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A PRACTICAL TREATISE, IN THREE VOLUMES, FOR ELECTRICAL, CIVIL AND MECHANICAL ENGINEERS

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VECTORS FOR ELECTRICAL ENGINEERS

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LONDON:
CHAPMAN & HALL LTD
11 HENRIETTA STREET, W.C.2
1933

Printed in Great Britain by
The Whitefriars Press Ltd.
London and Tonbridge.

PREFACE

In April, 1893, Professor A. E. Kennelly, of Harvard and Massachusetts, published a paper entitled " Impedance " (*Trans. A.I.E.E.*, April, 1893, Vol. X., pp. 175–216) in which he stated :—

" Any combination of resistances, non-ferric inductances, and capacities, carrying harmonically alternating currents, may be treated by the laws of unvarying currents, if the inductances are considered as resistances of the form $pl\sqrt{-1}$, and the capacities of the form $-\frac{1}{kp}\sqrt{-1}$, the algebraic operations being then performed according to the laws controlling ' complex quantities.' "

This was the first publication dealing with the subject. The operator $\sqrt{-1}$ and complex quantities were, of course, well known to mathematicians as a branch of pure mathematics, and the operator $\sqrt{-1}$ had previously been named j, but Kennelly first applied these quantities to the problems of alternate current electrical engineering. The method was adopted and applied very extensively by Dr. C. P. Steinmetz, and to-day all communication engineers are well versed in it. This is not so, however, with power engineers. The last chapter in several text-books on Electrical Engineering deals with the application of Complex Quantities, but throughout the text no use is made of this powerful method of solving problems.

The present book is the result of an attempt on the part of the Author to apply the methods of complex quantities generally. No great originality can be claimed for the book, as the same problems were dealt with by Steinmetz many years ago by quite similar methods. There is, however, as far as the author is aware, no book which contains the whole subject, from an elementary exposition of the methods and manipulation of complex algebra to its application to such diverse engineering problems as bridge networks, transformers, electrical machinery and power transmission.

The underlying idea in the treatment is the close association of the vector diagram with the complex algebra. Thus in some problems the vector diagram is drawn first and the algebraic expressions derived from it, while in others it is more convenient to find the algebraic expressions first, and to draw the vector diagrams from them. The fundamental principles of electricity and magnetism are assumed to be known, as well as the constructional details of the apparatus and machinery dealt with.

One small note on nomenclature in connection with the process of inversion. The mathematician describes the process which throughout the text has been called inversion, as inversion *and* reflection. Thus in Fig. 18, p. 17, the circle OD′P′ would be called by the mathematician the invert of the line p_1Rp_2, and the circle ODP_1 would be called the reflection of the circle OD′P′. It is, however, more convenient to call the circle ODP_1, the invert of the line p_1Rp_2, and this has been done throughout.

To Mr. H. E. Lowry, Head of the Department of Mathematics at the Woolwich Polytechnic, I am indebted for assistance in the work on pp. 106 and 107, and for the circular locus proof under (iii.), p. 145. The diagrams are the work of Mr. D. W. Hopkins of the City and Guilds (Eng.) College. For the reading of the proofs and for working out the examples I am indebted to Mr. D. Connelly of the Electrical Engineering Department of the Woolwich Polytechnic.

E. M.

WOOLWICH POLYTECHNIC,
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CONTENTS

VECTORS FOR ELECTRICAL ENGINEERS

CHAPTER I

VECTOR DIAGRAMS AND VECTOR ALGEBRA

(1) Vector Diagrams

ALTHOUGH it is assumed that the reader knows the fundamental principles of electricity and magnetism and their applications both to direct and alternating current problems, it is thought desirable to start from the most elementary consideration of the production of an alternating electromotive force, in order to make quite clear the connection between alternating quantities, vectors and complex quantities.

Imagine a rectangular single-turn coil of wire AA′B′B rotating in a uniform magnetic field B lines per sq. cm. about an axis OO′ through the centre of AB and A′B′ at a speed of n revolutions per second. Let the length of the coil (AA′ = BB′) be l cms., and the radius (OA = OB = O′A′ = O′B′) be r cms. The coil is shown at a particular instant in Fig. 1, where (a) is a side view, and (b) a sectional end view, and the horizontal lines in (a) and the dots in (b) indicate the magnetic field, the direction being vertically upwards through the paper. In (a) take rectangular co-ordinates OX and OY, and let the angle AOX be θ radians. From A draw a perpendicular Ax to OX.

Then the magnetic flux Φ through the coil at the position drawn is given by

$$\begin{aligned}\Phi &= 2.\mathrm{A}x.l.\mathrm{B}\\ &= 2rl\mathrm{B}\sin\theta,\end{aligned}$$

and the electromotive force induced in the coil by its rotation in the field is

$$e = -\frac{d\Phi}{dt} = -\,2rl\mathrm{B}\,\frac{d\sin\theta}{dt}$$

in absolute electromagnetic units

or $$e = -2rl\mathrm{B}\frac{d \sin \theta}{dt} \times 10^{-8} \text{ volts}$$

in practical units.

If time is measured from the instant when OA is in the direction OX, and if ω is the angular velocity of OA in radians per second, then $\theta = \omega t$, and

$$e = -2rl\mathrm{B}\omega \cos \omega t \times 10^{-8} \text{ volts.}$$

In the position shown the flux through the coil is increasing, and by Lenz's law the direction of the e.m.f. induced is such that the current which would flow would cause flux to go downwards through the paper (Fig. 1 (*b*)), *i.e.*, would be in the direction A′ABB′. AA′B′B is the positive direction.

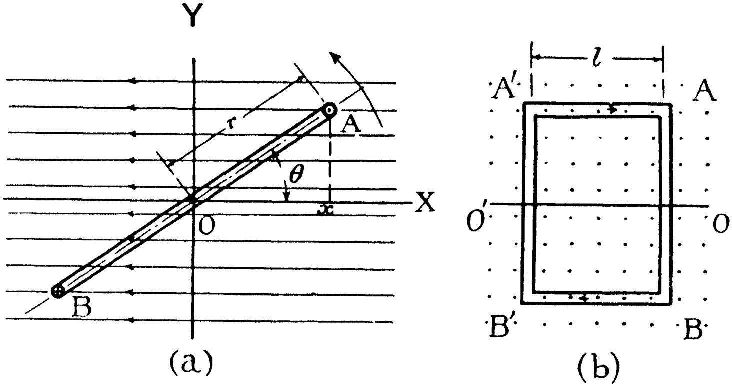


FIG. 1.—Induced electromotive force.

If, however, A′ABB′ is called the positive direction, the negative sign disappears from the equations, and the instantaneous electromotive force is

$$e = 2rl\mathrm{B}\omega \cos \omega t \times 10^{-8} \text{ volts.}$$

Instead of looking upon the e.m.f. as being produced by the alteration of flux through the coil, it can be looked upon as being produced by the cutting of the lines of flux by the coil sides. Evidently the sides AB and A′B′ cut no lines. Suppose θ changes by $d\theta$ (Fig. 2). The distance moved by AA′ is $rd\theta$, and the lines cut number Bl. $rd\theta \cos \theta$. The time taken for the movement is $d\theta/\omega$, and hence the rate of cutting lines is $\mathrm{B}rld\theta \cos \theta/(d\theta/\omega) =$

$Brl\omega \cos\theta$. BB′ cuts the same number and hence the e.m.f. produced is

$$e = 2rlB\omega \cos\theta \times 10^{-8} \text{ volts.}$$

From this expression the electromotive force in the direction ABB′A′ round the coil as the coil is rotated is plotted in Fig. 3 against the position of the coil, as determined by θ, *i.e.*, ωt.

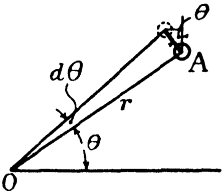


FIG. 2.

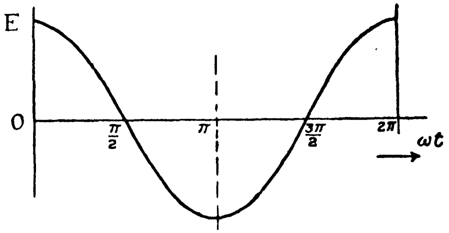


FIG. 3.

It should be noted that since there are n revolutions per second, the frequency f of the e.m.f. is given by

$$f = n$$

and the angular velocity by

$$\omega = 2\pi n = 2\pi f.$$

Now imagine (Fig. 4) a line OP of length $2rlB\omega \times 10^{-8}$ rotating in a counter-clockwise direction with angular velocity ω. Drop a perpendicular Px from P on to a reference line OX, chosen so that time is counted from the instant when OP coincides with OX.

Then
$$\begin{aligned} Ox &= OP \cos\omega t \\ &= 2rlB\omega \cos\omega t \times 10^{-8} \\ &= e. \end{aligned}$$

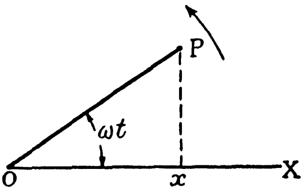


FIG. 4.

The length Ox gives, therefore, the value of the electromotive force induced in the coil at any instant, and it is found at once when the length of the rotating line OP, its angular velocity, and the reference line are known.

Suppose now (Fig. 5) that another rectangular single turn coil CDD′C′ is fixed to the same axis as the coil ABB′A′ and rotated with it in the uniform magnetic field. The angle between the two coils is ϕ radians, the radius OC is r', and the length CC′ is l'. The coils are connected together, and to slip rings, as is indicated in Fig. 5 (*b*).

By the same reasoning as before, reckoning time from the reference line OX, with the coil ABB′A′ in the same position, at an angle $\theta = \omega t$, the e.m.f. e' in the new coil is

$$e' = 2r'l'\mathrm{B}\omega \cos(\omega t + \phi) \times 10^{-8} \text{ volts,}$$

so that the e.m.f. in the two coils together, or the potential difference of the slip rings, is

$$\{\, 2rl\mathrm{B}\omega \cos \omega t + 2r'l'\mathrm{B} \cos(\omega t + \phi) \,\} \times 10^{-8} \text{ volts,}$$

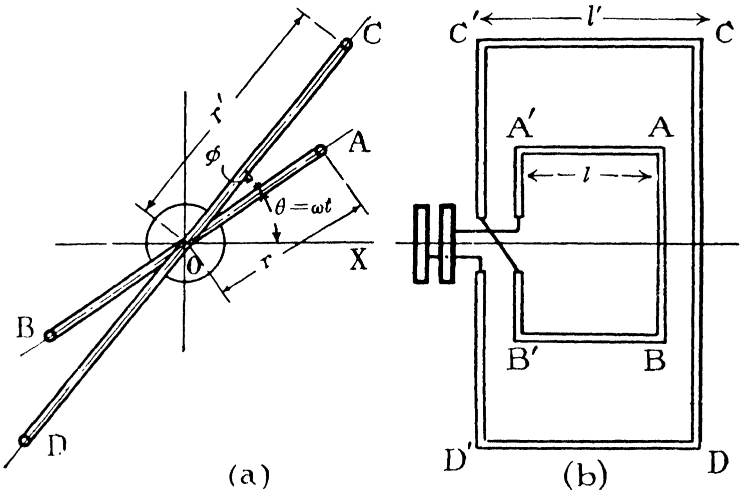


FIG. 5.—Multiple turn coil in uniform field.

or writing $2rl\mathrm{B}\omega \times 10^{-8} = \mathrm{E}_1$, and $2r'l'\mathrm{B} \times 10^{-8} = \mathrm{E}_2$, and the total instantaneous e.m.f. as e,

$$e = \mathrm{E}_1 \cos \omega t + \mathrm{E}_2 \cos(\omega t + \phi).$$

In Fig. 6 OP_1 is drawn equal in length to E_1 and OP_2 to E_2 and the angle $\mathrm{P}_2\mathrm{OP}_1 = \phi$, and the two lines OP_1 and OP_2 are supposed to rotate together with angular velocity ω. Then, if P_1x_1 and P_2x_2 are perpendiculars dropped on to OX, $\mathrm{O}x_1$ is the e.m.f. in the first coil at the instant for which the diagram is drawn, and $\mathrm{O}x_2$ is the e.m.f. in the second coil, and the total e.m.f. is $\mathrm{O}x_1 + \mathrm{O}x_2$.

Draw $\mathrm{P}_1\mathrm{Q}$ parallel to OP_2 and $\mathrm{P}_2\mathrm{Q}$ parallel to OP_1. Drop a perpendicular $\mathrm{Q}x_3$ on to OX and another P_1q on to $\mathrm{Q}x_3$. Clearly by the construction the triangles OP_2x_2 and P_1Qq are equal in all respects, hence $Ox_2 = P_1q = x_1x_3$.

Hence $Ox_3 = Ox_1 + Ox_2$ = total instantaneous e.m.f. in the two coils.

Join OQ. The projection of OQ on OX gives the instantaneous value of the combined e.m.f. in the two coils, and OQ is obtained from OP_1 and OP_2 by the same construction as that used to find the resultant of two forces (the parallelogram of forces) or of two velocities ; or in fact of any two vector quantities.

A third coil could be added in Fig. 5, and a third line OP_3 found for it in Fig. 6, and OP_3 added to OQ by the parallelogram law to give a new line, the projection of which would give the instantaneous e.m.f. in the three coils together, and the process could be continued indefinitely. The final resultant line would be found from all the other lines by a construction similar to that used in the polygon of forces.

Moreover, the process is not in the least limited to the very artificial case of coils rotated in a uniform magnetic field. Evidently the only requirement for the process to be valid is that the e.m.f. induced in a circuit shall be sinusoidal and of the same frequency. They can then be expressed in the form $E \cos(\omega t + \phi)$, and a line OP can be drawn for each. Nor is the process limited to e.m.f.'s in a circuit. It can apply equally well to currents. Lines, some representing currents and some electromotive forces, can be drawn on the same diagram and supposed to rotate together. The lengths of the projections of these lines on a fixed reference line give the instantaneous values of the individual quantities represented.

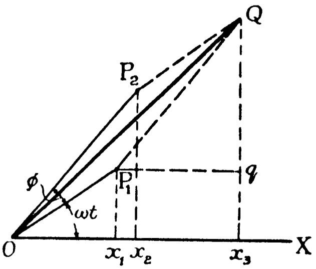


Fig. 6.—Addition of e.m.f.'s.

Such lines are commonly called vector electromotive forces, or vector currents, although they are not really vectors at all, and the drawing in which they are made is known as a vector diagram.

(2) **Vector Algebra and Geometry**

A scalar quantity is a quantity having magnitude and sense only.

A vector quantity is a quantity which has magnitude, direction and sense, and can be represented geometrically by a line of a certain length drawn in a certain direction with an arrow-head upon it to indicate the sense; this line is called a vector. Examples of vector quantities are force, displacement, velocity and acceleration.

. Vectors are added by the parallelogram construction. The lines drawn in a "vector" diagram to represent currents and voltages, have also magnitude direction and sense, and are added by the parallelogram construction, and they are called vectors by electrical engineers.

It should be borne in mind, however, that the actual (instantaneous) values of the current or voltage in these diagrams are given by the projections of the lines on to the reference line, wherein they differ from the true vectors of physics and mathematics.

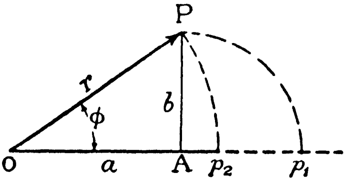


FIG. 7.—Construction of a vector.

(i.) A vector OP (Fig. 7) can be drawn in either of two ways. In the first a length OA $= a$ is drawn horizontally to the right, and a length AP $= b$ is drawn vertically upwards; that is, the vector OP is looked upon as the sum of two vectors, OA in the direction of the abscissa, and AP in the direction of the ordinate. The construction may be carried out in a slightly different manner. $Ap_1 = b$ may be marked off in the horizontal direction from A, and Ap_1 rotated through a right angle in a counter-clockwise direction to give AP. This geometrical construction can be written algebraically

$$\mathrm{OP} = a + jb,$$

if by jb is understood a rotation of b counter-clockwise through one right angle. *That is, an algebraic multiplication by j implies a geometric rotation through $\frac{\pi}{2}$.* This being so, a further multiplication by j must mean a further rotation through $\pi/2$, that is, multiplying twice by j rotates OB (Fig. 8) to OB_2, and similarly if again multiplied by j, a further rotation through $\pi/2$ takes place to OB_3, and if again, to OB.

These successive steps can be expressed as

$$\begin{aligned}
\mathrm{OB}_1 &= jb, \\
\mathrm{OB}_2 &= j\mathrm{OB}_1 = j(jb) = j^2b = -b, \\
\mathrm{OB}_3 &= j\mathrm{OB}_2 = j(j^2b) = j^3b = -jb, \\
\mathrm{OB}_4 &= j(\mathrm{OB}_3) = j(j^3b) = j^4b = b.
\end{aligned}$$

Thus algebraically j must be identified with $\sqrt{-1}$; geometrically j indicates that the quantity it multiplies in the algebraic expression is to be rotated counter-clockwise through 90°. Multiplication by $-j$ indicates a clockwise rotation (*i.e.*, in the negative direction) through 90°, or a positive rotation through three right angles. The identification of j with $\sqrt{-1}$ is obvious for the simple case considered; and is found by experience to lead to no inconsistent result, however complicated the expression and construction.

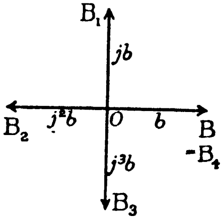


FIG. 8.—Use of operator j.

(ii.) The second method of drawing the vector OP is to mark off a length Op_2 equal to r along the abscissa, and rotate the line Op_2 through an angle ϕ to give OP. The algebraic expression corresponding to this construction is formed as follows :—

$$\begin{aligned}
\mathrm{OA} &= a = r\cos\phi, \\
\mathrm{AP} &= jb = jr\sin\phi, \\
\mathrm{OP} &= a + jb = r\cos\phi + jr\sin\phi, \\
&= r(\cos\phi + j\sin\phi).
\end{aligned}$$

Making use of the identity of j and $\sqrt{-1}$, the mathematicians show that

$$\cos\phi = \frac{\epsilon^{j\phi} + \epsilon^{-j\phi}}{2}\ ; \ \sin\phi = \frac{\epsilon^{j\phi} - \epsilon^{-j\phi}}{2j}.$$

Hence
$$\cos\phi + j\sin\phi = \epsilon^{j\phi}$$
and
$$\mathrm{OP} = r\epsilon^{j\phi}.$$

Thus $\epsilon^{j\phi}$ indicates that the quantity it multiplies is rotated counter-clockwise through an angle ϕ. $\epsilon^{j\phi}$ is usually abbreviated to $\angle\phi$ for a positive (counter-clockwise) rotation, and the vector is written

$$\mathrm{OP} = r\angle\phi.$$

For a negative (clockwise) rotation, $\mathrm{OP} = a - jb = r\epsilon^{-j\phi}$, and the abbreviation used to indicate this clockwise rotation through an angle ϕ is $\overline{\diagdown}\phi$.

(iii.) There are, therefore, two algebraic methods of expressing the vector OP,

$$\mathrm{OP} = a + jb$$
and
$$\mathrm{OP} = r\,\epsilon^{j\phi} \equiv r\,\angle\phi,$$

and these may readily be converted from one to the other by the relations

$$\left.\begin{array}{ll} a = r\cos\phi & b = r\sin\phi \\ r = \sqrt{a^2 + b^2} & \tan\phi = \dfrac{b}{a} \end{array}\right\}.$$

When expressed in the form $a + jb$, a is frequently called the *real* part of the vector, and b the *imaginary* part.

When expressed in the form $r\angle\phi$, r is called the *magnitude, modulus* or *size*, and ϕ the *angle* or *argument*.

When it is required to refer only to the magnitude of say a vector voltage, V, it is usual to write for this magnitude $|\,\mathrm{V}\,|$, or for a current, similarly, $|\,\mathrm{I}\,|$.

It is clear that two vectors $a + jb$ and $c + jd$ can only be equal if $a = c$ and $b = d$, for only in that case will the line OP, representing one, coincide with the line OP′, representing the other. Expressed in the form $r_1\angle\phi_1$ and $r_2\angle\phi_2$, the vectors can only be equal if $r_1 = r_2$ and $\hat{\phi}_1 = \hat{\phi}_2$, although the addition of any integral multiple of 2π to either ϕ_1 or ϕ_2 does not disturb the geometric equality.

(iv.) It is of interest, at this point, to indicate how the stationary vectors which have been already discussed are given rotation, so that they may be completely representative of a quantity which varies sinusoidally, such as the e.m.f. induced in the coils of Figs. 1 and 5.

Let the vector $r\angle\phi \equiv r\epsilon^{j\phi}$ be represented by R, where $|\,\mathrm{R}\,| = r$. If R rotates with an angular velocity ω radians per second, its projection on the reference line gives at all instants the value of the quantity represented, and so fulfils the above requirement. With this angular velocity, the angle which R makes with its original position after time t secs. is ωt radians, and in this position the vector is represented by $\mathrm{R}\epsilon^{j\omega t}$. It should be noted that

$$\mathrm{R}\epsilon^{j\omega t} \equiv |\,\mathrm{R}\,|\epsilon^{j\phi}\epsilon^{j\omega t} = |\,\mathrm{R}\,|\epsilon^{j(\omega t + \phi)}$$

but that if $\phi = 0$, the vector is *not* represented by $|\,\mathrm{R}\,|\epsilon^{j\omega t}$, but by $\mathrm{R}\epsilon^{j\omega t} = (|\,\mathrm{R}\,|\epsilon^{\circ})\epsilon^{j\omega t}$.

(v.) Graphically, the addition of a number of vectors is illustrated in Fig. 9, and is so well known as to need no long description. Clearly the vector resulting from the addition of $a_1 + ja_2$, $b_1 + jb_2$, $c_1 + jc_2$, $d_1 + jd_2$ . . . is

$$(a_1 + b_1 + c_1 + d_1 + \ . \ . \ . \ ) + j\,(a_2 + b_2 + c_2 + d_2 \ . \ . \ . \ ).$$

Hence the rule for the algebraic addition of vectors is "add the real parts and the imaginary parts separately." To subtract vectors, multiply the vectors to be subtracted by -1 (or reverse their directions) and add.

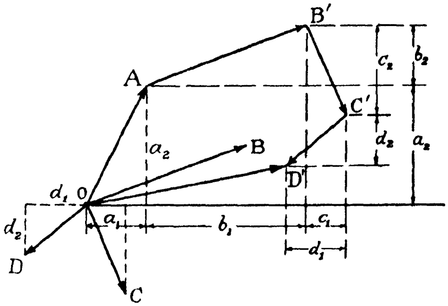


FIG. 9.—Addition of vectors.

If the vectors are in the form $r\angle\phi$ the graphical construction is unaltered; but the algebra is carried out by first converting to the form $a + jb$.

(vi.) Mathematicians have two vector products, the "dot" product and the "cross" product.

The "dot" product of two vectors $r_1 \angle\phi_1$ and $r_2 \angle\phi_2$ is a scalar quantity, and has the magnitude $r_1 r_2 \cos(\phi_1 - \phi_2)$. The "cross" product is a vector having a magnitude $r_1 r_2 \sin(\phi_1 - \phi_2)$, and a direction perpendicular to the plane containing the two vectors.

Neither of these products is, however, used by the engineer in manipulating his algebra for the purpose of constructing vector diagrams; but, instead, a straightforward algebraic multiplication.

For instance, the product of $(a + jb)$ and $(c + jd)$ is found as follows:—

$$(a + jb)(c + jd) = ac - bd + j(bc + ad).$$

The size of the product is

$$\begin{aligned} &\sqrt{(ac - bd)^2 + (bc + ad)^2} \\ &= \sqrt{(a^2c^2 - 2abcd + b^2d^2) + (b^2c^2 + 2abcd + a^2d^2)} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \text{the product of the sizes of the two vectors.} \end{aligned}$$

The angle is given by

$$\tan\phi = \frac{bc + ad}{ac - bd} = \frac{\dfrac{b}{a} + \dfrac{d}{c}}{1 - \dfrac{b}{a}\,\dfrac{d}{c}}$$

$$= \frac{\tan\phi_1 + \tan\phi_2}{1 - \tan\phi_1 \tan\phi_2}$$

$$= \tan(\phi_1 + \phi_2),$$

where ϕ_1 and ϕ_2 are the angles of the two vectors.

Thus the angle of the product is the sum of the angles of the two vectors.

This result is more simply arrived at by writing the vectors in the form $r\angle\phi$.

$$\begin{aligned}(r_1\angle\phi_1) \times (r_2\angle\phi_2) &= r_1\epsilon^{j\phi_1} \times r_2\epsilon^{j\phi_2}\\ &= r_1 r_2\epsilon^{j(\phi_1+\phi_2)}\\ &= r_1 r_2 \underline{\angle(\phi_1+\phi_2)}.\end{aligned}$$

Similarly for divisions, or finding ratios :—

$$\begin{aligned}\frac{r_1\angle\phi_1}{r_2\angle\phi_2} = \frac{r_1\epsilon^{j\phi_1}}{r_2\epsilon^{j\phi_2}} &= \frac{r_1}{r_2}\,\epsilon^{j(\phi_1-\phi_2)}\\ &= \underline{\frac{r_1}{r_2}\angle(\phi_1-\phi_2)}.\end{aligned}$$

Divide the sizes and subtract the angles.

Taking powers and extracting roots are simply special cases. For instance,

$$(r\angle\phi)^2 = r^2(\epsilon^{j\phi})^2 = r^2\epsilon^{j2\phi} = r^2\angle 2\phi$$

$$\sqrt{r\angle\phi} = \sqrt{r}\ \epsilon^{j\frac{\phi}{2}} = \sqrt{r}\angle\tfrac{\phi}{2}\,.$$

It is sometimes desirable to express a ratio such as $(a+jb)/(c+jd)$ in the form $A+jB$. This is done by multiplying numerator and denominator by $(c-jd)$. Thus

$$\frac{a+jb}{c+jd} = \frac{(a+jb)(c-jd)}{(c+jd)(c-jd)} = \frac{ac+bd}{c^2+d^2} + j\,\frac{bc-ad}{c^2+d^2}.$$

This process is known as " rationalising the denominator."

By the above rules, the reciprocal of $r\angle\phi$ is $\frac{1}{r}\overline{\diagdown}\phi$. Expressed in the algebraic form, the reciprocal of $(a+jb)$ is $\frac{a}{a^2+b^2} - j\frac{b}{a^2+b^2}$, which has a magnitude $\frac{1}{\sqrt{a^2+b^2}}$.

(vii.) The ratio of a " vector voltage " to a " vector current " is called a " vector impedance," or simply an impedance, and the ratio of a " vector current " to a " vector voltage " is called a " vector admittance," or simply an admittance ; but impedances and admittances differ from the current and voltage lines in the vector diagrams in that they are constant with regard to time for a given circuit, and there is no question of dropping perpendiculars in order to obtain instantaneous values. They are, however, algebraically of the same vector form as the current and voltage lines, and obey the same laws with regard to addition, multiplication, and division. Their essential difference can be emphasised by calling them *complex quantities.* Then if the voltage and current lines are called *engineers' vectors,* or simply *vectors,* and the true vectors of the mathematicians *real vectors,* there should be no confusion.

Engineers' vectors as well as impedances and admittances are represented by complex quantities.

(viii.) It is of interest to examine the exponential expressions for cosine and sine.

It is known that $\cos\phi = \frac{1}{2}\,\epsilon^{j\phi} + \frac{1}{2}\,\epsilon^{-j\phi}.$

In Fig. 10, $\text{OA} = \frac{1}{2}\,\epsilon^{j\phi}$

$$\text{OA}' = \tfrac{1}{2}\,\epsilon^{-j\phi},$$

and OB = the vector sum of OA and OA′

$$= \cos\phi,$$

and this is a scalar, since it has no angle, *i.e.*, it is drawn along the abscissa. That OB has a length equal to cos ϕ is made obvious by continuing OA to P, making AP = $\frac{1}{2}$. PB is evidently perpendicular to OB, and OB is evidently equal to cos ϕ.

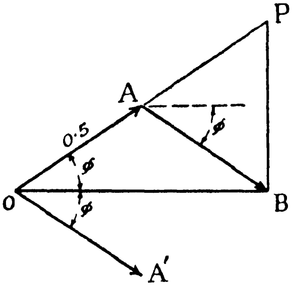


Fig. 10.—Cos ϕ.

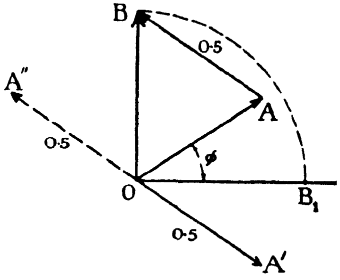


Fig. 11.— Sin ϕ.

Also, $\sin\phi = \dfrac{\epsilon^{j\phi} - \epsilon^{-j\phi}}{2j}.$

Multiplying numerator and denominator by $-j$ gives

$$\sin\phi = -j\left(\frac{\epsilon^{j\phi}}{2} - \frac{\epsilon^{-j\phi}}{2}\right).$$

In Fig. 11, $\text{OA} = \frac{1}{2}\,\epsilon^{j\phi}$, $\text{OA}' = \frac{1}{2}\,\epsilon^{-j\phi}$, and $\text{OA}'' = -\frac{1}{2}\,\epsilon^{-j\phi}$ AB is drawn equal to and parallel with OA″, so that

$$\text{OB} = \left(\frac{\epsilon^{j\phi}}{2} - \frac{\epsilon^{-j\phi}}{2}\right).$$

Finally, OB is multiplied by $-j$, by a negative (clockwise) rotation through 90°, and OB_1 results. This is obviously $\sin \phi$.

(ix.) A meaning can readily be found for the trigomometrical functions of complex quantities, whether circular or hyperbolic.

For instance,

$$\cos (a + jb) = \frac{\epsilon^{j(a+jb)} + \epsilon^{-j(a+jb)}}{2}$$
$$= \tfrac{1}{2}\epsilon^{-b} \epsilon^{ja} + \tfrac{1}{2}\epsilon^{b} \epsilon^{-ja}.$$

The first term is drawn (Fig. 12) as OA with a length $\frac{1}{2}\epsilon^{-b}$ at a

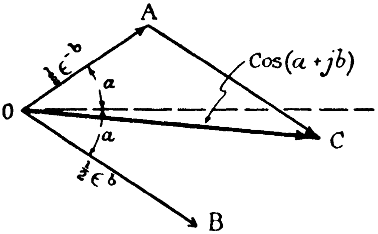


Fig. 12.—Cos $(a + jb)$.

positive angle a ; the second as OB, length $\frac{1}{2}\epsilon^{b}$ and negative angle a. The sum is found to be OC by drawing AC equal and parallel to OB. Thus the complex OC is $\cos (a + jb)$.

Again, $$\sin (a + jb) = \frac{\epsilon^{j(a+jb)} - \epsilon^{-j(a+jb)}}{2j}$$
$$= -j\,(\tfrac{1}{2}\epsilon^{-b}.\,\epsilon^{ja} - \tfrac{1}{2}\epsilon^{b}.\epsilon^{-ja}).$$

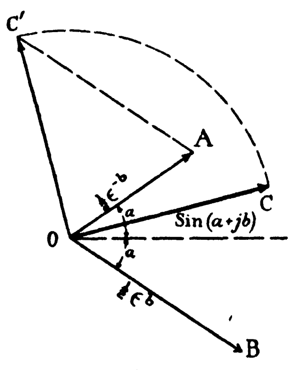


Fig. 13.—Sin $(a + jb)$.

In this case (Fig. 13) OA and OB are drawn as in Fig. 12, but AC′ is drawn equal and parallel to OB, but in the opposite direction to give OA − OB. Finally, OC′ is rotated clockwise through 90° to give OC, which is $\sin (a + jb)$. The remaining functions are formed from the cosine and sine by taking ratios and reciprocals.

Similarly for the hyperbolic functions,

$$\cosh (a + jb) = \tfrac{1}{2}(\epsilon^{a+jb} + \epsilon^{-(a+jb)})$$
$$= \tfrac{1}{2}\epsilon^{a}.\,\epsilon^{jb} + \tfrac{1}{2}\epsilon^{-a}.\,\epsilon^{-jb}.$$

In Fig. 14, OA is drawn with a length $\frac{1}{2}\epsilon^{a}$ at an angle b, and OB with a length $\frac{1}{2}\epsilon^{-a}$ at an angle $-b$. OB is added to OA by drawing AC equal and parallel to OB ; OC is cosh $(a+jb)$.

$$\begin{aligned}\sinh(a+jb) &= \tfrac{1}{2}\left(\epsilon^{a+jb} - \epsilon^{-(a+jb)}\right)\\ &= \tfrac{1}{2}\epsilon^{a}.\epsilon^{jb} - \tfrac{1}{2}\epsilon.^{-a}\,\epsilon^{-jb}.\end{aligned}$$

In this case (Fig. 14), AC′ is drawn in the opposite direction to OB, and OC′ is sinh $(a+jb)$.

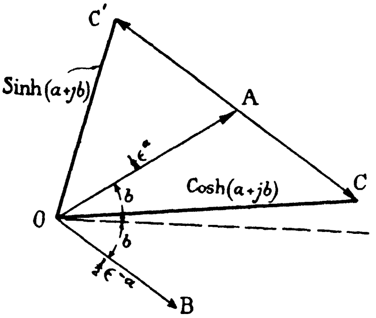


FIG. 14.—Cosh $(a+jb)$ and Sinh $(a+jb)$.

The relationship between the circular and hyperbolic functions is as follows,

$$\begin{aligned}\cos(a+jb) &= \cosh j(a+jb)\\ j\sin(a+jb) &= \sinh j(a+jb).\end{aligned}$$

for

$$\begin{aligned}\cos(a+jb) &= \tfrac{1}{2}\epsilon^{-b}\epsilon^{ja} + \tfrac{1}{2}\epsilon^{b}\epsilon^{-ja}\\ &= \tfrac{1}{2}\epsilon^{ja-b} + \tfrac{1}{2}\epsilon^{-(ja-b)}\\ &= \tfrac{1}{2}\epsilon^{j(a+jb)} + \tfrac{1}{2}\epsilon^{-j(a+jb)}\\ &= \cosh j(a+jb).\end{aligned}$$

and similarly for the other expression.

These relationships can also be deduced from the geometric constructions.

(x.) Rotating vectors can readily be differentiated and integrated. Let the vector R have length r, and be rotating with an angular velocity ω radians per second. Then

$$\mathrm{R} = r\epsilon^{j\omega t},$$

and by the usual rules

$$\frac{d\mathrm{R}}{dt} = j\omega r\epsilon^{j\omega t} = j\omega\mathrm{R}.$$

This result is obtained geometrically from Fig. 15, where OP is R, drawn with length r and angle ωt. In a small increment of time δt, P moves to P′, and the angle POP′ is $\delta(\omega t)$. The increment (vector) of OP is PP′ ($= \delta R$), having a length $r\delta(\omega t)$ and an angle $\omega t + \frac{\pi}{2}$. Thus, δR is written

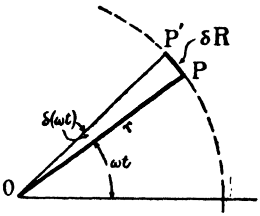


FIG. 15.—Vector differentiation.

$$\delta R = r\delta(\omega t) \,.\, \epsilon^{j\left(\omega t + \frac{\pi}{2}\right)}$$
$$= r\omega.\delta t \,.\, j\epsilon^{j\omega t}$$

since $\epsilon^{j\frac{\pi}{2}} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = j$

whence $\frac{\delta R}{\delta t} = j\omega r\epsilon^{j\omega t} = j\omega R$

and in the limit $\frac{dR}{dt} = jwR$

It follows that $\int R dt = \frac{R}{j\omega}$

$$\frac{d^2R}{dt^2} = -\,\omega^2 R$$

and so on.

(xi.) If one engineers' vector $V = a + jb$ represents the voltage across the terminals of some apparatus, and another $I = c + jd$ represents the current into the apparatus, then it is well known that the power into the circuit is $|V|.|I| \cos\theta$, where θ is the angle between the two vectors. This may be identified with the mathematical dot or scalar product. It may sometimes be conveniently read off a diagram as the quantity $(ac + db)$, due regard being paid to the sign of the quantity. If $\phi_1\ \phi_2$ are the angles of the two vectors, then the power

$$= \sqrt{a^2 + b^2}.\sqrt{c^2 + d^2}.\cos(\phi_1 - \phi_2)$$
$$= \sqrt{(a^2 + b^2)(c^2 + d^2)}.(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2)$$
$$= \sqrt{(a^2 + b^2)(c^2 + d^2)}.\left\{\frac{a}{\sqrt{a^2 + b^2}}\cdot\frac{c}{\sqrt{c^2 + d^2}} + \frac{b}{\sqrt{a^2 + b^2}}\cdot\frac{d}{\sqrt{c^2 + d^2}}\right.$$
$$= ac + bd.$$

(3) **Vector Loci**

The path traced by the extremity P of the complex quantity OP as the quantity is varied in any specified manner, is known as the locus of the quantity, and such locii are of great importance in engineering vector diagrams. They can indicate in a graphic manner current or potential changes resulting from changes of frequency, load or other circuit conditions.

(i.) If the imaginary part b of the complex $(a + jb)$ varies from $-\infty$ to $+\infty$, the locus of $(a + jb)$ is the vertical straight line drawn through R distant a horizontally from the pole O (Fig. 16). Then any line OP drawn from O to meet the line in P represents $(a + jb)$ for the particular value of $b = \text{RP}$. As b increases from $-\infty$ to $+\infty$ the point P travels up the vertical line, as is indicated by the arrow head, from p_1 to p_2.

The locus of $1/(a + jb)$ can be found by drawing OP′ at an angle $\text{P'OR} = -\widehat{\text{POR}} = -\phi$, and of length $1/\sqrt{a^2 + b^2}$, and by repeating this process for successive positions of the point P. R will give D also in the axis, and $(\text{OD}) = 1/a$.

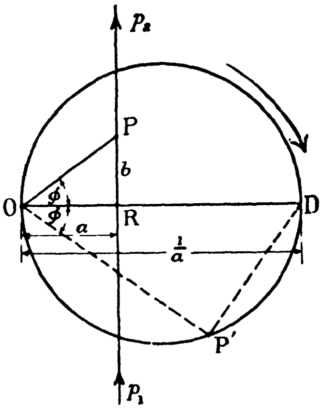


FIG. 16.—Inversion of a vertical straight line.

Now $\dfrac{\text{OP}'}{\text{OD}} = \dfrac{a}{\sqrt{a^2 + b^2}}$ by construction, and since $\tan\phi = \dfrac{b}{a}$,

$$\cos\phi = \frac{a}{\sqrt{a^2 + b^2}}.$$

$$\therefore \frac{\text{OP}'}{\text{OD}} = \cos\phi.$$

Hence the angle OP′D must be a right angle, and the locus of P′ is a circle with diameter OD. A little consideration will show that the circle is described in the clockwise direction from O as b increases from $-\infty$ to $+\infty$, that is as P moves from p_1 to p_2.

This process of obtaining the circle from the straight line is known as "inverting" the line. If the circle be "inverted" with a pole on the circumference, a straight line results. If instead of $1/(a + jb)$, the locus of $k/(a + jb)$ had been found, the process would still have been called "inversion," and k would be called the "constant of inversion."

(ii.) If the real part a of $(a + jb)$ varies from $-\infty$ to $+\infty$, the locus of $(a + jb)$ is $p_1 \mathrm{R} p_2$ with pole O as drawn in Fig. 17, where $\mathrm{OR} = jb$ and $\mathrm{RP} = a$ for a particular value.

To find a point on the locus of $1/(a + jb)$, draw OP′ at an angle ϕ below the axis OX equal to $\widehat{\mathrm{POX}}$ and of length $1/\sqrt{a^2 + b^2}$.

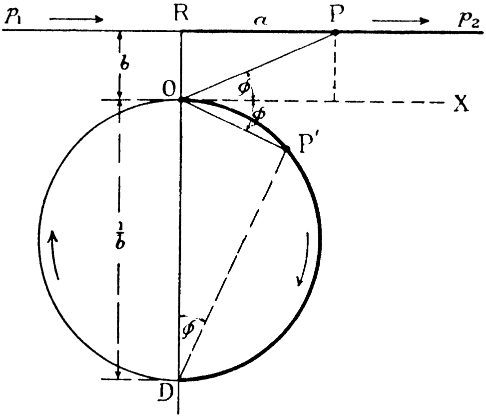


FIG. 17.—Inversion of a horizontal straight line.

By the same construction the length of OD drawn vertically downwards is $1/b$, and

$$\frac{\mathrm{OP}'}{\mathrm{OD}} = \frac{b}{\sqrt{a^2 + b^2}},$$

but since $\tan \phi = \dfrac{b}{a}$, $\sin \phi = \dfrac{b}{\sqrt{a^2 + b^2}}$ and $\dfrac{\mathrm{OP}'}{\mathrm{OD}} = \sin \phi$.

And hence, since the angle ODP′ is also ϕ, the angle OP′D is a right angle, and the locus of P′ a circle with diameter OD. The circle is described clockwise from O. If a is limited to positive values, the locus of $(a + jb)$ will be $\mathrm{R}p_2$, and that of $1/(a + jb)$ the semi-circle OP′D, drawn in heavy lines.

Fig. 18 illustrates a more general case of inversion from a straight line $p_1 R p_2$ (pole O). OR is drawn perpendicular to $p_1 p_2$, making an angle θ with the abscissa. OR is produced to D′, so that $\mathrm{OD}' = k/\mathrm{OR}$, and a circle (dotted) is drawn on OD′.

Then it may readily be proved as before that if any ray OPP′ is drawn to cut the line in P and the circle in P′, $OP' = k/OP$. If the direction of describing the line is from p' to p_2, that of describing the circle is counter-clockwise. Finally the "image" of the dotted circle in the axis Ox is formed by drawing OD at an angle θ below the axis, and drawing the circle ODP_1 with OD as diameter. This circle is the inverted straight line, and is described in the clockwise direction. If $P'\hat{O}X = P_1\hat{O}X$, then P_1, P′ and P are corresponding points.

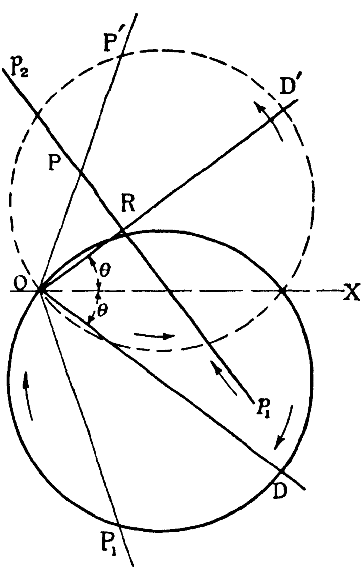


FIG. 18.—Inversion of any straight line.

A circle with a pole not on its circumference inverts to another circle. Let APB (Fig. 19) be the circle, and O the pole. Draw OAB through the centre C of the circle, and continue to B′ and A′, making $OB' = k/OB$ and $OA' = k/OA$. Through O draw any other line to meet the circle in P, and continue to P′, making $OP' = k/OP$. Then P′ lies on the circle drawn on A′B′ as diameter. Join PA, PB, P′A′ and P′B′. Now, since by construction $OB \cdot OB' = k$ and $OP \cdot OP' = k$, the points $PP'B'$ and B are concyclic, and it follows that

$$O\hat{P}B = P'\hat{B'}B.$$

Similarly, since $OA \cdot OA' = k = OP \cdot OP'$, the points $APP'A'$ are concylic, and

$$O\hat{P}A = P'\hat{A'}B.$$

Hence $O\hat{P}B - O\hat{P}A = P'\hat{B'}B - P'\hat{A'}B'$.

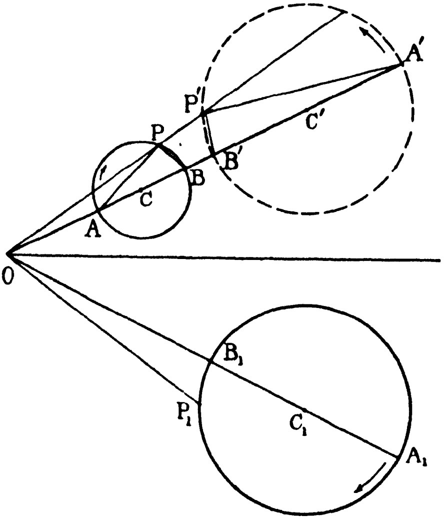


FIG. 19.—Inversion of any circle.

But $\widehat{OPB} - \widehat{OPA} = \widehat{APB}$ is a right angle, and, since $\widehat{P'B'B} = \widehat{B'P'A'} + \widehat{P'A'B'}$, $\widehat{B'P'A'}$ is also a right angle, and P′ lies on the circle drawn on A′B′ as diameter. Finally the image of A′P′B′ is found as $A_1P_1B_1$ by drawing OB_1A_1 at an angle below the axis equal to the angle of OB′A′ above the axis.

The expression for the first circle is of the form

$$c + jd + \frac{k_1}{a + jb},$$

where OA is the complex $c + jd$, and that of the inverted circle is

$$\frac{k_2}{c + jd + \dfrac{k_1}{a + jb}}.$$

(iii.) If α and β are constants and b varies, the locus of $(a + j\alpha b)(c + j\beta b)$ is a parabola.

Multiplying out gives the complex $ac - \alpha\beta b^2 + j\,(\alpha c + \beta a)\,b$, which, expressed in cartesian co-ordinates, gives

$$x = ac - \alpha\beta b^2$$
$$y = (\alpha c + \beta a)b,$$

whence

$$y^2 = \frac{(\alpha c + \beta a)^2}{\alpha\beta}(ac - x).$$

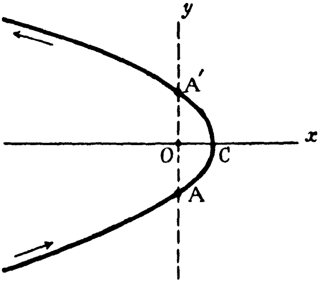


FIG. 20.

This parabola is drawn in Fig. 20; OC $= ac$, and OA $=$ OA′ $= (\alpha c + \beta a)\sqrt{\dfrac{ac}{\alpha\beta}}$.

If $a = c$ and $\alpha = \beta = 1$, OC $= a^2$ and OA $=$ OA′ $= 2a^2$. O is now at the focus of the parabola, which is the locus of $(a + jb)^2$.

If the origin is moved to the vertex C, the parabola is expressed by the relation $y^2 = p(-x)$, where $p = \dfrac{(\alpha c + \beta a)^2}{\alpha\beta}$.

As b increases from $-\infty$ to $+\infty$, the parabola is described in the counter-clockwise direction, as indicated by the arrows.

(iv.) Similarly, the locus of $(a + j\alpha b)/(c + j\beta b)$ is a circle. For

$$\frac{a + j\alpha b}{c + j\beta b} = \frac{(a + j\alpha b)(c - j\beta b)}{c^2 + \beta^2 b^2} = \frac{ac + \alpha\beta b^2 + j(\alpha c - \beta a)b}{c^2 + \beta^2 b^2}$$

and
$$x = \frac{ac + \alpha\beta b^2}{c^2 + \beta^2 b^2},$$

$$y = \frac{\alpha c - \beta a}{c^2 + \beta^2 b^2} \,.\, b.$$

If
$$b = o,\ x = \frac{a}{c} \text{ and } y = o.$$

If
$$b = \infty\ ,\ x = \frac{\alpha}{\beta} \text{and } y = o,$$

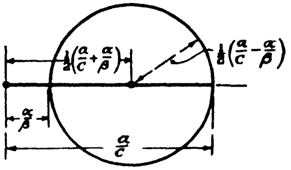


FIG. 21.

move the origin along the abscissa a distance $\frac{1}{2}\left(\frac{a}{c} + \frac{\alpha}{\beta}\right)$, *i.e.*,

write
$$x' = x - \frac{1}{2}\left(\frac{a}{c} + \frac{\alpha}{\beta}\right)$$
$$= \frac{ac + \alpha\beta b^2}{c^2 + \beta^2 b^2} - \frac{1}{2}\left(\frac{a}{c} + \frac{\alpha}{\beta}\right)$$
$$= \frac{1}{2}\left(\frac{a}{c} - \frac{\alpha}{\beta}\right)\frac{c^2 - \beta^2 b^2}{c^2 + \beta^2 b^2}.$$

Then
$$x'^2 + y^2 = \frac{1}{4}\left(\frac{a}{c} - \frac{\alpha}{\beta}\right)^2 \frac{(c^2 - \beta^2 b^2)^2}{(c^2 + \beta^2 b^2)^2}$$
$$+ \frac{(\alpha c - \beta a)^2}{(c^2 + \beta^2 b^2)^2} \,.\, b^2$$
$$= \frac{1}{4}\left(\frac{a}{c} - \frac{\alpha}{\beta}\right)^2 \,.\, \frac{(c^2 - \beta^2 b^2)^2 + 4c^2\beta^2 b^2}{(c^2 + \beta^2 b^2)^2}$$
$$= \frac{1}{4}\left(\frac{a}{c} - \frac{\alpha}{\beta}\right)^2.$$

Hence the locus is a circle of radius $\frac{1}{2}\left(\frac{a}{c} - \frac{\alpha}{\beta}\right)$, and centre on the axis distance $\frac{1}{2}\left(\frac{a}{c} + \frac{\alpha}{\beta}\right)$ from the pole, as indicated in Fig. 21.

(v.) The locus of cosh $(a + jb)$ when b is varied, is an ellipse. For cosh $(a + jb) = \cosh a \cos b + j \sinh a \sin b$.

Whence

$$x = \cosh a \cos b$$
$$y = \sinh a \sin b,$$

and

$$\frac{x^2}{\cosh^2 a} + \frac{y^2}{\sinh^2 a} = 1,$$

the equation of an ellipse with horizontal major axis 2 cosh a and vertical minor axis 2 sinh a (A′A and BB′ in Fig. 22).

If a is varied, elimination gives

$$\frac{x^2}{\cos^2 b} - \frac{y^2}{\sin^2 b} = 1.$$

This is the equation of a hyperbola (Fig. 22) in which OC = OC′ = cos b, and the angle θ of the asymptotes is tan b.

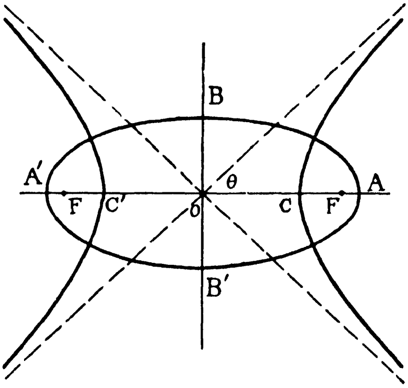


FIG. 22.—Loci of Cosh ($a + jb$) and Sinh ($a + jb$).

Similarly, sinh ($a + jb$) gives sinh ($a + jb$) = sinh a cos b + j cosh a sin b.

$$x = \sinh a \cos b$$
$$y = \cosh a \sin b.$$

With b varied

$$\frac{x^2}{\sinh^2 a} + \frac{y^2}{\cosh^2 a} = 1,$$

and with a varied

$$-\frac{x^2}{\cos^2 b} + \frac{y^2}{\sin^2 b} = 1.$$

The equations are again those of an ellipse and hyperbola, but, compared with those for the cosh, with x and y interchanged. If Fig. 22 be rotated counter-clockwise through 90°, the loci will be those of the sinh.

EXERCISES

(1) Draw the vectors (i.) $3 + j4$, (ii.) $-2 + j6$, (iii.) $-4 - j1$, (iv.) $2 - j2$, (v.) $4\epsilon^{j\frac{\pi}{5}}$, (vi.) $-3\epsilon^{j0\cdot 2}$, (vii.) $2\epsilon^{j14\cdot 0}$.

(2) Express the vectors of question (1), (i.) to (iv.) in the form $r\angle\phi$, (v.) to (vii.) in the form $a + jb$.

(3.) Add vectors (i.) and (iv.) and (ii.) and (v.). Subtract vector (iii.) from (iv.).

(4) Multiply vector (i.) by (ii.), and (i.) by (iv.). Divide vectors (ii.) by (iii.), and (v.) by (vii.).

(5) Find the reciprocals of (iii.) and (iv.), in each case algebraically and graphically.

(6) Find the square root of (ii.) and (iii.) and the square of (iv.).

(7) Simplify the following expressions :—

$$\text{(i.) } \frac{4+j6}{3-j4}, \quad \text{(ii.) } \frac{2-j4}{6+j7} \times \frac{3+j8}{j2}, \quad \text{(iii.) } \frac{(4+j2)^2}{2-j} \times \frac{3}{1+j10}.$$

(8) Find the values of the sin, cos, sinh and cosh of (i.) $(0{\cdot}7+j0{\cdot}6)$, (ii.) $(0{\cdot}7-j0{\cdot}6)$, (iii.) $(-0{\cdot}7+\mathrm{j}0{\cdot}6)$, (iv.) $(-0{\cdot}7-j0{\cdot}6)$.

(9) Draw the locus of $a+jb$ and of $1/(a+jb)$:—

(i.) When a is constant and equal to $0{\cdot}5$, and b varies from -5 to $+5$. Mark on each locus the values of $b=-5, -4, -3, -2, -1, 0, +1$, etc.

(ii.) When a varies from 0 to 5 and b remains constant and equal to 2. Mark on each locus the values of $a = 0, 1, 2, 3, 4$ and 5.

(10) Draw the locus of $\left(\dfrac{1}{2+j3+\dfrac{12}{4+jx}}\right)$, and find from the diagram the values of the expression when $x=-2$, 4 and 5.

(11) Draw the locus of $\cosh(a+jb)$:—

(i.) With a constant at 0, 0·25, 0·5, 0·75 and 1, and b varying.

(ii.) With b constant at 0, $\overline{0{\cdot}25}$, $\overline{0{\cdot}5}$, $\overline{0{\cdot}75}$ and $\overline{1{\cdot}0}$, and a varying.

From the drawing read off the values of :—

$$\cosh(0{\cdot}25+j2{\cdot}0),\ \cosh(-0{\cdot}75+j0{\cdot}5),$$
$$\sinh(0{\cdot}5+j4{\cdot}2),\ \sinh(-0{\cdot}25-j1{\cdot}5)$$

CHAPTER II

VECTOR DIAGRAMS OF SOME SIMPLE CIRCUITS

In applying the principles of the previous chapter to the circuit problems of the electrical engineer, it is desirable to have a clear understanding of the various terms and symbols to be adopted.

An alternating voltage of pure sine wave form is represented by the expressions $v = v_m \sin(\omega t + \phi)$ or $v = v_m \cos(\omega t + \phi)$, where v_m is the maximum instantaneous value of the voltage v. Now it has been shown previously that any quantity which varies sinusoidally may be represented vectorially by the expression $\mathrm{R}\epsilon^{j\omega t}$, where ω is the same as in the previous notation. Then if $|\mathrm{V}|$ is the magnitude of the vector, $|\mathrm{V}|$ and v_m will be synonymous; R will be replaced by V where $\mathrm{V} \equiv |\mathrm{V}|\epsilon^{j\phi}$, and $|\mathrm{V}|\epsilon^{j(\omega t+\phi)} \equiv \mathrm{V}\epsilon^{j\omega t}$ will be completely representative of the alternating voltage v.

The symbols to be used may be summarised as follows :—

$|\mathrm{V}|$ is a voltage magnitude synonymous with the v_m adopted in the algebraic treatment of alternating voltages, and is equal to the maximum instantaneous value of the voltage.

V is a voltage vector ; it represents the magnitude and position or phase of the " vector voltage " at a certain instant of time, and is given algebraically by $|\mathrm{V}|\epsilon^{j\phi}$ where

ϕ is the angle which V makes with the reference line when $t = 0$. This instant ($t = 0$) is chosen quite arbitrarily, but usually for convenience it is chosen so that the angle ϕ is zero for either the voltage or the current.

$\mathrm{V}\epsilon^{j\omega t}$ is a rotating voltage vector, the projection of which on the reference line or axis gives at all instants the actual value of the voltage represented.

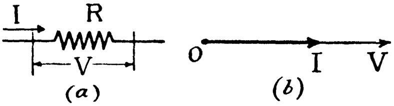
If in the expression $|\mathrm{V}|\epsilon^{j(\omega t+\phi)}$ ϕ is zero, it should be remembered that the vector representation is $\mathrm{V}\epsilon^{j\omega t}$ and *not* $|\mathrm{V}|\epsilon^{j\omega t}$. When the reference line is the x axis, instantaneous values are given algebraically by the *real* part of the rotating vector at the instant considered.

Thus $|\mathrm{V}|\epsilon^{j(\omega t+\phi)} = v_m \{\cos(\omega t+\phi) + j\sin(\omega t+\phi)\}$ the real part of which is $v_m \cos(\omega t+\phi)$, the instantaneous value.

Everything that has been said with regard to voltages applies also to currents, and in fact to any other sinusoidally varying quantity, such as flux and flux density ; and displacement, velocity and acceleration in simple harmonic motion.

One further note. Electrical engineers almost invariably use root-mean-square values of voltages and currents instead of maximum values, and draw their voltage and current vectors of lengths equal to the R.M.S. values, *i.e.*, $1/\sqrt{2}$ times the maximum values. If this is done it is obviously necessary to multiply the projections by $\sqrt{2}$ to obtain instantaneous values.

(i.) If an alternating P.D. $v_m \sin(\omega t+\phi)$ is established across a non-inductive resistance R (Fig. 23 (*a*)), the current which passes is found from the equality $i_m\mathrm{R}\sin(\omega t+\phi) = v_m \sin(\omega t+\phi)$. Similarly, if the instantaneous values of current and voltage are $v_m \cos(\omega t+\phi)$ and $i_m \cos(\omega t+\phi)$, $i_m\mathrm{R}\cos(\omega t+\phi) = v_m\cos(\omega t+\phi)$.



(*a*) (*b*)

FIG. 23.—Pure resistance.

It is possible, therefore, by multiplying the first equation by j and adding to the second, to write

$$v_m\{\cos(\omega t+\phi)+j\sin(\omega t+\phi)\} = i_m\mathrm{R}\{\cos(\omega t+\phi)+j\sin(\omega t+\phi)\}$$

i.e.,

$$|\mathrm{V}|\epsilon^{j(\omega t+\phi)} = |\mathrm{I}|\mathrm{R}\epsilon^{j(\omega t+\phi)}$$

$$|\mathrm{V}|\epsilon^{j\phi}\epsilon^{j\omega t} = |\mathrm{I}|\mathrm{R}\epsilon^{j\phi}.\epsilon^{j\omega t}$$

giving the rotating vector equation

$$\mathrm{V}\epsilon^{j\omega t} = \mathrm{IR}\epsilon^{j\omega t}$$

and the stationary vector equation

$$\mathrm{V} = \mathrm{IR}.$$

R is a pure number ; it is not complex, since it does not contain j and there is no angle between V and I ; *i.e.*, they are in phase.

The above merely indicates how the vector representation can be arrived at algebraically from the expression for the instantaneous values.

The vector diagram is drawn in Fig. 23 (*b*), when the reference line is taken along the x axis, and when $t = 0$ and $\phi = 0$, the length of the vector V being R times the length of the vector I.

(ii.) Suppose now that the voltage V is established across a non-inductive resistance R and a pure inductance L in series (Fig. 24 (*a*)). For instantaneous values the differential equation in this case is

$$v = \mathrm{R}i + \mathrm{L}\frac{di}{dt}.$$

Write as a solution $i = i_m \cos \omega t$, giving

$$v_1 = \mathrm{R}i_m \cos \omega t - \omega \mathrm{L} i_m \sin \omega t.$$

Now write as a solution $i = i_m \sin \omega t$, giving

$$v_2 = \mathrm{R}i_m \sin \omega t + \omega \mathrm{L} i_m \cos \omega t.$$

Multiplying the second solution throughout by j, and adding to the first, gives

$$\begin{aligned} v_1 + jv_2 &= \mathrm{R}i_m(\cos \omega t + j \sin \omega t) + j\omega \mathrm{L} i_m(\cos \omega t + j \sin \omega t) \\ &= (\mathrm{R} + j\omega \mathrm{L})\, i_m \epsilon^{j\omega t}, \end{aligned}$$

which can be written as the vector relationship

$$\mathrm{V} = (\mathrm{R} + j\omega \mathrm{L})\mathrm{I}.$$

If the differential equation had been written as the vector differential

$$\mathrm{V} = \mathrm{RI} + \mathrm{L}\frac{d\mathrm{I}}{dt}$$

this vector solution would have followed at once, since $\frac{d\mathrm{I}}{dt} = j\omega \mathrm{I}$. In fact there is, in general, no need to write the differential equation at all; the solution can be written down from an inspection of the circuit; but the above may serve as an indication of what is involved in so doing.

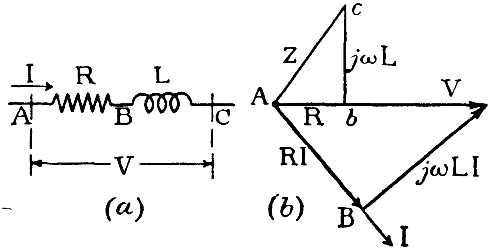


FIG. 24.—Resistance and inductance.

The vector diagram is drawn in Fig. 24 (*b*). A*b* is drawn horizontally, length equal to R, and *bc* vertically, length equal to ωL. AC is Z, the complex impedance between A and C. If V is drawn horizontally as AC, I is drawn at an angle below V equal to $c\hat{A}b$, and the length of I is $|\mathrm{V}|/\sqrt{\mathrm{R}^2 + \omega^2\mathrm{L}^2}$. AI is, in fact, Z inverted with a factor $|\mathrm{V}|$. Drop a perpendicular from C to AI meeting it in B. Then AB is RI and BC is $j\omega$LI; *i.e.*, AB and BC in Fig. 24 (*b*) are the voltage vectors of the voltages between the corresponding points in Fig. 24 (*a*), together adding to V.

(iii.) Next take the case of a resistance and condenser in series (Fig. 25 (*a*)). The vector differential equation may be written

$$\mathrm{V} = \mathrm{RI} + \frac{1}{\mathrm{C}}\int \mathrm{I}\,dt,$$

which, since $\int \mathrm{I}\,dt = \frac{\mathrm{I}}{j\omega}$, gives

$$\mathrm{V} = \mathrm{RI} + \frac{1}{\mathrm{C}}\cdot\frac{\mathrm{I}}{j\omega}$$

$$= \left(\mathrm{R} - \frac{j}{\omega\mathrm{C}}\right)\mathrm{I}.$$

Drawing A*b* horizontal = R (Fig. 25 (*b*)), and *bc* vertically downwards = $-j/\omega\mathrm{C}$, A*c* is the impedance of the resistance and capacity in series. Drawing V horizontally (= AC), I is drawn above the horizontal at an angle equal to $\widehat{c\mathrm{A}b}$, and of length $|\mathrm{V}|/\sqrt{\mathrm{R}^2 + \frac{1}{\omega^2\mathrm{C}^2}}$. Dropping from C a perpendicular to meet AI in B, AB = RI, and $\mathrm{BC} = -\frac{j}{\omega\mathrm{C}}\,\mathrm{I}$ are the voltages between the corresponding points in Fig. 25 (*a*), together adding to V.

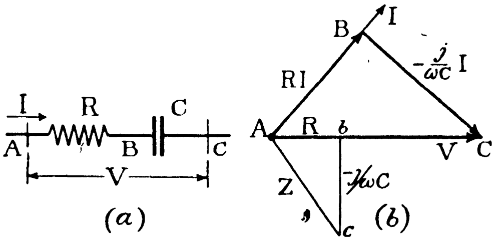


FIG. 25.—Resistance and capacity.

(iv.) The case of series resistance, inductance and capacity may be regarded as cases (ii.) and (iii.) combined, and the following vector relationship can be written down :—

$$\mathrm{V} = \left(\mathrm{R} + j\omega\mathrm{L} - \frac{j}{\omega\mathrm{C}}\right)\mathrm{I}.$$

In this expression the term $\left(\mathrm{R} + j\omega\mathrm{L} - \frac{1}{\omega\mathrm{C}}\right)$ represents the impedance of the circuit, R is the resistance, $\left(\omega\mathrm{L} - \frac{1}{\omega\mathrm{C}}\right)$ is known as the reactance, and the total impedance

$$\left\{ R + j\left(\omega L - \frac{1}{\omega C}\right) \right\}$$

is usually denoted by Z.

A little consideration of the identities already given will show that the same rules apply to the complex impedance Z, when in parallel or series, as are used for pure resistance in the same circumstances.

For if a voltage V be established across two impedances

$$Z_1 = R_1 + j\omega L_1 - \frac{j}{\omega C_1} \text{ and } Z_2 = R_2 + j\omega L_2 - \frac{j}{\omega C_2}$$

in series, then the total resistance of the circuit is $R_1 + R_2$, the total inductance is $L_1 + L_2$ and the total capacity is given by $\frac{1}{C_1} + \frac{1}{C_2}$, whence the total impedance is

$$\left\{ (R_1 + R_2) + j\omega(L_1 + L_2) - \frac{j}{\omega}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) \right\},$$

which is obviously $Z_1 + Z_2$, and the current I is found from the equation $V = (Z_1 + Z_2)I$.

Similarly, if the voltage V is established across two impedances Z_1 and Z_2 in parallel, if I_1 and I_2 are the currents in each, and I the total current, then $V = ZI$ where Z is the equivalent impedance of the circuit, and $I_1 = \frac{V}{Z_1}$ and $I_2 = \frac{V}{Z_2}$, but since $I = I_1 + I_2$, $\frac{V}{Z} = \frac{V}{Z_1} + \frac{V}{Z_2}$ or $\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$ which corresponds with the similar expression for pure resistances.

(v.) In the case of series resistance inductance and capacity (Fig. 26 (*a*)), the vector relationship is

$$V = \left(R + j\omega L - \frac{j}{\omega C}\right) I$$

In Fig. 26 (*b*), A*b* = R, *bc* = $j\omega L$, *cd* = $-\frac{j}{\omega C}$, and A*d* = Z the series impedance between A and D (Fig. 26 (*a*)).

AD = V drawn horizontally, AI = I, and is the inverse of A*d* with factor $|V|$. DB is drawn perpendicular to AI, and continued in the direction BD to C, making $BC = j\omega LI$, when $CD = -\frac{j}{\omega C}I$, and $AB = RI$.

If $| bc | = | cd |$, *i.e.*, if $\omega L = \frac{1}{\omega C}$ or $\omega = 1/\sqrt{LC}$, d falls on b,

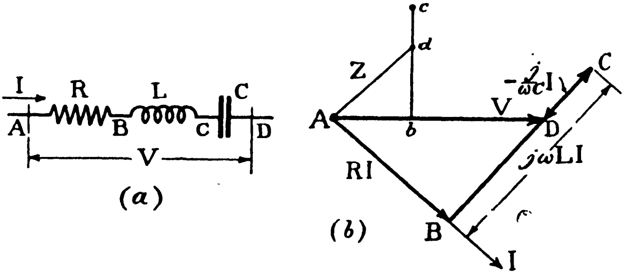


FIG. 26.—Resistance, inductance and capacity.

and I as well as V is horizontal, and $V = RI$. This is the case of resonance.

If $| bc | < | cd |$, *i.e.*, if $\omega L < \frac{1}{\omega C}$, I leads the voltage.

Suppose that the circuit remains the same, but that ω varies from a very small to a very large value. $j\left(\omega L - \frac{1}{\omega C}\right)$ then varies from $-\infty$ to $+\infty$ and the point d travels up the vertical line

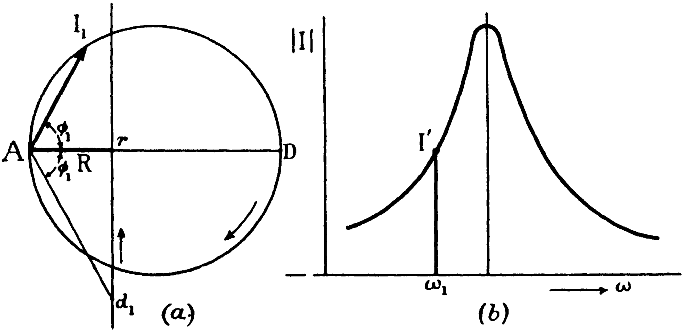


FIG. 27.—Simple resonance curve.

distant R horizontally from A, as is indicated in Fig. 27. Inverting this line with a factor $| V |$ gives the circle AI_1D, with diameter $= | V |/R$ which is the locus of the current vector. Thus, if the magnitude of the current is plotted against ω (as determined by the position of d) the resonance curve of Fig. 27 (b) results.

In the figure, $| rd_1 | = \omega_1 L - \frac{1}{\omega_1 C}$, $\widehat{I_1 A r} = \widehat{r A d_1} = \phi_1$ and $\omega_1 I' = | AI_1 |$

$$\tan\phi = \frac{d_1 r}{rA} = \frac{\omega L - \frac{1}{\omega C}}{R}.$$

If the values of L and C are such that a small change of ω produces an appreciable change of $\omega L - \frac{1}{\omega C}$, and R is small, then $\tan\phi$ will change rapidly with ω, the circle will be described with a small change of ω, and a sharp resonance curve will be obtained.

$$\frac{d\tan\phi}{d\omega} = \frac{L}{R} + \frac{1}{\omega^2 RC} = \frac{L}{R}\left(1 + \frac{1}{\omega^2 LC}\right).$$

Writing $\frac{1}{LC} = \omega_o{}^2$, where ω_o is the resonance value of ω,

$$\frac{d\tan\phi}{d\omega} = \frac{L}{R}\left(1 + \frac{\omega_o{}^2}{\omega^2}\right)$$

and at the resonance value

$$\frac{d\tan\phi}{d\omega} = \frac{2L}{R}$$

or

$$\frac{d\omega}{d\tan\phi} = \frac{R}{2L}.$$

$\frac{R}{2L}$ is known as the decay factor of the circuit; the smaller the decay factor the sharper the resonance curve. It is an important quantity in wireless circuits, and can be obtained from an experimentally drawn resonance curve, such as Fig. 27 (b), by a construction depending upon the derivation of the curve from a circle, and from the above relationship. In Fig. 28 the circle OD is drawn with diameter equal to the maximum value of the current ω_oP. Any point p on the resonance curve is projected to p' on the diameter OD, and this is rotated to p'' on the circle with O as centre. Op'' is evidently the vector current corresponding to p at an angular velocity ω, and the angle DOp'' is ϕ. Draw a horizontal line through s distant unity from O to meet Op'' in t. Then st is $\tan\phi$.

Mark off $\omega t'$ vertically $= st$. Then the curve through the points t' is the curve of $\tan\phi$ plotted against ω, and the slope at ω_0 is $\dfrac{d\omega}{d\tan\phi}$, *i.e.*, $\tan\psi$ is the decay factor $R/2L$.

Since in most circuits of interest the resonance is sharp, a large part of the circle is described with only a small change in ω, ω_0^2/ω^2, remains very nearly one, $\dfrac{d\tan\phi}{d\omega}$ is nearly constant $= 2L/R$, and the curve joining the points t' is nearly a straight line.

(vi.) Fig. 29 shows two coils coupled by a mutual inductance M. The coils are imagined to be wound on the same core in the same sense, and the positive directions of the currents I_1 I_2 are taken as shown in the same direction through the two coils. R_1 L_1 are the resistance and inductance of the first coil, and R_2 L_2 are the total resistance and inductance of the second circuit, which is closed. An alternating p.d. V is maintained across the first coil. The following vector relationship may be written down :—

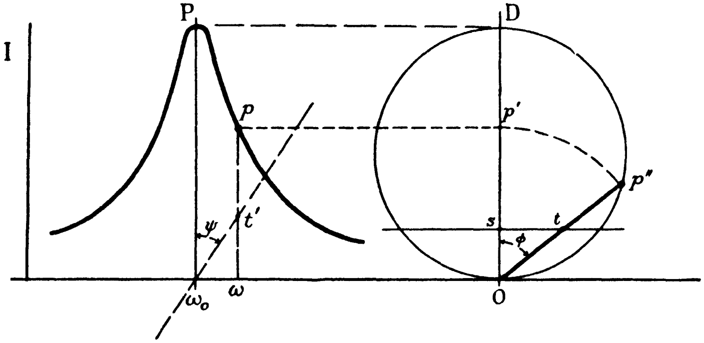


FIG. 28.—Determination of decay factor.

$$V = (R_1 + j\omega L_1)I_1 + j\omega M I_2 = Z_1 I_1 + j\omega M I_2$$
$$0 = (R_2 + j\omega L_2)I_2 + j\omega M I_1 = Z_2 I_2 + j\omega M I_1,$$

where Z_1 and Z_2 are written for the series impedances $R_1 + j\omega L_1$ and $R_2 + j\omega L_2$ respectively.

In Fig. 29 (*b*) OI_2 is drawn horizontally to represent I_2, $OA = R_2 I_2$, $AB = j\omega L_2 I_2$, and $OB = (R_2 + j\omega L_2)I_2 = Z_2 I_2$. OC is drawn equal and opposite to OB to represent $j\omega M I_1$ in agreement with the second equation. Hence, OI_1 to represent I_1 must be drawn at right-angles to COB as shown. OD along OI_1 is $R_1 I_1$, DF =

$j\omega L_1I_1$ and $OF = (R_1 + j\omega L_1)I_1 = Z_1I_1$. Finally, $FG = j\omega MI_2$, and $OG = V$ the applied voltage by the first equation, and the voltage required to give any desired secondary current is found.

Algebraically, substitution from the second equation into the first gives

$$I_1 = \frac{V}{Z_1 + \dfrac{\omega^2M^2}{Z_2}} = \frac{V}{Z'}$$

$$I_2 = -\frac{j\omega MV}{Z_1Z_2 + \omega^2M^2} = \frac{V}{Z''},$$

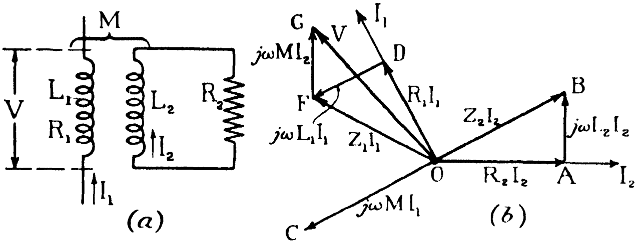


Fig. 29.—Simple coupled coils.

where Z' and Z'' are effective impedances determining the currents I_1 and I_2 from the applied voltage.

$$\begin{aligned} Z' &= Z_1 + \frac{\omega^2M^2}{Z_2} \\ &= R_1 + j\omega L_1 + \frac{\omega^2M^2}{R_2 + j\omega L_2} \\ &= R_1 + j\omega L_1 + \frac{\omega^2M^2}{R_2^2 + \omega^2L_2^2}(R_2 - j\omega L_2) \\ &= R_1 + \frac{\omega^2M^2}{R_2^2 + \omega^2L_2^2}R_2 + j\omega\left\{L_1 - \frac{\omega^2M^2}{R_2^2 + \omega^2L_2^2}L_2\right\}. \end{aligned}$$

This expression shows how the effective resistance of the first coil is increased, and the effective inductance is decreased by the presence of the second coil. With the arrangement shown the effective inductance can never become negative. Even if $R_2 = 0$, the effective inductance is $\left(L_1 - \dfrac{M^2}{L_2}\right)$. The greatest possible value of M (when there is no leakage) is $\sqrt{L_1L_2}$, which would make the effective inductance zero. Thus, I_1 always lags behind V, though the lag is smaller the smaller the value of R_2 and the better the coupling.

If R_1 and R_2 were both zero, it is clear from the vector diagram that I_1 and I_2 would be in exact phase opposition; if in addition there were no leakage, no voltage would be required to maintain the currents.

(vii.) If the secondary circuit contains a series condenser C_2 (Fig. 30) the expressions are the same except that $Z_2 = R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)$, and the vector diagram is drawn in the same manner, although AB may be drawn vertically downwards, depending upon the relative values of ωL_2 and $1/\omega C_2$. The locus of Z′ is interesting.

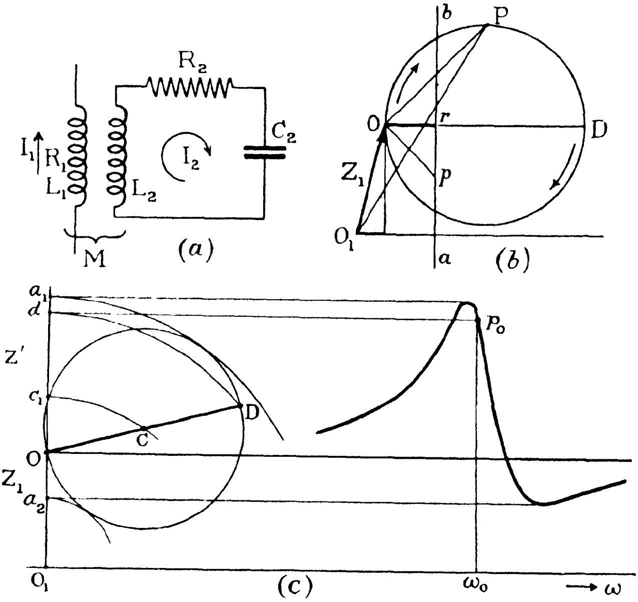


Fig. 30.—Resonance of oscillatory circuit.

$$Z' = Z_1 + \frac{\omega^2 M^2}{Z_2}$$

The vertical line $a\ p\ r\ b$ is the locus of Z_2, with pole O, and horizontal distance $Or = R_2$; this is inverted with a factor $\omega^2 M^2$ to give the circle OPD, any ray Op to the line becoming OP to the circle. Thus the circle is the locus (pole O) of $\omega^2 M^2/Z_2$.

Then Z_1 is added by finding a new pole O_1 such that $O_1O = Z_1 = R_1 + j\omega L_1$, and the circle with pole O_1 is the locus of Z'. O_1P is Z' for the particular ω value at which Op was drawn.

It has been assumed in this construction that ω^2M^2 and ωL_1 are constants, whereas, of course, they also vary with the frequency. But generally the resonance is sharp enough to make these variations quite insignificant.

If Z' is plotted against ω the " wiggle " type of curve of Fig. 30 is obtained, and this is also the shape of the curve of the observed voltage across the first coil when the current through the coil is kept constant but the frequency is varied.

This observed curve can be used to find the resonance frequency and decay factor of the secondary circuit. Draw a horizontal line through the extremities of the curve to meet a vertical line in O. This horizontal line is distant $| Z_1 |$ from the axis, which meets the vertical line in O_1. If there is any doubt about the position of this horizontal line, it can be determined by observation by open circuiting the secondary. Project the top and bottom points of the curve to the vertical to meet it in a_1 and a_2, and bisect a_1a_2 in c_1, and with centre O_1 draw arcs through a_1 a_2 and c_1. With centre O and radius a_1c_1, draw an arc to meet the arc through c_1 in C. With centre C and radius $CO = a_1c_1 = a_2c_1$ draw a circle. This passes through O, touches the arcs through a_1 and a_2, and is in fact the circle of Fig. 30 (*b*). To find the resonant frequency, with centre O_1 and radius O_1D (D is the extremity of the diameter of the circle), draw an arc to meet O_1a_1 in d, and project it to p_0 on the curve. The vertical through p_0 meets the horizontal in ω_0, which is the resonance value.

In the same way the ω values for various points on the circle can be found, and a curve of $\omega/\tan\phi$ constructed from which the decay factor of the secondary circuit can be found, as in case (v.).

(viii.) The " parallel resonance " case of Fig. 31 is of considerable importance as illustrating the principle of power-factor correction. The circuit consists of a series inductance L and resistance R (the load) in parallel with a condenser of capacity C across a supply of voltage V. I is the current supplied, made up of a current I_1 into the load and a current I_2 into the condenser.

$$V = (R + j\omega L)I_1 = -\frac{j}{\omega C} I_2$$

$$I = I_1 + I_2$$

are the vector relationships. From these the diagram Fig. 31 (*b*) is drawn.

$Oa = R$, $ab = j\omega L$, $Ob = R + j\omega L$, Ob is inverted with a factor V to give $OA = I_1$, $AB = I_2 = j\omega CV$ and $OB = OA + AB = I$. Produce AB to meet the axis in C. The original load phase angle $\phi_1 = \widehat{COA}$ has been reduced by the presence of the condenser to $\phi = \widehat{COB}$, and the original current $|I_1|$ has been reduced to $|I|$. By suitable choice of the value of C, the angle ϕ may be made as small as is desired. If ϕ is to be reduced to zero (unity power factor), it is necessary that

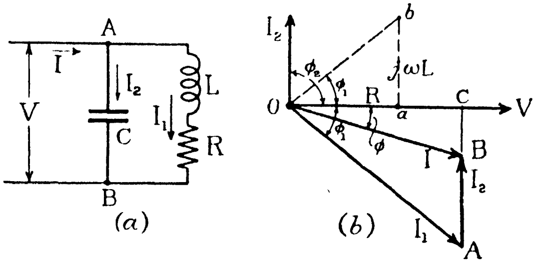


FIG. 31.—Power factor improvement.

$$|I_2| = |AC| = |I_1| \sin \phi_1$$

i.e.,
$$\omega CV = \frac{V}{\sqrt{R^2 + \omega^2 L^2}} \cdot \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} = \frac{\omega LV}{R^2 + \omega^2 L^2}$$

or
$$C = \frac{L}{R^2 + \omega^2 L^2}.$$

This is the case of resonance, when I and V are in phase. More generally,

$$I = I_1 + I_2 = V\left(\frac{1}{R + j\omega L} + j\omega C\right)$$
$$= V\left\{\frac{R}{R^2 + \omega^2 L^2} - j\,\frac{\omega L}{R^2 + \omega^2 L^2} + j\omega C\right\}.$$

If ϕ_1 is to be reduced to any desired value ϕ, generally given by the power factor $\cos \phi$, then C is determined from

$$\tan \phi = \frac{\omega C}{R}\left(R^2 + \omega^2 L^2\right) - \frac{\omega L}{R}.$$

EXERCISES

(1) If the voltage across a circuit is $(4 + j5)$ volts and the current through it is $(10 + j2)$ amperes, find the resistance and reactance of the circuit.

(2) A potential difference of 223 volts, frequency 50 cycles per second, is established across :—

(i.) An inductance of 0·5 henry having a resistance of 90 ohms.

(ii.) A condenser of 20 microfarads in series with a resistance of 50 ohms.

(iii.) The circuits of (i.) and (ii.) in series.

(iv.) The circuits of (i.) and (ii.) in parallel.

Find in each case the current flowing and its phase angle with the applied voltage, and find also the values of the voltage and the current at a time t after the voltage is a maximum and decreasing, giving t the values 0·002, 0·005, 0·010 and 0·015 second.

(3) A resistance of 10 ohms, an inductance of 5 millihenries and a condenser of 800 micro microfarads are connected in series across a voltage of 5 volts whose frequency is varied through the resonance value.

Find the current locus and from it draw a resonance curve.

(4) Two coils, A and B, A having an inductance of 0·2 henry and a resistance of 50 ohms, and B an inductance of 0·5 henry and a resistance of 200 ohms, are coupled by a mutual inductance of 0·25 henry. B is connected to a resistance of 100 ohms. Find the voltage, at an ω value of 1,000, that must be maintained across A in order that a current of 10 milliamperes may flow in B.

Draw a vector diagram to scale showing the currents and voltages in the coils.

(5) A single-phase 10 h.p. induction motor has an efficiency of 83 per cent., and is run off a 2,000-volt 50-cycle supply at a power factor of 0·79. What shunting condenser would be required to increase the power factor to 0·95 ?

Draw a vector diagram both with and without the condenser, and find the reduction of current effected.

CHAPTER III

BRIDGE NETWORKS

(i.) THE well-known Wheatstone bridge network of Fig. 32 (*a*) may be operated by alternating current if the usual primary cell is replaced by an alternating current supply of voltage V, and the usual D.C. galvanometer by a vibration galvanometer or telephone receiver. When the bridge is balanced B is at the same potential as D, and if P, Q, R and S are the resistances of the arms AB, BC, CD and DA in order, and if I_1 is the current through ABC and I_2 the current through ADC,

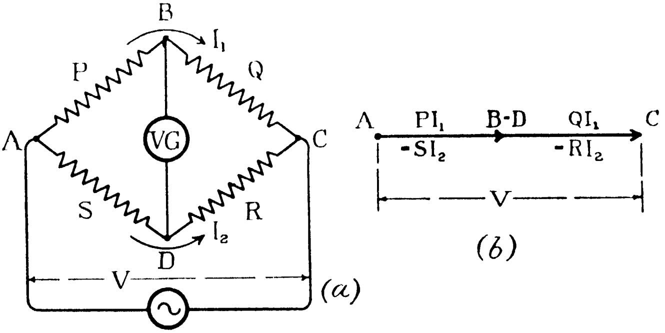


FIG. 32.—The Wheatstone Bridge.

$$I_1 = \frac{V}{P + Q}; \quad I_2 = \frac{V}{S + R},$$

$$I_1P = I_2S \text{ and } I_1Q = I_2R,$$

whence PR = SQ and the potential vector diagram is drawn as in Fig. 32 (*b*), where the lettering corresponds to that in Fig. 32 (*a*).

(ii.) Maxwell's inductance bridge (Fig. 33) can be treated in exactly the same manner. The balance conditions are most readily obtained by equating the products of the impedances of the opposite links of the bridge ; *i.e.*, by writing

$$Z_{AB} \cdot Z_{DC} = Z_{BC} \cdot Z_{AD},$$

where Z_{AB} is written for the impedance (complex) between A and B, etc.

This gives $$(R_1 + j\omega L_1)Q = (R_2 + j\omega L_2)P,$$
whence $$R_1 Q = R_2 P \text{ and } L_1 Q = L_2 P$$
or $$\frac{R_1}{R_2} = \frac{L_1}{L_2} = \frac{P}{Q}.$$

The vector diagram is drawn in Fig. 33 (*b*). Evidently from the balance conditions the impedances Z_{AB} and Z_{BC} must have the same angle, and the current I_1 through them is drawn as AI_1 at this angle behind the voltage AC. $Aa = R_1 I_1$, $aB = j\omega L_1 I_1$, $Bb = R_2 I_1$, and $bC = j\omega L_2 I_1$. The current I_2 through P and Q is in phase with V, and AD (D is the same point as B) $= PI_2$ and DC $= QI_2$.

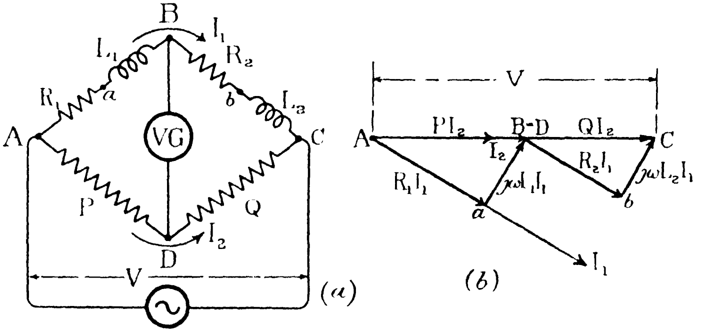


FIG. 33.—Maxwell's Inductance Bridge.

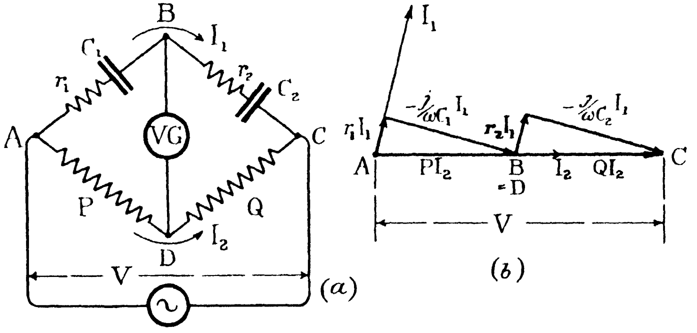


FIG. 34.—De Sauty's Capacity Bridge.

(iii.) The corresponding capacity bridge is due to De Sauty, and is shown in Fig. 34. The balance condition is

$$\left(r_1 - \frac{j}{\omega C_1}\right) Q = \left(r_2 - \frac{j}{\omega C_2}\right) P$$

or
$$\frac{r_1}{r_2}=\frac{C_2}{C_1}=\frac{P}{Q},$$

and the vector diagram is obtained in the same manner as before. With air condensers the resistances r_1 and r_2 are usually negligibly small and a measurement of C_1 in terms of C_2 can be made simply by adjusting P or Q.

(iv.) The Schering bridge is arranged as shown in Fig. 35. C_1 is the unknown capacity and r_1 its equivalent series resistance. This may be a short length of cable or an insulator to be tested at high voltages. C_2 is a standard condenser. P and Q are non-inductive resistances and C_3 a variable condenser usually very much larger than C_1 or C_2. The impedance from A to B or D is usually enormously greater than that between B or D and C, so that even though V is a high voltage, if C be earthed, the voltages on the DBC half of the bridge are quite small and there is no danger in manipulating P, Q and C_3.

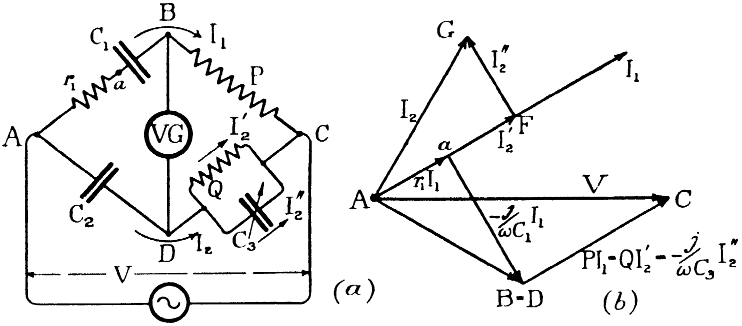


FIG. 35.—Schering's Capacity Bridge.

The impedances are

$$Z_{AB}=r_1-\frac{j}{\omega C_1},$$

$$Z_{BC}=P,$$

$$Z_{AD}=-\frac{j}{\omega C_2},$$

$$Z_{DC}=\frac{Q\left(-\frac{j}{\omega C_3}\right)}{Q-\frac{j}{\omega C_3}}=\frac{Q}{1+j\omega C_3Q},$$

and the balance condition

$$\left(r_1 - \frac{j}{\omega C_1}\right)\left(\frac{Q}{1 + j\omega C_3 Q}\right) = -\frac{j}{\omega C_2} P,$$

or $$Qr_1 - j\frac{Q}{\omega C_1} = -\frac{j}{\omega C_2} P + \frac{C_3}{C_2} QP,$$

giving $$r_1 = \frac{C_3}{C_2} P,$$

and $$C_1 = \frac{Q}{P} C_2.$$

The loss angle of the capacity C_1, *i.e.*, the departure of the angle of its impedance from 90 degrees, is given by

$$\tan\theta = \frac{r_1}{\dfrac{1}{\omega C_1}} = \omega C_1 r_1 = \omega Q C_3,$$

and since θ is always small, ωQC_3 is very nearly the loss angle and the power factor.

The vector diagram is drawn in Fig. 35 (*b*), though for the sake of clearness no attempt has been made to indicate the great differences in the voltages between A and B = D and between B = D and C. The currents through the arms ABC and ADC are called I_1 and I_2 as before, while I_2 divides into I_2' through Q and I_2'' through C_3. Evidently

$$I_2 = I_2' + I_2''.$$

Also, since $$QI_2' = PI_1$$

$$I_2' = \frac{P}{Q} I_1$$

and equating the voltage across C_3 to that across Q, gives

$$QI_2' = -\frac{j}{\omega C_3} I_2''$$

$$\begin{aligned} I_2'' &= j\omega C_3 Q I_2' \\ &= j\omega C_3 P I_1. \end{aligned}$$

I_1 leads the voltage V = AC by an angle determined by r_1, P and C_1, ($\tan\phi = 1/\{\omega C_1(r_1 + P)\}$), $Aa = r_1 I_1$ is the voltage across the resistance r_1, $aB = -\frac{j}{\omega C_1} I_1$ is that across the condenser C_1 and

$BC = PI_1$ that across P. Since D must coincide with B on the diagram, AB is the voltage across $C_2 = -\frac{j}{\omega C_2} I_2$. Hence $I_2 = j\omega C_2(AB)$, and is drawn as AG at right angles to AB. Drop a perpendicular GF on to AI_1. Then $AF = I_2' = \frac{P}{Q} I_1$ and $FG = j\omega C_3 PI_1 = I_2''$.

(v.) Inductance may be balanced by capacity in the opposite arm by two arrangements, one due to Maxwell, shown in Fig. 36, and the other due to Hay, shown in Fig. 37.

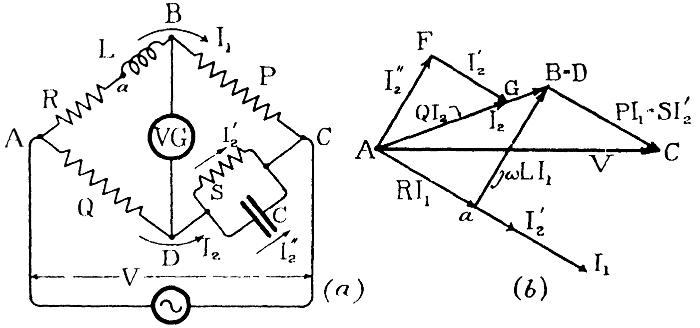


Fig. 36.—Maxwell's Capacity Bridge.

The balance condition for Fig. 36 is

$$(R + j\omega L)\left(\frac{S}{1 + j\omega CS}\right) = PQ,$$

i.e.,

$$RS + j\omega LS = PQ + j\omega CSPQ,$$

whence

$$R = \frac{PQ}{S}$$

and

$$L = PQC.$$

In the vector diagram, I_1 is drawn lagging behind $V(= AC)$ by an angle $\phi = \tan^{-1} \omega L/(R + P)$, $Aa = RI_1$, $aB = j\omega LI_1$ and $BC = PI_1$. Since $B = D$, DC must also be SI_2' and $-\frac{j}{\omega C} I_2''$; *i.e.,*

$$PI_1 = SI_2' = -\frac{j}{\omega C} I_2'',$$

so that

$$I_2' = \frac{P}{S} I_1 \text{ and } I_2'' = j\omega CPI_1.$$

AF is drawn perpendicular to AI_1 and of length $\omega CP|I_1|$ to give I_2''. FG is drawn parallel to AI_1 to meet AB in G. Then $FG = I_2'$ and $AG = AF + FG = I_2'' + I_2' = I_2$. Also $AB = QI_2$.

(vi.) Hay's bridge is similar, but the balancing capacity and resistance are in series instead of in parallel (Fig. 37).

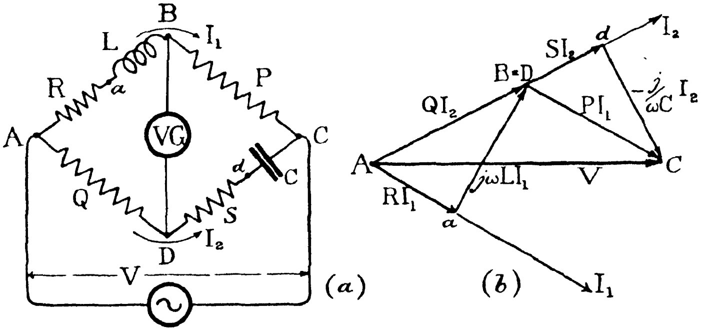


FIG. 37.—Hay's Capacity Bridge.

The balance conditions are obtained from

$$(R + j\omega L)\left(S - \frac{j}{\omega C}\right) = PQ$$

$$RS + \frac{L}{C} + j\omega LS - j\frac{R}{\omega C} = PQ,$$

$$RS + \frac{L}{C} = PQ,$$

$$\omega LS - \frac{R}{\omega C} = 0.$$

From these
$$L = \frac{PQC}{1 + \omega^2C^2S^2}$$

and
$$R = \frac{\omega^2C^2SPQ}{1 + \omega^2C^2S^2}.$$

It is seen that ω occurs in the expressions for finding the unknown quantities from the bridge constants. Unlike the previous cases, the bridge balance depends upon the frequency of the alternating supply, and this must be free from harmonics if silence is to be obtained at balance where telephone receivers are used instead of a vibration galvanometer. The bridge may be balanced on the fundamental, but the harmonics will be heard in the receivers.

The vector diagram is drawn in Fig. 37 (*b*), and is so straightforward as to need no comment.

(vii.) Anderson's method for the measurement of inductance involves a more complicated network, drawn in Fig. 38 (*a*). The inductance L, resistance R, constitutes the arm AB, while non-inductive resistances P and Q constitute the arms BC and AD. Between D and C are (i.) a non-inductive resistance S; and (ii.) a non-inductive resistance r in series with a capacity C. The vibration galvanometer is connected between B and the point D′ at the junction of r and C. When balance is obtained, B and D′ are always at the same potential. Let the currents be I_1 through ABC, I_2 from A to D, I_2' through DD′C and I_2'' from D to C through S. Then

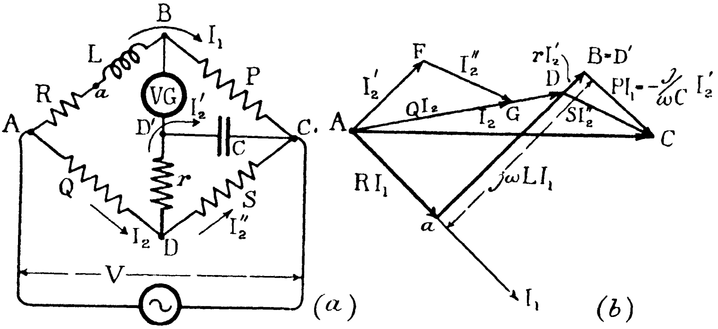


FIG. 38.—Anderson's Inductance Bridge.

$$I_1 = \frac{V}{R + j\omega L + P},$$

$$I_2 = \frac{V}{Q + \dfrac{S\left(r - \dfrac{j}{\omega C}\right)}{r + S - \dfrac{j}{\omega C}}}$$

$$I_2' = \frac{S}{r + S - \dfrac{j}{\omega C}} \cdot I_2$$

$$= \frac{VS}{Q\left(r + S - \dfrac{j}{\omega C}\right) + S\left(r - \dfrac{j}{\omega C}\right)}.$$

And the balance condition necessitates

$$(\mathrm{R} + j\omega \mathrm{L})\mathrm{I}_1 = \mathrm{QI}_2 + r\mathrm{I}_2',$$

i.e.,

$$\frac{\mathrm{R} + j\omega\mathrm{L}}{\mathrm{R} + j\omega\mathrm{L} + \mathrm{P}} = \frac{\mathrm{Q}}{\mathrm{Q} + \dfrac{\mathrm{S}\left(r - \dfrac{j}{\omega\mathrm{C}}\right)}{r + \mathrm{S} - \dfrac{j}{\omega\mathrm{C}}}} + \frac{r\mathrm{S}}{\mathrm{Q}\left(r + \mathrm{S} - \dfrac{j}{\omega\mathrm{C}}\right) + \mathrm{S}\left(r - \dfrac{j}{\omega\mathrm{C}}\right)}$$

$$= \frac{\mathrm{Q}\left(r + \mathrm{S} - \dfrac{j}{\omega\mathrm{C}}\right) + r\mathrm{S}}{\mathrm{Q}\left(r + \mathrm{S} - \dfrac{j}{\omega\mathrm{C}}\right) + \mathrm{S}\left(r - \dfrac{j}{\omega\mathrm{C}}\right)}.$$

Multiplying across

$$(\mathrm{R} + j\omega\mathrm{L})\left\{\mathrm{Q}\left(r + \mathrm{S} - \frac{j}{\omega\mathrm{C}}\right) + \mathrm{S}\left(r - \frac{j}{\omega\mathrm{C}}\right)\right\}$$

$$= \left\{\mathrm{R} + j\omega\mathrm{L} + \mathrm{P}\right\}\left\{\mathrm{Q}\left(r + \mathrm{S} - \frac{j}{\omega\mathrm{C}}\right) + r\mathrm{S}\right\},$$

i.e.,

$$(\mathrm{R} + j\omega\mathrm{L})\left(-\frac{j}{\omega\mathrm{C}}\mathrm{S}\right) = \mathrm{P}\left(\mathrm{Q}r + \mathrm{QS} + r\mathrm{S} - \frac{j}{\omega\mathrm{C}}\mathrm{Q}\right)$$

or

$$-j\frac{\mathrm{RS}}{\omega\mathrm{C}} + \frac{\mathrm{L}}{\mathrm{C}}\mathrm{S} = \mathrm{P}(\mathrm{Q}r + \mathrm{QS} + r\mathrm{S}) - j\frac{\mathrm{PQ}}{\omega\mathrm{C}}$$

whence

$$\mathrm{R} = \frac{\mathrm{PQ}}{\mathrm{S}}$$

and

$$\mathrm{L} = \mathrm{CP}\left[r\left(1 + \frac{\mathrm{Q}}{\mathrm{S}}\right) + \mathrm{Q}\right].$$

The vector diagram at balance is drawn in Fig. 38 (*b*). AC is the voltage V, Aa = RI_1, aB = $j\omega\mathrm{LI}_1$, and BC = PI_1. B = D′, hence $\mathrm{BC} = \mathrm{D'C} = -\dfrac{j}{\omega\mathrm{C}} \cdot \mathrm{I}_2' = \mathrm{PI}_1$.

$$\therefore \quad \mathrm{I}_2' = j\omega\mathrm{CPI}_1.$$

I_2' is drawn therefore at right angles to I_1, *i.e.*, in the same direction as aB, and since DD′ = $r\mathrm{I}_2'$, D must lie between a and D′ = B.

Join AD and DC. Then AD = QI_2 and DC = SI_2''. Draw AF =

I_2' parallel to DD′ and FG to meet AD in G parallel to DC. Then FG $= I_2''$ and AG $= I_2$.

(viii.) Self and mutual inductance may be compared by the arrangement of Fig. 39, due to Maxwell. The two coils of the mutual inductance are supposed to be wound on the same core in the same sense. Then if I_1 and I_2 are the currents through ABC and ADC respectively, that through the other coil of the inductometer is $I_1 + I_2$ in the direction indicated. If V is the voltage established across A and C,

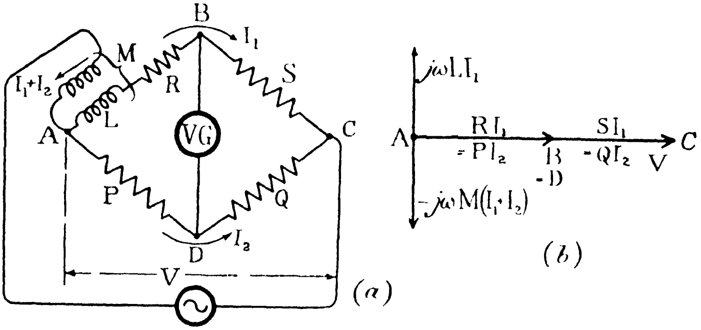


FIG. 39.—Maxwell's Self and Mutual Inductance Bridge.

$$V = (R + j\omega L + S)I_1 - j\omega M(I_1 + I_2)$$

and

$$V = (P + Q)I_2,$$

and from these

$$I_1(R + j\omega L + S - j\omega M) = V + j\omega M \frac{V}{P + Q},$$

and, since at balance the voltages at B and D are equal,

$$SI_1 = QI_2,$$

i.e.,

$$\frac{S\left(1 + j\dfrac{\omega M}{P + Q}\right)}{R + S + j\omega L - j\omega M} = \frac{Q}{P + Q}$$

$$(P + Q + j\omega M)S = Q(R + S + j\omega L - j\omega M),$$

whence

$$R = \frac{PS}{Q}$$

and

$$L = M\left(1 + \frac{S}{Q}\right).$$

Evidently, in order that a balance may be obtained, L must be greater than M. If L is actually less than M, a balance may be obtained by adding a known inductance L_0R_0 to the arm AB, and subtracting it from the values obtained. Or an unknown impedance L_xR_x may be added to the arm AB and a fresh balance obtained, when

$$R + R_x = \frac{PS}{Q}$$

and $$L + L_x = M\left(1 + \frac{S}{Q}\right),$$

and R_x and L_x are found by subtraction.

The vector diagram is the very simple one drawn in Fig. 39 (*b*). $j\omega LI_1 = j\omega M(I_1 + I_2)$, and the currents I_1 and I_2 are in phase with the voltage V. The bridge in its balanced condition is very similar to an ordinary Wheatstone bridge, the mutual inductance injecting into the arm AB a voltage which just counterbalances the self-inductive voltage developed across L.

(ix.) Campbell's modification of this last bridge consists in inserting an inductance in the arm AD.

Let the resistances and inductances be as shown in Fig. 40 (*a*). The balance conditions may be derived from the vector diagram, Fig. 40 (*b*).

A*a* is drawn $= R_1I_1$, $aF = j\omega L_1I_1$ and $FB = -j\omega M\ (I_1 + I_2)$. I_2 must be in phase with I_1 since $SI_1 = QI_2 = BC = DC$. And since $AB = AD = R_2I_2 + j\omega L_2I_2$, $Aa = R_2I_2$ and $aD = j\omega L_2I_2$.

Thus $$R_1I_1 = R_2I_2$$
$$SI_1 = QI_2,$$

$$\text{whence } R_1 = R_2\frac{S}{Q}$$

$$j\omega L_2I_2 = j\omega L_1I_1 - j\omega M(I_1 + I_2)$$
$$L_2\frac{S}{Q}I_1 = L_1I_1 - M\left(1 + \frac{S}{Q}\right)I_1,$$

$$\text{whence } L_1 = \frac{S}{Q}L_2 + M\left(1 + \frac{S}{Q}\right).$$

Suppose now that the unknown impedance R_xL_x is inserted in the arm AB and the mutual inductance and R_2 adjusted to M' and R_2', so that a balance is again obtained.

Then $$R_1 + R_x = R_2' \frac{S}{Q},$$

and $$L_1 + L_x = \frac{S}{Q} L_2 + M'\left(1 + \frac{S}{Q}\right),$$

and R_x and L_x are obtained by subtraction.

If L_1 is made equal to $\frac{S}{Q} L_2$,

$$L_x = M'\left(1 + \frac{S}{Q}\right).$$

The three coils comprising L_1 L_2 and the variable mutual are mounted together in Campbell's standard inductometer, and tapping points are arranged so that $\frac{S}{Q} L_2$ can be made as nearly as possible equal to L_1. S/Q is conveniently made equal to 1, 9 and 99, so that $L_x = 2M'$, 10 M′ and 100 M′. The first balance (with L_xR_x shorted) then gives a very small value for M which is used to correct for any departure from the exact relationship $L_1 = L_2S/Q$.

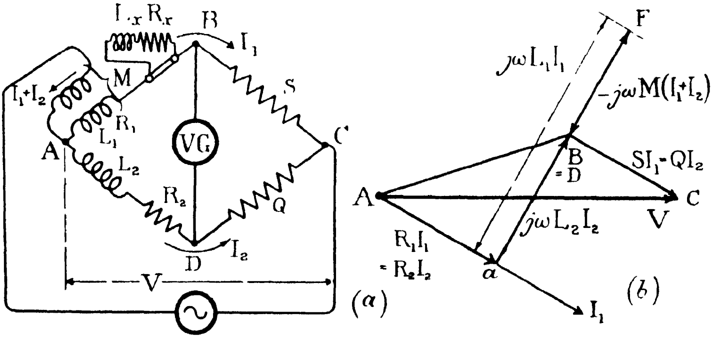


FIG. 40.—Campbell's modification of Maxwell's Bridge.

(x.) In the Heaviside-Campbell equal ratio bridge, there is mutual inductance to both arms AB and AD, as shown in Fig. 41, and the inductances LR in the two arms are equal, as well as the resistances Q in the arms BC and DC. For balance therefore the currents through ABC and ADC must be equal, and equating the voltage drops in the resistances and the voltage drops due to the inductances gives

$$R_x = r$$

and $\quad I_1(j\omega L_1 - 2j\omega M_1 + j\omega L_x) = I_1(j\omega L_1 + 2j\omega M_2),$

i.e., $\quad L_x = 2(M_1 + M_2),$

and $M_1 + M_2$ is the reading of the inductometer. It may be noticed that the result will be unaffected if there is mutual inductance between the two coils L in the arms AB and AD, as the voltages induced in the two arms will be equal.

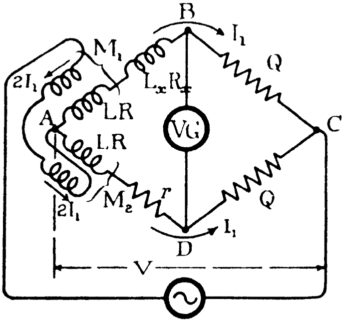


Fig. 41.—Heaviside-Campbell Inductance Bridge.

(xi.) As a final example the modified Carey-Foster bridge is drawn in Fig. 42 (*a*). In this the capacity C is measured in terms of the mutual M.

Since A and B are shorted, when balance is obtained, the three points AB and D on the vector diagram (Fig. 42 (*b*)) coincide. I_1 is evidently in phase with V, and AC = V = QI_1. I_2 must lead V, since $DC = \left(S - \frac{j}{\omega C}\right)I_2 = V$. A*d* is drawn = SI_2 and *d*C perpendicular to Ad $= -\frac{j}{\omega C}I_2$. The voltage drops in AD must add up to zero. They are A*f* = $(R + r)I_2$, *fg* $= j\omega LI_2$, *gh* $= -j\omega MI_1$ and *h*A $= -j\omega MI_2$. Hence it is clear that

$$QI_1 = SI_2 - \frac{j}{\omega C}I_2$$

$$(R + r)I_2 + j\omega LI_2 - j\omega MI_2 - j\omega MI_1 = 0.$$

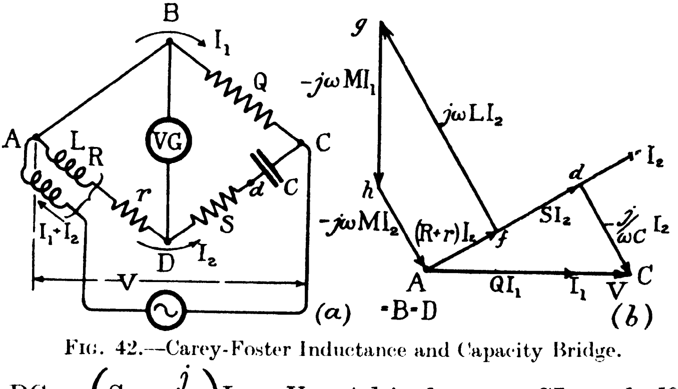


Fig. 42.—Carey-Foster Inductance and Capacity Bridge.

Substituting I_1 from the first in the second equation gives

$$R + r + j\omega L - j\omega M - j\omega M\frac{S}{Q} \cdot - \frac{M}{CQ} = 0,$$

whence $$C = \frac{M}{Q(R + r)}$$

and $$L = M\left(1 + \frac{S}{Q}\right).$$

In this treatment of some alternating current bridges, the only point of view that has been in mind is that of the vector algebra and vector diagrams involved. For practical details as to the arrangement of the bridges, for the choice of the best bridge for a particular measurement, for the relative accuracy of the various bridges and for the effect on the accuracy of the imperfections of the " non-inductive " resistances and " pure " mutual inductances, the reader is referred to " A.C. Bridge Methods," by B. Hague.

EXERCISES

1. In making measurements on a condenser by the Schering bridge (Fig. 35) at 20 kilovolts, 50 cycles, balance was obtained with $C_2 = 0{\cdot}055\ \mu F$, $C_3 = 1{\cdot}75\ \mu F$, $P = 200$ ohms and $Q = 700$ ohms. Find the capacity, the equivalent series resistance and the power factor of the condenser, and find the voltage across the variable condenser.

2. The arms of a balanced alternating-current bridge ABCD are as follows :—

AB. A resistance of 1,250 ohms in parallel with a condenser of a capacity 0·30 microfarad.
BC. A non-inductive resistance of 1,000 ohms.
AD. A non-inductive resistance of 500 ohms.
DC. An unknown inductive resistance.

Find the values of the inductance and resistance in the arm DC. Draw a vector diagram to scale showing the currents and potentials in the bridge when a potential difference of 2·5 volts, frequency 796 cycles per second, is established between the points A and C.

3. An alternating-current bridge ABCD has the following constants :—

Between	A & B.	A non-inductive resistance P ohms.
,,	B & C.	A capacitance C farads in series with a resistance r ohms.
,,	C & D.	A non-inductive resistance Q ohms.
,,	D & A.	An inductance L henries in series with a resistance R ohms.

Find the conditions necessary for balance.

If a voltage of 10 volts r.m.s. at $\omega = 5{,}000$ is maintained across A and C, and $P = 2{,}000$ ohms, $Q = 1{,}000$ ohms, $C = 0{\cdot}1$ microfarad and $r = 900$ ohms when the bridge is balanced, find the values of L and R, and draw a vector diagram to scale showing the voltages and currents in each part of the network.

4. In Anderson's bridge for measuring the inductance L and resistance R of an unknown impedance between the points A and B (Fig. 38), find R and L if balance is obtained when $Q = S = 1{,}000$ ohms, $P = 500$ ohms, $r = 200$ ohms and $C = 2\ \mu F$.

Draw a vector diagram showing the voltage and current at every point of the network when the voltage across AC is 10 volts and the frequency is 100 cycles per second.

CHAPTER IV

THE TRANSFORMER

(1) Air Core

THE simple case of two coupled coils has already been dealt with in Chapter II., and Fig. 29 shows the vector diagram.

Consider the case of Fig. 43. The primary coil has an impedance $Z_1 = R_1 + j\omega L_1$; the secondary coil has an impedance $Z_2 = R_2 + j\omega L_2$, and the mutual inductance between the two coils is M. Let the primary voltage be V_1, the secondary load Z, the secondary voltage (across the load Z) V_2, and the primary and secondary currents I_1 and I_2 respectively, considered positive as shown, when the coils are wound in the same sense on the same core. (The positive direction for I_2 is opposite to that taken in Fig. 29, and this accounts for the slight difference between the following equations and those of p. 30.) The vector equations are

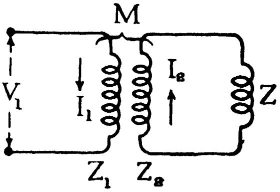


FIG. 43.—Air core transformer.

$$V_1 = Z_1I_1 - j\omega MI_2$$
$$O = (Z_2 + Z)I_2 - j\omega MI_1$$

whence

$$I_1 = \frac{V_1}{Z_1 + \dfrac{\omega^2M^2}{Z_2 + Z}} = \frac{V_1}{Z'}$$

$$I_2 = \frac{j\omega MV_1}{\omega^2M^2 + Z_1(Z_2 + Z)} = \frac{V_1}{Z''}$$

where Z′ and Z″ are effective impedances determining I_1 and I_2 respectively in terms of V_1.

$$Z' = Z_1 + \frac{\omega^2M^2}{Z_2 + Z}$$

$$Z'' = -\frac{j}{\omega M}\left\{\omega^2M^2 + Z_1(Z_2 + Z)\right\}$$

(i.) *Voltage Ratio.*

The voltage ratio of the transformer is found as follows,

$$V_2 = I_2Z = \frac{V_1}{Z''} \cdot Z$$

$$\therefore \frac{V_1}{V_2} = \frac{\omega^2M^2 + Z_1(Z_2 + Z)}{j\omega M Z}$$

$$= -j\omega\frac{M}{Z} - j\frac{Z_1Z_2}{\omega MZ} - j\frac{Z_1}{\omega M}.$$

If the reactances ωL_1 and ωL_2 are great compared with the resistances R_1 and R_2, and the latter are neglected, the ratio becomes

$$\frac{V_1}{V_2} = \frac{L_1}{M} - \frac{j\omega}{Z}\left(M - \frac{L_1L_2}{M}\right)$$

$$= \frac{L_1}{M} + j\omega\frac{L_1L_2}{MZ}\left(1 - \frac{M^2}{L_1L_2}\right),$$

or
$$V_2 = \frac{\frac{M}{L_1}V_1}{1 + j\omega\frac{L_2}{Z}\left(1 - \frac{M^2}{L_1L_2}\right)}$$

If $1 - \frac{M^2}{L_1L_2}$ is written as σ, σ is a "leakage coefficient," and

$M = \sqrt{L_1L_2(1 - \sigma)}$, $M/L_1 = \sqrt{\frac{L_2}{L_1}(1 - \sigma)}$

and
$$V_2 = \frac{\sqrt{\frac{L_2}{L_1}(1 - \sigma)}}{1 + j\omega\frac{L_2}{Z}\sigma}V_1.$$

If there is no leakage, *i.e.*, if $\sigma = 0$ ($M^2 = L_1L_2$),

$$V_2 = \sqrt{\frac{L_2}{L_1}}V_1,$$

and if the coils are of the same dimensions and shape, or have the same magnetic circuit, and the number of turns of wire are T_1 and T_2 respectively, then

$$V_2 = \frac{T_2}{T_1}V_1.$$

The voltage is " stepped up " (or stepped down) in the ratio of the number of turns.

However, there always is leakage, and then with no load ($Z = \infty$)

$$\frac{V_2}{V_1} = \sqrt{\frac{L_2}{L_1}}(1 - \sigma),$$

and the greater the load (the smaller Z) the greater the departure of the voltage ratio from this value.

In this connection is should be noticed that the influence of the inductance of the load on the voltage ratio is far greater than that of the resistance.

For writing $$Z = R + j\omega L$$

$$1 + j\omega\frac{L_2}{Z}\sigma = 1 + j\omega L_2\sigma\frac{R - j\omega L}{R^2 + \omega^2 L^2}$$

$$= 1 + \frac{\omega^2 L_2\sigma}{R^2 + \omega^2 L^2}L + \frac{j\omega L_2\sigma}{R^2 + \omega^2 L^2}R.$$

The second term, depending upon L, is added in the same straight line to 1, but the third, depending upon R is added at right-angles.

These results are of importance in the potential transformer.

In an alternative method of presentation M^2/L_1L_2 is written as τ^2, where τ is a " coupling coefficient," of maximum value unity with complete coupling, *i.e.*, with no leakage. In this method

$$\frac{M}{L_1} = \frac{\tau\sqrt{L_1 L_2}}{L_1} = \tau\sqrt{\frac{L_2}{L_1}}$$

and $$V_2 = \frac{\tau\sqrt{\frac{L_2}{L_1}}}{1 + j\omega\frac{L_2}{Z}(1 - \tau^2).}V_1.$$

(ii.) *Current Ratio.*

The current ratio is found in a similar manner.

$$\frac{I_1}{I_2} = \frac{Z''}{Z'} = \frac{-\frac{j}{\omega M}\left\{\omega^2 M^2 + Z_1(Z_2 + Z)\right\}}{Z_1 + \frac{\omega^2 M^2}{Z_2 + Z}}$$

$$= -\frac{j}{\omega M}(Z_2 + Z)$$

$$\frac{I_1}{I_2} \doteq -\frac{j}{\omega M}(j\omega L_2 + Z), \text{ neglecting resistances in } Z_1 \text{ and } Z_2$$

$$= \frac{L_2}{M} - \frac{j}{\omega M} Z = \sqrt{\frac{L_2}{L_1(1-\sigma)}} - \frac{j}{\omega M} Z$$

$$= \sqrt{\frac{L_2}{L_1(1-\sigma)}}\left(1 + \frac{Z}{j\omega L_2}\right) = \frac{1}{\tau}\sqrt{\frac{L_2}{L_1}} - \frac{j}{\omega M} Z$$

$$= \frac{1}{\tau}\sqrt{\frac{L_2}{L_1}}\left(1 + \frac{Z}{j\omega L_2}\right).$$

It is only when Z is zero, *i.e.*, when the secondary is short circuited, and when in addition there is no leakage, that the current ratio has the ideal value $\sqrt{\frac{L_2}{L_1}}$, or in the case of coils of identical magnetic circuits, the inverse ratio of the numbers of turns. This is of importance in the current transformer.

The flux linkages ($\varLambda_1$) through the primary coil are made up of two parts, L_1I_1, due to the current in the primary, and $-MI_2$, due to the current in the secondary. Thus

$$\varLambda_1 = L_1I_1 - MI_2$$

$$= L_1\frac{V_1}{Z'} - M\frac{V_1}{Z''}$$

$$= V_1\left\{\frac{L_1}{Z_1 + \dfrac{\omega^2M^2}{Z_2 + Z}} - \frac{M}{-\dfrac{j}{\omega M}\left\{\omega^2M^2 + Z_1(Z_2 + Z)\right\}}\right\}$$

$$= V_1\left\{\frac{L_1(Z_2 + Z) - j\omega M^2}{\omega^2M^2 + Z_1(Z_2 + Z)}\right\}$$

$$= \frac{V_1}{j\omega}\left\{\frac{\omega^2M^2 + j\omega L_1(Z_2 + Z)}{\omega^2M^2 + Z_1(Z_2 + Z)}\right\}$$

If the primary resistance R_1 is neglected, $Z_1 = j\omega L_1$, and

$$\varLambda_1 = \frac{V_1}{j\omega} \quad \text{or} \quad V_1 = j\omega\,\varLambda_1,$$

a result which could have been expected, since the e.m.f. produced by the alternating flux in the primary coil must, if the primary resistance is neglected, equal the supply voltage whatever the secondary load. Thus, with constant supply voltage, whatever primary and secondary currents are flowing they must so adjust themselves as to produce the same flux through the primary. (This conclusion will, however, be modified if it is necessary to take the primary resistance into account.)

This must be true also at no load, *i.e.*, when $Z = \infty$ or $I_2 = 0$, when the primary current is I_0. Thus

$$A_1 = L_1 I_1 - M I_2 = L_1 I_0$$

or

$$\begin{aligned} I_1 &= I_0 + \frac{M}{L_1} I_2 \\ &= I_0 + \sqrt{\frac{L_2}{L_1}(1-\sigma)}.\, I_2 \\ &= I_0 + \tau\sqrt{\frac{L_2}{L_1}}\, I_2. \end{aligned}$$

I_0 can be looked upon as the "magnetising" current, or the primary current necessary to produce the flux, while $\tau\sqrt{\frac{L_2}{L_1}}\, I_2$ is the primary current which is necessary to provide the "load." The total primary current is the vector sum of these two. If there is no leakage, and if the coils have identical magnetic circuits this equation becomes

$$I_1 = I_0 + \frac{T_2}{T_1} I_2.$$

(iii.) *Equivalent Networks.*

The transformer may be replaced for calculations by an equivalent T-circuit, as in Fig. 44, in which impedances Z_a, Z_b and Z_c can be found such that if the primary voltage V_1 is applied to A and C, and the secondary load Z is connected between B and C, the currents into and out of the T will be the same as the primary and secondary currents respectively.

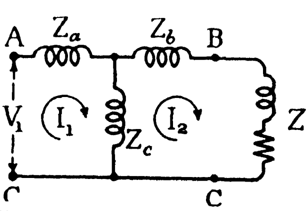


FIG. 44.—Equivalent T-circuit.

For the T

$$\begin{aligned} V_1 &= (Z_a + Z_c)\, I_1 - Z_c I_2 \\ 0 &= (Z_c + Z_b + Z)\, I_2 - Z_c I_1. \end{aligned}$$

Comparing these with the transformer equations, it is seen that they become identical if

$$Z_a + Z_c = Z_1$$

$$j\omega M \quad = Z_c$$

and
$$Z_c + Z_b = Z_2$$

i.e., if
$$Z_a = Z_1 - j\omega M = R_1 + j\omega\,(L_1 - M)$$

$$Z_b = Z_2 - j\omega M = R_2 + j\omega\,(L_2 - M)$$

and
$$Z_c = j\omega M.$$

Another equivalent network* can be obtained from the equation

$$I_1 = I_0 + \frac{M}{L_1} I_2.$$

Remembering that this equation was obtained on the assumption that $R_1 = 0$,

$$I_0 = \frac{V_1}{j\omega L_1}$$

$$\frac{M}{L_1} I_2 = \frac{V_1}{\frac{L_1}{M} Z''}$$

$$\begin{aligned} \frac{L_1}{M} Z'' &= -\frac{jL_1}{\omega M^2}\left\{\omega^2 M^2 + j\omega L_1(Z_2 + Z)\right\} \\ &= -j\omega L_1 + \frac{{L_1}^2}{M^2}(R_2 + j\omega L_2 + Z) \\ &= j\omega\left(\frac{{L_1}^2 L_2}{M^2} - L_1\right) + \frac{{L_1}^2}{M^2}(R_2 + Z) \\ &= (j\omega L_2\sigma + R_2 + Z)\frac{{L_1}^2}{M^2} \end{aligned}$$

If, therefore, as in Fig. 45, two impedance branches are drawn between B and C, the upper consisting of a pure inductance $\frac{{L_1}^2}{M^2}L_2\sigma$, a pure resistance $\frac{{L_1}^2}{M^2}R_2$ and the impedance $\frac{{L_1}^2}{M^2}Z$ in series, and the lower a pure inductance L_1, and if a voltage V_1 is applied across B and C, the currents flowing will be $\frac{M}{L_1}I_2$ in the upper

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* A. Russell: "Alternating Currents."

branch and I_0 in the lower branch. The vector sum of these two currents is I_1, and if a series resistance R_1 is added to the network at AB, and the voltage V_1 applied across A and C, the representation of the transformer is complete. If there is no leakage $\sigma = 0$ and the inductance $\frac{L_1^2}{M^2} L_2\sigma$ disappears, and $\frac{L_1^2}{M^2} = \frac{L_1}{L_2}$. If in addition the inductance L_1 is very large, and I_0 can be neglected in comparison with $\frac{M}{L_1} I_2$, the circuit reduces to the simple series one indicated in Fig. 46. The current is

$$I_1 = \sqrt{\frac{L_2}{L_1}}\, I_2.$$

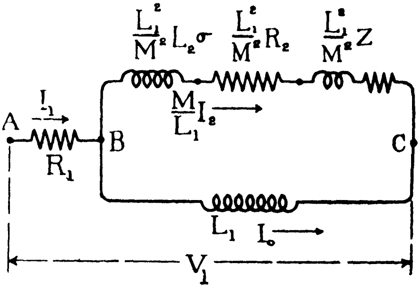


FIG. 45.—Equivalent network.

If, further, $R_1 = R_2 = 0$, the transformer behaves simply as if the primary voltage were applied to an impedance $\frac{L_1}{L_2}Z$, and the secondary current is $\sqrt{\frac{L_1}{L_2}}$ times the current flowing. This is the case of the ideal transformer; no resistance, no leakage, and no magnetising current.

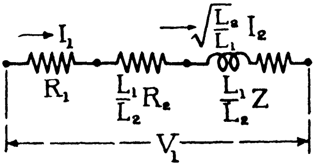


FIG. 46.

(iv.) *Vector locus*

Returning to the equation

$$I_1 = I_0 + \frac{M}{L_1} I_2$$

and substituting the value of $\frac{M}{L_1} I_2$ from the above, gives

$$I_1 = I_0 + \frac{V_1}{(j\omega L_2\sigma + R_2 + Z)\frac{L_1^2}{M^2}}$$

$$= I_0 + \frac{V_1}{\frac{L_1}{L_2(1-\sigma)}\left\{R_2 + j\omega L_2\sigma + Z\right\}}.$$

If the secondary load is non-inductive, say $Z = R$, and R is varied from 0 to ∞, the loci of the current vectors I_1 and I_2 are circles. The equation is now

$$I_1 = I_0 + \frac{V_1 \frac{L_2(1-\sigma)}{L_1}}{\left\{R_2 + R + j\omega L_2\sigma\right\}}$$

and the vector diagram of this equation is constructed in Fig. 47. V_1 is drawn horizontally as OV_1. $Oa = R_2$, $ab = j\omega L_2\sigma$, and $bp = R$. p travels along the horizontal from b to ∞ as R increases indefinitely. Inverting Op with a factor of inversion, $V_1\frac{L_2}{L_1}(1-\sigma)$ gives the arc OPB of the semi-circle OPBD, with vertical diameter OD.

$$|\,OD\,| = \frac{L_2}{L_1}(1-\sigma)\,V_1\,\frac{1}{\omega L_2\sigma} = \frac{1-\sigma}{\sigma}\,V_1\,\frac{1}{\omega L_1},$$

and $\qquad \widehat{V_1Ob} = \widehat{V_1OB}$, and $\widehat{V_1Op} = \widehat{V_1OP}$.

Thus

$$OP = \frac{M}{L_1} I_2$$

$$I_0 = \frac{V_1}{j\omega L_1}.$$

Draw $\qquad OO_1 = -\frac{V_1}{j\omega L_1}$, *i.e.*, $O_1O = \frac{V_1}{j\omega L_1} = I_0$,

when $\qquad O_1P = O_1O + OP = I_0 + \frac{M}{L_1} I_2 = I_1.$

It should be noted that

$$\frac{|\,OD\,|}{|\,I_0\,|} = \frac{\frac{1-\sigma}{\sigma}\,.\,V_1\,.\,\frac{1}{\omega L}}{\frac{V_1}{\omega L_1}}$$

$$= \frac{1-\sigma}{\sigma}.$$

If O_1O and O_1P are obtained experimentally, by an open circuit test and a test with a resistance load, and O_1O is produced to meet the perpendicular from P on OP in D, then the value of σ is found. If the secondary is short circuited, I_1 is equal to O_1B.

All the constants of the transformer may be found from open and short circuit tests. If the secondary is open circuited, an impedance measurement of the primary gives R_1 and L_1. If the primary is open circuited, the impedance of the secondary gives R_2 and L_2. If the secondary is short circuited, the effective impedance of the primary is $Z_1 + \frac{\omega^2 M^2}{Z_2}$, whence M is obtained. As a check the primary may be short circuited when the effective impedance of the secondary should be $Z_2 + \frac{\omega^2 M^2}{Z_1}$.

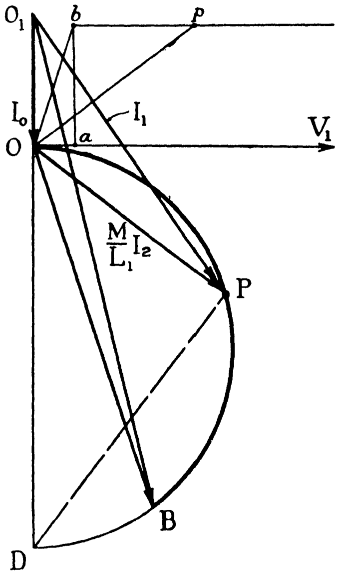


FIG. 47.—Vector diagram for equivalent network.

(2) Iron Core

The effects of introducing an iron core into the transformer are :—

(i.) The inductances of the windings are enormously increased owing to the very much greater flux associated with a given current.

(ii.) For the same reason the magnetising current is very much reduced.

(iii.) The leakage flux is very much reduced, owing to the far lower reluctance of the iron path than any air paths. What leakage flux there is has a " circuit " which is partly in air, and the reluctance of the leakage path is mainly due to the air.

(iv.) All reactances (ωL_1, ωL_2 and ωM) will have values which depend upon the values of the currents flowing, since the permeability of the iron is a function of the current. The reactances due to the leakage fluxes, however, will be constant, as their reluctances are mainly air reluctances which are constant.

(v.) Although the supply voltage is sinusoidal, the magnetising current is no longer sinusoidal owing to the shape of the hysteresis loop. The flux is, however, sinusoidal (neglecting the primary resistance voltage drop).

(vi.) The presence of the iron involves hysteresis and eddy current losses. Magnetisation no longer takes place without expenditure of energy.

(vii.) Since the inductances L_1 and L_2 arise from fluxes which have practically the same path, *i.e.*, the iron core, their ratio is equal to the square of the ratio of the numbers of turns of wire in the two windings ; *i.e.*, $L_1/L_2 = T_1^2/T_2^2$.

(i.) *Equivalent Circuits.*

The first three items involve no modification of the theory of the air core transformer to make it applicable to the iron core transformer. Item (iv.) can be dealt with in this way : Let the total inductance of the primary winding be separated into two parts : (*a*) that which is due to flux which also links the secondary winding —call this L_1 ; and (*b*) that which is due to flux which does not link the secondary winding—call this l_1. This is the leakage flux inductance, which as has been stated is a constant.

Similarly, let the secondary inductance be separated into two parts L_2 and l_2 ; L_2 due to flux which also links the primary and l_2 due to the leakage flux with a path whose reluctance is mainly air. It follows from these definitions that $L_1L_2 = M^2$, a relation which holds whatever the particular values of L_1 and L_2 corresponding to the currents flowing. It is further true that $L_1 >> l_1$ and $L_2 >> l_2$, and $L_1/L_2 = T_1^2/T_2^2$.

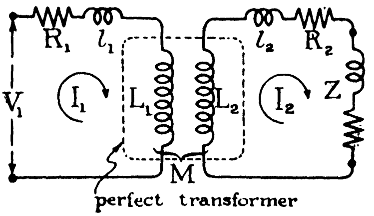


FIG. 48.—Iron core transformer.

The transformer is now drawn as in Fig. 48.

Under these circumstances

$$Z_1 = R_1 + j\omega l_1 + j\omega L_1$$
$$Z_2 = R_2 + j\omega l_2 + j\omega L_2,$$

and with a load Z

$$Z' = Z_1 + \frac{\omega^2 M^2}{Z_2 + Z},$$

$$= R_1 + j\omega l_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega l_2 + j\omega L_2 + Z},$$

$$= R_1 + j\omega l_1 + j\omega L_1 + \frac{\omega^2 M^2}{j\omega L_2\left(1 + \frac{R_2 + j\omega l_2 + Z}{j\omega L_2}\right)}$$

$$= R_1 + j\omega l_1 + j\omega L_1 - \frac{j\omega^2 M^2}{\omega L_2}\left(1 - \frac{R_2 + j\omega l_2 + Z}{j\omega L_2}\right),$$

on expansion by the Binomial Theorem, ωL_2 being much greater than $R_2 \omega l_2$ or Z at any but very small loads. Writing $M^2 = L_1 L_2$, this gives

$$Z' = R_1 + j\omega l_1 + \frac{L_1}{L_2}(R_2 + j\omega l_2 + Z),$$

$$= R_1 + j\omega l_1 + \frac{T_1^2}{T_2^2}(R_2 + j\omega l_2 + Z). \; . \; .$$

Similarly

$$Z'' = -\frac{j}{\omega M}\left\{\omega^2 M^2 + Z_1(Z_2 + Z)\right\}$$

$$= -\frac{j}{\omega M}\left\{\omega^2 M^2 + (R_1 + j\omega l_1 + j\omega L_1)\right.$$

$$\left.(R_2 + j\omega l_2 + j\omega L_2 + Z)\right\}$$

$$= -\frac{j}{\omega M}\left\{\omega^2 M^2 + j\omega L_1(R_2 + j\omega l_2 + Z)\right.$$

$$\left. + j\omega L_2(R_1 + j\omega l_1) + (R_1 + j\omega l_1)(R_2 + j\omega l_2 + Z) - \omega^2 L_1 L_2\right\}$$

and neglecting the product $(R_1 + j\omega l_1)(R_2 + j\omega l_2 + Z)$ as being small compared with the other terms,

$$Z'' = \sqrt{\frac{L_2}{L_1}}(R_1 + j\omega l_1) + \sqrt{\frac{L_1}{L_2}}(R_2 + j\omega l_2 + Z),$$

$$= \frac{T_2}{T_1}(R_1 + j\omega l_1) + \frac{T_1}{T_2}(R_2 + j\omega l_2 + Z).$$

The quantities L_1 L_2 and M which were awkward on account of their lack of constancy, have disappeared from the expressions for Z' and Z''.

The only reactance terms that appear in connection with the transformer windings are the leakage flux terms ωl_1 and ωl_2. It is probably for this reason that these reactances are sometimes referred to as though they were the total reactances of the windings, and l_1 and l_2 are sometimes referred to as though they were the total inductances of the windings; similarly $z_1 = R_1 + j\omega l_1$ and $z_2 = R_2 + j\omega l_2$ are called the impedances of the windings.

The approximations made in deriving the expressions for Z′ and Z″ involve ignoring the magnetising current. This is made evident by putting $Z = \infty$ (*i.e.*, open circuiting the secondary), when the expression for Z′ also becomes ∞ and in consequence the primary current is zero. Actually, however, the primary current is

$I_0 = V_1 \left\{ R_1 + j\omega(l_1 + L_1) \right\}$. This is, however, very small, and the error involved, except at very small loads, is quite insignificant.

Since $$I_1 = \frac{V_1}{Z'} \text{ and } I_2 = \frac{V_1}{Z''}$$

$$\frac{I_1}{I_2} = \frac{Z''}{Z'} = \frac{\frac{T_2}{T_1} R_1 + j\omega l_1 + \frac{T_1}{T_2}\left(R_2 + j\omega l_2 + Z\right)}{\left(R_1 + j\omega l_1\right) + \frac{T_1^2}{T_2^2}\left(R_2 + j\omega l_2 + Z\right)}$$

$$= \frac{T_2}{T_1}.$$

The e.m.fs. E_1 and E_2 induced in the two windings by the flux alternations must be proportional to the number of turns. That is

$$\frac{E_1}{E_2} = \frac{T_1}{T_2},$$

but the terminal potential differences depend to some extent on the currents flowing.

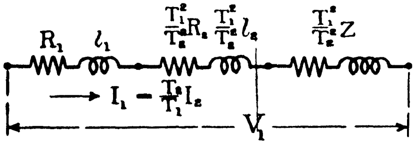


FIG. 49.—Equivalent circuit.

From the expression for Z′ the equivalent circuit of the transformer can be drawn as in Fig. 49. The voltage V_1 is applied across a simple series circuit containing R_1, l_1, and $\frac{T_1^2}{T_2^2}$ times R_2, l_2 and the load impedance Z. The latter terms are called the secondary and load impedance referred to the primary circuit. The current flowing through the circuit is $I_1 = \frac{T_2}{T_1} I_2$.

Comparing this circuit with that of Fig. 45, it is clear that the magnetising current has been ignored, since the coil L_1 is omitted. Otherwise the circuits are practically the same, when notice is taken of the new nomenclature. For writing the new symbols in Fig. 45,

i.e., for L_1 write $L_1 + l_1$,
for L_2 „ $L_2 + l_2$,
for M „ $\sqrt{L_1 L_2}$,

σ becomes

$$\sigma = 1 - \frac{M^2}{(L_1 + l_1)(L_2 + l_2)} = \frac{l_1 L_2 + l_2 L_1 + l_1 l_2}{(L_1 + l_1)(L_2 + l_2)}.$$

and the inductance $\frac{L_1^2}{M^2} L_2 \sigma$ becomes

$$\frac{(L_1 + l_1)^2}{L_1 L_2} \cdot \left(L_2 + l_2\right) \frac{l_1 L_2 + l_2 L_1 + l_1 l_2}{(L_1 + l_1)(L_2 + l_2)},$$

$$= \frac{L_1 + l_1}{L_1 L_2} \cdot (l_1 L_2 + l_2 L_1 + l_1 l_2),$$

$$= l_1 + \frac{L_1}{L_2} l_2 \quad \text{very nearly},$$

$$= l_1 + \frac{T_1^2}{T_2^2} l_2.$$

Also, since $\frac{L_1^2}{M^2}$ must be written $\frac{(L_1 + l_1)^2}{L_1 L_2}$, which is very nearly $\frac{L_1}{L_2} = \frac{T_1^2}{T_2^2}$, the remainder of the circuit is identical. It may be noted that very nearly

$$\sigma = \frac{l_1}{L_1} + \frac{l_2}{L_2}.$$

An equivalent circuit may also be drawn in which the impedances are referred to the secondary circuit.

For $$I_2 = \frac{V_1}{Z''}$$

$$= \frac{V_1}{\frac{T_2}{T_1}(R_1 + j\omega l_1) + \frac{T_1}{T_2}(R_2 + j\omega l_2 + Z)}$$

$$I_2 = \frac{\frac{T_2}{T_1}V_1}{\frac{T_2{}^2}{T_1{}^2}(R_1 + j\omega l_1) + (R_2 + j\omega l_2 + Z)}.$$

The equivalent circuit is drawn in Fig. 50. The applied voltage is $\frac{T_2}{T_1}V_1$, and the primary resistance and leakage inductance are "referred" by multiplication by $T_2{}^2/T_1{}^2$.

(ii.) *Regulation.*

These expressions and equivalent circuits indicate how the "regulation" of the transformer, *i.e.*, the drop of voltage for a given load, can be estimated from short circuit tests. If the effective impedance of the primary is measured with the secondary short circuited (in carrying out the test only just sufficient primary volts are applied to cause full load currents to flow), say Z_o, then Z_o is found from Z' by putting $Z = o$; *i.e.*,

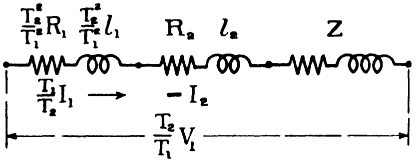


FIG. 50.—Circuit referred to secondary.

$$Z_o = R_1 + j\omega l_1 + \frac{T_1{}^2}{T_2{}^2}(R_2 + j\omega l_2)$$

and
$$I_2 = \frac{\frac{T_2}{T_1}V_1}{\left(\frac{T_2}{T_1}\right)^2 Z_o + Z}$$

or
$$I_2Z = \frac{T_2}{T_1}V_1 - \left(\frac{T_2}{T_1}\right)^2 Z_oI_2.$$

I_2Z is the voltage across the load, $\frac{T_2}{T_1}V_1$ is the no-load voltage, and the difference of these two is the regulation, generally expressed as a percentage of the no-load voltage.

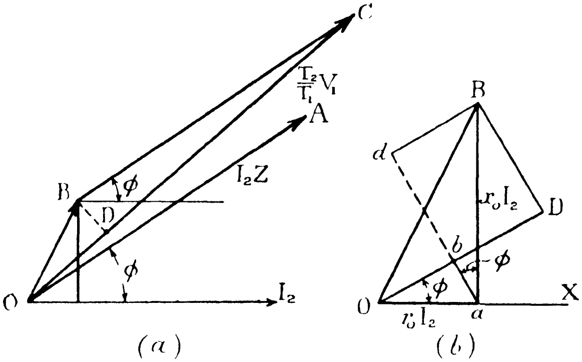
This expression is drawn in Fig. 51 for an inductive load. OI_2 is drawn horizontally, $OA = I_2Z$, the angle $I_2OA = \phi$ being that of the load, and $OB = \left(\frac{T_2}{T_1}\right)^2 Z_oI_2$ the angle I_2OB being that of Z_o.

BC is drawn equal and parallel to OA. Thus OC $= \frac{T_2}{T_1}V_1$, and the regulation

$$= \frac{|\,OC\,| - |\,OA\,|}{|\,OC\,|} \times 100\ \%.$$

And if BD is drawn perpendicular to OC, this very nearly equals $|\,OD/OC\,|$.

Writing $\left(\frac{T_2}{T_1}\right)^2 Z_o = r_o + jx_o$, and referring to Fig. 51 (*b*), which is an enlargement of the foot of Fig. 51 (*a*), $Oa = r_o I_2$, $aB = x_o I_2$.



(*a*) (*b*)

FIG. 51.—Regulation diagram.

Since OB is generally very much smaller than BC, the angle XOD will be very nearly ϕ. Drop *ab* perpendicular to OD and continue it to *d*, to meet B*d* drawn parallel to DO. Then the angle *da*B is also very nearly ϕ, and, considering only lengths,

$$\begin{aligned} OD &= Ob + bD \\ &= Ob + dB \\ &= r_o I_2 \cos\phi + x_o I_2 \sin\phi. \end{aligned}$$

and since BC is not very different from OC, the regulation

$$\doteqdot \frac{OD}{BC} = \frac{1}{|\,Z\,|}(r_o \cos\phi + x_o \sin\phi) \times 100\ \%.$$

If the load is non-inductive, $\phi = o$ and the regulation depends upon r_o and not at all upon x_o. On the other hand, if the load is very inductive, ϕ is large, $\sin\phi$ is large and $\cos\phi$ small, and the regulation depends more upon x_o than upon r_o. If the load is capacitative, ϕ is negative, sin ϕ is negative and the regulation may be negative ; *i.e.*, the voltage may rise as the load comes on.

These differences are brought out in the diagram of Fig. 52. OB is drawn as before equal to $(T_2/T_1)^2Z_oI_2$. With radius equal to $(T_2/T_1)V_1$ two circles are drawn, one $(C_1C_2C_3)$ with centre O, and one $(C_1'C_2'C_3')$ with centre B.

Three cases are drawn. In the first the load is inductive with an angle ϕ_1. $BC_1 = I_2Z_2 =$ the terminal voltage and $OC_1 =$ the secondary e.m.f. $= (T_2/T_1)V_1$. BC_1 is continued to meet the circle with centre B in C_1'. Evidently since $BC_1' = OC_1$, C_1C_1' is the fall of voltage. In the second case the load is non-inductive, BC_2 is the terminal voltage and the drop is C_2C_2'. In the third case the load is capacitative with a negative angle ϕ_3. BC_3 is the terminal voltage, and C_3C_3' the rise of voltage when the load comes on.

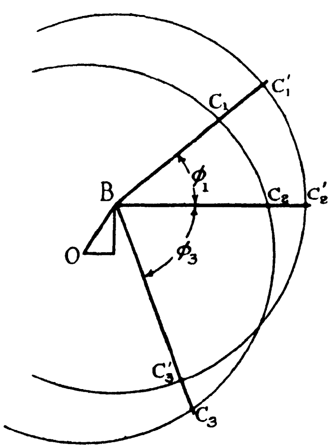


FIG. 52.—Kapp's Regulation Diagram.

This is Kapp's regulation diagram. As drawn, the regulation is very much greater than would be the case in practice.

The diagram could equally well have been drawn from Fig. 51, *i.e.*, referring all quantities to the primary side. The practical quantities are such that the result obtained would be nearly identical. Moreover, it does not matter from which side of the transformer the short circuit test is made. If the primary winding is short circuited the effective impedance Z_o' of the secondary is

$$Z_o' = R_2 + j\omega l_2 + \frac{T_2^2}{T_1^2}(R_1 + j\omega l_1)$$
$$= \frac{T_2^2}{T_1^2} Z_o$$

and the construction proceeds as before.

It is clear that if transformers are to be connected in parallel they must have the same regulation, otherwise they would develop unequal terminal voltages or load if they were equally loaded ; which results in their sharing the load unequally since the terminal voltages are the same. This is true both of the magnitude and phase angle of the regulation, although the phase angle differences are so small as not to matter in single phase transformers. The point must, however, be borne in mind when considering the parallel connection of three phase transformers.

(iii.) *Magnetising Current.*

Item (v.) of the differences noted between the air core and iron core transformer has now been disposed of. The magnetising current in a power transformer is so small as not to affect appreciably the vector diagram of the loaded transformer. Since this is so its lack of sine wave shape is of no importance.

On no load, however, the magnetising current is the only current flowing, and here the iron losses noted in item (vi.) are of importance. If a no-load test is carried out in which the current I_0 into the transformer at the supply voltage V_1 is measured, and the power in watts W_0 supplied is also measured, then the power factor $\cos\phi$ of the unloaded transformer can be found from

$$\cos\phi = \frac{W_0}{I_0 V_1}.$$

It is then possible to imagine the magnetising current replaced by an equivalent sine wave current, whose r.m.s. value is the same, and which lags behind the voltage by an angle ϕ determined from the above expression. If now this current is resolved (Fig. 53) into two components OI_p in phase with V_1 and OI_m at right-angles, I_p is the power component of the no-load current, and I_m the magnetising component.

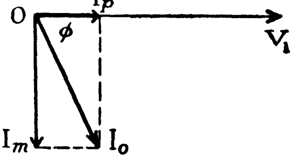


FIG. 53.

In the same way as with the air core transformer it can be shown that neglecting the primary resistance the magnetising current is a constant at all loads, since the flux must be constant. It follows that the iron losses are constant.

(iv.) *Flux Equation.*

Neglecting R_1, the flux and the supply voltage are related by the expression

$$V_1 = j\omega\Lambda_1 = j\omega\Phi T_1 \quad . \quad . \quad . \quad . \quad . \quad \text{(p. 52)}$$

If, as is usual, the equation is written in terms of the maximum value of the flux (Φ_m) the root mean square value of the voltage and the frequency, it becomes

$$\begin{aligned} |\mathrm{V}_1| &= 2\pi f \frac{\Phi_m}{\sqrt{2}} \mathrm{T}_1 \\ &= 4{\cdot}44 f \Phi_m \mathrm{T}_1 \end{aligned}$$

and if V_1 is in volts and Φ_m as usual in c.g.s. units,

$$|\mathrm{V}_1| = 4{\cdot}44 f \Phi_m \mathrm{T}_1 \times 10^{-8}.$$

This equation is of importance in the design of the iron core, as it gives the value of the flux required.

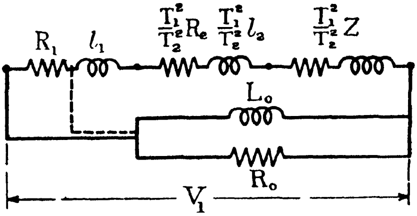


FIG. 54.—Equivalent network.

The equivalent circuit diagram of the transformer can now be modified as in Fig. 54 to take account of the magnetising and iron loss currents. An inductance L_o and a resistance R_o are connected in parallel across the supply. Since

$$\mathrm{I}_m = \frac{\mathrm{V}_1}{j\omega \mathrm{L}_o}, \qquad \mathrm{L}_o = \frac{|\mathrm{V}_1|}{\omega |\mathrm{I}_m|}$$

and since $\mathrm{I}_p = \frac{\mathrm{V}_1}{\mathrm{R}_o}, \qquad \mathrm{R}_o = \frac{|\mathrm{V}_1|}{|\mathrm{I}_p|}.$

These would be more correctly connected as shown by the dotted line (compare Fig. 45), but the error involved in the simpler connection shown is small.

The total losses at any load are made up of the constant iron losses W_o, and the copper losses which vary as the square of the load current.

Since at no load the currents are very small, the power required at the supply voltage is practically the iron loss. And since in the short circuit test the applied voltage is very small for full load currents to flow, the flux also is very small, and hence also the iron losses. Thus the power required for the short circuit test is practically the copper loss. In this way the open circuit and short circuit tests give all the information required to find the efficiency of the transformer at any load.

(3) **Instrument Transformers**

In transformers to be used in conjunction with voltmeters and ammeters in high voltage or large current circuits, the important consideration is that the ratio of transformation shall remain constant, while when the transformers are to be used with wattmeters, the phase angle departures are also of importance.

(i.) *Potential Transformer*

Instrument transformers are divided into two classes—potential transformers and current transformers. The theory of the potential transformer is the same as that of the power transformer already considered, and the regulation diagram will give the transformation ratio and the phase angle difference, neglecting magnetising current. This is not always justified, however, since the impedances of the voltmeter and wattmeter coils used in the secondary circuit are high, and the potential transformer is in consequence always very "lightly loaded."

The accurate expression for the voltage ratio is, (p. 50)

$$\frac{V_1}{V_2} = \frac{\omega^2 M^2 + Z_1(Z_2 + Z)}{j\omega M Z}.$$

Writing

$$Z_1 = z_1 + j\omega L_1 = R_1 + j\omega l_1 + j\omega L_1$$
$$Z_2 = z_2 + j\omega L_2 = R_2 + j\omega l_2 + j\omega L_2$$
$$\text{and } M = \sqrt{L_1 L_2}.$$

This gives

$$\frac{V_1}{V_2} = \frac{\omega^2 L_1 L_2 + (z_1 + j\omega L_1)(z_2 + j\omega L_2 + Z)}{j\omega\sqrt{L_1 L_2}\,.\,Z}$$
$$= \sqrt{\frac{L_1}{L_2}} + \frac{1}{Z}\left(\sqrt{\frac{L_2}{L_1}}\,.\,z_1 + \sqrt{\frac{L_1}{L_2}}\,.\,z_2\right) + \frac{z_1}{j\omega\sqrt{L_1 L_2}}\left(1 + \frac{z_2}{Z}\right).$$

If Z_0 is the short circuit impedance measured from the primary side,

$$Z_0 = z_1 + \frac{L_1}{L_2} z_2, \text{ and } \sqrt{\frac{L_2}{L_1}}\,.\,Z_0 = \sqrt{\frac{L_2}{L_1}}\,.\,z_1 + \sqrt{\frac{L_1}{L_2}}\,.\,z_2$$

and

$$\frac{V_1}{V_2} = \sqrt{\frac{L_1}{L_2}}\left\{1 + \sqrt{\frac{L_2}{L_1}}\,.\,\frac{Z_0}{Z} + \frac{z_1}{j\omega L_1}\left(1 + \frac{z_2}{Z}\right)\right\}.$$

Expressed in the form $r\angle\phi$, r gives the ratio of transformation, and ϕ the phase angle between V_1 and V_2.

If Z is large, z_2/Z can be neglected. The term $z_1/j\omega L_1$ depends upon the magnetising current, and the remainder of the expression has been obtained in the power transformer case.

Arranged in the form

$$\frac{V_1}{V_2} = \sqrt{\frac{L_1}{L_2}}\left\{1 + \frac{z_1}{j\omega L_1} + \frac{1}{Z}\left(\sqrt{\frac{L_2}{L_1}}\,.\,Z_0 + \frac{z_1 z_2}{j\omega L_1}\right)\right\}$$

the importance of having a large instrument impedance Z in order that the voltage ratio shall remain constant, is made clearer. This expression should be compared with the corresponding air core expression on p. 50, which was obtained on the assumption of no primary resistance.

(ii.) *Current Transformer.*

In the case of the current transformer the current ratio is found (p. 51) from

$$\begin{aligned}\frac{I_1}{I_2} &= -\frac{j}{\omega M}(Z_2 + Z)\\ &= -\frac{j}{\omega\sqrt{L_1 L_2}}(z_2 + j\omega L_2 + Z)\\ &= \sqrt{\frac{L_2}{L_1}} - \frac{j}{\omega\sqrt{L_1 L_2}}(z_2 + Z)\\ &= \sqrt{\frac{L_2}{L_1}}\left\{1 - \frac{j}{\omega L_2}(z_2 + Z)\right\}\end{aligned}$$

If $$Z = R + j\omega L,$$

$$\frac{I_1}{I_2} = \sqrt{\frac{L_2}{L_1}}\left\{1 + \frac{l_2 + L}{L_2} - j\,\frac{R_2 + R}{\omega L_2}\right\}$$

The leakage inductance and inductance of the instrument affect the ratio chiefly, while the influence of the resistances (of secondary and load) is chiefly on the phase angle difference. All quantities l_2 L R_2 and R should be kept small for the best results.

The primary current through the current transformer is quite independent of the secondary load. The magnetising ampere turns therefore, if the secondary were open circuited, would increase enormously, resulting in iron saturation and overheating through increased iron losses. The precaution should always be taken of shorting either the primary or the secondary terminals of the transformer before disconnecting the instrument.

(4) **Transformers in Parallel**

If two transformers are connected in parallel to the same supply voltage on the primary side and to the same load on the secondary side, the circuit diagram can be drawn as in Fig. 55. If the subscripts 1 and 2 refer to the primary and secondary respectively of the first transformer, and the subscripts 3 and 4 to the primary and secondary of the second transformer, and if M_{12} and M_{34} are the mutual inductances between the windings of the first and second transformers respectively, the circuital equations can be written as :—

$$V = Z_1I_1 - j\omega M_{12}I_2 \quad \ldots \ldots \quad \text{(i)}$$
$$0 = -j\omega M_{12}I_1 + Z_2I_2 + Z(I_2 + I_4) \quad \ldots \quad \text{(ii)}$$
$$V = Z_3I_3 - j\omega M_{34}I_4 \quad \ldots \ldots \quad \text{(iii)}$$
$$0 = -j\omega M_{34}I_3 + Z_4I_4 + Z(I_4 + I_2) \quad \ldots \quad \text{(iv)}$$

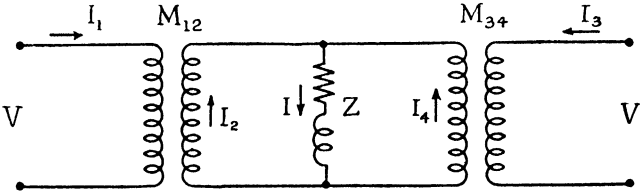


FIG. 55. Transformers in parallel.

Eliminating I_1 from (i) and (ii) gives

$$j\omega M_{12}V = \{\omega^2M_{12}^2 + Z_1(Z_2 + Z)\}I_2 + Z_1ZI_4$$

Approximating as before,

$$\begin{aligned}
&\omega^2M_{12}^2 + Z_1(Z_2 + Z) \\
&= \omega^2M_{12}^2 + (R_1 + j\omega l_1 + j\omega L_1)(R_2 + j\omega l_2 + j\omega L_2) + \\
&\qquad (R_1 + j\omega l_1 + j\omega L_1)Z \\
&\doteqdot j\omega L_2(R_1 + j\omega l_1) + j\omega L_1(R_2 + j\omega l_2) + j\omega L_1Z
\end{aligned}$$

and $$Z_1Z \doteqdot j\omega L_1Z$$

whence $$V = \left\{\sqrt{\frac{L_1}{L_2}}(R_2 + j\omega l_2 + Z) + \sqrt{\frac{L_2}{L_1}}(R_1 + j\omega l_1)\right\}I_2 + \sqrt{\frac{L_1}{L_2}}\cdot ZI_4$$

or $V = \{K_{12}(R_2 + j\omega l_2 + Z) + K_{21}(R_1 + j\omega l_1)\} I_2 + K_{12} Z I_4$. (v)

where $K_{12} = \sqrt{\frac{L_1}{L_2}} = \frac{T_1}{T_2}$, $K_{21} = \sqrt{\frac{L_2}{L_1}} = \frac{T_2}{T_1}$.

Writing also,

$$K_{34} = \sqrt{\frac{L_3}{L_4}} = \frac{T_3}{T_4} \text{ and } K_{43} = \sqrt{\frac{L_4}{L_3}} = \frac{T_4}{T_3}$$

and eliminating I_3 from (iii) and (iv) gives in the same way

$V = \{K_{34}(R_4 + j\omega l_4 + Z) + K_{43}(R_3 + j\omega l_3)\} I_4 + K_{34} Z I_2$. . (vi)

Writing $K_{12}(R_2 + j\omega l_2) + K_{21}(R_1 + j\omega l_1) = Z_{12}''$
and $K_{34}(R_4 + j\omega l_4) + K_{43}(R_3 + j\omega l_3) = Z_{34}''$

Z_{12}'' and Z_{34}'' are the impedances of the transformers for determining their secondary currents, and (v) and (vi) become

$$V = Z_{12}''I_2 + K_{12}Z(I_2 + I_4)$$

and $$V = Z_{34}''I_4 + K_{34}Z(I_2 + I_4)$$

giving for I_2 and I_4

$$I_2 = \frac{Z_{34}'' + (K_{34} - K_{12})Z}{Z_{34}''Z_{12}'' + (K_{12}Z_{34}'' + K_{34}Z_{12}'')Z} V \quad . \quad . \quad . \text{(vii)}$$

$$I_4 = \frac{Z_{12}'' + (K_{12} - K_{34})Z}{Z_{12}''Z_{34}'' + (K_{34}Z_{12}'' + K_{12}Z_{34}'')Z} V \quad . \quad . \quad \text{(viii)}$$

and the total load current is

$$I_2 + I_4 = \frac{Z_{12}'' + Z_{34}''}{Z_{12}''Z_{34}'' + (K_{34}Z_{12}'' + K_{12}Z_{34}'')Z} V.$$

If the transformers are identical, $Z_{12}'' = Z_{34}''$ and $K_{12} = K_{34}$ and

$$I_2 = I_4 = V/(Z_{12}'' + 2K_{12}Z)$$

and $$I = I_2 + I_4 = 2I_2 = V \Big/ \left(\frac{Z_{12}''}{2} + K_{12}Z\right)$$

as would be expected.

If, without having identical impedances, the turns ratios of the transformers are the same, so that $K_{12} = K_{34}$

$$I_2 = V \Big/ \left\{ Z_{12}'' + K_{12}Z\left(1 + \frac{Z_{12}''}{Z_{34}''}\right) \right\}$$

and $$I_4 = V \Big/ \left\{ Z_{34}'' + K_{34}Z\left(1 + \frac{Z_{34}''}{Z_{12}''}\right) \right\}$$

The primary currents are readily found by substitution from (vii) and (viii) in (i) and (iii) respectively.

Thus the primary current of the first transformer is found to be

$$I_1 = \frac{V}{Z_1} + V\frac{j\omega M_{12}}{Z_1} \cdot \frac{Z_{34}'' + (K_{34} - K_{12})Z}{Z_{34}''Z_{12}'' + (K_{12}Z_{34}'' + K_{34}Z_{12}'')Z}$$

Writing $Z_{12}' = R_1 + j\omega l_1 + K_{12}^2(R_2 + j\omega l_2) = K_{12}Z_{12}''$
and $Z_{34}' = R_3 + j\omega l_3 + K_{34}^2(R_4 + j\omega l_4) = K_{34}Z_{34}''$

and neglecting the first term $\frac{V}{Z_1}$, which is the magnetising current, and writing

$$j\omega M_{12}/Z_1 \doteq M_{12}/L_1 = \sqrt{L_2/L_1} = K_{21}$$

the expression becomes

$$I_1 = V\frac{Z_{34}' + (K_{34}^2 - K_{12}K_{34})Z}{Z_{34}'Z_{12}' + (K_{12}^2Z_{34}' + K_{34}^2Z_{12}')Z} \quad . \quad . \quad . \quad \text{(ix)}$$

and similarly

$$I_3 = V\frac{Z_{12}' + (K_{12}^2 - K_{34}K_{12})Z}{Z_{12}'Z_{34}' + (K_{34}^2Z_{12}' + K_{12}^2Z_{34}')Z} \quad . \quad . \quad . \quad \text{(x)}$$

The total current taken from the supply is

$$I_1 + I_3 = V\frac{Z_{12}' + Z_{34}' + (K_{12} - K_{34})^2Z}{Z_{12}'Z_{34}' + (K_{12}^2Z_{34}' + K_{34}^2Z_{12}')Z}.$$

If the transformers are identical,

$$I_1 = I_3 = \frac{V}{Z_{12}' + 2K_{12}^2Z}$$

and the total supply current is

$$I_1 + I_3 = \frac{V}{\frac{Z_{12}'}{2} + K_{12}^2Z}$$

as would be expected.

If only the ratio of turns is identical,

$$I_1 = \frac{V}{Z_{12}' + K_{12}^2Z\left(1 + \frac{Z_{12}'}{Z_{34}'}\right)}$$

and

$$I_3 = \frac{V}{Z_{34}' + K_{34}^2Z\left(1 + \frac{Z_{34}'}{Z_{12}'}\right)}.$$

If there is no load there will still in general be a primary and secondary current in addition to the primary magnetising current. These currents are found by putting $Z = \infty$ in equations (viii) to (x), and are

$$I_2 = -I_4 = \frac{(K_{34} - K_{12})V}{K_{12}Z_{34}'' + K_{34}Z_{12}''}$$

and

$$I_1 = \frac{(1 - K_{12}K_{43})V}{Z_{12}' + K_{12}^2K_{43}^2Z_{34}'}$$

$$I_3 = \frac{(1 - K_{34}K_{21})V}{Z_{34}' + K_{34}^2K_{21}^2Z_{12}'}.$$

With a knowledge of the currents flowing obtained from the above expressions, it is a simple matter by finding the dot products with the voltages concerned to determine the power expended in various parts of the circuit. The problem of most general interest in this connection is the manner in which the load is shared by the two transformers.

EXERCISES

(1) An air-core transformer has a primary inductance of 500 microhenries and a secondary inductance of 100 microhenries, and the ratio of resistance to inductance is 20,000 in each case at an ω value of 10^5. If the leakage coefficient is 0·2, find the voltage ratio of the transformer with the following loads :—

(i.) Open circuit.
(ii.) A non-inductive resistance of 10 ohms.
(iii.) A non-inductive resistance of 1 ohm.
(iv.) An inductive resistance with $L = 50 \times 10^{-6}$ henry, $R = 2$ ohms.
(v.) A condenser of 0·05 μF.
(vi.) A condenser of 0·2 μF.

Find also in each case the secondary current for a primary potential difference of 10 volts, and the current ratio.

(2) Find the various equivalent networks of the above transformer.

(3) Draw the vector locus of the transformer of Q. 1 with varying non-inductive load.

(4) The turns-ratio of a 50-cycle transformer is 6, and the primary

and secondary resistances are 1·0 and 0·04 ohms respectively, and the reactances are 6 ohms and 0·15 ohms respectively. Find

(i.) The voltage necessary on the high voltage side to produce 200 amps. in the short-circuited low-voltage winding.

(ii.) The power factor on short circuit.

(5) A 10 kVA, 400/800-volt, 50-cycle transformer tested from the low-voltage side, took 120 watts with 0·8 ampere and 400 volts on no load, and tested from the high-voltage side, took 50 watts with 6 amperes and 9 volts on short circuit.

Find (i.) the magnetising current and the iron loss current at normal voltage and frequency.

(ii.) The efficiency on full load at unity power factor.

(iii.) The secondary terminal voltage on full load current at power factors of unity, 0·7 lagging and 0·7 leading.

(6) A transformer has resistance and reactance drops of 3·0 per cent. and 6 per cent. respectively. Plot a curve showing how the voltage regulation on full load varies as the power factor changes from zero lagging to zero leading.

(7) Two 50-kVA transformers each having a voltage ratio of 4 are connected in parallel to supply loads of

(i.) 90 kW. at 0·9 lagging p.f.
(ii.) 60 kW. at 0·7 lagging p.f.
(iii.) 100 kW. at unity p.f.

Find in each case how the load will be shared.

One transformer has a resistance drop of 0·5 per cent. and a reactance drop of 8 per cent., and the other a resistance drop of 0·75 per cent. and a reactance drop of 4 per cent., all on full load.

(8) A 500-kVA transformer and a 250-kVA transformer are connected in parallel.

The 500-kVA transformer has a resistance drop of 1 per cent. and a reactance drop of 4 per cent., and an open circuit secondary potential difference of 500 volts. The corresponding figures for the 250-kVA transformer are 1·5 per cent., 6 per cent. and 510 volts.

Find (i.) the cross current in the secondaries on no load.

(ii.) The secondary currents in each transformer when the load is 700 watts at unity power factor.

CHAPTER V

THREE-PHASE CURRENTS

(1) Three Phase Circuits

In the first chapter the use of a rotating vector to represent an alternating current was developed from the generation of an alternating electro-motive force in a coil of wire rotated in a uniform magnetic field (Fig. 1). Suppose now that there are three equal coils of wire rigidly fixed to the same axle in such a manner that the coil sides are equally distributed round the circumference of the cylinder traced by each when they are revolved; in other words, the angle between the planes of adjacent coils is 60° (Fig. 56).

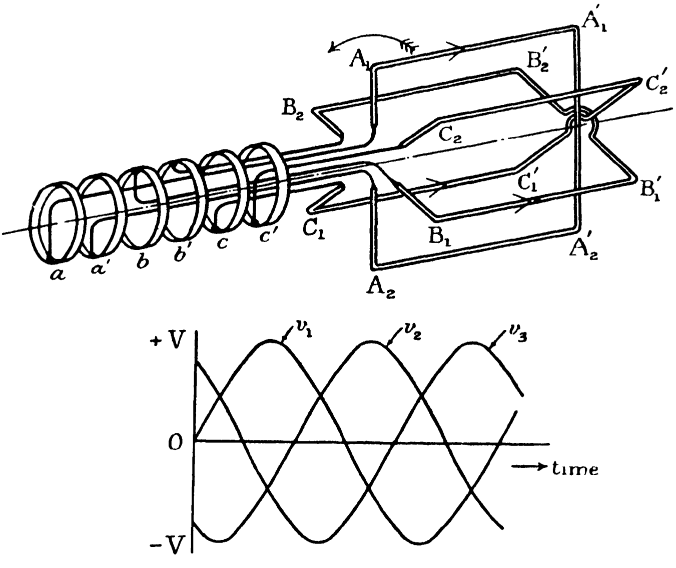


Fig. 56.—Three-phase generator.

Let the positive directions for induced e.m.fs. be $A_1A_1'A_2'A_2$, $B_1B_1'B_2'B_2$ and $C_1C_1'C_2'C_2$, where the letters have the significance indicated in the diagram drawn at a particular instant. It will be seen that the whole arrangement must rotate through 120° before the next coil occupies the place (both with regard to position and direction of induced e.m.f.) of the coil ahead of it in the diagram. Thus after rotation through 120°, B_1 lies where A_1 now lies, C_1 lies where B_1 now lies and A_1 lies where C_1 now lies. It follows that if OV_1, Fig. 59, represents the e.m.f. induced by the field in the A coil, OV_2 must represent the e.m.f. in the B coil and OV_3 the e.m.f. in the C coil, the positive directions being as defined above, and the angles V_1OV_2, V_2OV_3 and V_3OV_1 each being equal to 120°. Remembering that multiplying a vector by $\epsilon^{j\theta}$ rotates it through an angle θ in the positive direction, it is clear that the three e.m.f. vectors are related by the expressions :

$$V_1 = V_2\epsilon^{j120^\circ} = V_3\epsilon^{j240^\circ}$$
$$V_2 = V_3\epsilon^{j120^\circ}$$
$$V_3 = V_1\epsilon^{j120}$$

If the terminals of the three coils are brought out to six slip rings as shown in Fig. 56, the connections being such that the coil sides $A_1B_1C_1$ are connected to the slip rings *a*, *b*, *c*, while the coil sides A_2, B_2 and C_2 are connected to separate slip rings, *a′*, *b′* and *c′*, then OV_1 represents the voltage between *a* and *a′*, OV_2 the voltage between *b* and *b′*, and OV_3 the voltage between *c* and *c′*. The three coils are said to be " star " connected, when *a′*, *b′* and *c′* are connected together, and " mesh " connected when *b′* is connected to *a*, *c′* to *b* and *a′* to *c*. Each coil is referred to as a " phase," and the whole arrangement is a very elementary " three-phase generator," or generator of three-phase currents.

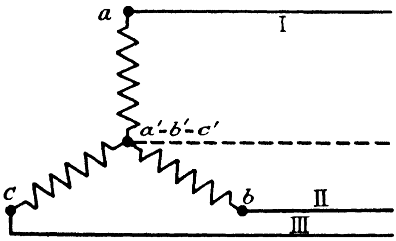


FIG. 57.—Star connection.

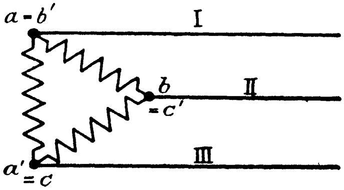


FIG. 58.—Mesh connection.

The star and mesh connections are indicated in Figs. 57 and 58 respectively. In each case a lead is taken from each of the slip rings *a*, *b* and *c*, and these three wires together are known as " the line " across which loads may be connected. The voltage between two of the wires is known as the line voltage, and the current in the wires as the line current. It is at once evident that in the star-connected arrangement the line current is the same as the phase current, and in the mesh-connected arrangement the line voltage is the same as the phase voltage, but that the line and phase voltages are different in the star arrangement, and the line and phase currents are different in the mesh arrangement.

In addition to the three wires in the star arrangement, a wire may also be taken out from the connected slip rings $a' = b' = c'$; this is generally earthed and called the neutral. The voltage from any wire to the neutral is the phase voltage.

The line voltages in the star arrangement are readily found in terms of the phase voltages.

The voltage from a to a' is V_1, and from b to $b' = a'$ is V_2. The voltage therefore from a to b is $V_1 - V_2$, and this is the voltage from Line I to Line II, say V_{12}.

Thus
$$\begin{aligned} V_{12} &= V_1 - V_2 \\ &= V_2\epsilon^{j120} - V_2 \\ &= V_2(\epsilon^{j120} - 1) \\ &= V_2(\cos 120^\circ + j \sin 120^\circ - 1) \\ &= V_2\left(-\tfrac{1}{2} + j\frac{\sqrt{3}}{2} - 1\right) \\ &= \sqrt{3}V_2\left(-\frac{\sqrt{3}}{2} + j\tfrac{1}{2}\right) \\ &= \sqrt{3}V_2(\cos 150^\circ + j \sin 150^\circ) \\ &= \sqrt{3}V_2\epsilon^{j150} \end{aligned}$$

Hence V_{12} is drawn as OV_{12} (Fig. 59), having a length equal to $\sqrt{3}$ times the length of OV_1, OV_2 or OV_3, and at an angle 150° in front of V_2 or 30° in front of V_1.

Similarly $\qquad V_{23} = \sqrt{3}V_3\epsilon^{j150}$

and $\qquad V_{31} = \sqrt{3}V_1\epsilon^{j150}$

As actually drawn in Fig. 59, if $|V|$ is the phase voltage magnitude (*i.e.*, V is taken as reference phase),

$$
\begin{aligned}
V_1 &= jV = V\epsilon^{j\frac{\pi}{2}}\\
V_2 &= V_1\epsilon^{-j120^\circ} = V\epsilon^{j(-120^\circ+\frac{\pi}{2})} = V\epsilon^{-j30^\circ}\\
V_3 &= V_1\epsilon^{j120} = V\epsilon^{j(120+90)} = V\epsilon^{j210^\circ} = V\epsilon^{-j150^\circ}\\
V_{12} &= \sqrt{3}V_2\epsilon^{j150} = \sqrt{3}V\epsilon^{j(150-30)} = \sqrt{3}V\epsilon^{j120^\circ}\\
V_{23} &= \sqrt{3}V_3\epsilon^{j150} = \sqrt{3}V\epsilon^{j(150-150)} = \sqrt{3}V\\
V_{31} &= \sqrt{3}V_1\epsilon^{j150} = \sqrt{3}V\epsilon^{(150+90)} = \sqrt{3}V\epsilon^{j240}
\end{aligned}
$$

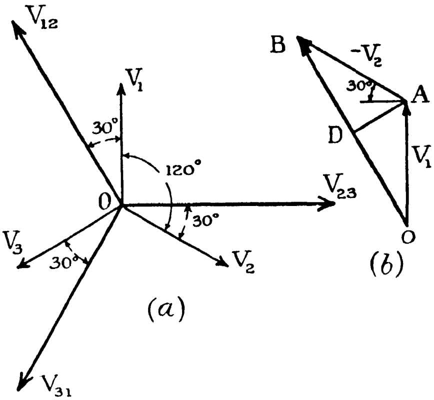


FIG. 59.—Phase and line voltages—star connection.

Note that $\quad V_1 + V_2 + V_3 = 0$
and $\quad V_{12} + V_{23} + V_{31} = 0$

These results are easily obtained geometrically, as indicated in Fig. 59 (*b*) for V_{12}. $OA = V_1$, $AB = -V_2$, and $OB = OA + AB = V_1 - V_2 = V_{12}$.

Draw AD perpendicular to OB. Since BA is drawn at an angle 30° below the horizontal, $\widehat{BAO} = 90 + 30 = 120°$. Also since $|AB| = |AO|$, $\widehat{ABD} = \widehat{AOD} = \frac{1}{2}(180 - 120) = 30°$.

$\therefore \quad |BD| = |DO| = |AB| \cos 30 = \dfrac{\sqrt{3}}{2}|V|$

$\therefore \quad |OB| = \sqrt{3}|V|$ and OB is drawn at an angle 30 to the vertical, or 120 to the horizontal.

Hence $V_{12} = OB = \sqrt{3}V\epsilon^{j120}$ as before.

In the mesh-connected generator (which is rarely met with in practice, largely owing to the danger of circulating currents in case of inequality of the phase voltages) the line currents will be the differences of the phase currents. Thus if in Fig. 58 the current from *a′* to *a* is I_a within the generator, (the current flows from *a* to *a′* in the external circuit) and that from *b′* to *b* is I_b, the current out to line I is clearly $I_a - I_b$; and similarly the current to line II is $I_b - I_c$, and that to line III is $I_c - I_a$. If the phase currents are balanced, *i.e.*, if they have the same magnitude and their vectors are displaced by angles of 120°, then the same geometrical conclusions come to with regard to line and phase voltages in the star case hold with regard to line and phase currents in the mesh case.

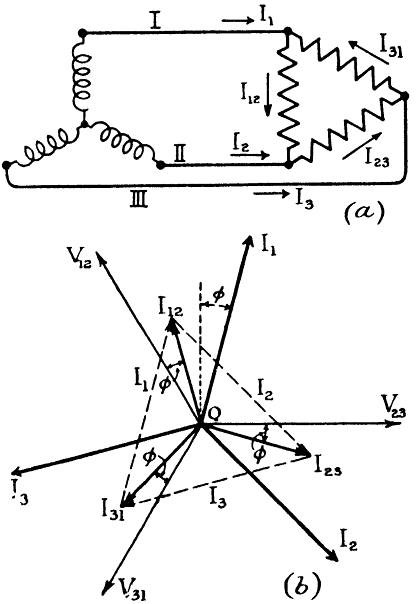


FIG. 60.—Balanced mesh-connected load.

Similar considerations hold in the case of a balanced mesh connected load on a star-connected alternator, indicated in Fig. 60 (*a*). Here the line currents are indicated by I_1, I_2 and I_3, with positive directions into the load, and the currents in the three phases of the load are I_{12}, I_{23} and I_{31} in sequence round the mesh, the positive directions agreeing with the positive directions of the line-to-line voltages. The vector diagram is drawn in Fig. 60 (*b*) for a load having an impedance $|Z|\angle\phi$. $|I_{12}| = |V_{12}|/|Z|$, and I_{12} is drawn as shown, lagging by an angle $\phi°$ behind V_{12}. Similarly I_{23} lags $\phi°$ behind V_{23} and I_{31} $\phi°$ behind V_{31}.

If V_l is the magnitude of the line voltage, in the diagram as drawn,

$$V_{23} = V_l$$

$$I_{12} = \frac{V_l}{Z}\epsilon^{j120} = \frac{V_l}{|Z|}\epsilon^{j(120-\phi)}$$

$$I_{23} = \frac{V_l}{|Z|}\epsilon^{-j\phi}$$

$$I_{31} = \frac{V_l}{|Z|}\epsilon^{j(240-\phi)}$$

The line currents are found as follows :—

$$
\begin{aligned}
I_1 &= I_{12} - I_{31} \\
&= \frac{V_l}{|Z|}\left(\epsilon^{j(120-\phi)} - \epsilon^{j(240-\phi)}\right) \\
&= \frac{V_l}{|Z|}\epsilon^{-j\phi}\left(\epsilon^{j120} - \epsilon^{j240}\right) \\
&= \frac{V_l}{|Z|}\epsilon^{-j\phi}(\cos 120 + j\sin 120 - \cos 240 - j\sin 240) \\
&= \frac{|V_l|}{Z}\epsilon^{-j\phi}\left\{-\frac{1}{2} + j\frac{\sqrt{3}}{2} - \left(-\frac{1}{2}\right) - j\left(-\frac{\sqrt{3}}{2}\right)\right\} \\
&= \frac{V_l}{|Z|}\epsilon^{-j\phi}\,.\,j\sqrt{3} \\
&= \sqrt{3}\,\frac{V_l}{|Z|}\epsilon^{j(90-\phi)}
\end{aligned}
$$

Similarly

$$I_2 = I_{23} - I_{12} = \sqrt{3}\,\frac{V_l}{|Z|}\epsilon^{j(330-\phi)}$$

$$I_3 = I_{31} - I_{23} = \sqrt{3}\,\frac{V_l}{|Z|}\epsilon^{j(210-\phi)}$$

The line current vectors are each equal in magnitude to $\sqrt{3}V_l/|Z|$, *i.e.*, to $\sqrt{3}$ times the phase current magnitude, and displaced from each other by angles of 120°, and lag behind the line voltage vectors by angles of $(30° + \phi)$.

Geometrically it is seen that the simplest way in which to obtain the line currents is to join the extremity of OI_{31} to that of OI_{12} to obtain I_1, the extremity of I_{12} to that of I_{23} to obtain I_2, and the extremity of I_{23} to that of I_{31} to obtain I_3.

The general argument and construction hold even when the loads are not equal. If $Z_{12} = |Z_{12}|\angle\phi_{12}$, $Z_{23} = |Z_{23}|\angle\phi_{23}$ and $Z_{31} = |Z_{31}|\angle\phi_{31}$ are the load, then

$$I_{12} = \frac{V_l\epsilon^{j120}}{Z_{12}} = \frac{V_l}{|Z_{12}|}\epsilon^{j(120-\phi_{12})}$$

$$I_{23} = \frac{V_l}{Z_{23}} = \frac{V_l}{|Z_{23}|}\epsilon^{-j\phi_{23}}$$

$$I_{31} = \frac{V_l\epsilon^{j240}}{Z_{31}} = \frac{V_l}{|Z_{31}|}\epsilon^{j(240-\phi_{31}}$$

And

$$\begin{aligned} I_1 &= I_{12} - I_{31} \\ I_2 &= I_{23} - I_{12} \\ I_3 &= I_{31} - I_{23} \end{aligned}$$

as before.

If the load is star connected (Fig. 61) the neutral wire may or may not be brought to the load. If it is, the voltage across each phase of the load is the phase voltage and the current into each phase of the load is the line current. Each phase can be treated as a single-phase circuit. The only question at issue is the value of the current in the neutral wire. If the load is balanced this is obviously zero; in any case, taking the positive direction outwards, the current in the neutral wire is $I_n = I_1 + I_2 + I_3$.

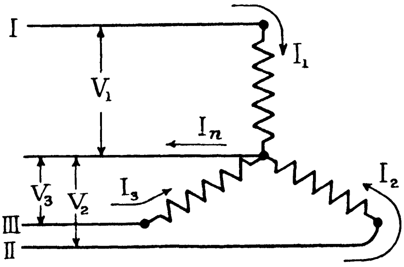


FIG. 61.—Star-connected Load.

If the neutral wire is *not* taken to the load and the load is balanced, exactly the same considerations hold as before. But if the load is not balanced the conditions are more complicated. In this case (Fig. 62):—

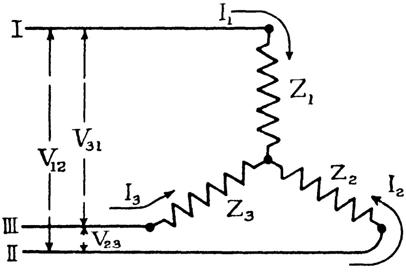


FIG. 62.—Star-connected Load, without Neutral.

$$\begin{aligned} I_1Z_1 - I_2Z_2 &= V_{12} \\ I_2Z_2 - I_3Z_3 &= V_{23} \\ I_3Z_3 - I_1Z_1 &= V_{31} \end{aligned}$$

also

$$\begin{aligned} I_1 + I_2 + I_3 &= 0 \\ V_{12} + V_{23} + V_{31} &= 0. \end{aligned}$$

These equations are readily simplified to

$$\begin{aligned} I_1 \left\{ \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \right\} &= \frac{V_{12}}{Z_1Z_2} - \frac{V_{31}}{Z_3Z_1} \\ I_2 \left\{ \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \right\} &= \frac{V_{23}}{Z_2Z_3} - \frac{V_{12}}{Z_1Z_2} \\ I_3 \left\{ \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} \right\} &= \frac{V_{31}}{Z_3Z_1} - \frac{V_{23}}{Z_2Z_3} \end{aligned}$$

and the currents are found when the impedances are known.

The power expended in the load is the sum of the powers expended in the three phases, and is readily found when the vector diagram has been drawn as $\Sigma(V_pI_p \cos \phi)$, or algebraically as the sum of the dot products of the phase voltages and currents: *i.e.* $\Sigma\ (V_p \cdot I_p)$ (see p. 14).

The power in a three-phase system is usually measured by the two-wattmeter method, in which the current coils of the two wattmeters are connected in two of the lines and the voltage coils are connected across these lines and the third. The total power is then given as the sum of the readings of the two wattmeters. While this is true whatever the wave form of the load, it can be proved by vector methods as follows.

Let the load be star connected and let W_1 be the reading of the wattmeter whose current coil carries the line current I_1, and whose voltage coil is across lines 1 and 2 and therefore has a current determined by V_{12}. Similarly let W_2 be the reading of the wattmeter having a current I_3 and voltage $V_{32} = -V_{23}$.

The wattmeters indicate the dot products of the currents through them and the voltages across them. Thus

$$W_1 = I_1 \cdot V_{12} \text{ and } W_2 = I_3 \,.\, V_{32}$$

Now the power P into the three phases of the star load is

$$P = I_1 \cdot V_1 + I_2 \cdot V_2 + I_3 \cdot V_3$$

$$\text{and } V_1 - V_2 = V_{12},\ V_2 - V_3 = V_{23} \text{ and } V_3 - V_1 = V_{31}$$
$$\text{and} \qquad -I_2 = I_1 + I_3$$

$$\begin{aligned} \therefore\ P &= I_1 \cdot V_1 - (I_1 + I_3) \cdot V_2 + I_3 \cdot V_3 \\ &= I_1 \cdot (V_1 - V_2) + I_3 \cdot (V_3 - V_2) \\ &= I_1 \cdot V_{12} + I_3 \cdot V_{32} = W_1 + W_2 \end{aligned}$$

If the load is balanced the power factor can be determined from the readings W_1 and W_2. For using the values obtained above from Fig. 59

$$I_1 = \frac{V_1}{Z} = \frac{|V|}{|Z|}\,\epsilon^{j(90.-\phi)}$$

$$I_3 = \frac{V_3}{Z} = \frac{|V|}{|Z|}\,\epsilon^{j(-150-\phi)}$$

$$V_{12} = \sqrt{3}V\epsilon^{j120}$$

$$V_{32} = -V_{23} = -\sqrt{3}V = \sqrt{3}V\epsilon^{j180}.$$

The angle between I_1 and V_{12} is therefore $(30 + \phi)$ and between I_3 and V_{32} is $(30 - \phi)$,

and $$W_1 = I_1 \cdot V_{12} = \sqrt{3}\frac{|V|^2}{|Z|}\cos(30 + \phi)$$

$$W_2 = I_3 \cdot V_{32} = \sqrt{3}\frac{|V|^2}{|Z|}\cos(30 - \phi)$$

whence $$W_1 + W_2 = 3\frac{|V|^2}{Z}\cos\phi$$

as would be expected,

and $$\frac{W_1}{W_2} = \frac{\cos(30 + \phi)}{\cos(30 - \phi)}$$

giving $$\tan\phi = \sqrt{3}\frac{W_2 - W_1}{W_1 + W_2}$$

or $$\cos\phi = \frac{1}{\sqrt{\left\{1 + 3\left(\frac{W_1 - W_2}{W_1 + W_2}\right)^2\right\}}}$$

It should be noted that if ϕ is greater than 60° W_1 is negative. The connections of one of the coils must be reversed in order to obtain a positive reading, and this reading subtracted from W_2 to obtain the power.

A similar proof holds in the case of a mesh-connected load.

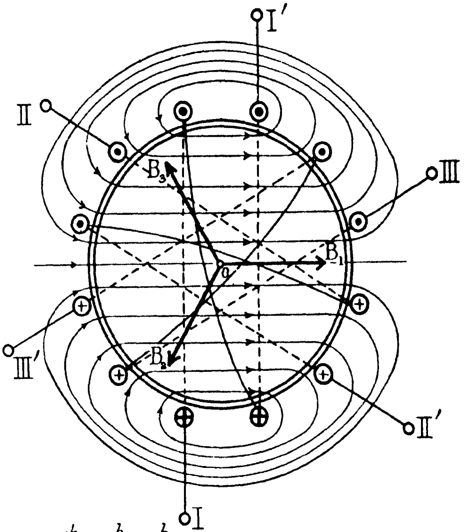
(2) Rotating and Travelling Fields

One of the most important of the applications of three-phase currents is in the production of rotating fields.

Let there be three equal coils I, II and III arranged with their axes meeting in a point O and making angles of 120° with each other (Figs. 63 and 64). Let the flux densities produced at O by currents in the coils be B_1, B_2 and B_3, as shown. The flux densities are true vector quantities ; they have magnitude sense and direction, and can be represented by real vectors. If B is the resultant flux density at O, then

$$\begin{aligned} B &= B_1 + B_2 + B_3 \\ &= |B_1| + |B_2|\epsilon^{-j120} + |B_3|\epsilon^{+j120}. \end{aligned}$$

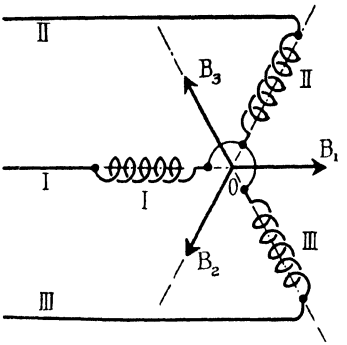
If equal currents flow in series through the three coils, then $|B_1| = |B_2| = |B_3|$, and $B = 0$. But if the current supply to the three coils is from the three phases of a three-phase supply, say $I_1 = I$, $I_2 = I\epsilon^{-j120}$, $I_3 = I\epsilon^{-j240}$, then taking real instantaneous values,



$$
\begin{aligned}
i_1 &= i_m \cos \omega t \\
i_2 &= i_m \cos (\omega t - 120) \\
i_3 &= i_m \cos (\omega t - 240) \\
&= i_m \cos (\omega t + 120)
\end{aligned}
$$

where

$$i_m = |I|.$$



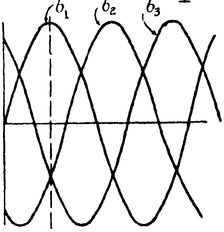


FIG. 63.—Production of rotating field.

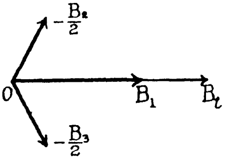


FIG. 64.

Hence if $|B_1| = Ki_m \cos \omega t$, at a particular instant, at the same instant

$$|B_2| = Ki_m \cos (\omega t - 120)$$

and

$$|B_3| = Ki_m \cos (\omega t + 120).$$

Hence at that instant

$$
\begin{aligned}
B &= Ki_m \cos \omega t + Ki_m \cos (\omega t - 120)\epsilon^{-j120} \\
&\qquad + Ki_m \cos (\omega t + 120)\epsilon^{+j120} \\
&= Ki_m \cos \omega t + Ki_m \cos (\omega t - 120)\{\cos 120 - j \sin 120\} \\
&\qquad + Ki_m \cos (\omega t + 120)\{\cos 120 + j \sin 120\} \\
&= Ki_m[\{\cos \omega t + \cos(\omega t - 120^\circ)\cos 120^\circ + \cos(\omega t + 120^\circ)\cos 120^\circ\} \\
&\qquad - j\{\cos (\omega t - 120^\circ) \sin 120^\circ - \cos (\omega t + 120^\circ) \sin 120^\circ\}]
\end{aligned}
$$

$$= \mathrm{K}i_m \{\cos \omega t + 2 \cos \omega t \cos^2 120^\circ - j2 \sin \omega t \sin^2 120^\circ\}$$

$$= \mathrm{K}i_m \left(\frac{3}{2} \cos \omega t - j \frac{3}{2} \sin \omega t\right)$$

$$= \frac{3}{2} \mathrm{K}i_m (\cos \omega t - j \sin \omega t)$$

$$= \frac{3}{2} \left| \mathrm{B}_1 \right| \epsilon^{-j\omega t} .$$

The flux density at the point O therefore has a constant magnitude one and a half times as great as the maximum flux due to one of the coils, and rotates in a clockwise direction with an angular velocity ω. It may be completely represented by a real rotating vector, which gives the actual flux density both in magnitude and direction, at any instant.

If a compass needle were placed at O, it would always try to point in the direction of the resultant flux, and would therefore try to rotate in a clockwise direction with an angular velocity ω. If a copper disc were pivoted at O, currents would be induced in it by the rotating field, and by Lenz's law the disc would try to move so as to prevent this ; that is, it would try to rotate clockwise with an angular velocity ω. These are the first considerations towards an understanding of the action of the induction motor and of armature reaction in alternators.

If the connections of any two coils are interchanged, the direction of rotation is reversed. For instance, if coil II is connected to phase III and coil III to phase II, the instantaneous currents in the three coils are

$$\begin{aligned} i_1 &= i_m \cos \omega t \\ i_2 &= i_m \cos (\omega t - 240) = i_m \cos (\omega t + 120) \\ i_3 &= i_m \cos (\omega t - 120) \end{aligned}$$

and

$$\mathrm{B} = \mathrm{K}i_m \cos \omega t + \mathrm{K}i_m \cos (\omega t + 120)\epsilon^{-j120} + \mathrm{K}i_m \cos (\omega t - 120)\epsilon^{+j120}$$

$$= \frac{3}{2}\mathrm{K}i_m \epsilon^{j\omega t}.$$

The rotation is now counter-clockwise.

The subject may also, and perhaps better for the purpose of alternating current machinery, be approached from the point of view of travelling fields.

Let there be an arrangement of flattened coils as indicated in Fig. 65 (*a*), and let it be required to find the flux distribution along the line *xx*. The conductors mainly concerned in producing this flux are shown in section at (*b*), and a current is supposed to be flowing in the direction shown by the arrow heads in (*a*) and by the dots and crosses in (*b*). Elementary consideration of the field (indicated by dotted lines in (*b*)), produced by the current, show that if ordinates proportional to the flux density are erected at points along *x*, the curve joining their extremities will be somewhat as shown at (*c*). In fact, it would not be difficult, by increasing the number of turns in each coil and spacing the conductors judiciously, to make the flux density curve approximate very closely to a sine wave.

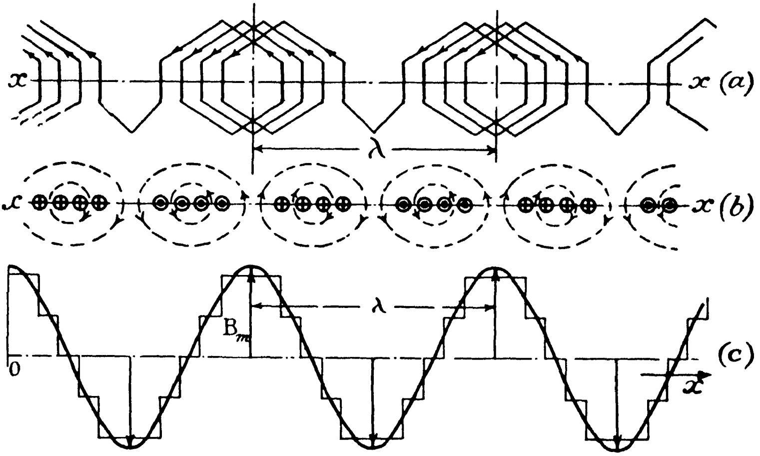


FIG. 65.—Field produced by flattened coils.

If this is done, and distances are measured to the right from O, the equation to the curve (*c*) is

$$b = b_m \cos\left(\frac{x}{\lambda} \cdot 2\pi\right)$$

where λ is the distance from the position of one maximum flux density to the position of the next maximum of the same sense, *i.e.*, the coil pitch, and b_m is the maximum value of the flux density.

If the current through the coils is alternating, the flux at each point alternates, having the maximum value indicated by the curve or by the above equation. If the current is $i_m \cos \omega t$, the flux density at any point x and any time t will be given by

$$b = b_m \cos\left(\frac{x}{\lambda}2\pi\right) \cos \omega t$$

where b_m depends upon i_m.

Suppose now that two other sets of coils are superposed upon the first at distances apart equal to $\frac{1}{3}\lambda$ as indicated in Fig. 66 (*a*), where the circles represent the coil sides, and where I, I′, I, I′ is the original coil system, and II, II′, II, II′ and III, III′, III, III′ those added. If direct currents are passed through the three systems in the same direction, the flux densities established along xx by the different coils will be

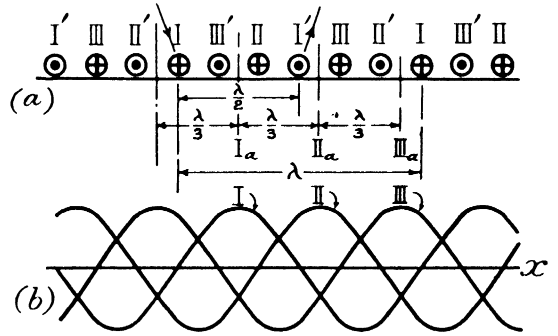


Fig. 66.—Fields produced by three coils.

$$b_I = b_m \cos\left(\frac{x}{\lambda}\cdot 2\pi\right)$$

$$b_{II} = b_m \cos\left(\frac{2\pi\left(x + \frac{2\lambda}{3}\right)}{\lambda}\right) = b_m \cos\left(\frac{2\pi x}{\lambda} + \frac{4}{3}\pi\right)$$

$$= b_m \cos\left(\frac{2\pi x}{\lambda} - \frac{2}{3}\pi\right)$$

$$b_{III} = b_m \cos\left(\frac{2\pi\left(x + \frac{\lambda}{3}\right)}{\lambda}\right) = b_m \cos\left(\frac{2\pi x}{\lambda} + \frac{2\pi}{3}\right)$$

as indicated in Fig. 66 (*b*), and the resultant flux at all points will be zero. The same will hold if an alternating current is passed in series through the three systems. But if one of the systems is fed from the first phase, the second from the second phase, and the third from the third phase of a three-phase supply, a very different state of affairs will hold.

For the instantaneous values of the currents in the three-coil system, write

$$\begin{aligned}
&i_m \cos \omega t \\
&i_m \cos (\omega t - 120) = i_m \cos \left(\omega t - \frac{2\pi}{3}\right) \\
&i_m \cos (\omega t - 240) = i_m \cos \left(\omega t + \frac{2\pi}{3}\right)
\end{aligned}$$

Then the instantaneous value of the total flux density at any point will be given by

$$\begin{aligned}
b &= b_1 + b_2 + b_3 \\
&= b_m \cos \left(\frac{2\pi x}{\lambda}\right) \cos \omega t \\
&\quad + b_m \cos \left(\frac{2\pi x}{\lambda} - \frac{2\pi}{3}\right) \cos \left(\omega t - \frac{2\pi}{3}\right) \\
&\quad + b_m \cos \left(\frac{2\pi x}{\lambda} + \frac{2\pi}{3}\right) \cos \left(\omega t + \frac{2\pi}{3}\right)
\end{aligned}$$

where b_m depends upon i_m. Hence

$$\begin{aligned}
b = {} & b_m \cos \left(\frac{2\pi x}{\lambda}\right) \cos \omega t \\
& + b_m \left\{ \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi}{3} + \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi}{3} \right\} \left\{ \cos \omega t \cos \frac{2\pi}{3} \right. \\
& \qquad\qquad\qquad\qquad \left. + \sin \omega t \sin \frac{2\pi}{3} \right\} \\
& + b_m \left\{ \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi}{3} - \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi}{3} \right\} \left\{ \cos \omega t \cos \frac{2\pi}{3} \right. \\
& \qquad\qquad\qquad\qquad \left. - \sin \omega t \sin \frac{2\pi}{3} \right\} \\
= {} & b_m \cos \left(\frac{2\pi x}{\lambda}\right) \cos \omega t \\
& + b_m \left\{ 2 \cos \frac{2\pi x}{\lambda} \cos \omega t \cos^2 \frac{2\pi}{3} + 2 \sin \frac{2\pi x}{\lambda} \sin \omega t \sin^2 \frac{2\pi}{3} \right\}.
\end{aligned}$$

And this, since $\cos^2 \frac{2\pi}{3} = \frac{1}{4}$ and $\sin^2 \frac{2\pi}{3} = \frac{3}{4}$

gives

$$b = b_m \left\{ \frac{3}{2} \cos \frac{2\pi x}{\lambda} . \cos \omega t + \frac{3}{2} \sin \frac{2\pi x}{\lambda} \sin \omega t \right\}$$

$$= \frac{3}{2} b_m \cos \left(\frac{2\pi x}{\lambda} - \omega t \right).$$

This expression shows that the flux distribution is travelling from left to right, since for any increase in t, an increase in x can be found which will leave $\left(\frac{2\pi x}{\lambda} - \omega t \right)$, and therefore b, unchanged.

Moreover if x is increased by λ the distribution is unchanged, and if t is increased by $2\pi/\omega$ the distribution is unchanged. Hence the flux travels a distance λ in a time $2\pi/\omega$, and its velocity c is therefore

$$c = \frac{\lambda \omega}{2\pi} = \lambda f$$

where f is the frequency of the currents. Thus b may be written

$$b = \frac{3}{2} b_m \cos \omega \left(t - \frac{x}{c} \right).$$

These expressions are of the same form as those of wave motions ; λ is identified with the wavelength and c with the velocity of propagation.

If there is another system of coils producing another travelling sinusoidal field of the same wavelength and velocity, but displaced from the first either in space or in time (or both), the resultant travelling field may be obtained by vector addition.

Suppose that the first field is

$$b_1 = b_{m1} \cos \omega \left(t - \frac{x}{c} \right)$$

and the second is

$$b_2 = b_{m2} \cos \omega \left(t + t' - \frac{x + x'}{c} \right)$$

$$= b_{m2} \cos \left\{ \omega \left(t - \frac{x}{c} \right) + \omega t' - \frac{\omega x'}{c} \right\}$$

$$= b_{m2} \cos \left\{ \omega \left(t - \frac{x}{c} \right) + \phi_t + \phi_x \right\}$$

where ϕ_t = phase angle due to time displacement

$= \omega t'$

ϕ_x = phase angle due to distance displacement

$$= -\frac{\omega x'}{c}.$$

The second travelling field could be produced by currents in a system of coils displaced a distance x' from the first, the currents lagging by a phase angle ϕ_t behind the corresponding currents in the first system.

Then, since b *at any position* is an alternating quantity, it may be found at any instant by projection from a rotating vector. Thus b_1 at a certain point may be represented by the engineers' vector OB_1 (Fig. 67), drawn below the horizontal at an angle $\phi = \frac{\omega x}{c}$. The horizontal is taken as the reference line ($t = 0$) and the vector is projected on to this horizontal to obtain instantaneous values. b_2 will be drawn as OB_2, making an angle with $B_1 = (\phi_t + \phi_x)$, and the resultant of the two will be OB_r found in the usual manner.

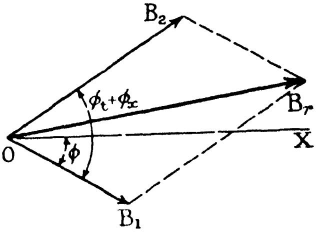


FIG. 67.

Evidently the maximum positive value of the wave b_1 has not yet reached the point considered, the maximum positive value of the wave b_2 has already passed the point, while the maximum positive value of the resultant has only just passed the point. Corresponding values at other points may be obtained either by (i.) a suitable (positive) rotation of the reference line OX, *i.e.*, through an angle $\omega x/c$, or by (ii.) a suitable negative rotation of the vector itself through an angle $\omega x/c$. The latter is the more useful method, and has been adopted in drawing the vector diagram of Fig. 67. If the rotating vector and the reference line rotate with the same angular velocity, the instantaneous values of the fluxes are always the same. This happens when $\omega x/c = \omega t$, or $x/t = c$; *i.e.*, when x is moving with the velocity of translation of the travelling flux.

If the systems of coils are arranged in a circle (Fig. 68), the circumference of which is an exact multiple p of the coil pitch λ, then

$$\lambda = \frac{2\pi}{p} r$$

and

$$x = \theta r$$

where r is the radius of the circle and θ is the angle at the centre

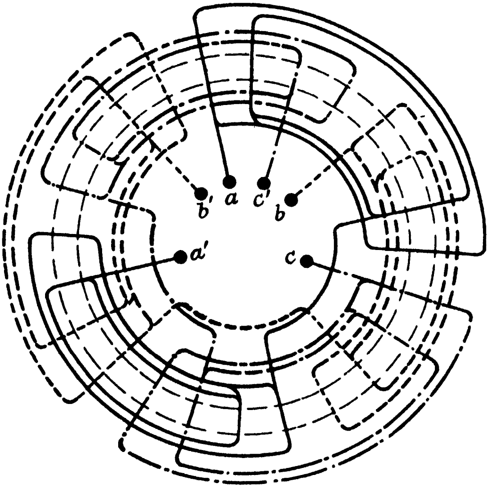


FIG. 68.—Production of rotating field.

subtended by the arc x'. The travelling field becomes a rotating field with an equation

$$b = \frac{3}{2} b_m \cos \omega \left(t - \frac{\theta r \, . \, 2\pi}{\frac{2\pi}{p} r\omega} \right)$$

$$= \frac{3}{2} b_m \cos (\omega t - p\theta).$$

The angular velocity is

$$\frac{c}{r} = \frac{\lambda\omega}{2\pi r} = \frac{\omega}{p}.$$

A displacement y in position of the travelling field becomes an angular displacement γ where $y = \gamma r$. The corresponding phase angle ϕ becomes

$$\phi = \frac{y}{\lambda} . 2\pi = \frac{\gamma r}{\frac{2\pi}{p} . r} . 2\pi$$

$$= p\gamma.$$

Thus in the equations for the rotating field, and in the vector diagrams, the actual positional angles are multiplied by p. $p\theta$ and $p\gamma$ are frequently referred to as "electrical angles."

p will be recognised as the number of pairs of poles produced by one phase.

EXERCISES

(1) A star-connected alternator has a line potential difference of 2,500 volts and supplies a balanced load taking a line current of 100 amperes at a power factor of 0·95. Draw a vector diagram to scale, show phase and line voltages and currents when the load is connected (*a*) in mesh, (*b*) in star.

(2) The three phases of a star-connected load are : (i.) A resistance of 20 ohms. (ii.) A resistance of 5 ohms in series with an inductance of 16 millihenries. (iii.) A resistance of 10 ohms in series with a condenser of 53 μF.

Find the phase voltages and currents and the total power absorbed when the load is connected to a 440-volt 50 cycle supply,

(*a*) With the neutral wire connected, in which case find also the current in the neutral wire.

(*b*) With no neutral wire.

(3) Find the currents and the power absorbed when the load of Q. 2 is connected in mesh to the same supply.

(4) Find in Q. 1, 2 and 3 the readings of two watt-meters connected in circuit to measure the power, if the current coils are connected in lines I. and III., and there is no neutral wire.

(5) Prove that the readings of two wattmeters give the power in a three-phase circuit in the case of a mesh-connected load and prove also the formula for tan ϕ.

CHAPTER VI

THE SYNCHRONOUS MACHINE

MODERN alternators have generally a rotating system of conductors excited by direct current, and a stationary system of conductors in which alternating electromotive forces are induced. The first system is known variously as the field system, the field or the rotor, and the second as the armature or the stator. In what follows the terms rotor and stator are used. The change over of terms necessary in the few special cases (generally very small machines) in which the " armature " rotates and the " field " is stationary will be obvious.

(1) The Electromotive Forces

(i) Let it be assumed that the field produced in the air gap of an alternator by the direct current through the coils of the stationary rotor has a sinusoidal distribution in space ; that the flux density can in fact be represented by an expression of the form

$$b = b_m \cos p\theta$$

as in the last chapter, where θ is the angle from some reference radius. p then is obviously the number of " pole pairs " the rotor produces.

The assumption of a sinusoidal field distribution is not by any means unjustified. In the non-salient pole type of rotor the field winding is purposely so arranged in the slots as to produce as nearly as possible this distribution of field, and even in the salient pole types the pole shoes are shaped to vary the air gap length with the same end in view. Further, whatever the actual distribution, it may be analysed into a number of sinusoidal fields by Fourier's analysis, having pitches of ratios 1, 3, 5, etc. The fundamental pitch is the one considered here, but similar considerations apply to the others, which cause induced e.m.f. of higher frequencies.

When the rotor is rotated clockwise with an angular velocity ω/p, the field rotates with it and is in every way the same as that produced by the three-phase coil system of the last chapter, *i.e.*, it is the same as would be caused if a stationary three-phase rotor replaced the rotating rotor excited by direct currents. The equation for the field at any angle θ and any time t can therefore be written

$$b = b_m \cos(\omega t - p\theta).$$

Let $\theta = 0$ at time $t = 0$, when the flux distribution for the existing rotor position will be as indicated by the dotted lines in Fig. 69. Let ABC . . . be conductors in the stator, and let the angles AOB, BOC, etc., be $2\gamma/p$ (*i.e.*, the electrical angle between A and B, B and C, etc., is 2γ). The fluxes at ABC . . . are represented in the vector diagram of Fig. 69 (*b*) by OB_a, OB_b, OB_c . . . at time $t = 0$. Let l be the length of the stator and r the radius. If the rotor moves through an angle $\delta\theta$ the flux cutting any conductor is $r\delta\theta lb$, and if the time taken is δt, the e.m.f. induced is

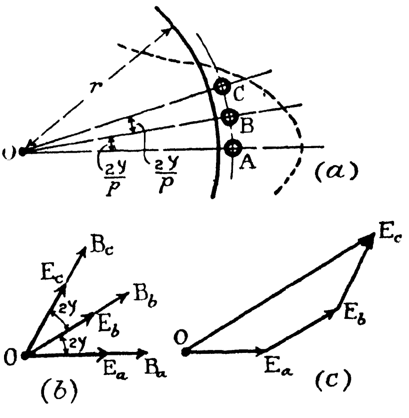


FIG. 69.—E.m.f. induced in distributed stator winding.

$$e = rlb \cdot \frac{\delta\theta}{\delta t}$$

$$= rl\frac{\omega}{p}b_m \cos(\omega t - p\theta)$$

since $\delta\theta/\delta t$ is the angular velocity of the rotor, and b is found from the above equation, putting in the correct value of θ for the conductor considered.

Thus for the conductors A, B, C, . . ., writing $rl\frac{\omega}{p}b_m = e$, the induced e.m.f.s are represented by OE_a, OE_b, OE_c, etc., each of length e, in the vector diagram.

Vectorially, these e.m.f.s are E_a, $E_b = E_a\epsilon^{j2\gamma}$, $E_c = E_a\epsilon^{j4\gamma}$, etc.

If there are n conductors per pole for each phase of the stator winding, the total vector e.m.f. E is the vector sum of these vectors, *i.e.*,

$$\begin{aligned} \mathrm{E} &= \mathrm{E}_a(1 + \epsilon^{j2\gamma} + \epsilon^{j4\gamma} + \ldots) \\ &= \mathrm{E}_a \frac{1 - \epsilon^{j2\gamma n}}{1 - \epsilon^{j2\gamma}} = \mathrm{E}_a \frac{(\epsilon^{-j\gamma n} - \epsilon^{j\gamma n})\epsilon^{j\gamma n}}{(\epsilon^{-j\gamma} - \epsilon^{j\gamma})\epsilon^{j\gamma}} \\ &= \mathrm{E}_a \frac{\sin n\gamma}{\sin\gamma} \cdot \epsilon^{j\gamma(n-1)}. \end{aligned}$$

The addition is indicated graphically in Fig. 69 (*c*).

The magnitude of the e.m.f. produced per coil side is thus $\mathrm{E}_a \sin n\gamma/\sin\gamma$. If all the conductors of the coil side were in the same slot, the e.m.f. would be $n\mathrm{E}_a$. The ratio of these two quantities is known as the *breadth* or *distribution factor*, and is

$$\frac{\sin n\gamma}{n \sin\gamma}$$

In practice there may be more than one conductor represented by A, B, C, etc., and in this case E_a must be multiplied by this number. Also the winding may not be symmetrically disposed as shown, in which case there will be a *winding factor* arrived at in a similar manner.

(ii) The magnitude of the e.m.f. per conductor has been found above to be

$$e = rl\frac{\omega}{p}b_m.$$

This is more usually expressed in terms of the total flux per pole, Φ. Evidently

$$\Phi = \int_{\theta=-\frac{\pi}{2p}}^{\theta=+\frac{\pi}{2p}} brld\theta$$

and

$$b = b_m \cos p\theta.$$

Hence

$$\begin{aligned} \Phi &= rlb_m \int_{-\frac{\pi}{2p}}^{+\frac{\pi}{2p}} \cos p\theta d\theta \\ &= 2\frac{rlb_m}{p}. \end{aligned}$$

Thus $$e = \frac{\omega \Phi}{2} \text{ c.g.s. units.}$$

The e.m.f. per coil will be twice this, the r.m.s. value will be found by dividing by $\sqrt{2}$, and ω may be written $2\pi f$. Thus

$$\begin{aligned} e_{rms} &= \frac{2\pi f \Phi}{\sqrt{2}} \\ &= 4{\cdot}44 f\Phi \times 10^{-8} \text{ volts.} \end{aligned}$$

Since each coil is repeated p times round the stator, the total phase e.m.f. is

$$= 4{\cdot}44\ pn\left(\frac{\sin n\gamma}{n \sin \gamma}\right) f\Phi \times 10^{-8} \text{ volts.}$$

The complete stator winding of a three-phase alternator is equivalent to the coil system of Fig. 69; although there are various methods of connecting the conductors, the result as far as the total e.m.f. produced is concerned is the same in all. It is therefore at once apparent that the e.m.f.s appearing at the different phase terminals will have the same vector diagram (Fig. 59 (*a*)) as the elementary three-phase alternator of Fig. 56.

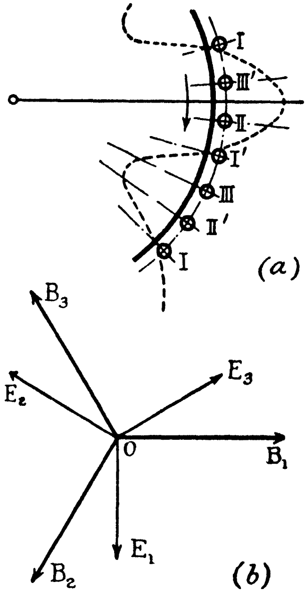


Fig. 70.—E.m.f.'s induced in three-phase winding.

(iii.) In Fig. 70 (*a*) the dotted lines represent the flux distribution for the rotor position for which the diagram is drawn, and the circles I I′, II II′, III III′ represent the coil sides. The rotor is rotating in a clockwise direction at an angular velocity ω/p (f/p revs. per second). OB_1 in Fig. 70 (*b*) is the flux vector for the rotor position of Fig. 70 (*a*). The flux cutting the conductors I and I′ is zero at this instant (the flux through the coil I I′ is a maximum). The e.m.f. induced in phase I is therefore zero, and the rotating vector of the e.m.f. is drawn as OE_1, at right angles to OB_1 lagging. The coil II II′ is $2\pi/3$ electrical degrees ahead (with regard to the rotating flux) of the coil I I′, hence the flux with regard to it is represented by the vector OB_2, and similarly OB_3 represents the flux distribution with regard to the coil III III′. The e.m.f. vectors OE_2 and OE_3 are drawn 90° behind OB_2 and OB_3 respectively.

It should be borne in mind that the diagram is drawn at a certain time instant, that indicated by the rotor position of Fig. 70 (*a*). The projections of the various vectors on to the horizontal represent the various flux values and e.m.f. values at that instant. When the whole diagram is rotated counter clockwise through an angle ωt the projections on to the horizontal give the instantaneous values of the various quantities at a time t after the instant considered. This counter-clockwise rotation of the vector diagram is necessitated by a clockwise movement of the rotor through an angle $\frac{\omega}{p}$.

OB_1, OB_2 and OB_3 really represent one and the same flux distribution, but with three different reference points, viz., the mid-points of the coils I I′, II II′ and III III′. In a consideration of the effects of loading the alternator with a balanced load, only one vector need be considered.

(2) Alternator Regulation

(i.) When the alternator is loaded (with a balanced load) two causes may be distinguished in the alteration of the terminal voltage of the machine.

(*a*) There is a fall of voltage owing to the resistance and " inductance " of the stator winding.

(*b*) The currents in the stator windings produce a rotating field which is combined with the rotating field of the rotor in producing the electromotive force. This is called " stator reaction " (or more generally " armature reaction ").

In the same way that, in the transformer, the " inductance " of the winding producing voltage drop at the terminals was found to be only that part due to leakage fluxes, so in the alternator, the " inductance " referred to in (*a*) is in reality only the leakage inductance ; that part due to flux accompanying the currents in the end connections and due to other fluxes not entering the air gap. It appears later that the part played by the main inductance is dealt with under (*b*).

It would, under (*b*), be more correct to say that the stator currents produce a rotating magneto-motive force. If the rotor is cylindrical and uniformly slotted round the circumference, this rotating magneto-motive force can safely be regarded as producing a sinusoidal rotating field of the type considered in the last chapter. The same holds approximately for all non-salient pole rotors. But the approximation is only rough in the case of salient pole machines, owing to the considerable variation of the length of the air gap across which the m.m.f. has to create the flux.

Effects due to partial saturation of the iron, and variations of reluctance round the circumference due to the slots and teeth, are also neglected in the present outline, which only holds rigidly for a perfectly smooth rotor and smooth stator constructed of iron whose magnetisation has a linear relationship with the m.m.f.

With this understood, the vector diagram for the alternator on load is drawn in Fig. 71.

OV is the terminal voltage, and OI the load current (determined by the impedance) lagging by an angle ϕ and determined by the load power factor. (Voltage and current are for one phase.) Vr drawn parallel to OI is the phase resistance drop rI, and rE drawn perpendicular to OI is the reactance drop jwlI. VE is thus the internal impedance or stator impedance drop.

OE is the e.m.f. generated by the resultant rotating field, and the latter is in consequence drawn as OB perpendicular to OE, and leading. The rotating field due to the stator currents is drawn as OB_s along OI, and the rotor field OB_r, found by drawing $\mathrm{B}_r\mathrm{B}$ parallel and equal to OB_s. The developed fields at the instant considered are drawn in Fig. 72. In the absence of a load the generated e.m.f. would be OE_r drawn perpendicular to OB_r and coinciding with OE. Since the e.m.f.'s generated must be proportional to the fluxes generating them, the ratio of OE_r to OB_r must be the same as the ratio of OE to OB. It follows that the triangles $\mathrm{OB}_r\mathrm{B}$ and $\mathrm{OE}_r\mathrm{E}$ are similar and that $\mathrm{E}_r\mathrm{E}$ is perpendicular to $\mathrm{B}_r\mathrm{B}$, *i.e.*, to OI, and that r,E and E_r lie on the same straight line. $\mathrm{E}_r\mathrm{E} = \mathrm{OE}_s$ is in fact the e.m.f. that would be produced by the stator flux OB_s acting alone.

The effect of the stator reaction may therefore be regarded as increasing the reactance drop from rE to $r\mathrm{E}_r$, and the impedance drop from VE to VE_r, and for this reason $r\mathrm{E}_r/\mathrm{I}$ is called the synchronous reactance and VE_r/I is called the synchronous impedance of the alternator.

The regulation may in fact be found from the synchronous impedance by means of a Kapp diagram in exactly the same manner as the regulation of the transformer is found from the effective impedance. It will moreover show the same characteristics; a maximum for inductive loads, becoming smaller and even negative as the load angle is reduced through zero to a negative value.

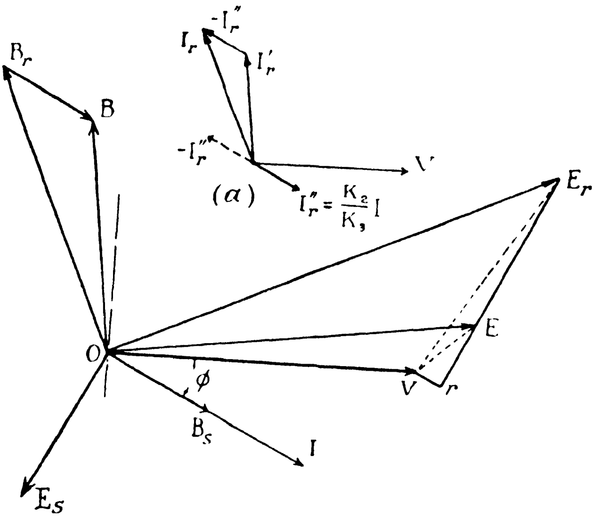


FIG. 71.—Vector diagram—alternator on load.

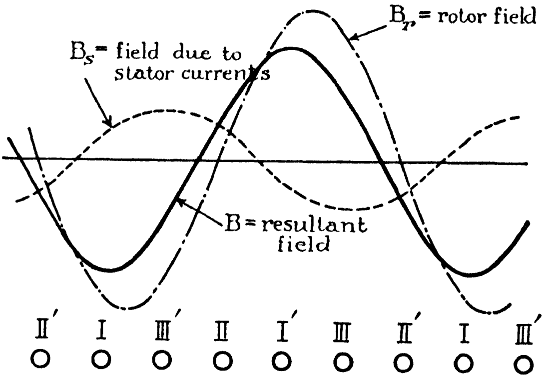


FIG. 72.—Component and resultant air-gap fluxes.

(ii.) Vector equations may readily be written down from Fig. 71, or from the considerations upon which Fig. 71 was drawn. Denoting the vectors OB_r, OE, etc., by B_r, E, etc.; the stator impedance by Z_i and the load impedance by Z,

$$I = V/Z$$

$$E = V + Z_i I$$

$$B = jK_1E,\ B_r = jK_1E_r \text{ and } B_s = jK_1E_s$$

also $$B_s = K_2I$$

and $$B_r + B_s = B$$

i.e. $$B_r = jK_1E - K_2I$$

$$= jK_1(V + Z_iI) - K_2I$$

Hence $$jK_1E_r = jK_1(V + Z_iI) - K_2I$$

$$E_r = V + Z_iI + j\frac{K_2}{K_1}I$$

i.e. $$E_r = V + \left(Z_i + j\frac{K_2}{K_1}\right)I$$

K_1 and K_2 are defined by the above equations. K_1 is the flux density per unit e.m.f. produced, and K_2 is the flux density produced by unit stator current. The ratio K_2/K_1 has evidently the dimensions of an impedance, and is measured in ohms. It is in fact that portion of the synchronous reactance due to stator reaction, while $Z_i + j\frac{K_2}{K_1}$ is the *synchronous impedance*, denoted by Z_s.

Further, since the flux linkages Λ of one phase winding (*i.e.*, the linkages of that part of the air-gap flux due to the stator current) is proportional to B_s, it may be written mB_s. It is also written LI, where L is the effective inductance of one phase of the three-phase system due to the main flux.

Thus $$\Lambda = mB_s = LI$$

and in writing $$B_s = K_2I$$

the value of K_2 is evidently

$$K_2 = \frac{L}{m}.$$

The e.m.f. induced in the winding by this alternating flux is

$$E_s = -\frac{d\Lambda}{dt}$$

$$= -jw\Lambda$$

$$= -jwmB_s.$$

But this has been written as

$$E_s = -\frac{jB_s}{K_1}.$$

Hence $$K_1 = \frac{1}{wm}$$

and the ratio $$\frac{K_2}{K_1} = wL.$$

Thus the synchronous impedance is

$$Z_s = Z_i + j\frac{K_2}{K_1}$$

$$= R + jwl + jwL = R + jw(l + L)$$

i.e., it is simply the *total effective impedance* of the phase of the stator winding, including reactances due to both the leakage and the main fluxes. It is to be compared with the open circuit impedance of the transformer.

With the ideal iron considered, it is possible to write

$$B_r = K_3I_r$$

where I_r is the rotor current, and the vector equation

$$B = B_r + B_s$$

may be written

$$K_3I_r' = K_3I_r + K_3I_r''$$

where $$K_3I_r' = B \text{ and } K_3I_r'' = B_s = K_2I.$$

That is $$I_r' = I_r + I_r''$$

or $$I_r = I_r' - I_r''.$$

The direct current in the rotating rotor is in this way regarded as being divided (vectorially) into two parts ; one part I_r' provides the flux from which the stator e.m.f. E is derived, and the other part, $-I_r''$ provides flux to neutralise the stator reaction flux ; $I_r' = B/K_3$ and $I_r'' = K_2I/K_3$. This division is indicated in Fig. 71 (*a*).

If the constants K_1, K_2 and K_3 can be determined, and the division of the rotor current can be carried out, then E may be found from I_r', and if further Z_i can be determined, the regulation for any current at any power factor is found, since

$$V = E - Z_iI.$$

On open circuit, $I = 0$ and $I_r'' = 0$,

$$V = E = E_r = -j\frac{B_r}{K_1} = -j\frac{K_3}{K_1}I_r;$$

an open circuit test determines K_3/K_1.

If the stator is short-circuited, $V = 0$ and

$$E = Z_i I$$

but

$$I_r' = j\frac{K_1}{K_3}E,$$

$$I_r'' = \frac{K_2}{K_3}I$$

and

$$I_r = I_r' - I_r''.$$

Hence

$$\begin{aligned} I_r &= j\frac{K_1}{K_3}E - \frac{K_2}{K_3}I \\ &= j\frac{K_1}{K_3}Z_i I - \frac{K_2}{K_3}I \\ &= j\frac{K_1}{K_3}\left(Z_i + j\frac{K_2}{K_1}\right)I \\ &= j\frac{K_1}{K_3}Z_s I. \end{aligned}$$

whence

$$|I_r| = \frac{K_1}{K_3}|Z_s| \times |I|$$

$|I_r|$, K_1/K_3 and $|I|$ are all known, and hence the magnitude of the synchronous impedance is found.

If the open circuit test and the short circuit test are carried out with the same value of exciting current I_r, then

$$|Z_s| = \frac{|V|}{|I|}$$

where $|V|$ is the terminal voltage produced in the open circuit test and $|I|$ is the phase current produced in the short circuit test.

Since

$$\begin{aligned} Z_s &= Z_i + j\frac{K_2}{K_1} \\ &= R + j\left(\omega l + \frac{K_2}{K_1}\right) \end{aligned}$$

the angle of the synchronous impedance can be found if the resistance of the winding is measured separately.

(iii.) Passing on from the machine with ideal iron to a consideration of the actual machine, the most notable difference is the curvature of the curve connecting the open circuit e.m.f. with the rotor current (Fig. 73) caused by partial saturation of the iron. The factor K_3 which has been taken as constant, decreases with increase of I_r. The electromotive forces generated, however, will be strictly proportional to the fluxes causing them, so that K_1 remains a constant. Experiments appear to show also that the stator reaction flux remains proportional to the stator current; K_2 remains constant.

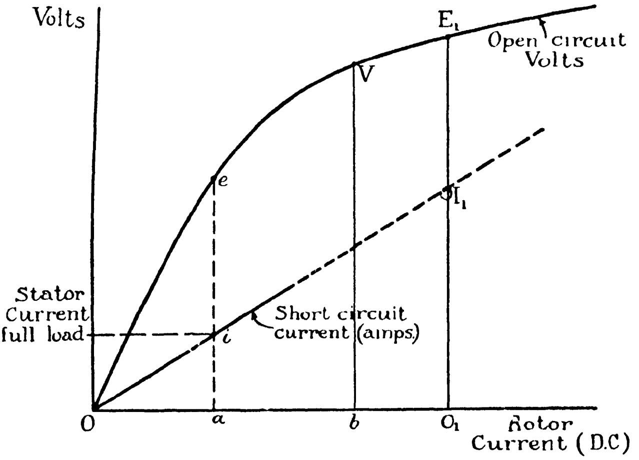


FIG. 73.—Open and short circuit characteristics of alternator.

Another practical departure from the ideal diagram may be noticed. The flux distribution B_s is drawn (Fig. 71) in phase with the current I producing it. Owing, however, to eddy current and hysteresis effects, B_s will actually lag behind I; the angle VOB_s will be larger, and in consequence the angles VOB_r and VOE less. Thus the synchronous impedance will be less than would be calculated from the ideal vector diagram.

(iv.) There are three well-known methods of determining the regulation without actually loading the machine. For each the experimental results required are the no-load (voltage) characteristic, *i.e.*, the open-circuit stator voltage with the rotor run at constant (normal) speed but with variable excitation current, and the short-circuit (current) characteristic, giving the stator current with the rotor running at constant (normal) speed, with the stator windings short-circuited, and with excitation current varied up to the value required to give full-load stator current. The former curve has the well-known shape of a magnetisation curve, the latter is very nearly a straight line (Fig. 73).

(*a*) In the first curve $I = 0$ and the measured voltage is E proportional to B, which in turn is a function of the magneto motive force or ampere turns or simply the current of the rotor.

In the second curve $V = 0$ and

$$E = \left|\left(Z_i + j\frac{K_2}{K_1}\right)\right| I$$
$$= |Z_s| I.$$

Thus ordinates on the curves having the same value of exciting current are related by the expression

$$|Z_s| = \frac{E}{I} = \frac{ae}{ai}.$$

where $|Z_s|$ is the magnitude of the synchronous impedance at that value (Oa) of the exciting current. It is commonly assumed that the short-circuit curve continues in a straight line to the full value of the exciting current (OO_1), and that

$$|Z_s| = \frac{O_1E_1}{O_1I_1}.$$

This value of the synchronous impedance is then used to determine the regulation as in the case of the transformer. The method is due to Behn-Eschenberg, leads to too high a value of the regulation, and is in consequence called the "pessimistic method."

The vector diagram for the short-circuit condition is drawn in Fig. 74, and it will be noticed that the reaction flux $OB_s = B_rB$ is nearly in phase opposition to the excitation flux OB_r. The resultant OB produces the e.m.f. OE required for the internal impedance Z_i.

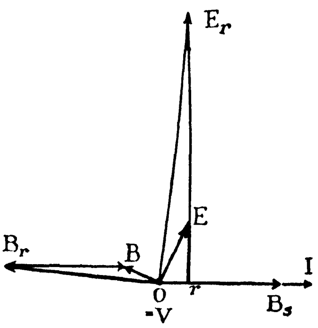


FIG. 74.—Vector diagram—short-circuit condition.

(*b*) The second method (Rothert's) neglects the internal impedance and assumes that all the excitation flux in the short-circuit test is required to overcome the stator flux. The method thus gives an optimistic value for the regulation.

Fluxes are all measured in terms of rotor m.m.f.'s, or simply rotor currents. Thus in Fig. 73, if *ai* is the full-load current on the short - circuit characteristic, it is assumed that O*a* is the rotor current required to overcome armature reaction. Then, Fig. 75, OV(= OE) is drawn horizontally, and OI is the stator current drawn at the phase angle ϕ determined by the load impedance. OB must be drawn as shown at right angles to OV, and OB_s of length = O*a* is drawn along OI. With centre B_s and radius OO_1 equal to full exciting current (Fig. 73), draw an arc to cut OB in B. Then OB is the resulting exciting current, and the corresponding voltage is found from Fig. 73 by marking off O*b* horizontally equal to OB, and erecting an ordinate *b*V to the open-circuit curve.

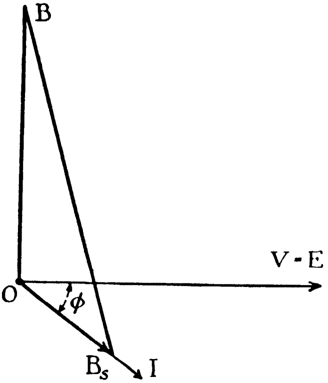


FIG. 75.—Vector diagram—short-circuit excitation.

(*c*) In the third method (Potier's) the effects of internal impedance and stator reaction are separated out and duly allowed for, and better results are accordingly obtained.

A third test is made. The machine is loaded with a highly inductive load so that full current flows at as nearly as possible zero power factor and the exciting currents required to give various terminal voltages are found. The vector diagram for the case of $\phi = 90°$ is drawn in Fig. 76, and it is apparent that, very nearly indeed, the reactance voltage rE_1 is in phase with the terminal voltage OV and is therefore added numerically, and the reaction flux OB_s is in phase opposition with the resultant flux OB, and is therefore subtracted numerically to find the rotor

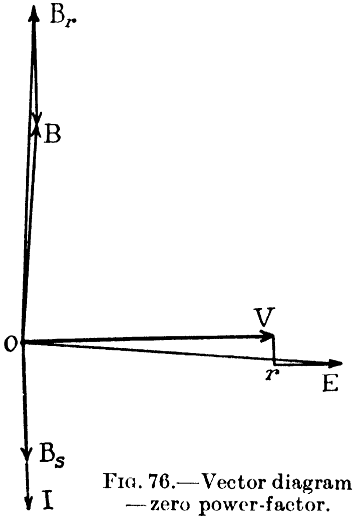


FIG. 76.—Vector diagram—zero power-factor.

flux. This latter relation also holds at short circuit, as is clear from Fig. 74.

The two characteristics, open-circuit OE, and full-load current at $\phi = 90°$ O_1V are drawn in Fig. 77. Evidently OO_1 is the short-circuit excitation for full-load current, can be determined from Fig. 73, and is made up of two parts, that required to provide the reactance drop (the resistance drop is neglected) and that required to overcome the armature reaction. These two effects may be assumed to be constant throughout the zero power factor characteristic.

At any value of the exciting current OC draw a perpendicular to meet the curve O_1V in V_c. Through V_c draw V_cA equal and parallel to O_1O, and through A draw AE_c parallel to the curve OE at O to meet OE in E_c. From E_c drop a perpendicular E_cD to meet AV_c in D. Through O_1 draw O_1E_1 parallel and equal to V_cE_c, and drop E_1D_1 perpendicular to OO_1. It is now clear that any point E on the OE curve can be transferred to the point V on the O_1V curve by drawing EV equal to and parallel with E_1O_1. It is evident also that at the excitation OC, DV_c is the exciting current required to overcome the armature reaction, AD is the exciting current to provide the voltage for the reactance drop, and E_cD is the reactance drop itself. The same triangle EDV appears all the way down the curves, and there is no necessity to obtain more than one point on the O_1V curve.

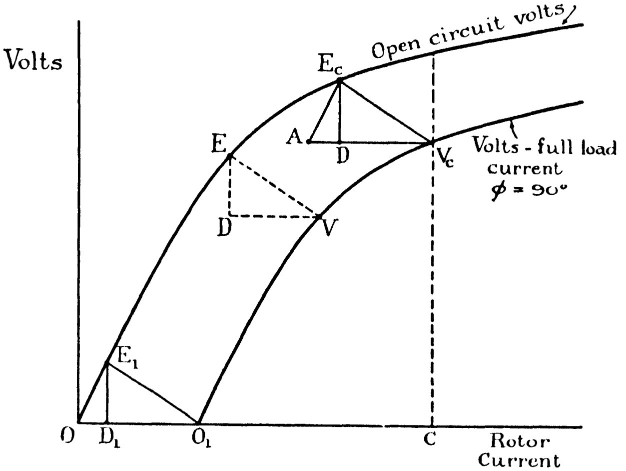


FIG. 77.—Determination of regulation—Potter's method.

Having now found the reactance drop and the exciting current for the armature reaction, the regulation at any power factor can be found by drawing the complete vector diagram, or by calculation.

(3) Electromotive Force and Regulations—Alternative Treatment

It has been supposed that the rotor flux has a sinusoidal distribution round the rotor circumference, and it has been seen that an ideal three-phase system of conductors produces a rotating sinusoidal field. As far as the electromotive forces produced in the stator conductors are concerned, therefore, the rotating rotor can be imagined to be replaced by a three-phase system as indicated in Fig. 78, the two systems, rotor and stator, constituting a stationary three-phase transformer. It should be possible to arrive at the results already obtained by a consideration of this transformer, without any further reference to rotating fields.

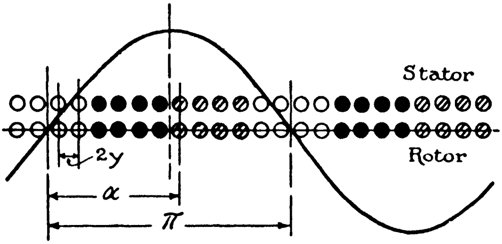


FIG. 73.—Three-phase winding on stator and rotor.

(i.) A slight simplification is obtained if the two systems are imagined to be exactly opposite each other, and in the figure phase I is indicated by the blank circles, phase II by the crossed circles, and phase III by the black circles. Taking the origin in the middle of a coil side of the first phase, let the rotor flux density of the phase be $b_m \sin \theta$. It is required to find the linkages with each of the stator phases. Calling the angular (electrical) displacement of the wires 2γ as before, let α be the displacement from the origin of the first wire of the coil side of the phase considered.

Then the value of the linkages Λ for any stator phase is given by the expression

$$\Lambda = \frac{rl}{p} b_m \left\{ \int_{\alpha}^{\pi+\alpha+(2n-2)\gamma} \sin \theta \, d\theta + \int_{\alpha+2\gamma}^{\pi+\alpha+(2n-4)\gamma} \sin \theta d\theta + \ldots . \int_{\alpha+(2n-2)\gamma}^{\pi+\alpha} \sin \theta d\theta \right\}$$

in which successive turns give the total flux through the successive turns of the phase winding, the first turn consisting of the two outer wires, the second of the next two wires, and so on, and the last of the two inner wires.

The series within the brackets

$$= 2\left\{\cos \alpha + \cos(\alpha + 2\gamma) + \ldots\ldots + \cos(\alpha + \overline{2n-2}\gamma)\right\}$$

$$= \text{Real part of}$$

$$2\left\{\epsilon^{j\alpha} + \epsilon^{j(\alpha+2\gamma)} + \ldots\ldots\ \epsilon^{j(\alpha+\overline{2n-2}\gamma)}\right\}$$

$$= \mathrm{R}\left[2\epsilon^{j\alpha}\left\{1 + \epsilon^{j2\gamma} + \ldots\ldots\ \epsilon^{j\overline{2n-2}\gamma}\right\}\right]$$

$$= \mathrm{R}\left[2\epsilon^{j\alpha}\frac{\epsilon^{j2n\gamma}-1}{\epsilon^{j2\gamma}-1}\right] = \mathrm{R}\left[2\epsilon^{j(\alpha+\overline{n-1}\gamma)}.\ \frac{\epsilon^{jn\gamma}-\epsilon^{-jn\gamma}}{\epsilon^{j\gamma}-\epsilon^{-j\gamma}}\right]$$

$$= 2\cos(\alpha + \overline{n-1}\gamma)\,.\,\frac{\sin n\gamma}{\sin\gamma}\,.$$

For the first phase $\alpha = -(n-1)\gamma$, and the linkages

$$\Lambda_{11} = 2\frac{rl}{p}b_m\frac{\sin n\gamma}{\sin\gamma}\,.$$

For the second phase $\alpha = (n-1)\gamma + 2n\gamma + 2\gamma = (3n+1)\gamma$ and the linkages $\Lambda_{12} = -2\frac{rl}{p}b_m \cos 4n\gamma \frac{\sin n\gamma}{\sin\gamma}$

For the third phase $\alpha = \pi + (n-1)\gamma + 2\gamma = \pi + (n+1)\gamma$ and the linkages $\Lambda_{13} = -2\frac{rl}{p}b_m \cos 2n\gamma \frac{\sin n\gamma}{\sin\gamma}$

As the wires are uniformly distributed round the stator,

$$3n \times 2\gamma = \pi,\ \gamma = \pi/6n, \text{ and}$$

$$\Lambda_{11} = \frac{rl}{p}b_m/\sin\gamma$$

$$\Lambda_{12} = -\frac{rl}{p}b_m/2\sin\gamma = -\frac{1}{2}\Lambda_{11}$$

$$\Lambda_{13} = -\frac{rl}{p}b_m/2\sin\gamma = -\frac{1}{2}\Lambda_{11}.$$

From symmetry it follows that a sinusoidal flux distribution of the same maximum value b_m due to the second rotor phase produces linkages $\Lambda_{21} = -\frac{rl}{p}b_m/2\sin\gamma$ in the first stator phase and similarly

$\Lambda_{31} = -\frac{rl}{p} b_m/2 \sin\gamma$ are the linkages produced in the first stator phase by a sinusoidal flux due to the third rotor phase.

In finding the total e.m.f. produced in any stator phase by the fluxes caused by the currents in the three rotor phases, due regard must be had to the time phase differences of these currents and fluxes. Thus if b_{m1}, b_{m2} and b_{m3} represent the instantaneous values of the maximum fluxes (maximum with regard to space) of the three phases, and

$$b_{m1} = b_m \cos\omega t$$

then

$$b_{m2} = b_m \cos(\omega t - 120^\circ)$$

and

$$b_{m3} = b_m \cos(\omega t + 120^\circ)$$

and the total instantaneous linkages through phase I of the stator are

$$\begin{aligned} \Lambda &= \Lambda_{11}\cos\omega t + \Lambda_{21}\cos(\omega t - 120) + \Lambda_{31}\cos(\omega t + 120) \\ &= \Lambda_{11}\left\{\cos\omega t - \frac{1}{2}\cos(\omega t - 120) - \frac{1}{2}\cos(\omega t + 120)\right\} \\ &= \frac{3}{2}\Lambda_{11}\cos\omega t \end{aligned}$$

The e.m.f. produced is $d\Lambda/dt$ and has therefore a maximum value

$$\begin{aligned} e_m &= \frac{3}{2}\omega|\Lambda_{11}| \\ &= \frac{3}{2} b_m 2\frac{rl}{p}\omega\frac{\sin n\gamma}{\sin\gamma}. \end{aligned}$$

Now $\frac{3}{2} b_m$ is the maximum value of the rotating rotor field and Φ the flux per pole is $2rl\left(\frac{3}{2}b_m\right)/p$. Thus for one coil

$$e_m = \omega\Phi\frac{\sin n\gamma}{\sin\gamma}$$

and for p coils, the r.m.s. value is

$$e_{rms} = 4{\cdot}44\, pn\left(\frac{\sin n\gamma}{n\sin\gamma}\right) f\Phi \times 10^{-8} \text{ volts}$$

as before.

(ii.) These results lead to some interesting considerations with regard to the *effective* inductance of a phase winding of a symmetrical three-phase system. The flux threading any phase winding will not be due to the current in that phase alone; there will be fluxes due to the currents in the other two phases, and in consequence the back e.m.f. produced will not be $j\omega \mathrm{LI}$, where L is the inductance of the winding, but some other quantity $j\omega \mathrm{L'I}$, where L′ takes account of the mutual inductances between the phases and has been referred to above as the *effective* inductance. Because of the phase differences (time) between the different phase currents, L′ is not necessarily a numeric but may be complex, but in a symmetrical system it *is* a numeric.

For in any symmetrical three-phase system, such as that of Fig. 63, let L be the self inductance of each coil and M the mutual inductance between each pair of coils.

The flux linkages through coil I are then

$$\begin{aligned}\Lambda &= \mathrm{LI}_1 - \mathrm{MI}_2 - \mathrm{MI}_3 \\ &= \mathrm{L}i_m \cos \omega t - \mathrm{M}i_m \cos (\omega t - 120) - \mathrm{M}i_m \cos (\omega t + 120) \\ &= (\mathrm{L} + \mathrm{M})i_m \cos \omega t \,. = (\mathrm{L} + \mathrm{M})\mathrm{I}_1.\end{aligned}$$

The effective inductance of the phase winding is thus simply (L+M), and once allowance has been made for this increase of the inductance, the presence of the coils of the remaining phases may be ignored in any calculations that are made, provided, of course, that the whole system is symmetrical, which necessarily involves a balanced load.

If the flux from the current in each stator phase winding of an alternator is assumed to be sinusoidal, it follows from the work of the preceding section that $\mathrm{M} = \frac{1}{2}\mathrm{L}$ for the main flux of the stator. Similar considerations apply to the leakage flux and to the alternating current resistance of the winding (in so far as this A.C. resistance is due to iron losses). So that if total stator impedance is written

$$r + j\omega(l + \mathrm{L})$$

it must be remembered that r and l are both somewhat greater than they would be for the phase winding alone, and that L is of the order 3/2 times the inductance of the winding alone. The *synchronous impedance* is the total *effective impedance*, and is greater than the isolated phase impedance for these reasons.

(iii.) With these considerations the three-phase transformer of Fig. 78 can be replaced by a single-phase transformer, the primary current of which is constant whatever the load conditions on the secondary, and the secondary impedance of which is the effective or synchronous impedance. The regulation diagram $OVrEE_r$ of Fig. 71 follows at once.

(4) Parallel Operation of Alternators and the Synchronous Motor

(i.) *Machine connected to constant voltage bus-bars.*

Let a constant potential difference V be maintained between bus-bars AB by a number of large alternators running in parallel, and let E_1 be the total electromotive force (E_r of the last section) of another alternator running in parallel, the synchronous impedance of which is Z_1. Let the current supplied to the load Z be I, of which $I - I_1$ comes from the bus-bars and I_1 from the alternator under consideration. The circuit diagram is given in Fig. 79.

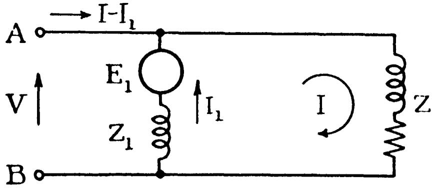


FIG. 79.—Alternator on constant voltage bus-bars.

Evidently $V = IZ$

and $V = E_1 - I_1Z_1$

or $I_1 = \dfrac{E_1 - V}{Z_1}.$

I_1 thus depends very largely on the phase angle between E_1 and V, as is shown in the three vector diagrams of Fig. 80. (*a*) shows E_1 and V in phase, (*b*) E_1 leading V, and (*c*) E_1 lagging behind V. The total electrical power supplied by the prime mover of the alternator is given by $|E_1| \times |I_1| \times \cos \widehat{E_1OI_1}$. This power is evidently greater in (*b*) than in (*a*), and is negative in (*c*), in which case the alternator is taking power from the bus-bars and running as a synchronous motor. Clearly then, if the prime mover gives a little more power, the alternator will tend to be driven a little faster, but immediately this starts E_1 is advanced on V, more electrical power is absorbed, and the prime mover slows down. Alternatively, if the prime mover gives a little less power, the alternator tends to slow down, E_1 lags behind V and the electrical power absorbed is reduced and the prime mover speeds up again. In this way the alternator is held in synchronism with the other alternators operating on the bus-bars.

It should be noted (by drawing out the corresponding vector diagrams) that if Z_1 were a pure resistance there would be no such synchronising action.

Taking V as the reference vector, let E_1 be advanced by an angle ψ,

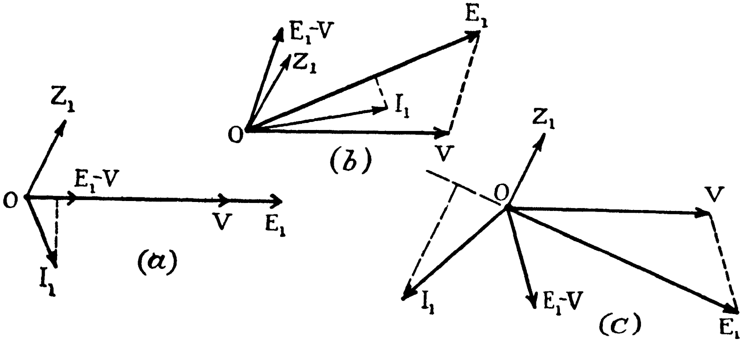


FIG. 80.—Vector Diagrams, Alternator e.m.f. (a) in phase with (b) leading and (c) lagging bus-bar voltage.

so that $E_1 = |E_1|\epsilon^{j\psi}$, and let the angle of Z_1 be ϕ_1, so that $Z_1 = |Z_1|\epsilon^{j\phi_1}$.

Thus

$$I_1 = \frac{|E_1|\epsilon^{j\psi} - |V|}{|Z_1|\epsilon^{j\phi_1}}$$

$$= \left|\frac{E_1}{Z_1}\right|\epsilon^{j(\psi-\phi_1)} - \left|\frac{V}{Z_1}\right|\epsilon^{-j\phi_1}$$

$$= \left\{\left|\frac{E_1}{Z_1}\right|\cos(\psi-\phi_1) - \left|\frac{V}{Z_1}\right|\cos\phi_1\right\}$$

$$+ j\left\{\left|\frac{E_1}{Z_1}\right|\sin(\psi-\phi_1) + \left|\frac{V}{Z_1}\right|\sin\phi_1\right\}$$

and

$$E_1 = |E_1|\cos\psi + j|E_1|\sin\psi.$$

The total electrical power therefore supplied by the prime mover of the alternator is the dot product of I_1 and E_1; *i.e.*,

$P_1 = (ac + bd)$ . . . (see p. 14)

$$= \frac{|E_1|^2}{|Z_1|}\cos(\psi - \phi_1)\cos\psi - \left|\frac{V}{Z_1}\right| \times |E_1| \cos\psi\cos\phi_1$$

$$+ \frac{|E_1|^2}{|Z_1|}\sin(\psi - \phi_1)\sin\psi + \left|\frac{V}{Z_1}\right| \times |E_1| \sin\psi\sin\phi_1$$

$$= \frac{|E_1|^2}{|Z_1|}\cos\phi_1 - \frac{|V| \times |E_1|}{|Z_1|}\cos(\psi + \phi_1)$$

and the net power, P_2, delivered by the alternator to the load is I_1V (dot product), *i.e.*,

$$P_2 = \left|\frac{VE_1}{Z_1}\right|\cos(\psi - \phi_1) - \left|\frac{V^2}{Z_1}\right|\cos\phi_1.$$

The first term in P_1 is constant, and disappears if $\phi_1 = \frac{\pi}{2}$, *i.e.*, if the stator resistance is neglected. It is in any case small. The second term varies as ψ is varied, but here again it may be noted that if $\phi_1 = 0$, *i.e.*, if the stator reactance is neglected, an increase of ψ from zero makes the same alteration to P_1 as an equal decrease, and there is no synchronisation.

The changes in P_1 are well exhibited by the diagram of Fig. 81. OV is drawn horizontally as before. A line D_1OD_2 is drawn making the angle VOD_2 equal to $-\phi_1$. On this line two circles are drawn as shown, with diameters $OD_1 = OD_2 = |V| \times |E_1|/|Z_1|$. The tangent to the two circles at O is the dotted line making an angle $(\pi/2 - \phi_1)$ with OV. Set out any value of E_1, E_1' say, at an angle ψ with V and let OE_1' cut one of the circles in p_1'. Join D_1p_1'. Then since $D_1\widehat{p_1'}O$ is a right angle, and (in upper circle) $D_1\widehat{O}p_1'$

$$= \{(\pi - (\phi_1 + \psi)\},$$

$$\cos D_1\widehat{Op_1'} = Op_1'/OD_1 = -\cos(\phi_1 + \psi)$$

$$\therefore \quad Op_1' = -OD_1\cos(\phi_1 + \psi)$$

$$= -\frac{|V| \times |E_1|}{|Z_1|}\cos(\phi_1 + \psi)$$

= (the second term of the expression for P_1).

In the lower circle, $D_2\widehat{O}p_1'' = (\phi_1 + \psi)$ (ψ is now negative) and

$$-Op_1'' = -\frac{|V| \times |E_1|}{|Z_1|}\cos(\phi_1 + \psi)$$

= (the second term of the expression for P_1).

Thus, wherever the vector E_1 is drawn, the chord cut off on a circle represents the second term of the power supplied from the

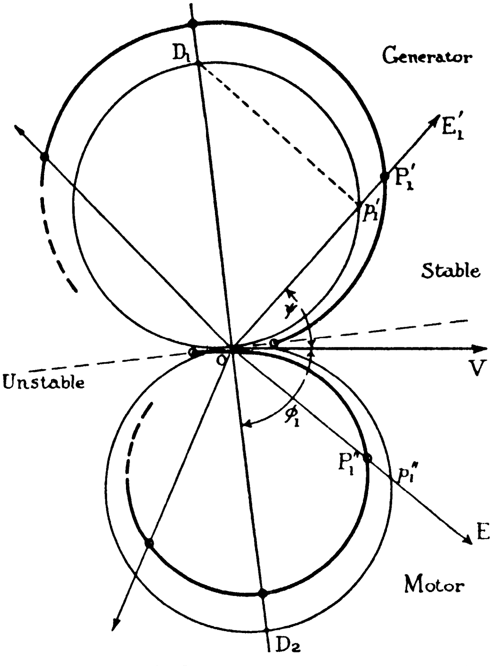


FIG. 81.—Power supplied to generator and delivered by motor.

prime mover, being read $+^{ve}$ in the upper circle and $-^{ve}$ in the lower circle. The constant term $|E_1|^2 \cos\phi_1/|Z_1|$ is allowed for by increasing each chord by this amount in the upper circle and reducing each chord by this amount in the lower circle. The heavily lined curves, approximating to circles, result.

Chords on the upper circle will be cut when ψ is $+^{ve}$ and exceeds $\left(\frac{\pi}{2} - \phi_1\right)$. Consider OE_1', cutting the heavy curve in P_1'. If ψ is increased OP_1' also increases, and *vice versa*, giving the synchronising action previously remarked upon. This is true as ψ increases to $\widehat{VOD_1}$. Beyond this, a further increase of ψ results in a lower power supplied to the load, and a still further increase in ψ. The machine speed goes on increasing, heavy currents flow owing to the phase opposition of E_1 and V, and the circuit breakers come out. The part of the curve to the left of OD_1 represents an unstable condition.

Take now the lower circle. The alternator will receive power from the bus-bars when $\psi < \left(\frac{\pi}{2} - \phi_1\right)$. When the e.m.f. is OE_1'' the machine is running as a synchronous motor, with power OP_1'' available for conversion to mechanical power. If the mechanical load increases the motor will slow down a little and ψ will be increased, *i.e.*, E_1'' lags a little more behind V. This results in an increase in OP_1'', the power drawn from the bus-bars, and the machine speed increases. Similarly, if the load decreases the power supplied also decreases. But if the mechanical load causes OE_1'' to lag behind OD_2, a further load increase will result in a decrease of power supplied, and the machine will slow right down, or fall out of step.

The whole region therefore to the left of D_1OD_2 represents an unstable condition, and the region to the right of D_1OD_2 a stable condition, in the upper part generating and in the lower part motoring.

When the machine is switched on to the bus-bars ψ is zero, and the ability to run in synchronism depends upon the change of P_1 with ψ. $dP_1/d\psi$ is called the *synchronising power.*

$$\frac{dP_1}{d\psi} = \left|\frac{VE_1}{Z_1}\right| \sin(\psi + \phi_1).$$

If $\phi_1 = 0$ and $\psi = 0$ there is no synchronising power.

When $\psi = 0$,

$$\frac{dP_1}{d\psi} = \frac{VE_1}{Z_1} \sin \phi_1$$

$$= VE_1 \cdot \frac{\omega L_1}{R_1^2 + \omega^2 L_1^2}$$

where L_1 is the effective inductance and R_1 the effective resistance of the stator winding.

$\frac{dP_1}{d\psi}$ is easily seen to be a maximum when $R_1 = \omega L_1$.

A diagram for P_2, the net power delivered by the alternator to the line, is drawn in Fig. 82.

$$P_2 = \left|\frac{VE_1}{Z_1}\right| \cos(\phi_1 - \psi) - \left|\frac{V^2}{Z_1}\right| \cos\phi_1$$

and the line D_2OD_1 is drawn with the angle $D_1OV = \phi_1$. The circles are drawn as before with diameter $|VE_1/Z_1|$, and the chord

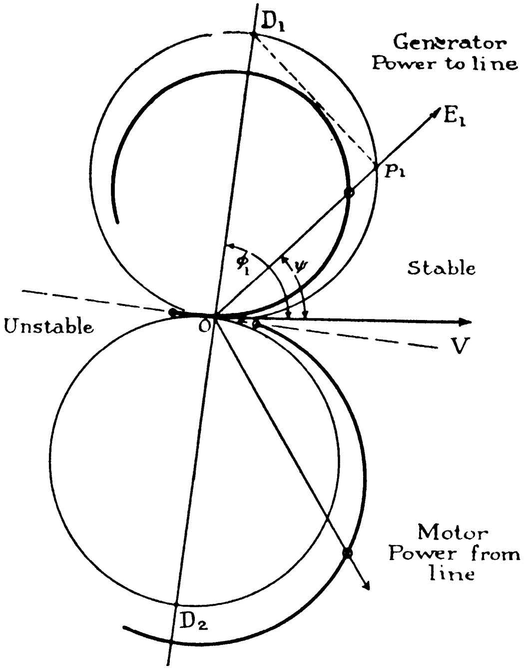


FIG. 82.—Power delivered by generator and supplied to motor.

OP, cut off by OE_1, is evidently the first term of P_2, and is positive in the upper circle and negative in the lower. The constant term $|V^2/Z_1| \cos\phi_1$ is consequently subtracted in the upper circle and added in the lower to give the heavily lined curves. The rays to these curves give the power delivered to line by the generator (upper curve), and the gross power taken by the motor (lower curve).

The manner in which the stator current varies with the rotor (field) current when the generator or motor is working under conditions of constant load is of considerable interest.

Neglecting losses, *i.e.*, putting $\phi_1 = \frac{\pi}{2}$, the expressions for P_1 and P_2 both give

$$P = \left| \frac{VE_1}{Z_1} \right| \sin \psi,$$

and this is to be constant.

V and Z_1 are in any case considered to be constant, so the condition is that

$$| E_1 | \sin \psi \text{ is constant.}$$

Consider first the case of the generator, when ψ is positive (Fig. 83), *i.e.*, OE_1 leads OV by an angle ψ. Drop E_1d perpendicular to OV. Then $E_1d = | OE_1 | \sin \psi$, which is constant. Hence the locus of E_1 is the horizontal straight line, and this is traversed from left to right as E_1, *i.e.*, as the rotor current is increased.

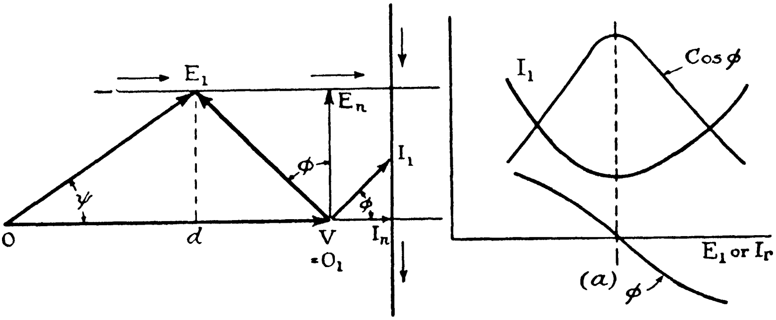


FIG. 83.—Generator excitation at constant load.

The current out of the stator is

$$I_1 = \frac{E_1 - V}{Z_1}.$$

Now $E_1 - V = \overline{VE_1}$,

and if a new pole O_1 is taken at V, then $O_1E_1 = (E_1 - V)$ and the locus of $(E_1 - V)$ is the same horizontal straight line. Since losses are being ignored, Z_1 may be written $j\omega L_1$, and division by $j\omega L_1$ rotates O_1E_1 clockwise through 90° to give O_1I_1. Similarly, for all other points on the horizontal line ; the locus of I_1 is a vertical line with pole O_1, traversed downwards as E_1 is increased. Let the vertical through O_1 meet the horizontal locus in E_n, and OO_1 continued meet the vertical locus in I_n. OE_n can be looked upon as the e.m.f. produced by the normal excitation, and O_1I_n as the resulting current, in phase with the bus-bar potential difference. The angle E_1OE_n = the angle $I_1O_1I_n$ is the phase angle for any other e.m.f. E_1.

The power output $= | V | \, | I_1 | \cos \phi$, and this is constant.

With an excitation below the normal, I_1 leads V and is greater than the normal. As the excitation is increased, I is reduced and the phase angle with it. As the excitation is increased beyond normal, I_1 again increases and lags behind V by an increasing angle.

If these changes are plotted against E_1 or the rotor current, the curves of Fig. 83a are obtained. The similarity in shape to simple resonance curves of series inductance capacity and resistance circuits plotted against ω will be noted; I_1 corresponds to the impedance, $\cos \phi$ to the current and ϕ to the phase angle between the current and the terminal potential difference.

Since
$$| O_1I_n | = \frac{| O_1E_n |}{Z_1} = \frac{| E_1 | \sin \psi}{Z_1}$$
$$= \frac{P}{| V |}$$

it is seen that the curves are sharper, *i.e.*, are described with a smaller change of the exciting current, the smaller the constant load P at which they are taken.

$$| E_nE_1 | = | V | - | E_1 | \cos \psi.$$

If ψ is small (the load small), $\cos \psi$ will not be very different from one, and

$$| I_nI_1 | = \frac{| V | - | E_1 |}{| Z_1 |}.$$

The curves of Fig. 83 (*a*), when plotted against E_1, are of the same form as the " resonance " curves.

Similar considerations hold for the synchronous motor, except that the phase changes are reversed, and the current leads for an over-excitation.

The vector diagram in this case is drawn in Fig. 84. ψ is negative and $I_1 = \dfrac{V - E_1}{Z_1}$,

where I_1 is the current from the mains into the motor. The locus of E_1 is a horizontal straight line as before, and $V - E_1 = E_1V$, which drawn from O is OE′. The horizontal line through E′ is the locus with pole O of $V - E_1$, and is traversed from right to left as E_1 is increased. Dividing by $Z_1 = j\omega L_1$ gives a vertical locus for I_1, traversed upwards as E_1 is increased, and so yielding leading currents when the excitation is greater than normal. The resulting curves are shown in Fig. 84 (*a*).

The ease with which synchronous motors may be caused to take leading currents by simply increasing the excitation is of considerable importance practically, as motors so run may be used in power factor correction.

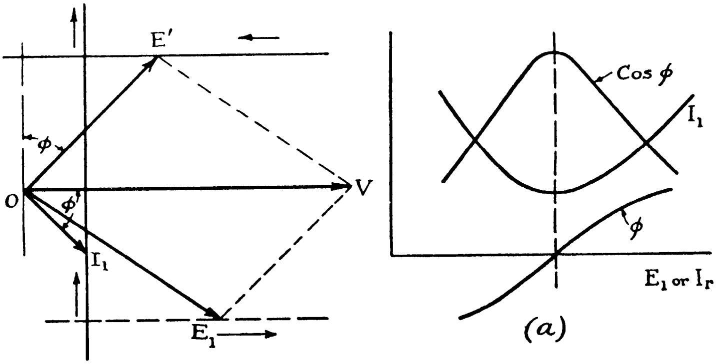


Fig. 84.—Synchronous motor excitation at constant load.

If the losses are taken into consideration the curves will be modified without altering their general shape. The conclusions arrived at are substantially unaltered.

(ii.) *Parallel Operation of Two Alternators.*

Suppose that two alternators are running in parallel to supply a load of impedance Z. Let V be the common terminal potential difference, and E_1 and E_2 the electromotive forces of the alternators and Z_1 and Z_2 their synchronous impedances. If I_1 and I_2

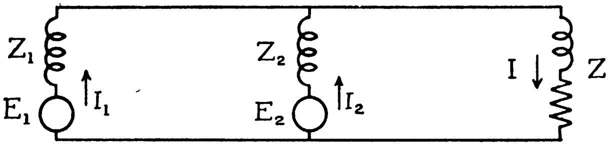


Fig. 85.—Two generators in parallel.

are the alternator currents as shown in Fig. 85, and $I = I_1 + I_2$ is the load current, then

$$\begin{aligned} E_1 - E_2 &= I_1Z_1 - I_2Z_2 \\ E_2 &= I_2Z_2 + IZ = I_2(Z_2 + Z) + I_1Z \\ E_1 &= I_1Z_1 + IZ = I_1(Z_1 + Z) + I_2Z. \end{aligned}$$

Eliminating I_2 from the first and second equations gives

$$I_1 = \frac{(E_1 - E_2)Z + E_1Z_2}{Z(Z_1 + Z_2) + Z_1Z_2}$$

and eliminating I_1 from the first and third equations gives

$$I_2 = \frac{(E_2 - E_1)Z + E_2Z_1}{Z(Z_1 + Z_2) + Z_1Z_2}.$$

The load current I is the sum of I_1 and I_2, and is

$$I = \frac{E_1Z_2 + E_2Z_1}{Z(Z_1 + Z_2) + Z_1Z_2}$$

and the terminal potential difference is

$$V = ZI = \frac{E_1Z_2 + E_2Z_1}{Z_1 + Z_2 + \frac{Z_1Z_2}{Z}}$$

also
$$I_1 = \frac{E_1 - V}{Z_1}, \qquad I_2 = \frac{E_2 - V}{Z_2}.$$

These results can be arrived at in a somewhat different manner by considering the currents due to each alternator separately, and finally adding these currents in each branch of the network to find the total current flowing.

In Fig. 86 (*a*) the currents due to the e.m.f. of the first alternator are shown; I_{11} through the alternator itself, I_{12} in the negative direction through the second alternator, and I_{1l} through the load. Fig. 86 (*b*) shows in a similar manner the currents due to the e.m.f. of the second alternator, I_{22}, I_{21} and I_{2l} through the second alternator, the first alternator and the load in order.

From an elementary consideration of the circuit impedances it is found from Fig. 86 (*a*) that

$$I_{11} = \frac{E_1}{Z_1 + \dfrac{Z_2 Z}{Z_2 + Z}}$$

$$I_{12} = \frac{Z}{Z_2 + Z} I_{11} = \frac{E_1 Z}{Z_1(Z_2 + Z) + Z_2 Z}$$

$$= \frac{E_1}{Z_1 + Z_2 + \dfrac{Z_1 Z_2}{Z}}$$

$$I_{1l} = \frac{Z_2}{Z_2 + Z} I_{11} = \frac{E_1 Z_2}{(Z_1 + Z_2)Z + Z_1 Z_2}$$

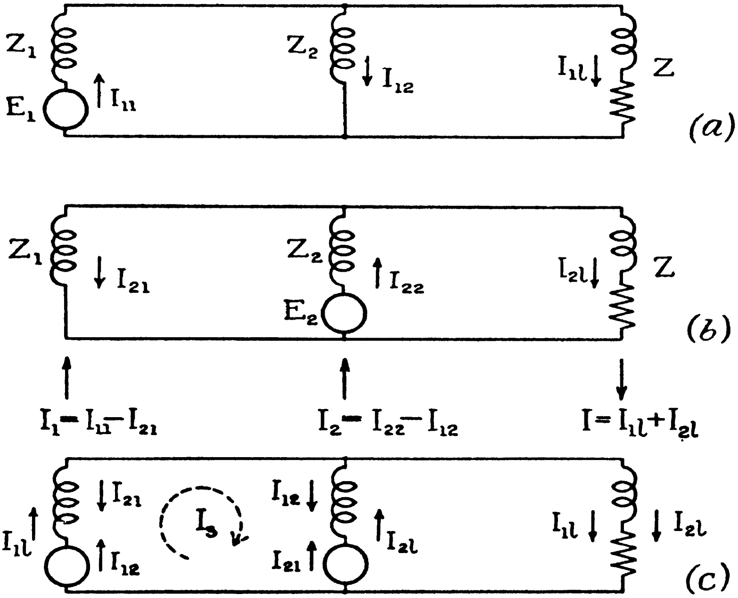


FIG. 86.—Two generators in parallel (alternative treatment).

and similarly from Fig. 85 (*b*),

$$I_{22} = \frac{E_2}{Z_2 + \dfrac{Z_1 Z}{Z_1 + Z}}$$

$$I_{21} = \frac{E_2}{Z_1 + Z_2 + \dfrac{Z_1Z_2}{Z}}$$

$$I_{2l} = \frac{E_2Z_1}{(Z_1 + Z_2)Z + Z_1Z_2}.$$

The total or net alternator and load currents are evidently given by

$$I_1 = I_{11} - I_{21}$$

$$I_2 = I_{22} - I_{12}$$

and $$I = I_{1l} + I_{2l}$$

and the values found from these expressions are identical with those found previously.

A further division may be made which is perhaps illuminating. The current I_{11} due to the e.m.f. of the first alternator may be regarded as being divided into two parts, I_{1l} through the load and I_{12} through the second alternator, and the current I_{22} is similarly regarded as made up of I_{2l} through the load and I_{21} through the first alternator. These currents are all indicated in Fig. 86 (*c*), and it appears that besides the load currents I_{1l} and I_{2l} contributed by the first and second alternators respectively, there is a current $(I_{12} - I_{21})$ which circulates round the two alternators in a clockwise direction. It is this circulating current which provides the synchronisation.

From this point of view the load current is

$$I = I_{1l} + I_{2l}$$
$$= \frac{E_1Z_2 + E_2Z_1}{(Z_1 + Z_2)Z + Z_1Z_2}$$

and the synchronising current is

$$I_s = I_{12} - I_{21}$$
$$= \frac{E_1 - E_2}{Z_1 + Z_2 + \dfrac{Z_1Z_2}{Z}}$$

and $I_1 = I_{1l} + I_s$, $I_2 = I_{2l} - I_s$.

The total loads on the alternators (including losses) are given by the dot products $I_1 \cdot E_1$ and $I_2 \cdot E_2$ respectively, while the contributions to the external load are given by $I_1 \cdot V$ and $I_2 \cdot V$ respectively.

Alternatively the powers contributed to the load may be looked upon as being $I_{1l}\cdot V$ and $I_{2l}\cdot V$ respectively, but then in addition the first alternator must be looked upon as supplying a power $I_s\cdot V$ to the second, or the second alternator as receiving a power $I_s\cdot V$ from the first.

Vector diagrams for a particular case are drawn in Fig. 87, in which (*a*) is the impedance diagram, (*b*) the voltage diagram and (*c*) the current diagram. E_2 is taken as the reference vector and

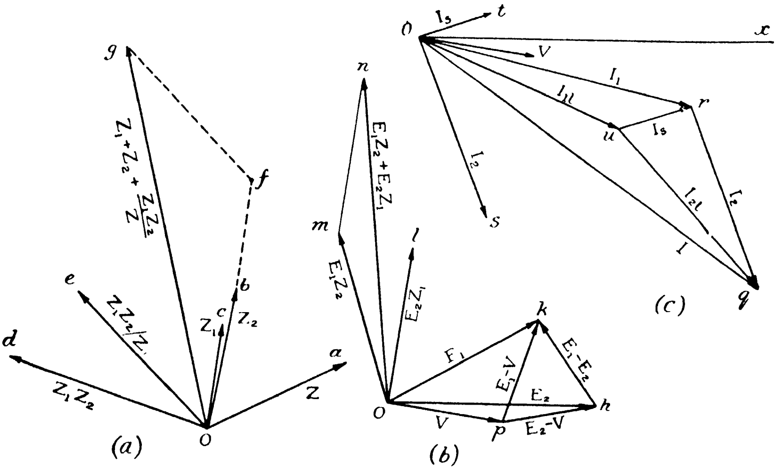


FIG. 87.—Vector diagrams for two generators in parallel.

$E_2 = 3{,}000$ volts; $E_1 = 2{,}500$ volts, leading E_2 by 30°. The synchronous impedances are taken as $Z_1 = 2 + j15$ ohms and $Z_2 = 4 + j20$ ohms, and the load impedance as $10 + j5$ ohms.

In Fig. 87 (*a*), $Oa = Z$, $Ob = Z_2$, $Oc = Z_1$, $Od = Z_1Z_2$, $Oe = Z_1Z_2/Z$ and $Og = Z_1 + Z_2 + Z_1Z_2/Z$.

In Fig. 87 (*b*), $Oh = E_2$, $Ok = E_1$, $Ol = E_2Z_1$, $Om = E_1Z_2$, $On = Om + Ol = E_1Z_2 + E_2Z_1$ and $Op = On/Og = V$. It should be noted that $hk = E_1 - E_2$, $pk = E_1 - V$ and $ph = E_2 - V$.

In Fig. 87 (*c*), $Oq=Op/Oa=V/Z=I$, $Or=pk/Oc=(E_1-V)/Z_1=I_1$ $Os = ph/Ob = (E_2 - V)/Z_2 = I_2 = rq$, $Ot = hk/Og = (E_1 - E_2)/(Z_1 + Z_2 + Z_1Z_2/Z) = I_s = ur$, $Ou = Or - ur = I_1 - I_s = I_1$ and $uq = ur + rq = I_s + I_2 = I_{2l}$.

From the diagrams, $V = 1{,}670$ volts, $I=150$ amps, $I_1=102$ amps,

$I_2 = 68$ amps and $I_s = 26{\cdot}7$ amps, and taking dot products (or the length of one vector multiplied by the length of the projection of the other upon it),

$$\begin{aligned}
\text{Power } (I_1 \cdot E_1) &= 192 \text{ kW.}\\
(I_2 \cdot E_2) &= 71 \text{ kW.}\\
(I_1 \cdot V) &= 171 \text{ kW.}\\
(I_2 \cdot V) &= 53 \text{ kW.}\\
\text{Total load} = I_1V + I_2V &= 224 \text{ kW.}\\
(I_s \cdot V) &= 40 \text{ kW.}\\
(I_{1l} \cdot V) &= 131 \text{ kW.}\\
(I_{2l} \cdot V) &= 93 \text{ kW.}
\end{aligned}$$

Thus the first machine supplies 171 kW. and the second 53 kW. to the load.

Or, alternatively, the first machine supplies 131 kW. to the load and 40 kW. to the second machine, and the second machine supplies 93 kW. to the load but receives 40 kW. from the first machine.

The total load is 224 kW., which agrees quite well with the value of $I^2R = 150^2 \times 10 = 225$ kW.

These results can be obtained algebraically as follows :—

$$E_2 = 3{,}000,\ E_1 = 2{,}500 \angle 30 = 2{,}160 + j1{,}250.$$

$$Z = 10 + j5,\ Z_1 = 2 + j15,\ Z_2 = 4 + j20.$$

$$V = \frac{(2{,}160 + j1{,}250)(4 + j20) + 3{,}000(2 + j15)}{(2 + j15) + (4 + j20) + \dfrac{(2 + j15)(4 + j20)}{10 + j5}}$$

$$= 1{,}660 - j216 \qquad |V| = 1{,}670 \text{ volts.}$$

$$I = \frac{V}{Z} = \frac{(1{,}660 - j216)}{10 + j5} = 124 - j83{\cdot}5 \text{ amps.}$$

$$|I| = 150 \text{ amps.}$$

$$I_1 = \frac{E_1 - V}{Z_1} = \frac{(2{,}160 + j1{,}250) - (1{,}660 - j216)}{2 + j15} = 100{\cdot}4 - j20.$$

$$|I_1| = 102 \text{ amps.}$$

$$\begin{aligned}
I_2 = I - I_1 &= (124 - j83{\cdot}5) - (100{\cdot}4 - j20)\\
&= 23{\cdot}6 - j63{\cdot}5 \qquad |I_2| = 68 \text{ amps.}
\end{aligned}$$

$$I_s = \frac{E_1 - E_2}{Z_1 + Z_2 + \dfrac{Z_1Z_2}{Z}} = \frac{2{,}160 + j1{,}250 - 3{,}000}{(2 + j15) + (4 + j20) + \dfrac{(2 + j15)(4 + j20)}{10 + j5}}$$

$$= 25{\cdot}2 + j9{\cdot}3$$

$$|I_s| = 26{\cdot}7 \text{ amps.}$$

The powers are obtained as the dot products (see p. 14).

Total power given by prime mover of first alternator

$$= (I_1 \cdot E_1) = 100{\cdot}4 \times 2{,}160 - 20 \times 1{,}250 = 192 \text{ kW.}$$

Power given to load

$$= (I_1 \cdot V) = 100{\cdot}4 \times 1{,}660 + 20 \times 216 \quad = 171{\cdot}3 \text{ kW.}$$

$$\text{Efficiency} = \frac{171{\cdot}3}{192} \quad = 89{\cdot}3\%.$$

Total power of second alternator

$$= (I_2 \cdot E_2) = 23{\cdot}6 \times 3{,}000 \quad = 70{\cdot}8 \text{ kW.}$$

Power given to load

$$= (I_2 \cdot V) = 23{\cdot}6 \times 1{,}660 + 63{\cdot}5 \times 216 \quad = 52{\cdot}9 \text{ kW.}$$

Total power to load

$$= (I \cdot V) = 124 \times 1{,}660 + 83{\cdot}5 \times 216 \quad = 224 \text{ kW.}$$

Synchronising power

$$= (I_s \cdot V) = 25{\cdot}2 \times 1{,}660 - 9{\cdot}3 \times 216 \quad = 39{\cdot}9 \text{ kW.}$$

EXERCISES

(1) A 3-phase 16-pole alternator has a star-connected winding with 144 slots and 10 conductors per slot. Find the speed if the frequency is to be 50 cycles per second. If the rotor flux is sinusoidally distributed and has a total value per pole of 5×10^6 lines, find the phase and line voltages.

(2) A 3-phase star-connected alternator rated at 2,000 kVA, 12,000 V has a synchronous impedance per phase of $1{\cdot}5 + j30$ ohms. Find the percentage regulation for a load of 1,800 kW at power-factors of (i.) 0·8 leading, (ii.) unity, (iii.) 0·8 lagging.

(3) Two 3-phase star-connected alternators have each a synchronous impedance of $1{\cdot}2 + j25$ ohms, and run in parallel to share equally a load 750 kW at 3,000 volts and 0·8 power factor. If the stator current of the first alternator is 140 amps. and lags behind the terminal voltage, find (i.) the stator current of the second alternator, (ii.) the power factor at which each machine is operating, (iii.) the e.m.f. of each machine, (iv.) the synchronising current.

(4) A 5,000 kVA alternator with a synchronous impedance of $1 + j10$ ohms is running in parallel with 10,000-volt bus-bars, and develops an e.m.f. of 11,000 volts. Find the maximum external load that the machine can supply.

(5) A 1,000-volt 3-phase star-connected synchronous motor with a synchronous impedance of $0{\cdot}2 + j2{\cdot}2$ ohms per phase takes 250 kW and develops a line e.m.f. of 1,250 volts. Find the line current and the power factor.

CHAPTER VII

THE INDUCTION MACHINE

(1) The Induction Motor

(i.) The most usual form of large induction motor, and the simplest from the point of view of the treatment by vector analysis, is the type having both stator and rotor wound with a three-phase system of coils. The stator is exactly similar to that of the alternator considered in the last chapter and the rotor is similarly wound, so that when the rotor is stationary, the induction motor is simply the rather special three-phase transformer of Fig. 78. It is again assumed that currents in each phase of the system give rise to a flux in the air gap which has a sinusoidal distribution round the circumference, and it follows as before that when the three phases are excited from a three-phase supply a rotating flux is produced. The angular velocity of this flux is ω/p and its revolutions per second $n_1 = f/p$, where f is the supply frequency, $\omega = 2\pi f$ and $2p$ is the number of poles developed by each phase.

In consequence of the transformer action, currents are produced in the rotor by the rotating flux, and these currents experience a torque which, by Lenz's law, tends to rotate the rotor in the same direction as the flux. If the rotor rotates in consequence with a speed n_2 revs. per second, the speed with which the stator flux now cuts the rotor conductors is reduced from n_1 to $(n_1 - n_2)$ revs. per second and the frequency (f_2) of the induced rotor currents is accordingly $(n_1 - n_2)p$.

$$\text{Let } s = \frac{n_1 - n_2}{n_1} = \frac{f_2}{f}$$

so that $$f_2 = sf$$

and $$n_2 = n_1(1 - s).$$

s is called the *slip* and n_1 the *synchronous speed.*

f is the synchronous frequency, and $f_2 = sf$ the frequency of the currents in the rotor.

The currents of frequency sf in the rotor produce a rotating field of speed $n_r = sf/p$ with regard to the rotor itself. With regard to the stator, the speed of the field is n_r plus the speed of the rotor n_2. The actual speed of the rotor flux with regard to the stator is thus

$$\begin{aligned} n_r + n_2 &= sf/p + n_1(1 - s) \\ &= sn_1 + n_1(1 - s) = n_1, \end{aligned}$$

or the frequency of the e.m.f. induced in the stator by the currents in the rotating rotor is f, the synchronous, *i.e.*, the supply frequency.

It follows that the reaction of the rotor on the stator is exactly the same as it would be if the currents in the rotor were of frequency f instead of sf and the rotor were stationary ; in fact, that the rotor reaction is independent of the rotor speed. Under these circumstances the induction motor can be looked upon as a static transformer. The e.m.f. acting in the rotor due to the stator currents will, however, be only s times that which would be acting were the rotor stationary, and the reactance of the rotor windings to the currents of frequency sf flowing in them will of course be only s times that at a frequency f.

If these points are borne in mind and Z_1 and Z_2 are the effective phase impedances of the stator and rotor windings respectively, and M is the effective mutual inductance between a stator and a rotor phase winding, and if V is the phase supply voltage, and I_1 and I_2 are the phase currents in the stator and rotor respectively, the following equations can be written down :—

$$\left.\begin{aligned} V &= Z_1 I_1 - j\omega M I_2 \\ O &= -js\omega M I_1 + Z_2 I_2 \end{aligned}\right\}$$

where

$$\left.\begin{aligned} Z_1 &= R_1 + j\omega L_1 \\ Z_2 &= R_2 + js\omega L_2 \end{aligned}\right\}$$

R_1 and R_2 being the stator and rotor effective phase resistances and L_1 and L_2 the stator and rotor effective phase inductances.

These equations give the following solutions for I_1 and I_2 :—

$$I_1 = \frac{V}{Z_1 + \dfrac{s\omega^2 M^2}{Z_2}} = \frac{V}{R_1 + j\omega L_1 + \dfrac{\omega^2 M^2}{\dfrac{R_2}{s} + j\omega L_2}}$$

$$I_2 = \frac{j\omega M V}{\omega^2 M^2 + \dfrac{Z_1 Z_2}{s}} = \frac{j\omega M V}{\omega^2 M^2 + (R_1 + j\omega L_1)\left(\dfrac{R_2}{s} + j\omega L_2\right)}.$$

The power taken by the motor can be regarded as being used in three ways :—

(i.) In losses in the stator $= | I_1 |^2 R_1$.

(ii.) In losses in the rotor $= | I_2 |^2 R_2$.

(iii.) In mechanical power supplied to the load, and in windage and friction.

The total power supplied to the motor is the dot product $(V \cdot I_1)$, and the power supplied to the rotor (*i.e.*, items (ii.) and (iii.)) is :—

$$P_r = (V \cdot I_1) - |I_1|^2 R_1.$$

I_1 can be written, by rationalising $\omega^2 M^2/(R_2/s + j\omega L_2)$,

$$I_1 = \frac{V}{R_1 + \dfrac{\omega^2 M^2 R_2/s}{R_2^2/s^2 + \omega^2 L_2^2} + j\left(\omega L_1 - \dfrac{\omega^2 M^2}{R_2^2/s^2 + \omega^2 L_2^2}\,\omega L_2\right)}$$

and, rationalising the denominator, to find the in-phase part of I_1, and multiplying by V gives

$$(V \cdot I) = V^2 \frac{R_1 + \dfrac{\omega^2 M^2}{R_2^2/s^2 + \omega^2 L_2^2} \cdot \dfrac{R_2}{s}}{\left(R_1 + \dfrac{\omega^2 M^2 R_2/s}{R_2^2/s^2 + \omega^2 L_2^2}\right)^2 + \left(\omega L_1 - \dfrac{\omega^2 M^2}{R_2^2/s^2 + \omega^2 L_2^2}\omega L_2\right)^2}$$

Further

$$|I_1|^2 R_1 = \frac{V^2 R_1}{\left(R_1 + \dfrac{\omega^2 M^2 R_2/s}{R_2^2/s^2 + \omega^2 L_2^2}\right)^2 + \left(\omega L_1 - \dfrac{\omega^2 M^2}{R_2^2/s^2 + \omega^2 L_2^2}\omega L_2\right)^2}$$

and

$$P_r = V^2 \frac{\dfrac{\omega^2 M^2}{R_2^2/s^2 + \omega^2 L_2^2} \cdot \dfrac{R_2}{s}}{\left(R_1 + \dfrac{\omega^2 M^2 R_2/s}{R_2^2/s^2 + \omega^2 L_2^2}\right)^2 + \left(\omega L_1 - \dfrac{\omega^2 M^2}{R_2^2/s^2 + \omega^2 L_2^2}\omega L_2\right)^2}$$

$$\text{Now } I_2 = \frac{j\omega M V}{\omega^2 M^2 + (R_1 + j\omega L_1)(R_2/s + j\omega L_2)}$$

$$= V \frac{j\omega M/(R_2/s + j\omega L_2)}{R_1 + j\omega L_1 + \dfrac{\omega^2 M^2}{R_2/s + j\omega L_2}}$$

and

$$|\,I_2\,|^2 = V^2 \frac{\dfrac{\omega^2M^2}{R_2^{\,2}/s^2 + \omega^2L_2^{\,2}}}{\left(R_1 + \dfrac{\omega^2M^2R_2/s}{R_2^{\,2}/s^2 + \omega^2L_2^{\,2}}\right)^2 + \left(\omega L_1 - \dfrac{\omega^2M^2}{R_2^{\,2}/s^2 + \omega^2L_2^{\,2}}\omega L_2\right)^2}$$

Thus $$P_r = |\,I_2\,|^2 \cdot \frac{R_2}{s}.$$

When the rotor is stationary $s = 1$ and the whole of the power supplied to the rotor is used in heating the rotor. But as the rotor speeds up and the slip becomes smaller and smaller, the power used in heating the rotor is still given by $|\,I_2\,|^2R_2$, and the remaining power, *i.e.*

$$|\,I_2\,|^2\left(\frac{R_2}{s} - R_2\right) = |\,I_2\,|^2 R_2\frac{1-s}{s},$$

is used mechanically.

(ii.) The vector diagrams for the stator and rotor currents can be drawn from the equations for I_1 and I_2. If the stator losses are neglected, a considerable simplification results, and the diagrams will first be drawn under this condition. Putting $R_1 = 0$ in the equations, the following result :—

$$I_1 = \frac{V}{j\omega L_1 + \dfrac{\omega^2M^2}{j\omega L_2 + \dfrac{R_2}{s}}} = \frac{V}{j\omega L_1 + \dfrac{\omega^2L_1L_2(1-\sigma)}{R_2/s + j\omega L_2}}$$

$$I_2 = \frac{j\omega MV}{\omega^2M^2 - \omega^2L_1L_2 + j\omega L_1\dfrac{R_2}{s}}$$

$$= \frac{\dfrac{M}{L_1}V}{j\omega L_2 - j\omega\dfrac{M^2}{L_1} + \dfrac{R_2}{s}}$$

$$= \frac{\dfrac{M}{L_1}V}{j\omega L_2\sigma + \dfrac{R_2}{s}} = \frac{\sqrt{\dfrac{L_2}{L_1}(1-\sigma)}}{j\omega L_2\sigma + R_2/s}V.$$

where σ is a leakage factor defined by

$$\sigma = 1 - \frac{M^2}{L_1 L_2},$$

as in the case of the transformer (p. 50).

The loci of I_1 and I_2 as R_2/s varies with the load are both circular.

(*a*) The locus of I_2 is drawn in Fig. 88 from

$$I_2 = \frac{\frac{M}{L_1} V}{j\omega L_2 \sigma + \frac{R_2}{s}}.$$

Taking OV = V as the reference vector, OA is drawn $= j\omega L_2 \sigma$ and AB $= R_2/s$. As R_2/s increases from O to ∞, B moves horizontally to the right from A, and inverting AB with a factor of inversion MV/L_1, the semi-circle OI_2D is obtained, described from D counter-clockwise as R_2/s increases from zero. Rays drawn from O to the circle give the rotor currents for various values of R_2/s. The diameter OD is the value the rotor current would have if the resistance R_2 were zero or the slip ∞, and cannot be realised practically. If the rotor is at a standstill, $s = 1$ and the rotor resistance has a value which can be drawn as AB_s, giving a rotor current OI_{2s}. This is the largest value the rotor current can have. As the rotor speeds up and load is taken, the extremity of the current vector travels round the circle from I_{2s} to O. At O the rotor is rotating at synchronous speed ($s = 0$ and $R_2/s = \infty$), again an ideal case because of the rotor losses, such as windage friction, copper and iron. Actually when running light the rotor current will have some such value as OI_2, and the portion of the locus realisable in practice is from I_{2s} to I_2.

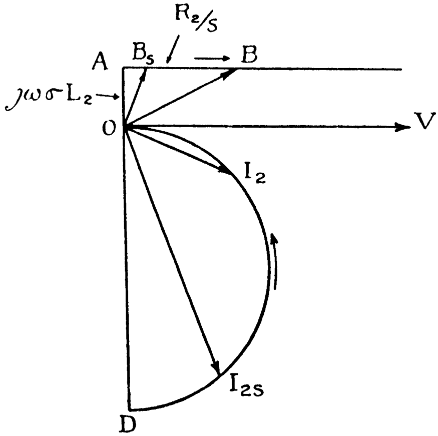


FIG. 88.—Rotor current locus (simplified diagram).

$$|\,OD\,| = \frac{MV}{\omega L_1 L_2 \sigma} = \frac{\sqrt{L_1 L_2 (1-\sigma)}}{\omega L_1 L_2 \sigma}\,V$$

$$= \frac{\sqrt{1-\sigma}}{\sigma} \cdot \frac{1}{\omega\sqrt{L_1 L_2}} \cdot V.$$

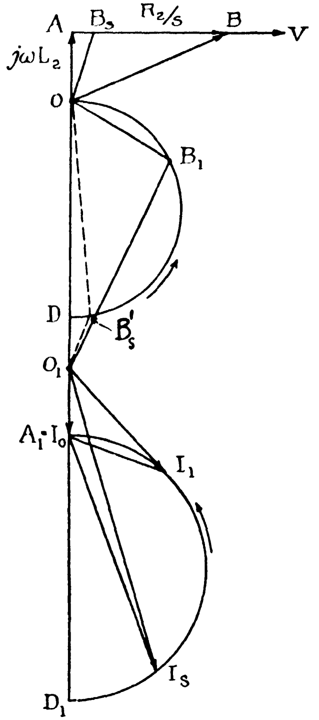


FIG. 89.—Stator current locus (simplified diagram).

(*b*) The vector diagram of the stator current is drawn from the equation

$$I_1 = \frac{V}{j\omega L_1 + \dfrac{\omega^2 L_1 L_2 (1-\sigma)}{j\omega L_2 + \dfrac{R_2}{s}}}$$

in Fig. 89.

$OA = j\omega L_2$, $AB = R_2/s$ and $OB = j\omega L_2 + R_2/s$. As R_2/s increases from zero to ∞ , B moves horizontally to the right from A. AB inverted with a factor of inversion $\omega^2 L_1 L_2 (1-\sigma)$ gives the locus of

$$\omega^2 L_1 L_2 (1-\sigma)/(j\omega L_2 + R_2/s),$$

that is the circle DB_1O of diameter

$$|\,OD\,| = \omega^2 L_1 L_2 (1-\sigma)/\omega L_2 = \omega L_1 (1-\sigma)$$

$j\omega L_1$ is added by using a new pole O_1 on OD produced, O_1O being made equal to $j\omega L_1$.

Finally the locus of I_1 is obtained by inverting the circle DB_1O from the pole O_1, with a factor V to give the circle $D_1I_1A_1$, where

$$|\,O_1D_1| = V/|\,O_1D\,| = V/|\,O_1O\,| - |\,DO\,| = V/\omega L_1 - \omega L_1(1-\sigma)$$
$$= V/\omega L_1 \sigma$$

$$|\,O_1A_1| = V/\omega L_1.$$

The diameter $|\,A_1D_1\,| = |\,O_1D_1\,| - |\,O_1A_1\,|$

$$= \frac{V}{\omega L_1}\left(\frac{1-\sigma}{\sigma}\right)$$

The ratio of O_1A_1 to $O_1D_1 = \sigma$.

O_1A_1 is the stator current I_0 with $R_2/s = 0$, *i.e.*, at no load,

$O_1A_1 = V/j\omega L_1$, as would be expected; the rotor would have no effect on the stator current, as it would be rotating at synchronous speed and so would have no currents induced in it.

O_1D_1 is the stator current with infinite slip.

$O_1D_1 = V/j\omega L_1\sigma$, and again the rotor has no effect.

O_1I_s is the stator current with the rotor at standstill, determined by marking $AB_s =$ the rotor resistance and inverting OB_s to OB_s and O_1B_s' to O_1I_s.

O_1I_o corresponds to the magnetising current of the transformer.

Vectorially $OI_1 - OI_o = I_oI_1$, with locus the circle $I_oI_1D_1$ with pole I_o.

Algebraically,

$$I_1 - I_o = V\left\{\frac{1}{j\omega L_1 + \dfrac{\omega^2 L_1 L_2(1-\sigma)}{j\omega L_2 + \dfrac{R_2}{s}}} - \frac{1}{j\omega L_1}\right\}$$

$$= V\left\{\frac{-\omega^2 L_1 L_2(1-\sigma)}{\left(j\omega L_2 + \dfrac{R_2}{s}\right)(-\omega^2 L_1{}^2) + j\omega^3 L_1{}^2 L_2(1-\sigma)}\right\}$$

$$= V\,\frac{\dfrac{L_2}{L_1}(1-\sigma)}{\dfrac{R_2}{s} + j\omega L_2\sigma}$$

and $$I_2 = \frac{V\sqrt{\dfrac{L_2}{L_1}(1-\sigma)}}{\dfrac{R_2}{s} + j\omega L_2\sigma}.$$

Thus $$\frac{I_1 - I_o}{I_2} = \sqrt{\frac{L_2}{L_1}(1-\sigma)}, \text{ a constant.}$$

It follows that by a suitable adjustment of scale the rotor currents can be read off the circle $I_oI_1D_1$ in the vector diagram of the stator currents.

Thus the rotor current is

$$I_oI_1 \times \sqrt{\frac{L_1}{L_2(1-\sigma)}}$$

when the stator current is O_1I_1. Fig. 89 should be compared with Fig. 88 in this respect. OI_2 and OI_{2s} in Fig. 88 correspond to I_0I_1 and I_0I_s in Fig. 89.

(*c*) Considerable information with regard to the running of the induction motor can readily be obtained from the circle diagram.

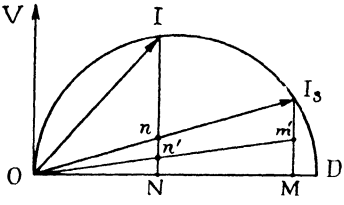


FIG. 90.—Power diagram for induction motor.

In Fig. 90 the primary current locus is redrawn turned through 90° in the positive direction, *i.e.*, taking the diameter of the circle as the reference vector, and omitting the magnetising current. If R is the effective copper resistance referred to the stator the heat losses for any current OI are I^2R. Draw IN and I_sM perpendicular to OD and let OI_s cut the first perpendicular at *n*. Then, considering lengths only.

$OI^2 = ON \cdot OD$ by a well known property of the circle,

i.e., $I^2R = ON \cdot OD \cdot R = k \times ON$, where *k* is some constant, and the heat losses can be found from ON.

But $$\frac{nN}{ON} = \frac{MI_s}{OM}$$

$$\therefore \quad nN = \frac{ON}{OM} . MI_s$$

$$= \frac{\text{losses with current I}}{\text{losses at standstill}} \times MI_s.$$

Now MI_s is the power component of the standstill current OI_s, *i.e.*, the component in phase with V. Thus if the length MI_s is taken to represent the losses at standstill to a certain scale, then to the same scale *n*N gives the losses with the current I. The total power with the current I is represented to the same scale by NI, and it follows, since other losses are being ignored, that the length *n*I represents the output power of the motor.

The efficiency is the ratio of the output power to the total power, *i.e.*, the ratio I*n*/IN.

In exactly the same way it is readily shown that if a point *m′* is found in I_sM so as to divide the heat losses at standstill into two portions, I_sm', due to one part of the circuit and *m′*M due to another part, and if m' is joined to O, cutting nN in n', then, with current I flowing, nn' represents the heat loss in the first part of the circuit and n'N the heat loss in the second part of the circuit. This construction is useful when the stator losses are taken into account.

(iii.) If the stator losses are not neglected, the vector diagram for the primary current can be drawn from the expression

$$I_1 = \frac{V}{R_1 + j\omega L_1 + \dfrac{\omega^2 M^2}{\dfrac{R_2}{s} + j\omega L_2}}$$

In Fig. 91, V is taken as the reference vector, $Oa = j\omega L_2$, $ab = R_2/s$ and Ob inverted with factor $\omega^2 M^2$ gives the circle Db_1O on the vertical diameter OD, described counter-clockwise from D as R_2/s increases from 0 to ∞ .

A new pole O_1 is found by making $O_1O = R_1 + j\omega L_1$, and rays O_1b_1 give the values of the denominator of the above expression.

Finally, inversion of the circle $Db'O$ with pole O_1 and a factor V gives the circle $G'b'O'$ with diameter $O'D'$, also described in the counter-clockwise direction as R_2/s increases, and this is the locus of I_1.

$O_1O' = V/(R_1 + j\omega L_1) = I_0$, the no-load current, *i.e.*, the current that would flow if the rotor were rotating at synchronous speed.

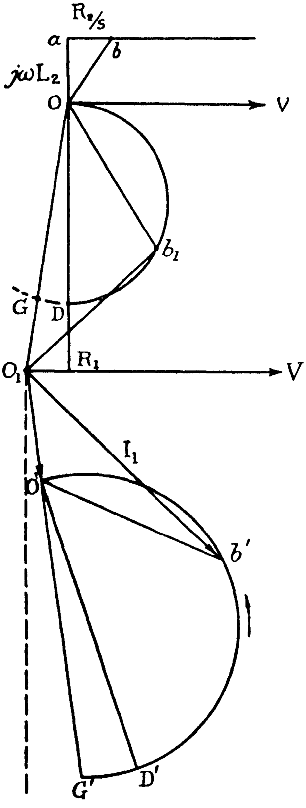


FIG. 91.—Induction motor vector diagram.

$$I_1 - I_0 = \frac{V}{R_1 + j\omega L_1 + \dfrac{\omega^2 M^2}{\dfrac{R_2}{s} + j\omega L_2}} - \frac{V}{R_1 + j\omega L_1}$$

$$= -\frac{V\omega^2 M^2}{\left\{\left(R_1 + j\omega L_1\right)\left(\dfrac{R_2}{s} + j\omega L_2\right) + \omega^2 M^2\right\}\left(R_1 + j\omega L_1\right)}$$

and $$I_2 = \frac{j\omega MV}{\left(R_1 + j\omega L_1\right)\left(\frac{R_2}{s} + j\omega L_2\right) + \omega^2 M^2}.$$

Then $$I_1 - I_0 = \frac{j\omega M}{R_1 + j\omega L_1} I_2.$$

If R_1 is much less than ωL_1, this expression becomes

$$I_1 - I_0 = I_2\sqrt{\frac{L_2}{L_1}(1 - \sigma)}$$

as before, and the circle D′*b*′O′ can represent the rotor current approximately. Actually $(R_1 + \omega L_1)/j\omega M$ has a small negative angle, and the chief effect of R_1 is to bring the diameter of the I_2 circle a little nearer the vertical than is the diameter of the $(I_1 - I_0)$ circle.

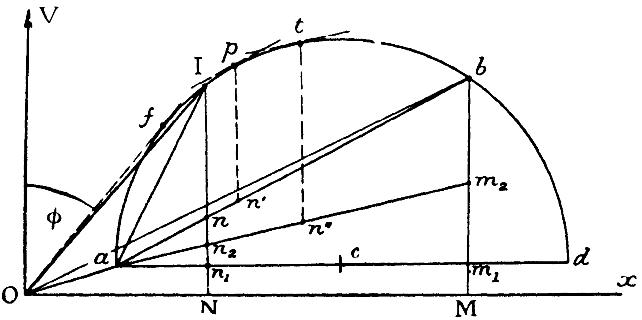


FIG. 92.—Induction motor circle diagram.

Fig. 92 shows the approximate circle diagram of the induction motor as it is usually drawn. The voltage vector OV is drawn vertically and O*a* = the no-load current I_0, and O*b* = the standstill current I_s. The circle is drawn to pass through *a* and *b* with diameter *ad* and its centre on the horizontal through *a*. The points on this diagram should be compared with the corresponding points O_1, O′, *b*′, D′ of Fig. 91.

If the no-load current is written $V/(R_1 + j\omega L_1)$ as before, the resistance R_1 includes components to account for such losses as eddy current, hysteresis, friction and windage in the rotor, as well as the no-load copper and iron losses in the stator.

If I is any point of the circle between *a* and *b*, OI is the primary current and $\widehat{VOI}$ the phase angle ϕ. The power factor is $\cos\phi$, and is a maximum when I is at f, where the tangent to the circle from O touches the circle. cos $\widehat{VOf}$ is the maximum power factor.

aI represents the rotor current and approximately the current scales are in the ratio $I_1/I_2 = \sqrt{L_2(1-\sigma)/L_1}$, or again approximately $= (T_2/T_1)\sqrt{1-\sigma}$, where T_2 and T_1 are the turns on the rotor and stator respectively.

ab is the "power line." If the vertical through I cuts ab in n, the diameter ad in n_1, and the horizontal axis Ox in N, In is the output power to the same scale that IN is the total input power, and nn_1 is the total copper loss and n_1N the no-load loss regarded as constant. If a tangent to the circle is drawn parallel to ab to touch the circle at p, then Op is the current at which the induction motor gives its maximum output pn', where pn' is a vertical cutting ab in n'.

The overall efficiency of the motor is the ratio In/IN.

If a vertical is drawn through b to meet the circle diameter in m_1, and the axis in M, m_1M is the no-load loss, and bm_1 the heat losses at standstill, *i.e.*, the heat losses in addition to the no-load heat losses. If a point m_2 is found in bm, so that the ratio bm_2/m_2m_1 equals the ratio, rotor heat losses/stator heat losses,

$$= \frac{|I_2|^2R_2}{|I_1|^2R_1} = \frac{L_1}{L_2(1-\sigma)} \cdot \frac{R_2}{R_1} = \frac{T_1^2}{T_2^2(1-\sigma)} \cdot \frac{R_2}{R_1}$$

and am_2 meets IN in n_2, then $nn_2 \doteqdot$ rotor heat loss and $n_2n_1 \doteqdot$ stator heat loss at the current I.

It should be noted that approximately

$$\frac{|Oa|}{|Od|} = \sigma \quad . \quad . \quad . \quad . \quad (\text{p. } 130)$$

which enables the leakage coefficient to be found.

Also *approximately* (accurately if O, a, d, were in one straight line),

$$\widehat{VOf} = \widehat{Ocf}$$

and the maximum power factor is given by

$$\begin{aligned}\cos\phi &= fc/Oc = \tfrac{1}{2}(ad)/(Oa + \tfrac{1}{2}(ad)) \\ &= \frac{\tfrac{1}{2}(Od - Oa)}{Oa + \tfrac{1}{2}(Od - Oa)} \\ &= \frac{Od - Oa}{Od + Oa} = \frac{Od - \sigma Od}{Od + \sigma Od} \\ &= \frac{1-\sigma}{1+\sigma}.\end{aligned}$$

Since $In_2 \propto$ power to rotor $= |\,I_2\,|^2R_2/s$ and $nn_2 \propto$ heat loss in rotor $= |\,I_2\,|^2R_2$, the slip is found from the ratio

$$\frac{nn_2}{In_2} = s.$$

And since output $\propto In$, and torque

$$= \frac{\text{output}}{\text{speed}} \propto \frac{In}{(1-s)n} = \frac{In_2 - nn_2}{n(1-s)} = \frac{In_2}{n} . \frac{\left(1 - \frac{nn_2}{In_2}\right)}{1-s}$$
$$= \frac{In_2}{n} \propto In_2,$$

since n the synchronous speed is constant.

Thus In_2 represents the torque to some scale, and bm_2 represents the standstill or starting torque to the same scale.

If a tangent to the circle parallel to am_2 touches the circle at t and a vertical is drawn cutting am_2 in n'', tn'' is the maximum torque the motor can develop. If I is to the left of t an increase of load with consequent increase of current gives an increase of torque and the running of the motor is stable. If I is to the right of t the reverse is the case. An increase of load results in a decrease of torque, and the conditions are unstable. The working part of the circle is therefore from a to t.

(iv.) The manner in which the torque varies with the rotor resistance is of some importance. It has been seen (p. 128) that the mechanical output of the rotor is $|\,I_2\,|^2R_2(1-s)/s$, the angular velocity of the rotor is $\omega(1-s)$, and that if the small primary resistance is neglected,

$$I_2 = \frac{\sqrt{\frac{L_2}{L_1}(1-\sigma)}}{j\omega\sigma L_2 + R_2/s} V.$$

The torque developed is therefore

$$G = \frac{\text{mechanical power}}{\text{angular velocity}}$$
$$= \frac{|\,I_2\,|^2R_2(1-s)/s}{\omega(1-s)} = \frac{|\,I_2\,|^2R_2}{\omega s}$$

and $$|\,I_2\,|^2 = \frac{\frac{L_2}{L_1}(1-\sigma)}{\omega^2\sigma^2 L_2{}^2 + \frac{R_2{}^2}{s^2}} V^2.$$

Thus writing $\omega\sigma L_2 = X_2$, the rotor reactance,

and $$A = \frac{L_2}{\omega L_1}(1-\sigma)V^2,$$

then $$G = \frac{AR_2 s}{R_2{}^2 + s^2 X_2{}^2}.$$

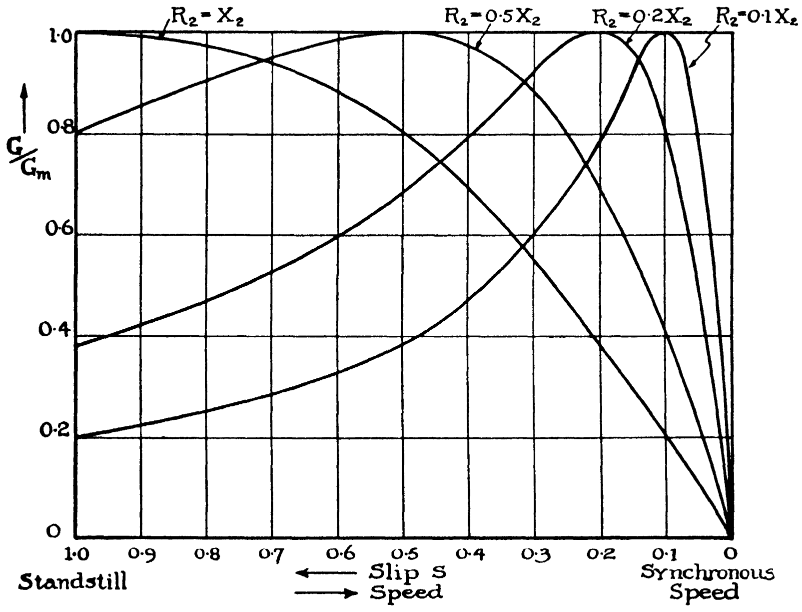


FIG. 93.—Torque-speed curves at various values of rotor resistance.

For G to be a maximum at any given value of s, $(R_2{}^2 + s^2X_2{}^2)/R_2 s$ must be a minimum, *i.e.*,

$$\frac{d}{dR_2}\left(\frac{R_2}{s} + \frac{sX_2{}^2}{R_2}\right) = \frac{1}{s} - \frac{sX_2{}^2}{R_2{}^2} = 0$$

or $$R_2 = sX_2,$$

and the maximum torque G_m is

$$G_m = A/2X_2$$

and $$\frac{G}{G_m} = \frac{2sR_2X_2}{R_2{}^2 + s^2X_2{}^2}.$$

Fig. 93 gives some curves plotted from this expression showing how the ratio G/G_m varies with s and the speed for various values of R_2/X_2. When $R_2 = X_2$ maximum torque is obtained at standstill; this explains why, since R_2 is usually much less than X_2, a large starting torque can be obtained by the insertion of rotor resistance by bringing the terminals of the rotor winding to slip rings.

(2) **The Induction Generator**

The manner in which the torque developed by the rotor as the speed increases from standstill has been studied in the preceding pages. The maximum speed attained is a little below the synchronous speed owing to the inherent power losses in the rotor. But if the rotor is driven by supplying the machine with mechanical power, the rotor speed can be raised to synchronism and beyond, *i.e.*, the slip can be given a zero value and negative values. A negative slip implies an effective negative resistance (R_2/s) in calculating the mechanical power ($I_2^2R_2/s$) developed by the machine; or what comes to the same thing, $I_2^2R_2/s$, with negative values of s, is a measure of the mechanical power given to the machine, and largely converted into electrical power. The electrical power is drawn from the stator; the frequency by the same considerations as before is f, and energy is pumped back to the supply as in regenerative braking.

All the formulæ developed above apply to the case of negative slip; the vector diagrams apply also with suitably extended loci. For instance, in the diagram of Fig. 91, it has been seen that b moves horizontally from a to the right as the slip (positive) decreases from ∞ to 0, reaching $+\infty$ as the slip reaches $+0$. As the speed is increased through synchronism, the slip changes from $+0$ to -0, and b moves from $+\infty$ to $-\infty$. With further increases in the speed the numerical value of the negative slip increases, the numerical value of R_2/s decreases and b moves along the horizontal from $-\infty$, to the right towards a. While this happens, the circle, Db_1O is continued from O counter-clockwise and the circle $D'b'O'$ is continued from O' counter-clockwise. When b' lies to the left of the dotted vertical through O_1 the product $I_1 \cdot V$ is negative, and power is supplied *from* the machine *to* the bus-bars.

Fig. 92 extended in this way gives Fig. 94, in which the points a, b, M, m_1 and m_2 are fixed as before, and in which OI indicates the stator current flowing with a negative slip. I, N, n_1, n_2, n is drawn perpendicular to Ox, and

OV $=$ bus-bar voltage.
OI $=$ stator current,
Ia $=$ rotor current (to another scale).
In $=$ total mechanical power supplied to the rotor.
n_2n $=$ rotor heat losses.
n_1n_2 $=$ stator heat losses.
Nn_1 $=$ " constant " losses.
IN $=$ electrical power delivered to bus-bars.
Slip $= nn_2/\mathrm{I}n_2$.

As in the case of the motor there is a region of stability and one of instability. If the speed is increased beyond a certain value, the torque required decreases and the machine races, delivering less and less power as the rotor speeds up. This is made clear by a consideration of the changes in the various quantities involved as I describes the circle in the counter-clockwise direction.

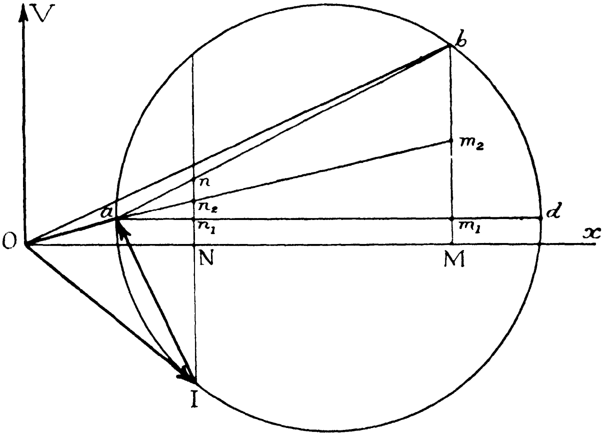


FIG. 94.—Induction generator circle diagram.

EXERCISES

(1) A 3-phase induction motor with a star-connected rotor has a voltage of 60 volts between slip-rings when normal voltage is applied to the stator and the rotor is at a standstill and open-circuited. The standstill impedance of the rotor winding per phase is $0{\cdot}6 + j4{\cdot}0$ ohms. Find the rotor current (i.) when at standstill, the slip-

rings being connected to a star-connected resistance of 5 ohms per phase ; (ii.) when running with 4 per cent. slip with the slip-rings short circuited.

(2) A 3-phase 4-pole 50-cycle induction motor has a rotor (star-connected) impedance (at standstill) of $0{\cdot}04 + j0{\cdot}20$ ohms per phase. Plot the torque/speed curve, and find (i.) the speed at which maximum torque is developed, (ii.) the resistance to be added to obtain maximum torque at starting. Find also the percentage by which the insertion of this resistance will reduce the starting current.

(3) A 6-pole induction motor operates on a 3-phase supply at 50 cycles per sec. The maximum torque obtainable is 100 lbs. ft., and this occurs when the speed is 880 r.p.m. Find the torque developed when the slip is 5 per cent. The rotor resistance per phase is 0·6 ohm.

(4) Draw the circle diagram for a 5 h.p. 200-volt, 50-cycle, 4-pole, 3-phase, star-connected induction motor from the following test data :—

No load : line voltage = 200 volts ; line current = 5 amps. ; input = 350 watts.

Short circuit : line voltage = 100 volts, line current = 26 amps. ; input = 1,700 watts.

Estimate from the diagram the line current, power factor, slip, efficiency, torque and speed at full load. The stator and rotor copper losses at standstill are equal.

CHAPTER VIII

SINGLE-PHASE MOTORS

(1) Induction Motors

ALTHOUGH a single-phase stator winding produces a pulsating and not a rotating field, yet the pulsating field can be regarded as being made up of two equal fields rotating in opposite directions. Each field will produce currents in the rotor and tend to rotate it in opposite directions. The machine, therefore, will not start up from standstill, but if it is started in some way, the forward torque produced by the field rotating in the same direction as the rotor is greater than the backward torque produced by the field rotating in the opposite direction, and the rotor will continue to rotate, and can take a load.

The two rotating fields can be regarded as being produced by separate windings. In Fig. 95 the single-phase stator winding of the machine is regarded as being divided into two equal windings, a_1 and b_1, and there are added, geometrically at angles of 120°, two other windings a_2b_2 and a_3b_3, each divided into two equal halves carrying oppositely directed currents as indicated by the arrows. These additional windings are purely imaginary, but would have no influence on the working of the machine, if they actually existed, since the combined effect of the currents in them is zero.

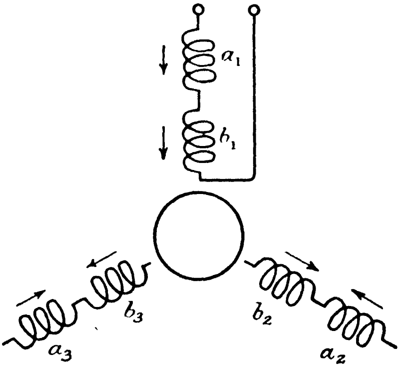


FIG. 95.—Single-phase motor. Equivalent field system.

Now imagine that the currents in a_2 and b_2 are 120° out of phase with those in a_1 and b_1, and the currents in a_3 and b_3 are 240° out of phase with those in a_1 and b_1. Then the currents in coils a_1 a_2 and a_3 are " three-phase " and produce a field rotating in, say, a clockwise direction, and the currents in b_1, b_2 and b_3 are also " three-phase," and since two of the phases are reversed, produce a field rotating in a counter-clockwise direction.

Let the rotor rotate in a clockwise direction with a speed n_r and let the synchronous speed be n_s. Then with regard to the "*a*" field, the slip is

$$s = \frac{n_s - n_r}{n_s} = 1 - \frac{n_r}{n_s}$$

and with regard to the "*b*" field, the slip is

$$s' = \frac{n_s + n_r}{n_s} = 1 + \frac{n_r}{n_s} = 2 - s.$$

The rotor currents produced in virtue of these slips may be regarded as having a separate existence in the rotor, or the machine may be regarded as divided into two separate machines with the two rotors fixed to the same shaft as indicated in Fig. 96, and the stator windings, a_1, a_2, a_3 for one machine and b_1, b_2, b_3 for the other, supplied in series from a three-phase supply of phase voltage V.

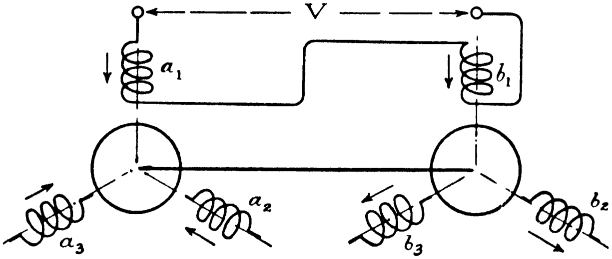


FIG. 96.—Single-phase motor. Equivalent field system.

The problem is now reduced to that of two three-phase machines in series, the *a* machine developing a forward, and the *b* machine a backward torque, and can be solved by the methods developed in the preceding chapter.

Although the frequency of the currents in the *a* rotor is sf and that of the currents in the *b* rotor is $(2 - s)f$, the frequency of the reaction currents in the stator is in each case f, by the same reasoning as before (p. 125). The circle diagram for constant voltage V_a across the first machine is readily found as before, but the diagram for constant voltage V_b across the second machine involves a slight modification. In this case *b* (Fig. 91) moves to the left from the standstill value as *s* decreases from unity, and the parts of the circles described are those below the standstill points. Fig. 92 is modified as shown in Fig. 97. I lies to the right of b and n lies on ab produced to cut the vertical through I. The heat losses are increased to nn_1 and of the power required $\mathrm{I}n_1$ comes from the stator and $n\mathrm{I}$ from the rotor. This power $n\mathrm{I}$ is mechanical power from the rotor of the a machine.

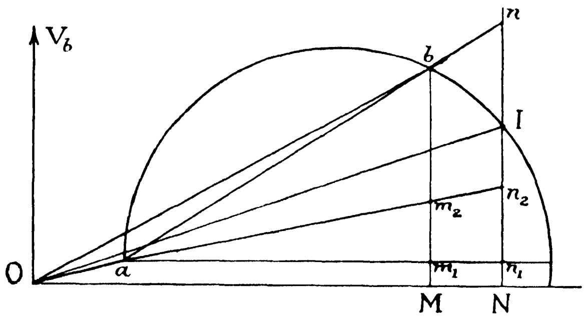


FIG. 97.—Three-phase induction motor circle diagram—driven reversed.

The voltages V_a and V_b, however, are not constant. But if Z_1 is the impedance of each of the windings a and b and M the mutual inductance of each with the rotor winding, the effective stator-phase impedances Z_a and Z_b of the two machines are given by

$$Z_a = Z_1 + \frac{\omega^2 M^2}{R_2/s + j\omega L_2}$$

and

$$Z_b = Z_1 + \frac{\omega^2 M^2}{R_2/(2-s) + j\omega L_2}$$

as before (p. 126), where R_2 and L_2 are the rotor resistance and inductance, and the equations

$$I_1 = \frac{V_a}{Z_a} = \frac{V_b}{Z_b}$$

$$V = V_a + V_b$$

$$I_1 = \frac{V}{Z_a + Z_b}$$

enable all these quantities to be found, and hence the two rotor currents I_{2a} and I_{2b} by the formula on p. 128.

The mechanical power developed by the first rotor is

$$|I_{2a}|^2 R_2 (1 - s)/s.$$

From the same formula that for the second rotor (replacing s by $2 - s$) is $-|I_{2b}|^2R_2(1-s)/(2-s)$, and this power must be supplied by the first rotor. Thus the net mechanical power available for the external load is

$$|I_{2a}|^2R_2(1-s)/s - |I_{2b}|^2R_2(1-s)/(2-s).$$

(ii.) If the resistances of the stator windings are neglected

$$I_{2a} = \frac{MV_a/L_1}{R_2/s + j\omega L_2\sigma}, \quad I_{2b} = \frac{MV_b/L_1}{R_2/(2-s) + j\omega L_2\sigma}$$

as on p. 128, and a further simplification results if the magnetising current is neglected in each case. Write $I_o = V_a/j\omega L_1$.

Then

$$I_1 - I_o = V_a\left\{\frac{1}{j\omega L_1 + \dfrac{\omega^2M^2}{R_2/s + j\omega L_2}} - \frac{1}{j\omega L_1}\right\}$$

$$= V_a\Big/\frac{L_1^2}{M^2}\left\{\frac{R_2}{s} + j\omega\sigma L_2\right\}$$

and similarly

$$I_1 - I_o = V_b\Big/\frac{L_1^2}{M^2}\left(\frac{R_2}{2-s} + j\omega\sigma L_2\right).$$

Thus neglecting I_o, and using these expressions in those for I_{2a} and I_{2b}, it appears that

$$I_{2a} = I_{2b} = \frac{L_1}{M}I_1$$

and

$$Z_a = \frac{L_1^2}{M^2}\left\{\frac{R_2}{s} + j\omega\sigma L_2\right\}$$

$$Z_b = \frac{L_1^2}{M^2}\left\{\frac{R_2}{2-s} + j\omega\sigma L_2\right\}$$

and

$$I_1 = \frac{V}{\dfrac{L_1^2}{M^2}\left\{\dfrac{R_2}{s} + \dfrac{R_2}{2-s} + j2\omega\sigma L_2\right\}}$$

with a circular locus.

Since $I_{2a} = I_{2b} = I_2$, say, the power available for the external load is

$$|I_2|^2R_2\{(1-s)/s - (1-s)/(2-s)\}$$

and if an approximate circle diagram is drawn for the two three-phase motors together (*i.e.*, for the single-phase motor), as in Fig. 92 this power is represented by In.

The total rotor power is

$$|I_2|^2R_2\{1/s + 1/(2 - s)\} \quad . \quad . \quad . \quad (\text{p. } 128).$$

and this is represented by In_2.

Thus

$$\frac{In}{In_2} = \frac{(1 - s)/s - (1 - s)/(2 - s)}{1/s + 1/(2 - s)}$$

$$= (1 - s)^2$$

and

$$s = 1 - \sqrt{\frac{In}{In_2}}.$$

Also the torque

$$= \frac{\text{mechanical output}}{\text{speed}}$$

$$\propto \frac{In}{1 - s}$$

$$= \sqrt{In \, . \, In_2}.$$

It is not literally correct to write, as has been done above, $I_{2a} = I_{2b}$ since the frequency of the first current is sf and that of the second is $(2 - s)f$. It is, however, true of the effects of these currents on the stator winding, since the reaction in each case is of frequency f.

(iii) Even if the simplifying assumptions of negligible stator resistance and magnetising current are not made, the stator current has a circular locus. For

$$I_1 = V/(Z_a + Z_b)$$

$$= V/\left\{2Z_1 + \omega^2M^2\left(\frac{1}{R_2/s + j\omega L_2} + \frac{1}{R_2/(2 - s) + j\omega L_2}\right)\right\}.$$

Writing $\dfrac{1}{R_2/s + j\omega L_2} + \dfrac{1}{R_2/(2 - s) + j\omega L_2} = x + jy$

it is clear that if $x + jy$ is a circle, then I_1 has a circular locus.

This equation gives

$$\frac{R_2}{s}+\frac{R_2}{2-s}+j2\omega L_2=\left\{\frac{R_2{}^2}{s(2-s)}-\omega^2L_2{}^2+j\omega L_2\left(\frac{R_2}{s}+\frac{R_2}{2-s}\right)\right\}(x+jy)$$

and putting

$$R_2/s + R_2/(2 - s) = \lambda$$

$$R_2{}^2/s(2 - s) = R_2\lambda/2$$

and
$$\begin{aligned}\lambda + j2\omega L_2 &= (R_2\lambda/2 - \omega^2L_2^2 + j\omega L_2\lambda)\,(x + jy)\\ &= (R_2\lambda/2 - \omega^2L_2^2)\,x - \omega L_2\lambda y\\ &\quad + j\,\{(R_2\lambda/2 - \omega^2L_2^2)y + \omega L_2\lambda x\}.\end{aligned}$$

Equating real and imaginary parts gives

$$\lambda = (R_2\lambda/2 - \omega^2L_2^2)\,x - \omega L_2\lambda y$$

and $2\omega L_2 = \omega L_2\lambda x + (R_2\lambda/2 - \omega^2L_2^2)y$

whence $\lambda\,(1 - R_2x/2 + \omega L_2y) = -\,\omega^2L_2^2x$

and $\lambda\,(\omega L_2x + R_2y/2) = 2\omega L_2 + \omega^2L_2^2y$

and eliminating λ,

$$-\,\omega L_2x(\omega L_2\,x + R_2y/2) = (2 + \omega L_2y)\,(1 - R_2x/2 + \omega L_2y),$$

giving

$$\omega^2L_2^2(x^2 + y^2) + 3\omega L_2y - R_2\,x + 2 = 0,$$

the equation of a circle which can be written

$$(x - R_2/2\omega^2L_2^2)^2 + (y + 3/2\omega L_2)^2 = (R_2^2 + \omega^2L_2^2)/4\omega^4L_2^4.$$

showing that the centre of the circle is

$x = R_2/2\omega^2L_2^2$, $y = -\,3/2\omega L_2$ and that the radius is

$$\sqrt{R_2^2 + \omega^2L_2^2}/(2\,\omega^2L_2^2).$$

(2) **Commutator Motors**

The single-phase induction motor is not inherently self-starting and is essentially a constant speed machine ; good starting torque and easy speed control are, however, possible with commutator motors and many forms have been devised for use where these considerations are important. The natural approach to an understanding of these machines is from the direct current series motor, in which the supply voltage V is related to the current I and the sum of the armature and field resistances R by the expression

$$V = RI + E.$$

Here E is the back e.m.f. generated by the armature rotating with a speed n_1 revs. per sec. in the field of Φ lines produced by the poles, and if $2p$ = the number of poles, $2a$ = the number of armature paths and N = the total number of armature conductors.

$$E = (p/a)Nn_1\Phi \times 10^{-8}.$$

Let n be the angular (electrical) velocity of rotation, *i.e.*, $n =$

$2\pi n_1 p$, and let the flux Φ be related to the current I by which it is produced by the expression $\Phi = k\mathrm{I}$, then

$$\mathrm{E} = \frac{\mathrm{N}k}{2\pi a} \times 10^{-8}\, 2\pi n_1 p \,.\, \mathrm{I} = n\mathrm{MI}$$

where $\mathrm{M} = \mathrm{N}k \times 10^{-8}/(2\pi a)$ can be regarded as the effective mutual inductance between the field winding system and the armature. M is not constant as k varies with the current, owing to the non-linear shape of the magnetisation curve of the iron, but in A.C. commutator motors the fluxes are kept low to minimise commutation difficulties and the variation of M with I is not serious.

In this way the D.C. series motor equation can be written

$$\mathrm{V} = \mathrm{RI} + n\mathrm{MI} = (\mathrm{R} + n\mathrm{M})\mathrm{I}$$

or

$$\mathrm{VI} = \mathrm{RI}^2 + n\mathrm{MI}^2$$

from which it appears (*a*) that part (RI^2) of the power VI given to the motor is used in heating the conductors, and part $n\mathrm{MI}^2$ is used in rotating the armature against the torque of the load. Further, since V is constant the smaller the load the smaller I and the larger n must be. (*b*) That in consequence of the mechanical load the effective resistance of the machine may be regarded as increased by an amount $n\mathrm{M}$ ohms.

If the direction of the current through the armature is reversed, the direction of the flux through the armature is also reversed, and the direction of the torque is in consequence unchanged. Thus there is no fundamental reason why a D.C. series motor should not be run off an A.C. supply. It is necessary to laminate the field system to minimise eddy current effects, and the difficulties of commutation make special arrangements essential except for the very smallest machines. The voltage V applied to the machine and hence the speed can readily be varied without loss by means of multiple tappings on the secondary winding of a transformer, and the starting torque is high.

A.C. commutator motors usually have no salient poles in the field system; the field winding is arranged on a stator similar to that of an induction motor. The stator slots can easily accommodate the compensating windings used in some designs.

(i.) *Simple Series Motor*

Consider first the simple series machine indicated in Fig. 98. Let $\mathrm{Z}_1 = \mathrm{R}_1 + j\omega\mathrm{L}_1$ be the impedance of the field winding and

$Z_2 = R_2 + j\omega L_2$ be the impedance of the armature winding from brush to brush.

There will be no brush to brush electromotive force owing to transformer action from the alternating field, as the e.m.f.'s induced in the armature conductors cancel each other with the symmetrical disposition of the brushes shown. This is made clear by a consideration of the e.m.f.'s induced in each turn of the gramme ring armature of Fig. 99, and is true in all cases. The light arrows show the direction of the e.m.f.'s induced when the field is increasing in the direction indicated. The same diagram also shows that there will be an induced current in the coils short-circuited by the brushes, and this leads to difficulty in commutation.

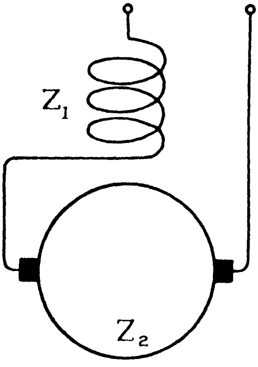


FIG. 98.—Simple series motor.

There will, however, be a brush to brush e.m.f. owing to the rotation of the armature in the field; the individual coil e.m.f.'s are shown by the heavy arrows. This can be taken account of as above by an increase in the effective armature resistance of nM_{12}, where M_{12} is the effective mutual inductance between field and armature, and n is the angular (electrical) velocity of rotation of the armature. Here it may be noted that although the armature rotates, the field associated with the armature currents is stationary because of the fixed position of the brushes. In consequence the frequency of all currents and electromotive forces is the supply frequency f.

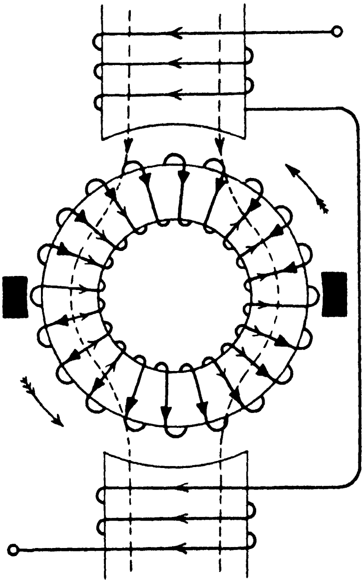


FIG. 99.—Gramme-ring armature.

The vector equation for the motor can now be written

$$\begin{aligned} V &= Z_1 I + Z_2 I + nM_{12} I \\ &= \{R_1 + R_2 + nM_{12} + j\omega(L_1 + L_2)\} I \end{aligned}$$

From this the locus of the current for varying load is readily drawn in Fig. 100, friction losses being included in the mechanical load. OA $= j\omega(L_1 + L_2)$, AB $= R_1 + R_2$ and BC $= nM_{12}$. C moves horizontally to the right from B as the speed increases. The line OC inverted with O as pole and with factor V gives the circle DSIO, the portion SIO of which is the locus of the current I. OB inverted gives the standstill current OS and OC inverted gives the current OI of phase angle Φ with OV. In exactly the same way as with the induction motor diagram, if perpendiculars SM, I*n*N are dropped from S and I on to OD, SM = the standstill heat losses, *n*N = the heat losses with current OI and I*n* = the mechanical output. BC is evidently proportional to the speed, and the torque is proportional to I*n*/BC.

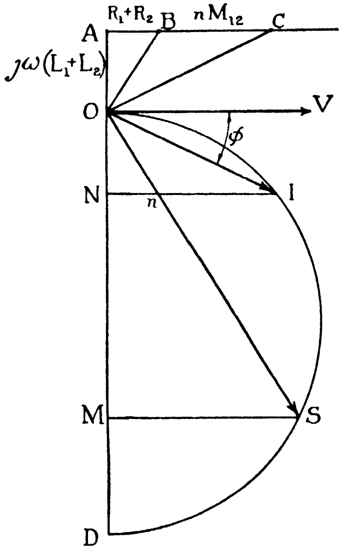


FIG. 100.—Series motor vector diagram.

(ii.) *Compensated Series Motor.*

In Fig. 99 the currents in the armature produce a field with an axis along the brush axis, and this field in combination with the field of the stator winding produces a resultant field whose axis is *not* along the brush axis, and in consequence sparking at the brushes results, as in a direct current machine. This effect may be overcome by winding a compensating winding on the stator to produce a field along the brush axis equal and oppositely directed to that produced by the armature, as is indicated in Figs. 101 and 102. From a consideration of Fig. 102 it will be clear that the e.m.f.'s in the armature conductors produced by the alternations of flux along

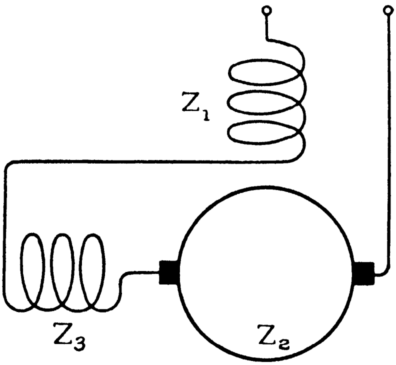


FIG. 101.—Compensated series motor.

the brush axis due to the compensating winding do not cancel out, but have a resultant tending to produce a current from brush to brush through any external circuit. In other words, there is a mutual inductance M_{23}, between the compensating winding and the armature winding, and if the current in the former is I_3, the e.m.f. induced in the latter is $j\omega M_{23}I_3$. There is, however, no net rotational e.m.f. due to the compensating winding.

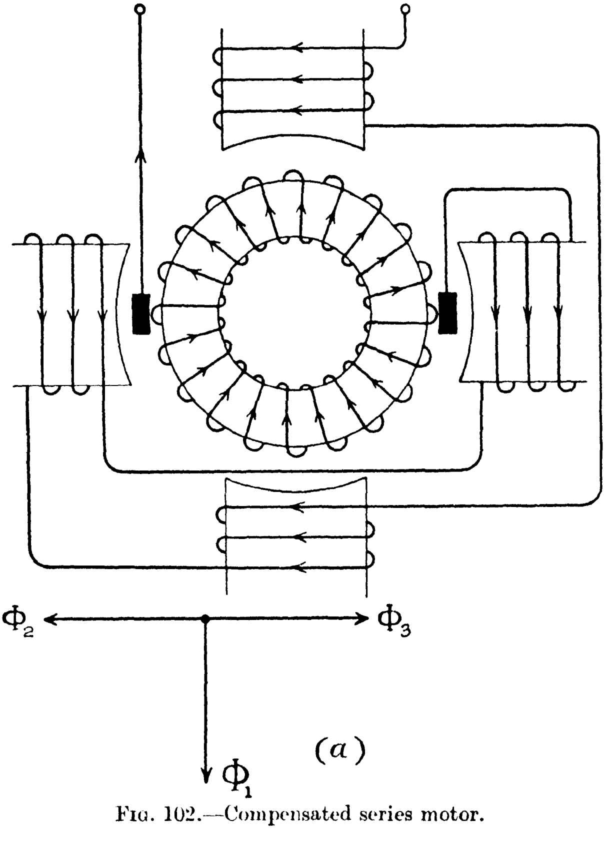


FIG. 102.—Compensated series motor.

The fluxes produced by the currents in the different windings are indicated in Fig. 102 (*a*). Φ_1 is the main or exciting field, Φ_2 the armature field, and Φ_3 the compensating field. Each, of course, is alternating with a frequency f. If I_1, I_2 and I_3 are the currents, then, as far as the effect on the armature is concerned,

Φ_1 gives rise to the rotational e.m.f. $nM_{12}I_1$
Φ_3 to the transformer e.m.f. $j\omega M_{23}I_3$, and
Φ_2 to the reaction e.m.f. $j\omega L_2I_2$.

In the motor under consideration, $I_1 = I_2 = I_3$, and if the impedance of the compensating winding is $Z_3 = R_3 + j\omega L_3$, the motor equation is

$$\begin{aligned} V &= (Z_1 + Z_2 + Z_3)\, I + nM_{12}I - 2j\omega M_{23}I \\ &= (R_1 + R_2 + R_3 + nM_{12})\, I + j\omega\, (L_1 + L_2 + L_3 - 2M_{23})I. \end{aligned}$$

If there is no leakage and $L_2 = L_3$, $L_2 + L_3 - 2M_{23} = 0$. In any case the main effect on the circle diagram is greatly to decrease the length OA and in consequence to increase the circle diameter, and as the lengths of OS and OI are not altered to the same extent, a considerable improvement in power factor results.

The compensating coil need not be connected in the main circuit; it may simply be shorted as shown in Fig. 103. The effect is exactly the same in annulling the cross field of the armature and improving the power factor. The effective impedance of the armature is now (see p. 30):

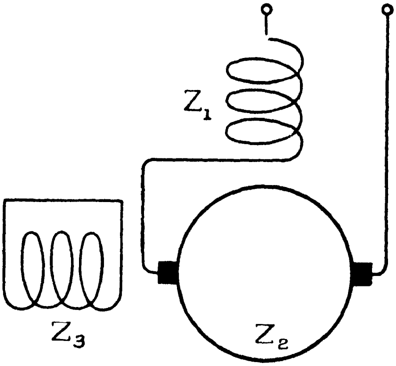


FIG. 103.—Compensated series motor.

$$\begin{aligned} Z_2' &= Z_2 + \frac{\omega^2 M_{23}^2}{Z_3} \\ &= R_2 + \frac{\omega^2 M_{23}^2}{R_3^2 + \omega^2 L_3^2} R_3 + j\omega \left\{ L_2 - \frac{\omega^2 M_{23}^2}{R_3^2 + \omega^2 L_3^2} L_3 \right\}. \end{aligned}$$

The general effect is seen by neglecting R_3^2 in comparison with $\omega^2 L_3^2$, and taking the case of no leakage, *i.e.*, $M_{23}^2 = L_2L_3$, when the expression becomes

$$Z_2' = R_2 + \frac{L_2}{L_3} R_3.$$

(iii.) *Repulsion Motors.*

Instead of short-circuiting the compensating coil the winding of the armature may be made to act as the secondary of a transformer by short-circuiting the brushes, as shown in Fig. 104. No direct connection is now made to the armature from the supply and the machine is known as a repulsion motor. The winding of impedance Z_1 giving a field at right angles to the brush axis and causing no armature currents by transformer action is called the *field winding*, while the winding of impedance Z_3, producing a field along the brush axis which causes currents to be induced in the armature winding is called the *transformer winding.*

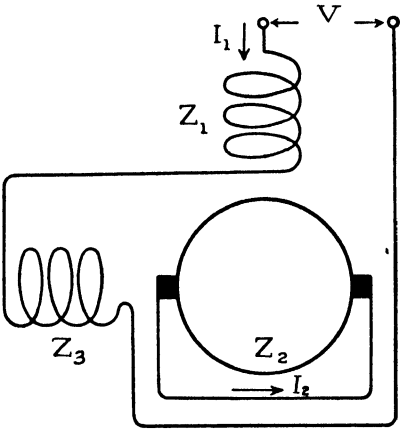


Fig. 104.—Repulsion motor.

Let the current in Z_1 and Z_3 be I_1 and that in the armature of impedance Z_2 be I_2. The armature back e.m.f. is $nM_{12}I_1$ as before and the voltage equations are :—

$$V = (Z_1 + Z_3)\, I_1 - j\omega M_{23}I_2$$
$$0 = Z_2I_2 + (nM_{12} - j\omega M_{23})I_1$$

whence the equation for the supply current is

$$V = \left(Z_1 + Z_3 + \frac{\omega^2 M_{23}{}^2 + j\omega M_{23}nM_{12}}{Z_2}\right) I_1$$

and the equation for the armature current is

$$V = \left(-\,j\omega M_{23} + \frac{(Z_1 + Z_3)Z_2}{j\omega M_{23} - nM_{12}}\right) I_2.$$

While the vector locus diagrams can readily be drawn from these expressions by the methods used before, a considerable simplification results if resistances are neglected. Putting $R_1 = R_2 = R_3 = 0$, and writing also

$$\sigma = 1 - M_{23}{}^2/(L_2L_3) \text{ and } \sigma' = 1 - M_{12}{}^2/(L_1L_2)$$

the equation for I_1 becomes

$$V = I_1 \left\{ j\omega L_1 + j\omega L_3 - j\omega(1-\sigma)L_3 + \sqrt{\frac{L_3}{L_2}(1-\sigma)}\,n\sqrt{(1-\sigma')\,L_1L_2} \right\}$$
$$= I_1 \left\{ j\omega(L_1 + \sigma L_3) + n\sqrt{(1-\sigma)\,(1-\sigma')\,L_1L_3} \right\}.$$

In Fig. 105, OA $= j\omega(L_1 + \sigma L_3)$, AC $= n\sqrt{(1-\sigma)(1-\sigma')L_1L_3}$, and the line OC inverted gives the circular locus of I_1. The phase angle ϕ between OV and OI_1 is equal to the $\widehat{OCA}$, and

$$\cot\phi = \frac{n}{\omega}\,\frac{\sqrt{(1-\sigma)(1-\sigma')L_1L_3}}{L_1 + \sigma L_3}$$

and if leakage is neglected, $\sigma = \sigma' = 0$ and

$$\cot\phi = \frac{n}{\omega}\sqrt{\frac{L_3}{L_1}} = \frac{n\,T_3}{\omega\,T_1}$$

which fixes the scale to which the speed is read off from AC. For a given load the power factor is fixed by the position of I_1 on the circle, and the speed is proportional to the turns ratio T_1/T_3.

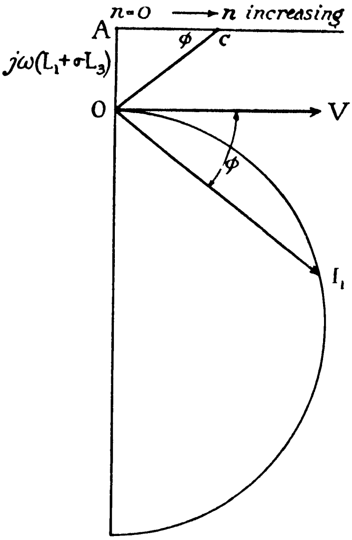


Fig. 105.—Repulsion motor vector diagram of field current.

With the same assumptions the expression for the armature current gives

$$V = \frac{-\omega^2\sigma L_2L_3 - \omega^2L_1L_2 + j\omega nL_2\sqrt{(1-\sigma)(1-\sigma')L_1L_3}}{j\omega\sqrt{(1-\sigma)L_2L_3} - n\sqrt{(1-\sigma')L_1L_2}}\,I_2$$

whence

$$I_2 = V\left\{\frac{\sqrt{(1-\sigma)L_3/L_2}}{j\omega(L_1 + \sigma L_3) + n\sqrt{(1-\sigma)(1-\sigma')L_1L_3}} + \frac{\sqrt{(1-\sigma')L_1/L_2}}{\frac{\omega^2}{n}(L_1 + \sigma L_3) - j\omega\sqrt{(1-\sigma)(1-\sigma')L_1L_3}}\right\}.$$

If the two terms of this expression are written as I_2' and I_2'' respectively, it is clear that the locus of each is a circle, and since

$$\frac{I_2'}{I_2''} = -\frac{j\omega}{n}\sqrt{\frac{1-\sigma}{1-\sigma'}\cdot\frac{L_3}{L_1}}$$

it appears that the vectors I_2' and I_2'' are always at right angles. It follows that the locus of I_2 is also a circle.

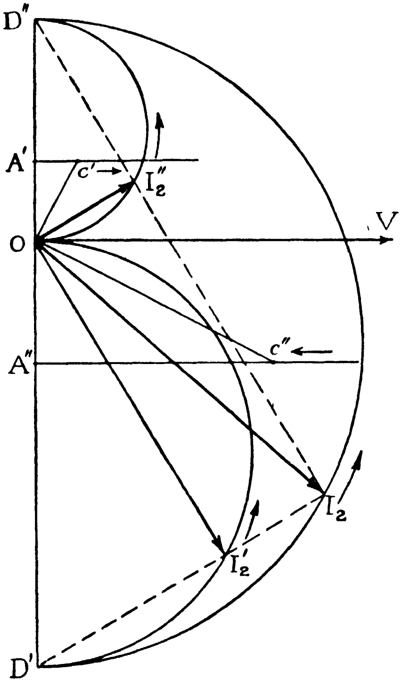


FIG. 106.—Repulsion motor vector diagram of armature current.

These circles are drawn in Fig. 106. OA' $= j\omega(L_1 + \sigma L_3)$, A'C' $= n\sqrt{(1-\sigma)(1-\sigma')L_1L_3}$, and C' moves to the right as n increases. The line A'C' inverted with factor $\sqrt{(1-\sigma)L_3/L_2}$ gives the semicircle D'I_2'O as the locus of I_2'. Similarly, OA'' $= -j\omega\sqrt{(1-\sigma)(1-\sigma')L_1L_3}$, A''C'' $= \omega^2(L_1 + \sigma L_3)/n$, C'' moves to the left as n increases, and the line C''A'' inverted with factor $\sqrt{(1-\sigma')L_1/L_2}$ gives the semicircle OI_2''D'' as the locus of I_2''. Since OI_2' and OI_2'' are always at right angles, their sum is found as the diagonal OI_2 of the rectangle O$I_2''I_2I_2'$, and since the angle D''I_2D' is a right angle, I_2 lies in the semicircle with D'D'' as diameter. Thus OI_2 is the armature current.

It may also be noted that

$$I_2' = \sqrt{(1-\sigma)L_3/L_2}\,.\,I_1 \text{ and } I_2'' = \frac{jn}{\omega}\sqrt{(1-\sigma')L_1/L_2}\,.\,I_1\,;$$

I_2' is the component of the current induced by the transformer field. It is in phase with I_1 and the exciting field, and the torque is due to this component. I_2'' is the component produced by the rotation of the armature. At synchronous speed $\omega = n$, and with equal leakage the ratio of the sizes of the two components is the ratio of the turns on the transformer and field windings. The component I_2'' is in quadrature with the stator current, and the fluxes produced are 90° apart in space. Together therefore they produce a rotating field which is constant in magnitude only when

$$\omega/n = \sqrt{(1-\sigma')L_1/L_2}.$$

The two stator windings of Fig. 104 produce fields threading the armature conductors, and these fields are in reality combined and have no separate existence. The operation of the machine will therefore be unaffected if the windings themselves are continued in a single winding whose axis makes an angle γ with the brush axis as is indicated in Fig. 107. If M is the mutual inductance between stator and armature windings when $\gamma = 0$, then the mutuals M_{12} and M_{23} of the previous work become M sin γ and M cos γ respectively, the vector equations are

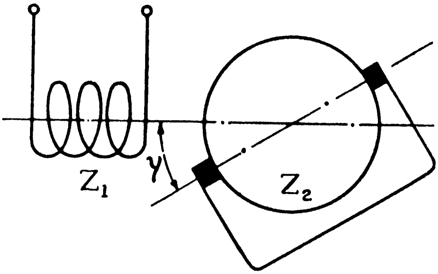


FIG. 107.—Repulsion motor.

$$V = Z_1I_1 - j\omega M \cos\gamma \,.\, I_2$$
$$O = Z_2I_2 + (nM \sin\gamma - j\omega M \cos\gamma)I_1$$

and the analysis is exactly as before.

The chief point of interest is that as γ can readily be altered by rocking the brush axis, and as the speed depends upon the ratio L_1/L_3 in the two winding machines, and therefore upon tan γ in the present single winding machine, a second easy means of speed adjustment and reversal of the direction of rotation is obtained.

There are many other modifications of the circuit arrangements of commutator motors, but the principle involved in all is covered by the considerations examined above, and the analysis for any particular type can readily be carried out in the same manner. The analysis is illustrative rather than exact. Apart from the effect of the shape of the magnetisation curve on the values of the inductances, it has been assumed throughout that the flux arising from any current is in phase with that current, whereas, owing to eddy current and hysteresis losses, the flux will lag behind the current. The values of the resistance and the inductance of the circuit can be considered to be suitably modified to take account of this effect on the circuit itself, but in order to take account of its effect on other circuits, it would be necessary to imagine the mutual inductance to be a complex quantity with a negative angle, instead of a scalar quantity.

CHAPTER IX

TRANSMISSION

(1) Line Equations

THE propagation of electromagnetic waves along any line is determined by the four primary constants of the line—resistance, inductance, leakance and capacity.

Considering a single-phase line consisting of a pair of wires, go and return, let these constants per mile of loop be R ohms, L henries, G mhos and C farads in order. Then in a short length, δx of the line AB, A′B′ (Fig. 108) the resistance of the wires AB and B′A′ is $R\delta x$, and the inductance is $L\delta x$, and the leakance between the wires AB and A′B′ is $G\delta x$ and the capacity between them is $C\delta x$. If the propagation is taking place from left to right and the potential difference between A and A′ is V volts and the current at A and A′ is I amperes, then the potential difference between B and B′ will be $V - (R + j\omega L)\delta x I$ and the current at B and B′ will be $I - (G + j\omega C)\delta x V$, on the assumption that in estimating the changes of voltage and current the current and voltage respectively can be assumed to be constant over the length δx, which is more and more nearly justified as δx is taken smaller and smaller, and is quite correct in the limit when δx is indefinitely small. Thus if δV and δI are written for the changes of potential difference and current between AA′ and BB′

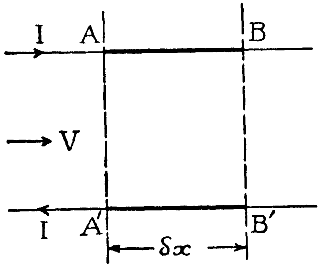


FIG. 108.—Element of single-phase line.

$$\delta V = -(R + j\omega L)\delta x I$$
$$\delta I = -(G + j\omega C)\delta x V$$

and in the limit

$$\frac{dV}{dx} = -(R + j\omega L)I \quad \ldots\ldots\ldots \quad \text{(i)}$$

$$\frac{dI}{dx} = -(G + j\omega C)V \quad \ldots\ldots\ldots \quad \text{(ii)}$$

Differentiating the first and substituting from the second gives

$$\frac{d^2V}{dx^2} = (R + j\omega L)(G + j\omega C)V \quad . \; . \; . \; . \; . \quad \text{(iii)}$$

and similarly

$$\frac{d^2I}{dx^2} = (R + j\omega L)(G + j\omega C)I \quad . \; . \; . \; . \; . \quad \text{(iv)}$$

These equations are solved by the expressions

$$V = A \cosh Px + B \sinh Px \quad . \; . \; . \; . \quad \text{(v)}$$

$$I = C \cosh Px + D \sinh Px \quad . \; . \; . \; . \quad \text{(vi)}$$

where $P = \sqrt{(R + j\omega L)(G + j\omega C)}$

as may be verified by substitution. The constants A, B, C and D are determined by the terminal conditions of the line, but they are not all independent. For differentiating (v),

$$\frac{dV}{dx} = AP \sinh Px + BP \cosh Px$$

$$= -(R + j\omega L)I \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; \text{by (i)}$$

$$= -(R + j\omega L)C \cosh Px - (R + j\omega L)D \sinh Px \quad . \quad \text{by (vi)}$$

Since this must be true for any value of x, the coefficients of the cosh and the sinh terms must be equal, *i.e.*,

$$AP = -(R + j\omega L)D$$

and $$BP = -(R + j\omega L)C$$

Thus $$C = -\frac{BP}{R + j\omega L} = -B\sqrt{\frac{G + j\omega C}{R + j\omega L}}$$

$$D = -\frac{AP}{R + j\omega L} = -A\sqrt{\frac{G + j\omega C}{R + j\omega L}}.$$

Writing

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$C = -B/Z_0$, $D = -A/Z_0$, and equations (v) and (vi) become

$$V = A \cosh Px + B \sinh Px \quad . \; . \; . \; . \; . \quad \text{(vii)}$$

$$I = -\frac{B}{Z_0} \cosh Px - \frac{A}{Z_0} \sinh Px \quad . \; . \; . \; . \quad \text{(viii)}$$

P is termed the *propagation constant* of the line, and Z_0 its *characteristic impedance*, or alternatively its *surge* or *natural impedance.*

(i.) If the line is *infinitely long*, both V and I must vanish when $x = \infty$, which is only possible when $A = -B$ for as $x \rightarrow \infty$ $\sinh x \rightarrow \cosh x$. When $x = 0$, $V = V_s$ the sending-end voltage, and $V_s = A$.

Then in this case

$$V = V_s(\cosh Px - \sinh Px)$$
$$= V_s \epsilon^{-Px} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(ix)}$$
$$I = \frac{V_s}{Z_0} \epsilon^{-Px} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(x)}$$

From these equations it appears that, whatever the value of x, the ratio $V/I = Z_0$, which explains the terms characteristic or natural impedance.

P is in general complex, and so may be written

$$P = \alpha + j\beta,$$

and (ix) gives

$$V = V_s \epsilon^{-(\alpha + j\beta)x} = V_s \epsilon^{-\alpha x} \,.\, \epsilon^{-j\beta x}$$

The factor $\epsilon^{-\alpha x}$ determines the diminution in size or attenuation of the vector voltage as x is increased ; the factor $\epsilon^{-j\beta x}$ determines the change of phase angle. Since the wavelength λ is given by the value of x which corresponds to a phase angle change of 2π,

$$\beta\lambda = 2\pi$$

or

$$\lambda = \frac{2\pi}{\beta}.$$

α is called the *attenuation constant* of the line, and β the *wavelength constant*, and these, together with the propagation constant P and the characteristic impedance Z_0, are known as the four secondary constants of the line.

(x) gives similarly

$$I = \frac{V_s}{Z_0} \epsilon^{-\alpha x} \,.\, \epsilon^{-j\beta x}$$
$$= I_s \epsilon^{-\alpha x} \,.\, \epsilon^{-j\beta x}$$

where $I_s = V_s/Z_0$ is evidently the value of the current at the sending end.

If V_s is taken as reference vector, in Fig. 109, $OV_s = V_s$, $OI_s = I_s$ where $\widehat{V_sOI_s} = -\phi$, where ϕ is the angle of Z_0 ; $OV = V$, where $|OV| = |V_s|\epsilon^{-\alpha x}$ and $\widehat{V_sOV} = -\beta x$ and $OI = I$, where

$$|OI| = |V_s/Z_0|\epsilon^{-\alpha x} = |I_s|\epsilon^{-\alpha x}$$

and $\widehat{VOI} = -\phi$, $\widehat{I_sOI} = -\beta x$. The loci of I and V as x varies are evidently logarithmic spirals.

It has been seen that at any point in the line the ratio of V to I is Z_0. If therefore the line is cut at any point and closed with an impedance equal to Z_0, the potential and current distribution in the now finite line will be unaffected. That is, a line closed with its characteristic impedance behaves as though it were infinitely long.

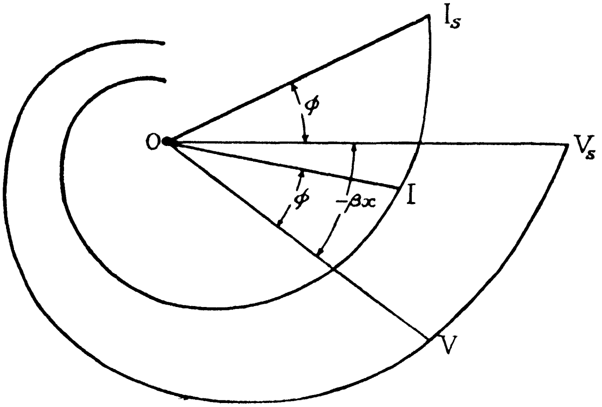


FIG. 109.—Vector diagram for infinitely long line.

(ii.) If the line is *not infinitely long*, let it have a length l, and let the voltage and current at the receiving end be V_r and I_r respectively.

Then when $x = 0$, $V = V_s = A$ and $I = I_s = -B/Z_0$ and when $x = l$, $V = V_r$ and $I = I_r$, and equations (vii) and (viii) become

$$V_r = V_s \cosh Pl - I_s Z_0 \sinh Pl \quad . \quad . \quad . \quad . \quad \text{(xi)}$$

$$I_r = I_s \cosh Pl - \frac{V_s}{Z_0} \sinh Pl \quad . \quad . \quad . \quad . \quad . \quad \text{(xii)}$$

which enable the receiving end potential and current to be determined in terms of the sending end potential and current.

Similarly if distances are measured from the receiving end of the line, when $x = 0$, $V = V_r = A$ and $I = I_r = -B/Z_0$, and when $x = -l$, $V = V_s$ and $I = I_s$, and equations (vii) and (viii) become

$$V_s = V_r \cosh Pl + I_r Z_0 \sinh Pl \quad . \quad . \quad . \quad . \quad \text{(xi.a)}$$

$$I_s = I_r \cosh Pl + \frac{V_r}{Z_0} \sinh Pl \quad . \quad . \quad . \quad . \quad . \quad \text{(xii.a)}$$

which enable the sending end potential and current to be determined in terms of the potential and current required at the receiving end.

The values cosh Pl and sinh Pl can be found from Kennelly's tables or chart atlas of the hyperbolic functions of complex numbers,* or may be expanded in the form ($\cosh\alpha\cos\beta + j\sinh\alpha\sin\beta$) and ($\sinh\alpha\cos\beta + j\sinh\beta\cos\alpha$) respectively and thence found, or they may be found graphically as indicated on p. 13.

The manner in which a typical problem can be solved graphically is indicated in Fig. 110, where the sending end voltage and current are found when the receiving end values are known.

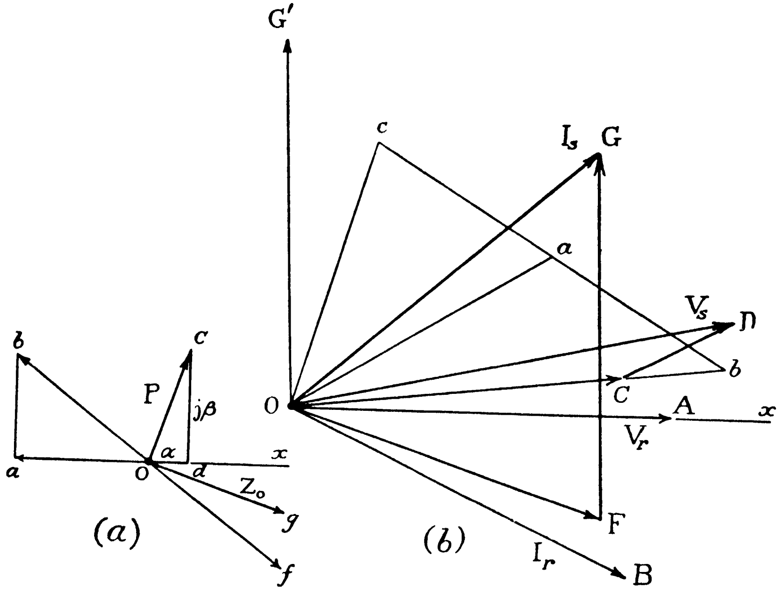


FIG. 110.—Vector diagram for line of finite length.

Pl, and hence αl and βl, and Z_0 are found in Fig. 110 (a). Neglecting G,

$$Pl = l\sqrt{j\omega CR - \omega^2 LC}$$

and in the figure $Oa = -\omega^2 LC$, $ab = j\omega CR$, $Ob = (j\omega CR - \omega^2 LC)$, $|Oc| = l\sqrt{|Ob|}$, $x\widehat{O}c = \frac{1}{2}\, x\widehat{O}b$ and hence $Oc = Pl = \alpha + j\beta = Od + jdc$. Thus $|Od| = \alpha$ and $|dc| = \beta$.

$Z_0 = \sqrt{(R + j\omega L)/j\omega C} = \sqrt{L/C - jR/\omega C}$, $Of = L/C - jR/\omega C$, and $|Og| = \sqrt{|Of|}$, $x\widehat{O}g = \frac{1}{2}x\widehat{O}f$ and $Og = Z_0$.

* See E. Mallett, "Telegraphy and Telephony," p. 394.

In Fig. 110 (*b*) the values of cosh Pl and sinh Pl are found as in Fig. 14, p. 13. $|Oa| = \frac{1}{2}\epsilon^{\alpha l}$, $x\widehat{O}a = \beta l$, $|ab| = |ac| = \frac{1}{2}\epsilon^{-\alpha l}$, and the angle cab makes with the horizontal $= -\beta l$. Then $Ob = \cosh Pl$ and $Oc = \sinh Pl$.

$OA = V_r$, taken as reference vector, and $OB = I_r$.

$OC = V_r \cosh Pl$, $CD = I_r Z_0 \sinh Pl$, and $OD = V_s$ by equation (xi.a).

$OF = I_r \cosh Pl$, $OG' = FG = V_r \sinh Pl/Z_0$, and $OG = I_s$ by equation (xii.a).

If $I_r = O$, then $I_s = V_r \sinh Pl/Z_0$. OG′ therefore is the charging current of the cable.

Powers are readily obtained from the diagram as the dot products of the voltages and corresponding currents.

If greater accuracy is required, cosh Pl and sinh Pl can be expanded in series, and their values obtained by vector algebra. If the total series impedance of the line is written

$$Z = (R + j\omega L)l$$

and the total shunting admittance is written

$$Y = (G + j\omega C)l$$

Then $Z_0 = \sqrt{Z/Y}$ and

$$\cosh Pl = \cosh \sqrt{ZY} = 1 + \frac{ZY}{2\,!} + \frac{Z^2Y^2}{4\,!} + \dots$$

$$\sinh Pl = \sinh \sqrt{ZY} = \sqrt{ZY} + \frac{(\sqrt{ZY})^3}{3\,!} + \frac{(\sqrt{ZY})^5}{5\,!} + \dots$$

$$Z_0 \sinh Pl = \sqrt{Z/Y} \sinh \sqrt{ZY} = Z\left(1 + \frac{ZY}{3\,!} + \frac{Z^2Y^2}{5\,!} + \dots\right)$$

$$\sinh Pl/Z_0 = \sqrt{Y/Z} \sinh \sqrt{ZY} = Y\left(1 + \frac{ZY}{3\,!} + \frac{Z^2Y^2}{5\,!} + \dots\right)$$

and equations (xi) and (xii) become

$$V_r = V_s\left(1 + \frac{ZY}{2} + \frac{Z^2Y^2}{24} + \dots\right) - I_sZ\left(1 + \frac{ZY}{6} + \frac{Z^2Y^2}{120} + \dots\right) \quad \dots \text{(xi.b)}$$

$$I_r = I_s\left(1 + \frac{ZY}{2} + \frac{Z^2Y^2}{24} + \dots\right) - V_sY\left(1 + \frac{ZY}{6} + \frac{Z^2Y^2}{120} + \dots\right) \quad \dots \text{(xii.b)}$$

and equations (xi.a) and (xii.a) become

$$V_s = V_r\left(1+\frac{ZY}{2}+\frac{Z^2Y^2}{24}+\ldots\right)+I_rZ\left(1+\frac{ZY}{6}+\frac{Z^2Y^2}{120}+\ldots\right)$$

$$I_s = I_r\left(1+\frac{ZY}{2}+\frac{Z^2Y^2}{24}+\ldots\right)+V_rY\left(1+\frac{ZY}{6}+\frac{Z^2Y^2}{120}+\ldots\right)$$

For short lines, the first terms only of the series need be included, and it is rare in power problems at ordinary frequencies that more than the first two terms are necessary. In any case the summation of the series may be made graphically.

(iii.) The line can alternatively be replaced for the purposes of calculation by an equivalent T or by an equivalent Π, as indicated in Fig. 111 (*a*) and Fig. 111 (*b*) respectively.

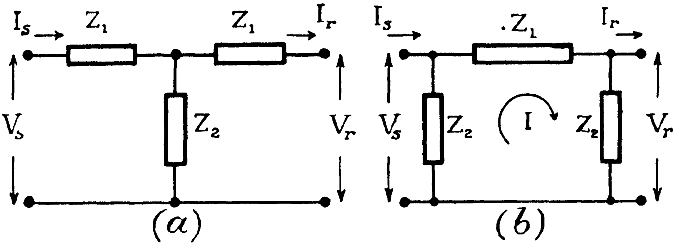


FIG. 111.—Equivalent T and Π networks.

The equations for the T are

$$V_s = I_s(Z_1+Z_2) - I_rZ_2 \quad \ldots\ldots \quad \text{(xiii)}$$

$$0 = I_r(Z_1+Z_2) - I_sZ_2 + V_r \quad \ldots \quad \text{(xiv)}$$

(xiii) rearranged is

$$I_r = I_s\frac{Z_1+Z_2}{Z_2} - \frac{V_s}{Z_2}$$

and this is identical with (xii) if

$$\frac{Z_1+Z_2}{Z_2} = \cosh Pl, \text{ and } Z_2 = \frac{Z_0}{\sinh Pl}$$

i.e., if

$$\left.\begin{aligned} Z_1 &= \frac{Z_0}{\sinh Pl}(\cosh Pl - 1) \\ &= Z_0 \tanh\frac{Pl}{2} \\ Z_2 &= \frac{Z_0}{\sinh Pl} \end{aligned}\right\} \quad \ldots \quad \text{(xv)}$$

Eliminating I_r from (xiii) and (xiv) gives

$$V_r = V_s \frac{Z_1 + Z_2}{Z_2} - I_s\left(\frac{Z_1^2}{Z_2} + 2Z_1\right)$$

and this is identical with (xi) if

$$\frac{Z_1 + Z_2}{Z_2} = \cosh Pl, \text{ and } \frac{\sinh Pl}{Z_o} = \frac{Z_1^2}{Z_2} + 2Z_1$$

which lead again to the values in (xiv).

Thus equations (xv) give the values Z_1 and Z_2 for the equivalent T of the line.

The equations for the Π (Fig. 111 (*b*)) are

$$\begin{aligned} V_s &= (I_s - I)Z_2 \\ O &= I(Z_1 + 2Z_2) - I_sZ_2 - I_rZ_2 \\ O &= I_rZ_2 - IZ_2 + V_r \end{aligned}$$

which, eliminating I, give

$$I_r = I_s \frac{Z_1 + Z_2}{Z_2} - \frac{V_s}{Z_2^2}(Z_1 + 2Z_2) \quad . \quad . \quad . \quad . \quad \text{(xvi)}$$

$$V_r = - V_s + I_sZ_2 - I_rZ_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xvii)}$$

and these are readily shown to be identical with (xi) and (xii) if

$$\left.\begin{aligned} Z_1 &= Z_0 \sinh Pl \\ Z_2 &= \frac{Z_0}{\sinh Pl}(\cosh Pl + 1) \\ &= Z_0 \coth \frac{Pl}{2} \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(xviii)}$$

Equations (xviii) therefore give the values of the impedances in the equivalent Π of the line.

(iv.) Nominal T and Π.

With the comparatively short lines usually met with in power transmission, expressions for the impedances and the equivalent T and Π networks can be considerably simplified. In the equivalent T expressions (xv), if Pl is small, $Pl/2$ can be written for $\tanh Pl/2$ and Pl for $\sinh Pl$, and

$$Z_1 = Z_o\frac{Pl}{2} = \sqrt{\frac{Z}{Y}}. \frac{\sqrt{ZY}}{2} = \frac{Z}{2}$$

$$Z_2 = \frac{Z_o}{Pl} = \sqrt{\frac{Z}{Y}} . \frac{1}{\sqrt{ZY}} = \frac{1}{Y}$$

where $Z = (R + j\omega L)l$ and $Y = (G + j\omega C)l$ as before. Referring to Fig. 111 (*a*), Z_1 is now half the total series impedance of the loop, and $1/Z_2$ the total shunting admittance from line to line.

Similarly the equivalent Π expressions (xviii) give

$$Z_1 = Z$$
$$Z_2 = 2/Y$$

and in Fig. 111 (*b*), Z_1 is the total series impedance, and $1/Z_2$ half the total shunting admittance.

These simplified networks are known as the *nominal* T and Π respectively.

The extent of the approximations made in these nominal networks is indicated by solving them for the terminal currents and potentials. For the nominal T, equations (xiii) and (xiv) become

$$V_s = I_s\left(\frac{Z}{2} + \frac{1}{Y}\right) - I_r\frac{1}{Y}$$
$$0 = I_r\left(\frac{Z}{2} + \frac{1}{Y}\right) - I_s\frac{1}{Y} + V_r$$

which give

$$V_r = V_s\left(1 + \frac{ZY}{2}\right) - I_sZ\left(1 + \frac{ZY}{4}\right)$$
$$I_r = I_s\left(1 + \frac{ZY}{2}\right) - V_sY.$$

These should be compared with (xi.b) and (xii.b).

Similarly, the equations for the nominal Π, from equations (xvi) and (xvii), are

$$I_r = I_s\frac{Z + 2/Y}{2/Y} - \frac{V_s}{4/Y^2}(Z + 4/Y)$$
$$V_r = -V_s + 2I_s/Y - 2I_r/Y$$

which give

$$V_r = V_s\left(1 + \frac{ZY}{2}\right) - I_rZ$$
$$I_r = I_s\left(1 + \frac{ZY}{2}\right) - V_sY\left(1 + \frac{ZY}{4}\right).$$

(v.) The formulæ for the determination of the current at the sending end and at the receiving end of a line of length l, when an e.m.f. E_s is applied to the line at the sending end through an impedance Z_s, and the line is closed at the receiving end by an impedance Z_r, can now be determined by the use of either the equivalent T or Π. Using the equivalent T, the circuit diagram is as shown in Fig. 112, and

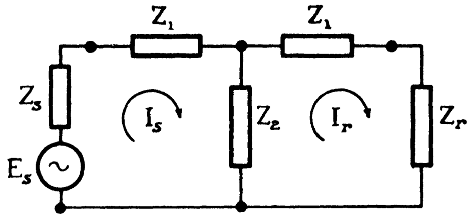


FIG. 112.—Equivalent T network.

$$E_s = I_s(Z_s + Z_1 + Z_2) - I_r Z_2$$
$$0 = I_r(Z_1 + Z_2 + Z_r) - I_s Z_2$$

which give, on substitution from (xv) and simplification,

$$I_r = \frac{E_s}{(Z_s + Z_r)\cosh Pl + \left(Z_0 + \frac{Z_s Z_r}{Z_0}\right)\sinh Pl}$$

$$I_s = E_s \frac{Z_0 \cosh Pl + Z_r \sinh Pl}{Z_0(Z_s + Z_r)\cosh Pl + (Z_0^2 + Z_s Z_r)\sinh Pl}.$$

(2) **Resonance**

If a transmission line is open-circuited or very lightly loaded and is long enough, the voltage at the receiving end may quite readily be larger than that at the sending end. This fact is often referred to as the Ferranti effect.

If the line is open-circuited $I_r = 0$, and from (xii) $I_s Z_0 = V_s \tanh Pl$, which inserted in (xi) gives

$$\begin{aligned} V_r &= V_s/\cosh Pl \\ &= V_s/(\cosh \alpha l \cos \beta l + j \sinh \alpha l \sin \beta l) \\ \text{and } |V_s/V_r| &= (\cosh^2 \alpha l \cos^2 \beta l + \sinh^2 \alpha l \sin^2 \beta l)^{\frac{1}{2}} \\ &= (\cosh^2 \alpha l - \sin^2 \beta l)^{\frac{1}{2}} \end{aligned}$$

Thus $|V_s/V_r|$ is a minimum when

$$\frac{d}{dl}(\cosh^2 \alpha l - \sin^2 \beta l) = 0,$$

i.e., when $\sin 2\beta l = \frac{\alpha}{\beta} \sinh 2\alpha l.$

This expression may be solved graphically, and if α is much less than β, may give several possible values of l. If $\alpha = \beta$ there is no solution. This is the case of a line with no inductance and no leakance. The phenomenon depends upon the line having inductance.

Generally speaking, power lines are shorter than even the first resonance length for the fundamental frequency. As an example consider an open line with resistance 0·406 ohms per mile, inductance 2·13 millihenries per mile, capacitance 0·014 microfarads per mile and negligible leakance. At 50 cycles per second, $\omega L = 0{\cdot}67$ and $\omega C = 4{\cdot}4 \times 10^{-6}$ and

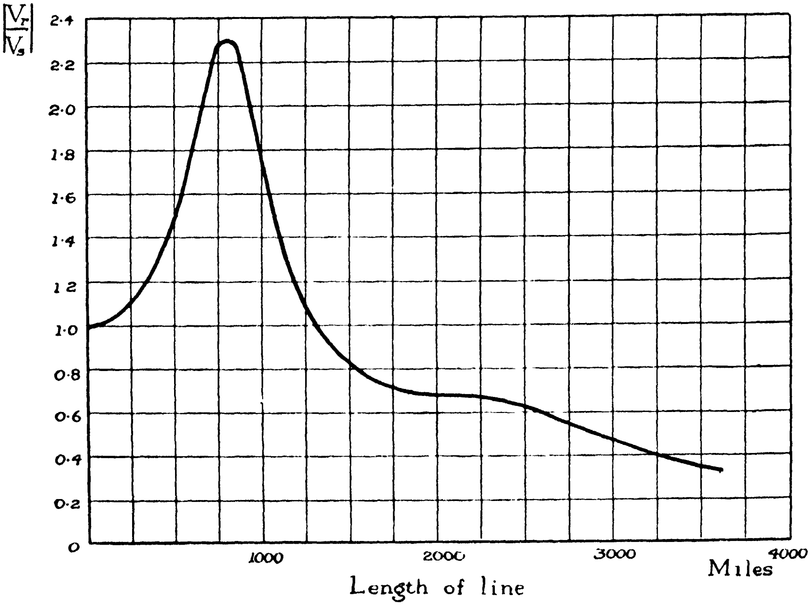


FIG. 113.—Variation of receiving to sending voltage ratio, with length of line.

$$P = \sqrt{(0{\cdot}406 + j0{\cdot}67)j4{\cdot}4 \times 10^{-6}}$$
$$= 10^{-3}\sqrt{j1{\cdot}79 - 2{\cdot}95} = (0{\cdot}5 + j1{\cdot}8)10^{-3}$$

Thus $\alpha = 0{\cdot}5 \times 10^{-3}$ and $\beta = 1{\cdot}8 \times 10^{-3}$.

The curve of Fig. 113 was calculated from the expression

$$\frac{|V_r|}{|V_s|} = \frac{1}{\sqrt{\cosh^2 \alpha l - \sin^2 \beta l}}$$

using these values.

It is seen that only on a length of 800 miles of the line is the full resonance effect found, and then the ratio $|V_r/V_s|$ is about 2·3. But even with a 200-mile line the effect will be noticed, as the ratio is then 1·06.

(3) Three-phase Lines

(i.) *Equivalent Star Network.*—A symmetrical balanced three-phase line can be regarded as made up of three individual loops, A*a*, B*b* and C*c*, as indicated in Fig. 114. For the sum of the currents in the conductors *a*, *b* and *c* is at any instant nil, and if these conductors are regarded as having zero resistance and as being infinitely close to each other without contact, their presence will not affect the electrical conditions in the conductors A, B and C. Since the system is supposed to be balanced, a knowledge of the electrical conditions in one loop, say A*a*, is sufficient for a complete knowledge of the conditions in the three-phase line.

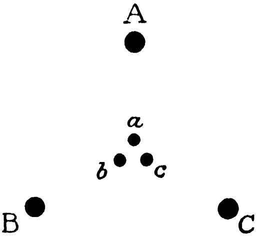


FIG. 114.—Equivalent star network—three-phase-line.

The resistance per mile of the loop A*a* is evidently R, the conductor resistance per mile.

In finding the effective inductance of the loop, account must be taken of the loop-to-loop mutual inductance. Let

L_1 = the inductance per mile of each loop such as A*a* when isolated.

M = the mutual inductance per mile between each pair of loops, *i.e.*, A*a* to C*c*, C*c* to B*b*, and B*b* to A*a*.

L = the *effective* inductance per mile of each loop such as A*a* when forming part of the three-phase system.

Then, by exactly the same considerations as those examined on p. 109,

$$L = L_1 + M.$$

Now consider the effective inductance L_2 of any two loops, such as A*a* and *c*C when supplied in series from a single-phase source. Imagine current to flow down A, up *a*, down *c* and up C. Clearly

$$L_2 = L_1 + 2M + L_1 = 2(L_1 + M)$$

and since the currents in a and c mutually cancel each other's effect, L_2 = the inductance of the loop AC when isolated. Thus

$$L = L_1 + M = \tfrac{1}{2}L_2,$$

or the inductance to be ascribed to the loop Aa is half the inductance of the loop AC when isolated, or as it is often put, is the inductance per wire of the loop AC.

Analogous considerations with regard to the capacity to be ascribed to the loop Aa leads to a similar result; the effective capacity is twice that of the isolated loop AC, or is equal to that of a single isolated wire to earth.

The leakance also can be regarded as that of a single isolated wire to earth.

The system Aa, Bb, Cc can be regarded as an equivalent star network of the three-phase line, and from the foregoing considerations the four primary constants R, L, C and G for each phase of the star can be found, and the symmetrical three-phase transmission line with balanced load can be dealt with by a consideration of one phase only of the equivalent star. The problem has been reduced to a single-phase one, and can be dealt with by the formulæ and networks of the preceding section.

The following formulæ are useful in calculating the inductance and capacity of overhead conductors :—

The inductance per mile of loop of conductors of radius r inches, distance apart d inches is given in millihenries by

$$L = 0{\cdot}160 + 1{\cdot}482 \log_{10} \frac{d}{r}$$

and the capacity between the two conductors per mile is given in microfarads by

$$C = \frac{0{\cdot}0194}{\log_{10} \frac{d}{r}}$$

provided in each case that d is much greater than r.

The leakance is usually negligible. For instance, if $d = 100''$ and $r = 0{\cdot}2''$, $\log_{10} d/r = 2{\cdot}7$ and $L = 4{\cdot}160$ mH per mile and $C = 0{\cdot}00719\,\mu F$ per mile. In the equivalent star therefore, $L = 2{\cdot}08$ mH and $C = 0{\cdot}01438\,\mu F$. In the equivalent nominal T at 50 cycles per second the series reactance $2{\cdot}08 \times 2\pi \times 50 \times 10^{-3}\, l = 0{\cdot}65\, l$ ohms and the shunting reactance is $10^6/(2\pi \times 50 \times 0{\cdot}01438\, l) = 222{,}000/l$

ohms, and on short lines the inductance effect predominates and the shunting impedance can be ignored without serious error.

Calculation therefore on short overhead lines can be made by replacing the line by its resistance and equivalent inductance, and the problem becomes the same as that of the regulation of a transformer (Figs. 51 and 52, pp. 63 and 64), considering r_0 as the resistance of the line, x_0 its reactance, I_2 the load current and $\cos\phi$ its power factor. The magnitude of the voltage difference between the sending and receiving ends of the line is given approximately by

$$I_2 r_0 \cos\phi + I_2 x_0 \sin\phi$$

(ii.) *Cable Capacities.* In a three-core cable the capacity effects are more complicated owing to earth capacities between the conductors and the sheath. In this case Maxwell's capacity coefficient equations are :—

$$\begin{aligned}
q_0 &= k_{00}v_0 + k_{01}v_1 + k_{02}v_2 + k_{03}v_3 \\
q_1 &= k_{10}v_0 + k_{11}v_1 + k_{12}v_2 + k_{13}v_3 \\
q_2 &= k_{20}v_0 + k_{21}v_1 + k_{22}v_2 + k_{23}v_3 \\
q_3 &= k_{30}v_0 + k_{31}v_1 + k_{31}v_2 + k_{33}v_3
\end{aligned}$$

where the q's are charges on the conductors and the v's are potentials above earths, and the subscripts are $_0$ for the sheath and $_{1\,2\,3}$ for the three conductors.

By the mutual nature of the phenomenon

$$k_{12} = k_{21}, \text{ etc.,}$$

and by symmetry

$$\begin{aligned}
k_{11} &= k_{22} = k_{33} \\
k_{12} &= k_{23} = k_{31} \\
k_{01} &= k_{02} = k_{03}
\end{aligned}$$

and since $\qquad q_0 = 0 \text{ and } v_0 = 0$

and $\qquad v_1 + v_2 + v_3 = 0$

the equations give

$$\begin{aligned}
q_1 &= k_{11}v_1 + k_{12}(v_2 + v_3) \\
&= (k_{11} - k_{12})v_1, \text{ etc.}
\end{aligned}$$

If $v_0 = v_1 = v_2 = v_3$ there will be no charge on the internal conductors ; and the equations give

$$k_{01} + k_{11} + 2k_{12} = 0$$

The " charging current " is

$$i_1 = \frac{dq}{dt} = (k_{11} - k_{12}) \frac{dv_1}{dt}$$

and it is necessary to find $(k_{11} - k_{12})$.

If $q_1 = q$ and $q_2 = -q$ and there are no other charges the equations give

$$q = k_{10}v_0 + k_{11}v_1 + k_{12}v_2 + k_{13}v_3$$
$$-q = k_{10}v_0 + k_{12}v_1 + k_{11}v_2 + k_{13}v_3$$
$$\therefore \qquad 2q = (k_{11} - k_{12})(v_1 - v_2)$$

and the capacity C_{12} between conductors 1 and 2 is

$$C_{12} = \frac{q}{v_1 - v_2} = \tfrac{1}{2}(k_{11} - k_{12}).$$

Thus if the capacity between any two conductors is measured with the others insulated, the equivalent star capacity is twice the value found.

A model of the capacities of the cable can be drawn as in Fig. 115, where C_c and C_o are partial capacities, C_c between conductors, and C_o between conductors and sheath.

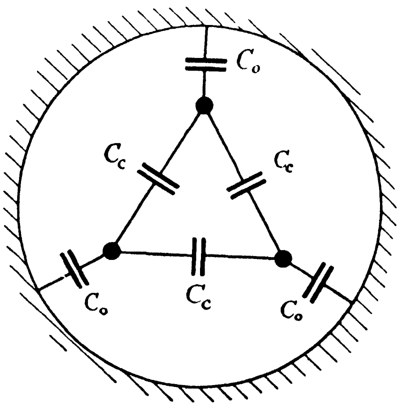


FIG. 115.—Equivalent capacities of 3-phase cable.

The capacity C_{12} between conductors 1 and 2 with the others insulated is evidently

$$C_{12} = C_c + \frac{C_c}{2} + \frac{C_o}{2}$$
$$= \tfrac{1}{2}(3C_c + C_o)$$

Thus $$k_{11} - k_{12} = 3C_c + C_o$$

The capacity C_{10} between conductor 1 and all the others earthed is from the second of Maxwell's equations, putting $v_0 = v_2 = v_3 = 0$

$$C_{10} = k_{11}$$

and for the model of Fig. 115 is

$$C_{10} = 2C_c + C_o$$

Thus $$k_{11} = 2C_c + C_o$$

also $$k_{01} + k_{11} + 2k_{12} = 0$$

Thus
$$\left.\begin{aligned} k_{11} &= 2C_c + C_o \\ -k_{12} &= C_c \\ -k_{01} &= C_o \end{aligned}\right\}$$

give the relationships between the partial capacities of Fig. 115 and Maxwell's capacity coefficients.

The charging current is

$$i_1 = (k_{11} - k_{12})\frac{dv_1}{dt}$$
$$= (3C_c + C_o)\frac{dv_1}{dt}$$

or vectorially, if V is the line voltage, remembering that v_1 is voltage above earth,

$$I = j\omega(3C_c + C_o)\frac{V}{\sqrt{3}}$$

The equivalent star capacity is evidently $k_{11} - k_{12} = 3C_c + C_o$.

In the case of the overhead line the earth capacities were neglected, and it was stated that the equivalent star capacity was twice that of the isolated wire to wire capacity. From the above it appears that when the wires are arranged in a three-phase system, the partial capacity between conductors is two-thirds the capacity between the conductors when isolated.

The accurate calculation of the capacities of underground cables is very complicated, and it is usual to obtain the required values by measurement as indicated above. On short cables the capacity effects predominate, and the inductance can be neglected without serious error.

EXERCISES

(1) A three-phase overhead transmission line has copper conductors of diameter 0·324 in. spaced 5 ft. apart.

Find the resistance, inductance and capacity of the equivalent star, and, neglecting leakance, find the propagation constant, the attenuation and wave-length constants, and the characteristic impedance at 50 cycles.

Take $\rho = 1{\cdot}594 \times 10^{-6}$ ohms/cm./cm.^{2}

(2) If a 500-mile length of the line of Q. 1 is closed with its characteristic impedance, find the voltage distribution along the line when 50 kilovolts 50 cycles is applied at the sending end.

(3) Find the length of the line of Q. 1 which would show the maximum Ferranti effect. Plot a curve showing the ratio V_r/V_s for an open-circuited line of various lengths.

(4) Find the charging current in a 60-mile length of the line of Q. 1 when open-circuited, and when 50 kilovolts, 50 cycles is the sending end voltage.

(5) A three-core cable 120 miles long has the following equivalent star constants :—

$R = 0{\cdot}17$ ohms, $L = 0{\cdot}7$ mH, $C = 0{\cdot}32\ \mu F$, all per mile.

The three-phase load at the receiving end is 40,000 kilowatts at a power factor of 0·9 lagging and a voltage of 66,000 volts, 50 cycles. Find the voltage and current at the sending end, the charging current and the power lost in the line.

SOLUTIONS TO EXERCISES

Chapter I. pp. 20–21

2. (i.) $5\underline{/53^\circ\ 8'}$. (ii.) $6{\cdot}325\underline{/108^\circ\ 26'}$. (iii.) $4{\cdot}123\overline{\backslash 165^\circ\ 58'}$. (iv.) $2{\cdot}828\overline{\backslash 45^\circ}$. (v.) $3{\cdot}236 + j2{\cdot}351$. (vi.) $-2{\cdot}941 - j0{\cdot}596$. (vii.) $0{\cdot}273 + j1{\cdot}981$.

3. (i.) $5 + j2$. (ii.) $1{\cdot}236 + j8{\cdot}351$. (iii.) $6 - j1$.

4. (i.) $-30 + j10$. (ii.) $14 + j2$. (iii.) $0\ 118 - j1{\cdot}530$. (iv.) $2\epsilon^{-j13{\cdot}37}$.

5. (i.) $0{\cdot}235 + j0{\cdot}059$. (ii.) $0{\cdot}25 + j0{\cdot}25$.

6. (i.) $1{\cdot}471 + j2{\cdot}04$. (ii.) $0{\cdot}248 - j2{\cdot}016$. (iii.) $0 - j8$.

7. (i.) $-0{\cdot}48 + j1{\cdot}76$. (ii.) $1{\cdot}423 - j1{\cdot}505$. (iii.) $2{\cdot}662 - j0{\cdot}214$.

8. (i.) $0{\cdot}764 + j0{\cdot}487$ (ii.) $0{\cdot}764 - j0{\cdot}487$
(iii.) $-0{\cdot}764 + j0{\cdot}487$ (iv.) $-0{\cdot}764 - j0{\cdot}487$
(v.) $0{\cdot}907 - j0{\cdot}410$ (vi.) $0{\cdot}907 + j0{\cdot}410$
(vii.) $0{\cdot}907 + j0{\cdot}410$ (viii.) $0{\cdot}907 - j0{\cdot}410$
(ix.) $0{\cdot}626 + j0{\cdot}709$ (x.) $0{\cdot}626 - j0{\cdot}709$
(xi.) $-0{\cdot}626 + j0{\cdot}709$ (xii.) $-0{\cdot}626 - j0{\cdot}709$
(xiii.) $1{\cdot}031 + j0{\cdot}428$ (xiv.) $1{\cdot}031 - j0{\cdot}428$
(xv.) $1{\cdot}031 - j0{\cdot}428$ (xvi.) $1{\cdot}031 + j0{\cdot}428$

10. (i.) $0{\cdot}119 - j0{\cdot}113$. (ii.) $0{\cdot}241 - j0{\cdot}103$. (iii.) $0{\cdot}255 - j0{\cdot}124$.

11. (i.) $0{\cdot}431 + j0{\cdot}229$ (ii.) $1{\cdot}134 - j0{\cdot}396$
(iii.) $0{\cdot}251 + j0{\cdot}987$ (iv.) $-0{\cdot}178 - j1{\cdot}029$

CHAPTER II. p. 34

1. Resistance 0·481 ohms. Reactance 0·404 ohms.

2. (i.) 0·834 amps. (R.M.S.). $\phi = -60^\circ\ 11'$.

Instantaneous values : (*a*) 180 volts, + 1·08 amps.
(*b*) 0 volts, + 1·02 amps. (*c*) − 223 volts, − 0·59 amps.
(*d*) 0 volts, − 1·02 amps.

(ii.) 0·928 amps. (R.M.S.). $\phi = +72^\circ\ 34'$.

Instantaneous values : (*a*) 180 volts, − 0·416 amps.
(*b*) 0 volts, − 1·24 amps. (*c*) − 223 volts, − 0·392 amps.
(*d*) 0 volts, + 1·24 amps.

(iii.) 1·23 amps. (R.M.S.). $\phi = +0^\circ\ 54'$.

Instantaneous values : (*a*) 180 volts, + 1·27 amps.
(*b*) 0 volts, − 0·025 amps. (*c*) − 223 volts, − 1·59 amps.
(*d*) 0 volts, + 0·025 amps.

(iv.) 0·699 amps. (R.M.S.). $\phi = 11^\circ\ 4'$.

Instantaneous values : (*a*) 180 volts, + 0·675 amps.
(*b*) 0 volts, − 0·189 amps. (*c*) − 223 volts, − 0·972 amps.
(*d*) 0 volts, + 0·189 amps.

4. 3·48 volts.

5. 1,560 μF, 0·96 amps.

CHAPTER III. pp. 47–48

1. $C_1 = 0{\cdot}1925\ \mu F$. $r_1 = 6{,}364$ ohms. $\cos\phi = 0{\cdot}933$.
$V = 224{\cdot}7$ volts.

2. $L = 0{\cdot}15$ henry. $R = 400$ ohms.

3. $$R = \frac{PQ\omega^2 rC^2}{1 + \omega^2 r^2 C^2} \qquad L = \frac{PQC}{1 + \omega^2 r^2 C^2}$$

$R = 374$ ohms. $L = 0{\cdot}1662$ henry.

4. $R = 500$ ohms. $L = 1{\cdot}4$ henry.

SOLUTIONS TO EXERCISES

CHAPTER IV. pp. 72–73

1.

Section	Z	$\frac{V_2}{V_1}$	$\frac{I_2}{I_1}$	I_2	$\lvert I_2 \rvert$
(i.)	∞	0·392	0	0	0
(ii.)	10	0·286	1·28	$0{\cdot}286 + j0{\cdot}0082$	0·286
(iii.)	1	0·077	1·92	$0{\cdot}742 - j0{\cdot}208$	0·770
(iv.)	$2 + j5$	0·095	1·29	$0{\cdot}32 - j0{\cdot}284$	0·429
(v.)	$-j200$	0·275	0·105	$-0{\cdot}0032 + j0{\cdot}0178$	0·018
(vi.)	$-j50$	0·374	0·50	$-0{\cdot}0014 + j0{\cdot}097$	0·097

4. $V_1 = 388$ volts. Power-factor 0·209.

5. (i.) $I_m = 0{\cdot}742$ amps. $I_p = 0{\cdot}3$ amps.
   (ii.) Efficiency = 96·7 per cent.
   (iii.) (*a*) 731 volts. (*b*) 731 volts. (*c*) 775 volts.

CHAPTER V. p. 91

	Phase	Z	Phase Volts	Phase Current	Current in neutral	Power absorbed
2 (*a*).	I.	20	$V_1 = 440$	$I_1 = 22$	$I_n = 44{\cdot}9 - j18{\cdot}64$	29,557
	II.	$5 + j5$	$V_2 = -220 - j381$	$I_2 = -60{\cdot}1 - j16{\cdot}1$		
	III.	$10 - j60$	$V_3 = -220 + j381$	$I_3 = -6{\cdot}77 - j\,2{\cdot}54$		
2 (*b*).	I.	20	$V_1 = +220 - j127{\cdot}2$	$I_1 = 18{\cdot}5 - j\,3{\cdot}42$	*Nil*	8,365
	II.	$5 + j5$	$V_2 = -220 - j127{\cdot}2$	$I_2 = -13{\cdot}84 + j\,0{\cdot}149$		
	III.	$10 - j60$	$V_3 = j254{\cdot}3$	$I_3 = -4{\cdot}66 + j\,3{\cdot}28$		
3.	I.	20	$V_{12} = 440$	$I_{12} = 22$	*Nil*	29,557
	II.	$5 + j5$	$V_{23} = -220 - j381$	$I_{23} = -60{\cdot}1 - j16{\cdot}1$		
	III.	$10 - j60$	$V_{31} = -220 + j381$	$I_{31} = -6{\cdot}77 - j\,2{\cdot}54$		

4. Q. 1 (*a*) $W_1 = 237{\cdot}5$ kW. $W_2 = 186{\cdot}4$ kW.
   (*b*) $W_1 = 237{\cdot}5$ kW. $W_2 = 186{\cdot}4$ kW.
   Q. 2 (*b*) $W_1 = 8{,}140$ watts $W_2 = 225$ (without neutral).
   Q. 3 $W_1 = 36{,}124$ watts $W_2 = -6567$ watts.

CHAPTER VI. p. 124

1. Speed 375 r.p.m. $V_p = 4{,}430$ volts. $V_l = 7{,}670$ volts.
2. (i.) − 32·1 per cent. (ii.) 5·1 per cent. (iii.) 21·2 per cent.
3. (i.) 176·5 amps. (ii.) (*a*) 0·893. (*b*) 0·708.
   (iii.) $4{,}725 + j3{,}049, \lvert 5{,}000 \rvert$ volts, $6{,}262 + j2{,}976, \lvert 6{,}933 \rvert$ volts.
   (iv.) $7{\cdot}12 + j12{\cdot}94$ amps.
4. 9,790 kW.
5. 557·5 amps. at p.f. 0·949 leading.

CHAPTER VII. pp. 139–140

1. (i.) 5·03 amps. (ii.) 2·23 amps.

2. (i.) 1,200 r.p.m. (ii.) 0·16 ohms. (iii.) 27·8 per cent.

3. 71 lbs. ft.

4. (i.) 29·6 amps. (ii.) 0·86 lagging. (iii.) 0·256. (iv.) 54·6 per cent. (v.) 201 lbs. ft. (vi.) 1,120 r.p.m.

CHAPTER IX. pp. 171–172

1. $R = 0{\cdot}482$ ohms/mile. $L = 2{\cdot}483$ mH/mile. $C = 0{\cdot}0151$ μF/mile. $P = (0{\cdot}566 + j2{\cdot}02)10^{-3}$. $\alpha = 0{\cdot}566 \times 10^{-3}$. $\beta = 2{\cdot}02 \times 10^{-3}$. $Z_0 = (2{\cdot}34 - j31{\cdot}1)10^3$.

2.

Length	0	100	200	300	400	500
Voltage kV	$50\underline{/0}$	$47{\cdot}26\diagdown\overline{11^\circ 35'}$	$44{\cdot}66\diagdown\overline{23^\circ 10'}$	$42{\cdot}25\diagdown\overline{34^\circ 45'}$	$39{\cdot}25\diagdown\overline{46^\circ 21'}$	$37{\cdot}68\diagdown\overline{57^\circ 55'}$

3. Value of l for $V_{max} = 715$ miles.

4. $I_{ch} = -\ 1{\cdot}27 + j5{\cdot}14$ amps. $|I_{ch}| = 5{\cdot}29$ amps.

5. $V_s = (74{\cdot}42 + j17{\cdot}87)10^3$ volts. $|V_s| = 76{\cdot}56$ k. volts.
$I_s = 507 + j579$ amps. $|I_s| = 769{\cdot}6$ amps.
$I_{ch} = -\ 150{\cdot}9 + j1{,}042$ amps. $|I_{ch}| = 1{,}053$ amps.
Power lost = 8,090 kW.

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