots,$$

$$\sin x = \frac{x}{v} \cos^{n-1} v \sin v$$
$$- \frac{\frac{x}{v}\left(\frac{x}{v} - 1\right)\left(\frac{x}{v} - 2\right)}{1\cdot 2\cdot 3} \cos^{n-3} v \sin^3 v + \cdots$$
$$= x \cos^{n-1} v \frac{\sin v}{v}$$
$$+ \frac{x(x-v)(x-2v)}{1\cdot 2\cdot 3\cdot} \cos^{n-3} v \left(\frac{\sin v}{v}\right)^3 + \cdots.$$

Now let n become indefinitely large and v at the same time approach zero in such a way that nv is always equal to x. Then, since $\lim\limits_{v\to 0} \frac{\sin v}{v} = 1$,

$$\text{(132)} \qquad \cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \cdots,$$

$$\text{(133)} \qquad \sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots,$$

where $\underline{|2} = 2 \cdot 1$, $\underline{|3} = 3 \cdot 2 \cdot 1$, $\underline{|4} = 4 \cdot 3 \cdot 2 \cdot 1$, etc., these symbols being read *factorial two, factorial three, factorial four,* etc., respectively. They are sometimes written $2!$, $3!$, $4!$, etc.

The foregoing series for the sine and cosine are convergent for all finite values of x. That is, if we substitute any finite number for x in the series, the sum of the terms of the series, as we take more and more terms, approaches a definite limit, which is the value of the function for the given value of x. For illustration, if we set $x = 1$ in (133), we get

$$\sin 1 = 1 - \frac{1}{\underline{|3}} + \frac{1}{\underline{|7}} - \frac{1}{\underline{|9}} + \cdots,$$

and as the number of the terms on the right increases the sum of these terms approaches a definite value, which is the sine of one radian.

To find series for the tangent and cotangent we divide the sine and cosine series into each other, obtaining

$$(134) \qquad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots,$$

$$(135) \qquad \cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots.$$

The series for $\tan x$ is convergent if x is less in absolute value than $\pi/2$; the series for $\cot x$ is convergent if x is different from zero but less in absolute value than π.

It should be noted that the radian measure of x is to be used in all of these series. Tables of the natural values of the trigonometric functions may be computed from the series.

Exercise 41

1. Compute sin 5° correct to four decimal places and check with the tables. $\left(x = \frac{5}{180}\pi = \frac{\pi}{36} = 0.087265.\right)$

2. Compute cos 10° correct to four decimal places.

3. Find sin 15° and cos 15° from the results of examples 1 and 2.

4. Find sin 30° by means of the sine series.

5. Find cos 95°, sin 85°.

6. Find $\sin 1^{(r)}$, $\cos 1^{(r)}$.

7. Find the series for $\sec x$ by dividing the cosine series into 1.

8. Find the series for $\csc x$.

9. Prove $\sin^2 x - \cos^2 x = 1$ by means of series.

10. Prove $\sin 2x = 2 \sin x \cos x$ by means of series.

92. The number *e*. The limit of the quantity $(1 + 1/n)^n$ as n becomes indefinitely large is denoted by e. Let us find an approximate value of this number. Expanding by the binomial theorem, we get

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{1\cdot 2}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\frac{1}{n^3} + \cdots$$

$$= 1 + 1 + \frac{1\left(1 - \frac{1}{n}\right)}{1\cdot 2} + \frac{1\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{1\cdot 2\cdot 3} + \cdots.$$

Now let n approach infinity. Then $1/n$ approaches zero, and consequently

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots.$$

It is readily seen that e lies between 2 and 3. For certainly e is greater than 2. But

$$\frac{1}{\lfloor 3} = \frac{1}{3 \cdot 2} < \frac{1}{2 \cdot 2} = \frac{1}{2^2},$$

$$\frac{1}{\lfloor 4} = \frac{1}{4 \cdot 3 \cdot 2} < \frac{1}{2 \cdot 2 \cdot 2} = \frac{1}{2^3},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Thus,

$$e < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots.$$

But the terms on the right, after the first term, constitute a geometric series whose ratio is $\frac{1}{2}$, and whose sum is therefore $1/(1 - \frac{1}{2}) = 2$. Thus e is less than $1 + 2$, that is, less than 3.

To find out more exactly the value of e we note that the second term in the series is found by dividing the first by one, the third term is found by dividing the second by two, the fourth by dividing the third by four, etc. This suggests the following scheme for finding the value of e:

$$\begin{array}{r}
1)1.00000 \\
2)1.00000 \\
3)0.50000 \\
4)0.16667 \\
5)0.04167 \\
6)0.00833 \\
7)0.00139 \\
8)0.00017 \\
\underline{0.00002} \\
2.71825
\end{array}$$

This value is correct to four decimal places.

The number e is used as the base of the natural system of logarithms, as was mentioned in section 28.

93. The exponential series. Let us now find a series for e^x. We have

$$\left[\left(1+\frac{1}{n}\right)^n\right]^x = \left(1+\frac{1}{n}\right)^{nx} = 1 + nx\frac{1}{n}$$

$$+\frac{nx(nx-1)}{1\cdot 2}\frac{1}{n^2}+\frac{nx(nx-1)(nx-2)}{1\cdot 2\cdot 3}\frac{1}{n^3}+\cdots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{1\cdot 2} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{1\cdot 2\cdot 3} + \cdots.$$

Letting n approach infinity, we get

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots,$$

which is called the **exponential series,** the quantity e^x being called the **exponential function.** The exponential function is very important in both theoretical and applied mathematics. The exponential series converges for all values of x. Obviously for $x = 1$ it reduces to the series for e.

Values of e^x for various values of x are given in tables. (See Note, p. 26.)

94. Relations connecting the trigonometric functions with the exponential function. So far the trigonometric functions sin x, etc., and also the exponential function e^x, have only been defined for real values of x. That is, sin 2 and $e^{1/3}$, for example, have the perfectly definite meanings of *the sine of two radians* and *the cube root of e,* respectively; such expressions as sin $(i/4)$ and e^{3i} are meaningless. If, however, we define the trigonometric and exponential functions by means of the series developed in the preceding sections, then they have definite values for imaginary and even complex values of x. For instance, the value of sin $(i/4)$ may be found by substituting $i/4$ in the sine series. We find

$$\sin\frac{i}{4} = \frac{i}{4} - \frac{1}{3!}\cdot\frac{i^3}{4^3} + \frac{1}{5!}\cdot\frac{i^5}{4^5} - \frac{1}{7!}\cdot\frac{i^7}{4^7} + \cdots$$
$$= 0.25261i.$$

Exercise 42

Find the values of the following:

1. $\cos\,(i/10)$.
2. $\sin\,(\pi i/4)$.
3. $e^{\pi i/2}$.
4. $e^{-\pi i/4}$.
5. $\sin i$.
6. $\sin\,(1 + i)$. *Suggestion.* Expand by the formula for $\sin\,(u + v)$.

We can now establish some interesting relations connecting the trigonometric functions with the exponential function. In the exponential series let us replace x by iv. We get

$$e^{iv} = 1 + iv + \frac{i^2v^2}{2!} + \frac{i^3v^3}{3!} + \frac{i^4v^4}{4!} + \frac{i^5v^5}{5!} + \cdots$$
$$= 1 + iv - \frac{v^2}{2!} - \frac{iv^3}{3!} + \frac{v^4}{4!} + \frac{iv^5}{5!} - \cdots,$$

since $i^2 = -1$, $i^3 = -i$, etc.

Separating the real and imaginary terms of the foregoing series, we get

$$e^{iv} = 1 - \frac{v^2}{2!} + \frac{v^4}{4!} - \cdots + i\left(v - \frac{v^3}{3!} + \frac{v^5}{5!} - \cdots\right).$$

But referring to (132) and (133) we see that

$$e^{iv} = \cos v + i \sin v. \tag{136}$$

By the same process, or by changing v into $-v$, we obtain

$$e^{-iv} = \cos v - i \sin v. \tag{137}$$

Subtracting (133) from (132), and dividing by $2i$, we get

$$\sin v = \frac{e^{iv} - e^{-iv}}{2i}. \tag{138}$$

Adding (132) and (133) and dividing by 2, we get

$$\cos v = \frac{e^{iv} + e^{-iv}}{2}. \tag{139}$$

By division we find that

$$\tan v = \frac{e^{iv} - e^{-iv}}{i(e^{iv} + e^{-iv})}. \tag{140}$$

Formulas (138), (139), and (140) are called **Euler's formulas.** The formulas for cot v, sec v, and csc v are given by the reciprocals of the fractions in (140), (139), and (138) respectively.

Exercise 43

1. By means of formulas (138) and (139) prove that

(*a*) $\cos^2 v + \sin^2 v = 1$,
(*b*) $\sin 2v = 2 \sin v \cos v$,
(*c*) $\cos 2v = \cos^2 v - \sin^2 v$.

2. Making use of formulas (138) and (139), prove the formulas for

(*a*) $\sin (u + v)$, (*b*) $\cos (u + v)$,
(*c*) $\sin (u - v)$, (*d*) $\cos (u - v)$.

3. Using the results of section 92, prove that

$$1 + \tan^2 v = \sec^2 v.$$

4. By using (136) show how e^{iv} may be plotted as a complex number.

5. Plot the following as complex numbers:

(a) $e^{\pi i/4}$, (b) $2e^{\pi i/3}$,
(c) $5e^{2\pi i/3}$, (d) $e^{-\pi i/6}$,
(e) $3e^{\pi i/2}$, (f) $e^{\pi i}$,
(g) $e^{2\pi i}$.

6. Making use of the relation

$$e^{u+iv} = e^u e^{iv} = e^u(\cos v + i \sin v),$$

plot the following as complex numbers:

(a) $e^{1+\pi i/6}$, (b) $e^{2-\pi i/4}$,
(c) $e^{-1+\pi i}$, (d) $e^{-2+2\pi/3}$.

95. The hyperbolic functions. The hyperbolic functions receive their name from their connection with a curve called the hyperbola, a connection somewhat similar to that which the circular or trigonometric functions bear to the circle. The hyperbolic functions are suggested by (138), (139), and (140) of the preceding section. In fact, the **hyperbolic sine of v** (written **sinh v**), the **hyperbolic cosine of v** (**cosh v**), and the **hyperbolic tangent of v** (**tanh v**) are defined as follows:

$$\sinh v = \frac{e^v - e^{-v}}{2},$$

$$\cosh v = \frac{e^v + e^{-v}}{2},$$

$$\tanh v = \frac{e^v - e^{-v}}{e^v + e^{-v}}.$$

The **hyperbolic cotangent of** v **(coth** v**)**, the **hyperbolic secant of** v **(sech** v**)**, and the **hyperbolic cosecant of** v **(csch** v**)** are defined as the reciprocals of tanh v, cosh v, and sinh v, respectively. These definitions of the hyperbolic functions are to hold whether v is real or complex.

It is readily seen that

$$\begin{aligned} \sin iv &= i \sinh v, & \sinh iv &= i \sin v, \\ \cos iv &= \cosh v, & \cosh iv &= \cos v, \\ \tan iv &= i \tanh v, & \tanh iv &= i \tan v, \\ \cot iv &= -i \coth v, & \coth iv &= -i \cot v, \\ \sec iv &= \operatorname{sech} v, & \operatorname{sech} iv &= \sec v, \\ \csc iv &= -i \operatorname{csch} v, & \operatorname{csch} iv &= -i \csc v. \end{aligned}$$

The hyperbolic functions play an important rôle in many branches of science. Among the topics where they find application are electricity, radioactivity, heat, sound, light, wave motion in fluids, theory of elasticity, etc. They are useful in geology, and even in geography, where they are employed in map making. Moreover, they are extremely useful in pure mathematics.

The hyperbolic functions satisfy certain identities analogous to those satisfied by the circular or trigonometric functions. For example,

$$(141) \qquad \sinh^2 v = \left(\frac{e^v - e^{-v}}{2}\right)^2 = \frac{e^{2v} - 2 + e^{-2v}}{4},$$

$$(142) \qquad \cosh^2 v = \left(\frac{e^v + e^{-v}}{2}\right)^2 = \frac{e^{2v} + 2 + e^{-2v}}{4}.$$

Subtracting (141) from (142) gives

$$(143) \qquad \cosh^2 v - \sinh^2 v = 1.$$

Exercise 44

Prove the following identities:

1. $1 - \tanh^2 v = \operatorname{sech}^2 v.$
2. $\coth^2 v - 1 = \operatorname{csch}^2 v.$
3. $\sinh 2v = 2 \sinh v \cosh v.$
4. $\cosh 2v = \cosh^2 v + \sinh^2 v.$
5. $\sinh (u + v) = \sinh u \cosh v + \cosh u \sinh v.$
6. $\sinh (u - v) = \sinh u \cosh v - \cosh u \sinh v.$
7. $\cosh (u + v) = \cosh u \cosh v + \sinh u \sinh v.$
8. $\cosh (u - v) = \cosh u \cosh v - \sinh u \sinh v.$
9. $\tanh (u + v) = \dfrac{\tanh u + \tanh v}{1 + \tanh u \tanh v}.$
10. $\tanh (u - v) = \dfrac{\tanh u - \tanh v}{1 - \tanh u \tanh v}.$
11. $\sinh u + \sinh v = 2 \sinh \tfrac{1}{2}(u + v) \cosh \tfrac{1}{2}(u - v).$
12. $\sinh u - \sinh v = 2 \cosh \tfrac{1}{2}(u + v) \sinh \tfrac{1}{2}(u - v).$
13. $\cosh u + \cosh v = 2 \cosh \tfrac{1}{2}(u + v) \cosh \tfrac{1}{2}(u - v).$
14. $\cosh u - \cosh v = 2 \sinh \tfrac{1}{2}(u + v) \sinh \tfrac{1}{2}(u - v).$

96. Series for the hyperbolic functions. If we take the series

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \cdots$$

and replace x by ix, we get

$$\sin ix = ix + i\frac{x^3}{\underline{|3}} + i\frac{x^5}{\underline{|5}} + \cdots.$$

But $\sin ix = i \sinh x$, and consequently $\sinh x = \dfrac{1}{i} \sin ix$.

Therefore

$$\mathbf{\sinh x = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \cdots.} \tag{144}$$

This series, and also the series for $\cosh x$ and $\tanh x$ given in the next exercise, converge for all values of x.

Exercise 45

1. Prove

$$(a)\ \cosh x = 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \cdots,$$

$$(b)\ \tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \cdots.$$

2. Deduce the series for $\sinh x$, $\cosh x$, and $\tanh x$ immediately from the definitions of the hyperbolic functions by using the exponential series.

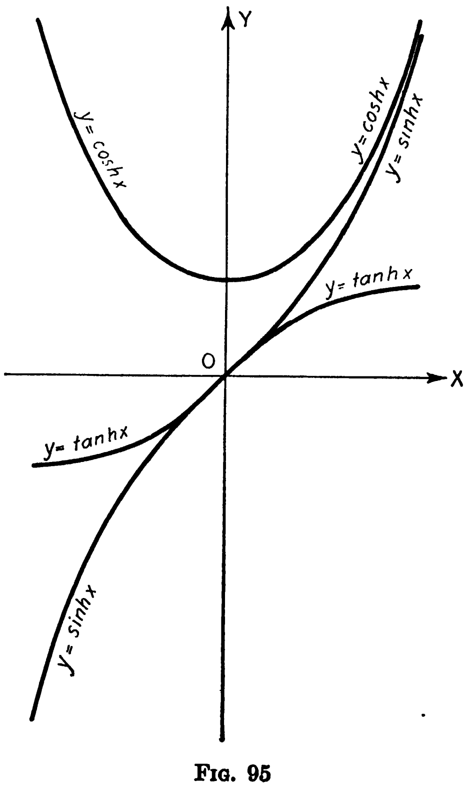


FIG. 95

97. Graphs of the hyperbolic functions. The hyperbolic functions may be plotted by using tables. (See Note, p. 26.) The graphs of $y = \sinh x$, $y = \cosh x$, $y = \tanh x$ are shown in Fig. 95. The student should verify these graphs and also plot the other three hyperbolic functions.*

98. The inverse hyperbolic functions. From the equation $x = \sinh y$ we may express y in terms of x by using the inverse notation

$$y = \sinh^{-1} x,$$

which is read *y equals the inverse hyperbolic sine of x*.

An interesting relation may be obtained as follows:

$$x = \sinh y = \frac{e^y - e^{-y}}{2} = \frac{1}{2}\left(e^y - \frac{1}{e^y}\right).$$

Clearing of fractions, we get

$$2xe^y = e^{2y} - 1,$$

or

$$e^{2y} - 2xe^y - 1 = 0,$$

which is a quadratic equation in e^y. Solving, we obtain

$$e^y = x \pm \sqrt{x^2 + 1},$$

in which we must use the plus sign before the radical, since e^y can not be negative. Taking the Napierian or natural logarithm of each side of the equation, we see that

$$y = \log_e (x + \sqrt{x^2 + 1}) = \sinh^{-1} x.$$

Similarly,

$$\cosh^{-1} x = \log_e (x + \sqrt{x^2 - 1}),$$

$$\tanh^{-1} x = \frac{1}{2} \log_e \frac{1 + x}{1 - x}.$$

* For more information concerning the hyperbolic functions the reader is referred to McMahon's *Hyperbolic Functions*, and to the *Smithsonian Tables of Hyperbolic Functions*.

99. The logarithmic series. Let

$$u^v = e^z.$$

Taking the natural logarithm of each side of the equation, we get

$$v \log_e u = z.$$

Consequently

$$u^v = e^{v \log_e u}.$$

Now from the exponential series,

$$e^x = 1 + x + \frac{x^2}{|\underline{2}} + \frac{x^3}{|\underline{3}} + \cdots,$$

(see section 93) we have

$$u^v = e^{v \log_e u} = 1 + v \log_e u + \frac{v^2}{|\underline{2}} (\log_e u)^2 + \cdots.$$

Replacing u by $1 + x$ gives

$$(1 + x)^v = 1 + v \log_e (1 + x) + \frac{v^2}{|\underline{2}} [\log_e (1 + x)]^2 + \cdots.$$

But by the binomial theorem,

$$(1 + x)^v = 1 + vx + \frac{v(v - 1)}{1 \cdot 2} x^2 + \frac{v(v - 1)(v - 2)}{1 \cdot 2 \cdot 3} x^3 + \cdots.$$

Equating the coefficients of v in the two expressions gives

$$\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots. \tag{145}$$

This series is not well adapted for the computation of logarithms because it converges so slowly; that is, we should have to take a great number of terms in order to get the value of the logarithm. For example, if we wished to find the logarithm of 2 we should place $x = 1$ and get

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

But it would be necessary to take 1,000 terms in order to find $\log_e 2$ correct to three decimal places. Since series (145) is so impractical, a more rapidly converging series for computing logarithms has been devised.

In (145) replace x by $-x$. This gives

$$\log_e (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots. \tag{146}$$

Subtracting (146) from (145), we obtain

$$\begin{aligned} \log_e (1 + x) - \log_e (1 - x) &= \log_e \frac{1 + x}{1 - x} \\ &= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right), \end{aligned} \tag{147}$$

which converges if x is numerically less than one. Now let

$$\frac{1 + x}{1 - x} = \frac{m}{n}, \qquad \text{or} \qquad x = \frac{m - n}{m + n}, \tag{148}$$

and x will be numerically less than one for all positive values of m and n.

By substituting (148) into (147) we obtain

$$\begin{aligned} \log_e \frac{m}{n} = \log_e m - \log_e n = 2\Bigg[&\frac{m - n}{m + n} \\ &+ \frac{1}{3}\left(\frac{m - n}{m + n}\right)^2 + \frac{1}{5}\left(\frac{m - n}{m + n}\right)^5 + \cdots\Bigg], \end{aligned} \tag{149}$$

which converges for all positive values of m and n.

This is the series used in computing tables of logarithms. Of course it is not necessary to find the logarithms of all numbers by means of series. For example, if we have found $\log_e 2$ and $\log_e 5$, we can add them to obtain $\log_e 10$ or subtract them in the proper order to find $\log_e 2.5$ and $\log_e 0.4$.

To find the Briggsian or common logarithm (base 10) of a number, use the following formula:

$$\log_{10} N = \frac{\log_e N}{\log_e 10} = \log_e N \times \log_{10} e. \tag{150}$$

This formula can be proved as follows:

Suppose that

$$\log_{10} N = L.$$

Then

$$N = 10^L.$$

Taking natural logarithms of both sides, we get

$$\log_e N = L \log_e 10 = \log_{10} N \log_e 10,$$

from which we get the first part of (150).

To get the second part of (150) we set

$$\log_e N = K.$$

Then

$$N = e^K,$$

and taking common logarithms, we get

$$\log_{10} N = K \log_{10} e = \log_e N \log_{10} e.$$

Incidentally we have proved that

$$\log_{10} e = \frac{1}{\log_e 10}.$$

Exercise 46

Find

1. $\log_e 2$. *Suggestion.* Set $m = 2$, $n = 1$ in (149).
2. $\log_e 3$. *Suggestion.* Set $m = 3$, $n = 2$ in (149) and use the result of problem 1.
3. $\log_e 5$.
4. $\log_e 10$.
5. $\log_{10} 5$.
6. $\log_{10} \frac{1}{5}$.
7. $\log_{10} \frac{2}{3}$.

INDEX

ANSWERS TO EXERCISES

Exercise 1, pages 19–20

8. $0.01472°$. **9.** $58' \, 30''$. **10.** $57.29583°$, $3437.75'$, $206265''$. **11.** $-41° \, 30' \, 00.6''$. **12.** $76° \, 55' \, 32.52''$. **13.** $39^g \, 39' \, 65.1''$.

Exercise 2, pages 22–23

1. 40 mils. **2.** 22.5 ft. **3.** 1650 ft. **4.** 62.4 m. **5.** 83.3 mils. **6.** 1200 ft. **7.** 43.3 mils. **8.** 50 ft. below. **9.** 20 ft. **10.** 2333 ft. **11.** 177.8 mils, 266.7 mils, 2.96 mils, 0.049 mils, 160 mils. **12.** $33' \, 45''$, $2° \, 48' \, 45''$, $5° \, 37' \, 30''$. **13.** 0.625^g, 3.125^g, 6.25^g.

Exercise 3, pages 28–31

1. $\pi/18$, $7\pi/36$, $-\pi/4$, $7\pi/18$, $5\pi/6$, $-14\pi/9$, $\pi/10$, $20\pi/9$, $7\pi/120$, $11\pi/80$, $641\pi/240$.

2. $0.17^{(r)}$, $0.32^{(r)}$, $6.46^{(r)}$, $2.41^{(r)}$, $2.49^{(r)}$.

3. $60°$, $45°$, $30°$, $108°$, $84°$, $54°$, $42°$.

6. $17\pi/18$, $49\pi/36$, $25\pi/12$, $30479\pi/81000$; $17/9$, $49/18$, $25/6$, 0.75257.

7. (*a*) $4/\pi$, $11/\pi$, $-7/\pi$, $16/\pi$. (*b*) $2/3$, $1/2$, $8/5$, $16/15$, 3, $4/3$.

8. $1\frac{2}{3}^{(r)}$, $\frac{1}{15}^{(r)}$.

9. 3.54 ft.

10. $80.52°$, $1.4^{(r)}$.

11. $1.4^{(r)}$, $80.21°$.

12. $154.93°$, $2.704^{(r)}$.

13. 1583.45 ft.

14. $1.44686°$.

15. 1.1520 mi.

16. 92,610,000 mi.

17. 240,000 mi.

18. 25,240,000,000,000 mi.

19. 173 mi.

20. $142° \, 30'$.

21. 0.0375 in.

22. (*a*) 0.208 in. (*b*) 0.048 in.

Exercise 4, pages 32–33

1. 300, 18000. **2.** $972°$ per sec., $16.96^{(r)}$ per sec. **3.** $6°$ per min. **4.** 8π in. per hr. **5.** $5\frac{1}{2}°$ per min. **6.** $2.64^{(r)}$ per min., $2.5212°$

per sec. **7.** 76.39. **8.** 15° per hr. **9.** 17.2788 mi. per min. **10.** $31^{(r)}$ per sec. **11.** 18.5 mi. per sec. **12.** $5.16^{(r)}$ per sec. **13.** $16\pi^{(r)}$ per sec., 16π ft. per min.

Exercise 5, pages 38–40

2. (0, 0). **4.** (*a*) $\sqrt{65}$, $\sqrt{34}$, $3\sqrt{13}$, 7, $\sqrt{257}$, 5. (*b*) $\sqrt{19}$, 6.77, $\sqrt{7}$. **5.** $(4, \pm 3)$. **6.** $(\pm\sqrt{5}, -2)$. **7.** $(-3, 0)$. **8.** (*a*) $(3, 3\sqrt{3})$. (*b*) $(3\sqrt{3}, 3)$. (*c*) (6, 0). (*d*) $(3\sqrt{3}, -3)$. (*e*) $(0, -6)$. (*f*) $(-3, -3\sqrt{3})$. (*g*) $(-3\sqrt{2}, 3\sqrt{2})$.

Exercise 6, pages 52–54

	sin *v*	cos *v*	tan *v*	cot *v*	sec *v*	csc *v*
1. (*a*)	$1/\sqrt{2}$	$1/\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
(*b*)	$-3/5$	$-4/5$	$3/4$	$4/3$	$-5/4$	$-5/3$
(*c*)	$-2/\sqrt{13}$	$3/\sqrt{13}$	$-2/3$	$-3/2$	$\sqrt{13}/3$	$-\sqrt{13}/2$
(*d*)	$-5/\sqrt{26}$	$-1/\sqrt{26}$	5	$1/5$	$-\sqrt{26}$	$-\sqrt{26}/5$
(*e*)	$\sqrt{21}/5$	$-2/5$	$-\sqrt{21}/2$	$-2/\sqrt{21}$	$-5/2$	$5/\sqrt{21}$
(*f*)	$-9/41$	$40/41$	$-9/40$	$-40/9$	$41/40$	$-41/9$
2. (*a*)	$1/2$	$\pm\sqrt{3}/2$	$\pm 1/\sqrt{3}$	$\pm\sqrt{3}$	$\pm 2/\sqrt{3}$	2
(*b*)	$\pm\sqrt{5}/3$	$2/3$	$\pm\sqrt{5}/2$	$\pm 2/\sqrt{5}$	$3/2$	$\pm 3/\sqrt{5}$
(*c*)	$\pm 2/\sqrt{29}$	$\mp 5/\sqrt{29}$	$-2/5$	$-5/2$	$\mp\sqrt{29}/5$	$\pm\sqrt{29}/2$
(*d*)	$3/4$	$\pm\sqrt{7}/4$	$\pm 3/\sqrt{7}$	$\pm\sqrt{7}/3$	$\pm 4/\sqrt{7}$	$4/3$
(*e*)	$\pm 2/\sqrt{29}$	$\pm 5/\sqrt{29}$	$2/5$	$5/2$	$\pm\sqrt{29}/5$	$\pm\sqrt{29}/2$
(*f*)	$\pm 3/5$	$4/5$	$\pm 3/4$	$\pm 4/3$	$5/4$	$\pm 5/3$
(*g*)	$\pm\sqrt{3}/2$	$-1/2$	$\mp\sqrt{3}$	$\mp 1/\sqrt{3}$	-2	$\pm 2\sqrt{3}$
(*h*)	$\pm\sqrt{15}/4$	$-1/4$	$\mp\sqrt{15}$	$\mp 1/\sqrt{15}$	-4	$\pm 4/\sqrt{15}$
(*i*)	$\pm 2/\sqrt{5}$	$\pm 1/\sqrt{5}$	0.5	2	$\pm\sqrt{5}$	$\pm\sqrt{5}/2$
(*j*)	-0.8	∓ 0.6	$\pm 1\frac{1}{3}$	± 0.75	$\mp 1\frac{2}{3}$	-1.25
3. (*a*)	$\sqrt{3}/2$	$\pm 1/2$	$\pm\sqrt{3}$	$\pm 1/\sqrt{3}$	± 2	$2/\sqrt{3}$
(*b*)	$\pm\sqrt{2}/2$	$\sqrt{2}/2$	± 1	± 1	$\sqrt{2}$	$\pm\sqrt{2}$
(*c*)	$\pm 1/2$	$\pm\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$	$\pm 2/\sqrt{3}$	± 2
(*d*)	$\pm 1/\sqrt{2}$	$1/\sqrt{2}$	± 1	± 1	$\sqrt{2}$	$\pm\sqrt{2}$

4. (*a*) $\sin v = \pm a/\sqrt{a^2+b^2}$, $\cos v = \pm b/\sqrt{a^2+b^2}$, $\cot v = b/a$,
$\sec v = \pm\sqrt{a^2+b^2}/b$, $\csc v = \pm\sqrt{a^2+b^2}/a$.
(*b*) $\cos v = \pm(m^2-n^2)/(m^2+n^2)$, $\tan v = \pm 2mn/(m^2-n^2)$,
$\cot v = \pm(m^2-n^2)/2mn$, $\sec v = \pm(m^2+n^2)/(m^2-n^2)$,
$\csc v = (m^2+n^2)/2mn$.

5. (*a*) $\frac{2}{5}$ or $1\frac{3}{5}$. (*b*) $\frac{1}{5}$ or $1\frac{4}{5}$. (*c*) $\frac{2}{3}$ or $-2\frac{2}{3}$. (*d*) $\frac{1}{5}$ or $\frac{4}{5}$.

6. $\sin v = -1/3$, $\cos v = \pm 2\sqrt{2}/3$, $\tan v = \pm 1/(2\sqrt{2})$.

7. $\sin v = \pm\sqrt{3}/2$, $\cos v = 1/2$, $\tan v = \pm\sqrt{3}$.

8. $\sin v = \pm\sqrt{33}/7$, $\cos v = 4/7$, $\tan v = \pm\sqrt{33}/4$.

9. If either function is 1 the other is 0.

11. $1/\sin v$. 12. $1 + 2(1 \pm \sqrt{1+\tan^2 v})/\tan^2 v$.

13. $(2+\cot^2 v)/(1-\cot^4 v)$.

Exercise 7, pages 65–70

1. (*a*) .2250	(*h*) .0175	(*o*) .4663
(*b*) .1219	(*i*) .0349	(*p*) 1.4281
(*c*) .3746	(*j*) .9998	(*q*) .4663
(*d*) .5592	(*k*) .9563	(*r*) 1.4281
(*e*) .8910	(*l*) .7660	(*s*) 1.0306
(*f*) .9945	(*m*) .9135	(*t*) 11.4737
(*g*) .9563	(*n*) 1.0000	(*u*) 3.0716
2. (*a*) 22°	(*g*) 19°	(*m*) $33^\circ\ 16'$
(*b*) 75°	(*h*) 69°	(*n*) $68^\circ\ 12'$
(*c*) 42°	(*i*) 22°	(*o*) $24^\circ\ 05'$
(*d*) 78°	(*j*) 47°	(*p*) $68^\circ\ 11'$
(*e*) 49°	(*k*) $14^\circ\ 16'$	(*q*) $28^\circ\ 02'$
(*f*) $40^\circ\ 18'$	(*l*) $47^\circ\ 05'$	(*r*) $80^\circ\ 06'$

3. $\sin A = 1/\sqrt{5}$, $\cos A = 2/\sqrt{5}$, $\tan A = 1/2$,
$\cot A = 2$, $\sec A = \sqrt{5}/2$, $\csc A = \sqrt{5}$.

4. $a = 80.4$, $b = 60.3$.

5. $a = 72$, $b = 27\sqrt{3}$.

6. (*a*) $\cos 45^\circ$	(*c*) $\cot 75^\circ$	(*e*) $\sec 23^\circ$
(*b*) $\sin 60^\circ$	(*d*) $\csc 38^\circ$	(*f*) $\tan 9^\circ$
7. (*a*) $\cos 22^\circ$	(*c*) $\cot 39^\circ$	(*e*) $\csc 2^\circ$
(*b*) $\sin 20^\circ$	(*d*) $\tan 35^\circ$	(*f*) $\sec 26^\circ$

8. (*a*) 45°. (*b*) 30°. (*c*) $22\frac{1}{2}^\circ$. (*d*) 40°.

9. (*a*) 0.3633 (*e*) 6.0296 (*i*) 0.8227
(*b*) 0.9317 (*f*) 0.9013 (*j*) 0.3770
(*c*) 0.3132 (*g*) 1.3976 (*k*) 0.2759
(*d*) 0.7402 (*h*) 0.3567 (*l*) 0.4612

10. (*a*) $41^\circ\ 12'$ (*d*) $37^\circ\ 39'$ (*g*) $12^\circ\ 36'$ (*j*) $25^\circ\ 11'$
(*b*) $15^\circ\ 25'$ (*e*) $47^\circ\ 08'$ (*h*) $77^\circ\ 24'$ (*k*) $71^\circ\ 34'$
(*c*) $35^\circ\ 54'$ (*f*) $63^\circ\ 50'$ (*i*) $42^\circ\ 35'$ (*l*) $24^\circ\ 42'$

11. (*a*) 0.4107 (*e*) 0.7620 (*i*) 0.7472
(*b*) 0.4313 (*f*) 3.1391 (*j*) 0.9195
(*c*) 0.3435 (*g*) 1.5590
(*d*) 0.5941 (*h*) 0.3113

12. (*a*) $31^\circ\ 10'$ (*d*) $58^\circ\ 19'$ (*g*) $86^\circ\ 11'$ (*j*) $26^\circ\ 12'$
(*b*) $65^\circ\ 18'$ (*e*) $58^\circ\ 51'$ (*h*) $38^\circ\ 00'$
(*c*) $36^\circ\ 26'$ (*f*) $13^\circ\ 25'$ (*i*) $45^\circ\ 32'$

15. (*a*) $b = 12.649$, $A = 25^\circ\ 23'$, $B = 64^\circ\ 37'$, area $= 37.947$.
(*b*) $c = 5.8310$, $A = 30^\circ\ 58'$, $B = 59^\circ\ 02'$, area $= 7\frac{1}{2}$.
(*c*) $B = 60^\circ$, $a = 2\sqrt{2}$, $c = 4\sqrt{3}$, area $= 6\sqrt{3}$.

16. 151.1 ft. **17.** $1^\circ\ 26'$. **18.** 157.09 ft.
19. 713.58 ft. **20.** $46^\circ\ 01'$. **21.** 71.5 ft.
23. $19^\circ\ 17'$, $8^\circ\ 32'$. **24.** 1417.8 ft. **25.** 42.657 ft.
26. 273.9 ft. if the balloon is between the two points, 2554.7 ft. if not.

Exercise 8, pages 78–79

1. (*a*) 2 (*d*) 0 (*g*) 9–10 (*j*) 0
(*b*) 1 (*e*) 9–10 (*h*) 8–10 (*k*) 9–10
(*c*) 3 (*f*) 9–10 (*i*) 7–10 (*l*) 1

2. (*a*) 1.30103 (*e*) 7.30103–10
(*b*) 8.30103–10 (*f*) 4.30103–10
(*c*) 3.30103 (*g*) 9.30103–10
(*d*) 9.30103–10 (*h*) 6.30103

3. (*a*) 1.54407 (*e*) 4.54407
(*b*) 9.54407–10 (*f*) 2.54407
(*c*) 0.54407 (*g*) 8.54407–10
(*d*) 5.54407–10 (*h*) 7.54407–10

4. (a) 0.60206
(b) 0.77815
(c) 0.90309
(d) 0.95424
(e) 1.00000
(f) 1.07918
(g) 1.14613
(h) 1.17609
(i) 1.20412
(j) 1.25527
(k) 1.30103
(l) 1.32222
(m) 1.38021
(n) 1.39794
(o) 1.43136
(p) 1.44716
(q) 1.47712
(r) 1.50515
(s) 1.55630
(t) 1.54407
(u) 1.60206
(v) 1.62325
(w) 1.65321
(x) 1.68124
(y) 1.69020
(z) 1.69897
(1) 1.74819
(2) 1.77815
(3) 1.80618
(4) 1.84510
(5) 1.85733
(6) 1.87506
(7) 1.90309
(8) 1.90849
(9) 2.09691
(10) 2.10721
(11) 2.14613
(12) 2.32222
(13) 2.38560
(14) 2.38917

Exercise 9, pages 83–84

1. (a) 1.83251
(b) 1.83442
(c) 2.55509
(d) 2.55678
(e) 3.77401
(f) 3.60271
(g) 9.84510−10
(h) 8.97313−10
(i) 6.53782−10
(j) 5.37236
(k) 3.83149
(l) 2.83149
(m) 1.83149
(n) 0.83149
(o) 9.83149−10
(p) 4.73501
(q) 4.58876
(r) 3.80023
(s) 9.54073−10
(t) 3.94298−10

2. (a) 5.0000
(b) 57.600
(c) 863.00
(d) 18270
(e) 0.64980
(f) 0.92770
(g) 0.000,000,578,8
(h) 4299.0
(i) 0.069890
(j) 4.3646
(k) 0.049074
(l) 32.671
(m) 0.001,576,4
(n) 30.734
(o) 0.066567
(p) 154000
(q) 1,427,700
(r) 0.61700

3. (*a*) 1489.3 (*b*) 25.41 (*c*) 55.044 (*d*) 15697 (*e*) 231740 (*f*) 18676 (*g*) 11222 (*h*) 1.2480 (*i*) 0.019974 (*j*) 16346 (*k*) 0.88704 (*l*) 0.000,000,361,97

4. (*a*) 5.25 (*b*) 38.485 (*c*) 51.356 (*d*) 73.820 (*e*) 7.3388 (*f*) 89.834 (*g*) 0.0129 (*h*) 0.16763 (*i*) 0.008,860,6 (*j*) 0.078171

Exercise 10, pages 86–87

3. (*a*) 0.16474 (*b*) 3953.19 (*c*) 434.42 (*d*) 16.150 (*e*) 0.028644

4. (*a*) 185190 (*b*) 176,330,000,000 (*c*) 78125×10^{-14} (*d*) 12335 (*e*) 42839×10^{-30} (*f*) 32546×10^{-11} (*g*) 1.9129 (*h*) 3.9362 (*i*) 76.910 (*j*) 1.2023 (*k*) 0.62869 (*l*) 212.54

5. 26,442,000,000,000 mi.

6. 369,693,101.

Exercise 11, pages 88–89

1. (*a*) 0.69315. (*b*) 0.43429.

2. (*a*) 1.09861. (*b*) 0.47712.

3. 13225.

4. (*a*) 6 min. (*b*) 16.33°. (*c*) 0.03698.

Exercise 12, page 90

1. (*a*) 3. (*b*) 3.3688.

2. $x = 3.655$, $y = 2.517$.

3. $1 + \dfrac{\log l - \log a}{\log r}$.

4. $\dfrac{\log A - \log P}{\log (1 + r)}$.

5. 11.18 yrs.

6. 17.67 yrs.

7. (*a*) $\dfrac{\log (1 + rS) - \log R}{\log (1 + r)}$. (*b*) $-\dfrac{\log (1 - rV/R)}{\log (1 + r)}$.

Exercise 13, page 92

1. (*a*) 9.68557–10
(*b*) 9.93307–10
(*c*) 9.99040–10
(*d*) 0.54250
(*e*) 0.51466
(*f*) 9.24632–10
(*g*) 9.32747–10
(*h*) 9.71184–10
(*i*) 9.81694–10
(*j*) 9.05497–10
(*k*) 9.13078–10
(*l*) 9.77592–10
(*m*) 0.23101
(*n*) 9.65331–10
(*o*) 9.84933–10
(*p*) 9.32567–10
(*q*) 9.71646–10
(*r*) 9.17870–10
(*s*) 9.22613–10
(*t*) 0.14465

2. (*a*) 20° 13′ 30″
(*b*) 8° 49′ 30″
(*c*) 63° 41′ 22″
(*d*) 1° 37′ 43″
(*e*) 57° 00′ 04″
(*f*) 58° 35′ 00″
(*g*) 11° 00′ 00″
(*h*) 69° 46′ 30″
(*i*) 57° 37′ 44″
(*j*) 40° 25′ 48″
(*k*) 38° 12′ 26″
(*l*) 52° 51′ 13″
(*m*) 39° 11′ 45″
(*n*) 81° 10′ 30″
(*o*) 81° 13′ 30″
(*p*) 32° 59′ 56″
(*q*) 49° 25′ 30″
(*r*) 51° 41′ 23″
(*s*) 88° 24′ 20″
(*t*) 89° 49′ 10″

Exercise 14, page 97

1. (*a*) 8.26212–10
(*b*) 7.96979–10
(*c*) 7.86856–10
(*d*) 7.09719–10
(*e*) 8.28942–10
(*f*) 7.03193–10

2. (*a*) 0° 24′ 33″
(*b*) 0° 25′ 09″
(*c*) 89° 43′ 42″
(*d*) 0° 01′ 49″
(*e*) 1° 35′ 58″
(*f*) 0° 05′ 55″

3. 2.86264

4. (*a*) 1.88822
(*b*) 9.99994–10
(*c*) 9.99990–10
(*d*) 1.62441

5. 88° 22′ 15″

6. (*a*) 89° 01′ 43″
(*b*) 89° 07′ 12″
(*c*) 89° 54′ 37″
(*d*) 0° 09′ 51″

7. (*a*) 7.86895–10
(*b*) 7.96146–10
(*c*) 7.76027–10
(*d*) 6.30863–10
(*e*) 6.07908–10
(*f*) 6.44432–10

Exercise 15, pages 101–107

1. (a) $a = 39.938$, $c = 66.363$, $B = 53°$.
(b) $a = 46.972$, $b = 69.638$, $A = 34°$.
(c) $c = 28.601$, $A = 53° 31' 51''$, $B = 36° 28' 09''$.
(d) $b = 32.27$, $A = 29° 49' 22''$, $B = 60° 10' 38''$.
(e) $a = 294.00$, $c = 308.17$, $A = 72° 33' 30''$.
(f) $a = 0.024033$, $b = 0.015505$, $B = 32° 49' 42''$.
(g) $c = 0.8942$, $A = 16° 42' 37''$, $B = 73° 17' 23''$.
(h) $c = 1948.9$, $a = 467.49$, $B = 76° 07' 15''$.
2. 161.45 ft., 32° 36′ 26″, 57° 23′ 34″.
3. 282.22 ft.
4. 86.036 yds.
5. 316.85 ft.
6. 48° 35′ 25″, 26.4575 ft.
7. 63.143 ft.
8. 120.57 ft.
9. 53.271 ft.
10. 300.4 in.
11. (a) 38.398 mi., (b) 48.728 mi., (c) 41.224 mi.
12. 53.99 mi.
13. 755.55 ft.
14. 185.9 in.
15. $d \tan A$, $d \sec A = d/\cos A$
16. 16° 06′.
17. 391.36 ft.
18. 3.4785 ft.
19. 2.916 sq. ft.
20. 68.27 sq. in., 384.12 sq. in.
21. 1.373 sq. in.
22. 135.8 sq. in.
23. 1260.9 ft.
25. 86.60 mi.
26. 94.868 mi., 122.47 mi.
27. 42.67 ft., 150 ft., 416.67 ft., 2400 ft., 9600 ft.
28. 27 mi.
29. 12.25 mi.
30. $\frac{2}{3}$ of the radius = 2640 mi.
34. 10′ 44″.
35. 0° 03′ 35″.
36. 0° 07′ 10″.
37. 65.450 ft.
38. 238900 mi., 347.
39.* 1° 54′.
40. 55 in.
41. 92.2 ft.
42. (a) 1° 54′ 34″. (b) 0.00033″.
43. 433740 mi.
44. 39,706,000 mi.
45. 5.2 mi.
46. 2.6″.

* The diameter of the earth should be taken as 7920 mi instead of 3960 mi.

Exercise 16, pages 108–109

1. sides = 9.8265 in., angle = $99° 30'$, area = 47.619 sq. in.
2. 6.3990 in.
3. $21° 57' 40''$, $79° 01' 10''$, $79° 01' 10''$.
4. $41° 24' 35''$, 793.71 sq. ft.
5. 225 sq. ft. each, 21.213 ft., 2545.56 sq. ft.
6. $122° 05' 26''$.
7. 3.7017 in.
8. 59.864 in.
9. (*a*) 16.180 in., 15.388 in., 769.42 sq. in.
   (*b*) 21.928 in., 20.606 in., 1390.9 sq. in.
   (*c*) 21.600 in., 21.334 in., 1441.5 sq. in.
10. 1.6578 cu. in.
11. (*a*) 587.8 ft., 23777 sq. ft.
    (*b*) 618.0 ft., 29389 sq. ft.
    (*c*) 726.5 ft., 36327 sq. ft.
    (*d*) 649.8 ft., 32492 sq. ft.
12. 6.351 ft.
13. 15.347 ft., 12.416 ft.

Exercise 17, pages 111–113

1. (*a*)	$C = 70°$,	$b = 29.544$,	$c = 28.191$.
(*b*)	$B = 28° 21'$,	$C = 116° 39'$,	$a = 20.537$.
(*c*)	$B = 74° 02' 00''$,	$C = 35° 58' 00''$,	$b = 8.1851$.
(*d*)	$A = 50°$,	$a = 58.34$,	$c = 38.078$.
(*e*)	$A = 95° 45'$,	$B = 40° 27'$,	$C = 43° 48'$.
(*f*)	$A = 39° 33'$,	$C = 78° 27'$,	$b = 36.049$.

5. 210.99 ft. if the balloon is between the two tents, 623.73 ft. if not.
6. 204.90 ft.
7. 252.18 ft.
8. 40.40 ft.
9. 104.77 ft.
10. 409.38 ft.
11. 14.4.
12. 4.9147 ft.

Exercise 18, pages 114–115

1. 39° 26.0′ W. **2.** 50° 15′ N., 64° 07.5′ W.
3. 43° 17′ N., 71° 50.0′ W. **4.** 52° 04.8′ W.

Exercise 19, pages 116–118

1. 4° 19′ S., 150. **2.** 48° 04′ N., 183.85. **3.** 2° 34′, 102.78.
4. 37° 59′ N., N. 18° W. **5.** S. 56° 17′ E., 203.79.
6. 403.89, S. 58° 40′ E. **7.** 38° 36′ N., 54° 50′ W.
8. 33° 52′ N., 34° 47′ W. **9.** S. 28° 47′ W., 1293.
10. 14° 37′ N., 115° 58′ E.; 14° 37′ N., 121° 44′ E.; 19° 31′ N., 115° 58′ E.; 19° 31′ N., 121° 44′ E.

Exercise 20, pages 121–124

1. S. 28° 11′ W., 14.18 knots. **2.** 4089.5 lbs.
3. 167.98 lbs., 109.64 lbs. **4.** 5.83 mi. per hr., N. 30° 58′ W.
5. (*a*) 53° 08′ with bank. (*b*) 15 min.
6. (*a*) 71.51 mi. per hr., S. 57° 06′ 42″ E. (*b*) S. 32° 37′ 34″ E., 68.37 mi. per hr. **7.** 24.166 ft. per sec., 24° 27′.
8. (*a*) N. 15° 15′ 20″ E. or N. 15° 15′ 20″ W., 22.8035 ft. per sec. (*b*) If the man walks toward the front of the barge, the direction of his motion is N. 4° 37′ 10″ E. or N. 4° 37′ 10″ W., and his velocity relative to the earth is 27.66 ft. per sec. Another solution is obtained if he walks toward the rear.
9. (*a*) 90° 58′ 00″. (*b*) 1.50 sec.
10. 15.72 ft. per sec. **11.** − 8.6037 lbs., 122.87 lbs.
12. 28.844 lbs., 123° 41′ with positive X-axis.
13. 111.14 lbs., 109° 07′ 40″ with positive X-axis.
14. 28.87 lbs.
15. 500.64 lbs., 31° 45′ 14″ with positive X-axis.
17. 0.3038 acres.
18. (*a*) 1007.1 ft. (*b*) − 609.27, 1031.9.
19. 414.5 yds.

Exercise 21, pages 127–128

1. (*a*) amp. = 5, per. = 2π, freq. = $1/(2\pi)$.
(*b*) 4.97500, 4.90035, 4.77670, 4.60530, 4.38790, 2.70150, $-$ 2.08075, $-$ 5, 0.
2. (*a*) 7.84056, 7.36848, 6.60272, 5.57368, 4.32240, $-$ 3.32920, $-$ 5.22912, $-$ 6.71264, 3.26464, 6.47216, 2.47216, $-$ 2.47216, $-$ 6.47216, $-$ 8. (*b*) 10π.
3. $x = 4 \cos 3t$.
4. (*a*) 8. (*b*) 0.4243, 0.6, 0.4243, 0.6, 0.4243, 0.04708, 0.5563.
5. $x = 10 \cos \pi t$. **7.** $x = a \cos 640\pi t$.
8. per. = $2\pi \sqrt{l/g}$, freq. = $\dfrac{1}{2\pi} \sqrt{g/l}$.
9. (*a*) $s = 0.5236 \cos 3.276t$, amp. = 0.5236 ft., per. = 1.92 sec., freq. $\doteq$ 0.521.
(*b*) $s = -$ 0.519 ft., 0.505 ft., $-$ 0.482 ft., 0.229 ft.

Exercise 22, pages 136–137

2. (*a*) 92°. (*b*) 215°. (*c*) 130°. (*d*) 30°. (*e*) 20°. (*f*) 65°.
3. (*a*) sin 5°. (*b*) $-$ sin 13°. (*c*) sin 20°. (*d*) cos 35° 26′. (*e*) $-$ csc 2° 32′. (*f*) $-$ csc 34° 18′ 09″.
4. (*a*) cos v. (*b*) cot v. (*c*) $-$ cot v. (*d*) csc v. (*e*) $-$ sec v. (*f*) $-$ sec v.
5. (*a*) 0.57358. (*b*) $-$ 0.40674. (*c*) $-$ 3.7321. (*d*) 1.5617. (*e*) 1.2868. (*f*) $-$ 1.0257. (*g*) 2.5745. (*h*) 0.97622.
6. $\frac{1}{2}v$ is in 2d quadrant, $2v$ may be in any quadrant.
7. (*a*) $\cos v = \pm \sqrt{3}/2$, $\tan v = \pm 1/\sqrt{3}$, $\cot v = \pm \sqrt{3}$, $\sec v = \pm 2/\sqrt{3}$, $\csc v = 2$.
(*b*) $\sin v = \pm \sqrt{3}/2$, $\cos v = \pm \frac{1}{2}$, $\cot v = -1/\sqrt{3}$, $\sec v = \pm 2$, $\csc v = \pm 2/\sqrt{3}$.
(*c*) $\sin v = \pm 2\sqrt{2}/3$, $\tan v = \pm 2\sqrt{2}$, $\cot v = \pm \sqrt{2}/4$, $\sec v = \pm 3$, $\csc v = \pm 3\sqrt{2}/4$.
(*d*) $\sin v = 1/3$, $\cos v = \pm 2\sqrt{2}/3$, $\tan v = \pm \sqrt{2}/4$, $\cot v = \pm 2\sqrt{2}$, $\sec v = \pm 3\sqrt{2}/4$.
(*e*) $\sin v = \pm 4/5$, $\cos v = 3/5$, $\tan v = \pm 4/3$, $\cot v = \pm 3/4$, $\csc v = \pm 5/4$.

(*f*) $\sin v = \pm 5/\sqrt{26}$, $\cos v = \pm 1/\sqrt{26}$, $\cot v = \pm 1/5$, $\sec v = \pm \sqrt{26}$, $\csc v = \pm \sqrt{26}/5$.
(*g*) $\sin v = \pm 2\sqrt{6}/5$, $\tan v = \pm 2\sqrt{6}$, $\cot v = \pm \sqrt{6}/12$, $\sec v = 5$, $\csc v = \pm 5\sqrt{6}/12$.
(*h*) $\cos v = \pm \sqrt{11}/6$, $\tan v = \pm 5/\sqrt{11}$, $\cot v = \pm \sqrt{11}/5$, $\sec v = \pm 6/\sqrt{11}$, $\csc v = -6/5$.
8. $14° 51' 20''$ or $165° 08' 40''$.
10. 0.

Exercise 23, pages 148–153

1. $\cos 15° = \frac{1}{4}(\sqrt{6} + \sqrt{2})$, $\tan 15° = 2 - \sqrt{3}$, $\cot 15° = 2 + \sqrt{3}$.
2. $\sin 75° = \frac{1}{4}(\sqrt{6} + \sqrt{2})$, $\cos 75° = \frac{1}{4}(\sqrt{6} - \sqrt{2})$, $\tan 75° = 2 + \sqrt{3}$, $\cot 75° = 2 - \sqrt{3}$.
3. $\sin 60° = \frac{1}{2}\sqrt{3}$, $\cos 60° = \frac{1}{2}$, $\tan 60° = \sqrt{3}$, $\cot 60° = \frac{1}{3}\sqrt{3}$, $\sec 60° = 2$, $\csc 60° = \frac{2}{3}\sqrt{3}$.
4. $\sin 150° = \frac{1}{2}$, $\cos 150° = -\frac{1}{2}\sqrt{3}$, $\tan 150° = -\frac{1}{3}\sqrt{3}$, $\cot 150° = -\sqrt{3}$, $\sec 150° = -\frac{2}{3}\sqrt{3}$, $\csc 150° = 2$.
5. $\sin 22° 30' = \frac{1}{2}\sqrt{2 - \sqrt{2}}$, $\cos 22° 30' = \frac{1}{2}\sqrt{2 + \sqrt{2}}$, $\tan 22° 30' = \sqrt{2} - 1$, $\cot 22° 30' = \sqrt{2} + 1$, $\sec 22° 30' = \sqrt{4 - 2\sqrt{2}}$, $\csc 22° 30' = \sqrt{4 + 2\sqrt{2}}$.
6. $\sin \frac{1}{2}v = \pm \sqrt{66}/12$, $\cos \frac{1}{2}v = \pm \sqrt{78}/12$, $\tan \frac{1}{2}v = \pm \sqrt{11/13}$, $\cot \frac{1}{2}v = \pm \sqrt{13/11}$, $\sec \frac{1}{2}v = 2\sqrt{78}/13$, $\csc \frac{1}{2}v = \pm 2\sqrt{66}/11$.
8. (*a*) $2 \cos 3v \sin v$. (*b*) $2 \sin 2v \cos v$. (*c*) $2 \cos 7v \cos 2v$. (*d*) $2 \sin v \cos \frac{1}{2}v$. (*e*) $2 \sin 20° \sin 3°$. (*f*) $2 \cos 25° \sin v$. (*g*) $2 \sin 40° \sin v$. (*h*) $\sqrt{2} \sin (v - 45°) = \sqrt{2} \cos (v - 45°)$. (*i*) $-\cot \frac{1}{2}(v - u) = \cot \frac{1}{2}(u - v)$.
15. $\sin (u + v) = \pm (9 \pm 8\sqrt{10})/13\sqrt{13}$,
$\cos (u + v) = \pm 6(\pm 2\sqrt{10} - 1)/13\sqrt{13}$,
$\tan (u + v) = (9 + 8\sqrt{10})/6(2\sqrt{10} - 1)$
or $(-9 + 8\sqrt{10})/6(2\sqrt{10} + 1)$,
$\cot (u + v) = 6(2\sqrt{10} - 1)/(9 + 8\sqrt{10})$
or $6(2\sqrt{10} + 1)/(-9 + 8\sqrt{10})$,
$\sec (u + v) = \pm \sqrt{13}(\pm 2\sqrt{10} + 1)/18$,
$\csc (u + v) = \pm 13\sqrt{13}(9 \pm 8\sqrt{10})$.
17. (*b*) $\cos v \cos u \cos w - \sin v \sin u \cos w$
$- \sin v \cos u \sin w - \cos v \sin u \sin w$.

(c) $\dfrac{\tan v + \tan u + \tan w - \tan v \tan u \tan w}{1 - \tan v \tan u - \tan v \tan w - \tan u \tan w}$.

(d) $\dfrac{\cot v \cot u \cot w - \cot v - \cot u - \cot w}{\cot v \cot u + \cot v \cot w + \cot u \cot w - 1}$.

(e) $\sin v \cos u \cos w + \cos v \sin u \cos w - \cos v \cos u \sin w + \sin v \sin u \sin w$.

(f) $\sin v \cos u \cos w - \cos v \sin u \cos w - \cos v \cos u \sin w - \sin v \sin u \sin w$.

(g) $\cos v \cos u \cos w + \sin v \sin u \cos w + \sin v \cos u \sin w - \cos v \sin u \sin w$.

Exercise 24, page 155

1. $\sqrt{2} \sin (u - v)$, $v = 45°$.
2. $\sqrt{5} \sin (u + v)$, $v = 63° 26'$.
3. (a) $13 \cos (u + 22° 37')$. (b) $2 \cos (u - 60°)$. (c) $\sqrt{2} \cos (u - 45°)$. (d) $\sqrt{13} \sin (u - 33° 41')$. (e) $\sin (u + 60°)$. (f) $1.552 \cos (u - 75° 04')$.
4. $1.2997 \cos (u + 73° 43' 13'')$.
5. (a) $x = 5 \cos (2t - 36° 52')$. (b) amplitude $= 5$, phase angle $= 36° 52'$.
6. (a) $x = \sqrt{53} \cos (\pi t/4 - 15° 57')$. (b) amplitude $= \sqrt{53}$, phase angle $= 15° 57'$.
7. $\sqrt{m^2 + n^2} \cos (u - v)$, where $v = \arctan (n/m)$.

Exercise 25, pages 166–168

1. (a)	$A = 33° 09' 54''$,	$a = 435.71$,	$c = 787.53$.
(b)	$A = 65° 49' 48''$,	$a = 122.13$,	$b = 83.560$.
(c)	$B = 15° 57'$,	$b = 5.4420$,	$c = 17.865$.
(d)	$B = 1° 02' 00''$,	$a = 9.3684$,	$b = 0.18134$.
(e)	$C = 50° 08' 29''$,	$a = 196.01$,	$b = 211.01$.
(f)	$A = 61° 16' 00''$,	$a = 103.46$,	$c = 113.99$.
(g)	$B = 111° 11' 18''$,	$a = 102.16$,	$b = 491.04$.
(h)	$A = 147° 11' 18''$,	$b = 432.11$,	$c = 268.58$.
(i)	$B = 42° 12' 46''$,	$a = 514.76$,	$c = 1024.9$.
(j)	$A = 57° 18' 35''$,	$a = 413.43$,	$c = 168.44$.
(k)	$A = 42° 07' 42''$,	$a = 18.940$,	$c = 26.964$.

2. 1.5861 mi., 1.6961 mi., 1.299 mi. (if the balloon is between the observers).

3. Let $AB = a$, the longer base, and let l and l' be the legs adjacent to A and B respectively. Then

$$l = \frac{(a-b)\sin B}{\sin(A+B)}, \quad l' = \frac{(a-b)\sin A}{\sin(A+B)}, \quad l = 9.3830, \quad l' = 6.6547.$$

4. Let a be adjacent to B, b adjacent to A. Then

$$a = \frac{d \sin A}{\sin(A+B)}, \quad b = \frac{d \sin B}{\sin(A+B)}, \quad a = 12.669, \quad b = 5.3487.$$

5. 2.394 mi., 1 mi. **6.** 84.842 yds. **7.** 1072.3 ft.

8. 1062.4 ft., 532.5 ft. **10.** 74.34 yds., 54.26 yds.

11. Incorrectly stated.

12. 22.845. **13.** 8.752 ft.

Exercise 26, pages 173–174

1.	(*a*) $A = 57° 59' 54''$,	$C = 23° 36' 36''$,	$c = 29.526$.
	(*b*) $B = 13° 55' 38''$,	$C = 35° 30' 40''$,	$b = 135.96$.
	(*c*) $A = 104° 32' 20''$,	$B = 40° 01' 52''$,	$a = 5888.4$;
	$A' = 4° 36' 04''$,	$B' = 139° 58' 08''$,	$a' = 487.98$.
	(*d*) $A = 137° 18' 50''$,	$C = 4° 12' 32''$,	$a = 14131$.
	(*e*) $A = 63° 08' 22''$,	$B = 67° 32' 42''$,	$b = 89.532$;
	$A' = 116° 51' 38''$,	$B' = 13° 49' 26''$,	$b' = 23.147$.
	(*f*) $A = 26° 01' 43''$,	$B = 37° 38' 41''$,	$a = 160.55$.
	(*g*) $A = 103° 22' 00''$,	$C = 48° 48' 44''$,	$a = 67.733$;
	$A' = 20° 59' 28''$,	$C' = 131° 11' 16''$,	$a' = 24.939$.
	(*h*) $B = 52° 40' 10''$,	$C = 90°$,	$b = 88.234$.
	(*i*) $A = 134° 37' 20''$,	$C = 25° 08' 12''$,	$a = 94.371$;
	$A' = 4° 53' 44''$,	$C' = 154° 51' 48''$,	$a' = 11.315$.
	(*j*) $A = 23° 50' 59''$,	$C = 136° 51' 16''$,	$c = 96.651$;
	$A' = 156° 09' 01''$,	$C' = 4° 33' 14''$,	$c' = 11.221$.
	(*k*) $B = 24° 17' 32''$,	$C = 134° 24' 56''$,	$c = 31.079$;
	$B' = 155° 42' 28''$,	$C' = 3° 59' 00''$,	$c' = 3.0225$.
	(*l*) No solution.		

2. $d \cos A \pm \sqrt{4a^2 - d^2 \sin^2 A}$ if A is opposite a,
$-d \cos A \pm \sqrt{4a^2 - d^2 \sin^2 A}$ if the supplement of A is oppo-

site a; 124.06 if $25° 45'$ is opposite 36, 8.770 if the supplement of $25° 45'$ is opposite 36.

3. 4.449 in. or 0.843 in.
4. 20976 ft.
5. 6308.0 ft.
6. 530.94 ft.

Exercise 27, pages 175–177

1. (a) $A = 101° 00' 33''$, $B = 31° 42' 51''$, $c = 36.946$.
   (b) $A = 57° 49' 33.5''$, $B = 58° 32' 15.5''$, $c = 300.95$.
   (c) $A = 38° 52' 40''$, $B = 8° 49' 00''$, $c = 43.019$.
   (d) $B = 10° 33' 05''$, $C = 22° 18' 07''$, $a = 734.17$.
   (e) $A = 98° 45' 20''$, $C = 68° 45' 10''$, $b = 8.5425$.
   (f) $B = 23° 15' 06''$, $C = 30° 12' 44''$, $a = 578.29$.
   (g) $A = 23° 27' 57''$, $C = 19° 00' 51''$, $b = 823.12$.
   (h) $A = 46° 42' 21''$, $C = 49° 48' 17''$, $b = 0.13516$.
   (i) $B = 71° 04' 45''$, $C = 40° 33' 35''$, $a = 0.005,321,5$.
2. 30.414 mi. 3. 15.18 mi. 4. 12.725 mi. 5. 257.47 ft.
6. $\sqrt{4a^2 + d^2 - 8ad \cos v}$, $\sqrt{a^2 + d^2 - 2ad \cos v}$.
7. 27.096 in. 8. 3669 ft.

Exercise 28, pages 182–183

1. (a) $A = 44° 04' 50''$, $B = 101° 44' 32''$, $C = 34° 10' 40''$.
   (b) $A = 66° 22' 08''$, $B = 90° 39' 36''$, $C = 22° 58' 14''$.
   (c) $A = 30° 41' 40''$, $B = 99° 25' 14''$, $C = 49° 53' 10''$.
   (d) $A = 45° 11' 06''$, $B = 51° 04' 16''$, $C = 83° 44' 38''$.
   (e) $A = 27° 12' 08''$, $B = 21° 42' 32''$, $C = 131° 05' 20''$.
   (f) $A = 42° 51' 40''$, $B = 111° 26' 22''$, $C = 25° 41' 58''$.
   (g) $A = 19° 26' 20''$, $B = 150° 18' 32''$, $C = 10° 15' 12''$.
   (h) $A = 33° 32' 32''$, $B = 50° 40' 46''$, $C = 95° 46' 42''$.
   (i) $A = 139° 56' 16''$, $B = 2° 58' 44''$, $C = 37° 05' 00''$.
   (j) $A = 53° 33' 52''$, $B = 26° 05' 06''$, $C = 100° 21' 08''$.
   (k) No solution.
2. A is 83.64 mi. south of C and 175.04 mi. east or west of C.
3. 2d town is N. 60° 22′ 42″ E. from 1st, 3d town is S. 74° 25′ 28″ E. from 1st; or 2d town is N. 60° 22′ 42″ W. from 1st, 3d town is S. 74° 25′ 28″ W. from 1st.

4. 11.926 in. 5. $60° 37'$. 6. 11.832.
7. $50(3 - \sqrt{3})/3$, $50(\sqrt{3} - 1)$, $100(3 - \sqrt{3})/3$.
8. (*a*) 3.9686 in. (*b*) The center of the circle is located at the intersection of the bisectors of the angles of the triangle. Also, the circle is tangent to the sides of the triangle at points which are 13.5 in. from the intersection of sides 17 and 22, 8.5 in. from the intersection of sides 22 and 12, 3.5 in. from the intersection of sides 12 and 17. The center of the circle can therefore be located by erecting at any one of these points a line 3.9686 in. long perpendicular to the side.

Exercise 29, pages 183–184

1. 819.9 ft. 2. $33° 30' 26''$, $50° 42' 14''$, $95° 44' 20''$.
3. 783.36. 4. 15.416 ft. 5. $25° 19' 50''$.
6. $13° 03' 43''$. 7. 37.619 ft. 8. 45.611.
9. $3a/4$. 10. $2\sqrt{2}/3$. 11. $4\sqrt{34}/9$, $3\sqrt{41}/16$, ∞.
12. 238610 mi. 13. 867500 mi. 14. 2.2875 mi.
16. (*a*) $B = 1° 37' 40''$, $C = 177° 07' 00''$, $a = 127.05$.
(*b*) No solution.
(*c*) $A = 87° 00'$, $a = 1030$, $b = 1031$.
(*d*) $A = 2° 30' 48''$, $C = 176° 38' 57''$, $c = 19.994$;
$A' = 177° 29' 12''$, $C' = 1° 40' 33''$, $c' = 10.004$.

Exercise 30, pages 186–187

1. For exercise 25, problem 1: (*a*) 156030. (*b*) 4941.11. (*c*) 36.402. (*d*) 0.79680. (*e*) 34042. (*f*) 4073.0. (*g*) 21189. (*h*) 31445. (*i*) 177260. (*j*) 33975. (*k*) 130.04.
For exercise 26, problem 1: (*a*) 913.07. (*b*) 16956. (*c*) 6,678,150, 553420. (*d*) 6,724,150. (*e*) 2933.9, 758.52. (*f*) 16078. (*g*) 828.10, 304.91. (*h*) 2968.7. (*i*) 919.41, 110.23. (*j*) 912.58, 105.97. (*k*) 101.01, 9.8232. (*l*) No solution.
For exercise 27, problem 1: (*a*) 479.38. (*b*) 36490. (*c*) 120.36. (*d*) 34521. (*e*) 268.21. (*f*) 41338. (*g*) 64448. (*h*) 0.003,277,9. (*i*) 0.000,010,126.
For exercise 28, problem 1: (*a*) 6212.3. (*b*) 118.47.

(*c*) 74.734. (*d*) 275.89. (*e*) 15,925,500. (*f*) 0.094045.
(*g*) 55961. (*h*) 17422×10^5. (*i*) 360.13. (*j*) 483.06.
(*k*) No solution.

2. $a^2\sqrt{3}/4$. **3.** 38.940 sq. chains.

4. 57.67 rods, 96.11 rods, 134.56 rods.

5. 26.274, 22° 12′, 30° 53′ 28″, 126° 54′ 32″.

6. 173.87 sq. yds. **9.** 15581.

Exercise 31, page 190

1. (*a*) $A = 20° 15' 39''$, $B = 115° 47' 30''$, $C = 43° 56' 45''$.
(*b*) $A = 45° 19' 32''$, $B = 6° 52' 18''$, $C = 127° 48' 00''$.
(*c*) $A = 43° 22' 34''$, $B = 60° 14' 57''$, $C = 76° 22' 30''$.
(*d*) $A = 44° 46' 02''$, $B = 52° 46' 50''$, $C = 82° 27' 11''$.
(*e*) $c = 14.379$, $A = 75° 45'$, $B = 40° 15'$.
(*f*) $a = 101.32$, $B = 35° 46' 13''$, $C = 78° 33' 47''$.

2. 0.07310.

Exercise 32, pages 197–198

1. The aeroplane must be pointed N. 82° 15′ W. on the return trip. **3.** 15.18 lbs., 44° 25′. **4.** 30° 47′, 41° 11′, 108° 02′. **5.** 30° with vertical and from front to back of window. **6.** 10.959 mi. per hr., up stream at angle of 43° 11′ with bank. **7.** (*a*) Up stream at angle of 21° 18′ with bank. (*b*) 41.3 min. **8.** 49° 27′ 30″. **9.** N. 75° 50′ E. **10.** (*a*) 144 mi., at end of 3 hr. (*b*) 147 mi., at end of 2.5 hr. **11.** (*a*) N. 10° 38.9′ E. on outward trip, S. 10° 38.9′ E. on homeward trip. (*b*) At end of 1 hr. 4 min. 40 sec. **12.** (*a*) S. 54° 0.6′ W. (*b*) N. 42° 49.9′ E. (*c*) 126.3 mi., at end of 1 hr. 0 min. 56 sec.

Exercise 33, pages 198–207

1. 107.11 ft. **2.** 38° 56′ 30″. **3.** (*a*) 19.272 mi. (*b*) 14.079 mi. **4.** 74° 10′ 24″, 64° 25′ 00″, 41° 24′ 36″. **5.** 1622.6 sq. rds. **6.** 154.65 ft. **7.** 36.5 mi. per hr., N. 18° 21′ W. **8.** 954.4 lbs. **9.** $0.2588a$. **10.** Yes. **11.** 37° 31′, 52° 50′, 89° 38′. **12.** 751.5 ft. **13.** 278.20 ft., 12466 sq. ft. **14.** 48° 26′ 21″.

15. 9632.3 ft. **16.** 55.705 ft. **17.** $67^\circ\ 07'\ 11''$.
18. $8^\circ\ 40'\ 52''$. **19.** 70186 sq. ft. **20.** $51^\circ\ 20'\ 26''$.
21. 120,524,000 mi. or 61,009,300 mi. **22.** 19.416.
23. 64.27 sq. ft. **24.** 22.6 mi. **25.** 4.892 mi. **26.** $-1/51$.
27. 1735.5 sq. yds. **28.** 66.23 rods, $4^\circ\ 05'\ 30''$, $104^\circ\ 54'\ 30''$.
29. 58.50 mi. **30.** 60°. **31.** 885.22 ft. **32.** 163.78 ft.
33. $c/\sqrt{4-\sqrt{3}}$. **34.** 1.283 mi., 1.205 mi. **35.** 23.99 ft.
36. $l/(\cot A + \cot B)$ or $l \sin A \sin B/\sin (A + B)$.
37. 2201.6 ft. **38.** 0.27 in., 0.90 in., 2.01 in., 3.41 in., 5.14 in., 7.13 in., 9.18 in., 11.33 in., 13.46 in., 15.50 in., 17.39 in., 19.13 in., 20.56 in., 21.79 in., 22.79 in., 23.45 in., 23.87 in., 24.00 in.
42. 6.58 in. **43.** $105^\circ\ 27'$. **45.** N. $22^\circ\ 30'$ W., 7.84 mi. per hr.

Exercise 34, page 219

5. $x = \pi/4 + n\pi \quad (n = 0, \pm 1, \pm 2, \cdots)$.
6. $x = \pi/4 + n\pi \quad (n = 0, \pm 1, \pm 2, \cdots)$.
13. (*a*) π. (*b*) 4π. (*c*) 4π. (*d*) 2π. (*e*) $2\pi/3$. (*f*) $\pi/2$. (*g*) 2. (*h*) 4. (*i*) 3.

Exercise 35, pages 226–229

2. (*a*) $\pi/6$, $n\pi + (-1)^n\pi/6$. (*b*) $3\pi/4$, $2n\pi \pm 3\pi/4$. (*c*) 0, $n\pi$. (*d*) $\pi/2$, $n\pi/2$. (*e*) $\pi/3$, $n\pi + \pi/3$. (*f*) $\pi/4$, $n\pi + \pi/4$.
(*g*) $0.61548^{(r)}$ $(= 35^\circ\ 15'\ 52'')$, $n\pi + (-1)^n 0.61548$.
(*h*) $0.38358^{(r)}$ $(= 21^\circ\ 58'\ 40'')$, $n\pi + (-1)^n 0.38358$.
4. (*a*) $\pm 2\sqrt{2}$. (*b*) $\pm \sqrt{5}/2$. (*c*) $\pm \sqrt{6}/3$.
11. (*a*) ∞, $\pm 7\sqrt{2}/8$. (*b*) $\pm (2\sqrt{2}/7 \pm 2\sqrt{10}/9)$. (*c*) $-1/9$. (*d*) $\pm (3 - \sqrt{5})/2$. (*e*) $\pm (\sqrt{7} \pm \sqrt{6})/6$. (*f*) $\pm 1/\sqrt{m}$.

Exercise 36, pages 234–236

1.* (*a*) 0°, 180°. (*b*) 210°, 330°.
(*c*) 45°, 225°. (*d*) $63^\circ\ 26'$, $243^\circ\ 26'$.
(*e*) $75^\circ\ 58'$, $255^\circ\ 58'$. (*f*) 90°, 210°, 330°.
(*g*) 0°, 90°.
(*h*) 45°, 225°, $161^\circ\ 34'$, $341^\circ\ 34'$.

(*i*) 90°, 270°, 26° 34′, 206° 34′.
(*j*) 0°, 180°, 45°, 225°.
(*k*) 45°, 225°, 161° 34′, 341° 34′.
(*l*) 18° 26′, 108° 26′, 198° 26′, 288° 26′, 67° 30′, 157° 30′, 247° 30′, 337° 30′.

2.* (*a*) 120°, 240°, 45°, 135°, 225°, 315°.
(*b*) 26° 34′, 206° 34′.
(*c*) 0°, 180°.
(*d*) 90°, 270°, 210°, 330°.
(*e*) 139° 28′, 355° 18′.
(*f*) 126° 13′ 19″, 174° 25′ 35″.
(*g*) 31° 57′ 02″, 133° 16′ 08″.
(*h*) 15°, 285°.
(*i*) 120°.
(*j*) 90°, 270°, 45°, 135°, 225°, 315°.
(*k*) 30°, 60°, 120°, 150°, 210°, 240°, 300°, 330°.
(*l*) 0°, 45°, 135°, 225°, 315°.

3.* (*a*) 8°, 168°.
(*b*) 50° 36′, 230° 36′, 129° 24′, 309° 24′.
(*c*) 30°, 60°, 120°, 150°, 210°, 240°, 300°, 330°.
(*d*) 0°, 180°, 220° 39′, 319° 21′.
(*e*) 30°, 150°, 180°, 210°, 330°.
(*f*) 240°, 300°.
(*g*) 70°, 190°, 310°, 330°.
(*h*) 45°, 225°.

4. $r = \pm \sqrt{x^2 + y^2}, \quad v = \tan^{-1}(y/x)$.

5. $r = \pm \sqrt{x^2 + y^2 + z^2}$, $v = \tan^{-1}(y/x)$, $u = \tan^{-1}(\sqrt{x^2 + y^2}/z)$ $= \cos^{-1}(z/\sqrt{x^2 + y^2 + z^2})$.

6. $u = 76° \ 10'$, $v = 15° \ 30'$.

7. $u = \tan^{-1}(a/b) + \cos^{-1} \frac{1}{2}\sqrt{a^2 + b^2}$,
$v = \tan^{-1}(a/b) - \cos^{-1} \frac{1}{2}\sqrt{a^2 + b^2}$.

9. (*a*) 0.450. (*b*) 1.9346.

* In the answers to problems 1–3 only positive angles less than 360° are given. Any one of the solutions plus or minus a multiple of 360° is also a solution.

(*c*)* 1.114. (*d*)* 0.480.
(*e*)* 2.69625 (for $\log_{10} x$). (*f*)* $-$ 3.1832.
(*g*)* 0, $-$ 3.9407. (*h*) $\pm$ 0.8241.
(*i*) 0.3990 (for $\log_{10} x$). (*j*) 2.863.
(*k*) 1.6175. (*l*) 0, $\pm$ 0.948.
10. (*a*)* 0.5081, $-$ 1.4718. (*b*)* $-$ 3.14231.

* Other solutions exist.

Exercise 37, pages 242–244

1. (*a*) $2\sqrt{2}$, 315°. (*b*) $2\sqrt{2}$, 225°. (*c*) $2\sqrt{2}$, 135°. (*d*) $2\sqrt{5}$, 243° 26′. (*e*) 3, 90°. (*f*) 3, 0°. (*g*) $0.1\sqrt{61}$, 39° 48′. (*h*) 7, 0°. (*i*) 5, 180°. (*j*) 1, 90°. (*k*) 1, 270°. (*l*) 5/4, 90°. (*m*) $\sqrt{13}/6$, 33° 41′. (*n*) $2\sqrt{17}$, 165° 58′. (*o*) $5\sqrt{2}$, 351° 52′. (*p*) 2, 330°. (*q*) 3, 48° 11′.

2. (*a*) $\sqrt{2} + i\sqrt{2}$. (*b*) $\frac{3}{2}\sqrt{3} + \frac{3}{2}i$. (*c*) $-\frac{5}{2}\sqrt{3} + \frac{1}{2}i$. (*d*) $-2\sqrt{2} - 2i\sqrt{2}$. (*e*) $\frac{3}{8} - \frac{3}{8}i\sqrt{3}$. (*f*) $3 + 3i\sqrt{3}$. (*g*) 3. (*h*) -3. (*i*) $2 - 2i$. (*j*) $2\sqrt{2} + 2i\sqrt{2}$. (*k*) $-1 - i\sqrt{3}$. (*l*) $\dfrac{\sqrt{3}}{2} - \frac{1}{2}i$. (*m*) $-\frac{5}{2}\sqrt{2} + \frac{5}{2}i\sqrt{2}$.

3. (*a*) $7 - 11i$. (*b*) $3 + 4i$. (*c*) $-2 - 2i$. (*d*) $2.2 + 1.7i$. (*e*) $-7 - 3i$. (*f*) $6 - 4i$. (*g*) $(\sqrt{3} - 1)i$. (*h*) $1 + (1 - \sqrt{3})i$. (*i*) $-3.38885 + 7.05018i$.

Exercise 38, pages 245–246

The abbreviation **cis** v is used for **cos** $v + i$ **sin** v.

1. (*a*) $6i$. (*b*) 10 cis 195°. (*c*) 12 cis 50°. (*d*) $2(1 + i\sqrt{3})$. (*e*) $\frac{7}{8}\sqrt{2}(1 - i)$. (*f*) $\frac{3}{2}i$. (*g*) $\frac{2}{3}$ cis ($-$ 78°). (*h*) 6 cis 165°. (*i*) $\frac{9}{2}(1 + i\sqrt{3})$.

2. (*a*) $2\sqrt{3} - 2 + 2i(1 + \sqrt{3})$. (*b*) 244/225. (*c*) $11 - 7i$. (*d*) $175i$. (*e*) $-1 + \sqrt{3} + i(1 + \sqrt{3})$. (*f*) $2i$. (*g*) $-2 + 2i$. (*h*) $44 - 128i$. (*i*) $0.6 - 3i$. (*j*) $\frac{1}{4}(\sqrt{3} + i)$. (*k*) $\frac{3}{4}[\sqrt{3} - 1 + i(\sqrt{3} + 1)]$. (*l*) $\frac{1}{2}(\sqrt{3} + i)$. (*m*) $\frac{3}{5}(3 - i)$. (*n*) $\frac{3}{2} + i$. (*o*) $\frac{2}{2}\frac{5}{6}(1 + 5i)$. (*p*) $6(-1 + i)$. (*q*) $-i$. (*r*) $\frac{1}{17}(-9 + 5i)$. (*s*) $\frac{1}{25}[6 + 4\sqrt{5} + (8 - 3\sqrt{5})i]$. (*t*) $1 - \sqrt{3} - i(1 + \sqrt{3})$. (*u*) $4\sqrt{2}$ cis 15°.

Exercise, pages 250–251

$$\text{cis}\,\frac{360°}{p},\quad \text{cis}\,\frac{720°}{p},\quad \cdots,\quad \text{cis}\,\frac{(p-1)360°}{p},\quad 1.$$

Exercise 39, pages 252–253

1. (a) $27 \text{ cis } 75°$. (b) $-1/16$. (c) $(512/16807)(1+i\sqrt{3})$. (d) $16(-1+i)$. (e) $2(1-i\sqrt{3})$. (f) $-3-4i$. (g) $-415+234i$. (h) i. (i) $16(1+i)$. (j) 32. (k) -8.
2. $2 \text{ cis } 15°$, $2 \text{ cis } 135° = \sqrt{2}(-1+i)$, $2 \text{ cis } 255°$.
3. $(\sqrt{2}/2)(-\sqrt{3}+i)$, $(\sqrt{2}/2)(1-i\sqrt{3})$.
4. 1, $\frac{1}{2}(-1 \pm i\sqrt{3})$.
5. $\pm(\sqrt{2}/2)(1 \pm i)$.
6. $3 \text{ cis } 22.5°$, $3 \text{ cis } 112.5°$, $3 \text{ cis } 292.5°$.
7. $2 \text{ cis } 54°$, $2 \text{ cis } 126°$, $2 \text{ cis } 198°$, $-2i$, $2 \text{ cis } 342°$.
8. $\sqrt[3]{2} \text{ cis } 10°$, $\sqrt[3]{2} \text{ cis } 130°$, $\sqrt[3]{2} \text{ cis } 250°$.
9. $\pm\frac{1}{2}(1 \pm i\sqrt{3})$.
10. $\pm(\sqrt{2}/2)(\sqrt{3}-i)$.
11. $(1+i)/\sqrt[3]{2}$, $\sqrt[6]{2} \text{ cis } 165°$, $\sqrt[6]{2} \text{ cis } 285°$.
12. $\sqrt[6]{13} \text{ cis } 18°\,46'$, $\sqrt[6]{13} \text{ cis } 138°\,46'$, $\sqrt[6]{13} \text{ cis } 258°\,46'$.
13. $\sqrt[6]{13} \text{ cis } 41°\,14'$, $\sqrt[6]{13} \text{ cis } 101°\,14'$, $\sqrt[6]{13} \text{ cis } 161°\,14'$, $\sqrt[6]{13} \text{ cis } 221°\,14'$, $\sqrt[6]{13} \text{ cis } 281°\,14'$, $\sqrt[6]{13} \text{ cis } 341°\,14'$.
14. (a) 1, $\text{cis } 72°$, $\text{cis } 144°$, $\text{cis } 216°$, $\text{cis } 288°$.
    (b) $\pm(\sqrt{3} \pm i)$, $\pm 2i$.
    (c) $\sqrt[5]{10} \text{ cis } 36°$, $\sqrt[5]{10} \text{ cis } 108°$, $-\sqrt[5]{10}$, $\sqrt[5]{10} \text{ cis } 252°$, $\sqrt[5]{10} \text{ cis } 324°$.
    (d) $\sqrt[5]{235}$, $\sqrt[5]{235} \text{ cis } 72°$, $\sqrt[5]{235} \text{ cis } 144°$, $\sqrt[5]{235} \text{ cis } 216°$, $\sqrt[5]{235} \text{ cis } 288°$.
    (e) $\sqrt[7]{9}$, $\sqrt[7]{9} \text{ cis } 51\frac{3}{7}°$, $\sqrt[7]{9} \text{ cis } 102\frac{6}{7}°$, $\sqrt[7]{9} \text{ cis } 154\frac{2}{7}°$, $\sqrt[7]{9} \text{ cis } 205\frac{5}{7}°$, $\sqrt[7]{9} \text{ cis } 257\frac{1}{7}°$, $\sqrt[7]{9} \text{ cis } 308\frac{4}{7}°$.
    (f) $\sqrt[5]{8} \text{ cis } 36°$, $\sqrt[5]{8} \text{ cis } 108°$, $-\sqrt[5]{8}$, $\sqrt[5]{8} \text{ cis } 252°$, $\sqrt[5]{8} \text{ cis } 324°$.
15. $\text{cis } 72°$, $\text{cis } 144°$, $\text{cis } 216°$, $\text{cis } 288°$.
16. $\text{cis } 36°$, $\text{cis } 108°$, $\text{cis } 252°$, $\text{cis } 324°$.
17. $\sqrt[5]{a}$, $\sqrt[5]{a} \text{ cis } 72°$, $\sqrt[5]{a} \text{ cis } 144°$, $\sqrt[5]{a} \text{ cis } 216°$, $\sqrt[5]{a} \text{ cis } 288°$.

Exercise 40, pages 256–257

1. $\sin 1' = 0.000291$, $\cos 1' = 0.999{,}999{,}958$, $\tan 1' = 0.000291$, $\cot 1' = 3436.43$, $\sec 1' = 1.000{,}000{,}042$, $\csc 1' = 3436.43$.

2. 20.6748 ft. **3.** 2160 mi. **4.** 863950 mi.
5. 25,804,835,000,000 mi. **6.** $0° 45' 35''$. **7.** 92,700,000 mi.
8. 238000 mi. **9.** 6.34 ft., 232,500 mi.

Exercise 41, page 260

1. 0.0872. **2.** 0.9848. **3.** 0.2588, 0.9659. **4.** 0.5000.
5. $-$ 0.0872, 0.9962. **6.** 0.84147, 0.54030.

7. $1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots$.

8. $\frac{1}{x} + \frac{x}{3!} + \frac{7x^3}{3 \cdot 5!} + \frac{31x^5}{3 \cdot 7!} + \cdots$.

Exercise 42, page 263

1. 1.005004. **2.** $0.869i$. **3.** i. **4.** $(1 - i)/\sqrt{2}$. **5.** 1.1752.
6. $1.29847 + 0.63496i$.

Exercise 43, pages 264–265

To aid in plotting, the following co-ordinates with respect to the real and the imaginary axes are given:
5. (*a*) (0.707, 0.707). (*b*) (1, 1.732). (*c*) ($-$ 2.5, 4.330).
(*d*) (0.866, $-$ 0.5). (*e*) (0, 3). (*f*) ($-$ 1, 0). (*g*) (1, 0).
6. (*a*) (2.354, 1.359). (*b*) (5.225, $-$ 5.225). (*c*) ($-$ 0.368, 0).
(*d*) ($-$ 0.068, 0.117).

Exercise 46, page 272

1. 0.69315. **2.** 1.09861. **3.** 1.60944. **4.** 2.30259.
5. 0.69897. **6.** 9.30103–10. **7.** 9.82391–10.

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