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AN INTRODUCTION
TO
COORDINATE GEOMETRY

AN INTRODUCTION

TO

COORDINATE GEOMETRY

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PREFACE

THIS book, which is in harmony with the Mathematical Association Report on the Teaching of Higher Geometry in Schools, is intended as a three- or four-year course and assumes only an elementary knowledge of geometry and algebra at the outset. It is suitable for both scientists and mathematical specialists and covers all the metrical coordinate geometry needed for the G.C.E. and University Entrance Scholarship examinations.

In order that, from the start, the reader may be able to see where he is going, the opening chapters depart from the traditional arrangement and are so devised that he makes immediate use of what he learns, and is in a position to recognize the curves whose equations he meets in the exercises. To make the book more useful for those who have to study it with only occasional help from a teacher, an attempt has been made in many of the worked examples to explain how the problem can be approached rather than to present the solution as a polished product of omniscient cunning; and a number of historical notes are given. There has also been a reappraisal of the bookwork; a few topics (e.g. the director circle) have been relegated to the exercises but can be found by means of the index.

The mode of thought demanded by coordinate geometry does not come easily to many people, and there is a tendency in the early stages to adopt methods which seem easier at first but later have to be unlearnt. This has been guarded against where possible. A consistent notation has been used throughout, and particular care has been taken in introducing such topics as locus problems, pencils of lines, freedom equations, and the use of symmetric functions of roots.

Results printed in bold type are important, and many of them should be learnt. A number of mnemonic hints are given to make this easier.

Sections on determinants have been included because, although in various parts of the subject the working is simplified by their use,

they are often treated too late in an algebra course to be available when needed. Those who prefer to omit the topic, however, will find that nothing in the later chapters is thereby made unintelligible The sections on poles and polars can also be omitted if desired.

The later chapters need not be read in the order given. Chapter 11 can be taken any time after Chapter 8. Chapter 12—which it is hoped will serve as a work of reference to the properties of a number of famous curves as well as providing an unusual collection of examples for solution—can be begun with the help of Section 11.4 on polar coordinates as soon as Chapter 7 has been read, although for its full treatment the ideas of Chapter 9 are needed.

The exercises in Part I (Chapters 1–6) are largely numerical, and have been arranged to 'come out', though a few fractions and surds will be found among the answers. Bad methods, however, may lead to heavy arithmetic. The questions numbered 1 and 2 (and in general $2n-1$ and $2n$) are closely similar, so that the second can be used if the reader has had to get help with the first; the even-numbered questions can also be used for revision. The same principle holds in Chapters 7–11 of Part II, where, after the introduction of freedom equations, more general work becomes possible. Though here the parallelism between odd and even questions is less exact, either set can be effectively used to cover the bookwork.

In the miscellaneous exercises at the end of Chapters 4–10, the plan is somewhat different. Nos. 1–6 are roughly parallel with Nos. 7–12, and $12n+1$ to $12n+6$ with $12n+7$ to $12(n+1)$. Each set of six covers a fairly wide field, and some of the later questions in each exercise call for more ingenuity than those in the body of the chapters. In both types of exercise, however, an effort has been made to provide questions which are interesting, and those of the type 'find the equation . . .' have been avoided except as drill.

Questions taken from examination papers are indicated as follows:

O.L. G.C.E. Ordinary Level (or School Certificate) of the Oxford and Cambridge Schools Examination Board;

A.L. Advanced Level (or Higher Certificate) of the same Board;

S.L. Scholarship Level (or Higher Certificate Distinction Papers);

C.S. Entrance Scholarships to Cambridge University.

Thanks are due to the Oxford and Cambridge Schools Examination Board, to Dr. D. M. Brown of King's College, Cambridge, and to the Syndics of the Cambridge University Press, for permission to use these questions, and also to the staff of the University of London Press for their invaluable advice and help.

A. B.

June 1958

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PART I

CHAPTER 1

INTRODUCTION

1.1. What Coordinate Geometry is about

It is assumed that in geometry you have already learnt some of the properties of straight lines and circles, and that in algebra you have drawn easy graphs such as $y = 12/x$. In coordinate geometry we link these two subjects together and study geometry by means of algebra.

Ordinary elementary geometry (which we shall call pure geometry) is almost entirely concerned with straight lines and circles, but in coordinate geometry we can range more widely. You will already have met straight-line graphs and, though you may never have plotted a circular graph, you will at least know that an equation such as $y = x^2$ gives a U-shaped curve. We shall be able to investigate the properties of this curve and of others whose equations are fairly simple.

Again, in pure geometry, if we are given a problem which says "Prove that the locus of points which do such-and-such a thing is a circle", it is often necessary to wait for inspiration about where the centre of the circle will be before it is possible to prove that the locus really is a circle. In coordinate geometry, however, every line and every curve has its own equation, and the ordinary processes of algebra enable us to find the equation of any required locus, though when the equation of the locus has been obtained it may be necessary to find out what curve the graph of the equation is.

So we begin with two tasks before us:

1. Given some property of a moving point, to find the equation of the curve it traces out, and

2. Given an equation, to discover some geometrical properties of the curve it represents.

Each of these tasks is unlimited, and at first we shall have to confine ourselves to simple cases, particularly since a fair proportion of the time will be spent in learning *how* to do these two things. All the same, we shall be able to obtain a number of new results and it will be interesting to look at old results from a new viewpoint.

One thing, however, must be made clear at the outset. It is no part of our plan to abandon pure geometry completely; coordinate geometry may take us into regions where pure geometry alone would be ineffective, but wherever elementary pure geometry can help to obtain results we shall not hesitate to use it; and similarly in the exercises (except in rare cases where the contrary is indicated), you are meant to use pure geometry wherever it shortens the working.

1.2. Some Names and Conventions

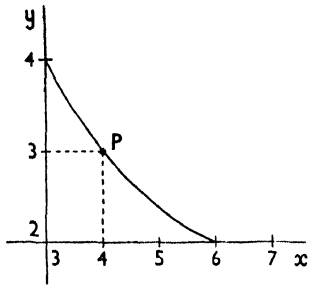

Fig. 1

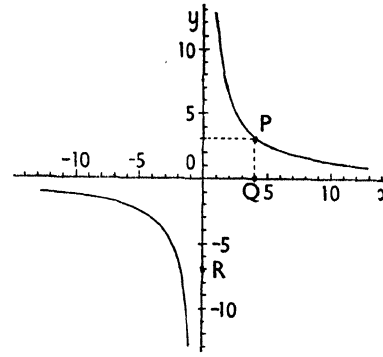

Fig. 2

Figs. 1 and 2 are both graphs of $y = \frac{12}{x}$, but there are two important differences between them. In each figure the values of x and y are marked on two perpendicular lines called **the coordinate axes** (or just **the axes**): the horizontal one, with values of x marked on it, is called the x-axis, and the vertical one the y-axis. In Fig. 1 these axes meet where $x = 3$ and $y = 2$; in Fig. 2 they meet where $x = 0$ and $y = 0$. In coordinate geometry **we always take our axes as in Fig. 2.** The point at which they meet, where $x = 0$ and $y = 0$, is called **the origin,** and in this book the letter ***O*** will always mean the origin unless anything is said to the contrary. The axes divide the paper into four compartments called **quadrants:** starting with the one in which x and y are both positive, and going counter-clockwise round the origin, they are called the first, second, third, and fourth quadrants.

The other difference between Figs. 1 and 2 is in the scales. In Fig. 1 the scales on the two axes are different; in Fig. 2 they are both the same. In coordinate geometry **we always use the same scale on both axes.**

ERRATA

Page 28, line 30. *For* (2, 3) *read* (3, 2).

Page 77, Qu. 14. *For* $x-3y+5=0$ *read* $x+3y+5=0$.

Page 84, line 28. *For* $\sin\alpha\cos\beta(\tan\beta+\tan\alpha)=g$ *read* $c\sin\alpha\cos\beta(\tan\beta+\tan\alpha)=g$.

Page 93, Qu. 10. *For* $4x^2-20x+y=0$ *read* $4x^2-20x-y=0$.

Page 101, Qu. 11(ii). *For* x^2 *read* $2x^2$.

Page 120, Qu. 6. *For* $\cos\theta$ *and* $\sin\theta$ *read* $2\cos\theta$ *and* $2\sin\theta$.

Page 127, Qu. 17, third line. *For* 4 *read* 2.

Page 128, line 7. *For* $\pm\sqrt{(l^2+m^2)}$ *read* $\pm t\sqrt{(l^2+m^2)}$.

Line 25. *For* l_1x+m_1y+n *read* lx_1+my_1+n.

Line 26. *For* l_2x+m_2y+n *read* lx_2+my_2+n.

Page 155, Equation (15). *For* $x(t^4-1)$ *read* $c(t^4-1)$.

Page 231, Qu. 28. *For* $\frac{x}{al}+\frac{y}{bm}-n=0$ *read* $\frac{x}{al}+\frac{y}{bm}-\frac{1}{n}=0$.

Page 289, Ex. 3c, No. 5. *For* $x-5y=11$ *read* $x+5y=11$.

Ex. 3c, No. 7. *For* $x-5y=13$ *read* $x+5y=13$.

Page 290, Ex. 4a, No. 5(ii). *For* $2x^2+2y^2-6x-10y=9$ *read* $2x^2+2y^2-6x-10y+9=0$.

On the curve shown, P is the point at which $x = 4$ and $y = 3$. The values of x and y are called the **coordinates** of the point, the value of x being called its x-coordinate, and the value of y its y-coordinate; and to say where the point is we simply write its coordinates inside a bracket, with a comma between them, like this: (4, 3). Notice that the x-coordinate comes first, and the y-coordinate second, because x comes before y in the alphabet.

The origin is thus the point (0, 0); any point on the x-axis has zero for its y-coordinate, for example Q is (4, 0); any point on the y-axis has zero for its x-coordinate, for example R is (0, −7). The x-coordinate is sometimes called the **abscissa** (the word means 'cut off', the 'sciss' part coming from the same root as in 'scissors'); the y-coordinate can be called the **ordinate.** We shall not use these two terms in this book, but you may meet them in problems taken from other sources.

As we are now using a mixture of algebra and geometry, it is important to distinguish between small letters and capitals. Small letters are used for algebraic expressions, each letter representing a number, while the chief use of capital letters is to represent points. Thus xy means the number x multiplied by the number y, while XY means the line joining the point X to the point Y, or else it measures the distance between these two points. In the same way xy^2, following the rules of algebra, means the number x multiplied by the square of the number y; but on the other hand, XY^2, following the customs of geometry, means the square of the distance between the points X and Y. One exception deserves mention: the x-axis and y-axis are often called Ox and Oy, or even $x'Ox$ and $y'Oy$ if it is desired to draw attention to the parts on both sides of the origin.

Two other conventions may usefully be mentioned here. A **line** will always mean a straight line. A **curve** will mean any kind of graph: it may therefore occasionally mean a straight line. Instead of speaking of "the curve whose equation is $y = 12/x$", we shall usually say "the curve $y = 12/x$".

Exercise 1A

1. (*a*) Draw two axes on graph-paper, and, taking ½ in. as the unit on each axis, mark the points A, B, C, D, E, F whose coordinates are (0, 4), (3, 0), (−3, 2), (3, −4), (−1, −4), (5, 2).

(*b*) In your figure, draw the line DF, find and mark the mid-point of this line (use any convenient means), and write down its coordinates.

(*c*) In your figure, can you find—

(i) three points in a straight line,

(ii) a line joining two of the points which is parallel to a line joining two others,

(iii) a line joining two points which is perpendicular to a line joining two others?

2. (*a*) Draw two axes on graph-paper, and, taking 0·1 in. as the unit, mark the points P $(-12, 15)$, Q $(8, 22)$, R $(22, 10)$, S $(-5, -5)$, T $(4, 0)$, U $(15, 2)$, V $(29, -10)$.

(*b*) In your figure, draw the line QR, find and mark the mid-point of this line, and write down its coordinates.

(*c*) In your figure, can you find

(i) three points in a straight line,

(ii) four points which form a square,

(iii) four points which form a parallelogram?

CHAPTER 2

FIRST IDEAS ABOUT CURVE-TRACING

2.1. What is meant by Curve-tracing

Before we have gone far with coordinate geometry we shall obtain the equation of a locus and want to know what the locus itself looks like. Although we can sometimes make a table of values and plot the curve, it will save a lot of trouble if we can find what a curve looks like when we are given its equation, without having to plot it. This process is called curve-tracing, and usually includes finding some exact information about the curve (such as that it goes through a certain point), and some more general information (such as that it is U-shaped, and which way up it is).

2.2. The Straight Line

You probably already know from experience that any equation such as $y = 2x-3$ gives a straight line when plotted. It is called a **first-degree equation,** because the terms in x and y are of the first degree. The equation $x-2y+8 = 0$ is of the first degree, but $\frac{2}{x}+y = 5$ is not, because before its degree is reckoned an equation must be written so that it contains no fractions with x or y in the denominator. So this last equation must be written $2+xy = 5x$, and the term xy is then of the second degree.

We shall later find good reasons why a first-degree equation always represents a line, but for the present you can simply accept the fact. If you are given a first-degree equation, an easy way to sketch the line which it represents is to find where it cuts the axes. For example, if the equation is $x-2y+8 = 0$, by putting $y = 0$ we find that it cuts the x-axis where $x+8 = 0$, i.e. at the point $(-8, 0)$, and by putting $x = 0$ we find that it cuts the y-axis where $-2y+8 = 0$, i.e. at the point $(0, 4)$. So a sketch of the line would be like Fig. 3.

There are two cases in which this method cannot be applied. One is when the equation contains only one of the letters x and y. Suppose

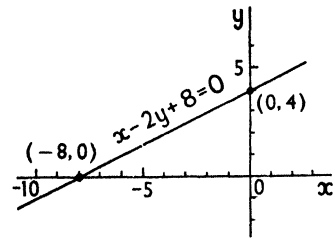

Fig. 3

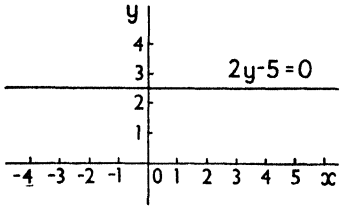

Fig. 4

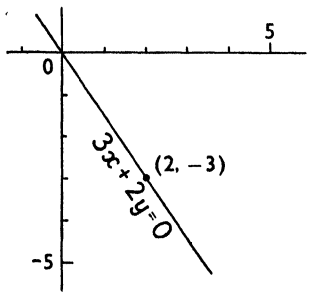

Fig. 5

we have the equation $2y-5=0$, which can be written $y=2\frac{1}{2}$. This tells us that y is always $2\frac{1}{2}$, but x can be anything. So the line is as shown in Fig. 4, and is parallel to the x-axis. In the same way, if the equation does not contain y, for example $x+1=0$, it represents a line parallel to the y-axis.

The other case is when the equation does not contain a **constant term,** for example $3x+2y=0$. In such a case $y=0$ when $x=0$, and so the line goes through the origin. It is easy to find another point on it—in this case $(2, -3)$ would do—and the line may be drawn by joining this point to the origin. (See Fig. 5.)

EXERCISE 2A

In Nos. 1–4, draw the lines represented by the three equations on one piece of graph-paper, using the same axes, and state whether any of the lines are parallel or perpendicular.

1. $3x+4y=12$, $4x-3y=0$, $y+1=0$.
2. $y-3=0$, $y+2=0$, $5x-y=0$.
3. $2x-3y+12=0$, $2x=3y$, $x=3$.
4. $y=6$, $x+2y=4$, $2x-y-8=0$.

In Nos. 5–6, draw the lines on one piece of graph-paper, using the same axes, and state what you notice about them.

5. $4x-3y=0$, $2x+3y=18$, $y=4$.
6. $2x+3y+6=0$, $5x+3y=30$, $5x+6y=0$.
7. What is the equation of the x-axis? Of the y-axis?
8. What are the equations of the lines which bisect the angles between the axes?

2.3. The Uses of an Equation

In 1.2 we considered the graph of $y=\frac{12}{x}$, and the equation was written in this form so that when a value of x was given, the value of

y could be found at once. For instance, if $x = 2$, $y = \frac{12}{2} = 6$, and so the point (2, 6) was on the graph. But the equation could equally have been written $xy = 12$, or $x = \frac{12}{y}$. This last form would have some advantage if we were given the y-coordinate of a point on the curve and wanted to find its x-coordinate. What is the particular use of the form $xy = 12$? The answer is that it tells us whether a given point is on the curve or not. Take, for example, the point (2·75, 4·4): if $x = 2{\cdot}75$ and $y = 4{\cdot}4$, then $xy = 2{\cdot}75 \times 4{\cdot}4 = 12{\cdot}1$, and so the point is not on the curve, although it is not far off. In the same way, (1·5, 8) is on the curve because $1{\cdot}5 \times 8 = 12$.

This use of the equation is much more important when the coordinates are not numbers but letters. If the point (h, k) lies on the curve $xy = 12$, then $hk = 12$; if we want to find a value of t such that $(t, t+1)$ lies on the curve $xy = 12$, then we have $t(t+1) = 12$, i.e. $t^2+t-12 = 0$, from which we can find that $t = 3$ or -4, and so on. We shall make use of this idea in the next section.

Since much of our later work will bring in letters rather than numbers it is a good thing to make a habit of writing all equations so that they contain no fractions, or at least no fractions with letters in the denominator. Thus the equation $y = \frac{2x+3}{3x-4}$ would be rewritten $3xy-2x-4y-3 = 0$. One advantage of this should be noticed. When the coordinates of the origin (0, 0), are substituted in the left-hand side of the second version of the equation, all the terms become zero except the last, the constant term. As the constant term is not zero, the equation is not satisfied and therefore the origin does not lie on the curve. **But if the constant term of such an equation is zero, the curve passes through the origin.**

EXERCISE 2B

1. Which of the following points lie on the line $3x-5y+2 = 0$? A (1, 1), B (17, 11), C (−19, −11), D (5n−4, 3n−2).

Find the value of n for which D lies on the line $x-2y+4 = 0$. What are then ts coordinates numerically? What does this tell you about the lines $x-2y+4 = 0$ and $3x-5y+2 = 0$?

2. Which of the following points lie on the line $5x+7y = 6$? E (11, −7), F (−20, 15), G (7n, 1−5n), H (18−7t, 5t−12).

Find the value of t for which H lies on the line $11x+6y = 32$. What does this ell you about the lines $11x+6y = 32$ and $5x+7y = 6$?

In Nos. 3–10, find the equation connecting h and k if the given point lies on the given curve.

3. (h, k), $y = x^2$.
4. (h, k), $x+y+1 = 0$.
5. $(-h, -k)$, $xy = 6$.
6. $(-h, k)$, $y = 2x^2$.
7. $(h+2, k-1)$, $3x+4y = 2$.
8. $(h+3, k+2)$, $y^2-5x-4y+19 = 0$.
9. $(2h, 2k)$, $y = x^2$.
10. $(2h, 2k)$, $xy = 12$.
11. Which of the curves mentioned in Nos. 3–6 goes through the origin?
12. Which of the curves mentioned in Nos. 7–10 goes through the origin?

2.4. Symmetry

The time-honoured juvenile sport of making 'ink-devils' consists in taking a piece of paper, spilling ink on it, folding it in two and squashing it flat so that the ink is squeezed on to both parts, unfolding it again, and admiring the result. Any figure obtained in this way is made up of two balanced halves: if you draw a line from one side of the figure to the other, at right angles to the fold, the fold will divide the line into two equal halves and these two halves will fit exactly on to one another when the paper is refolded. The whole figure is said to be **symmetrical** about the fold (*symmetrical* comes from two Greek words meaning 'like-measurement') and the fold is called the **axis of symmetry.**

Again, if the figure is cut in two along the fold and one half of it is held against a mirror so that the cut edge is in contact with the glass, the reflection or **image** in the mirror will supply the missing half. Also, if we imagine a mirror which has no thickness and which reflects on both sides, such a mirror could be placed with one edge along the fold of the uncut figure in such a way that each half of the figure coincided with the reflection of the other. In such a case (as this is a book on geometry and not on optics) we say that each half of the figure *is* the image of the other, and consequently that the whole figure is its own image in the fold. In more general terms, a figure is symmetrical about a certain line if it is its own image in that line.

It is easy to find the image of a point in the y-axis. To find the image of (5, 3), for example, we drop a perpendicular from it to the y-axis—this takes us to (0, 3) and we have moved 5 units to the left—and then produce this perpendicular the same distance on the other side of the y-axis. This takes us to $(-5, 3)$ which is therefore the image of (5, 3). In the same way (5, 3) is the image of $(-5, 3)$, and you can see that to reflect a point in the y-axis we have merely to change the sign of its x-coordinate: the image of (x, y) is $(-x, y)$.

We can now go on to find the image of any line or curve in the

y-axis. Take, for example, the line $x-2y+2=0$. The points $(-2, 0)$, $(2, 2)$, $(4, 3)$ all lie on this line. Their images in the y-axis are $(2, 0)$, $(-2, 2)$, $(-4, 3)$, and these three points lie on the line $(-x)-2y+2=0$, obtained from the equation of the given line by changing the sign of x. The result of substituting the coordinates of the new points in the new equation is the same as that of substituting the coordinates of the old points in the old equation because the sign of the x-coordinates has now been changed twice over. Verify this for yourself; then plot both lines in ink on graph-paper, fold the paper about the y-axis and look through it at the light to verify that the lines really are the reflections of each other.

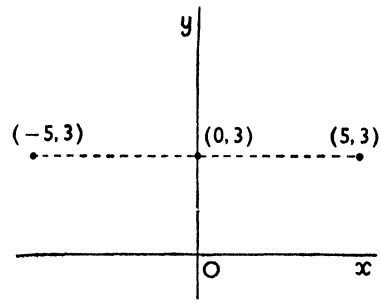

Fig. 6

To obtain the image of any curve in the y-axis, therefore, we have only to replace x by $(-x)$ in its equation. For example, the image of the curve $y=x^3$ in the y-axis is $y=(-x)^3$, which becomes $y=-x^3$ when simplified. If you have the least doubt about this, plot both curves for yourself.

The image of the curve $y=x^2$ in the y-axis is $y=(-x)^2$. This, however, simplifies to $y=x^2$ and turns out to be the same as the original curve. Hence the curve is symmetrical about the y-axis, because it is its own image.

The test is therefore as follows. **Reflect the curve in the y-axis by substituting $(-x)$ for x in its equation: if the new equation is (after simplification) the same as the old, the curve is symmetrical about the y-axis.**

By an argument like the one just given we can show that the image of a point in the x-axis is got by changing the sign of its y-coordinate and that the image of a curve in the x-axis is obtained by writing $(-y)$ for y in its equation. It follows that **a curve is symmetrical about the x-axis if its equation is unaltered when $(-y)$ is substituted for y.**

These are the two most easily recognized kinds of symmetry, but there are two other lines in which we can readily reflect a curve and about which we can therefore test for symmetry. They are the lines

which bisect the angles between the axes, the lines $x = y$ and $x+y = 0$.

Fig. 7 shows how to find the image of a point in the line $x = y$. In it, $ON = ON'$ and $NP = N'P'$, so that if the figure were folded about the line $x = y$, N would fall on N' and P on P', making P' the image of P. But ON is equal to the x-coordinate of P while ON' is equal to the y-coordinate of P'; in the same way NP and $N'P'$ are respectively equal to the y-coordinate of P and the x-coordinate of P'. Hence if P is (a, b), then P' is (b, a), and the image of a point in the line $x = y$ is got by interchanging its coordinates. Thus, in the figure, Q and Q' are the images of each other, their coordinates being (2, −7) and (−7, 2).

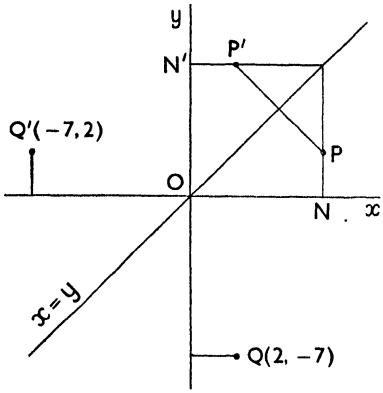

Fig. 7

If (2, −7) lies on a certain curve, then (−7, 2) lies on the image of that curve in the line $x = y$, and so to reflect a curve in this line we have only to exchange x and y in its equation. For example, the image of $3x-4y+12 = 0$ is $3y-4x+12 = 0$, that is to say $4x-3y-12 = 0$. The points (−4, 0), (0, 3) and (4, 6) lie on the first line while (0, −4), (3, 0) and (6, 4) lie on the second. The point (12, 12) lies on both and also on $x = y$.

To test the symmetry of a curve about $x = y$, therefore, we exchange x and y in its equation: the curve is symmetrical if the new equation is (after simplification) the same as the old. The curve $xy = 12$ is an example: the equation of its image is $yx = 12$ and this is the same as $xy = 12$.

In the same way we can show that reflection in the line $x+y = 0$ can be carried out by replacing x by $(-y)$ and y by $(-x)$. Symmetry about this line can be tested accordingly.

There is also another type of symmetry which a curve may possess. The curve $y = x^3$, for example, has the property that, if the line joining any point on it to the origin is produced an equal distance beyond the origin, the point so reached is also on the curve. Such a curve is said to be **symmetrical about the origin,** and it is easy to see

that **a curve is symmetrical about the origin if its equation is unaltered when $(-x)$, $(-y)$ are substituted for x, y.**

In the same way, a curve may be symmetrical about points other than the origin; a point about which a curve is symmetrical is called its **centre.**

Notice that a curve may be symmetrical about more than one line. For example, a circle is symmetrical about any line through its centre. Also, it may be possible to obtain one curve from another by reflection in any of several lines. Thus $y=-x^3$ can be obtained from $y=x^3$ by reflection in the x-axis or in the y-axis.

EXERCISE 2C

The equations in this exercise are given in such a form that you can easily plot the curves if you wish.

Write down the equations of the images of the following curves in the x-axis and also in the y-axis. Hence find out whether the curves are symmetrical about either or both of these axes. Which of them are symmetrical about the origin?

1. $y = x$.	2. $x = 0$.	3. $y = x^2$.
4. $y = x^4$.	5. $y = x^3$.	6. $y^2 = x^3$.
7. $y^3 = x$.	8. $y^3 = x$.	9. $y^2 = x^2$.
10. $y^4 = x^4$.	11. $y = (x+1)^2$.	12. $y^2 = (x+1)^2$.
13. $y = x^2+1$.	14. $y^2 = x-1$.	15. $y = 1/x$.
16. $y = 1/x^3$.	17. $y = 1/x^2$.	18. $y^2 = 1/x$.
19. $y^2 = x^2+1$.	20. $y^2 = 4-x^2$.	21. $y = \pm\sqrt{(1+x)}$.
22. $y = \pm\sqrt{x}$.		

Write down the equations of the images of the following curves in each of the lines $x = y$ and $x+y = 0$ and so investigate their symmetry about these lines. Get rid of fractions before you begin.

23. $y = \dfrac{11-2x}{2}$.	24. $y = \dfrac{3x-5}{3}$.	25. $y = x^2$.
26. $y = x^3$.	27. $y = \dfrac{2x+3}{x+2}$.	28. $y = \dfrac{3+x}{1-2x}$.
29. $y = 7/3x$.	30. $y = -5/2x$.	31. $y^2 = 9-x^2$.
32. $y^2 = x^2-9$.		

2.5. The Parabola

Fig. 8 is a sketch of the graph of $y = x^2$, a curve called a parabola. It is symmetrical about the y-axis; and the point at which the axis of symmetry cuts the curve (in this case the origin) is called the **vertex** of the parabola.

The graph of $y = 4x^2$, on the same scale, would be narrower—or taller, if you prefer to look at it that way. For a given value of x, the value of y would be four times as great for the second curve as

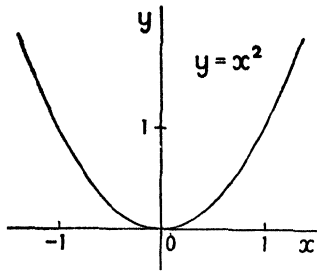

Fig. 8

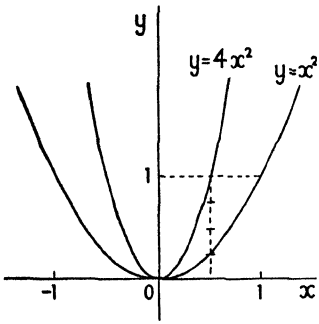

Fig. 9

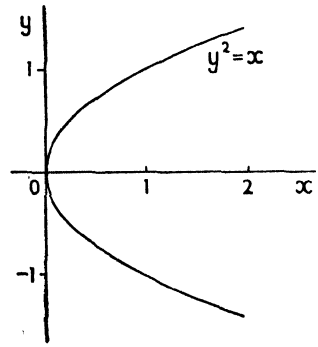

Fig. 10

for the first; or, as the equations of the two curves can be written $x = \pm\sqrt{y}$ and $x = \pm\frac{1}{2}\sqrt{y}$, for a given value of y the value of x would be half as great for the second as for the first. The two parabolas are shown in Fig. 9.

In the same way, the graph of, say, $y = \frac{1}{3}x^2$ would be wider and squatter than the first curve. From these two examples you can get a general idea of the curve $y = kx^2$ when k is any positive number.

If k is negative, the curve is upside down, as you can see by considering the case of $y = -x^2$. In between the cases when k is positive and those when k is negative is the case in which $k = 0$, when the parabola has gone quite flat and become the x-axis, $y = 0$.

The graph of $y^2 = x$ can be easily derived from that of $y = x^2$. Draw up a table of values for $y = x^2$ and then interchange x and y, so getting a table of values for $x = y^2$, or $y^2 = x$. Then plot your new values, putting Ox and Oy in their usual places, and verify that the new graph is as in Fig. 10.

We can also write its equation $y = \pm\sqrt{x}$: this shows at once its symmetry about the x-axis and also shows that the curve contains no points for which x is negative, because a negative number has no square root. Make a table of values from this version of the equation and verify that the points so obtained lie on the curve you have drawn. From what has been said in 2.4 it follows that the curves $y = x^2$ and $y^2 = x$ are the

reflections of one another in the line $x = y$. This also is easily tested.

To get parabolas of different sizes we can use the equation $y^2 = kx$. For a given value of x, y is then $\sqrt{k}$ times as big as before, and it follows that a large value of k gives a large parabola, a small value of k a small one. If k is negative the parabola moves over to the left-hand side of the y-axis, since only negative values of x are then possible.

In spite of all the different curves which can be obtained from these equations, **all parabolas are geometrically similar**—that is to say, any parabola can be made identical with a second parabola by redrawing the first one on a different scale. Parabolas therefore, though they may differ in size, are all the same shape, and a geometrical property possessed by one will as a rule be a property of all. But you must bear in mind that a parabola is not a closed curve (like a circle or a figure eight) and it is impossible to draw the whole of it. That is why the two curves in Fig. 9 do not look similar. If the part of the outer parabola there shown were redrawn on one-quarter the scale, it would coincide with a small portion of the inner one near the vertex.

A good way to convince yourself that parabolas are similar is to draw the graph of $y = x^2$ on one side of a piece of graph-paper, taking 1 in. as the unit, and then on the other side of the paper, using the same axes (look through the paper and see where they are), draw the graph of $y = 2x^2$, taking 2 in. as the unit, so that the second graph is on twice the scale of the first. On looking through the paper you will see that the two curves coincide.

In making this test you have turned one of the parabolas round, since the positive part of the x-axis on the back of the paper coincides with the negative part on the front, but this makes no difference because the curves are symmetrical about the y-axis. Apart from this change of sign, the numbers marked on both the axes on the front of the paper are twice as big as those which lie under them on the back: this and the fact that the equation of the curve on the back can be written $(2y) = (2x)^2$ lead to the coincidence of the two curves.

A practical example of a parabola is the section of the reflector of a searchlight, or of the surface of a liquid which is rotating uniformly. The path of a stone which has been thrown obliquely into the air is also very nearly a parabola.

2.6. The Rectangular Hyperbola

The graph of $y = 12/x$, shown in Fig. 2, is a curve called a **rectangular hyperbola.** It consists of two separate parts, called its two **branches,** lying in the first and third quadrants. The curve cannot cut the x-axis, because $12/x$ can never be zero and so y cannot be zero. The equation can also be written $x = 12/y$, and so the curve cannot cut the y-axis either, because, however large y is, x can never be zero. The curve gets nearer and nearer to the axes but never quite reaches them; they are therefore called the **asymptotes** of the curve, from three Greek words meaning 'not-falling-together'. As in the case of the parabola, it is impossible to draw the whole of the curve.

The curve has no symmetry about the axes, but it is symmetrical about the lines $x = y$ and $x+y = 0$. These lines are shown dotted in Fig. 11. The point at which the axes of symmetry meet (in this case the origin) is called the **centre** of the hyperbola. It satisfies the test of 2.4.

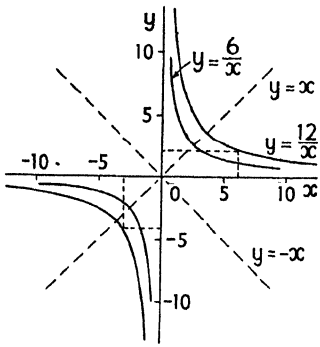

Fig. 11

A point on the graph of $y = 6/x$ has a y-coordinate half that of $y = 12/x$, for a given value of x; and, as the equation is also $x = 6/y$, the x-coordinate will be half that of a point on the first curve for a given value of y. The two curves are shown together in Fig. 11.

You will now have a general idea of what the graph of $y = k/x$ looks like for various positive values of k. The larger k is, the further the curve is from the axes. If k is negative, x and y must have opposite signs, and so the curve lies in the second and fourth quadrants and appears as in Fig. 12. Notice that the equation can be written $xy = k$; you will need this fact in the next section.

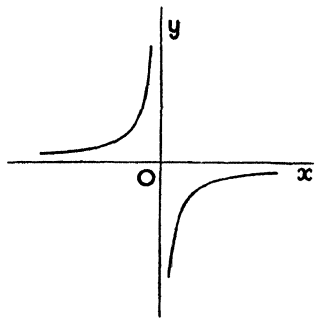

Fig. 12

All rectangular hyperbolas are

geometrically similar though they may differ in size, but there are other types of hyperbola (whose asymptotes are not at right angles) and these need not be similar to one another. You can test the similarity of two rectangular hyperbolas by drawing the graph of $y = 12/x$ taking 1 in. as the unit and, on another sheet of paper, $y = 3/x$ taking 2 in. as the unit. If you then place the two sheets together, face upwards, so that the axes coincide when you look through them at the light, you will find that the curves coincide too. Once again, you may see why, if you notice that $y = 3/x$ can be written $(2y) = 12/(2x)$.

Apart from graphs it is difficult to find a familiar example of a rectangular hyperbola. If a reading-lamp with a tilted circular shade is placed on a table so that some of the rays of light rise as they travel away from the lamp, the shadow of the shade which the lamp casts on the table will be part of a hyperbola, but the asymptotes of the curve will not generally be at right angles. If the shade is lowered so that the highest ray travels horizontally the shadow will be a parabola, and if it is lowered further the curve will be some form of ellipse, a curve which will be dealt with in Chapter 4.

2.7. How to shift a Curve

We said in 1.1 that our two broad aims were (1) to find the equation of a required curve and (2) to find the properties of a curve with a given equation. So far we have confined ourselves to a parabola with its vertex at the origin and a rectangular hyperbola with its centre at the origin, but it is clear that in carrying out (1) we may be concerned with two parabolas whose vertices are in different places, and on the other hand we shall very soon come across new equations and require to interpret them. So our next step is to find the equations of parabolas and rectangular hyperbolas which are given in other positions and to recognize such curves when we are given their equations.

Suppose we want to shift the parabola $y^2 = 4x$ so that its vertex moves to the point (3, 7), its axis of symmetry remaining parallel to the x-axis. Consider what the equation $y^2 = 4x$ means. If P is any point on the curve, V the vertex, and N the foot of the perpendicular from P to the axis of symmetry, the equation means that $PN^2 = 4\,VN$. If we move the parabola to its new position, and the coordinates of P are (x, y), then you can see from Fig. 13 that $PN = y-7$, and $VN = x-3$. So the equation of the parabola in its new position is

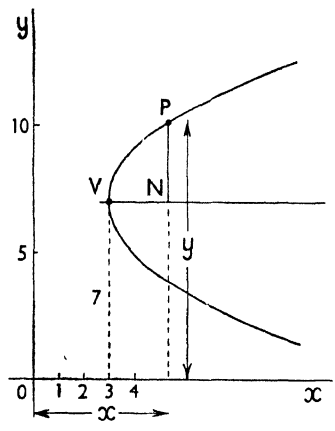

Fig. 13

$(y-7)^2 = 4(x-3)$. In the same way, if we shift the vertex to the point (a, b), the equation will be $(y-b)^2 = 4(x-a)$.

The problem occurs more often the other way round. We might be given the equation $(y+2)^2 = 6(x-1)$ and want to know what curve it represents. We can now say that it is a parabola, just like the parabola $y^2 = 6x$, with its vertex at the point $(1, -2)$.

Even if the equation is jumbled up, we can still do the same thing, as the following two examples show.

Example 1. Trace the curve $y^2-5x-4y+19 = 0$.

The fact that the equation contains a y^2 suggests that the curve is a parabola $y^2 = kx$ with its vertex shifted away from the origin. The equation of such a parabola is $(y-b)^2 = k(x-a)$, where (a, b) are the coordinates of the vertex. If we multiply this equation out, it is $y^2-2by+b^2 = kx-ka$. In order to make the given equation as much like this as possible, the y^2 has to stay on the left-hand side (L.H.S.) and so has the $-4y$, while the $-5x$ has to be transferred to the right-hand side (R.H.S.). Some of the constant term has to go on each side, but for the present we can put it all on the right. So we can write down $y^2-4y = 5x-19$. The L.H.S. has to be a perfect square, so we must add the square of half the coefficient of y (in this case 4) to both sides. This gives us $y^2-4y+4 = 5x-15$, and we can write it as $(y-2)^2 = 5(x-3)$. The curve is therefore a parabola just like $y^2 = 5x$, with its vertex at (2, 3), and we can easily sketch it.

Example 2. Trace the curve $xy+3x-4y = 0$.

We should not expect this to be a parabola, because there is no square term; but the equation does contain an xy term, so we can try to find a rectangular hyperbola $xy = k$ such that when its centre is shifted to some point (a, b) it will have the required equation. Its equation would be $(x-a)(y-b) = k$, by the same argument as we used for the parabola, and when this is multiplied out it is $xy-bx-ay+ab = k$. The xy is as it should be, but we want $+3x$

instead of $-bx$, and $-4y$ instead of $-ay$. So $a = 4$ and $b = -3$. We then need -12 on the L.H.S. instead of $+ab$, and so we add -12 to both sides of the given equation. It is then $xy+3x-4y-12 = -12$, and we can write it as $(x-4)(y+3) = -12$. The curve is therefore a rectangular hyperbola just like $xy = -12$, with its centre shifted to $(4, -3)$. We can sketch it more easily if we notice that, as the given equation has no constant term, the values $x = 0$, $y = 0$ satisfy it and the curve goes through the origin. The sketching is also made easier by drawing the asymptotes, which are parallel to the axes and pass through the centre. The final sketch is as in Fig. 14.

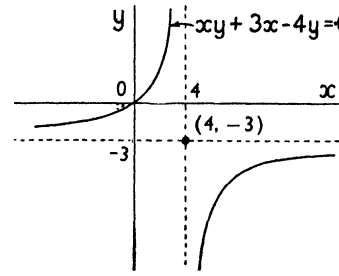

Fig. 14

There is no difficulty in plotting this curve and so verifying the truth of our deductions. The equation may be solved for y, giving $y = \dfrac{-3x}{x-4}$, and the table of values may be laid out as follows:

x	-12	-8	-4	0	2	3	4	5	6	8	12	16	20
$-3x$	36	24	12	0	-6	-9	-12	-15	-18	-24	-36	-48	-60
$x-4$	-16	-12	-8	-4	-2	-1	0	1	2	4	8	12	16
y	$-2{\cdot}25$	-2	$-1{\cdot}5$	0	3	9	*	-15	-9	-6	$-4{\cdot}5$	-4	$-3{\cdot}75$

The asterisk indicates that no value of y can be found when $x = 4$. The values in the table can be seen to agree with Fig. 14.

Example 3. Trace the curve $x^2+4x+2y = 0$.

[The first two examples were given to show you how to do such problems. This one shows you how to write out the solution.]

Comparing the given equation with $(x-a)^2 = k(y-b)$, we can write it

$$x^2+4x = -2y,$$

i.e. $$x^2+4x+4 = -2y+4,$$

i.e. $$(x+2)^2 = -2(y-2).$$

The curve is therefore a parabola like $x^2 = -2y$, with its vertex at

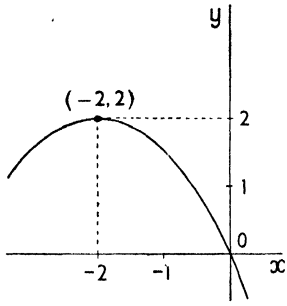

Fig. 15

(−2, 2). As the equation has no constant term, the curve passes through the origin, and its appearance is as shown. (See Fig. 15.)

Exercise 2D

Trace the following curves: that is, say what each curve is like, where its centre or its vertex is, and give a sketch of it.

1. $xy-3x-2y=0$.
2. $xy+x+6=0$.
3. $y^2-x-2y+3=0$.
4. $y=2x^2+12x+8$.
5. $y^2+4x-8y=0$.
6. $y^2-x+6y=0$.
7. $xy+x-3y+5=0$.
8. $xy-3x+6y=0$.

9. By tracing the curve $y=x^2-6x+13$, find the least value which $x^2-6x+13$ can have.

10. By tracing the curve $y=\dfrac{x+10}{x+2}$, find the range of values which $\dfrac{x+10}{x+2}$ can have for positive values of x.

11. Plot the curve of No. 7, from $x=-8$ to $x=12$, using $\frac{1}{4}$ in. as the unit, and so verify your results in No. 7.

12. Plot the curve of No. 8, from $x=-20$ to $x=10$, using $\frac{1}{5}$ in. as the unit, and so verify your results in No. 8.

CHAPTER 3

POINTS

3.1. Distance

The pure geometry which you have learnt has been concerned almost entirely with three things—distance, angle, and area—as you can see by recalling the theorems which you know. Of these three, the first two are the most important for our present purpose, and we must discover how they can be dealt with in coordinate geometry. Distance is closely associated with points, and will be considered forthwith, but angle depends more on lines and will be considered in Chapter 5.

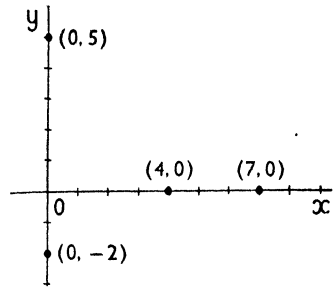

Fig. 16

If we are given two points on one of the axes, there is no difficulty in finding the distance between them. The distance between (4, 0) and (7, 0) is 3 units (i.e. 7−4), and the distance between (0, 5) and (0, −2) is 7 units (i.e. 5−[−2])—that is to say, the distance between two points on the x-axis is equal to the difference between their x-coordinates, and a similar result holds for points on the y-axis.

But before we go any further we must notice an important fact about the coordinates we are using. They are *directed* or *signed* numbers. With the axes in their usual positions the x-coordinate of a point represents its distance to the right or left of the origin, distances to the left being indicated by a *minus* sign and distances to the right by a *plus* sign or no sign at all. In the language of directed numbers this is usually expressed by saying that "the x-coordinate represents the distance of the point to the right of the origin", and the reader is supposed to understand that a distance of −3 units to the right means a distance of 3 units to the left. In the same way, the

y-coordinate of a point represents its distance *above* the origin.*

If A, B are the points $(a, 0)$, $(b, 0)$, then $b-a$ represents the distance of B to the right of A, and this is true in the language of directed numbers no matter whether a and b are positive or negative or whether a is greater or less than b. We call $b-a$ the **directed distance** from A to B, and we denote it by AB. You should notice that AB is then measured by a directed number and may be negative; in particular $AB = -BA$ because BA would be measured by $a-b$ and would represent the distance of A to the right of B. This is not what you are used to in pure geometry, where lengths have always been treated as signless numbers, but the difference need not cause any confusion: sometimes the sign of the distance will be useful, but if it is not we can merely ignore it.

More generally, **if A and B are two points on one of the axes, or if the line joining them is parallel to one of the axes, we shall mean by AB the directed distance from A to B.**

We can now go on to consider the distance between *any* two points. To avoid using too many different letters we will adopt a new notation and let the coordinates of the points be (x_1, y_1) and (x_2, y_2). These should be read as 'x-one', 'y-one', 'x-two', and 'y-two'. The small figure is called a **suffix**; it is only a label and does not mean that $x = 1$ or $x = 2$, and you must be careful to put it at the bottom of the letter and not at the top, because x^2 already means 'x squared'. We can call these two points P_1 and P_2 for short: this notation will be used without further explanation throughout the book.

If we draw lines through P_1, P_2 parallel to the y-axis to cut the x-axis in A, B respectively, and draw a line through P_1 parallel to the x-axis to cut P_2B (produced if necessary†) in C, then P_1P_2C is a right-angled triangle and by the theorem of Pythagoras

$$P_1P_2^2 = P_1C^2 + CP_2^2.$$

In this theorem the lengths were all supposed to be signless, whereas under our present convention P_1C and CP_2 are directed lengths and may be positive or negative. The theorem, however, only mentions their squares, and these (being necessarily positive) are indistinguishable from the squares of signless numbers. So no complications arise,

* You may find it useful here to read again the chapter on directed numbers in your algebra book, keeping the idea of coordinates particularly in mind. The results of that chapter will be assumed in what follows.

† We shall usually omit these words in future: they are to be understood where required.

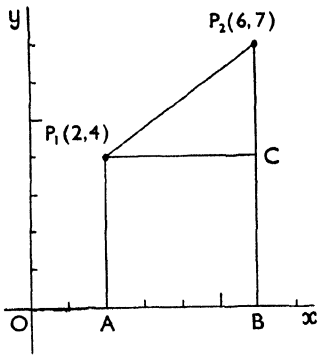

Fig. 17a

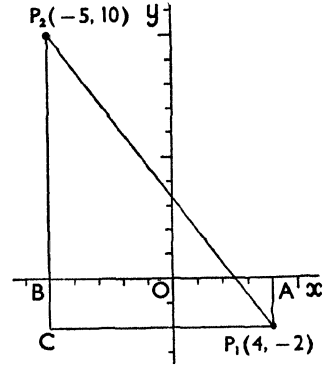

Fig. 17b

and if we can write down the values of P_1C and CP_2 we can immediately find the value of P_1P_2.

Many different figures are possible and two are shown above. Fig. 17a shows the case in which P_1 is (2, 4) and P_2 is (6, 7), while in Fig. 17b P_1 is (4, −2) and P_2 is (−5, 10). You should draw other cases of your own and verify that all stages of the following argument apply to each of them.

Directed lengths	General value	Value in 17a	Value in 17b
OA =	x_1	2	4
OB =	x_2	6	−5
AP_1 =	y_1	4	−2
BP_2 =	y_2	7	10
$AB = OB - OA$ =	x_2-x_1	6−2 = 4	−5−4 = −9
$P_1C = AB$ =	x_2-x_1	4	−9
$BC = AP_1$ =	y_1	4	−2
$CP_2 = BP_2 - BC$ =	y_2-y_1	7−4 = 3	10−(−2) = 12

$$\therefore \ P_1P_2^2 = (x_2-x_1)^2+(y_2-y_1)^2 \quad . \quad . \quad (1)$$

This formula must be learnt. Notice that it does not matter which way round the subtractions are carried out: the R.H.S. can equally be written $(x_1-x_2)^2+(y_1-y_2)^2$.

In Fig. 17a we get $P_1P_2^2 = 4^2+3^2 = 16+9 = 25$ and therefore $P_1P_2 = 5$.

In Fig. 17b, $P_1P_2^2 = (-9)^2+12^2 = 81+144 = 225$ and so $P_1P_2 = 15$.

We do not attempt to choose a positive or negative direction along the line P_1P_2 and so P_1P_2 represents a signless length which for practical purposes is the same as a positive one.

The general treatment of the distance of a point from a line must be left until Chapter 7, but the distance of a point from one of the *axes* is simpler. It is easy to see that, if a point is in the first quadrant, its x-coordinate is its distance from the y-axis, and its y-coordinate is its distance from the x-axis. If a point is in any other quadrant, at least one of its coordinates is negative and must have its sign changed if a positive distance is required. In this chapter we shall restrict points to the first quadrant if we are interested in their distances from the axes (see Exercise 3c, Nos. 11-14).

Example. The vertices of a triangle are P (3, 4), Q (4, −2), R (−7, −4). Determine whether the triangle is acute-angled, right-angled, or obtuse-angled.

A figure is not necessary, but may be helpful to you until you are used to the work.

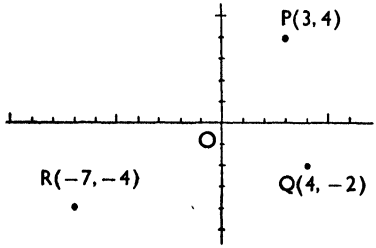

Fig. 18

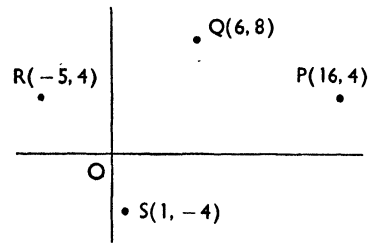

Fig. 19

$PQ^2 = (3-4)^2+(4+2)^2 = 1+36 = 37$,
$QR^2 = (4+7)^2+(-2+4)^2 = 121+4 = 125$, and
$PR^2 = (3+7)^2+(4+4)^2 = 100+64 = 164$.
PR is the greatest side, and so PR^2 must be compared with PQ^2+QR^2. $PR^2 = 164$, $PQ^2+QR^2 = 162$; hence PR is too long to make the triangle right-angled, and PQR is an obtuse-angled triangle.

Exercise 3A

1. Find the ten distances between the points O, P, Q, R, S shown in Fig. 19.
2. A, B, C, D are the points (8, 6), (−5, 9), (−1, −6), (3, −6). Find the ten distances between the five points A, B, C, D, O.

In Nos. 3–14, by finding the distances between the points, show that they are the vertices of the stated figure.

3. (7, 3), (5, 5), (−1, −3); an isosceles triangle.
4. (−4, −3), (4, −2), (0, 4); an isosceles triangle.
5. (4, 3), (−4, 1), (−1, −6), (7, −4); a parallelogram.
6. (3, −1), (−2, 3), (−4, −1), (1, −5); a parallelogram.
7. (−5, −2), (7, −1), (−1, 4), (3, −7); a parallelogram.
8. (−9, 1), (5, 2), (−3, 5), (−1, −2); a parallelogram.
9. (−3, 0), (−2, −1), (1, 2), (0, 3); a rectangle.
10. (8, 5), (1, −6), (−2, −1), (11, 0); a rectangle.
11. (3, −2), (7, 6), (−1, 2), (−5, −6); a rhombus.
12. (−4, −3), (−1, −14), (5, 4), (8, −7); a rhombus.
13. (−1, 0), (3, 1), (2, 5), (−2, 4); a square.
14. (−3, 2), (4, 7), (2, −5), (9, 0); a square.

In Nos. 15 and 16, what figure is formed by the points?

15. (5, 5), (10, −7), (−2, 1), (−3, −7).
16. (−16, 0), (−16, −25), (24, 5), (0, 12).
17. Determine whether the triangles with the given vertices are acute-angled, right-angled, or obtuse-angled.

(i) (3, −2), (4, 9), (8, 1); (ii) (−4, −3), (1, −9), (5, −6);
(iii) (−1, 3), (2, 1), (3, 9); (iv) (−2, −2), (4, 1), (8, −7).

18. Determine whether the triangles with the given vertices are acute-angled, right-angled, or obtuse-angled.

(i) (−4, 5), (2, −4), (15, 5); (ii) (−6, −1), (2, −7), (11, 5);
(iii) (−5, 9), (−5, −6), (2, −1); (iv) (−4, −1), (2, 7), (7, −9).

19. Show that the points (−2, 2), (2, −6), (5, 3) lie on a circle whose centre is (2, −1). What is its radius?
20. Show that the points (0, 8), (17, 1), (17, −9) lie on a circle whose centre is (5, −4). What is its radius?
21. Show that the points (20, 18), (−24, 6) and (12, −22) lie on a circle whose centre is (1, 1). Without further calculation, find twenty-one more points which lie on this circle and whose coordinates are whole numbers.
22. Show that the points (20, 6), (−7, 15), (4, −18) lie on a circle whose centre is (3, 0). Without further calculation, find twenty-one more points which lie on this circle and whose coordinates are whole numbers.
23. Show that each of the points (1, −2), (7, 7), (11, 13) is the same distance from A (0, 3) as from B (6, −1). On what must the first three points lie?
24. Show that each of the points (−3, −6), (0, 0), (2, 4) is the same distance from P (−3, 4) as from Q (5, 0). On what must the first three points lie?

3.2. Mid-point

Consider first two points A, B on the x-axis and let C be the point midway between them. Then $AC = CB$, this equation meaning not only that the two lengths are numerically equal but also that both have the same sign—for example, if C is to the right of A then B is to the right of C. If the coordinates of A, B, C are $(a, 0)$, $(b, 0)$ $(c, 0)$, this means that $c-a = b-c$ and hence $c = \frac{a+b}{2}$.

In Fig. 20a, $a = 2$, $b = 8$, $c = \frac{2+8}{2} = 5$; $AC = CB = 3$. In

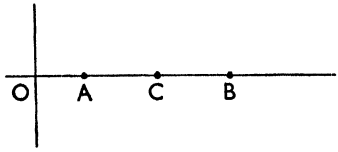

Fig. 20a

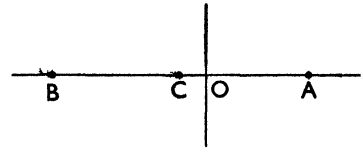

Fig. 20b

Fig. 20b, $a = 4$, $b = -6$, $c = \dfrac{4-6}{2} = -1$; $AC = CB = -5$. So in every case the mid-point of the line joining $(a, 0)$ and $(b, 0)$ is $\left(\dfrac{a+b}{2}, 0\right)$.

We are now able to find the coordinates of M, the mid-point of P_1P_2. Through P_1, P_2, M draw lines parallel to the y-axis to cut the x-axis in A, B, C respectively. Then by a theorem in pure geometry,* C is the mid-point of AB. Now the coordinates of A, B are $(x_1, 0)$, $(x_2, 0)$ and so the coordinates of C are $\left(\dfrac{x_1+x_2}{2}, 0\right)$, by what we have proved above.

Next, draw lines through P_1, P_2, M parallel to the x-axis to cut the y-axis in D, E, F respectively. Then in the same way we find that these are the points $(0, y_1)$, $(0, y_2)$ and $\left(0, \dfrac{y_1+y_2}{2}\right)$. But M has the x-coordinate of C and the y-coordinate of F, and so we have proved that

$$\textbf{the mid-point of } P_1P_2 \textbf{ is } \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \quad . \quad . \quad (2)$$

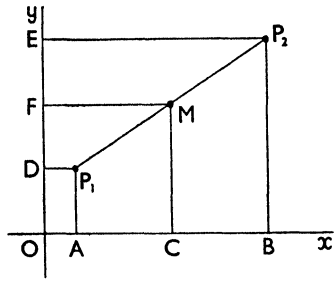

Fig. 21a

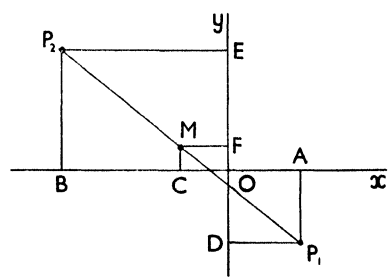

Fig. 21b

* If there are three or more parallel straight lines, and the intercepts made by them on any one straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.

This formula must be learnt. Two examples of mid-points are shown in Figs. 21a and 21b. In the first, P_1 is (1, 2), P_2 is (7, 6) and M is (4, 4); in the second, P_1 is (3, −3), P_2 is (−7, 5) and M is (−2, 1). The figures are drawn merely to illustrate the results of Formula (2); they are in no way necessary for its use.

Perhaps enough examples have now been given to convince you that directed distances in geometry have the convenient property of directed numbers in algebra; namely, that a result obtained from considering a case in which all the quantities are positive remains true even when some or all of the quantities are negative. It is usual, for that reason, to draw the easiest version of a figure as a guide to the calculations and argument in solving any problem: the result can then be relied on to be true in other cases as well. In the present chapter Formulæ (1) and (2) could have been obtained from Figs. 17a, 20a, and 21a only.

EXERCISE 3B

1. Verify result (2), using squared paper and any method which you fancy, for the following pairs of points.

(*a*) (2, 3) and (8, 15); (*b*) (7, 5) and (−3, 9);
(*c*) (6, 1) and (2, −11); (*d*) (−2, 5) and (10, −1);
(*e*) (−2, −8) and (−4, −6); (*f*) (3, 4) and (−7, −10).

2. Verify result (2), using squared paper and any method which you fancy, for the following pairs of points.

(*a*) (−4, 7) and (6, −9); (*b*) (−2, 8) and (−6, −2);
(*c*) (3, 3) and (7, −5); (*d*) (−2, −3) and (−6, −1).

3. Do Ex. 3a, No. 7 by showing that the two diagonals have the same mid-point.

4. Do Ex. 3a, No. 8 by showing that the two diagonals have the same mid-point.

5. The point (5, 5) is the mid-point of PQ. P is the point (17, −2). What are the coordinates of Q?

6. The point (2, −3) is the mid-point of AB. A is the point (7, 8). What are the coordinates of B?

7. $ABCD$ is a parallelogram. A, B, C are respectively (1, −3), (4, 2), (−5, 7). Find the coordinates of D.

8. $PQRS$ is a parallelogram. P, Q, R are respectively (1, −3), (−5, 7), (4, 2). Find the coordinates of S.

9. O, P_1, P_2 having their usual meanings, describe the position of the point (x_1+x_2, y_1+y_2) in relation to them.

10. It can be proved that the points which divide P_1P_2 into three equal parts are $A\left(\frac{2x_1+x_2}{3}, \frac{2y_1+y_2}{3}\right)$ and $B\left(\frac{x_1+2x_2}{3}, \frac{y_1+2y_2}{3}\right)$. Verify that A is the mid-point of P_1B, and that B is the mid-point of AP_2.

11. Apollonius' Theorem states that if M is the mid-point of the side BC of a triangle ABC, then $AB^2+AC^2 = 2AM^2+2BM^2$. Verify the theorem for the case in which A, B, C are the points (0, 0), (4, 7), (12, 3).

12. Verify Apollonius' Theorem (see No. 11) for the case in which A, B, C are the points (0, 0), (5, 3), (9, −3).

13. Prove Apollonius' Theorem (see No. 11) by taking axes so that the x-axis lies along BC, with the origin at M. The points A, B, C can then be taken as (x_1, y_1), $(-a, 0)$, $(a, 0)$.

14. Prove Apollonius' Theorem (see No. 11) by taking the origin at A and the x-axis along AM. The points B, C can then be taken as (b, h), $(c, -h)$.

3.3. Locus Problems

Example 1. A, B, C are the points (4, 0), (0, 3), (−5, −5); find the locus of the point P for which $PA^2+PB^2 = 2PC^2-75$.

The words 'find the locus' mean that we have first to find the equation of the locus (i.e. the equation satisfied by the coordinates x, y of any point on the locus), then to find what kind of curve the equation represents and how it is situated, and finally to sketch the curve. Every point on the locus satisfies a certain condition (in this case, that $PA^2+PB^2 = 2PC^2-75$), and in order to get the equation of the locus we have to express this condition in terms of the coordinates of a point on the locus: it is therefore necessary at some stage of a locus problem to say "Let (x, y) be any point on the locus", and wherever possible we do so at the beginning. We can now go ahead with the solution of this example.

Let (x, y) be any point on the locus. Then $PA^2 = (x-4)^2+y^2$, $PB^2 = x^2+(y-3)^2$, $PC^2 = (x+5)^2+(y+5)^2$.

$\therefore (x-4)^2+y^2+x^2+(y-3)^2 = 2(x+5)^2+2(y+5)^2-75$,

$\therefore 2x^2+2y^2-8x-6y+25 = 2x^2+2y^2+20x+20y+25$,

$\therefore 14x+13y = 0$, and this is the equation of the locus. It represents a line through the origin, joining it to (13, −14), and is shown in Fig. 22.

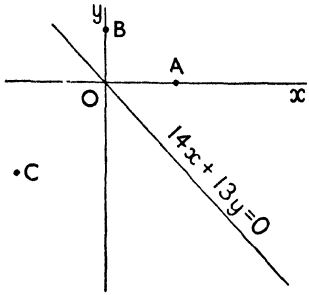

Fig. 22

Example 2. Find the locus of points which are 4 units further from the origin than from the point (4, 4).

Let (x, y) be any point on the locus; its distance from the origin is $\sqrt{(x^2+y^2)}$, and its distance from (4, 4) is $\sqrt{\{(x-4)^2+(y-4)^2\}}$.

Hence $\sqrt{(x^2+y^2)} = \sqrt{\{(x-4)^2+(y-4)^2\}}+4$.

This is the equation of the locus, but it is not in a form which tells us what the curve is like, and the chief purpose of this example is to show how to deal with square

roots in equations. The only way to get rid of a square root is to square both sides of the equation: doing this, we obtain

$$x^2+y^2 = (x-4)^2+(y-4)^2+8\sqrt{\{(x-4)^2+(y-4)^2\}}+16.$$

This still contains one square root, and so we must rewrite the equation so that the square root is on one side and all the remaining terms on the other, simplify the result, and square both sides again. Thus,

$$\begin{aligned}
&x^2+y^2-(x-4)^2-(y-4)^2-16 = 8\sqrt{\{(x-4)^2+(y-4)^2\}},\\
\therefore\ &8x+8y-48 = 8\sqrt{\{(x-4)^2+(y-4)^2\}},\\
\therefore\ &x+y-6 = \sqrt{\{(x-4)^2+(y-4)^2\}},\\
\therefore\ &(x+y)^2-12(x+y)+36 = (x-4)^2+(y-4)^2,\\
\therefore\ &x^2+2xy+y^2-12x-12y+36 = x^2+y^2-8x-8y+32,\\
\therefore\ &2xy-4x-4y+4 = 0,\\
\therefore\ &xy-2x-2y+2 = 0.
\end{aligned}$$

This is the equation of the locus, in the form in which it is usually given: notice that the second-degree term is written first, then the terms of first degree, and finally the constant term. In order to trace the curve, we proceed as in 2.7, Example 2.

Comparing the equation with $(x-a)(y-b) = k$, which when expanded is $xy-bx-ay+ab = k$, we see that $a = b = 2$, so that $ab = 4$. We therefore add 2 to each side of the equation and write it as $(x-2)(y-2) = 2$. Hence the curve is a rectangular hyperbola just like $xy = 2$, with its centre shifted to (2, 2). (See Fig. 23.)

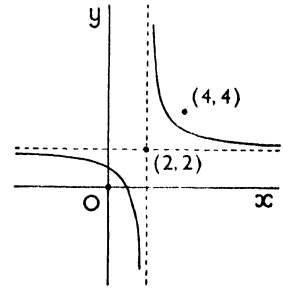

Fig. 23

Whenever an equation is simplified by squaring both sides, the resulting equation may yield solutions which do not satisfy the original equation, and in this case it is clear that any point on the lower (left-hand) branch of the hyperbola is *nearer* to the origin than to (4, 4). Hence the locus consists of the upper branch only, while the points of the lower branch satisfy the equation

$$\sqrt{(x^2+y^2)} = \sqrt{\{(x-4)^2+(y-4)^2\}}-4,$$

which (as you can easily verify) yields the same equation when the square roots are got rid of. The whole curve is the locus of points whose distances from (0, 0) and (4, 4) differ by 4 units.

Example 3. A variable point P on the parabola $y^2 = 4x$ is joined to the origin. Find the locus of the mid-point of OP.

It is possible to begin this problem by saying "Let (x, y) be any point on the locus", but you might find this confusing because x and y have already been mentioned in the equation $y^2 = 4x$, which is not the equation of the required locus. When doing examples in which the equation of a curve is given, and in some others which we shall meet later, it is better to distinguish the coordinates of a point on the locus by giving them a suffix.

Let (x_0, y_0) be any point on the locus. It is then the mid-point of OP, where P (x, y) is a point on the curve $y^2 = 4x$. But the mid-point of OP is $\left(\frac{x}{2}, \frac{y}{2}\right)$, and so $x_0 = \frac{x}{2}$, $y_0 = \frac{y}{2}$. These equations are more useful when written as $x = 2x_0$, $y = 2y_0$. Then, because P lies on the curve $y^2 = 4x$, substituting these values of x and y we get $(2y_0)^2 = 4(2x_0)$, which simplifies to $y_0^2 = 2x_0$.

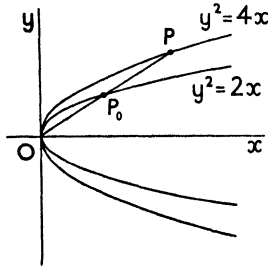

Fig. 24

We have now found that the coordinates (x_0, y_0) of any point on the locus are connected by the equation $y_0^2 = 2x_0$. Hence, if we had called them (x, y) instead of (x_0, y_0), they would have been connected by the equation $y^2 = 2x$. So this is the equation of the locus. It represents another parabola with its vertex at the origin, lying inside the given parabola, as in Fig. 24.

There are two important things to notice about this example. One is the way we start with a point (x_0, y_0) on the locus, and the way we get rid of the suffixes again at the end. The other is the contrast between the method employed here and the way in which you would construct the locus if this were a problem in pure geometry. Since in many problems the locus-condition is expressed in the language of pure geometry (in terms of distances, mid-points, angles, lines, parallels and so on), it is natural to think of the locus as being constructed by means of geometrical instruments. In this problem, if you were given a graph of $y^2 = 4x$ and wanted to construct the locus, you would draw a large number of lines (such as OP in Fig. 24) joining the origin to points on the graph, you would find the mid-point of each of these lines (e.g. P_0 in Fig. 24), and you would sketch a curve through all these mid-points. But in coordinate geometry

we do everything the other way round. We *begin* with a point P_0 on the locus, *from it* we find P, and *then* we use the fact that P is on the curve $y^2 = 4x$.

Example 4. A variable point P on the rectangular hyperbola $xy-2x-2y+2 = 0$ is joined to the point A (1, 1); PA is produced to Q so that $AQ = PA$. Find the locus of Q.

Let (x_0, y_0) be any point on the locus (we shall have to call this point Q instead of P_0). The mid-point of PQ is A, and so, if P is (x, y), the point $\left(\frac{x+x_0}{2}, \frac{y+y_0}{2}\right)$ is the same as (1, 1). Hence $x+x_0 = 2$, $y+y_0 = 2$.

We want x and y in terms of x_0 and y_0, so that we can use the fact that P lies on the given hyperbola, i.e. that $xy-2x-2y+2 = 0$. Now $x = 2-x_0$, $y = 2-y_0$, and therefore

$$(2-x_0)(2-y_0)-2(2-x_0)-2(2-y_0)+2 = 0.$$

This simplifies to $x_0y_0 = 2$, and is the equation connecting the coordinates (x_0, y_0) of any point on the locus. Hence, if we had called these coordinates (x, y) instead of (x_0, y_0), they would have been connected by the equation $xy = 2$, which is therefore the equation of the locus. It represents a rectangular hyperbola with its centre at the origin, similar to Fig. 2.

Example 5. $ABCD$ is a rectangle; find the locus of P if

$$PA.PC = PB.PD.$$

No axes or coordinates are mentioned in this example, and before we can do it we must take notice of the obvious but important fact that, **if a problem is expressed in terms of pure geometry, we can place the axes wherever we like.** In the example given, a little thought shows that we are likely to keep the working as simple as possible if we choose for our axes the axes of symmetry of the rectangle, since the locus will then be symmetrical about them. If we take a general position for A, say (a, b), we can then let B be $(-a, b)$, C $(-a, -b)$, and D $(a, -b)$.

If $P(x, y)$ is any point on the locus, then since $PA^2.PC^2 = PB^2.PD^2$,

$$\{(x-a)^2+(y-b)^2\}\ \{(x+a)^2+(y+b)^2\} = \{(x+a)^2+(y-b)^2\}\ \{(x-a)^2+(y+b)^2\}.$$

If this is multiplied out, we notice that the terms $(x-a)^2(x+a)^2$ and

$(y-b)^2(y+b)^2$ occur on both sides and need not be written down. We are therefore left with the equation

$(x-a)^2(y+b)^2+(x+a)^2(y-b)^2 = (x+a)^2(y+b)^2+(x-a)^2(y-b)^2$,

which can be written

$$\{(x+a)^2-(x-a)^2\}\ \{(y+b)^2-(y-b)^2\} = 0.$$

It reduces first to $16abxy = 0$, then to $xy = 0$, and is satisfied only when either $x = 0$ or $y = 0$. The locus therefore consists of the two coordinate axes.

As the question was worded in terms of pure geometry, the answer must be similarly worded: the locus consists of the axes of symmetry of the rectangle.

Exercise 3c

1. Find the locus of points which are the same distance from $(-2, 7)$ as from $(4, 3)$.

2. Find the locus of points which are the same distance from $(-4, 3)$ as from $(5, 0)$.

3. If A, B are the points $(3, 1)$, $(1, -2)$, find the locus of a variable point P such that $PA^2-PB^2 = 5$.

4. If A, B are the points $(5, -1)$, $(5, 6)$, find the locus of a variable point P such that $PA^2-PB^2 = 7$.

5. A, B, C are the points $(2, 0)$, $(-2, 0)$, $(1, 5)$. Find the locus of a variable point P such that $PA^2+PB^2 = 2PC^2$.

6. A, B, C, D are the points $(-2, 1)$, $(2, -1)$, $(-2, -1)$, $(2, 2)$. Find the locus of a variable point P such that $PA^2+PB^2 = PC^2+PD^2$.

7. A, B, C being the same as in No. 5, find the locus of a variable point P such that $PA^2+PB^2 = 2PC^2+8$.

8. A, B, C are the points $(-3, 5)$, $(5, -1)$, $(1, 5)$. Find the locus of a variable point P such that $PA^2+PB^2 = 2PC^2+8$.

9. A and B are the same as in No. 5. Try to find the locus of a variable point P such that $PA^2+PB^2 = 2PO^2+8$. What goes wrong? What has this to do with Apollonius' Theorem?

10. Try to do No. 8 for the case in which A, B, C are the points $(-3, 5)$, $(5, -1)$, $(1, 2)$. What goes wrong? What has this to do with Apollonius' Theorem?

11. P is a variable point in the first quadrant such that the sum of its distances from the axes is equal to its distance from $(-2, -2)$. Find the locus of P.

12. P is a variable point in the first quadrant such that the sum of its distances from the axes is equal to its distance from $(-4, 0)$. Find the locus of P.

13. Find the locus of a variable point in the first quadrant whose distance from the y-axis is equal to its distance from $(2, 0)$.

14. Find the locus of a variable point in the first quadrant whose distance from the x-axis is equal to its distance from $(0, 4)$.

15. A point moves so that its distances from $(-4, 4)$ and $(4, -4)$ differ by 8 units. Find its locus.

16. A point moves so that its distance from (2, 2) is 4 units less than its distance from (−2, −2). Find its locus.

17. A point in the first quadrant moves so that its distance from the origin is 6 units less than the sum of its distances from the axes. Find its locus.

18. A point moves in the first quadrant so that its distance from (−2, −2) is 2 units greater than the sum of its distances from the axes. Find its locus.

19. A point moves so that its distance from the origin is always 5 units. Find the equation of its locus. What curve must its locus be?

20. A point moves so that its distance from (3, 0) is always 3 units. Find the equation of its locus. What curve must its locus be?

21. A point P moves so that the sum of the squares of its distances from A (−4, 3) and B (4, −3) is 100 square units. Find the equation of its locus. What is the value of AB^2? What can you say about the angle APB and hence about the locus?

22. O, X are the points (0, 0), (6, 0); a point P moves so that $PO^2+PX^2 = 36$. Show that the locus of P is the same as in No. 20.

23. C and D are the points (0, 2), (0, −2); a point P moves so that $PC^2+PD^2 = 58$. Show that the locus of P has the same equation as in No. 19. Explain why this is so, with the help of Apollonius' Theorem.

24. A point moves so that the sum of the squares of its distances from A (3, 4) and B (3, −4) is 50 square units. Find the equation of its locus. If P is a point on the locus and C is the point (3, 0), what value does Apollonius' Theorem give for PC? Compare No. 20.

25. Q is the point (−4, 0); a point P moves so that $PQ^2 = PO^2+8PO$. Find the locus of P.

26. X is the point (−5, 0); a point P moves so that $PX^2 = PO^2+8PO+25$. Find the equation of the locus of P. Try to see what this equation represents.

27. A is the point (2, 6); P is a variable point on the x-axis. Use the method of Examples 3 and 4 of 3.3 to find the locus of the mid-point of AP. When you have obtained the answer you should be able to recognize quite easily that it is correct.

28. A is the point (2, 6); P is a variable point on the y-axis. Use the method of 3.3, Examples 3 and 4, to find the locus of the mid-point of AP.

29. A is the point (2, 6); P is a variable point on the x-axis. If PA is produced to Q so that $PA = AQ$, find the locus of Q. Use the method of 3.3, Example 4.

30. With the same data as in No. 28, if PA is produced to Q so that $PA = AQ$, find the locus of Q.

31. P is a variable point on the line $3x+4y = 7$, and A is the point (2, −1). If AP is produced to Q so that $AP = PQ$, find the locus of Q.

32. P is a variable point on the line $x-2y+3 = 0$, and A is the point (2, 4). If AP is produced to Q so that $AP = PQ$, find the locus of Q.

33. Find the locus of the mid-point of the line joining (6, 0) to a variable point of the parabola $y = x^2$.

34. Find the locus of the mid-point of the line joining (1, −2) to a variable point on the hyperbola $xy-2x+y = 6$.

35. P is a variable point on the rectangular hyperbola $xy = 6$; OP is produced to Q so that $OQ = 2OP$. Find the locus of Q.

36. P is a variable point on the parabola $y^2-2x-4y+2 = 0$, and H is the point (−2, 4). If HP is produced to Q so that $HQ = 2HP$, find the locus of Q.

37. Prove by means of coordinate geometry that the locus of points equidistant from two fixed points A and B is the perpendicular bisector of AB.

38. If A, B are fixed points, prove that the locus of a point P, which moves so that $AP^2 - BP^2$ is constant, is a line perpendicular to AB.

39. A rectangle, while remaining constant in area, varies so that two adjacent sides continue to lie along two fixed perpendicular lines; find the locus of the vertex which lies on neither of these lines.

40. A rectangle $ABCD$ varies while A and D remain fixed. Find the locus of a point P on BC, which moves so that BP^2 is always equal to the area of the rectangle.

CHAPTER 4

MORE CURVE-TRACING: CIRCLE AND ELLIPSE

4.1. Circle with given Centre and Radius

Since a circle is the locus of points which are at a given distance from a given point, we can use the results of Chapter 3 to find the equation of any circle. We will first take a numerical case and find the equation of the circle whose centre is C (3, -7) and whose radius is 5 units.

Let P (x, y) be any point on this circle: its distance from C is $\sqrt{\{(x-3)^2+(y+7)^2\}}$, and this must be equal to 5. The equation is therefore

$$\sqrt{\{(x-3)^2+(y+7)^2\}} = 5,$$

i.e. $$(x-3)^2+(y+7)^2 = 5^2 \quad . \quad . \quad . \quad . \quad (1)$$

i.e. $$(x^2-6x+9)+(y^2+14y+49) = 25,$$

i.e. $$x^2+y^2-6x+14y+33 = 0 \quad . \quad . \quad . \quad . \quad (2)$$

This equation differs in two respects from those which we met in Chapter 2. In the first place, if you wanted to plot its graph in order to satisfy yourself that it really does represent a circle, you would at once notice that it is difficult to rearrange equation (2) to give y in terms of x: the equation as it stands contains terms in both y and y^2, so that you would have to solve a quadratic equation. The same thing would happen if you wanted x in terms of y. If we want to solve for y in terms of x, we must write equation (2) as

$$y^2+14y = -(x^2-6x+33),$$

and complete the square on the L.H.S. by adding 7^2 to each side, thus,

$$y^2+14y+7^2 = 7^2-(x^2-6x+33),$$

i.e. $$(y+7)^2 = 16+6x-x^2 = (8-x)(2+x).$$

[Notice that this could have been obtained more easily from equation (1).] Hence

$$y = -7\pm\sqrt{\{(8-x)(2+x)\}},$$

and from this a table of values could be constructed.

In the second place, not every value of x gives values of y. From the standpoint of pure geometry this is obvious: the circle is a closed curve and therefore the coordinates of its points are confined between

certain limits. But let us see how the algebra itself leads to the same result. If x is below -2, then $2+x$ is negative and $8-x$ is positive, so that $(8-x)(2+x)$ is negative and has no square root: so no value can be found for y. The same is true when x is greater than 8, and it is only when x lies between -2 and 8 that values of y can be found. Equation (1) also makes this clear: neither $(x-3)^2$ nor $(y+7)^2$ can be negative, and so, since their sum is 25, neither of them can itself exceed 25. Now $(x-3)^2$ is the square of the difference between x and 3, so that this difference cannot exceed 5 and x must lie between $3-5$ and $3+5$, i.e. between -2 and 8. In the same way, the difference between y and -7 cannot exceed 5, and so y must lie between -12 and -2. These are, as we should expect, just the limits between which the given circle does lie.

The table of values, with x ranging from -2 to 8, can be set out as follows:

x	-2	$-1{\cdot}5$	-1	0	1	2	3	4
$8-x$	10	9·5	9	8	7	6	5	4
$2+x$	0	0·5	1	2	3	4	5	6
$(8-x)(2+x)$	0	4·75	9	16	21	24	25	24
$\sqrt{\{(8-x)(2+x)\}}$	0	2·18	3	4	4·58	4·90	5	4·90
$-7+\sqrt{\{(8-x)(2+x)\}}$	-7	$-4{\cdot}82$	-4	-3	$-2{\cdot}42$	$-2{\cdot}10$	-2	$-2{\cdot}10$
$-7-\sqrt{\{(8-x)(2+x)\}}$	-7	$-9{\cdot}18$	-10	-11	$-11{\cdot}58$	$-11{\cdot}90$	-12	$-11{\cdot}90$

The last two rows give the two values of y belonging to each value of x. Beyond the value $x = 3$, the values of y go back on their own tracks: y is the same when $x = 2$ and 4, 1 and 5, etc. You may like to plot these points on graph-paper, using $\frac{1}{2}$ in. as the unit, and see for yourself that they lie on a circle.

Another way of looking at equation (1) is useful. The distance of P (x, y) from the origin is $\sqrt{(x^2+y^2)}$, and so the equation of the circle whose centre is the origin and whose radius is 5 units is

$$x^2+y^2 = 25 \quad . \quad . \quad . \quad . \quad (3)^*$$

Using the method of 2.7, we can shift this circle so that its centre lies at (3, -7) by substituting $x-3$ for x, and $y+7$ for y, in equation (3).

* Notice that the tests of 2.4 show that this equation represents a curve whi has all the types of symmetry mentioned there.

In the same way, **the equation of the circle whose centre is (x_1, y_1) and whose radius is r units, is**

$$(x-x_1)^2+(y-y_1)^2 = r^2 . \quad . \quad . \quad . \quad (4)$$

This result must be learnt.

When the brackets are multiplied out it becomes

$$(x^2-2x_1x+x_1^2)+(y^2-2y_1y+y_1^2) = r^2.$$

Remembering that x_1 and y_1 are fixed numbers, not variables like x and y, we can rearrange this equation so that the second-degree terms come first, then the first-degree terms, and finally the constant term, thus,

$$x^2+y^2-2x_1x-2y_1y+(x_1^2+y_1^2-r^2) = 0 \quad . \quad . \quad (5)$$

and, bearing in mind that $(x_1^2+y_1^2-r^2)$ is simply a number, we see that the equation has the same form as (2).

4.2. The General Equation of a Circle

Equations (2), (3), (5) have in common the fact that the second-degree terms consist of x^2+y^2 and of nothing else; there is no term in xy. As (5) can be made to represent *any* circle by a suitable choice of x_1, y_1, and r, we may ask whether an equation of the second degree will always represent a circle provided that its second-degree terms consist simply of x^2+y^2. Let us take the most general case of such an equation: it will contain, besides x^2+y^2, terms in x and y and a constant term, though any of these last three may be zero in a particular instance such as (3). The equation is customarily written

$$x^2+y^2+2gx+2fy+c = 0 \quad . \quad . \quad . \quad (6)*$$

In order to make it look as much like (4) as possible, we write it

$$(x^2+2gx \quad)+(y^2+2fy \quad)= -c,$$

* You might well suppose that a and b would be more reasonable letters to use as the coefficients of x and y. The reason for the curious choice of letters in equation (6) is that in more advanced work the general equation of the second degree contains *three* variables, x, y, z, and every term is of the second degree. Such an equation will contain square terms x^2, y^2, z^2, and also product terms in yz, zx, xy. The squares are given the coefficients a, b, c; the letters d, e are omitted (being used for other things); and the letters f, g, h are used in the coefficients of the products. Of these products, yz omits x, zx omits y, and xy omits z: the coefficients are therefore attached in this order, and a factor 2 is included because *half* the coefficient is often required in such processes as completing the square. The general equation then stands as

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0.$$

If in this we replace z by 1, we get

$$ax^2+by^2+c+2fy+2gx+2hxy = 0,$$

which, when written in another order, is

$$ax^2+2hxy+by^2+2gx+2fy+c = 0.$$

In the case under discussion, $a = b = 1$, and $h = 0$.

and complete the square in each bracket by adding g^2+f^2 to each side, thus,

$$(x^2+2gx+g^2)+(y^2+2fy+f^2) = g^2+f^2-c,$$

i.e. $$(x+g)^2+(y+f)^2 = (g^2+f^2-c) \quad . \quad . \quad (7)$$

The L.H.S. now represents the square of the distance of (x, y) from $(-g, -f)$, and so (7)—and therefore (6)—represents a circle whose centre is $(-g, -f)$ and whose radius is $\sqrt{(g^2+f^2-c)}$.

Three points should be noticed here:

1. It is not *necessary* for the second-degree terms to be x^2+y^2 in order that an equation of the second degree shall represent a circle: the equation could be multiplied throughout by any constant k, and the second-degree terms would then be kx^2+ky^2. What is essential is that the coefficients of x^2 and of y^2 shall be *equal*—for example, the equation of the circle whose centre is $(\frac{1}{2}, \frac{3}{2})$, and whose radius is $\frac{2}{3}$, is most conveniently written

$$18x^2+18y^2-18x-54y+37 = 0.$$

2. When using results depending on (6), do not forget that g and f are *half* the coefficients of x and y *when the coefficients of* x^2 *and* y^2 *have been reduced to* 1.

3. If (g^2+f^2-c) is zero, the radius is zero and the equation is said to represent a **point circle,** because only one real point—namely, the centre—can be found to satisfy it. If (g^2+f^2-c) is negative, no points can be found which satisfy the equation, because the L.H.S. of (7) cannot be negative. Such an equation, which might be obtained as a 'solution' to a locus problem, does not represent a real curve at all, and would show that there were no points satisfying the required condition. This fact should serve as a reminder that the solution of a locus problem is not completed by merely finding an equation.

We may now summarize the results of this section so far as follows: **In order that an equation of the second degree may represent a circle, the coefficients of x^2 and y^2 must be equal, and there must be no term in xy. The equation**

$$x^2+y^2+2gx+2fy+c = 0 \quad . \quad . \quad . \quad (6)$$

represents a circle whose centre is $(-g, -f)$ and whose radius is $\sqrt{(g^2+f^2-c)}$. These results should be learnt.

If the coordinates of a point are substituted in the L.H.S. of (6), the quantity obtained is known as the **power** of the point with respect to the circle. The power is evidently zero if the point is on the circle. What does it represent in the general case?

A comparison of (6) and (7) shows that the power is also equal to

$(x+g)^2+(y+f)^2-(g^2+f^2-c)$, and here the first two terms represent the square of the distance of the point from the centre, while the third represents the square of the radius. The power of a point P with respect to a circle with centre C and radius r is therefore PC^2-r^2; it is thus positive if the point is outside the circle, negative if inside. Moreover, if the point is outside the circle, it is easy to see from a diagram that the power of the point is equal to the square of the tangent from it to the circle.

EXERCISE 4A

In Nos. 1 and 2, without assuming any of the results of this chapter, find the equations of the circles having the given centres and radii.

1. (i) (0, 0), 3; (ii) (−1, −2), 3;
   (iii) (3, 3), 3; (iv) (3, −4), 5.
2. (i) (0, 0), 2; (ii) (5, −2), 2;
   (iii) (−1, 1), 4; (iv) (−4, 0), 4.

In Nos. 3 and 4, without assuming any of the results of this chapter, show that each of the equations represents a circle, and find its centre and radius.

3. (i) $x^2+y^2 = 16$; (ii) $x^2+y^2+4x-10y+28 = 0$;
   (iii) $x^2+y^2+8x = 0$; (iv) $x^2+y^2-10x+24y = 0$.
4. (i) $x^2+y^2 = 9$; (ii) $x^2+y^2+16x-30y = 0$;
   (iii) $x^2+y^2+2x-10y-10 = 0$; (iv) $x^2+y^2-12x+20 = 0$.

In Nos. 5 and 6, using the results of 4.1, find the equations of the circles having the given centres and radii. Express each equation in a form which contains no fractions.

5. (i) (0, 0), $\frac{1}{2}$; (ii) $(\frac{3}{2}, \frac{5}{2})$, 2;
   (iii) $(\frac{1}{2}, -\frac{2}{3})$, $\frac{5}{6}$; (iv) $(\frac{3}{2}, 1)$, $\frac{3}{2}$.
6. (i) (0, 0), $\frac{2}{3}$; (ii) $(2, -\frac{5}{2})$, $\frac{3}{2}$;
   (iii) $(-\frac{5}{4}, 3)$, $3\frac{1}{4}$; (iv) $(\frac{1}{5}, \frac{2}{5})$, 1.

In Nos. 7 and 8, using the results of 4.2, find the centre and radius of the circle represented by each equation. Plot the curves on graph-paper, and so verify your conclusions.

7. (i) $x^2+y^2-2x-4y-4 = 0$; (ii) $x^2+y^2+5x-12y = 0$;
   (iii) $4x^2+4y^2-12x-16y-11 = 0$; (iv) $5x^2+5y^2-4x+8y-1 = 0$.
8. (i) $x^2+y^2+4x+6y-12 = 0$; (ii) $x^2+y^2-x+2y-1 = 0$;
   (iii) $9x^2+9y^2+12x-18y+4 = 0$; (iv) $5x^2+5y^2-5x-2y-1 = 0$.
9. For what value of k does the equation
$$(x-y)^2+k(x-2)(y-1) = 0$$
represent a circle, and what are then its centre and radius?

Draw in one figure the lines $x-y = 0$, $x-2 = 0$, $y-1 = 0$ and the circle which you have found. You should find that two of the lines are tangents to the circle at the points where it is cut by the third line. This is also true of the curves obtained by taking other values of k, but the general treatment of the problem must be left until Chapter 10.

10. For what value of k does the equation
$$x^2+4y^2+8x+12+k(2x^2+y^2-2y-3) = 0$$
represent a circle, and what are then its centre and radius?

(The curves $x^2+4y^2+8x+12 = 0$ and $2x^2+y^2-2y-3 = 0$ are two ellipses; the circle goes through the four points in which they cut one another. Other values of k would give other ellipses through these four points.)

11. Two circles touch each other externally if the distance between their centres is equal to the sum of their radii; they touch internally if the distance between their centres is equal to the difference between their radii. Use these facts to determine which of the following pairs of circles touch each other, and whether they touch externally or internally.

(i) $x^2+y^2-2x-8=0$, $x^2+y^2-8x-8y+28=0$;
(ii) $x^2+y^2+2y=0$, $x^2+y^2-8x+7=0$;
(iii) $x^2+y^2-4x-2y-13=0$, $x^2+y^2-6x-4y+5=0$;
(iv) $x^2+y^2+2x+2y=0$, $x^2+y^2-12x+4=0$.

12. Determine whether the following pairs of circles touch each other, and if so, how.

(i) $x^2+y^2+6x+6y-6=0$, $x^2+y^2-10x-6y+8=0$;
(ii) $x^2+y^2+2x+2y-11=0$, $x^2+y^2-10x-6y-83=0$;
(iii) $x^2+y^2+12x-16y=0$, $x^2+y^2-18x+32=0$;
(iv) $x^2+y^2-6x=0$, $x^2+y^2-2x-14y-56=0$.

13. A circle has centre $(1, -2)$; find its equation if it—

(*a*) passes through the centre of the circle $x^2+y^2+6x-2y+6=0$,
(*b*) touches the given circle externally,
(*c*) touches it internally.

14. Find the equations of the two circles concentric with $x^2+y^2+4x-4y=0$ which touch the circle $x^2+y^2-2x+4y-31=0$. Is the contact internal or external?

15. Are the points $A(-4, 8)$, $B(-2, 9)$, $C(0, 10)$, $D(2, 11)$ inside, on, or outside the circle $x^2+y^2-6x-8y=25$?

How could you, with very little working, have deduced the facts about A and C from those about B and D?

16. Are the points $P(6, -2)$, $Q(8\frac{1}{2}, 5)$, $R(5, 13)$, $S(8, 2)$ inside, on, or outside the circle $x^2+y^2+6x-10y=96$?

17. Find the lengths of the tangents from $(1, -2)$ and $(-1, 6)$ to the circle $4x^2+4y^2+28x-8y=63$.

18. Find the points on the y-axis from which the tangents to the circle $x^2+y^2-6x-2y-8=0$ are 4 units in length.

19. The circle $x^2+y^2+2gx+2fy+c=0$ cuts the x-axis at A and B. Obtain an equation giving the x-coordinates of A and B, and show that the power of O with respect to the circle is equal to $OA.OB$.

20. A circle with centre (h, k) touches the y-axis at T; find its equation. Write down the power of O with respect to the circle and show that it is equal to OT^2.

4.3. Conditions needed to fix a Circle

The general equation of a circle contains three constants, g, f, c, and the circle is completely fixed if we are given enough information to determine their values. For example, if a circle has a radius of 4 units and its centre is at $(1, -2)$, then we know that

$$\begin{aligned} g^2+f^2-c &= 4^2, \\ -g &= 1, \\ -f &= -2, \end{aligned}$$

and these three equations enable us to find the values of g, f, c, and so to find the equation of the circle. To fix a circle, then, we must

be given sufficient facts to provide three simultaneous equations connecting g, f, c; and some simple examples will now be given.

Example 1. Find the equation of the circle passing through the points (1, 2), (3, −4), (5, −6).

Let the equation be

$$x^2+y^2+2gx+2fy+c = 0;$$

then as the coordinates of the three points satisfy it, we have

$$1^2+2^2+2g+4f+c = 0,$$
$$3^2+4^2+6g-8f+c = 0,$$

and $$5^2+6^2+10g-12f+c = 0;$$

and these when simplified reduce to

$$2g+4f+c+5 = 0 \quad . \quad . \quad . \quad \text{(i)}$$
$$6g-8f+c+25 = 0 \quad . \quad . \quad . \quad \text{(ii)}$$

and $$10g-12f+c+61 = 0. \quad . \quad . \quad . \quad \text{(iii)}$$

Subtracting (i) from (ii), and (ii) from (iii), we have

$$4g-12f+20 = 0 \quad . \quad . \quad . \quad \text{(iv)}$$

and $$4g-4f+36 = 0. \quad . \quad . \quad . \quad \text{(v)}$$

Subtracting (iv) from (v), we find that

$$8f+16 = 0,$$

so that $$f = -2;$$

substituting this in (iv), $4g+44 = 0$,

so that $$g = -11;$$

and substituting these results in (i),

$$-22-8+c+5 = 0,$$

so that $$c = 25.$$

The equation of the required circle is therefore

$$x^2+y^2-22x-4y+25 = 0.$$

This result is easily checked by drawing. The centre of the circle is (11, 2) and its radius is $\sqrt{(11^2+2^2-25)} = 10$ units; we could plot the three points on graph-paper, draw the circle with this centre and radius, and verify that it passes through them. The circle is known as the **circumcircle** of the triangle formed by the three points, and its centre is the **circumcentre.**

Example 2. Find the equation of the circle whose centre lies on the line $x+3y = 5$, and which passes through the points (3, 1) and (6, 4).

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c = 0;$$

then its centre is $(-g, -f)$, and so, as this lies on the line,

$$g+3f+5=0. \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Also, as the two points lie on the circle,

$$6g+2f+c+10=0 \quad . \quad . \quad . \quad \text{(ii)}$$

and

$$12g+8f+c+52=0. \quad . \quad . \quad . \quad \text{(iii)}$$

Subtracting (ii) from (iii), and dividing the result by 6,

$$g+f+7=0. \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

Subtracting (iv) from (i), $2f-2=0$,

$$\therefore f=1;$$

substituting this in (iv), $g=-8$;

substituting these in (ii), $c=36$.

Hence the equation of the circle is

$$x^2+y^2-16x+2y+36=0.$$

The centre is $(8, -1)$, and this certainly lies on the required line; the radius is $\sqrt{29}$, or approximately 5·4, and the correctness of this may be checked either by drawing or by calculating the distances of the given points from the centre.

Exercise 4B

1. Find the equation of the circle which passes through the origin and the points (3, 6), (4, 4). Check your result by drawing.

2. Find the equation of the circle which passes through the points (0, 9), (3, 0), and (5, 4). Check your answer by drawing.

In Nos. 3 and 4, find the circumcentre of the triangle with the given points as vertices. Then draw the triangle on graph-paper, construct the perpendicular bisectors of the sides, and so verify your answer.

3. (0, 8), (1, 0), (9, 14). 4. (3, 0), (0, 9), (16, 13).

5. If two circles are concentric, in what way do their equations differ?

Find the equation of the circle which passes through (7, 6) and is concentric with $x^2+y^2-2x+3y-4=0$.

6. Two circles cut the x-axis in the same two points; in what way do their equations differ? (Consider the equation which gives these two points.)

Find the equation of the circle which passes through (9, 1) and cuts the x-axis in the same points as $x^2+y^2-10x+4y+17=0$.

7. Find the equation of the circle with respect to which the powers of $(-2, 0)$ $(0, 0)$ and $(0, 2)$ are respectively 16, 0 and -12.

8. Find the equation of the circle such that the tangents to it from (0, 0), (9, 3) and (4, 8) are all of length 3 units.

9. Show that a circle can be drawn through the four points A (2, 0), B (7, 0), C (6, 8), D (0, 4). Hence verify in this case the truth of Ptolemy's Theorem, which says that if $ABCD$ is a cyclic quadrilateral then $AB.CD+AD.BC=AC.BD$.

10. Show that a circle can be drawn through the four points A (0, 6), B (0, 1), C (3, 0), D (6, 4), and verify Ptolemy's Theorem (see No. 9) in this case.

11. Find the equation of the circle whose centre is on the x-axis, and which passes through the points A $(-3, -4)$, B (12, 16). Find where this circle cuts the y-axis, and so verify the theorem which says that, if two chords AB, CD of a circle cut at a point O, then $AO.OB=CO.OD$.

12. Find the equation of the circle whose centre is on the line $2x = 25$, and which passes through the points A (7, 1) and B (14, 2). Find where this circle cuts the x-axis, and so verify the theorem which says that, if two chords AB, CD of a circle are produced to meet at O, then $OA.OB = OC.OD$.

13. The x-axis is a tangent to a circle whose centre is (x_1, y_1), where y_1 is positive: what is the radius of the circle? What is the equation of the circle?

A circle is drawn to touch the x-axis and to pass through the points (2, 2), (4, 4). Find its equation; there are two possible answers.

Verify in each of these cases the theorem that, if OT is a tangent from a point O to a circle, and another line through O cuts the circle in P and Q, then $OP.OQ = OT^2$.

14. The y-axis is a tangent to a circle whose centre is (x_1, y_1), x_1 being positive: what is the radius of the circle? What is the equation of the circle?

A circle is drawn to touch the y-axis and to pass through the points (4, 2), (20, 10). Find its equation; there are two possible answers.

Verify in each of these cases the same theorem as in No. 13.

4.4. The Ellipse

Let us take a circle of radius a, with its centre at the origin, and from any point P' on it drop a perpendicular $P'M$ to the x-axis. Now let us take a point P on $P'M$ so that

$$\frac{MP}{MP'} = \frac{b}{a},$$

where b is a fixed number less than a, and P and P' are both on the same side of the x-axis. Then as P' moves round the circle, P will move round a closed curve called an ellipse, which touches the circle at the points A, A' in which it is cut by the x-axis. The ellipse cuts the y-axis in two points B, B' whose distance from the origin is b. $A'A$ is called the **major axis**, $B'B$ the **minor axis**, and the two axes together are called the **principal axes** of the ellipse. The lengths OA, OB are called the **semi-axes**; A and A' are the **vertices** of the ellipse.

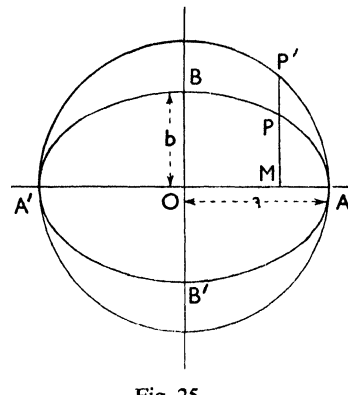

Fig. 25

To obtain the equation of the curve, we let P be (x, y), so that $OM = x$, $MP = y$. Now, by Pythagoras, $OM^2 + MP'^2 = a^2$, and we can write this as

$$\frac{OM^2}{a^2} + \frac{MP'^2}{a^2} = 1.$$

But $\frac{MP'}{a} = \frac{MP}{b}$, and so $\frac{OM^2}{a^2} + \frac{MP^2}{b^2} = 1$. Hence the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad (8)$$

Notice that this equation does fit the facts. As each of the terms of the L.H.S. is a perfect square, each must be positive or zero, and so, since their sum is 1, neither x^2/a^2 nor y^2/b^2 can be greater than 1. So x must lie between $-a$ and a, while y must lie between $-b$ and b. Furthermore, the equation is unaltered if $-x$ is substituted for x, or $-y$ for y, or both, and so the curve is symmetrical about the axes and about the origin. The principal axes are thus the axes of symmetry of an ellipse, and meet at its **centre.**

Notice that ellipses are *not* all the same shape. The major and minor axes may have any lengths we please: if the major axis is much longer than the minor, we get a long thin ellipse, while if the axes are almost equal the curve is nearly circular. The case in which $b = a$, when the equation becomes $x^2+y^2 = a^2$, shows that a circle itself may be regarded as a special kind of ellipse.

Ellipses are frequently met with in practice: if a tumbler of water is tilted, the surface of the water will meet the glass in an ellipse; the curve is seen in any perspective drawing or photograph of a circle, and it is also the path of the planets as they move round the sun.

Example 1. Trace the curve $4x^2+9y^2 = 36$.

As equation (8) has 1 for its constant term, we divide both sides of the given equation by 36 and get $x^2/9+y^2/4 = 1$. Comparing this with equation (8) we see that it represents an ellipse whose centre is the origin, whose major axis goes from $(-3, 0)$ to $(3, 0)$ and is therefore 6 units in length, and whose minor axis goes from $(0, -2)$ to $(0, 2)$ and is 4 units in length.

Example 2. Trace the curve $4x^2+y^2 = 1$.

Although the constant term is 1, this equation does not look like equation (8), but the resemblance is greater if we write it as

$$\frac{x^2}{(\frac{1}{2})^2} + \frac{y^2}{1^2} = 1,$$

and so it represents an ellipse whose centre is the origin and whose axes go from $(-\frac{1}{2}, 0)$ to $(\frac{1}{2}, 0)$ and from $(0, -1)$ to $(0, 1)$. As the second of these is the longer it is called the major axis, while the first

is the minor axis. The curve is shown in Fig. 26: it could be obtained either as above from the circle $x^2+y^2 = \frac{1}{4}$ by making $MP/MP' = 2$, or else by dropping a perpendicular $P'N$ on to the y-axis from a variable point P' on the circle $x^2+y^2 = 1$ and letting P be the mid-point of $P'N$.

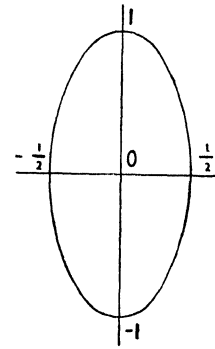

Fig. 26

Example 3. Trace the curve

$$9x^2+16y^2-36x+32y-92 = 0.$$

The only difference between this and the equation of a circle is that the coefficients of x^2 and y^2 are unequal. We therefore modify slightly the procedure of 4.2 and begin by writing the equation in the form

$$9(x^2-4x\quad)+16(y^2+2y\quad) = 92.$$

Completing the square in each bracket, we obtain

$$9(x-2)^2+16(y+1)^2 = 144.$$

Writing this in the form

$$\frac{(x-2)^2}{4^2}+\frac{(y+1)^2}{3^2} = 1,$$

we see that the curve is an ellipse with centre $(2, -1)$ and semi-axes of length 4 and 3, the major axis being parallel to Ox.

Exercise 4C

In Nos. 1–6 sketch the ellipses, marking the coordinates of the ends of their principal axes. Then plot the curves on graph-paper, to make yourself familiar with their actual shapes.

1. $x^2/25+y^2/9 = 1$.
2. $x^2/16+y^2/36 = 1$.
3. $x^2+4y^2 = 16$.
4. $25x^2+4y^2 = 400$.
5. $9x^2+4y^2 = 1$.
6. $x^2+9y^2 = 1$.

In Nos. 7–10, find the coordinates of the centre and the lengths of the semi-axes, stating the direction of the major axis.

7. $x^2+4y^2-4x+8y+4 = 0$.
8. $4x^2+9y^2-16x-18y = 0$.
9. $8x^2+y^2+8x-4y+2 = 0$.
10. $18x^2+4y^2-12x+12y+3 = 0$.

In Nos. 11–18, find the equation of the ellipse with the given centre and semi-axes, the first-mentioned axis being parallel to Ox. Clear the equation of fractions.

11. $(0, 0)$, $3\sqrt{2}$, 3.
12. $(0, 0)$, $2\sqrt{3}$, $3\sqrt{2}$.
13. $(0, 0)$, $\frac{1}{2}$, $\frac{2}{3}$.
14. $(0, 0)$, $\frac{2}{3}$, $\frac{1}{2}$.
15. $(1, -2)$, 4, 3.
16. $(-3, 1)$, 2, 3.
17. $(\frac{1}{2}, \frac{1}{2})$, 2, $\frac{3}{2}$.
18. $(-\frac{1}{3}, \frac{1}{2})$, $\frac{4}{3}$, $\frac{3}{2}$.

4.5. Locus Problems

There are no new principles in the following problems, and the remarks and examples of 3.3 apply equally here: the only new feature is that most of the loci are circles or ellipses.

Example 1. Find the locus of the mid-points of perpendiculars from points on the circle $x^2+y^2-4y = 0$ to the line $y+2 = 0$.

The given circle has its centre at (0, 2) and a radius of 2 units. It is therefore as shown in Fig. 27, where the line $y+2 = 0$ is also marked.

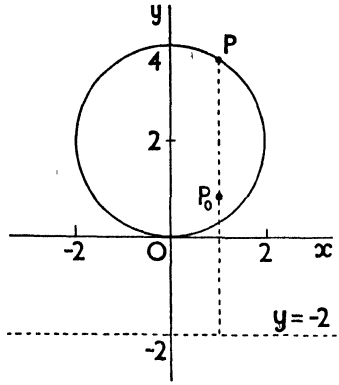

Fig. 27

Now let P_0 (x_0, y_0) be a point on the locus (remember the remarks in 3.3, Example 3!); it is then the mid-point of the perpendicular from a point P on the circle to the line $y = -2$, and to solve the problem we must find the coordinates of P in terms of those of P_0. As $y = -2$ is parallel to the x-axis, any perpendicular to it must be parallel to the y-axis, so that P and P_0 have the same x-coordinate. P_0 is at a height y_0 above the x-axis, and therefore at a height of y_0+2 above the line $y = -2$. So P must be at a height of y_0+2 above P_0, and its y-coordinate is therefore $2y_0+2$. Hence, if P is (x, y), then $x = x_0$ and $y = 2y_0+2$, so that $y-2 = 2y_0$. But P lies on the circle $x^2+(y-2)^2 = 4$, and so $x_0^2+(2y_0)^2 = 4$, which simplifies to $x_0^2/4+y_0^2 = 1$.

This is the equation connecting the coordinates (x_0, y_0) of any point on the locus. Hence, if we had called these coordinates (x, y) instead of (x_0, y_0), they would have been connected by the equation

$$x^2/4+y^2 = 1,$$

which is therefore the equation of the locus. It represents an ellipse like the one in Fig. 25, the major axis going from $(-2, 0)$ to $(2, 0)$ and the minor axis from $(0, -1)$ to $(0, 1)$.

Example 2. A point moves so that its power with respect to a fixed circle is proportional to its distance from a fixed line; show that its locus is a circle.

The remarks of 3.3, Example 5, apply here with new force, since the choice of axes determines not the coordinates but the equations of the data. What we are given is a circle and a line: the geometrical figures, not the equations. The axes which we choose, and the equations to which they give rise, are in the nature of scaffolding which we erect in order to carry out the working and which must be removed again to complete the job.

The locus will be symmetrical about the line which is perpendicular to the given line and passes through the centre of the given circle; we therefore take this axis of symmetry as one of the coordinate axes. The choice of the other axis is less important. Two obvious choices are to let it either coincide with the given line or pass through the centre of the given circle, and if you were making the choice yourself you would have to decide whether you preferred a 'nice' line and a 'nasty' circle or whether you would rather have the adjectives the other way round.

Taking the given line as the y-axis, we can let the circle be $x^2+y^2+2gx+c = 0$, so that its centre is on the x-axis. If $P\ (x, y)$ is on the locus, then

$$x^2+y^2+2gx+c = kx,$$

where k is a constant. From the form of this equation it is clear that the locus is a circle.

EXERCISE 4D

1. A and B are the points $(1, -3)$, $(3, 1)$. Find the locus of P if $PA^2+PB^2 = 18$.
2. A and B are the points $(-1, 1)$, $(5, -1)$. Find the locus of P if $PA^2+PB^2 = 52$.
3. From a variable point P on the ellipse $x^2/100+y^2/25 = 1$, a perpendicular PN is drawn to the y-axis. Find the locus of the mid-point of PN.
4. From a variable point P on the ellipse $4x^2+y^2 = 36$, a perpendicular PM is drawn to the x-axis. Find the locus of the mid-point of PM.
5. F is the point $(16, 0)$; a point P moves so that FP is four-fifths of the distance of P from the line $x = 25$; find the locus of P. You may assume that the x-coordinate of P is less than 25.
6. F is the point $(0, 9)$; a point P moves so that FP is three-fifths of the distance of P from the line $y = 25$. Find the locus of P. You may assume that its y-coordinate is less than 25.
7. A is the point $(12, 0)$; a point P moves so that AP^2 is equal to twice the distance of P from the y-axis; find the locus of P. You may assume that the x-coordinate of P is positive.
8. B is the point $(0, 3)$; a point P moves so that BP^2 is equal to four times the distance of P from the x-axis; find the locus of P which may be assumed to be in the first or second quadrant.
9. PT is a tangent from the point $P\ (x, y)$ to the circle $x^2+y^2+2gx+2fy+c = 0$ whose centre is C. Show that

$$PT^2 = x^2+y^2+2gx+2fy+c.$$

Find the locus of a point which moves so that the length of a tangent from it to $x^2+y^2-26x+14y+178 = 0$ is three times the length of a tangent from it to $x^2+y^2+6x-2y-62 = 0$.

10. A circle has radius r, and its centre is at $C(h, k)$; PT is a tangent to it from the point $P(x, y)$. Show that

$$PT^2 = PC^2-r^2 = (x-h)^2+(y-k)^2-r^2.$$

Find the locus of a point which moves so that the square of the length of a tangent from it to $x^2+y^2+4x+2y = 0$ is twice the square of the length of a tangent from it to $x^2+y^2-14x-4y+28 = 0$.

11. Find the locus of a point which moves so that the length of the tangent from it to the circle $x^2+y^2 = 4$ is half its distance from the y-axis.

12. Find the locus of a point which moves so that its power with respect to the circle $x^2+y^2 = 4$ is minus three times the square of its distance from the x-axis.

13. Find the locus of a point which moves so that the sum of its distances from (4, 0) and (−4, 0) is 10 units. (For the method, see 3.3, Example 2.)

To verify your result, stick two stout pins into a drawing-board so that they are $4\frac{1}{2}$ in. apart, take a piece of linen thread or fine wire and tie it *tightly* round them, forming a loop 9 in. in circumference. Now take a piece of graph-paper, mark on it two axes crossing near the middle of the paper, and pin it to the drawing-board; with $\frac{1}{2}$ in. as the unit, stick a pin in each of the points (4, 0) and (−4, 0), place the loop of thread over the pins, place the point of a sharp pencil in the loop and run it round the loop keeping the thread taut. The pencil will trace out the required locus, and the coordinates of points on it can be compared with those shown in a table of values made from the equation.

14. Find the locus of a point which moves so that the sum of its distances from (15, 0) and (−15, 0) is 34 units.

Verify your result as in No. 13, the two pins being 6·4 in. apart, the x-axis being parallel to the long edge of the paper, and the unit being 0·2 in. [The error in drawing the locus is likely to be greater in this case than in No. 13.]

15. Find the locus of points which are twice as far from (−12, −9) as from the origin.

16. Find the locus of points which are one and a half times as far from $(13\frac{1}{2}, 18)$ as from (6, 8).

17. A, B, C are the points (−4, 1), (1, −4), (3, 3). Find the locus of points P for which $PA^2+PB^2 = PC^2$.

18. With the data of No. 17, find the locus of points P for which $PA^2+PB^2 = 3PC^2$.

19. With the data of No. 17, find the locus of points P for which $PA^2+PB^2 = 2PC^2$.

20. With the data of No. 17, what can you say about the numbers α, β, γ if the locus of points P, such that $\alpha PA^2+\beta PB^2 = \gamma PC^2$, is a straight line?

21. A, B, C are the points (2, 4), (2, −4), (10, 0). Find the locus of a point P for which $5PA^2+5PB^2 = 2PC^2$.

22. With the data of No. 21, show that there are no positions of P for which $3PA^2+3PB^2 = PC^2$.

23. A variable point $P(x, y)$ on the circle $x^2+y^2 = 4$ is joined to a variable point X on the x-axis so that OP is always equal to PX. Find the locus of Q, the mid-point of PX. [Find the coordinates of X in terms of those of P, let Q be (x_0, y_0), then find x and y in terms of x_0 and y_0.]

24. With the data of No. 23, if PX is produced to R so that $PX = XR$, find the locus of R.

25. Find the locus of the mid-point of the join of (−3, 0) to a variable point on the curve $4x^2+9y^2-24x = 108$.

26. Find the locus of the mid-point of the join of (10, 4) to a variable point on the circle $x^2+y^2 = 36$.

27. Find the locus of a point which moves so that the sum of the squares of its distances from two fixed points is constant.

28. Find the locus of a point which moves so that the ratio of its distances from two fixed points is constant.

29. A point moves so that its powers with respect to two given circles are unequal but in a fixed ratio. Show that it traces out a circle.

30. Find the locus of points at which two given circles subtend equal angles. [Use similar triangles.]

31. A point moves so that its power with respect to a given circle is equal to the square of its distance from a given line. Show that it traces out a parabola.

32. A point moves so that its power with respect to a given circle is equal to half the square of its distance from a given line. What curve does it trace out?

EXERCISE 4M (Miscellaneous examples arranged in sets of 6)

1. Prove that the lines which join the mid-points of opposite sides of the quadrilateral $P_1P_2P_3P_4$ bisect each other.

2. Show that the circle whose centre is $\left(-\frac{a}{2}, \frac{b+1}{2}\right)$ and which passes through (0, 1) cuts the x-axis in points whose coordinates satisfy the equation $x^2+ax+b = 0$.

By this means, using compasses and graph-paper, solve the equation $x^2-6x+3 = 0$.

3. Find the equations of two circles which touch both axes and pass through (2, 9). What is their other point of intersection?

4. Find the locus of a point P which moves so that its distance from the point (3, 0) is equal to the length of the tangent from it to the circle $x^2+y^2 = 4$.

5. Find the locus of the centre of a variable circle which touches the y-axis and the circle $x^2+y^2 = 4$.

6. It is required to draw the ellipse $x^2/a^2+y^2/b^2 = 1$, where a is greater than b, by means of pins and thread as in Exercise 4d, No. 13. By considering special positions of the thread, show that the pins must be placed at $(\pm\sqrt{(a^2-b^2)}, 0)$ and that the loop of thread must have a perimeter $2\{a+\sqrt{(a^2-b^2)}\}$.

What are the corresponding results if b is greater than a?

---

7. Show that the triangle whose angular points are $(1, -1)$, $(3, 1)$, $(2-\sqrt{3}, \sqrt{3})$ is equilateral.

8. What geometrical fact does the equation $(x+1)^2+(y-3)^2 = x^2+y^2$ tell you about the points which satisfy it?

9. One end of a diameter of the circle $x^2+y^2-10x-12y+43 = 0$ is (2, 3). Find the other end.

10. A rod of constant length c moves so that it always has one end on each of the coordinate axes; find the locus of its mid-point.

11. Find the locus of a point whose distance from the line $x = 6$ is equal to the length of the tangent from it to the circle $x^2+y^2 = 4$.

12. Find the equation of the ellipse the ends of whose axes are $(-1, 3)$, $(3, -2)$, $(7, 3)$, $(3, 8)$.

---

13. The equation $y^2 = x^2(4-x^2)$ represents a curve like a figure 8.
(i) Find what symmetry the curve possesses about the axes.
(ii) By finding where it cuts the axes, sketch the curve.

14. Show that (4, 1) lies on the circle $x^2+y^2-2x+6y-15 = 0$, and that (5, -7) lies outside it.

15. A variable circle touches the y-axis and passes through the point (4, 0). Find the locus of its centre.

16. Points A and B move on Ox and Oy so that $OA+OB = 2c$; find the locus of the mid-point of AB.

17. A point moves so that the sum of the squares of its distances from the sides of a rectangle is constant. Find its locus.

18. P is a variable point on the ellipse $x^2/a^2+y^2/b^2 = 1$; PN is the perpendicular to the major axis AA'.
Prove that $\dfrac{PN^2}{AN.NA'}$ is constant for all positions of P.

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19. $ABCD$ is a rectangle; P is any point in its plane. Prove that $PA^2+PC^2 = PB^2+PD^2$.

20. The tangents from the points (0, 1), (4, 0), (0, 2) to a circle are each of length 2. Find the equation of the circle.

21. Points A and B move on Ox and Oy so that $OA.OB = c^2$; find the locus of the mid-point of AB.

22. Find the equations of the circles which touch the x-axis and the lines $x = a$, $x = b$.

23. Show that the locus of points whose powers with respect to two given circles differ by a constant is a straight line perpendicular to that joining the two centres.

24. Find the equation of the locus of a point whose distance from the point (6, 0) is always half its distance from the y-axis.
Find the points on the locus at which $y = 0$, and those at which $y = \pm 3$. Make a rough sketch of the locus. [O.L.]

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25. A certain curve is symmetrical about both axes. If (2, 3), (-1, 4), (5, -2) lie on it, what other points must lie on it?

26. Prove that the circles $x^2+y^2+3x-y = 0$, $x^2+y^2+x-7y+10 = 0$ are equal and that they touch each other. [O.L.]

27. A point moves so that the sum of the squares of its distances from the corners of a square is constant. Find its locus.

28. A line through the origin cuts the circles $x^2+y^2 = 1$, $x^2+y^2 = 4$ at A and B, so that A is the mid-point of OB. Lines through A and B, parallel to Ox and Oy respectively, meet at C. Prove that C lies on the ellipse $x^2+4y^2 = 4$.
[Let B be (x_0, y_0); find the coordinates of C in terms of those of B, and use the fact that B lies on the larger circle.]

29. Find the locus of a point which moves so that its distance from a fixed point is equal to its distance from a fixed line.

30. Variable points Q, R move, one one each of two fixed lines AB, AC, so that AQ remains equal to AR. Find the locus of a point P on QR such that PQ^2+PR^2 is constant.

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31. If a curve is symmetrical about two given perpendicular lines, prove that it is symmetrical about the point in which they meet.

32. Find the equations of the circles which touch $x^2+y^2 = 16$, $x^2+y^2 = 36$, and $y = 5$.

33. Use the fact that $3^2+4^2 = 5^2$ to write down the coordinates of eight points on the circle $x^2+y^2 = 25$. Hence find eight points on the ellipse $9x^2+25y^2 = 225$ whose coordinates are whole numbers or simple fractions.

34. Find the locus of points from which the tangents to the two circles $x^2+y^2 = 36$ and $x^2+y^2-6x+4y = 0$ are equal.

Sketch the two circles and the locus, showing any interesting features.

35. Find the locus of a point which moves so that its distance from a fixed point is proportional to the length of the tangent from it to a fixed circle.

36. Variable points Q, R, one on each of two fixed lines AB, AC, move so that AQ remains equal to AR. Find the locus of a point P on QR such that PQ^2-PR^2 is constant.

CHAPTER 5

LINES

5.1. Equation of the Line through Two Given Points

So far we have assumed that any equation of the first degree represents a line and that any line can be represented by such an equation. The second of these statements will be proved in this section and the first in 5.3.

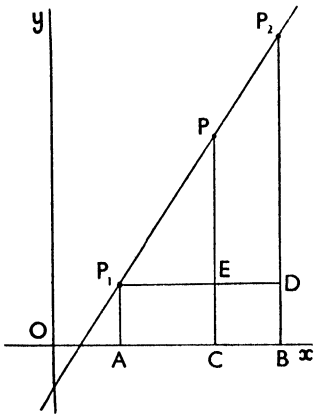

Fig. 28

Suppose we are given two points P_1, P_2 and we want to find the equation of their **join**—that is to say, of the line through them. Let P be any point on this line, which we shall first suppose is not parallel to either of the axes.

Let lines through P_1, P_2, and P parallel to the y-axis meet the x-axis at A, B, C and meet the line through P_1 parallel to the x-axis at P_1, D, E. One possible set of positions for these points is shown in Fig. 28, and there $OA = x_1$, $OB = x_2$, $OC = x$, $AP_1 = y_1$, $BP_2 = y_2$, $CP = y$; so that

$$P_1D = OB - OA = x_2 - x_1,$$
$$P_1E = OC - OA = x - x_1,$$
$$DP_2 = BP_2 - AP_1 = y_2 - y_1,$$

and

$$EP = CP - AP_1 = y - y_1.$$

Now the triangles P_1EP and P_1DP_2 are equiangular, and so their corresponding sides are in the same ratio. Hence

$$\frac{EP}{P_1E} = \frac{DP_2}{P_1D},$$

so that

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \quad . \quad . \quad . \quad . \quad (1)$$

and this when simplified is the required equation. The simplified version is $(y_2-y_1)x-(x_2-x_1)y+(x_2y_1-x_1y_2) = 0$, but no attempt

need be made to remember it. Bearing in mind that the contents of the brackets are fixed numbers and not variables, you will see that it is of the first degree. Equation (1) must be learnt.

Notice that in obtaining equation (1) we have followed the course mentioned in 3.2 and drawn the simplest case, in which all coordinates were positive as also were the four quantities $y-y_1$, $x-x_1$, y_2-y_1 and x_2-x_1. A little thought will show that, if P had been taken on the other side of P_1, both $y-y_1$ and $x-x_1$ would have been negative and so their ratio would still have been positive. As the triangles would still have been equiangular, the equation would still have been satisfied.

Example. Find the equation of the line joining (10, −2) and (−2, 7).

Here $x_1 = 10$, $y_1 = -2$, $x_2 = -2$, $y_2 = 7$, and so the equation of the line is $\frac{y+2}{x-10} = \frac{7+2}{-2-10}$, which simplifies to $\frac{y+2}{x-10} = -\frac{3}{4}$, and finally to $3x+4y-22 = 0$. This is the required equation.

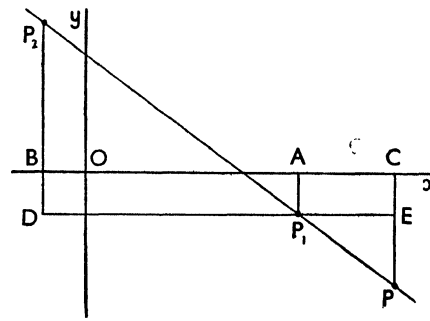

Fig. 29

Fig. 29 illustrates this case. In it $OA = 10$, $OB = -2$, $OC = x$, $AP_1 = -2$, $BP_2 = 7$, $CP = y$ (negative), $P_1D = OB-OA = -2-10$ (bear in mind that P_1D means the distance of D to the *right* of P_1), $P_1E = OC-OA = x-10$, $DP_2 = BP_2-AP_1 = 7+2$, $EP = CP-AP_1 = y+2$ (EP means the height of P *above* E, and is negative, but y is more negative than −2) and

$$\frac{EP \text{ (negative)}}{P_1E \text{ (positive)}} = \frac{DP_2 \text{ (positive)}}{P_1D \text{ (negative)}},$$

both fractions being negative. So we get $\frac{y+2}{x-10} = \frac{7+2}{-2-10}$, just as by substitution in the formula. Substitution in the formula is easier and is to be preferred because it is less likely to lead to mistakes, but this analysis of Fig. 29 has been included to show that the argument leading to equation (1) is still valid in a less obvious case.

The case of a line parallel to one of the axes has already been

considered in 2.2: it occurs when P_1 and P_2 have equal x-coordinates or equal y-coordinates and so is readily recognized. The equation is then best written down by inspection: thus the line joining $(-3, 2)$ and $(-3, -6)$ is $x+3 = 0$.

5.2. Gradients, Parallels, Perpendiculars

The fraction on the R.H.S. of equation (1) of 5.1 does not contain the variables x and y but only the coordinates of the fixed points P_1 and P_2. Moreover, the value of this fraction does not depend upon the particular points P_1 and P_2 chosen: if two other points P_3 and P_4 were chosen on the same line, $\frac{y_4-y_3}{x_4-x_3}$ would be equal to $\frac{y_2-y_1}{x_2-x_1}$. This can be seen by drawing a line through P_3 parallel to the x-axis and another through P_4 parallel to the y-axis: if these meet in F, the triangles P_1DP_2 and P_3FP_4 are equiangular so that FP_4/P_3F is equal to DP_2/P_1D, and these are the two fractions referred to.

The value of any of these equal fractions is called the **gradient** of the given line and we shall often denote it by the Greek letter γ (gamma).* It represents the increase in y as we go along the line, divided by the corresponding increase in x. The line in Fig. 28 has a positive gradient, for x and y increase together; the line in Fig. 29 has a negative gradient because, as x increases, the increase in y is negative—that is to say, it decreases. A line whose gradient is zero is parallel to the x-axis; as the gradient increases the line gets steeper, and a line with a very large gradient is nearly parallel to the y-axis. If a line is actually parallel to the y-axis, its gradient is said to be infinite.

It follows from what has been said that the equation of the line with gradient γ which passes through the point (x_1, y_1) can be written

$$\frac{y-y_1}{x-x_1} = \gamma \quad . \quad . \quad . \quad . \quad (2)$$

though this equation will as a rule require simplification. For example the line with gradient $\frac{2}{3}$, passing through $(4, -1)$, is $\frac{y+1}{x-4} = \frac{2}{3}$, which reduces to $2x-3y-11 = 0$. In the same way, the line with gradient

* Many textbooks use the letter m for the gradient, but this is confusing because the same letter is also used with a different meaning in the equation $lx+my+n = 0$ (see 5.3). The letter γ is already firmly established in technical use, e.g. in photography, where it represents the gradient of a graph relating the logarithm of the density of a negative to the logarithm of the exposure given.

a, passing through $(0, b)$, is $(y-b)/(x-0) = a$, which can be written $y = ax+b$. This form of equation is useful in applied mathematics since it can be used to express many physical laws. The experimental data when plotted will then very nearly lie on a straight line, and the values of a and b may be found by drawing the line which seems best to fit the plotted points. In geometry, however, the equation in this form is of little importance because we are rarely concerned with the point in which a line meets the y-axis; also since many of the equations with which we have to deal do not express y directly in terms of x, it is best to get used to the equation of a line set out in an orderly fashion—the x-term first, then the y-term, and finally the constant term, the whole being equated to zero.

Another definition of the gradient, which is equivalent to the one already given, is that it is the tangent of one of the angles between the line and the x-axis, the angle being that through which a line, originally lying along or parallel to the x-axis, would have to be turned counter-clockwise to make it parallel to the given line. When **the angle which a line makes with the x-axis** is mentioned, it is normally to be interpreted as having the meaning just given. If this angle is acute the gradient is positive, if it is obtuse the gradient is negative. If two lines are parallel, they make equal angles with the x-axis and their gradients are therefore equal. Conversely, if the gradients are equal, the lines make equal angles with Ox and are therefore parallel.

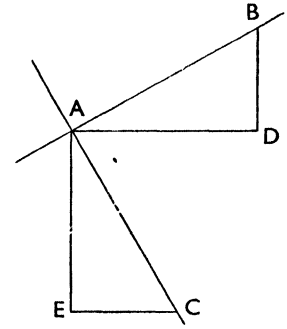

Fig. 30

If two lines are at right angles but are not parallel to the axes, one of them must have a positive gradient and the other a negative, since the difference between two acute angles or two obtuse angles cannot be a right angle. Suppose that two perpendicular lines meet at A, that B is a point on the line with positive gradient, C a point on the other line, that B and C are both the same distance from A, and that the x-coordinates of B and C are each greater than that of A, as in Fig. 30. Lines are drawn through A and C parallel to the x-axis, through A and B parallel to the y-axis, forming triangles ADB, AEC. As $\angle BAC = \angle DAE = 90°$, it follows that $\angle BAD = \angle CAE$, and so the triangles are congruent, since their hypotenuses are equal. Hence $DB = EC$, $AD = EA$. If the gradients of the two lines are

γ, γ', then $\gamma = DB/AD$, $\gamma' = -EA/EC$, and so $\gamma\gamma' = -1$: thus if the gradient of one line is $\frac{2}{3}$, that of the other will be $-\frac{3}{2}$. The argument is reversible, with slight modifications, and if $\gamma\gamma' = -1$ it follows that lines with gradients γ and γ' are perpendicular.

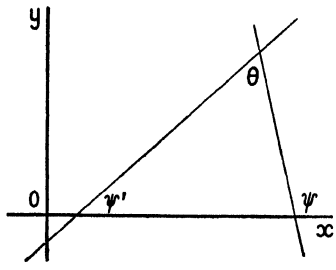

Fig. 31

Another way of obtaining this result is shown in Fig. 31. Suppose two lines have gradients γ, γ', and make angles ψ (psi), ψ' with the x-axis, so that $\gamma = \tan\psi$, $\gamma' = \tan\psi'$. One of the angles between the lines is then $\theta = \psi - \psi'$ in the case illustrated, because of the theorem about an exterior angle of a triangle. Since ψ and ψ' are measured by directed numbers, the same is true of θ, and it may sometimes be necessary to discard a negative sign. To find the value of θ, we could find the values of ψ and ψ' from tangent tables and then subtract them, but it is usually better to use the formula for the tangent of the difference of two angles. Thus $\tan\theta = \dfrac{\tan\psi - \tan\psi'}{1+\tan\psi\tan\psi'} = \dfrac{\gamma-\gamma'}{1+\gamma\gamma'}$, and so **the angle between the lines is** $\tan^{-1}\left(\dfrac{\gamma-\gamma'}{1+\gamma\gamma'}\right)$. From the awkwardness of this result you will rightly conclude that coordinate geometry is not as a rule well suited to problems about angles, but one case at least is simple: if $\theta = 90°$, $\tan\theta$ is infinite, hence the denominator of the fraction is zero and $\gamma\gamma' = -1$.

Two results of this section should be remembered, namely that **two lines are parallel if their gradients are equal, perpendicular if the product of their gradients is -1.** Two other methods of obtaining this last result will be found in Exercise 5a, Nos. 43, 44.

The case of a line parallel to the y-axis needs special consideration, since its infinite gradient cannot be substituted in any formula. It is easy to see from a figure that the angle between it and a line of gradient γ' is $\tan^{-1}(1/\gamma')$, and that the two are perpendicular if $\gamma' = 0$.

Example 1. Find the equations of the altitudes of the triangle whose vertices are A (2, 4), B (2, 0), C (5, 3), and show that they meet in a point.

A figure is not necessary for the calculation, but as you may find

that it makes the working easier to follow, the points are shown in Fig. 32.

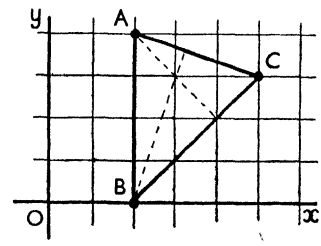

Fig. 32

The gradient of BC is $\frac{3-0}{5-2} = 1$, and so the gradient of the altitude through A, which is perpendicular to BC, is -1. Its equation is therefore $\frac{y-4}{x-2} = -1$, which simplifies to

$$x+y-6 = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Similarly, the gradient of CA is $\frac{4-3}{2-5} = -\frac{1}{3}$; the gradient of the altitude through B is therefore 3 and its equation is $\frac{y-0}{x-2} = 3$,

i.e. $$3x-y-6 = 0 \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

If we were to try writing down the gradient of AB without looking at the figure, we should get $\frac{0-4}{2-2} = -\frac{4}{0}$. This shows that the gradient is infinite and that AB is parallel to the y-axis; hence the altitude through C is parallel to the x-axis and all points on it have the same y-coordinate, namely that of C. Its equation is accordingly

$$y-3 = 0. \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

This last result is obvious from the figure.

At the point where the second and third altitudes meet, their equations are satisfied simultaneously, so that

$$3x-y = 6$$

and $$y = 3,$$

and from these we see that $x = 3$. Hence they meet at (3, 3). These coordinates satisfy equation (i), and so the point (3, 3) lies on that altitude too. Hence the three altitudes meet in a point. The point is known as the **orthocentre** of the triangle.

Example 2. Show that the points A (0, 2), B (3, -7), C (9, 0) are the corners of an isosceles triangle whose largest angle is $\tan^{-1}(\frac{15}{8})$.

The lengths of the sides are given by

$$BC = \sqrt{(6^2+7^2)} = \sqrt{85},$$

$$CA = \sqrt{(9^2+2^2)} = \sqrt{85},$$

and $$AB = \sqrt{(3^2+9^2)} = \sqrt{90}.$$

Hence the triangle is isosceles; AB is the greatest side and so $\angle BCA$ is the greatest angle.

The gradient of BC is 7/6, that of CA is $-2/9$, and so the angle between them is

$$\tan^{-1}\left(\frac{\frac{7}{6}+\frac{2}{9}}{1-\frac{7}{6}\cdot\frac{2}{9}}\right) = \tan^{-1}\left(\frac{63+12}{54-14}\right) = \tan^{-1}\frac{75}{40} = \tan^{-1}\frac{15}{8}.$$

Example 3. Show that the three points $(-7, 5)$, $(-3, 2)$, and $(9, -7)$ are **collinear,** i.e. that they lie in a straight line.

The gradient of the line joining the first two is $(2-5)/(-3+7)$ or $-\frac{3}{4}$; the gradient of the line joining the second and third is $(-7-2)/(9+3) = -\frac{3}{4}$ also. Hence the points lie in a straight line.

Example 4. Find the equation of the circle on P_1P_2 as diameter.

It is of course possible to find the centre and radius of the circle, and so to obtain its equation, but the working can be shortened, and the equation written more neatly, by using the fact that the angle in a semicircle is a right angle.

Let P be any point on the circle. The gradients of PP_1 and PP_2 are $(y-y_1)/(x-x_1)$ and $(y-y_2)/(x-x_2)$. As these lines are at right angles,

$$\frac{y-y_1}{x-x_1}\cdot\frac{y-y_2}{x-x_2} = -1,$$

and hence **the equation of the circle on P_1P_2 as diameter is**

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0.$$

EXERCISE 5A

(The equations of the lines should be written in the form "x-term$\pm y$-term$\pm$constant term $= 0$".)

In Nos. 1 and 2, find the gradients of the lines joining the given pairs of points.

1. (i) (7, 1), (4, 7); (ii) (−3, −2), (1, 1); (iii) (0, 0), (x_1, y_1).
2. (i) (−3, −5), (−7, 1); (ii) (−3, 3), (4, 3); (iii) $(a, 0)$, $(0, b)$.

In Nos. 3 and 4, find the equations of the joins of the given pairs of points.

3. (i) (1, 3), (5, 7); (ii) (−5, −2), (1, −1);
   (iii) (−3, −2), (−3, 4).
4. (i) (1, 6), (7, 4); (ii) (2, −4), (5, 4); (iii) (5, −1), (8, −1).

In Nos. 5, 6, find the equations of the sides, and the tangents of the angles, of the triangle with corners at the given points.

5. A (−3, −1), B (1, −3), C (4, 6).
6. D (2, −2), E (−3, 3), F (−1, −1).

7–12. Do Exercise 3a, Nos. 7, 8, 9, 10, 13, 14, by means of gradients.

In Nos. 13, 14, find the equations of the lines which pass through the given points and have the given gradients.

13. (i) $(4, -1), \frac{8}{2}$; (ii) $(3, 2), -\frac{4}{5}$; (iii) $(-2, 3), 0$.
14. (i) $(5, -4), 2$; (ii) $(-2, -3), -\frac{1}{3}$; (iii) $(3, 8), \infty$.

In Nos. 15, 16, write down and simplify the equations of the circles on the lines joining the given points as diameter. (Use Example 4.)

15. (i) $(0, 0), (5, -7)$; (ii) $(-3, 0), (11, 2)$; (iii) $(9, -4), (-1, 6)$.
16. (i) $(p, q), (-p, -q)$; (ii) $(p, q), (-q, p)$; (iii) $(1, -2), (8, 4)$.

In Nos. 17, 18, find the equations of the medians of the triangle with the given vertices and show that they meet in a point. (A **median** is a line joining a vertex of a triangle to the mid-point of the opposite side. The point in which they meet is called the **centroid.**)

17. A (2, 13), B (4, 3), C (12, 5).
18. D (−7, 1), E (−1, 3), F (5, −1).

In Nos. 19, 20, find the equations of the perpendicular bisectors of the given pairs of points. (Find first the gradient of the join and the coordinates of the mid-point.)

19. (i) (1, 7), (5, 1); (ii) (−8, 2), (4, 2).
20. (i) (−2, −3), (3, 4); (ii) (−1, −2), (−1, 8).

In Nos. 21, 22, find the equations of the perpendicular bisectors of the sides of the triangle with the given vertices, and show that they meet in a point. Compare your results with Exercise 3a, Nos. 19, 20.

21. A (−2, 2), B (2, −6), C (5, 3).
22. D (0, 8), E (17, 1), F (17, −9).
23. A, B, C are the points (3, 9), (1, 1), (7, 3). Find the equations of the lines which pass through the mid-point of BC and are parallel to AB and AC. Verify that these lines pass respectively through the mid-points of AC and AB.
24. D, E, F are the points (1, 6), (−3, 2), (9, −2). Find the equations of the lines through the mid-point of EF parallel to DE and DF, and verify that they pass respectively through the mid-points of DF and DE.
25, 26. With the data of Exercise 4b, Nos. 9, 10, verify that $\angle ADB = \angle ACB$.

In Nos. 27, 28, find the equations of the altitudes of the triangle with the given vertices and show that they meet in a point.

27. A (0, −6), B (8, 2), C (5, 4).
28. D (5, 12), E (−7, 0), F (−1, −3).
29. Choose two of the points (6, −1), (−2, 3), (2, −3), $(1\frac{1}{2}, -4)$ and show that their join is at right angles to the join of the other two. Show that this is true whichever two points are chosen. Verify that the four points lie on a rectangular hyperbola. [This illustrates a property of a rectangular hyperbola, namely that, if three points on it are the corners of a triangle, then the orthocentre of the triangle is also on the curve.]
30. Show that the six lines joining the points (−3, −3), (9, 1), (2, $4\frac{1}{2}$), $(1\frac{1}{2}, 6)$ form three pairs of perpendicular lines. Verify that the four points lie on a rectangular hyperbola. [See note on No. 29.]
31. Show that the three points A (−7, −5), B (−3, −2), C (9, 7) are collinear.
32. Show that the three points D (−4, 8), E (1, −2), F (4, −8) are collinear.
33. A, B are the points (−4, 0), (4, 0). A point P moves so that the product of the gradients of AP and PB is $-\frac{1}{4}$; find the locus of P.
34. C, D are the points (−2, 0), (2, 0). A point P moves so that the product of the gradients of CP and PD is $-\frac{9}{4}$; find the locus of P.
35. A, B are the points (3, 2), (−3, −2). A point P moves so that the sum of the gradients of AP and PB is zero; find the locus of P.
36. C, D are the points (0, 8), (10, 0). Find the locus of a point P which moves so that the sum of the gradients of CP and PD is zero.

37. Choose any two points A, B on the rectangular hyperbola $xy = 6$ [for example, $(-1, -6)$ and $(1\frac{1}{2}, 4)$, but choose some others of your own], find the equation of the line AB, find the points C, D in which this line cuts the asymptotes, and verify that the mid-point of AB is the same as the mid-point of CD. [This is a well-known property of a hyperbola.]

38. Do No. 37, but use the hyperbola $xy+10 = 0$.

39. If P_1 and P_2 were exchanged in Fig. 28, the line joining them would be unaltered, but equation (1) would become $(y-y_2)/(x-x_2) = (y_1-y_2)/(x_1-x_2)$. Show that this still simplifies to the same result as before.

40. If P_1 and P were exchanged in Fig. 28, the line joining them would be unaltered but its equation would be $(y_1-y)/(x_1-x) = (y_2-y)/(x_2-x)$. Show that this still simplifies to the same result as before.

41. Show that the equation of any line can be written $x\cos\alpha + y\sin\alpha = p$, where p is positive and measures the length of the perpendicular to the line from O.

[Let this perpendicular meet the line at A; let PB be the perpendicular from any point P on the line to the x-axis, BC that from B to OA, and PD that from P to BC. Let α be the angle which OA makes with Ox.

[Show that, regarding OA as being positive, the directed lengths OC, DP are equal to $x\cos\alpha$ and $y\sin\alpha$. Deduce the result.

[The case in which α is acute is easiest and may be used for the drawing, but make sure that the steps in your proof are still true for other values of α.]

42. Do No. 41 as follows: Let the perpendicular from O to the line make an angle α with Ox, and let the line meet Ox and Oy at X and Y. Find the coordinates of X and Y in terms of p and α; then find the equation of the line joining them.

In Nos. 43, 44, obtain the relation between the gradients of perpendicular lines by the method indicated.

43. Let two perpendicular lines meet at A, let AD be a line of length 1 unit, parallel to Ox, and let the line through D, perpendicular to AD, meet the two given lines at B and C. If the gradients of the two given lines are γ, γ', one of these must be negative, but in what follows, except the final conclusion, we ignore signs and regard γ, γ' and all lengths as signless.

(i) Show that triangles BDA, ADC are similar. Hence show that $BD.DC = 1$.

(ii) What are the lengths of BD, DC?

(iii) Bearing in mind the actual signs of γ, γ', what do you deduce from (i) and (ii)?

44. Let two perpendicular lines, with gradients γ, γ', meet at P_1. Let Q, R be two points, one on each line, both having x_1+1 as its x-coordinate.

Write down the values of P_1Q^2, P_1R^2, QR^2; then apply Pythagoras' Theorem to triangle P_1QR, and simplify the result.

5.3. The General Equation of the First Degree

An equation of the first degree can contain nothing but a term in x, a term in y, and a constant term; and they are best written in this order. The most general equation of this form can be written

$$lx+my+n = 0 \quad . \quad . \quad . \quad . \quad (3)$$

and any required line can be obtained by giving suitable values to the constants l, m, n. For instance, equation (1) of 5.1 can be written

$$(y_2-y_1)x-(x_2-x_1)y+(x_2y_1-x_1y_2) = 0,$$

in which $l = y_2 - y_1$, $m = -(x_2 - x_1)$, $n = x_2y_1 - x_1y_2$; $y = ax + b$ can be written $ax - y + b = 0$, in which $l = a$, $m = -1$, $n = b$; $y = 2\frac{1}{2}$ can be written $2y - 5 = 0$, in which $l = 0$, $m = 2$, $n = -5$; and so on. Of course, the values of l, m, n which give a particular line are not unique: they may be multiplied by any number we please, for example in order to avoid fractions. Thus $2x/3 - y/4 + 5 = 0$, in which $l = \frac{2}{3}$, $m = -\frac{1}{4}$, $n = 5$, would be more conveniently written $8x - 3y + 60 = 0$, in which $l = 8$, $m = -3$, $n = 60$.

We can now show that any equation of the first degree represents a straight line. Let the equation be $lx + my + n = 0$, and suppose that m is not zero. Choose a point P_1 (x_1, y_1) which satisfies the equation. There is no difficulty in doing this: we may choose any value for x_1, and then equation (3) will give the value of y_1. Suppose also that P (x, y) is *any* point which satisfies the equation. Substituting these coordinates, we have

$$lx + my + n = 0$$

and

$$lx_1 + my_1 + n = 0,$$

whence, on subtraction,

$$l(x - x_1) + m(y - y_1) = 0,$$

which is the same as

$$\frac{y - y_1}{x - x_1} = -\frac{l}{m},$$

and this, by comparison with equation (2), is the equation of a line through P_1 with gradient $-l/m$. Hence **$lx + my + n = 0$ represents a line with gradient $-l/m$.** Because it represents a line, an equation is said to be **linear** if it is of the first degree.

The case in which $m = 0$ must not be forgotten: the equation can then be written $x + n/l = 0$, so that all points satisfying it have the same x-coordinate and therefore lie on a line parallel to the y-axis. In the same way, if $l = 0$ the equation represents a line parallel to the x-axis.

Consider now the two lines $lx + my + n = 0$ . . . . (3)

and $l'x + m'y + n' = 0$. . . . (4)

whose gradients are $-l/m$ and $-l'/m'$. The gradients are equal if $\frac{l}{m} = \frac{l'}{m'}$, that is to say if $\frac{l}{l'} = \frac{m}{m'}$.

Hence **the lines $lx + my + n = 0$, $l'x + m'y + n' = 0$ are parallel if** $\frac{l}{l'} = \frac{m}{m'}$ . . . . . (5)

Now let (3) and (4) represent any two *parallel* lines, and let each of the fractions in (5) be denoted by k, so that $l'k = l$, $m'k = m$. Then, if (4) is multiplied throughout by k it becomes

$$lx+my+n'k = 0,$$

which is the same as equation (3) except for the constant term.

Thus $4x-10y+7 = 0$ and $6x-15y-2 = 0$ are parallel, by (5), and the second can be written $4x-10y-\frac{4}{3} = 0$. Hence **the equations of two parallel lines can be written so that their only difference is in their constant term.**

If $$\frac{l}{l'} = \frac{m}{m'} = \frac{n}{n'} \quad . \quad . \quad . \quad . \quad (6)$$

and each fraction is denoted by k as before, we have $l'k = l$, $m'k = m$, and $n'k = n$, so that if (4) is multiplied throughout by k it becomes identical with (3). Hence **(6) is the condition for (3) and (4) to represent the same line.**

Example 1. Find the equation of the line through $(3, -5)$ parallel to $2x-7y-10 = 0$.

First Method. The required equation can be written

$$2x-7y \quad = 0,$$

and all we have to do is to fill in the constant term. This can be done by finding the value of $2x-7y$ when $x = 3$ and $y = -5$, namely 41, and changing its sign, giving $2x-7y-41 = 0$ as the required line.

This line passes through $(3, -5)$ because substitution in the L.H.S. gives $41-41$.

Second Method. The line through the origin parallel to the given line is $2x-7y = 0$. Shifting this so that the point which was the origin moves to $(3, -5)$, we find that the required equation is

$$2(x-3)-7(y+5) = 0,$$

and this again simplifies to $2x-7y-41 = 0$.

Next consider the case in which (3) and (4) represent perpendicular lines. The product of the gradients is -1 if $\frac{ll'}{mm'} = -1$, which we may also write as $ll'+mm' = 0$. Hence the lines $\boldsymbol{lx+my+n = 0}$, $\boldsymbol{l'x+m'y+n' = 0}$ **are perpendicular if** $\boldsymbol{ll'+mm' = 0}$ . . (7)

If this condition is satisfied, we have $\frac{m}{l'} = \frac{-l}{m'}$; and if the value of each of these fractions is denoted by k, it follows that (4), when multiplied throughout by k, becomes

$$mx-ly+kn' = 0.$$

Hence **the equation of a line perpendicular to a given line can be so**

written that in it the coefficients of x and y are obtained by exchanging those in the given equation and changing the sign of one of them.

Thus any line perpendicular to $2x-7y-10=0$ can be written $7x+2y\ldots=0$, where the constant term has still to be filled in.

It is easily verified that statement (7) and the deduction from it are still true if the lines are parallel to the axes.

Example 2. Find the equation of the line through $(3, -5)$ perpendicular to $2x-7y-10=0$.

Either of the methods of Example 1 can be used. Using the first we find that, when $x=3$ and $y=-5$, $7x+2y=11$, and the required equation is $7x+2y-11=0$.

5.4. Blended Coefficients: A Useful Device

If we wish to find the point in which two lines meet, we have to solve their equations simultaneously, as in Example 1 of 5.2. Now let us think geometrically about the *process* of solving two simultaneous equations, for example

$$7x+12y+1=0 \quad . \quad . \quad . \quad \text{(i)}$$

and

$$5x+2y-19=0 \quad . \quad . \quad . \quad \text{(ii)}$$

both of which happen to be satisfied by $(5, -3)$. The usual method of solving these equations is to multiply (ii) by 6 and subtract the result from (i), giving

$$-23x+115=0 \quad . \quad . \quad . \quad \text{(iii)}$$

The purpose of this is clear enough, but why is it justifiable? Because, if x, y are such that $7x+12y+1=0$ and $5x+2y-19=0$, then $(7x+12y+1)-6(5x+2y-19)$ will be equal to $0-6(0)$, and so will itself be zero. Or, using geometrical language, if the point (x, y)—in this case $(5, -3)$—lies on the lines $7x+12y+1=0$ and $5x+2y-19=0$, then it also lies on the line

$$(7x+12y+1)-6(5x+2y-19)=0.$$

This particular line is convenient because it is parallel to the y-axis, but it is equally true that $(5, -3)$ lies on the line

$$(7x+12y+1)+k(5x+2y-19)=0 \quad . \quad . \quad \text{(iv)}$$

whatever the value of k. By taking different values of k (which is called the **parameter** in the new equation) we can make (iv) represent different lines all passing through the point of intersection (the **meet**) of (i) and (ii).

We can of course please ourselves whether we attach the parameter

to the coefficients of the first or the second equation, and all that we have so far said of (iv) would be equally true of the line

$$k(7x+12y+1)+(5x+2y-19) = 0 \quad . \quad . \quad (v)$$

It is worth noticing, however, that (iv) can be made to represent (i) (when $k = 0$) but never (ii), and that the reverse is true of (v). If we want to be able to avoid exceptions of this kind we must use an equation such as

$$k(7x+12y+1)+k'(5x+2y-19) = 0,$$

but the extra clumsiness of this is usually more nuisance than it is worth. It is immaterial what letter is used for the parameter: in practice k and the Greek letter λ (lambda) are most commonly met with.

There is no phrase in common use for describing the process of obtaining (iv) or (v) from (i) and (ii), but since (iv) can be written

$$(7+5k)x+(12+2k)y+(1-19k) = 0,$$

we shall say for convenience that it is got by **blending the coefficients** in (i) and (ii). Ignoring the exception mentioned above, we can then make the following general statement.

The equation got by blending the coefficients in the equations of two intersecting lines always represents a line through their meet, and can be made to represent any such line by a suitable choice of the value of the parameter.

Proof. Let the lines be $lx+my+n = 0 \quad . \quad . \quad . \quad (8)$

and $l'x+m'y+n' = 0 \quad . \quad . \quad . \quad (9)$

then we have first to show that

$$lx+my+n+k(l'x+m'y+n') = 0 \quad . \quad . \quad . \quad (10)$$

represents a line through the point of intersection of (8) and (9). Let this point of intersection be $P_0\ (x_0, y_0)$; since it satisfies both (8) and (9), we have $lx_0+my_0+n = 0$ and $l'x_0+m'y_0+n^1 = 0$, and hence $lx_0+my_0+n+k(l'x_0+m'y_0+n^1) = 0$, showing that the coordinates x_0, y_0 satisfy (10). Also (10) represents a line, since it is of the first degree. Hence it represents a line through the point of intersection of (8) and (9).

Secondly, we have to show that (10) can be made to represent *any* line through P_0. This is best done by showing that the line (10) can be made to pass through *any* chosen point $P_1\ (x_1, y_1)$, which we will suppose to be different from P_0. Now (10) passes through P_1 if $lx_1+my_1+n+k(l'x_1+m'y_1+n^1) = 0$, i.e. provided

$$k = -\frac{lx_1+my_1+n}{l'x_1+m'y_1+n^1}.$$

The denominator of this fraction is not zero unless P_1 lies on the second given line. Hence values of k can be found to make (10) represent *any* line through the point of intersection of (8) and (9) except (9) itself.

Notice what happens if (8) and (9) do not meet but are parallel. Let their gradient be γ; then $l = -\gamma m$ and $l' = -\gamma m'$, so that the gradient of (10) is

$$-\frac{l+kl'}{m+km'} = \frac{\gamma m+\gamma km'}{m+km'} = \gamma \text{ also.}$$

Hence in this case (10) will give any line parallel to the given two except (9) itself. The fact is not often useful in practice, since there are easier ways of writing down the equation of a line parallel to a given line.

Example 1. Find the equation of the line joining the point of intersection of the lines $x+9y-4 = 0$ and $2x+y-5 = 0$ to the point (7, 3).

Any line through the point of intersection of the lines can be written $x+9y-4+k(2x+y-5) = 0$, and this passes through (7, 3) if $7+27-4+k(14+3-5) = 0$, i.e. if $30+12k = 0$, so that $k = -\frac{5}{2}$. The required line is therefore

$$x+9y-4-\tfrac{5}{2}(2x+y-5) = 0,$$

which is better written as

$$2(x+9y-4)-5(2x+y-5) = 0,$$

and this, on removing the brackets and changing the sign, becomes

$$8x-13y-17 = 0.$$

Example 2. Find the orthocentre of the triangle whose sides are

$$20x+10y-77 = 0 \quad . \quad . \quad . \quad \text{(i)}$$
$$5x-3y+11 = 0 \quad . \quad . \quad . \quad \text{(ii)}$$
$$x+5y+11 = 0. \quad . \quad . \quad . \quad \text{(iii)}$$

The orthocentre is the point in which the altitudes of the triangle meet, so we will first find the line through the point of intersection of (ii) and (iii) at right angles to (i). Any line through the point of intersection of (ii) and (iii) is

$$(5x-3y+11)+k(x+5y+11) = 0$$

which may be written

$$(k+5)x+(5k-3)y+(11k+11) = 0,$$

and this, by (7) of 5.3, is at right angles to (i) if

$$20(k+5)+10(5k-3) = 0,$$

i.e. if $7k+7=0$, so that $k=-1$. The altitude at right angles to (i) is therefore $(-1+5)x+(-5-3)y+(-11+11)=0$, which is the same as

$$x-2y=0 \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

The next altitude is found in the same way. Any line through the point where (i) and (iii) meet is

$$(20x+10y-77)+k'(x+5y+11)=0,$$

i.e. $$(k'+20)x+(5k'+10)y+(11k'-77)=0,$$

and this is perpendicular to (ii) if $5(k'+20)-3(5k'+10)=0$, from which $k'=7$. The altitude is therefore

$$(7+20)x+(35+10)y+(77-77)=0,$$

which simplifies to

$$3x+5y=0 \quad . \quad . \quad . \quad . \quad \text{(v)}$$

Solving (iv) and (v), we see that the orthocentre is the origin.

You may like to work this problem out for yourself by first finding the coordinates of two of the vertices: you will find that the numbers are awkward and the method clumsier than that given above. In a simpler case there might be little to choose between the two methods so far as simplicity of working is concerned, but this device of blending the coefficients in two equations is of great importance in coordinate geometry. It is not difficult to see, by an argument similar to that already used, that, by blending the coefficients in the equations of any two curves whatever, we get the equation of a curve passing through all their points of intersection. If, for example, the two curves are intersecting circles, we can obtain by this means the equations of all the circles through their points of intersection. Several cases of this kind will be found in Exercise 6b, and the subject will be developed further in Chapter 10.

The method of this section also enables us sometimes to recognize a set of concurrent lines (lines which all pass through the same point), as the next example shows.

Example 3. Show that the line $(t+2)x+y+(7t+3)=0$ passes through a fixed point for all values of t, and find the coordinates of the point.

The equation can be written $2x+y+3+t(x+7)=0$ and is therefore satisfied for all values of t provided

$$2x+y+3=0$$

and

$$x \quad +7=0,$$

from which $x = -7, y = 11$. The line therefore always goes through $(-7, 11)$.

In general it may be said that, if the coefficients in the equation of a line are linear functions of a parameter, the line passes through a fixed point for all values of the parameter. This will be useful later.

EXERCISE 5B

Write down the gradients of the following lines:

1. (i) $x+2y+3 = 0$; (ii) $3x-7y+4 = 0$; (iii) $3x = 4y+6$; (iv) $y = 2x+5$; (v) $y = 2$.
2. (i) $4x-6y+9 = 0$; (ii) $3x+y-17 = 0$; (iii) $2x = 6y-3$; (iv) $y = 7-3x$; (v) $2y+3 = 0$.

Write down the equation of the line:

3. Through (3, 5) parallel to (i) $2x-3y+7 = 0$; (ii) $2x-5 = 0$; (iii) $5y+1 = 0$.
4. Through $(-2\frac{1}{2}, 1)$ parallel to (i) $5x+y-8 = 0$; (ii) $3y-4 = 0$; (iii) $x+1 = 0$.
5. Through (1, −4) perpendicular to (i) $2x-5y+8 = 0$; (ii) $3x-4 = 0$; (iii) $2y = 7$.
6. Through $(0, \frac{4}{3})$ perpendicular to (i) $x+3y-1 = 0$; (ii) $2y = 1$; (iii) $x+5 = 0$.
7. Through $(at^2, 2at)$ at right angles to $x-ty+at^2 = 0$.
8. Through $(ct, c/t)$ at right angles to $x+t^2y-2ct = 0$.
9. Find the coordinates of the points where the lines through (3, −2) parallel and perpendicular to $4x-8y+1 = 0$ cut the line $x-5 = 0$. [You do not need to find the equations of the two lines.]
10. Find the points in which the lines through (−2, −3) parallel and perpendicular to $2x+5y-9 = 0$ cut the line $y-7 = 0$.
11. Find the acute angles between the following pairs of lines. (i) $x-y+2 = 0$, $7x-y+21 = 0$; (ii) $(2+\sqrt{3})x+y = 0$, $x-y-1 = 0$.
12. Find the obtuse angles between the following pairs of lines. (i) $3x+y+4 = 0$, $x+2y-5 = 0$; (ii) $x+(2+\sqrt{3})y = 0$, $x+y-2 = 0$.
13. Show that $x+y-2 = 0$ and $7x-y+3 = 0$ make equal angles with $x-3y+5 = 0$.
14. Show that $2x-y+4 = 0$, $11x-2y-7 = 0$, and $x-3y+5 = 0$ form an isosceles triangle with the third line as base.
15. Find x_1 and y_1 if $(7x_1-1)x+(y_1-12)y-(x_1+12y_1-12) = 0$ is the same line as $4x-2y-3 = 0$.
16. Find x_1 and y_1 if $(5y_1-1)x+(5x_1+3)y-(x_1-3y_1-2) = 0$ is the same line as $6x-3y+5 = 0$.
17. Draw, in the same diagram, the lines represented by the equation $2x-y-2+k(x+y-7) = 0$ when k has the values $-1, 0, 1$.
18. Draw, in the same diagram, the lines represented by the equation $x+2y-11+k(x-y+4) = 0$ when k has the values 0, 2, 4.
19. Verify that $3x+5y+1 = 0$, $4x+7y+1 = 0$, and $x+y+1 = 0$ all go through $(-2, 1)$, and find the value of λ for which $3x+5y+1+\lambda(4x+7y+1) = 0$ reduces to the third line.
20. Verify that $2x+5y-3 = 0$, $6x+15y-4 = 0$, $14x+35y-1 = 0$ are all parallel, and find the value of λ for which $14x+35y-1+\lambda(6x+15y-4) = 0$ reduces to the first line.

21. Find the equation of the line joining the origin to the point where $3x-y+8=0$ meets $2x+5y-4=0$.

22. Find the equation of the line joining the point (1, 2) to the meet of the lines $x+2y-10=0$ and $3x-5y-3=0$.

23. Find the equation of the line through the meet of $2x-7y+3=0$ and $4x-y=1$, parallel to $12x+10y+3=0$.

24. Find the equation of the line with gradient 2, concurrent with $4x+3y-2=0$ and $6x+5y+1=0$.

In Nos. 25, 26, find the orthocentre of the triangle with the given sides.

25. $3x-7y=0$, $6x-2y+27=0$, $8x+8y-5=0$.

26. $x-y-6=0$, $5x+y=0$, $4x+16y-19=0$.

In Nos. 27, 28, draw the triangle formed by the given lines and find its orthocentre.

27. $x+y=0$, $x-3y=0$, $x-2y-1=0$.

28. $2x-3y=0$, $x+6y=0$, $2x+y-11=0$.

29. Find the points in which the line $3x-4y+5=0$ is met by the lines $4x+y-6=0$ and $4x+y-6+\lambda(3x-4y+24)=0$, and show that the distance between them is proportional to λ.

30. Find the points in which the lines $x-2y=0$ and $12x+5y+\lambda(x-2y)=0$ meet the line $12x+5y-58=0$, and show that the distance between them is inversely proportional to λ.

31. Express k in terms of γ, the gradient of $3x-y+7+k(x-2y+2)=0$. Show that the only unobtainable gradient is that of $x-2y+2=0$ itself.

32. Express k in terms of a if $2x-5y+6+k(x-3y-2)=0$ passes through $(a, 0)$. Show that the only unobtainable value of a is that for which $(a, 0)$ lies on $x-3y-2=0$.

In Nos. 33–36, find the coordinates of the point through which the given line passes for all values of k.

33. $(2k-3)x+(3k+1)y+6k+13=0$.

34. $(3k-1)x+(2k-1)y+1=0$.

35. $(k+5)x-y+k+1=0$.

36. $2kx+3y-12=0$.

37. Show that the join of $(at^2, 2at)$ and $(a/t^2, -2a/t)$ passes through a fixed point on Ox for all values of t.

38. Show that the join of $(1+t, 0)$ and $(0, 2+2/t)$ passes through a fixed point for all values of t.

39. Find whether $5x-6y+3=0$, $6x+7y+8=0$, $32x+32y+41=0$ are concurrent by obtaining the equation of the line through the meet of two of them, parallel to the third.

40. Find the equation of the line joining the meet of $2x-7y+2=0$ and $3x+y-5=0$ to the meet of $2x-7y+6=0$ and $3x+y-1=0$.

41. Show that $x^2+y^2-2x+3y+\lambda(x^2+y^2-4)=0$ represents a circle through the common points of $x^2+y^2-2x+3y=0$ and $x^2+y^2=4$, unless $\lambda=-1$. What does it represent if $\lambda=-1$?

42. Find the equation of the circle through the common points of the two ellipses $x^2+4y^2=16$ and $4x^2+y^2-5x-5y=24$.

Exercise 5M (Miscellaneous Examples)

1. A rhombus has the coordinate axes as diagonals; if one side is $lx+my+n=0$, what are the others?

2. Two circles, centres C and D, intersect at P. Show that the angle at which the circles intersect (i.e. the angle between the tangents at P) is $180°-\angle CPD$.

Hence (using the cosine formula or otherwise) find the angles at which the following pairs of circles intersect.

(i) $x^2+y^2+2y-2=0$, $x^2+y^2-4x-4y+4=0$;

(ii) $x^2+y^2+2x-4y-7=0$, $x^2+y^2-4x+6y-9=0$;

(iii) $x^2+y^2+6x-4y-12=0$, $x^2+y^2-8x-4y-44=0$.

3. M and N are two points, one on each axis, and move so that MN is of constant length c. The lines through M and N at right angles to their respective axes meet at P. Find the locus of P.

4. Without finding the coordinates of any points, obtain the eq ıation of the line joining the origin to the meet of the first and second of the three lines

$$5x-7y+3=0, \quad x+2y-5=0, \quad 6x-5y-2=0.$$

Do the same with the second and third lines, and so show that the three given lines are concurrent.

5. Find the equations of two perpendicular lines through the origin which form an isosceles triangle with $2x+3y+13=0$.

6. Find the orthocentre of the triangle formed by the lines

$$sx+(s+1)y+s(s+1)=0, \quad tx+(t+1)y+t(t+1)=0, \quad x=0.$$

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7. A rhombus has $x=y$, $x+y=0$ as diagonals; if one side is $lx+my+n=0$, what are the others?

8. Two circles cutting at right angles (see No. 2) are said to be **orthogonal.** Show that the circles $x^2+y^2+2gx+2fy+c=0$ and $x^2+y^2+2g'x+2f'y+c'=0$ are orthogonal if $2gg'+2ff'=c+c'$.

9. A point moves so that its distance from a fixed line is equal to the length of the tangent from it to a fixed circle. Show that its locus is a parabola.

10. P_1,P_2 have their usual meaning; Q is (x_1+x_2, y_1+y_2). Prove by means of gradients that OP_1QP_2 is a parallelogram, and so obtain the coordinates of the mid-point of P_1P_2.

11. The equation of one side of an equilateral triangle is $x-y=0$, and one vertex is $(2+\sqrt{3}, 5)$. Prove that the equation of a second side is $y+(2-\sqrt{3})x=6$ and find the equation of the third side.

12. From each corner of a parallelogram a perpendicular is drawn to the diagonal which does not pass through it, and these perpendiculars are produced to form a second parallelogram; show that its diagonals are perpendicular to the sides of the first parallelogram.

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13. AB, AC are two lines such that $\angle BAC=45°$. From a variable point D on AB a line is drawn, perpendicular to AB, to meet AC at E. F is a point on DE, the length DF remaining constant. If the line through E, parallel to AB, meets AF at P, find the locus of P.

14. Use the method of 5.4 to show that, if the circles $x^2+y^2+2gx+2fy+c=0$ and $x^2+y^2+2g'x+2f'y+c'=0$ intersect, then by a suitable choice of the value of λ the equation $x^2+y^2+2gx+2fy+c-\lambda(x^2+y^2+2g'x+2f'y+c')=0$ can be made to represent *any* circle through their common points, with the exception of the second circle itself.

What does the equation represent when $\lambda=1$?

15. Find the values of l and m for which the equation

$$(3x+y-41)^2-(x+7y-7)(lx+my+1)=0$$

represents a circle. Plot the circle together with the lines $lx+my+1=0$, $x+7y-7=0$ and $3x+y-41=0$.

16. Find the coordinates of the other corners of two squares which have the points (2, 3), (5, 7) as the ends of a side.

17. A,B,C,D are the points $(-1, 1)$, $(1, 1)$, $(1, c)$, $(-1, -c)$; AC and BD meet at M. Find the coordinates of M and show that, if a line through M parallel to AD meets AB at L and CD at N, then M is the mid-point of LN.

18. Obtain the condition $ll'+mm' = 0$ for perpendicular lines by the following method. The lines $lx+my+n = 0$ and $l'x+m'y+n' = 0$ are respectively parallel to $lx+my = 0$ and $l'x+m'y = 0$. The second two lines join the origin to $(m, -l)$ and $(m', -l')$. Apply Pythagoras' Theorem to the triangle formed by these three points.

19. A line is reflected in one of the axes; the image is then reflected in the line $x = y$. Show that the second image is at right angles to the original line.

20. Prove that the circles $x^2+y^2+ax+by = 0$, $x^2+y^2 = c^2$ touch each other if $a^2+b^2 = c^2$, and find the coordinates of the point of contact.

Two circles pass through the origin and the point (1, 0) and touch the circle $x^2+y^2 = 4$. Find the coordinates of the points of contact. [A.L.]

21. If the equation $(3x-4y)(x+2y)-(3x+y-30)(lx+my) = 0$ represents a circle, find the values of l and m. Plot the circle together with the lines $3x-4y = 0$, $x+2y = 0$, $3x+y-30 = 0$, $lx+my = 0$.

22. Find the coordinates of the other corners of a square which has (1, 5), (8, 8) as the ends of a diagonal.

23. AU, BV are two parallel lines to which AB is perpendicular, and O is any point on AB. Q and R are variable points, one on each of AU and BV, such that QOR is a right angle. Find the locus of the foot of the perpendicular from O to QR. [Take O as origin and discard a factor x^2+y^2.]

24. Prove that the locus of points whose powers with respect to two circles are equal is a straight line. (It is known as the **radical axis** of the two circles.)

Show that, if the two circles meet, their radical axis is their common chord.

[Consider their points of intersection.]

25. Verify that (10, 2), (11, 5), (7·6, −2) all lie on the curve $5x^2-6xy+5y^2 = 400$. By investigating the symmetry of the curve find nine more points on it. Find also where the axes of symmetry meet the curve and sketch it after plotting all the points obtained.

26. If r is the radius of the circle through $(a, 0)$, $(-a, 0)$, $(0, b)$; and r' that of the circle through $(b, 0)$, $(-b, 0)$, $(0, a)$; show that $r/r' = a/b$.

27. Show that the equation of any circle can be written in the form $x^2+y^2+2gx+2fy+c = 0$.

Find the equation of the circle which touches the axis of x at the point (3, 0) and passes through the point (1, 4). Prove that this circle subtends an angle of about 79° 37′ at the origin. [A.L.]

28. Show that the lines $2x-y-5 = 0$, $x+y+4 = 0$, $x-2y+3 = 0$ form an isosceles triangle whose vertical angle is $\tan^{-1}\frac{3}{4}$.

29. A is a fixed point on the x-axis, Q a variable point on the y-axis. Show that the locus of P, such that PA and PQ are respectively perpendicular to AQ and the y-axis, is a parabola.

30. (i) P, Q are any two points on a rectangular hyperbola. Perpendiculars PA, QB are drawn to one asymptote; PC, QD to the other. Prove that PQ, AD, BC are parallel.

(ii) The asymptotes of a rectangular hyperbola are given, also one point

P on the curve. Show how to construct the point in which any line through P meets the curve again.

31. A certain curve is similar in appearance to the two hyperbolas $xy \pm 2 = 0$ taken together and has the symmetry which would be expected from this fact. In giving its equation as

$$x^4y^2+2x^3y^4+3x^2y^2-4x^2-4y^2 = 0,$$

two mistakes have been made, both in the same term. Find the term and show what it should be.

32. Show that the locus of a point which moves so that its powers with respect to two circles are in a fixed ratio is itself a circle unless the ratio is 1 : 1, when it is a straight line.

33. A semicircle is described on a diameter joining $(-2, 0)$ and $(2, 0)$, and lies in the first two quadrants. Find the locus of centres of circles inscribed in this semicircle.

The equation of the locus represents a curve having points outside the semicircle: explain how these points are related to the condition which gave rise to the locus.

34. Find an expression for the distance between the points whose coordinates are (x_1, y_1) and (x_2, y_2).

Prove that the points (2, 3), (5, −6), (11, 1) are the corners of an isosceles triangle whose largest angle is an angle whose cosine is $\frac{8}{17}$. [A.L.]

35. Prove that the equations of the diagonals of the parallelogram whose sides are

$$lx+my+n \pm p = 0 \text{ and } l'x+m'y+n' \pm p' = 0$$

are

$$p'(lx+my+n) \pm p(l'x+m'y+n') = 0$$

36. **Trisection of an angle.** Let the angle to be trisected be the angle between the side AB and diagonal BD of a rectangle $ABCD$. Draw the rectangular hyperbola through C with asymptotes AB, AD; draw the circle with centre C and radius twice CA. These two curves meet in four points: let E be the one whose perpendicular EF to AB produced is as short as possible; then $\angle AFD$ is one-third of $\angle ABD$.

Use coordinate geometry to prove that DF is parallel to CE; then let DF meet BC at G, let H be the mid-point of FG, and use pure geometry to prove that $GF = CE$, $BD = BH = GH = HF$. Finally make use of isosceles triangles to prove the required result.

This construction was given by Pappus of Alexandria about A.D. 300.

CHAPTER 6

CONICS

6.1. The Plane Section of a Cone

The curves which we have met already are called conic sections or **conics,** because they can be obtained as the curve of intersection of a plane and a cone.

Consider a right circular cone generated by a line which passes through a fixed point V and rotates about a fixed axis through V so that the angle which it makes with the axis is always equal to α. The generating line or **generator** is supposed to be unlimited in length so that the conical surface extends on both sides of the vertex V and forms two parts called **sheets** meeting there. It is, of course, impossible to draw more than a portion of the figure.

Suppose now that the axis is inclined at an angle β to the horizontal and that the cone is cut by a horizontal plane. Four cases are possible and are shown in Fig. 33 a–d, which represent a vertical section

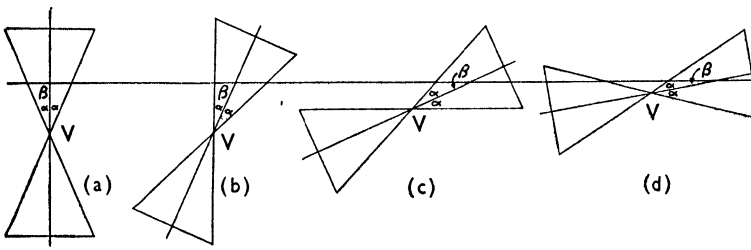
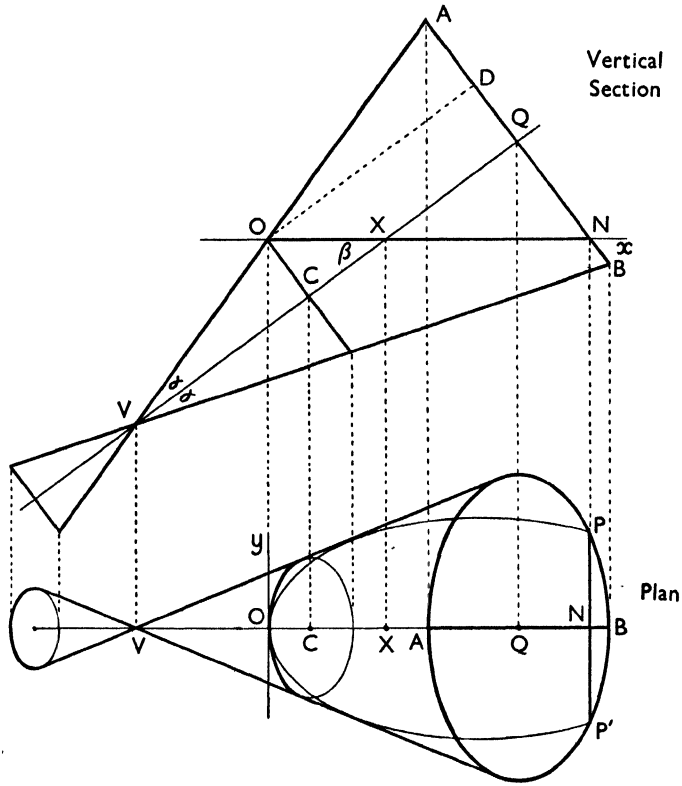


Fig. 33

through the axis of the cone. In (a) $\beta = 90°$, and since the plane is at right angles to the axis it cuts the cone in a circle. In (b) β is acute but greater than α, and the cone is still cut in a closed curve, though this is no longer a circle. In (c) $\beta = \alpha$, so that the cone has just one horizontal generator and as this never cuts the plane the curve cannot be closed. In (d) β is less than α, so that the plane cuts both sheets of the cone and the curve of intersection has two branches.

In order to find the equation of the curve of intersection we have illustrated case (b) more fully in Figs. 34 and 35; the calculations which follow, however, apply equally to the other cases. Fig. 34



Figs. 34 and 35

again shows a section by a vertical plane through the axis, so that the cone appears as two lines with the axis bisecting the angle between them, and the horizontal plane appears as a single line. Fig. 35 represents a plan, or view from vertically above the cone. The horizontal plane cannot therefore be shown, and the only parts of the figure which are supposed to lie in it are the lines Oy and PNP' and the curve POP' whose equation we require. The line ON also lies in the plane, but it is indistinguishable from a part of the line VQ, which does not. Fig. 35 shows the true lengths of these horizontal lines.

The vertical plane through the axis meets the horizontal plane in a line, and this line [in all but case (c)] meets the cone in two points; one of these we choose for our origin O, as shown in Figs. 34 and 35, and we take the line as our x-axis. Our y-axis is then the horizontal

line through O at right angles to it. Let the plane through O at right angles to the axis of the cone meet the axis at C, so that C is the centre of the circular section in which this plane meets the cone, and let $VO = c$. Let $P(x, y)$ be any point on the required curve of intersection; then, as the whole figure is symmetrical about the vertical plane through the axis, the reflection of P in that plane will also be on the required curve. This point $(x, -y)$ we shall call P'. The plane through P and P' at right angles to the axis will meet the cone in a circle; let Q be its centre and let AB be the diameter at right angles to PP', meeting PP' at N, as shown in the figures, so that O and A lie on the same generator of the cone. Then, by the property of intersecting chords of a circle,

$$PN^2 = AN.NB \quad . \quad . \quad . \quad . \quad (1)$$

Let D be the foot of the perpendicular from O to AB so that OD is parallel to CQ.

We can now proceed to the calculation. $ON = x$, $NP = y$, $\angle AOD = \alpha$, $\angle DON = \beta$, $DN = x \sin\beta$, $OD = x\cos\beta$, $AD = OD \tan\alpha = x \tan\alpha \cos\beta$,

$$\therefore AN = AD+DN = x(\sin\beta + \tan\alpha\cos\beta) \quad . \quad . \quad . \quad (2)$$

Also $VC = c\cos\alpha$, $\therefore VQ = VC+OD = c\cos\alpha + x\cos\beta$,

$$\therefore AB = 2VQ\tan\alpha = 2(c\sin\alpha + x\tan\alpha\cos\beta),$$

$$\therefore NB = AB-AN = 2c\sin\alpha - x(\sin\beta - \tan\alpha\cos\beta). \quad (3)$$

Substituting (2) and (3) in equation (1) we obtain, as the equation of the curve of intersection,

$$y^2 = 2cx\sin\alpha(\sin\beta + \tan\alpha\cos\beta) - x^2(\sin^2\beta - \tan^2\alpha\cos^2\beta),$$

which can be written

$$y^2 = 2cx\sin\alpha\cos\beta(\tan\beta + \tan\alpha) - x^2\cos^2\beta(\tan^2\beta - \tan^2\alpha). \quad (4)$$

If we let $\sin\alpha\cos\beta(\tan\beta + \tan\alpha) = g$

and $\cos^2\beta(\tan^2\beta - \tan^2\alpha) = p$,

(4) becomes $y^2 = 2gx - px^2 \quad . \quad . \quad . \quad . \quad (5)$

in which g is positive, but p is positive, zero or negative according as β is greater than, equal to, or less than α.

In case (a), where $\beta = 90°$, (4) becomes $y^2 = 2cx\sin\alpha - x^2$, that is,

$$x^2+y^2-2cx\sin\alpha = 0,$$

and this represents a **circle** with centre $(c\sin\alpha, 0)$ and radius $c\sin\alpha$—in fact the circle, centre C, which now lies in the horizontal plane.

In case (b), where $\beta > \alpha$, (5) can first be written

$$px^2+y^2-2gx = 0,$$

then $$p(x-g/p)^2+y^2 = g^2/p,$$

and finally $$\frac{p^2(x-g/p)^2}{g^2}+\frac{py^2}{g^2}=1;$$

and since p is positive this represents an **ellipse** with semi-axes g/p and $g/\sqrt{p}$, whose centre is $(g/p, 0)$.

In case (c), where $\beta=\alpha$, p is zero and (5) becomes the **parabola**

$$y^2=2gx.$$

Lastly, in case (d), p is negative. If we write

$$q=-p=\cos^2\beta\,(\tan^2\alpha-\tan^2\beta),$$

(5) can be written $$qx^2-y^2+2gx=0$$

and then $$\frac{q^2(x+g/q)^2}{g^2}-\frac{qy^2}{g^2}=1,$$

but this is not an ellipse because of the negative coefficient of y^2. If we let $g/q=a$ and $g/\sqrt{q}=b$, the equation can be written

$$\frac{(x+a)^2}{a^2}-\frac{y^2}{b^2}=1,$$

and of this we can say that it represents a curve exactly like

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad . \quad . \quad . \quad . \quad (6)$$

but shifted through a distance a parallel to the x-axis in the negative direction. We also know, from Fig. 33(d), that the curve has two branches. It is in fact called a **hyperbola,** and we shall investigate it further in 6.2.

A plane section of a cone is thus a circle, ellipse, parabola, or hyperbola in the four cases shown in Fig. 33. The last three names (in their Greek form) were given to the curves by Apollonius of Perga (about 250 B.C.), who obtained the geometrical equivalent of (5). He obtained a rectangle whose length was the abscissa x and area equal to the square on the ordinate (y^2), and compared its breadth with $2g$. In the case of the parabola, the two are equal and one can be applied (*paraballein*) to the other; in the case of the ellipse, the breadth of the rectangle falls short (*elleipein*) of $2g$, while in the hyperbola it overshoots it (*huperballein*).

For the sake of completeness, we can also consider the cases in which the plane passes through the vertex of the cone. If $\beta>\alpha$, the plane meets the cone at its vertex and nowhere else; if $\beta=\alpha$, the plane is tangential to the cone, and its points of contact lie along one of the generators; and if $\beta<\alpha$, the plane meets the cone in a pair of generators crossing at the vertex. A pair of lines, or **line-pair,** can thus also be regarded as a conic section and represented by a

single equation. All the conic sections, except the parabola, have a centre and are called **central conics.**

6.2. The Hyperbola

It will first be convenient to give a formal definition of an asymptote.

When a point P moves so that its distance from a given fixed point increases without limit—i.e. if at least one of its coordinates **tends to infinity**—the point itself is said to tend to infinity, and we denote this symbolically by writing $P \to \infty$. We can now say that **if, as a point $P \to \infty$ along a curve, its distance from a fixed line tends to zero, the line is an asymptote to the curve.**

We will now see what can be found out about the curve

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad (6)$$

which we have already called a hyperbola. Its equation differs only slightly from that of an ellipse, and the tests of 2.4 show that the new curve is still symmetrical about both axes, but if we write its equation in the form

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1 \quad . \quad . \quad . \quad . \quad (7)$$

we can soon see that it differs from an ellipse in several ways.

In the first place, (7) shows that $\frac{x^2}{a^2}$ cannot be less than 1, since the L.H.S. cannot be negative. Hence no part of the curve can lie between the lines $x = -a$ and $x = a$. Putting $x = \pm a$ in (7) we find $y = 0$, and so the curve goes through $(a, 0)$ and $(-a, 0)$.

Secondly, (7) shows that y^2/b^2 is less than x^2/a^2, so that y/b is numerically smaller than x/a. This means that, for any given value of x, the y-coordinates of the points on the curve are numerically smaller than the y-coordinates of the points on the lines

$$\frac{y}{b} = \frac{x}{a} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$\frac{y}{b} = -\frac{x}{a} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and so the curve lies between these lines and the x-axis.

The two lines (8) and (9) go through the origin and through the points (a, b) and $(a, -b)$ respectively. We have therefore shown that the curve is confined to the unshaded regions of Fig. 36, that it goes

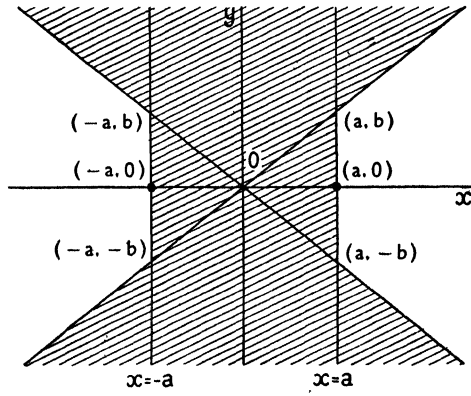


Fig. 36

through $(a, 0)$ and $(-a, 0)$, but that it does not meet the boundary lines of these regions anywhere else.

We can get more information about the relationship between the curve and the two lines (8) and (9) by writing (6) in the form $\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=1$, from which $\frac{x}{a}-\frac{y}{b}=\frac{1}{\frac{x}{a}+\frac{y}{b}}$, and hence

$$y=\frac{bx}{a}-\frac{b}{\frac{x}{a}+\frac{y}{b}}. \quad . \quad . \quad . \quad (10)$$

This form of the equation would be useless for plotting the curve, since y occurs on both sides. In it, we are to regard x and y as the coordinates of a point which is already known to lie on the curve, and then the last term of (10) shows how far y falls short of $\frac{bx}{a}$, i.e. how far the point is "vertically" below the line (8).

Confining our attention to the part of the curve in the first quadrant, where x and y are both positive, let us notice that $\frac{x}{a}+\frac{y}{b}$ is greater than $\frac{x}{a}$, and so the last term of (10) is less than $\frac{b}{x/a}$, or $\frac{ab}{x}$. The "vertical" distance of the point below the line (8) is therefore less than ab/x, and its perpendicular distance from the line is even less.

Now the behaviour of ab/x is already known: as x increases through positive values it becomes continually smaller, and it can be made as

small as we please by taking large enough values of x, though it never actually reaches zero; in other words, as $x \to \infty$, $ab/x \to 0$. Hence, if a point moves round the curve so that $x \to \infty$, (10) shows that its distance from the line (8) tends to zero: $y/b = x/a$ is therefore an asymptote.

We have shown this to be true in the first quadrant. As the curve is symmetrical about the x-axis it follows that the line $y/b = -x/a$ is an asymptote in the fourth quadrant, while its symmetry about the y-axis shows that the curve again approaches those two asymptotes in the second and third quadrants. The shape of the curve depends upon the actual values of a and b. In Fig. 36 we have supposed b to be less than a; the asymptotes therefore make an angle of less than 45° with the x-axis and the curve is confined within an acute angle. If a and b are equal, the asymptotes will be inclined at 45° to the x-axis and will therefore be at right angles to each other. The curve will then be a **rectangular hyperbola** and can be shown to be the same shape as the curve $xy = k$ (see Exercise 6a, Nos. 17, 18, and Exercise 7c, No. 21). If b is greater than a, the curve will lie within an obtuse angle. Fig. 37 shows the hyperbolas $\frac{x^2}{9} - \frac{y^2}{4} = 1$, $\frac{x^2}{4} - \frac{y^2}{4} = 1$, $x^2 - \frac{y^2}{16} = 1$, which illustrate the three cases. The dotted rectangle in each of these diagrams is formed by the lines $x = \pm a$ and $y = \pm b$; they meet at $(\pm a, \pm b)$ (any combination of signs being taken), and

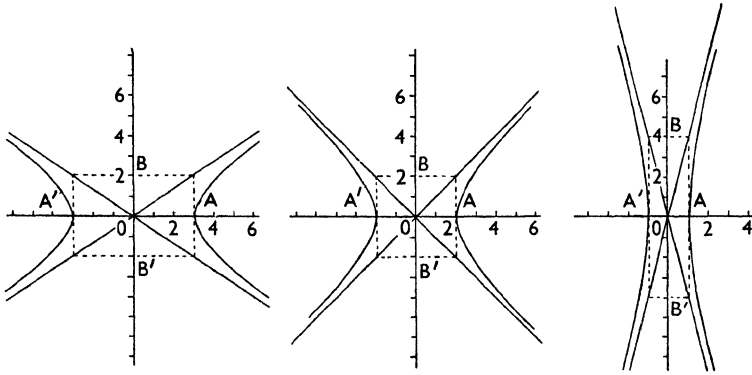


Fig. 37

these four points lie on the asymptotes $\frac{x}{a} = \pm\frac{y}{b}$. When it is required to sketch a hyperbola whose equation is given, it is helpful to mark the corners and one pair of sides of this rectangle and so to draw the asymptotes; the curve can then be easily sketched.

If the points $(a, 0)$, $(-a, 0)$, $(0, b)$, $(0, -b)$ are denoted by A, A', B, B', then $A'A$ is called the **transverse axis** of the hyperbola and $B'B$ (or sometimes the whole y-axis) is called its **conjugate axis.** O is called the **centre** of the curve and A, A' its **vertices.** B, B' do not lie on the curve and are unnamed.

The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0 \quad . \quad . \quad . \quad . \quad (11)$$

which can also be written $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, can be seen to represent a hyperbola with the same asymptotes as before, but lying between these asymptotes and the y-axis. Its vertices are $(0, \pm b)$ and its transverse axis is the conjugate axis of the hyperbola (9). The two hyperbolas are said to be **conjugate** to each other (from the Latin, meaning "yoked together").

Exercise 6A

In Nos. 1 and 2, with the notation of 6.1 and the given values of c, α, β, find the equation of the curve of intersection of the cone and plane. You may find it helpful to rewrite the argument of 6.1, using the particular values given. In parts (i) and (iii) find also the centre of the curve.

1. $c = 100$ units, $\tan\alpha = \frac{3}{4}$, $\tan\beta =$ (i) $\frac{4}{3}$, (ii) $\frac{3}{4}$, (iii) $\frac{5}{12}$.

2. $c = 6$ units, $\tan\alpha = 1$, $\tan\beta =$ (i) 7, (ii) 1, (iii) $\frac{1}{7}$.

3. If equation (4) of 6.1 represents a rectangular hyperbola, prove that $\tan^2\alpha = 2\tan^2\beta + 1$.

4. What can be said about the angle α if it is possible to cut a cone in a rectangular hyperbola? (Use the result of No. 3.)

If $\alpha = 60°$, what must β then be?

5. In equation (4), put $c = 0$, $\alpha = \tan^{-1}3$, $\beta = 45°$, and simplify the result. Describe in words how the plane meets the cone in this case.

6. In equation (4), put $c = 0$, $\alpha = 45°$, $\beta = 60°$, and simplify the result. What does the equation represent?

7. Plot, using the same axes and 1 in. as unit, the circle $y^2 = 4x - x^2$, the ellipse $y^2 = 4x - \frac{1}{2}x^2$, the parabola $y^2 = 4x$, and the hyperbola $y^2 = 4x + \frac{1}{2}x^2$. Plot particularly carefully in the neighbourhood of the origin and notice that the curves are closely alike there but show marked differences of shape elsewhere.

8. Plot carefully, using the same axes and 1 in. as unit, the circle $2y^2 = 13x - 2x^2$,

the ellipse $2y^2 = 11x - x^2$, the parabola $2y^2 = 9x$, and the hyperbola $2y^2 = 7x + x^2$. Take values of x from 0 to 6. Notice how the curves differ in shape although all of them pass through the origin and the points $(2, \pm 3)$.

9. Sketch roughly in the same figure the curves given by $\frac{x^2}{\lambda+50} + \frac{y^2}{\lambda} = 1$ when $\lambda = 50, 6\frac{1}{4}, -1, -25, -49$.

10. Sketch roughly in the same figure the curves given by $\frac{x^2}{\lambda} - \frac{y^2}{20-\lambda} = 1$ when $\lambda = 4, 10, 16, 20\frac{1}{4}, 36$.

11. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is shifted so that its centre is at $(a, 0)$. What is the equation of the new curve?

In the new equation, let $a = b^2/2c$; then let b tend to infinity, keeping c constant. In the limit, what does the curve become? The final curve can thus be regarded as a limiting case of an ellipse.

12. Do No. 11, starting with the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and shifting it so that its centre is at $(-a, 0)$.

13. Put $a = 3$, $b = 2$ in the argument of 6.2 and so show that the vertical distance of any point on the hyperbola $x^2/9 - y^2/4 = 1$ from the nearer asymptote is less than $6/x$. Then plot part of the curve and its asymptotes and verify the result for several points.

14. By an argument like that of 6.2, show that the horizontal distance of any point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ from the nearer asymptote is less than a^2/x, and verify this result by plotting the hyperbola $x^2 - y^2 = 9$.

15. A point on the line $x/a - y/b = 0$ and another on $x/a - y/b = t$ have the same y-coordinate; find the difference between their x-coordinates and show that it and t both become zero together.

Find the x-coordinate of the point at which the second line meets the hyperbola $x^2/a^2 - y^2/b^2 = 1$, and show that this coordinate tends to infinity as $t \to 0$.

What does this tell you about the hyperbola and the line $x/a - y/b = 0$?

16. It can be shown that, if q is small compared with p^2, $\sqrt{(p^2+q)} \simeq p + q/2p$. Use this to show that, as a point in the first quadrant on the hyperbola $y^2/b^2 = x^2/a^2 - 1$ tends to infinity, its vertical distance from $x/a - y/b = 0$ tends to zero.

17. Plot the hyperbola $x^2 - y^2 = 4$ and its asymptotes, taking $\frac{1}{2}$ in. as the unit. On a separate sheet, but with the same scale, plot the hyperbola $xy = 2$. It should be possible to make the two curves and their asymptotes coincide by placing one sheet on the other and looking through them at the light.

18. On one sheet of graph-paper plot the hyperbola $xy = 50$, using 0·1 in. as unit, and on another sheet draw a pair of coordinate axes crossing near the centre of the paper, mark them with the same scale, and draw the lines $x \pm y = 0$. Place the first sheet on the second, so that the axes of the first lie along the lines $x \pm y$ of the second, and reproduce the curve on the lower sheet by pricking through with a compass point or by bearing heavily with a pencil.

Read off the coordinates of various points on the new curve and find the value of $x^2 - y^2$ at each of them.

In Nos. 19 and 20 make neat sketches of the given hyperbolas and their asymptotes; then check your sketch by making a very brief table of values.

19. (i) $4x^2-y^2=4$; (ii) $x^2-y^2=4$; (iii) $4x^2-9y^2+36=0$.

20. (i) $4x^2-y^2+4=0$; (ii) $x^2-y^2+1=0$; (iii) $9x^2-4y^2=36$.

21. Find the locus of a point which moves so that its distance from the line $5x=16$ is four-fifths of its distance from the point (5, 0).

22. Find the locus of a point which moves so that its distance from the line $5y=9$ is three-fifths of its distance from the point (0, 5).

23. If A and B are the points $(-5, 0)$, $(5, 0)$, find the locus of P such that $AP-BP=8$. What is the locus if $BP-AP=8$?

24. With the notation of No. 23 find the locus if AP and BP differ by 6 units.

25. A point moves so that the length of the tangent from it to a fixed circle is twice its distance from a fixed line. Prove that its locus is a hyperbola.

26. A point moves so that its power with respect to a circle is twice the square of its distance from a diameter. Find its locus.

6.3. Curve-Tracing: Summary of Results

It will be convenient to recall here the different types of curve we have now met, and the features of their equations by which they may be recognized. If you refer back to 2.7, 4.1, and 4.4, you will see that so far, whenever we have shifted a curve, the second-degree terms in its equation have been unaltered, though the first-degree terms and the constant term have changed. Now to make a shift we substitute $x-x_1$ for x, and $y-y_1$ for y in the equation of the curve, x_1 and y_1 being two numbers which do not vary with x or y but depend only on the particular way in which the curve is to be shifted. So a term x^2 in the equation becomes $(x^2-2x_1x+x_1^2)$, a term xy becomes $(xy-y_1x-x_1y+x_1y_1)$, and a term y^2 becomes $(y^2-2y_1y+y_1^2)$. But the second-degree parts of these expressions are still respectively x^2, xy, and y^2, and so, provided an equation contains no terms of higher degree than the second, the second-degree terms are not altered by shifting the curve, except that we can if we please multiply all the terms of an equation by some constant. For example, if the curve $xy=1$ is shifted by substituting $x-\frac{1}{8}$ for x, $y-\frac{3}{8}$ for y, the new equation is

$$xy-\frac{3}{8}x-\frac{1}{8}y=\frac{61}{64}$$

and the second-degree term is unaltered, but it is more satisfactory to write

$$64xy-24x-8y=61.$$

As a shift of a second-degree curve does not introduce any new second-degree terms or remove any old ones, we can classify the curves we have met so far as follows:

Second-degree terms	Type of curve
Multiple of x^2 only	Parabola with axis parallel to y-axis
Multiple of y^2 only	Parabola with axis parallel to x-axis
Multiple of xy only	Rectangular hyperbola with asymptotes parallel to coordinate axes
Multiple of x^2+y^2 only	Circle
Terms in x^2 and y^2, having unequal coefficients with the same sign, but no term in xy	Ellipse with principal axes parallel to coordinate axes
Terms in x^2 and y^2 having coefficients with opposite sign, but no term in xy	Hyperbola with principal axes parallel to coordinate axes

It is natural to ask what would be represented in the other possible cases—that is to say, when there is a term in xy together with one in x^2 or y^2 or both. The answer, which we shall not prove until Chapter 10, is that the curve would still be a parabola, ellipse, or hyperbola, but the axes of symmetry would no longer be parallel to the coordinate axes.

There are, however, two respects in which we must qualify the statements made above; they are best indicated by examples.

Example 1. Trace the curve $x^2-4y^2-2x-24y-35 = 0$.

This appears to be a hyperbola with axes parallel to the coordinate axes, and we therefore try to put its equation into the form $(x-\quad)^2-4(y-\quad)^2 = \quad$. With this in view we write it as $(x^2-2x\quad)-4(y^2+6y\quad) = 35$, and obtain, after completing the squares and simplifying slightly, $(x-1)^2-4(y+3)^2 = 0$, showing that the curve is exactly like $x^2-4y^2 = 0$, but shifted so that its centre is at $(1, -3)$. But this last equation can be written $x = \pm 2y$ and represents, not a hyperbola with centre at O, but two lines passing through O. The original equation therefore represents a pair of lines with gradients $\pm\frac{1}{2}$, passing through $(1, -3)$.

We have already seen in 6.1 that a line-pair is a conic section, occurring when the plane passes through the vertex of the cone, and that any parallel plane will cut the cone in a hyperbola. It now appears that the equation of a line-pair is indistinguishable at first sight from that of a hyperbola. Because of the close relationship between the two, a line-pair is often called a degenerate hyperbola,

the word **degenerate** being used to show that some of the properties of the curve have been lost.

In the same way, a pair of parallel lines (or the same line twice over) can be regarded as a degenerate parabola, while a point may be a degenerate circle or ellipse. The equations $y^2 = 1$ (or $y^2 = 0$), $x^2+y^2 = 0$, and $x^2+2y^2 = 0$ illustrate these possibilities: according to the table above they should represent a parabola, circle, and ellipse, but in fact they are degenerate cases. Notice that a pair of parallel lines is degenerate in a special way: it is not a section of a cone but of a cylinder, which is a degenerate cone.

Example 2. What does the equation $x^2+4y^2-2x+12y+14 = 0$ represent?

At first sight it appears to represent an ellipse, but if we complete the squares as in Example 1, we obtain

$$(x-1)^2+4(y+\tfrac{3}{2})^2 = -4.$$

Since the L.H.S. is the sum of two squares, it cannot be negative and so no real points can be found to satisfy the equation.

This serves as a reminder of the fact, which we met in 4.2, that it may not be possible to satisfy a given second-degree equation. The table given above must therefore be interpreted to mean that the equation represents the curve specified, *provided that it is satisfied by at least one real point.*

Exercise 6B

Trace the following curves: name them, state the chief facts about them, and sketch them.

1. $2xy-x+2y-6 = 0$.
2. $x^2+y^2-2y-8 = 0$.
3. $x^2+4x-2y+10 = 0$.
4. $4x^2+y^2+4x = 24$.
5. $x^2+y^2+2x-6y+10 = 0$.
6. $x^2-y^2+3x-3y = 0$.
7. $y^2-x+3y = 0$.
8. $xy+2y-8 = 0$.
9. $x^2+y^2-3x+4y = 0$.
10. $4x^2-20x+y = 0$.
11. $x^2-y^2-10x-8y = 0$.
12. $x^2+y^2-10x-8y+41 = 0$.
13. $x^2+2y^2 = 12y$.
14. $y^2+x+2y-8 = 0$.
15. $y^2-5y = 6$.
16. $xy+2x-y = 2$.
17. $4x^2-y^2+4x+2y = 0$.
18. $4x^2+y^2+4x+2y+2 = 0$.
19. $x^2-4y^2-2x-8y+1 = 0$.
20. $x^2+4y^2+6x-16y = 0$.
21. $xy-3x+2y = 6$.
22. $x^2-4x+4 = 0$.
23. $9x^2+16y^2 = 18x+32y$.
24. $9x^2-25y^2 = 18x+50y$.

In Nos. 25–30, sketch the given curves in the same figure. It may be helpful to re-read 5.4 first.

25. $y^2 = 4a(x+a)$, when $a = -2, -1, 1, 2$. Notice where the curves meet the axes.

26. $x^2-4y+2kx=0$, when $k=-2, -1, 0, 1, 2$. On what curve do the vertices lie?

27. $x^2+8y-16+\lambda(y^2-2y)=0$, when $\lambda=-4, -1, 0, 1, 4$. Draw also the lines represented by $y^2-2y=0$.

28. $4x^2+9y^2-576-\lambda(y^2+8y)=0$, when $\lambda=0, 5, 9, 13, 18$. Draw also the lines represented by $y^2+8y=0$.

29. $\lambda x^2-2y^2+9x+18=0$, when $\lambda=-2, -\frac{9}{8}, 0, \frac{1}{2}, \frac{9}{8}, 2$, noting the intersections with the axes.

30. $x^2-4y^2+12x+36-\lambda(x^2-y^2-12y-36)=0$, when $\lambda=0, 1, 2, 2\frac{1}{2}, 3, 4, 5$. Also draw the lines represented by $x^2-y^2-12y-36=0$.

6.4. Tangents

Having obtained equations for all the conic sections, we must now see how to find the equations of tangents to them. As in most cases the equation of the curve gives two values of y for each value of x, and vice versa, we shall suppose that both coordinates of a point on the curve are given, and our task will be to find the equation of the tangent at that point.

The circle presents no difficulties: we already know from pure geometry that a tangent to a circle is at right angles to the radius drawn through the point of contact, and this property enables us to find the equation of the tangent. The method is sometimes a good one, particularly in a numerical case, although other ways of obtaining the equation will be given later.

Example 1. Find the equation of the tangent at A (2, 6) to the circle $x^2+y^2=40$; also find the equations of the two tangents at right-angles to this.

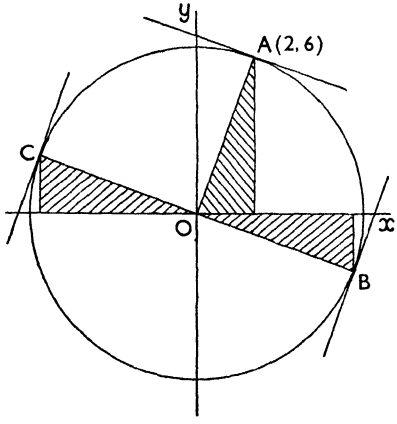


Fig. 38

The coordinates of A are readily seen to satisfy the equation of the circle. The gradient of OA is 3 and so the gradient of the tangent at A is $-\frac{1}{3}$. Its equation is therefore

$$x+3y=2+3.6,$$

i.e. $$x+3y=20.$$

To find the equations of the tangents at right angles to this, we draw the radii OB, OC at right angles to OA. The shaded triangles in Fig. 38 are then congruent and so B is (6, −2),

C is $(-6, 2)$. The required tangents are the lines through B and C with gradient 3, and their equations are therefore

$$3x-y = \pm 20.$$

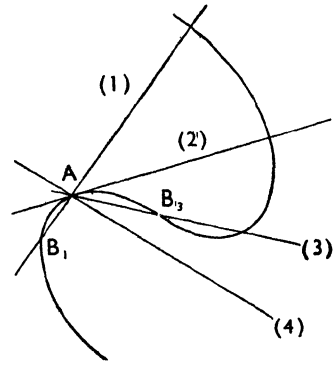


Fig. 39

The method used here can be applied only to a circle, and before we go any further we must put into mathematical terms our intuitive idea of what is meant by a tangent to a curve. A line drawn through a point A of a smooth curve usually cuts the curve again in one or more points, though there may be exceptions as in line (4) of Fig. 39. If the line is rotated about A, it must sooner or later cut the curve somewhere, and for some positions of the line one of the points of intersection (which we will call B) will be near to A, as in the line (1) of the figure. If the line is rotated from this position, the point B can be made to move along the curve towards A until a position of the line is reached in which B is no longer distinct from A, as in line (2). The line is then said to be a tangent to the curve at A, or to **touch** the curve at A. Usually, any further rotation of the line will make B reappear on the other side of A as in line (3), though cases can occur (when the equation of the curve is of the third degree or higher) in which the conditions of line (4) arise immediately after tangency.

In this explanation we have started with a variable line through A, and from it have obtained a variable point B on the curve. This is perhaps the easiest way of thinking about a tangent and we shall consider it further in 6.6, but for our immediate purpose it is simpler if, having been given A, we start with a variable point B on the curve and find what happens to the gradient of AB as B approaches A along the curve.

Let us, for example, take the ellipse

$$x^2+4y^2 = 25$$

and the point A $(3, 2)$ on it. We know that A is on the curve because

$$3^2+4.2^2 = 25. \quad . \quad . \quad . \quad \text{(i)}$$

If now B (x_1, y_1) is any point on the curve,

$$x_1^2+4y_1^2 = 25. \quad . \quad . \quad . \quad \text{(ii)}$$

The gradient of AB is $(y_1-2)/(x_1-3)$, and if we know the values of

x_1 and y_1 we can find the value of the gradient, but the expression $(y_1-2)/(x_1-3)$ can tell us nothing of what happens to the gradient as B approaches A *along the curve* unless we can contrive to make use of equation (ii): to substitute $x_1 = 3$, $y_1 = 2$ would merely give the meaningless fraction 0/0. But we find from (i) and (ii) by subtraction that

$$(x^2-3^2)+4(y_1^2-2^2) = 0;$$

$$\therefore (x_1-3)(x_1+3)+4(y_1-2)(y_1+2) = 0;$$

$$\therefore \frac{y_1-2}{x_1-3} = -\frac{x_1+3}{4(y_1+2)},$$ provided that neither of the factors by which we have divided is zero. Hence, if B is on the curve and is distinct from A, the gradient of AB is $-(x_1+3)/4(y_1+2)$.

Now there is no doubt about the value of this expression as B approaches A: it approaches the value $-(3+3)/4(2+2)$, or $-\frac{3}{8}$, as B approaches A in any manner whatever. We conclude that, as the line AB rotates about A, the point B becomes merged with A when the gradient of AB is $-\frac{3}{8}$. The tangent is thus the line through A with this gradient, and its equation is

$$3x+8y = 3.3+8.2$$

i.e. $$3x+8y = 25.$$

In mathematical language, what we have done is to find the **limit** of the gradient of AB as B **tends** to A along the curve; this is then the gradient of the tangent at A. The process is the same as that used in calculus to find dy/dx, and if you can differentiate the equation of the curve you can obtain the gradient of the tangent more quickly by that means.

Example 2. Find the equation of the tangent to the parabola $9y = 2x^2$ at the point at which $x = 3$.

As $y = 2x^2/9$, $dy/dx = 4x/9$, and so the gradient at the given point is 4/3. By substitution in the equation $9y = 2x^2$ we find that when $x = 3$, $y = 2$; the tangent is therefore the line through (3, 2) with gradient 4/3, namely $4x-3y = 6$.

Example 3. Find the equation of the tangent to the conic $2x^2-xy+y-8 = 0$ at the point (3, 5).

First Method, using calculus.

Differentiating the equation, we have

$$4x-y-x\frac{dy}{dx}+\frac{dy}{dx} = 0,$$

$$\therefore \frac{dy}{dx} = \frac{4x-y}{x-1}.$$

At the required point the gradient is therefore

$$\frac{4.3-5}{3-1} = \frac{7}{2},$$

and the tangent is $7x-2y = 7.3-2.5$

i.e. $7x-2y = 11.$

Second Method

As the coefficients of xy and y are odd, we shall avoid fractions by multiplying the equation through by 2. Let A be (3, 5), and let $B(x, y)$ be any point on the conic. Then

$$4x^2-2xy+2y-16 = 0 \quad . \quad . \quad . \quad \text{(i)}$$

Also A lies on the curve because

$$4.3^2-2.3.5+2.5-16 = 0 \quad . \quad . \quad . \quad \text{(ii)}$$

Notice that we do not simplify (ii).

The gradient of AB is $(y-5)/(x-3)$. Now from (i) and (ii), by subtraction,

$$4(x^2-3^2)-2(xy-3.5)+2(y-5) = 0,$$

and this we rewrite

$$4(x-3)(x+3)-\{(x-3)(y+5)+(x+3)(y-5)\}+2(y-5) = 0,$$

and then $(x-3)\{4(x+3)-(y+5)\}-(y-5)\{(x+3)-2\} = 0,$

from which $\frac{y-5}{x-3} = \frac{4x-y+7}{x+1}.$

The gradient of AB is therefore $\frac{4x-y+7}{x+1}$. Letting x tend to 3 (which we write $x \to 3$) and $y \to 5$, we find that the gradient of the tangent at A is $\frac{4.3-5+7}{3+1}$, or $\frac{7}{2}$, and the equation of the tangent is $7x-2y = 11$ as before.

The conic used in this example, though outside the classification of 6.3, is actually a hyperbola and was chosen in order to show how the method deals with square terms, product terms and first-degree terms.

When the point of contact of the required tangent is not given numerically, the equation of the tangent can be written in several ways, of which one is to be preferred as being easier to remember and also more useful. The most general case will illustrate this.

Example 4. Find the equation of the tangent at the point (x', y') to the conic

$$ax^2+2hxy+by^2+2gx+2fy+c = 0. \quad \text{(12)}$$

As P_1 (x_1, y_1) lies on the conic,

$$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c = 0 \quad . \quad . \quad (13)$$

We can find the gradient by either of the methods of Example 3; for brevity calculus will be used, but if you cannot follow the working you should work through the other method yourself in order to arrive at the same result.

Differentiating (12), we have

$$2ax+2hy+2hx\frac{dy}{dx}+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx} = 0,$$

from which $\dfrac{dy}{dx} = -\dfrac{ax+hy+g}{hx+by+f}$.

The equation of the tangent at P_1 is therefore

$(ax_1+hy_1+g)x+(hx_1+by_1+f)y=(ax_1+hy_1+g)x_1+(hx_1+by_1+f)y_1$.

But by virtue of (13), the R.H.S. is equal to $-(gx_1+fy_1+c)$; hence the equation of the tangent can be written

$$(\boldsymbol{ax_1+hy_1+g})\boldsymbol{x}+(\boldsymbol{hx_1+by_1+f})\boldsymbol{y}+\boldsymbol{gx_1+fy_1+c} = 0 \quad (14)$$

This result is sufficiently important to be remembered. Rewriting it, we obtain

$$ax_1x+h(xy_1+x_1y)+by_1y+g(x+x_1)+f(y+y_1)+c = 0.$$

A comparison of this with the equation of the conic gives the following rule.

To obtain the equation of the tangent at (x_1, y_1) to a conic, in the equation of the conic replace

$$\begin{aligned} x^2 &\text{ by } x_1x, \\ 2xy &\text{ by } x_1y+xy_1, \\ y^2 &\text{ by } y_1y, \\ 2x &\text{ by } x+x_1, \\ 2y &\text{ by } y+y_1, \end{aligned}$$

and leave the constant term unaltered.

By this means the equation of the tangent in Example 1 can be written down at sight as $2x+6y = 40$, which reduces to the result obtained there; or the tangent at P_1 to the hyperbola $x^2/a^2-y^2/b^2 = 1$ can at once be given as $\dfrac{x_1x}{a^2}-\dfrac{y_1y}{b^2} = 1$. One warning is necessary, however: if the equation of a tangent is required in the course of solving a problem, or if an examination question asks you to *write down* the equation of a tangent, it is sufficient to make use of the rule given; but if you are asked to *find* or *obtain* the equation, it should be worked out as in one of the examples above, or by some other method.

from which $x = -7$, $y = 11$. The line therefore always goes through $(-7, 11)$.

In general it may be said that, if the coefficients in the equation of a line are linear functions of a parameter, the line passes through a fixed point for all values of the parameter. This will be useful later.

EXERCISE 5B

Write down the gradients of the following lines:

1. (i) $x+2y+3=0$; (ii) $3x-7y+4=0$; (iii) $3x=4y+6$; (iv) $y=2x+5$; (v) $y=2$.
2. (i) $4x-6y+9=0$; (ii) $3x+y-17=0$; (iii) $2x=6y-3$; (iv) $y=7-3x$; (v) $2y+3=0$.

Write down the equation of the line:

3. Through (3, 5) parallel to (i) $2x-3y+7=0$; (ii) $2x-5=0$; (iii) $5y+1=0$.
4. Through $(-2\frac{1}{2}, 1)$ parallel to (i) $5x+y-8=0$; (ii) $3y-4=0$; (iii) $x+1=0$.
5. Through (1, −4) perpendicular to (i) $2x-5y+8=0$; (ii) $3x-4=0$; (iii) $2y=7$.
6. Through $(0, \frac{4}{3})$ perpendicular to (i) $x+3y-1=0$; (ii) $2y=1$; (iii) $x+5=0$.
7. Through $(at^2, 2at)$ at right angles to $x-ty+at^2=0$.
8. Through $(ct, c/t)$ at right angles to $x+t^2y-2ct=0$.
9. Find the coordinates of the points where the lines through (3, −2) parallel and perpendicular to $4x-8y+1=0$ cut the line $x-5=0$. [You do not need to find the equations of the two lines.]
10. Find the points in which the lines through (−2, −3) parallel and perpendicular to $2x+5y-9=0$ cut the line $y-7=0$.
11. Find the acute angles between the following pairs of lines. (i) $x-y+2=0$, $7x-y+21=0$; (ii) $(2+\sqrt{3})x+y=0$, $x-y-1=0$.
12. Find the obtuse angles between the following pairs of lines. (i) $3x+y+4=0$, $x+2y-5=0$; (ii) $x+(2+\sqrt{3})y=0$, $x+y-2=0$.
13. Show that $x+y-2=0$ and $7x-y+3=0$ make equal angles with $x-3y+5=0$.
14. Show that $2x-y+4=0$, $11x-2y-7=0$, and $x-3y+5=0$ form an isosceles triangle with the third line as base.
15. Find x_1 and y_1 if $(7x_1-1)x+(y_1-12)y-(x_1+12y_1-12)=0$ is the same line as $4x-2y-3=0$.
16. Find x_1 and y_1 if $(5y_1-1)x+(5x_1+3)y-(x_1-3y_1-2)=0$ is the same line as $6x-3y+5=0$.
17. Draw, in the same diagram, the lines represented by the equation $2x-y-2+k(x+y-7)=0$ when k has the values $-1, 0, 1$.
18. Draw, in the same diagram, the lines represented by the equation $x+2y-11+k(x-y+4)=0$ when k has the values 0, 2, 4.
19. Verify that $3x+5y+1=0$, $4x+7y+1=0$, and $x+y+1=0$ all go through $(-2, 1)$, and find the value of λ for which $3x+5y+1+\lambda(4x+7y+1)=0$ reduces to the third line.
20. Verify that $2x+5y-3=0$, $6x+15y-4=0$, $14x+35y-1=0$ are all parallel, and find the value of λ for which $14x+35y-1+\lambda(6x+15y-4)=0$ reduces to the first line.

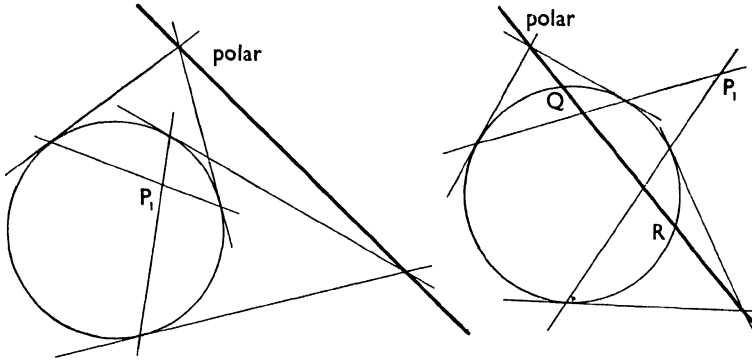


Fig. 40

whole line is called the polar of P_1. The rules of 6.5 thus enable us to write down at sight the equation of the polar of any point wo any conic.

If the coordinates of P_1 are substituted in (14), the result is (13), the condition for P_1 to lie on the conic (12). Hence **if a point lies on its polar wo a conic, it also lies on the conic, and its polar is the tangent at it to the conic.**

Example 5. Show that $x+2y-13=0$ is a tangent to the circle $x^2+y^2+2x+6y-70=0$, and find the point of contact.

The polar of P_1 wo the circle is

$$(x_1+1)x+(y_1+3)y+x_1+3y_1-70=0 \quad . \quad . \quad (i)$$

and, by (6) of 5.3, this is the same as the given line if

$$\frac{x_1+1}{1}=\frac{y_1+3}{2}=\frac{x_1+3y_1-70}{-13} \quad . \quad . \quad . \quad (ii)$$

ıickest way to find x_1 and y_1 from (ii) is to use the theorem in

ıt, if $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, these fractions are equal to $\frac{pa+qc+re}{pb+qd+rf}$

r are any numbers whatever. Thus the three fractions of

ı to

$$\frac{+1)+3(y_1+3)-(x_1+3y_1-70)}{1+3(2)-(-13)}=\frac{80}{20}=4.$$

4, and $y_1+3=2.4$;

$$\therefore x_1=3, \quad y_1=5.$$

t $x+2y-13=0$ is the polar of (3, 5). All that

remains is to verify that (3, 5) lies on its polar—i.e. that $3+10-13=0$—and it follows that the given line is a tangent to the circle, the point of contact being (3, 5)

An alternative method will be found in 6.6, Example 1.

Exercise 6C

In Nos. 1, 2, obtain the equation of the tangent to the given curve at the given point.

1. (i) $xy+4x-y=5$, (2, −3); (ii) $2x^2-y^2+6x+4y+4=0$, (−1, 4); (iii) $y^2-2x-2y-3=0$, (−2, 1).
2. (i) $x^2+4y^2-4x+8y-1=0$, (−1, −1); (ii) $xy+x-3y-1=0$, (2, 1); (iii) $x^2-5x-3y+4=0$, (4, 0).

In Nos. 3, 4, write down the equation of the tangent to the given curve at the given point.

3. (i) $xy+3=0$, (−3, 1); (ii) $y=2x^2$, (−2, 8); (iii) $x^2+y^2-10x+8y+12=0$, (0, −2); (iv) $x^2/25+y^2/100=1$, (3, 8); (v) $x^2-y^2+3x-2y-4=0$, (1, 0).
4. (i) $xy=24$, (−3, −8); (ii) $y=3x^2-11x-15$, (5, 5); (iii) $x^2/25-y^2/9+1=0$, $(\frac{15}{4}, \frac{15}{4})$; (iv) $x^2+y^2-2x+4y-24=0$, (−1, 3); (v) $x^2+2y^2-5x+4=0$, (3, −1).

In Nos. 5–8, use the rule for writing down the equation of the tangent to the given curve at the given point; find what the given equation represents, and explain the result.

5. $x^2-4y^2-4x+8y=0$, (0, 0) and (6, 3).
6. $xy-x+3y=3$, (−3, 0) and (−3, 3).
7. $xy+3x-2y=6$, (2, −3).
8. $4x^2+y^2-8x+4y+8=0$, (1, −2).

In Nos. 9–12, show that the given line is a tangent to the given conic, and find the point of contact.

9. (i) $3x+4y-24=0$, $xy=12$; (ii) $x=2y$, $x^2+y^2-2x-6y+5=0$.
10. (i) $2x+y=9$, $xy+x-y=9$; (ii) $x+y=1$, $x^2+y^2-8x+2y+15=0$.
11. (i) $3x-y=9$, $x^2-4y=0$; (ii) $x-2y+2=0$, $x^2-4y^2+3x+2y+2=0$.
12. (i) $x+4y+4=0$, $y^2=x$; (ii) $x+1=0$, $x^2+4y^2+4x+3=0$.
13. A, B, C are the points (−5, 0), (4, −3), (3, 4). Verify that the circumcircle of ABC is $x^2+y^2=25$, and write down the equations of the tangents to it at A, B, and C. If they cut the opposite sides of the triangle at L, M, N, find the coordinates of these points and show that they are collinear.
14. Show that the tangent at A (14, 8) to the circle $x^2+y^2-36x-2y+260=0$ passes through the origin. If the circle cuts the x-axis at B, C, verify that $OA^2=OB.OC$.
15. Verify that A (8, 6) lies on the parabola $y^2=4(x+1)$. If the tangent at A cuts the x-axis at B, show that $BO=OA$.
16. The tangent at any point A on the parabola $y^2=kx$ cuts the x-axis at B. Show that the mid-point of AB lies on the y-axis.
17. Find the equation of the line through (4, 0) perpendicular to the tangent at (9, 12) to the parabola $y^2=16x$, and show that it meets that tangent at a point on the tangent at the vertex of the parabola.
18. Write down the equations of the tangents to the parabola $y^2=4(x+1)$ at its vertex and the point (15, 8). Without finding the coordinates of the point at which these tangents meet, find the equation of the line joining this point to the origin and show that it is perpendicular to the second tangent.

19. Write down the equation of the tangent at A (5, 3) to the rectangular hyperbola $xy = 15$. If it cuts the asymptotes at B and C, show that $BA = AC$.

20. Write down the equation of the tangent at A (5, 2) to the hyperbola $x^2-4y^2 = 9$. If it cuts the asymptotes at B and C, show that $BA = AC$.

21. Verify that the point $(t+1/t,\ t-1/t)$ lies on the hyperbola $x^2-y^2 = 4$, and write down the equation of the tangent at it. If the tangent cuts the asymptotes at A and B, show that $OA.OB = 8$.

22. A variable tangent to a rectangular hyperbola whose centre is O cuts the asymptotes at A and B. Prove that the area of $\Delta\ OAB$ is constant.

23. Find the equation of the tangent at A (1, 1) to the ellipse $16x^2+9y^2 = 25$. Find the values of p and q if the curve $9y = x^3-4x^2+px+q$ passes through A and has the same tangent there as the ellipse. [O.L.]

24. Find the equation of the tangent to the hyperbola $xy = 1$ at the point $(2, \frac{1}{2})$. Find the numerical values of a and b which make the curve $4y = x^2+ax+b$ pass through the point $(2, \frac{1}{2})$ and touch the hyperbola at that point. [O.L.]

25. Write down the equation of the tangent to the ellipse $x^2/25+y^2/16 = 1$ at $(3, 3\frac{1}{5})$. Find the foot of the perpendicular from (3, 0) to this tangent, and verify that it lies on the circle $x^2+y^2 = 25$.

26. A, B are the points $(\pm 3, 0)$, and C is $(3, 3\frac{1}{5})$. Show that AC and BC are equally inclined to the tangent at C to the ellipse $x^2/25+y^2/16 = 1$.

27. Draw a circle and mark a point P inside or outside it; through P draw any line cutting the circle at A_1 and B_1; draw the tangents at A_1, B_1, and let them meet at Q_1; repeat with a second line through P, a third, and so on; finally, verify that Q_1, Q_2, Q_3 . . . are in a straight line.

28. Find the equation of the locus of a point P whose polar wo the general conic passes through a fixed point P_1.

Describe the locus in words.

29. Prove that, if a point is such that its polar wo an ellipse is at right angles to the line joining it to the centre, it must lie on one of the axes.

30. Show that the polars of a point wo two conjugate hyperbolas are parallel and equidistant from the centre.

31. The polar of a point P wo a parabola meets the axis at Q. Prove that the distance of Q from the vertex is equal to the distance of P from the tangent at the vertex.

32. Prove that, if parabolas $y^2 = 4ax$ are drawn for different values of a, the polars of P_1 wo all of them pass through a fixed point on the x-axis, and find its coordinates.

33. Show that the polars of a fixed point, wo the circles given by the equation $x^2+y^2+2gx+2fy+c+\lambda(x^2+y^2+2g'x+2f'y+c') = 0$ for different values of λ, all go through another fixed point.

34. Show that the polars of (7, 1), wo the circles given by the equation $x^2+y^2-50+\lambda(3x-5y-16) = 0$ for different values of λ, all go through (7, 1) itself. Explain why.

35. Prove that, if the polars of P and Q wo a circle meet at R, the line joining R to the centre is perpendicular to PQ.

36. If Q lies on the polar of P wo a circle, prove that PQ^2 is equal to the sum of the powers of P and Q wo the circle.

In Nos. 37–40, find the locus of P if PP_0 is perpendicular to the polar of P wo the given conic, showing that it is a rectangular hyperbola. It is called the **hyperbola of Apollonius** for the point P_0 wo the given conic.

37. $x^2/a^2+y^2/b^2 = 1$.

38. $x^2/a^2-y^2/b^2 = 1$.

39. $xy = c^2$.

40. $y^2 = 4ax$.

6.6. Intersection of Line and Conic

The coordinates of the points in which a line meets a conic satisfy both the equation of the line and that of the conic; they are therefore found by treating these as simultaneous equations and solving them in the way which is explained in algebra books. Our present purpose is to look at two special cases.

In 6.4, to obtain the gradient of the tangent at A (3, 2) to the ellipse

$$x^2+4y^2 = 25 \quad . \quad . \quad . \quad . \quad \text{(i)}$$

we started with any other point B on the ellipse and found the limit of the gradient of AB as B tends to A along the curve. Let us now approach the problem a different way, by starting with a line through A and finding the coordinates of the other point B in which it meets the ellipse.

Let the gradient of the line be γ, so that its equation can be written

$$y = 2+\gamma(x-3) \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Substituting this expression for y in (i) we obtain, as the equation giving the x-coordinates of A and B,

$$x^2+4\{2+\gamma(x-3)\}^2 = 25,$$

which can be written

$$(1+4\gamma^2)x^2+8\gamma(2-3\gamma)x-3(3+16\gamma-12\gamma^2) = 0 \quad . \quad \text{(iii)}$$

This equation looks unpromising but, since we know that one of its roots is the x-coordinate of A, namely 3, it follows that $x-3$ must be one factor of the L.H.S. Using this hint we can rewrite (iii) as $(x-3)\{\quad x+\quad\} = 0$, where in the second bracket the coefficient of x and the constant term have to be filled in. A comparison with the coefficient of x^2 in (iii) shows that the required coefficient of x is $1+4\gamma^2$; and the required constant term is $3+16\gamma-12\gamma^2$ in order to agree with the constant term in (iii). We do not need to use the x-term of (iii) except as a check.

Our new version of (iii) is now

$$(x-3)\{(1+4\gamma^2)x+(3+16\gamma-12\gamma^2)\} = 0,$$

from which we see that the x-coordinate of B is $-\dfrac{3+16\gamma-12\gamma^2}{1+4\gamma^2}$, and this can be written $3-\dfrac{2(3+8\gamma)}{1+4\gamma^2}$. Substituting in (ii), we find that the y-coordinate of B is $2-\dfrac{2\gamma(3+8\gamma)}{1+4\gamma^2}$. The line (ii) thus meets the ellipse (i) in the points A (3, 2) and $B\left(3-\dfrac{2(3+8\gamma)}{1+4\gamma^2},\ 2-\dfrac{2\gamma(3+8\gamma)}{1+4\gamma^2}\right)$.

Now if we rotate the line until $3+8\gamma = 0$, making $\gamma = -3/8$, B will coincide with A and the line will therefore become a tangent. The equation (iii) then reduces to $(x-3)^2 = 0$ and is said to have 3 as a **double root,** or to have **two equal roots.** To draw attention to the double root it is usual to say that the line and ellipse then meet in **two coincident points,** or that they **meet twice** at A. It may be helpful to the imagination if you think of two coincident points as being two points very close together, but you must bear in mind that this is not literally true: the phrase is no more than a geometrical way of stating the occurrence of equal roots in the equation giving x or y.

A little thought, aided by experimental examples if necessary, will show that the connection which we have found between tangents and equal roots is quite general, and we can say that **if a line is a tangent to a conic, it meets the conic in two coincident points.**

Will the converse of this be true? If, on eliminating one of the variables between the equations of a line and of a conic, we obtain an equation with two equal roots, can we conclude that the line is a tangent to the conic? The answer is that **the converse is true provided the conic is not degenerate** but is a genuine curve.

For let the line and the conic meet twice at A. Suppose we rotate the line about A: this may be expected to alter the equation giving the x-coordinates of the points of intersection, but one root will continue to be the x-coordinate of A, since A is still a point of intersection. If, as seems likely, the other root is altered by the rotation, the new line meets the conic at two points A and B, and B can be brought into coincidence with A by rotating the line back to its original position. The line is then a tangent, by what we said in 6.4. But it may happen that, when the line is rotated, there is no change in the roots of the equation, and the line continues to meet the conic at A only. The only ways in which this can happen are (a) that the conic consists only of the point A: it is a circle of zero radius or an ellipse with both axes zero; or (b) that it consists of two lines meeting at A. These two cases are illustrated in Exercise 6d, Nos. 15, 16; they are easily remembered if you think of the point-circle or ellipse as being a very small closed curve, the pair of lines as being a hyperbola with its vertices very close to the centre, and the line as actually being a tangent to these curves.

Example 1. Show that $2x+9y = 26$ is a tangent to the ellipse $x^2+9y^2 = 52$ and find the coordinates of the point of contact.

Eliminating y we obtain $x^2+9\left(\frac{26-2x}{9}\right)^2 = 52$;

$\therefore\ 9x^2+26^2-104x+4x^2 = 52.9,$

$\therefore\ x^2-8x+16 = 0;$

$\therefore\ (x-4)^2 = 0.$

As this has a double root, the line is a tangent to the ellipse and touches it where $x = 4$.

Substituting $x = 4$ in the equation of the line we find $y = 2$. Hence the point of contact is (4, 2).

(An alternative method was given in 6.5, Example 5.)

Example 2. Find the equations of the tangents from the origin to the circle $x^2+y^2-6x-2y+2 = 0$.

First Method. Any line through the origin is $y = \gamma x$. This meets the circle where $x^2+\gamma^2x^2-6x-2\gamma x+2 = 0,$

i.e. $(1+\gamma^2)x^2-2(3+\gamma)x+2 = 0.$

The line will be a tangent if this equation has equal roots. Now it is shown in algebra books that the equation $ax^2+2bx+c = 0$ has equal roots if $b^2 = ac$. Hence the line will be a tangent if

$$(3+\gamma)^2 = 2(1+\gamma^2),$$

i.e. if $$\gamma^2-6\gamma-7 = 0,$$

from which $$\gamma = 7 \text{ or } -1.$$

The tangents are therefore $y = 7x$ and $x+y = 0$.

Second Method. The points of contact of the required tangents lie on the polar of O, namely

$$3x+y-2 = 0.$$

This meets the circle where

$$x^2+(2-3x)^2-6x-2(2-3x)+2 = 0,$$

i.e. $$10x^2-12x+2 = 0,$$

from which $$x = 1 \text{ or } \tfrac{1}{5}.$$

Substituting in the equation of the polar we find that the two points are $(1, -1)$ and $(\frac{1}{5}, \frac{7}{5})$; the tangents at these points can be written down and are found to be $y = 7x$ and $x+y = 0$. Alternatively we can write down the equations of the lines joining the two points to O.

The other special case of intersection, looked at algebraically, has a slight resemblance to that in which coincident points occur. From a geometrical standpoint, however, the two are very different, as the next example shows.

Example 3. Find the points of intersection of

$$x^2-4y^2 = 36 \quad . \quad . \quad . \quad . \quad \text{(i)}$$

and

$$x+2y = 2. \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

From (ii), $2y = 2-x; \quad . \quad . \quad . \quad . \quad \text{(iii)}$

substituting in (i), $x^2-(2-x)^2 = 36;$

$$\therefore \ -4+4x = 36;$$

$$\therefore \quad x = 10.$$

Hence from (iii), $y = -4$.

Thus once again the equations have only one solution, (10, −4), but in this case there is no question of double roots.

The line is not a tangent. In fact (i) represents a hyperbola whose asymptotes are $x+2y = 0$ and $x-2y = 0$, and (ii) represents a line parallel to one of these asymptotes. Such a line meets the hyperbola in only one point. If you sketch the curve and line you will more easily see the reason for this: a point which moves round one branch of the curve continually approaches one asymptote and recedes from the other; so, if it meets a line parallel to an asymptote once, it cannot meet it a second time; further, the other branch is on the other side of the asymptote and hence cannot cut the line at all.

The example leads us to ask in what circumstances the equation obtained by eliminating one variable between the equations of a line and a conic is not a quadratic but of the first degree only. Once again the answer can be found by rotating the line about the single point in which it meets the curve. Thus in Example 3 we could replace the line (ii) by another through (10, −4) having gradient γ, and on eliminating y obtain

$$(1-4\gamma^2)x^2+16\gamma(2+5\gamma)x-20(5+16\gamma+20\gamma^2) = 0.$$

The important feature of this equation is that the coefficient of x^2 is zero for certain values of γ (namely $\pm\frac{1}{2}$), and that the coefficient of x is *not* zero for these values of γ.

Now it is shown in algebra books that, if the roots of the equation $ax^2+2bx+c = 0$ are x_1 and x_2,

$$x_1+x_2 = -\frac{2b}{a}.$$

In the situation we are considering, as γ tends to certain values, $a \to 0$ but b does not. Hence $x_1+x_2 \to \pm\infty$. But one root remains constant (in the case given, $x = 10$) and so the numerical value of the other root increases without limit. Thus, **as the gradient**

of the line approaches that for which the coefficient of x^2 vanishes, one of the points of intersection tends to infinity.

Since the circle and ellipse are closed curves, the only conics to which this can happen are the parabola and hyperbola. The case of the hyperbola has been illustrated in Example 3: the line was parallel to an asymptote and the least degree of rotation would have made it meet the curve again at a distant point. One of the points of intersection of a line and a parabola is very distant if the line is nearly parallel to the axis of the curve, while if it is exactly parallel it meets the parabola in one point only: this can be illustrated by the parabola $y^2 = x$ and the line $y = 2$.

6.7. Mid-points of Chords

The property of the roots of a quadratic quoted above is sometimes useful in problems about the mid-point of a chord of a conic.

Example 1. Find the coordinates of the mid-point of the chord $2x-y+3 = 0$ of the ellipse $4x^2+9y^2 = 36$.

Since the equation of the chord can be written $y = 2x+3$, it meets the ellipse where

$$4x^2+9(2x+3)^2 = 36,$$

i.e.
$$40x^2+108x+45 = 0.$$

If the roots of this are x_1, x_2, the x-coordinate of the mid-point of the chord is $\frac{1}{2}(x_1+x_2)$.

But $x_1+x_2 = -108/40 = -27/10$;

$\therefore$ the x-coordinate of the mid-point is $-27/20$.

By substitution in the equation of the chord we find $y = 3/10$.

Hence the mid-point is $\left(-\frac{27}{20}, \frac{3}{10}\right)$.

Sometimes the reverse process is needed: in order to solve a problem it may be necessary to find the equation of the chord of a given conic which has a given mid-point. This can be easily written down with the aid of the following theorem.

Theorem. The polar of the mid-point of a chord of a conic is parallel to that chord.

To be perfectly general, we shall again suppose that the equation of the conic is

$$ax^2+2hxy+by^2+2gx+2fy+c = 0,$$

and that the ends of the chord are P_1, P_2, so that

$$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c = 0$$

and
$$ax_2^2+2hx_2y_2+by_2^2+2gx_2+2fy_2+c = 0.$$

Then by subtraction,

$$a(x_1^2-x_2^2)+2h(x_1y_1-x_2y_2)+b(y_1^2-y_2^2)$$
$$+2g(x_1-x_2)+2f(y_1-y_2)=0,$$

and this can be written

$$(x_1-x_2)\{a(x_1+x_2)+h(y_1+y_2)+2g\}$$
$$+(y_1-y_2)\{h(x_1+x_2)+b(y_1+y_2)+2f\}=0.$$

Hence

$$\frac{y_1-y_2}{x_1-x_2}=-\frac{a(x_1+x_2)+h(y_1+y_2)+2g}{h(x_1+x_2)+b(y_1+y_2)+2f}=-\frac{ax_0+hy_0+g}{hx_0+by_0+f},$$

where P_0 (x_0, y_0) is the mid-point of P_1P_2. But the L.H.S. of this equation represents the gradient of P_1P_2, while the R.H.S. represents, by (14), the gradient of the polar of P_0. The polar of P_0 wo the conic is thus parallel to the chord of which P_0 is the mid-point.

Example 2. A line meets the hyperbola $xy = c^2$ at A,B, and meets the coordinate axes at X,Y. Show that AB and XY have the same mid-point.

This can be proved in various ways, but to make use of the theorem just proved we can start by letting the mid-point of AB be $P_1(x_1, y_1)$. Then, by the theorem, the line AB passes through P_1 and is parallel to the polar of P_1, namely $y_1x+x_1y = 2c^2$. The equation of AB is therefore $y_1x+x_1y = 2x_1y_1$. This meets the axes at $(2x_1, 0)$ and $(0, 2y_1)$, these being therefore X and Y. The mid-point of XY is thus (x_1, y_1), the same as the mid-point of AB.

EXERCISE 6D

In Nos. 1, 2, solve the equations where possible, and interpret them geometrically.

1. (i) $x^2+2y^2+3x-2y = 10$, $2x+3y = 7$;
   (ii) $y^2+4x-2y = 19$, $x+y = 7$;
   (iii) $4x^2-25y^2+30x = 9$, $2x-5y = 7$;
   (iv) $x^2-2y^2 = 18$, $x+y = 1$.
2. (i) $x^2+y^2-6x+8y = 0$, $x+2y = 7$;
   (ii) $xy-3x+3y = 3$, $x-3y+3 = 0$;
   (iii) $x^2-y^2+4x = 1$, $3x+2y+1 = 0$;
   (iv) $x^2-4y^2+8y = 1$, $x+2y = 3$.

3–6. Do Exercise 6c, Nos. 9–12, by the method of

In Nos. 7–12, show that the line is a tangent to the conic for all values of t or θ, and find the coordinates of the point of contact.

7. $x-ty+at^2 = 0$, $y^2 = 4ax$.
8. $x+t^2y-2ct = 0$, $xy = c^2$.
9. $(1-t^2)x+2ty = a(1+t^2)$, $x^2+y^2 = a^2$.
10. $(t^2+1)x-(t^2-1)y = 2at$, $x^2-y^2 = a^2$.
11. $bx\cos\theta + ay\sin\theta = ab$, $b^2x^2+a^2y^2 = a^2b^2$.
12. $bx\sec\theta - ay\tan\theta = ab$, $b^2x^2-a^2y^2 = a^2b^2$.

13. Find the value of r if the line $3x+4y = 10$ is a tangent to the circle $x^2+y^2 = r^2$. What does this tell you about the distance of the origin from the line?

14 Show that the equation of any circle through (1, 0) and (2, 0) can be written $x^2+y^2-3x+2fy+2 = 0$, and find the values of f for which the circle touches the line $x+y = 3$.

15. By solving the simultaneous equations

$$4x^2-y^2+4x+2y = 0, \quad 2yx-2y+y+2 = 0,$$

show that the loci represented by them meet in two coincident points when $y \neq \pm 2$, but that when $y = \pm 2$ any point satisfying the second equation also satisfies the first.

Show that the first equation represents a line-pair, find the point of intersection of its two members, and verify that its coordinates satisfy the second equation.

16. Solve the simultaneous equations

$$9x^2+4y^2+12x-12y+13 = 0, \quad 6yx-6y+4y+9 = 0,$$

showing that the loci represented by them meet in two coincident points for all values of y.

Show that the point of intersection is the only point which satisfies the first equation.

17. Find the equations of the tangents from (1, 7) to the circle $x^2+y^2 = 25$. What is the angle between the tangents?

18. Two tangents can be drawn from the point $(-2, 0)$ to the parabola $y^2 = 8x$. Find their equations and show that they are perpendicular. [O.L.]

19. Two tangents can be drawn from the origin to the hyperbola $4x^2-9y^2+24x+36y = 36$. Use both methods of Example 2 of 6.6 to find them, and explain the result.

20. Prove that the y-axis is a tangent to the circle $x^2+y^2-4x-6y+9 = 0$ and find the coordinates of the point of contact of the other tangent from the origin.

21. Find the values of t for which the line $x-ty+at^2 = 0$ is a tangent to the circle $5(x^2+y^2) = 16a^2$.

Hence, using No. 7, find the equations of the common tangents of the circle and the parabola $y^2 = 4ax$.

22. Show that the line $t^2(x-1)+t(y-2)+1 = 0$ touches the parabola $y^2-4x-4y+8 = 0$ for all values of t. Find the common tangent of this parabola and $y^2 = 4x$. Draw a sketch showing the two curves and the common tangent.

In Nos. 23–26, find the relation between l, m, n, if the line $lx+my+n = 0$ is a tangent to the given curve.

23. $y^2 = 4ax$. 24. $xy = c^2$.

25. $x^2+y^2 = a^2$. 26. $x^2/a^2-y^2/b^2 = 1$.

27. A circle and a rectangular hyperbola have the same centre. Show that the locus of points whose polars wo the hyperbola touch the circle is a concentric circle.

28. A circle touches the asymptotes of a rectangular hyperbola. Show that the locus of points whose polars wo the hyperbola touch the circle is another rectangular hyperbola.

In Nos. 29, 30, find the coordinates of the mid-point of the given chord of the given conic.

29. $4x-3y = 5$, $4x^2-y^2 = 4$.

30. $2x+5y = 3$, $y^2 = 4x+5$.

31. Show that mid-points of chords of the ellipse $x^2/a^2+y^2/b^2 = 1$ which are parallel to $lx/a+mv/b = 0$ lie on the line $x/al-v/bm = 0$.

32. Show that mid-points of chords of the hyperbola $x^2/a^2-y^2/b^2 = 1$ which are parallel to $lx/a+my/b = 0$ lie on the line $x/al+y/bm = 0$.

33. A chord AB of a hyperbola cuts the conjugate hyperbola at C, D. Show that AB and CD have the same mid-point.

34. Two rectangular hyperbolas have the same centre, and the axes of each are the asymptotes of the other. If chords of the two hyperbolas bisect each other, prove that they are perpendicular.

In Nos. 35–38, find the locus of mid-points of chords of the given conic which pass through the fixed point (h, k).

35. The circle $x^2+y^2 = a^2$.

36. The parabola $y^2 = 4ax$.

37. The hyperbola $x^2/a^2-y^2/b^2 = 1$.

38. The rectangular hyperbola $xy = c^2$.

39. Find the locus of mid-points of chords of the parabola $y^2 = 4x$ which touch the parabola $y^2+8x = 0$.

40. A variable tangent to the parabola $y^2+2x = 0$ meets the axes at X and Y. Find the locus of the mid-point of XY.

41. Find the radius of the circle with centre on the y-axis which meets the ellipse $x^2/a^2+y^2/b^2 = 1$ in four coincident points at B $(0, b)$. This circle is called the circle of curvature at B. [Notice that, since for each value of y there are two values of x, only two equal roots are required from the equation for y.]

42. Do No. 41 for the hyperbola $x^2/a^2-y^2/b^2+1 = 0$.

43. A line through the origin is a tangent to the cubic curve $8y = x^3-4x^2$ at a point P on it. Find the equation of the line and the coordinates of P. [Proceed as in the first method of 6.6, Example 2, but reject the zero root.]

44. Find the values of a for which the line $x = y$ is a tangent to the curve $y^2 = x(x+4)(x+a)$, giving the coordinates of the point of contact in each case.

Exercise 6M

1. Find the equations of the circles that touch the circle $x^2+y^2 = 4$ and the lines $y = 0$ and $x = 7$.

2. Verify that $A(-1, 3)$ lies on the circle $x^2+y^2-5x+8y = 39$.

If $ABCD$ is a square inscribed in the circle, find the coordinates of B, C, D.

3. If $\frac{1}{u}+\frac{1}{v} = \frac{1}{f}$, show that the line joining $(u, 0)$ and $(0, v)$ passes through (f, f).

4. A is the point (4, 10); a variable line through A cuts the axes at X and Y. Find the locus of the centre of the circle OXY.

5. A and B are two points on the parabola $y^2 = 16x$ and C is the mid-point of AB. The equation of the line AB is $4x-y-24 = 0$. Find the coordinates of C.

The line CD is drawn parallel to the axis of the parabola to cut the parabola at the point D. Find the equation of the tangent at D and show that it is parallel to AB. [O.L.]

6. Show that the locus of the mid-point of a line joining a fixed point A to a variable point on a hyperbola is a hyperbola having asymptotes parallel to those of the given curve and having its centre midway between A and that of the given curve.

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7. Show that the circles $x^2+y^2-8x-6y+16 = 0$ and $x^2+y^2 = 16$ cut at right angles, and find the length of their common chord.

8. A square is inscribed in the circle $x^2+y^2-2x+4y-93 = 0$ with its sides parallel to the axes of coordinates. Find the coordinates of its four vertices.

Find also the equation of the circle inscribed in the square. [A.L.]

9. The equations of the sides of a triangle are $x+ay-a^2 = 0$, $x+by-b^2 = 0$, $x+cy-c^2 = 0$; find the coordinates of the orthocentre.

10. A and B are two points, one on each of the axes. If the line AB moves so that the area of $\triangle OAB$ is always $2c^2$, find the locus of the mid-point P of AB, and show that AB touches the locus at P.

11. Find the points, other than the origin, where the perpendicular lines $y = \gamma x$, $y = -x/\gamma$ meet the parabola $y^2 = 4ax$. Show that the tangents at these points meet on the line $x+4a = 0$.

12. L, M are the points $(-a, 0)$, $(a, 0)$; find the locus of a point P which moves so that the sum of the angles which PL, PM make with the positive direction of the x-axis is 90°.

What would be the locus if P moved so that $\angle PLM+\angle PML = 90°$?

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13. A point moves so that the tangent from it to a fixed circle is equal to the difference between its distances from two perpendicular diameters. Show that its locus is a rectangular hyperbola and draw a sketch showing the locus in relation to the given circle.

14. The points (2, 3) and $(-3, -4)$ are opposite corners of a square. Find the equations of its sides.

15. A and B are variable points, one on each axis, and move so that $1/OA+1/OB = 1/c$. Find the locus of the foot of the perpendicular from O to AB.

16. A line is drawn through $P\ (t, 1/t)$, at right angles to the tangent at P to the hyperbola $xy = 1$, and meets the lines $x = y$, $x+y = 0$ at Q, R. Show that P is the mid-point of QR.

17. Sketch the parabolas $y^2 = 32x$ and $x^2 = 108y$, and find the angles at which they intersect.

18. P is a variable point on the hyperbola $x^2/a^2-y^2/b^2 = 1$, and PN is the perpendicular from it to the transverse axis AA'. Prove that $PN^2/AN.A'N$ is constant for all positions of P.

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19. Two circles, each of diameter $3a$, touch one another. Prove that the locus of a point, which moves so that the sum of the four tangents from it to the circles is $10a$, is an ellipse, and find its semi-axes.

20. The circles $x^2+y^2+2x-3y+c = 0$ and $x^2+y^2+2gx+2fy+17 = 0$ touch at the point (1, 2). Find the values of g, f, c.

21. Find the locus of the foot of the perpendicular from O to AB if A and B move on the axes so that

$$1/OA^2+1/OB^2 = 1/c^2.$$

22. Show that a square can be drawn having one vertex at (6, −1) and all its sides touching the circle $x^2+y^2-2x-4y = 12$.

Find the other vertices.

23. Two perpendicular lines through the origin meet the rectangular hyperbola $xy+gx+fy = 0$ again at L and M; show that LM is perpendicular to the tangent at O.

24. A, A' are $(a, 0)$, $(-a, 0)$; P_0 is any point on the hyperbola $x^2/a^2-y^2/b^2 = 1$. Find (in terms of x_0, y_0) the coordinates of the points L, L' at which P_0A, P_0A' respectively meet the y-axis.

Find the coordinates of the point M in which the tangent at P_0 meets the y-axis.

Write down the coordinates of the mid-point of LL' and use the fact that P_0 lies on the hyperbola to show that this mid-point is M.

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25. Show that the expression $x_1^2+y_1^2+2gx_1+2fy_1+c$ represents the square of the tangent from the point (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+c = 0$.

Prove that the squares of tangents drawn from any point on the circle $x^2+y^2+2x-16y = 0$ to the two circles $x^2+y^2-10y+2\lambda(x-3y) = 0$, $x^2+y^2-10y+2\lambda'(x-3y) = 0$ are in the constant ratio $\lambda-1 : \lambda'-1$. [A.L.]

26. The lines $3x-4y+1 = 0$, $2x-y-1 = 0$ are sides of a parallelogram, and the point (6, 6) is a vertex. Find the equations of the diagonals. [A.L.]

27. A point moves so that its power wo a given circle is equal to the sum of the squares of its distances from two perpendicular tangents. Show that its locus is a straight line.

How is this line related to the chord of contact of the tangents?

28. A, B are the points (1, 2), (1, -2) on the parabola $y^2 = 4x$. If the perpendiculars from any point P on AB to the tangents at A and B meet them at Q and R, prove that QR is a tangent to the parabola.

29. Through a point P, a line is drawn parallel to an asymptote of a hyperbola. If it cuts the hyperbola at Q and the polar of P at R, show that Q is the mid-point of PR.

30. A, B are the points $(2a, 0)$, $(-2a, 0)$; find the locus of P if $AP.BP = OP^2$.

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31. A is the point (2, 0); Q is a variable point on the line $x = 2$ and P is taken on OQ such that $OQ.PQ = AQ^2$. Find the locus of P.

32. The sides of a parallelogram are parallel to $x+y = 0$ and $x-2y = 0$. If the ends of one diagonal are $(-1, 3)$ and (0, 1), find the equation of the other diagonal.

33. A, B are $(a, 0)$, $(0, b)$; a variable line parallel to AB meets Ox at C, Oy at D. Find the locus of the point of intersection of AD and BC.

34. P is any point on the tangent at (0, 4) to the ellipse $x^2/25+y^2/16 = 1$; S and S' are the points (3, 0), $(-3, 0)$. Prove that the tangent at P to the circle PSS' is also a tangent to the ellipse.

35. The polar of any point P wo a parabola meets the axis at A; the perpendicular from P to the polar meets the axis at B; show that the mid-point of AB is the same for all positions of P.

If the perpendicular from P to the axis cuts the parabola at two points, show that these points lie on the circle on AB as diameter.

36. AB, CD are two lines bisecting each other at right angles. Prove that the locus of P such that $PA.PB = PC.PD$ is a rectangular hyperbola.

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37. Prove that the equation of one common tangent to the curves $x^2+y^2 = 5a^2$ and $y^2 = 5ax$ is $2y = x+5a$. Sketch the curves and write down the equation of the other common tangent.

38. The equations of the sides BC, CA, AB of a triangle ABC are $y = 0$, $x+y = 4$, $x+2y = 5$; the line $y = 2x$ meets BC at A', CA at B', and AB at C'. Find the coordinates of the middle points of AA', BB', CC'. Verify that these middle points all lie on a straight line, and find its equation. [A.L.]

39. If the circle $x^2+y^2 = a^2$ meets the line $lx+my+n = 0$ in two points A and B, prove that the equation $x^2+y^2-a^2+k(lx+my+n) = 0$ represents a circle through A and B whatever the value of k.

By choosing k so that the centre of the new circle lies on the given line, find the equation of the circle on AB as diameter.

40. A chord PQ of the hyperbola $xy = c^2$ passes through the fixed point (h, k). Prove that the locus of the remaining vertices of the parallelogram formed

by drawing straight lines through P and Q parallel to the asymptotes is the hyperbola $xy+c^2 = kx+hy$. [A.L.]

41. Write down the equation of the polar of P_1 wo the ellipse $x^2/a^2+y^2/b^2 = 1$ and show that, if the line OP_1 meets this polar at Q and the ellipse at R, $OP_1 . OQ = OR^2$.

[Notice that the result can be written $OP_1/OR = OR/OQ$; hence show by the aid of similar triangles that the three lengths concerned can be replaced by their projections on the x-axis: this makes the calculation easier.]

42. A, B are the points $(a, 0)$, $(0, b)$; C, D are variable points on Ox, Oy such that $OC+OD = OA+OB$. Find the locus of the meet of AD and BC.

[When the locus equation has been obtained, show that it can be written in the form $pq = 0$, where p and q are linear functions, and hence that the locus is the two lines $p = 0, q = 0$, one of which is AB.]

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43. Write down the equation of the tangent at $(-\frac{5}{2}, \frac{5}{2})$ to the circle $2x^2+2y^2 = 25$ and show that it is also a tangent to the ellipse $x^2/16+y^2/9 = 1$. Write down the equations of the other three common tangents to the circle and ellipse.

44. The sides of the triangle DEF are parallel to the sides of the triangle ABC, D, E, F corresponding to A, B, C. The coordinates of A, B, C, E, F are $(4, 7)$, $(3, 2)$, $(7, 4)$, $(-8, -8)$, $(-6, -7)$. Find the coordinates of D.

The point $(5, 5)$ is inside the triangle ABC. Find the coordinates of the point similarly placed inside the triangle DEF. [A.L.]

45. P is the point $(t, 1/t)$ on the rectangular hyperbola $xy = 1$; the circle on OP as diameter cuts the curve again at Q. Prove that the y-coordinate of Q is the cube root of the x-coordinate of P.

46. Two variable perpendicular lines pass through a fixed point P_0. One meets the x-axis at X, the other meets the y-axis at Y. Find the locus of the foot of the perpendicular from P_0 to XY. [If a third-degree equation is obtained, reject the factor $(x-x_0)^2+(y-y_0)^2$.]

47. Find the locus of the centre of a variable circle which cuts off intercepts of lengths a and b on the axes.

48. In No. 42, find the locus if $1/OC-1/OD = 1/OA-1/OB$.

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49. A variable point Q is taken on the line $x = 1$; P is taken on OQ such that $OP . OQ = 1$. Find the locus of P.

50. A line ON of length p makes an angle α with Ox. Find the equation of the line through N perpendicular to ON as follows. Let P be any point on the required line, let $OP = r$, and let OP make an angle θ with Ox. (i) Write down the coordinates of P in terms of r and θ; (ii) express p in terms of r, α, θ and write down the trigonometrical expansion of the result; (iii) substitute the results of (i) in (ii).

51. A is the point (h, k), B is $(h+kt, k-ht)$. Show that AB is perpendicular to AO.

A circle is drawn through A with centre B and cuts the x-axis at P and Q. Find the equation giving the lengths of OP and OQ, and deduce that $OP . OQ$ is independent of t. Give a geometrical reason for the value of $OP . OQ$.

52. Show by the second method of 6.4, Example 3, that the equation of the chord P_1P_2 of the ellipse $x^2/a^2+y^2/b^2 = 1$ can be written

$$\frac{x(x_1+x_2)}{a^2}+\frac{y(y_1+y_2)}{b^2} = 1+\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}.$$

If P_0 is its mid-point, use this result to express (i) its gradient, (ii) its equation, in terms of x_0 and y_0.

53. P, Q are two points on a rectangular hyperbola, centre C; if PQ is perpendicular to the transverse axis, prove that CQ is perpendicular to the tangent at P.

54. A is a fixed point, P is a variable point on the polar of A wo a parabola. Show that the locus of the mid-point of AP is a tangent to the parabola.

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55. On the line through O and P_1, a point P is taken so that $OP.OP_1 = k^2$. Find the coordinates of P.

If P traces out the circle $x^2+y^2+2gx+c = 0$, show that the locus of P_1 is a circle unless $c = 0$, when it is a straight line. The locus is known as the **inverse** of the original circle wo O.

56. If P_1, P_2, P_3 lie on a line AB not through O, prove that the lines $x_1x+y_1y+1 = 0$, $x_2x+y_2y+1 = 0$, $x_3x+y_3y+1 = 0$ all pass through a point C such that OC is perpendicular to AB.

57. Find (i) the equation of the perpendicular bisector of P_1P_2; (ii) the coordinates of the image of P_1 in the line $lx+my+n = 0$.

58. A circle is drawn on the major axis of an ellipse as diameter; if points P on the ellipse and Q on the circle are such that PQ is perpendicular to the major axis, show that the tangents at P and Q to their respective curves meet on the major axis.

59. A line parallel to the conjugate axis of a hyperbola is drawn through a variable point P on the curve and meets the asymptotes at U, V. Prove that $UP.PV$ is constant.

60. A and B are the points $(c, 0)$, $(-c, 0)$; find the locus of P if $\angle PAB = 2\angle PBA$.

PART II

CHAPTER 7

FREEDOM EQUATIONS, LINES, DETERMINANTS

7.1. Two kinds of Equation

So far, each line or curve has been represented by a single equation satisfied by the coordinates (x, y) of every point on it. As an equation of this sort exercises a constraint upon the values of x and y, it is sometimes called the **constraint equation** of a curve, and we saw in 2.3 that one of its important uses is to test whether a given point lies on the curve or not.

But the constraint equation has serious drawbacks if we want to find or make use of points on the curve. The obvious way to obtain such a point is to choose a value of x and use the equation to find the value of y (or *vice versa*), but this may lead to difficulties. For example, the equation

$$x^3+y^3 = 3xy \quad . \quad . \quad . \quad . \quad (1)$$

can be seen to represent a curve symmetrical about the line $x = y$ and passing through the origin, but if we try to express one co-ordinate in terms of the other we are faced with a cubic equation to be solved. This makes it difficult to plot the curve from its constraint equation.

Even if the equation is of only the second degree, difficulties may still remain. For example, the equation of the tangent at P_1 to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad (2)$$

is

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2} = 1 \quad . \quad . \quad . \quad . \quad (3)$$

but (3) gives little information about tangents to (2) unless we know that P_1 really does lie on the ellipse—that is to say, that the equation

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad (4)$$

s also satisfied. Any property of the tangent is therefore bound up with the properties of the *pair* of equations (3) and (4); if we try to dispense with (4) by writing it as

$$y_1 = \pm\frac{b}{a}\sqrt{(a^2-x_1^2)}$$

and substituting this result in (3), we obtain a very clumsy equation; and we cannot get rid of the $\pm$ sign which it contains without losing one-half of the ellipse.

To avoid these difficulties we need another way of providing information about the points of a curve. Previously we began by imagining that every point on the x-axis was labelled with a number (actually the number measuring its distance from the origin in terms of some chosen unit), and when we wanted to refer to a point on the x-axis we did so by stating the number on its label, calling this the value of x at that point; we made similar arrangements on the y-axis, and so gave a double label to any point in the plane. But now let us temporarily ignore these coordinates and, given a curve in which we are interested, let us label every point of the *curve* with a number. We can do this in any way we like; the number attached to a point P can, for example, measure the distance round the curve to P from some agreed starting-point A, or the angle which AP makes with some chosen direction; or it can represent something much less simple than these, provided only that it enables us to distinguish one point of the curve from another. The number is called the **parameter*** of the point, and the way in which the numbers are distributed is called the **parametric representation** of the curve. When we want to refer to a point on the curve we do so by stating the number on its label; in other words, by stating the value of its parameter.

The coordinates of a point on the curve can then be expressed in terms of its parameter, and by choosing the representation suitably we may be able to make these expressions simple. Let us take as an example the circle whose constraint equation is

$$x^2+y^2 = a^2 \quad . \quad . \quad . \quad . \quad (5)$$

To obtain a simple parametric representation, take any point A on the curve, say $(-a, 0)$, and consider the gradient of the chord joining A to any other point P on the circle. Denote the gradient by t; then different positions of P will always give different values of t, and so t will serve as a parameter for the points of the circle.

Since AP passes through A with gradient t, its equation is

$$tx-y+at = 0 \quad . \quad . \quad . \quad . \quad (6)$$

* We have already used this word in 5.4 where, in effect, every line through the meet of two given lines was given a different label.

Eliminating y between (5) and (6) we find

$$(1+t^2)x^2+2at^2x-a^2(1-t^2) = 0 \quad . \quad . \quad (7)$$

as the equation giving the x-coordinates of A and P. But we already know that $x = -a$ is one of these, so $x+a$ must be one factor of the L.H.S. of (7). We have already met a similar situation in 6.6. Proceeding as we did there, we easily factorize (7) as

$$(x+a)\{(1+t^2)x-a(1-t^2)\} = 0,$$

giving $x = a(1-t^2)/(1+t^2)$ at P.

Substituting this result in (6), we find $y = 2at/(1+t^2)$, so that P is given by the pair of equations

$$x = \frac{a(1-t^2)}{1+t^2}, \; y = \frac{2at}{1+t^2} \quad . \quad . \quad . \quad (8)$$

These are called **freedom equations** of the circle, since in them t is free to take any value, and the coordinates given by them always satisfy (5). They are also called **parametric equations** and are said to express the circle parametrically.

Notice that (8) can be written

$$x:y:a = 1-t^2:2t:1+t^2.$$

This form will often be used, as it is easier for the printer and has other merits, one of which will be seen in 7.6, Examples 1 and 2.

From what has been said, it is clear that there is no limit to the number of ways in which a given curve can be represented by a pair of freedom equations: in dealing with (5) for example, we could have taken A anywhere on the circle and we could have given other geometrical interpretations to the parameter. As a result, some representations may be expected to be better than others, and it is worth setting down a few conditions which the freedom equations should satisfy if they are to be most useful.

(*a*) No two points on the curve should have the same parameter.

(*b*) By letting the parameter take all possible values (letting it also tend to infinity if necessary), we should be able to obtain all the points of the curve.

(*c*) The coordinates should be simple and easily-handled functions of the parameter.

(*d*) In applying freedom equations to a particular problem, we should if possible choose a parameter representing a measurement in which we are interested.

Thus a representation such as $x = t$, $y = \pm\sqrt{(a^2-t^2)}$ satisfies neither (*a*) nor (*c*), while if the $\pm$ sign is dropped it satisfies (*a*) but not (*b*). On the other hand (8), which was seen to satisfy (*a*) and (*c*),

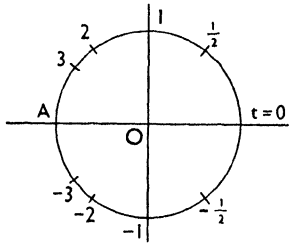


Fig. 41

also satisfies (*b*) because, by a suitable choice of the gradient of AP, we can make P coincide with any point of the circle except A itself; and $P \to A$ as $t \to \infty$. Fig. 41 shows the circle and the values of the parameter at some of its points.

Before we go on to apply these ideas to particular problems, let us look briefly at the advantages and disadvantages of freedom equations. One advantage is that they always enable us, by giving the parameter a number of different values, to make a table of values of x and y, and so to plot the curve. In the same way, if an arbitrary point on the curve is required, there is no difficulty in assigning coordinates to it: any point on (5), for example, can be called

$$\left(\frac{a(1-t^2)}{1+t^2}, \frac{2at}{1+t^2}\right).$$

We can also find the gradient of the curve at any point, since $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$; thus, if a curve is given by $x = t^2+t$, $y = t^2-t$, we have $dy/dx = (2t-1)/(2t+1)$, giving the gradient of the curve at the point with parameter t. Further, if we want a tangent, or any other line connected with a point on a curve, the coefficients in the equation of such a line will contain only one variable, the parameter of the point.

But apart from these advantages (which we had sought) there are two others. Suppose, first, that we want to find the points in which two curves intersect. If we are given their constraint equations it is necessary to eliminate one of the variables, but if we are given the freedom equations of one curve and the constraint equation of the other, no elimination is needed. Thus the intersections of the circle (8) and the line

$$3x+4y = 5a \quad . \quad . \quad . \quad . \quad (9)$$

may be obtained as follows. If the point given by (8) lies on (9),

$$3a\frac{1-t^2}{1+t^2}+8a\frac{t}{1+t^2} = 5a,$$

$$\therefore 4t^2-4t+1 = 0,$$

from which $t = \frac{1}{2}$, this being a double root. The line therefore meets the circle in two coincident points where $t = \frac{1}{2}$. But from (8), when

$t = \frac{1}{2}$, $x = \frac{3a}{5}$ and $y = \frac{4a}{5}$. Hence the line touches the circle at $\left(\frac{3a}{5}, \frac{4a}{5}\right)$. This is a simple example, but clearly the method has a wide application.

The other advantage comes through being able to make the freedom equations satisfy condition (*d*) above. For example, if we want to find the distance between the two points in which a line cuts a conic, we can use the constraint equation of the conic and freedom equations for the line, choosing as the parameter of any point P on the line its directed distance from some chosen point A of the line. Proceeding as in the previous paragraph, we shall obtain an equation whose roots are the distances from A to the two points of intersection, and the difference between the roots will be the distance apart of the two points. This method is far more direct than using the constraint equations to find the coordinates of the points and then finding the distance by means of the formula of 3.1.

The chief disadvantage of freedom equations is that we cannot give rules, as in 6.3, for saying what curve they represent, since an unlimited number of pairs of freedom equations can all represent the same curve. Unless the equations happen to be familiar we are often obliged to obtain the corresponding constraint equation in order to identify the curve. As the constraint equation has to be satisfied by the given coordinates for all values of t, it can be obtained by eliminating t between the two freedom equations, but this may not always be easy to do. A second disadvantage is that it may be difficult to find the points in which two curves meet if *both* curves are given parametrically.

Exercise 7A

1. Find the coordinates of the point (other than O) at which the line $y = tx$ meets Descartes' *Folium*, $x^3+y^3 = 3xy$, and so obtain freedom equations for the curve.

From these, make a table of values of x and y, taking $t = -10, -2, -1{\cdot}2, -1, -0{\cdot}8, -0{\cdot}5, -0{\cdot}1, 0, 0{\cdot}1, 0{\cdot}5, 1, 2, 10$, and plot the curve.

2. Find the coordinates of the point (other than O) at which the line $y = tx$ meets the curve $x(x^2+y^2) = 3x^2-y^2$. Hence make a table of values of x and y for the curve, taking the values $t = 0, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1, \pm 1\frac{1}{2}, \pm\sqrt{3}, \pm 2, \pm 3$.

Plot the curve and verify that, if Q is $(2, 0)$ and P is any point on the curve, the angle which OP makes with Ox is one-third the angle which QP makes. The curve is Maclaurin's *Trisectrix*.

3. Find freedom equations for the circle $x^2+y^2 = a^2$ by the method of 7.1, letting A be $(0, -a)$ and t the gradient of the chord AP.

4. Find the point in which the line $x-y = at$ meets the hyperbola $x^2-y^2 = a^2$, and so obtain freedom equations for the curve.

Use them to make a table of values of the value for the hyperbola $x^2-y^2 = 4$, and plot it.

In Nos. 5–8, choose a suitable value of a, make a table of values, and plot the curve.

5. The *astroid*: $x = a\cos^3\theta$, $y = a\sin^3\theta$.

6. The *cardioid*: $x = a(\cos 2\theta + \cos\theta)$, $y = a(\sin 2\theta + \sin\theta)$.

7. The *cycloid*: $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$, θ being measured in radians.

8. The *archimedean spiral*: $x = a\theta\cos\theta$, $y = a\theta\sin\theta$. Choose a small value of a, say 0·002, and measure θ in degrees.

In Nos. 9–12, show that the point given by the freedom equations satisfies the constraint equation. Sketch the curve and mark the sketch to show how the value of the parameter changes as a point moves round the curve.

9. $x = ct$, $y = c/t$; $xy = c^2$.

10. $x = at^2$, $y = 2at$; $y^2 = 4ax$.

11. $x = a\cos\theta$, $y = b\sin\theta$; $x^2/a^2+y^2/b^2 = 1$.

12. $x = a\sec\theta$, $y = b\tan\theta$; $x^2/a^2-y^2/b^2 = 1$.

In Nos. 13–14, show that both pairs of freedom equations represent the same curve, by finding its constraint equation. Do they represent the whole of it?

13. $x = 5+2t$, $y = \frac{2}{3}(2+t)$; $x = \sec^2\theta$, $y = \frac{3}{4}\tan^2\theta$.

14. $x = t(t+4)$, $y = t+2$; $x = -4\sin^2\theta$, $y = 2\cos\theta$.

In Nos. 15–18, S is $(a, 0)$ and P is the point $(at^2, 2at)$ on the parabola $y^2 = 4ax$.

15. Prove that SP is equal to the distance of P from the line $x+a = 0$.

16. Write down the equation of the tangent at P to the parabola. If it meets the x-axis at X, show that PX is bisected by the y-axis.

17. A tangent is drawn to the parabola at P; show that the foot of the perpendicular to it from S lies on the y-axis.

18. A line is drawn through P, at right angles to the tangent there, and meets Ox at N. If M is the foot of the perpendicular from P to Ox, show that MN is independent of t.

In Nos. 19–20, P is the point $(ct, c/t)$ on the rectangular hyperbola $xy = c^2$.

19. Write down the equation of the tangent at P to the hyperbola and find the coordinates of the points X, Y at which it meets Ox, Oy. Show that P is the midpoint of XY.

20. Show that OP and the tangent at P are equally inclined to an asymptote.

21. P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse $x^2/a^2+y^2/b^2 = 1$; S and S' are $(c, 0)$, $(-c, 0)$, where $c^2 = a^2-b^2$. Find the value of $SP+S'P$, showing that it is independent of θ.

22. P is the point $(a\sec\theta, b\tan\theta)$ on the hyperbola $x^2/a^2-y^2/b^2 = 1$; S and S' are $(c, 0)$, $(-c, 0)$, where $c^2 = a^2+b^2$. Find the value of $S'P-SP$, showing that, apart from sign, it is independent of θ.

In Nos. 23–26, find the equation giving the parameters of the points at which the curve with the given freedom equations meets the line $lx+my+n = 0$. Find also the relation between l, m, n if the line is a tangent.

23. $x = at^2$, $y = 2at$.

24. $x = ct$, $y = c/t$.

25. $x = a(1-t^2)/(1+t^2)$, $y = 2at/(1+t^2)$.

26. $x = a(1+t^2)/2t$, $y = a(1-t^2)/2t$; show that these satisfy $x^2-y^2 = a^2$.

7.2. Freedom Equations of a Line

We saw in 5.1 that if P_1 (x_1, y_1) lies on a given line, the ratio of the two quantities $x-x_1$ and $y-y_1$ is constant for all points P (x, y) on

the line, and we used this fact to obtain the constraint equation of the line. Let us now suppose that the ratio is $p:q$, so that

$$\frac{x-x_1}{p} = \frac{y-y_1}{q}.$$

Let each of these fractions be denoted by t; then

$$x-x_1 = pt \text{ and } y-y_1 = qt.$$

So we obtain the freedom equations

$$x = x_1+pt, \quad y = y_1+qt \quad . \quad . \quad . \quad (10)$$

to represent the line through (x_1, y_1) with gradient q/p.

In this representation the parameter is proportional to the directed distance P_1P, for

$$P_1P = \pm\sqrt{\{(x-x_1)^2+(y-y_1)^2\}} = \pm\sqrt{(p^2t^2+q^2t^2)};$$

i.e.

$$P_1P = \pm t\sqrt{(p^2+q^2)} \quad . \quad . \quad . \quad (11)$$

and the sign is at present ambiguous because we have not decided which is to be the positive direction along the line. But you can see from (10) that all the points at which t is positive lie on one side of P_1, and all those at which t is negative on the other; for instance, if p is positive, positive values of t give those points of the line at which $x>x_1$, while negative values of t give those at which $x<x_1$. As soon as we choose the positive direction along the line, therefore, the sign in (11) is settled.

If we start from another point $Q_2(x_2, y_2)$ and draw a line Q_2Q parallel to P_1P, the coordinates of Q can be given in terms of a parameter u by

$$x = x_2+pu, \quad y = y_2+qu,$$

and it is worth noticing that t and u have like or unlike signs according as P_1P and Q_2Q are in the same or opposite senses.

Example 1. Find freedom equations for the line $3x-5y = 2$.

We begin by choosing a point on the line. Any point will do, but if we can find one whose coordinates are whole numbers, so much the better. Let us take (4, 2).

The gradient of the line is $\frac{3}{5}$; hence as a point moves along it the increments in x and y are in the ratio 5 : 3, and the freedom equations can be written

$$x = 4+5t, \quad y = 2+3t \quad . \quad . \quad . \quad (12)$$

Notice that if we substitute these values of x and y in the L.H.S. of the equation of the line we obtain

$$3(4+5t)-5(2+3t),$$

which reduces to 2 and is equal to the R.H.S.

Notice also that the equations $x = 4-5t$, $y = 2-3t$ would have done equally well.

Example 2. Find the distances from A (4, 2) to the points in which $3x-5y = 2$ meets the hyperbola $xy = 24$. Are these points on the same or opposite sides of A?

Using the freedom equations (12) to give the coordinates of a point P on the line, we see that $AP = t\sqrt{34}$. Here we have arbitrarily chosen the positive sign in front of the square root: a directed distance AP is thus to be regarded as having the same sign as t, and reference to (12) shows that this means that motion along the line in the direction of increasing x and y is to be regarded as positive.

Now if P lies on the hyperbola, $(4+5t)(2+3t) = 24$;

$$\therefore\ 15t^2+22t-16 = 0;$$
$$\therefore\ (15t-8)(t+2) = 0;$$
$$\therefore\quad t = \frac{8}{15} \text{ or } -2.$$

These values of t give the two points of intersection; the corresponding distances are $\frac{8}{15}\sqrt{34}$ and $-2\sqrt{34}$. As these have opposite signs, the points are on opposite sides of A.

Example 3. Find the angles which the tangents from (1, 5) to the hyperbola $x^2-y^2 = 25$ make with the x-axis.

Here we are concerned with finding gradients, so it will be better to denote a gradient by γ than by q/p.

Any point P on the line through (1, 5) with gradient γ is given by

$$x = 1+t, \quad y = 5+\gamma t,$$

and the values of t which make it lie on the hyperbola are given by

$$(1+t)^2-(5+\gamma t)^2 = 25,$$

i.e. by $$(\gamma^2-1)t^2+2(5\gamma-1)t+49 = 0.$$

If γ is the gradient of a tangent from (1, 5), the two roots of this equation are equal,

$$\therefore\ 12\gamma^2+5\gamma-25 = 0;$$
$$\therefore\ (3\gamma+5)(4\gamma-5) = 0;$$
$$\therefore\quad \gamma = -\tfrac{5}{3} \text{ or } \tfrac{5}{4}.$$

The tangents therefore make angles of $-\tan^{-1}\frac{5}{3}$ and $\tan^{-1}\frac{5}{4}$ with the x-axis.

The freedom equations of a line can be put in another form which gives the distances along the line more directly. Let the line pass through $P_1(x_1, y_1)$ and let the positive direction along the line make an angle θ with the positive x-axis, θ being measured counter-clockwise as in 5.2. Then the coordinates of any point $P\ (x, y)$ on the line are given by

$$x = x_1 + r\cos\theta, \quad y = y_1 + r\sin\theta \quad . \quad . \quad (13)$$

where r measures the directed distance from P_1 to P. This result follows from Fig. 42, which shows as usual the simplest case.

Although (13) brings in the distance P_1P explicitly, (10) is often to be preferred on the ground that algebraic functions are easier to handle than trigonometric. Certainly the examples so far given are done more easily using (10) than (13), as you can verify for yourself. The next example shows (13) at its best.

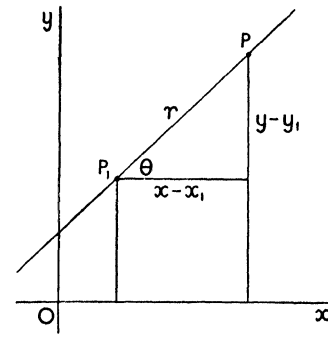


Fig. 42

Example 4. A variable line through P_1 meets the circle $x^2+y^2 = a^2$ at Q and R. Prove that $P_1Q.P_1R$ is constant.

Let the line make an angle θ with the x-axis, so that the coordinates of any point on it are given by (13). If this point lies on the circle,

$$(x_1 + r\cos\theta)^2+(y_1 + r\sin\theta)^2 = a^2;$$

$$\therefore \quad r^2+2r(x_1\cos\theta + y_1\sin\theta)+(x_1^2+y_1^2-a^2) = 0.$$

The roots of this equation give the two lengths P_1Q and P_1R; we only require their product and it is shown in algebra books that this can be obtained by dividing the constant term by the coefficient of the square term. Hence

$$P_1Q.P_1R = x_1^2+y_1^2-a^2$$

and this value does not depend upon θ. It is, of course, the power of P_1 wo the circle.

For comparison we give an alternative solution.

Let the line have gradient q/p, so that the coordinates of any point on it are given by (10) and the distance of this point from P_1 is $t\sqrt{(p^2+q^2)}$. If the point lies on the circle,

$$(x_1+pt)^2+(y_1+qt)^2 = a^2;$$

$$\therefore (p^2+q^2)t^2+2(px_1+qy_1)t+(x_1^2+y_1^2-a^2) = 0.$$

If the roots of this are α and β, then

$$\alpha\beta = \frac{x_1^2+y_1^2-a^2}{p^2+q^2},$$

$$P_1Q = \alpha\sqrt{(p^2+q^2)}, \quad P_1R = \beta\sqrt{(p^2+q^2)},$$

and so $P_1Q.P_1R = x_1^2+y_1^2-a^2$ as before.

It is sometimes useful to write the freedom equations of a line in a different way, depending upon the formula for the coordinates of the point P dividing P_1P_2 in a given ratio. We shall denote the fraction P_1P/PP_2 by the Greek letter λ (lambda), so that P divides P_1P_2 in the ratio $\lambda:1$.

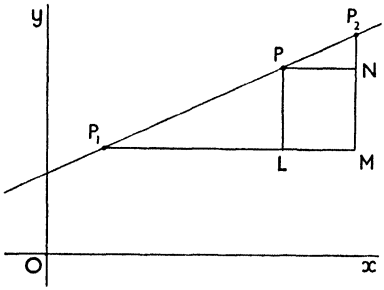


Fig. 43

In Fig. 43, $P_1L = x-x_1$, $PN = x_2-x$, $LP = y-y_1$, $NP_2 = y_2-y$, and as triangles P_1LP and PNP_2 are similar,

$$\frac{P_1L}{PN} = \frac{LP}{NP_2} = \frac{P_1P}{PP_2};$$

$$\therefore \frac{x-x_1}{x_2-x} = \frac{y-y_1}{y_2-y} = \lambda.$$

Hence
$$x = \frac{x_1+\lambda x_2}{1+\lambda}, \quad y = \frac{y_1+\lambda y_2}{1+\lambda} \quad . \quad . \quad . \quad (14)$$

give the coordinates of the point dividing P_1P_2 in the ratio $\lambda:1$, and are freedom equations of the line through P_1 and P_2, since P always lies on this line. This result is known as the **section formula** and should be learnt.

Notice that since $\lambda = P_1P/PP_2$, if λ is positive P_1P and PP_2 have the same sign and P therefore lies between P_1 and P_2 as in Fig. 43, whereas if λ is negative they have opposite signs and P lies outside the segment P_1P_2. P is said to divide P_1P_2 internally in the first case, externally in the second. If λ lies between 0 and -1, PP_1 is smaller than PP_2, so that P lies on P_2P_1 produced, but if λ is less than -1 the reverse is true and P lies on P_1P_2 produced. The values of λ at various points of the line are shown in Fig. 44. There is no position of P for which $\lambda = -1$, though λ tends to this value as P becomes very distant; also, though no value of λ can make P coincide with P_2, P_2 is the limiting position of P as $\lambda \to \infty$.

Example 5. In what ratio is the line joining A $(-2, 1)$ and B $(3, -3)$ cut by the lines $x+2y+1 = 0$ and $2x+2y-1 = 0$? Does it cut the lines $4x+5y = 2$ and $2x+y = 3$?

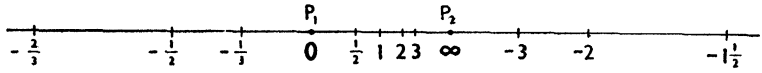


Fig. 44

The point P dividing AB in the ratio $\lambda:1$ is given by

$$x = \frac{-2+3\lambda}{1+\lambda}, \quad y = \frac{1-3\lambda}{1+\lambda},$$

and if P lies on the first of the given lines,

$$\frac{-2+3\lambda}{1+\lambda} + \frac{2(1-3\lambda)}{1+\lambda} + 1 = 0;$$

$$\therefore -2+3\lambda+2(1-3\lambda)+1+\lambda = 0,$$

from which $\lambda = \frac{1}{2}$.

The first line therefore divides AB internally in the ratio 1:2.

If P lies on the second line,

$$2(-2+3\lambda)+2(1-3\lambda)-(1+\lambda) = 0,$$

from which $\lambda = -3$. This line therefore divides AB externally in the ratio 3:1.

If we substitute the coordinates of P in the equation of the third line we obtain

$$4(-2+3\lambda)+5(1-3\lambda) = 2(1+\lambda),$$

from which $\lambda = -1$. But there is no point on AB at which $\lambda = -1$ and so AB does not cut $4x+5y-2 = 0$. The two lines must therefore be parallel. This can also be seen by finding the constraint equation of AB, which is in fact $4x+5y+3 = 0$.

Finally, substituting the coordinates of P in the equation of the fourth line we find

$$2(-2+3\lambda)+(1-3\lambda) = 3(1+\lambda),$$

which simplifies to the contradictory statement

$$-3 = 3.$$

This is not a new 'proof' that minus three is equal to plus three. We are not *proving* anything; we are trying to *find* something, a value of λ, and we have failed. There is therefore no finite value of λ for which P lies on the required line. In our search we have tried every point on the line AB except B itself. It is easy to verify that B does lie on $2x+y = 3$, and that is evidently what the contradiction means. Just as, in 6.6, Example 3, the coefficient of the term of highest degree in a quadratic equation became zero when one root became

infinite, so here the contradictory statement can be regarded as a linear equation in λ, but the coefficient of λ has become zero in an attempt to yield an infinite root.

Example 6. A is the point (3, 1), B is any point on $3x+2y = 12$. Show that the ellipse $x^2+2y^2 = 12$ cuts AB in two points which divide it internally and externally in the same ratio.

Before we obtain the freedom equations of AB it is convenient to express $3x+2y = 12$ parametrically by letting B be the point $(4-2t, 3t)$. The point P dividing AB in the ratio $\lambda : 1$ is then given by

$$x = \frac{3+\lambda(4-2t)}{1+\lambda}, \quad y = \frac{1+3t\lambda}{1+\lambda}$$

and lies on the ellipse if

$$\{3+\lambda(4-2t)\}^2+2(1+3t\lambda)^2 = 12(1+\lambda)^2.$$

Bearing in mind that this is to be regarded as an equation in λ, we can simplify it to

$$(4-16t+22t^2)\lambda^2 = 1,$$

from which $\lambda = \pm \dfrac{1}{\sqrt{(4-16t+22t^2)}}$.

The two values of λ are thus numerically equal but opposite in sign, showing that the ellipse cuts AB in two points which divide it internally and externally in the same ratio. The given line is easily seen to be the polar of A wo the ellipse. A similar result would be true of any point and its polar wo any conic.

EXERCISE 7B

In Nos. 1, 2, obtain freedom equations for two of the given lines and use them to find the three points of intersection.

1. $x+y = 1$, $2x-y = 3$, $x-3y+2 = 0$.
2. $2x+y = 1$, $3x-2y = 1$, $x+3y = 1$.

In Nos. 3, 4, write down the coordinates of the point dividing the line joining the given points in the given ratio.

3. (i) (4, 3), (10, −12), 2 : 1; (ii) (1, 0), (3, 3), −1 : 2;
   (iii) (−8, 7), (7, −8), 2 : 3.
4. (i) (3, −5), (−5, 7), 1 : 3; (ii) (−6, −1), (9, 9), 4 : 1;
   (iii) (17, 5), (5, −4), −5 : 2.

In Nos. 5, 6, show that the three points are collinear, and find the ratio in which the third divides the join of the first two.

5. (−28, 31), (11, 5), (44, −17). 6. (83, 21), (−2, −44), (32, −18).

In Nos. 7, 8, find where possible the ratios in which the lines divide the join of the points. Explain any unusual cases.

7. $A\,(-3, 4)$, $B\,(5, -2)$; (i) $3x+2y+7=0$; (ii) $3x+4y+5=0$; (iii) $3x+y+5=0$; (iv) $3x+5y-5=0$.

8. $A\,(-3, 1)$, $B\,(12, 7)$; (i) $2x-5y+11=0$; (ii) $2x-3y+5=0$; (iii) $2x-3y-3=0$; (iv) $x-2y-2=0$.

9. M_1 is the mid-point of the side P_2P_3 of the triangle $P_1P_2P_3$; G is the point dividing P_1M_1 in the ratio 2 : 1. Find the coordinates of G in terms of those of P_1, P_2, P_3, and show that G also lies on the other two medians of the triangle. G is called the **centroid** of the three points P_1, P_2, P_3.

10. A, B, C, D are the points $(-2, 1)$, $(-4, 3)$, $(6, -1)$, $(4, -3)$.

Find (i) E in AB so that $AE = EB$,

(ii) F in CE so that $CF = 2.FE$,

(iii) G in DF so that $DG = 3.GF$,

and verify that G is the mid-point of the line joining the mid-points of AB, CD. It is called the centroid of A, B, C, and D.

Show that *any* four points A, B, C, D lead in the same way to a point G having the same final property.

11. Find the length of the chord $5x-2y=84$ of the hyperbola $x^2-4y^2=144$.

12. Find the length of the chord $3x+2y=12$ of the ellipse $9x^2+16y^2=144$.

In Nos. 13, 14, find the equations of the tangents from the given point to the given curve.

13. $(1, 4)$, $2x^2+y^2=12$. 14. $(4, -1)$, $x^2+8x+32y=0$.

15. Verify that $A\,(2, 1)$ lies on the line $3x-4y=2$; hence find freedom equations for the line.

Find an expression for the distance from A at which $3x-4y=2$ cuts the line $x+y-10+\lambda(2x-3y)=0$, and explain the result.

16. Verify that $A\,(3, -2)$ lies on $5x+12y+9=0$ and find freedom equations for this line.

Find an expression for the distance from A at which $5x+12y+9=0$ cuts the line $4x+9y-3+\lambda(2x+5y+2)=0$.

Deduce from your answer (i) the value of λ for which the two lines are parallel; (ii) the distance from A at which the first line cuts $4x+9y=3$.

17. A line through O cuts the curve $(x^2+y^2)^2-(4x+1)(x^2+y^2)+4x^2=0$ at A and B as well as at O itself. Use equations (13) to show that AB is of length 4 units. The curve is a *limaçon* (see Chapter 12).

18. If a line through O cuts the curve $x(x-4)(x^2+y^2)+4y^2=0$ again at A and B, show by using equations (13) that AB is of constant length 4 units. The curve is a *conchoid* (see Chapter 12).

19. A line through P_1 meets the ellipse $x^2/a^2+y^2/b^2=1$ at Q and R. A parallel line through the centre O meets the curve at U. Obtain an equation whose roots give the lengths of P_1Q and P_1R, and so prove that

$$P_1Q.P_1R/OU^2 = x_1^2/a^2+y_1^2/b^2-1.$$

If the tangents from a point A to an ellipse meet it at S and T, and if parallel lines through the centre C meet it at L and M, prove that $AS/AT = CL/CM$.

20. PS, PT are two tangents to a hyperbola, centre C; L, M are points on the conjugate hyperbola such that CL and CM are parallel to, and in the same sense as, PS and PT. Prove that PC, SL, TM are concurrent.

(Proceed as in No. 19.)

In Nos. 21–24, use equations (10) to prove, for the given conic, that if the line through P_1 parallel to the polar of P_1 cuts the conic at L and M, P_1 is the mid-point of LM. This is equivalent to the theorem of 6.7.

21. $xy=c^2$ 22. $y^2=4ax$.

23. $x^2/a^2-y^2/b^2=1$. 24. $x^2/a^2+y^2/b^2=1$.

7.3. Distance of a Point from a Line

Let the point be $P_1(x_1, y_1)$ and the line $lx+my+n = 0$. The gradient of the line is $-l/m$ and so the gradient of a perpendicular line is m/l.

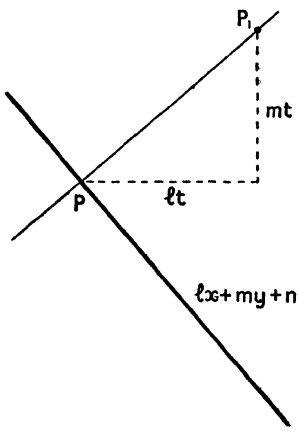


Fig. 45

Hence any point P on the perpendicular from P_1 is given by $x = x_1-lt$, $y = y_1-mt$, and the distance $P_1P = \pm\sqrt{(l^2+m^2)}$. If P lies on the given line,

$$l(x_1-lt)+m(y_1-mt)+n = 0,$$

and $$t = \frac{lx_1+my_1+n}{l^2+m^2}.$$

Hence **the distance from P_1 to the line $lx+my+n = 0$ is**

$$\pm\frac{lx_1+my_1+n}{\sqrt{(l^2+m^2)}} \quad . \quad . \quad (15)$$

That is to say, if the R.H.S. of the equation of a line is zero, we obtain the distance of a point from it by substituting the coordinates of the point in the L.H.S. of the equation and dividing by the square root of the sum of the squares of the coefficients of x and y.

The choice of the sign depends, as in (11) of 7.2, upon which direction along the perpendicular is selected as positive.

If $Q_2(x_2, y_2)$ is some other point and Q is given by $x = x_2-lu$, $y = y_2-mu$, then Q is on the line if $u = \frac{lx_2+my_2+n}{l^2+m^2}$. If P_1 and Q_2 are on the same side of the line, P_1P and Q_2Q are in the same sense, hence t and u have like signs, and so l_1x+m_1y+n has the same sign as l_2x+m_2y+n; if P_1 and Q_2 are on opposite sides of the line, these two expressions have opposite signs. We can tell by this means whether two points are on the same or opposite sides of a line.

Other methods of obtaining formula (15) will be found in Exercise 7c, Nos. 35–38.

Example 1. Find the equations of the bisectors of the angles between the lines $x-y+2 = 0$ and $x+7y-9 = 0$.

Any point on either bisector is equidistant from the two lines. Now the distance of $P(x, y)$ from the first line is $\pm\frac{x-y+2}{\sqrt{(1^2+1^2)}}$ and from

the second is $\pm \dfrac{x+7y-9}{\sqrt{(1^2+7^2)}}$. If these two distances are equal,

$$\pm \frac{x-y+2}{\sqrt{2}} = \pm \frac{x+7y-9}{5\sqrt{2}},$$

so that the equations of the bisectors are

$$5(x-y+2) = \pm(x+7y-9).$$

Notice that this equation is a blend of the original two.

To determine which of the two signs is to be taken for each bisector it is best to start by sketching the two lines, as in Fig. 46. The expression $x-y+2$ is positive at the origin, where $x = 0$ and $y = 0$; it will therefore be positive at all points which lie on the same side of the line $x-y+2 = 0$ as the origin—that is to say, positive on the right of the line and negative on the left. In the same way we can see that $x+7y-9$ is positive above the line $x+7y-9 = 0$ and negative below. Thus at all points of the line p these two expressions have opposite signs, while at all points of q they have like signs. The equation of p is accordingly $5(x-y+2) = -(x+7y-9)$, or $6x+2y+1 = 0$, while that of q is $5(x-y+2) = +(x+7y-9)$, which reduces to $4x-12y+19 = 0$.

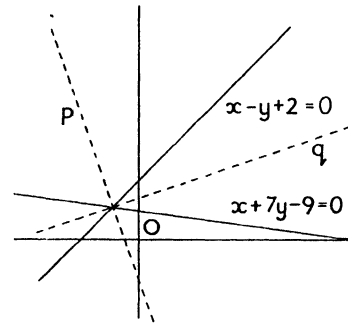


Fig. 46

It would also be possible to distinguish between the two bisectors by the fact that the gradient of $6x+2y+1 = 0$ is negative while that of the other is positive. Notice that these gradients show that the lines are at right angles, as they should be.

Example 2. Find the equation of the ellipse whose major and minor axes lie along the lines $4x+3y-10 = 0$ and $3x-4y+5 = 0$ and are of lengths 6 and 4 respectively.

If the principal axes of such an ellipse had been the coordinate axes, the equation would have been $\dfrac{x^2}{9}+\dfrac{y^2}{4} = 1$, or $4x^2+9y^2 = 36$. Put in words, this equation means that, at any point on the curve, four times the square of the distance from the minor axis added to nine times the square of the distance from the major axis comes to 36.

This verbal formula is readily applied to the axes given. The dis-

tance of (x, y) from the minor axis $3x-4y+5 = 0$ is $\pm\frac{1}{5}(3x-4y+5)$, and from the major axis is $\pm\frac{1}{5}(4x+3y-10)$. The required equation is therefore

$$\tfrac{4}{25}(3x-4y+5)^2+\tfrac{9}{25}(4x+3y-10)^2 = 36,$$

which reduces to

$$36x^2+24xy+29y^2-120x-140y+20 = 0.$$

EXERCISE 7C

In Nos. 1, 2, find the distance of the given point from the given line.

1. (i) $(0, 0)$, $5x-12y+26 = 0$; (ii) $(3, 5)$, $6x-5y+7 = 0$;
   (iii) $(-1, 4)$, $3x+4y = 8$; (iv) $(7, -2)$, $3x = 11$;
   (v) (x_1, y_1), $x\cos\alpha + y\sin\alpha = p$.
2. (i) $(2, -8)$, $4x-5y = 7$; (ii) $(2, 5)$, $15y = 8(x+1)$;
   (iii) $(7, 1)$, $4y+9 = 0$; (iv) $(-3, 2)$, $3x+4y+1 = 0$;
   (v) $(a, 0)$, $x-ty+at^2 = 0$.
3. Which of the points $A(-23, -16)$, $B(-17, -12)$, $C(-8, -5)$, $D(5, 4)$, $E(11, 8)$ are on the same side of $5x-7y+2 = 0$ as the origin?
4. Which of the lines (*a*) $x-4y = 5$, (*b*) $4x+2y = 9$, (*c*) $3x-3y = 22$, passes between $(4, -3)$ and $(9, 1)$?
5. If $(2, 1)$ and $(1, -4)$ lie on the same side of the line $2x-3y+c = 0$, what can be said about the value of c?
6. Which of the points $(1, 3)$, $(3, 3)$, $(3, 4)$ lies inside the triangle with sides $4x+3y = 23$, $5x = 2y$, $x = 5y$?
7. Sketch the three lines $2x-y-2 = 0$, $x+3y-12 = 0$, $x+y-10 = 0$, and on your sketch label the parts of the plane in which the three left-hand sides of the equations have the signs $(+, +, +)$, $(-, +, +)$, $(+, -, +)$, $(+, +, -)$, $(+, -, -)$, $(-, +, -)$, $(-, -, -)$. Is there a point at which the signs are $(-, -, +)$?
8. Sketch the triangle formed by the lines $2x-y = 12$, $y = 4$, $x+y = 5$ and write down three functions of x and y which have the property that all are positive at all points within the triangle but at least one is negative at any point outside the triangle.
9. Find the equations of the bisectors of the angles between $x+2y+3 = 0$ and $11x-2y+9 = 0$. Which bisects the angle in which the origin lies?
10. Find the equations of the bisectors of the angles between $2x+16y-1 = 0$ and $4x-7y-5 = 0$. Which bisects the acute angle between the lines?
11. Find the incentre of the triangle whose sides are $x+y = 15$, $x-7y = 39$, $7x-y+31 = 0$.

[The **incentre** is the centre of the circle which lies inside the triangle and touches all its sides. The circle is called the **incircle.**]

12. The lines $6x-7y = 21$, $2x+9y = 41$, $9x-2y+28 = 0$ form a triangle. Find the centre of the ecircle lying outside the third side.

[An **ecircle** is a circle touching the three sides of a triangle, two sides being produced. Its centre, which lies outside the triangle, is an **ecentre.**]

13. Show that the point $(-1, -1)$ is equidistant from the three lines $3x+4y+2 = 0$, $5x+12y+4 = 0$, $12x-5y-6 = 0$. Determine whether the point is the incentre or one of the ecentres of the triangle.
14. Show that a circle can be inscribed in the quadrilateral whose sides are $8x+y = 16$, $4x+7y = 0$, $4x-7y = 8$, and $x+8y+25 = 0$.
15. Find the incentre and the three ecentres of the triangle formed by the axes and the line $4x-3y = 12$.

16. Find the equation of the incircle of the triangle with corners at (2, 0), (8, 0), and (5, 4).

17. Find the distance between $3x-4y+8=0$ and $3x-4y=7$.

18. Find the equations of the two lines which are parallel to $4x+3y=1$ and at a distance 2 from it.

19. Find the coordinates of the image of (3, 4) in the line $x+y=10$.

20. The equation of one bisector of the angles between two lines is $7x-4y+1=0$. If one of the lines is $3x+4y=11$, find the equation of the other.

21. P is a variable point on the curve $x^2-y^2=c^2$; if X, Y are the distances of P from the lines $x+y=0$, $x-y=0$, show that (with a suitable choice of sign) $2XY=c^2$.

22. By finding its distance from the centre, calculate the length of the chord $3x+4y=19$ of the circle $x^2+y^2-4x+6y=156$.

23. Show that the line $lx+my+n=0$ is a tangent to the circle $x^2+y^2+2gx+2fy+c=0$ provided $(lg+mf-n)^2=(l^2+m^2)(g^2+f^2-c)$.

24. Prove that the straight line $lx+my+n=0$ touches the circle $(x-a)^2+(y-b)^2=r^2$ if $(al+bm+n)^2=r^2(l^2+m^2)$.

Find the equations of the common tangents of the circles $x^2+y^2-2x=0$, $x^2+y^2+8x=0$. [A.L.]

25. Find the equation of the ellipse whose axes terminate at the points (0, 0), (5, 0), (8, 4), (3, 4).

26. Find the equation of a hyperbola if its transverse axis, of length 2, lies along $5x-12y=0$, and the axis of the conjugate hyperbola, of length 1, lies along $12x+5y=0$.

In Nos. 27–30, trace the given curves, which are easily recognizable from their equations as being conics. Do not multiply the equations out: they give more information as they stand.

27. $(3x-4y-12)(4x+3y-12)=25$.

28. $(3x-4y-12)^2=20(4x+3y-12)$.

29. $(x+2y-2)^2+(4x-2y-1)^2=20$.

30. $(3x+9y+2)^2-(6x-2y+3)^2=360$.

31. By first writing the L.H.S. in the form $a(x+y)^2+b(x-y)^2$ show that $29x^2-42xy+29y^2=50$ is the equation of an ellipse, and find the lengths of its axes.

32. By writing the factors of the L.H.S. in the form $(p+q)$ and $(p-q)$, where p and q are functions of x and y, show that $(5x+y)(x+5y)=72$ represents a hyperbola, and find the length of its transverse axis.

33. ABC is a triangle in which $AB=AC$. A point P moves between AB and AC in such a way that the product of its distances from these two lines is equal to the square of its distance from BC. Prove that the locus of P is a circle touching AB and AC at B and C.

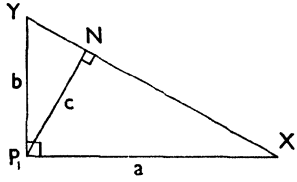
34. If, with the data of No. 33, the distances of P from the three lines are related as before but P does not lie between AB and AC, show that its locus is a hyperbola touching AB and AC at B and C.

Nos. 35–38 give alternative proofs of formula (15).

35. Let the line $lx+my+n=0$ make an angle θ with the x-axis. Let the line $y=y_1$ through P_1 meet the first line at X.

Find P_1X; show that the distance required is $P_1X\sin\theta$; show that $\tan\theta=-l/m$ and hence find the value of $\sin\theta$; finally evaluate the required distance.

36. If P_1XY is a triangle right-angled at P_1, with P_1N the altitude from P_1, and if $P_1X=a$, $P_1Y=b$, $P_1N=c$, show by measuring the area of the triangle in two ways that $c=\pm\dfrac{ab}{\sqrt{(a^2+b^2)}}$.



Now let XY be the line $lx+my+n = 0$ and let P_1X, P_1Y be parallel to the axes as in the figure; find expressions for a and b, and hence c.

37. Write down the equation of the circle with centre P_1 and radius r. Eliminate y between this and the equation $lx+my+n = 0$, and obtain the equation giving the x-coordinates of the points in which the line meets the circle. Write down the condition that this equation shall have equal roots; rewrite the condition so that it expresses r^2 in terms of the other letters, and simplify.

[It helps to keep the algebra under control if you write the expression my_1+n in brackets wherever it occurs and avoid expanding its square.]

38. Let the perpendicular from O to the given line be of length p and make an angle α with Ox. If its foot is A, the original line passes through A and is at right angles to OA; hence, show that its equation can be written $x\cos\alpha + y\sin\alpha = p$.

Next, shift the line so that P_1 moves to O and write the new equation in the form $x\cos\alpha + y\sin\alpha = p_1$, showing that $p_1 = -(x_1\cos\alpha + y_1\sin\alpha - p)$. What is the geometrical meaning of p_1?

Now let $x\cos\alpha + y\sin\alpha = p$ represent the same line as $lx+my+n = 0$, so that

$$\frac{\cos\alpha}{l} = \frac{\sin\alpha}{m} = \frac{-p}{n}.$$

Show that each of these fractions is equal to $\pm\dfrac{1}{\sqrt{(l^2+m^2)}}$, the sign depending on the sign of n (in what way ?), and so deduce the length of the perpendicular.

7.4. Area of a Triangle

The area of a triangle having given angular points is occasionally required, particularly in examinations. If the coordinates of the angular points are given numerically the area can easily be found from first principles by enclosing the triangle in a rectangle whose sides are parallel to the axes.

Example. Find the area of the triangle whose vertices are A (5, 2), B (−3, 5), C (−1, −2).

This triangle can be enclosed in the rectangle $BDEF$ formed by the lines $x+3 = 0$, $y+2 = 0$, $x = 5$, $y = 5$, as shown in Fig. 47 The area of the rectangle is $8 \times 7 = 56$ sq. units, and the rectangle is made up of the given triangle and three others.

Now $\triangle BDC = \frac{1}{2}\times 2\times 7 = 7$ sq. units,

$\triangle CEA = \frac{1}{2}\times 6\times 4 = 12$ sq. units,

and $\triangle AFB = \frac{1}{2}\times 8\times 3 = 12$ sq. units,

making a total of 31 sq. units. Hence $\triangle ABC = 56-31 = 25$ sq. units.

This method is not easily extended to the general case, and in order

to find the area of the triangle $P_1P_2P_3$ (where these letters have their usual meanings) we must first give a meaning to the *sign* of the area, so measuring it by a directed number. The convention used (which suits the algebra best) is as follows. Imagine yourself walking along the sides of the triangle in the given order of the vertices, from P_1 to P_2, from P_2 to P_3, and straight back again from P_3 to P_1: if, as you walk along any side, the remaining vertex is on your left, the area counts as positive; if it is on your right, the area is negative. These two cases are briefly expressed by saying that the triangle is described in a counterclockwise or a clockwise direction.

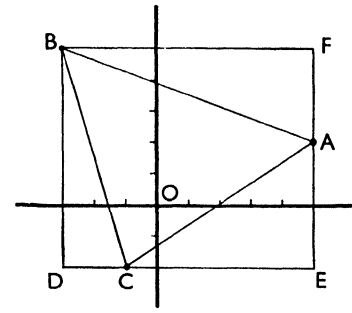


Fig. 47

Consider two equal perpendicular lines LM, MN, so placed that N is on your left as you go from L to M. If the increments in x and y on going from L to M are u, v, then it is readily seen that, whatever the sign of these increments, the increments in x and y on going from M to N are $-v$, u. The case in which u and v are positive is shown in Fig. 48; you should draw figures showing the three other possibilities yourself.

Now take the three points P_1, P_2, P_3, and let Q be the foot of the perpendicular from P_3 to P_1P_2. Since Q lies on P_1P_2, its coordinates are given by

$$x = x_1+s(x_2-x_1), \quad y = y_1+s(y_2-y_1)$$

for a suitable value of s. Since QP_3 is at right angles to P_1P_2, the coordinates of P_3 are given by

$$x = x_1+s(x_2-x_1)-t(y_2-y_1),$$
$$y = y_1+s(y_2-y_1)+t(x_2-x_1) \qquad (16)$$

for a suitable value of t; P_3 is on the left of P_1P_2 if t is positive, on the right if it is negative; and if the length of QP_3 takes its sign from that of t, it is equal to

$$t\sqrt{\{(y_2-y_1)^2+(x_2-x_1)^2\}},$$

the square root being regarded as positive. Then, using the sign convention mentioned,

$$\triangle P_1P_2P_3 = \tfrac{1}{2}P_1P_2 \cdot QP_3 = \tfrac{1}{2}t\{(y_2-y_1)^2+(x_2-x_1)^2\}.$$

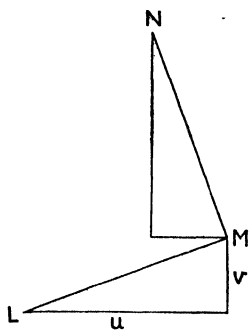


Fig. 48

To finish the calculation we have now only to find the value of t, using the fact that (16) is to represent P_3. Substituting x_3, y_3 for x, y in (16) and rearranging slightly, we find

$$t(y_2-y_1)-s(x_2-x_1)+(x_3-x_1) = 0$$

and

$$t(x_2-x_1)+s(y_2-y_1)-(y_3-y_1) = 0,$$

from which, eliminating s,

$$t\{(y_2-y_1)^2+(x_2-x_1)^2\} = (x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1)$$
$$= x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2).$$

Hence $\triangle \boldsymbol{P_1P_2P_3} = \frac{1}{2}\{\boldsymbol{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)}\}$ (17)

Thus, in the example given, the area of $\triangle ABC$ is

$$\tfrac{1}{2}\{5(5+2)-3(-2-2)-1(2-5)\} = \tfrac{1}{2}(35+12+3) = 25,$$

as before; the triangle was described counterclockwise, and (17) gives it a positive area.

If we are not interested in the sign of the area, or if we are willing to assume certain consequences of the sign convention, the expression for the area can be obtained by simpler means. The two easiest are given in Exercise 7d, Nos. 21 and 22.

The area of a triangle is zero if either its base or its height is zero, and in either case the three vertices are collinear. Thus (17) gives an easy test for collinearity.

Exercise 7D

In Nos. 1–10, find the area of the triangle with the given points in the given order as vertices.

1. (5, 3), (−3, 2), (3, −4).
2. (3, −7), (−3, 1), (2, 5).
3. (4, −1), (3, −5), (−1, 3).
4. (0, 4), (−4, −5), (3, 2).
5. (3, 4), (2, 2), (4, 6).
6. (−1, 5), (7, −3), (3, 1).
7. (0, 0), (x_1, y_1), (x_2, y_2).
8. (x, y), (a, a), $(-a, -a)$,
9. (a^2, a), (b^2, b), (c^2, c).
10. $(bc, b+c)$, $(ca, c+a)$, $(ab, a+b)$.

11. Use the test for collinearity to find the ratio in which the join of (−8, −5) and (7, 5) cuts the join of (−7, 5) and (3, 0).

12. Show that the points A (11, −1), B (−1, 8), C (31, −16), D (−17, 20) are collinear by showing that the point dividing AB in the ratio $\lambda : 1$ is collinear with C and D.

13. Find the locus of a point P which moves so that the area of $\triangle PAB$ is 3 square units, where A and B are (4, 1) and (−5, 7).

14. A, A' are the points (a, a), $(-a, -a)$; find the locus of P if the area of $\triangle PA'A$ is equal to the product of the distances of P from the axes.

The sign convention

15. (i) If ABC is a triangle and T is any point in its plane, draw figures and shade any negative areas to show that, with the given sign convention,

$$\triangle ABC = \triangle TBC + \triangle TCA + \triangle TAB.$$

[Illustrate the case in which T is inside, and two different types of case in which it is outside the triangle.]

(ii) The area of a quadrilateral $ABCD$ is *defined* to be $\triangle ABC+\triangle ACD$. Draw figures to show that (*a*) if the quadrilateral does not cross itself, this definition gives the usual meaning to its area; (*b*) if the quadrilateral crosses itself and AC is parallel to DB, the area is zero; (*c*) in all cases the area is the same as $\triangle BCD+\triangle BDA$.

If the coordinates of the vertices are given in order, the area can therefore be found by two applications of formula (17), and there is no need to draw the quadrilateral.

16. Three parallel lines l, m, n are given, and u, v, w are three lines at right angles to them; l, m, n are met by u in A, D, E, by v in F, T, B, and by w in G, C, H. If the area of a rectangle follows the same sign convention as that of a triangle, draw figures and show by suitable shading that, whatever the order of the lines, $AEHG = ADTF+TDEB+TBHC+TCGF$.

In Nos. 17–20, find the area of the quadrilateral, the order of the vertices being as given (see No. 15).

17. $(-3, 2), (2, -1), (5, 1), (3, 5)$. 18. $(-1, -1), (1, 4), (3, 5), (4, 0)$.

19. $(-3, -4), (4, 2), (-1, 4), (0, -5)$. 20. $(-1, -2), (4, 4), (7, 2), (-2, 1)$.

In Nos. 21–24, obtain formula (17) by the method indicated.

21. Find the equation of P_2P_3; use (15) to write down the distance of P_1 from it; with this as the height of the triangle and P_2P_3 as base obtain an expression for the area. Pay no regard to the sign.

22. Draw a triangle $P_1P_2P_3$ and drop perpendiculars P_1X_1, P_2X_2, P_3X_3 to the x-axis. Show that in your figure, having regard to sign, the area of $\triangle P_1P_2P_3$ is the sum of the areas of the trapezia $P_1P_2X_2X_1$, $P_2P_3X_3X_2$, $P_3P_1X_1X_3$. Write down the areas of these trapezia by the usual rule of pure geometry; hence obtain an expression for the area of $\triangle P_1P_2P_3$. This method fails to show that the formula is true in all cases.

23. Shift the triangle $P_1P_2P_3$ so that P_1 moves to O; write down the coordinates of A and B, the new positions of P_2, P_3. Let OA, OB have lengths a, b, and make angles α, β with Ox; express the coordinates of A and B in terms of a, b, α, β. Now show that the area of $\triangle OAB$, including its sign, is equal to $\frac{1}{2}ab \sin(\beta-\alpha)$; expand this in terms of the sine and cosine of α and β, and hence express the area in terms of the coordinates of A and B. The result is also the area of $\triangle P_1P_2P_3$ since this is congruent to $\triangle OAB$.

24. Let ABC be the triangle, having vertices (x_1, y_1), (x_2, y_2), (x_3, y_3); use the notation of No. 16 and let u, v, w be parallel to Ox. Use No. 15 to show that $\triangle ABC = \frac{1}{2}(TDEB+TBHC+TCGF)$; use No. 16 to simplify the R.H.S. and hence to show that

$$\triangle ABC = \tfrac{1}{2}\{(x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1)\}.$$

From this, obtain formula (17).

7.5 Determinants

Many results in coordinate geometry, including the expression for the area of a triangle, can be written more compactly by means of the determinant notation. In this, $a_1b_2-a_2b_1$ is written $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and is called a **2-row determinant,** the quantities a_1, b_1, a_2, b_2 being called its **elements.**

Thus $\begin{vmatrix} 2 & 3 \\ 5 & 8 \end{vmatrix} = 16-15 = 1$, $\begin{vmatrix} -6 & 10 \\ 9 & -15 \end{vmatrix} = 90-90 = 0$, and the

equation of P_1P_2 can be written $\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix} x - \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} y + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = 0.$

Larger determinants can also be defined, the number of columns being always equal to the number of rows: thus $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is a **3-row determinant**; but before showing how it is expanded we must first explain the term **cofactor.** The cofactor of a given element of a 3-row determinant is defined as follows:

(i) omit from the determinant the row and column containing the given element, so obtaining a 2-row determinant;

(ii) choose from the array $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ the sign occupying the position of the given element, and attach that sign to the 2-row determinant obtained in (i). Thus the cofactor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ and that of c_2 is $-\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$. A 3-row determinant is then expanded by choosing any row or column, multiplying the elements of that row or column by their respective cofactors, and adding the results.

Thus $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, if expanded in terms of its first row, becomes $a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$, while the expansion in terms of the second column is

$$-b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

You should verify that, when the 2-row determinants are multiplied out, both these expansions become $a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2$, and are therefore equal.

Many properties of determinants are proved in algebra books; the two given below are of immediate importance in coordinate geometry.

I. If all the elements of a row (or column) are multiplied by the same quantity, the value of the determinant is multiplied by that quantity. Thus, for example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

as can be seen by expanding in terms of the third row.

II. The value of a determinant is not altered by adding to the elements of a row (or column) equal multiples of the corresponding elements of another row (or column). The multiples may be negative if we wish.

Thus $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2+ka_3 & b_2+kb_3 & c_2+kc_3 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

You should verify this by expanding both determinants completely.

Example 1. Evaluate $\begin{vmatrix} 29 & 38 & 40 \\ 24 & 32 & 34 \\ 19 & 26 & 28 \end{vmatrix}$.

Direct expansion would involve inconveniently large numbers, but these can be avoided by using Property II. Subtract the elements of the second row from those of the first: the determinant is equal to $\begin{vmatrix} 5 & 6 & 6 \\ 24 & 32 & 34 \\ 19 & 26 & 28 \end{vmatrix}$, and to $\begin{vmatrix} 5 & 6 & 6 \\ 5 & 6 & 6 \\ 19 & 26 & 28 \end{vmatrix}$ on subtracting the third row from the second. Again subtracting the second from the first we have $\begin{vmatrix} 0 & 0 & 0 \\ 5 & 6 & 6 \\ 19 & 26 & 28 \end{vmatrix}$, and, expanding in terms of the first row, this reduces to zero.

An important consequence of Property II, which is illustrated here, is that **the value of a determinant is zero if two rows or columns are identical.**

Example 2. Factorize the determinant $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$.

In the explanation, we shall let R and C stand for row and column, and C_1-C_2 will mean that the second column has been subtracted from the first.

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ x-y & y & z \\ x^3-y^3 & y^3 & z^3 \end{vmatrix}, \quad C_1-C_2,$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^3-y^3 & y^3-z^3 & z^3 \end{vmatrix}, \quad C_2-C_3,$$

$$= \begin{vmatrix} x-y & y-z \\ x^3-y^3 & y^3-z^3 \end{vmatrix}, \text{ expanding in terms of } R_1,$$

$$= (x-y)(y-z) \begin{vmatrix} 1 & 1 \\ x^2+xy+y^2 & y^2+yz+z^2 \end{vmatrix}, \text{ by Prop. I,}$$

$$= (x-y)(y-z)(z^2+yz-xy-x^2),$$

$$= (x-y)(y-z)(z-x)(x+y+z).$$

Notice that we could have seen at once that $x-y$ was a factor, since the first two columns are identical when $x = y$ and hence the determinant is zero when $x-y$ is zero. Similar considerations apply to the second and third factors too.

EXERCISE 7E

Evaluate:

1. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$ 2. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$ 3. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix}$ 4. $\begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 5 & 1 & 4 \end{vmatrix}$

Expand completely:

5. $\begin{vmatrix} 1 & c & -b \\ -c & 1 & a \\ b & -a & 1 \end{vmatrix}$ 6. $\begin{vmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{vmatrix}$ 7. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ 8. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Use Properties I and II to evaluate:

9. $\begin{vmatrix} 1 & 2 & 8 \\ 1 & 3 & 12 \\ 1 & 4 & 16 \end{vmatrix}$ 10. $\begin{vmatrix} 2 & 7 & 9 \\ 3 & 1 & 2 \\ 5 & 8 & 11 \end{vmatrix}$ 11. $\begin{vmatrix} 67 & 40 & 21 \\ 39 & 27 & 14 \\ 81 & 50 & 26 \end{vmatrix}$ 12. $\begin{vmatrix} 13 & 16 & 23 \\ 30 & 37 & 53 \\ 39 & 48 & 70 \end{vmatrix}$

13. $\begin{vmatrix} 0 & b & c \\ a & 0 & c \\ a & b & 0 \end{vmatrix}$ 14. $\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix}$ 15. $\begin{vmatrix} a & x & a \\ b & y & b \\ c & z & c \end{vmatrix}$ 16. $\begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix}$

Express in factors:

17. $\begin{vmatrix} a & a & a \\ a & b & b \\ a & b & c \end{vmatrix}$ 18. $\begin{vmatrix} a & b & c \\ b & b^2 & b \\ c & b & a \end{vmatrix}$ 19. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$ 20. $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

Solve the equations:

21. $\begin{vmatrix} x & a & a \\ b & x & b \\ 1 & 1 & 1 \end{vmatrix} = 0.$ 22. $\begin{vmatrix} a-x & b & c \\ c & b-x & a \\ a & b & c-x \end{vmatrix} = 0.$

7.6. Applications of Determinants

(i) Parallel Lines

We have seen in 5.3 that the lines $lx+my+n = 0, l'x+m'y+n' = 0$ are parallel if $l/l' = m/m'$. This condition can now be written

$$\begin{vmatrix} l & m \\ l' & m' \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad (18)$$

(ii) Area; Collinearity

The area of $\triangle P_1P_2P_3$ was found in 7.4 to be

$$\begin{aligned} &\tfrac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\} \\ &= \tfrac{1}{2}(x_2y_3-x_3y_2-x_1y_3+x_3y_1+x_1y_2-x_2y_1) \\ &= \tfrac{1}{2}\left(\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} - \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right). \end{aligned}$$

Now the contents of the last bracket are the expansion of $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ in terms of its third column.

Hence $\triangle P_1P_2P_3 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ . . . . . (19)

a result which is easily remembered.

In particular, **the condition that P_1, P_2, P_3 should be collinear is**

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad (20)$$

(iii) Join of two Points; Chords

If a point P moves so that it remains collinear with P_1 and P_2, its locus is the line P_1P_2.

Hence **the equation of P_1P_2 is** $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ . . . (21)

This method of obtaining the equation of the join of two points has the merit that Properties I and II sometimes enable simplification to be done at an early stage of the working.

Example 1. Find the equation of the line joining

$$\left(\frac{a(1-t_1^2)}{1+t_1^2}, \frac{2bt_1}{1+t_1^2}\right) \text{ and } \left(\frac{a(1-t_2^2)}{1+t_2^2}, \frac{2bt_2}{1+t_2^2}\right).$$

The required equation is

$$0 = \begin{vmatrix} x & y & 1 \\ \dfrac{a(1-t_1^2)}{1+t_1^2} & \dfrac{2bt_1}{1+t_1^2} & 1 \\ \dfrac{a(1-t_2^2)}{1+t_2^2} & \dfrac{2bt_2}{1+t_2^2} & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x & y & 1 \\ a(1-t_1^2) & 2bt_1 & 1+t_1^2 \\ a(1-t_2^2) & 2bt_2 & 1+t_2^2 \end{vmatrix},$$ since by Property I we have here multiplied zero by $(1+t_1^2)(1+t_2^2)$;

$$= \begin{vmatrix} x/a & y/b & 1 \\ 1-t_1^2 & 2t_1 & 1+t_1^2 \\ 1-t_2^2 & 2t_2 & 1+t_2^2 \end{vmatrix},$$ by Property I, dividing the first two columns by a, b;

$$= \begin{vmatrix} x/a & y/b & 1 \\ -(t_1^2-t_2^2) & 2(t_1-t_2) & t_1^2-t_2^2 \\ 1-t_2^2 & 2t_2 & 1+t_2^2 \end{vmatrix},$$ by Property II;

$$= \begin{vmatrix} x/a & y/b & 1 \\ -(t_1+t_2) & 2 & t_1+t_2 \\ 1-t_2^2 & 2t_2 & 1+t_2^2 \end{vmatrix},$$ by Property I, dividing R_2 by t_1-t_2,

$$= 2(1-t_1t_2)x/a+2(t_1+t_2)y/b-2(1+t_1t_2).$$

The equation of the line is thus

$$(1-t_1t_2)\frac{x}{a}+(t_1+t_2)\frac{y}{b} = 1+t_1t_2.$$

This example can be interpreted as finding the equation of the chord joining the points with parameters t_1, t_2 on the curve whose freedom equations are $x = \dfrac{a(1-t^2)}{1+t^2}$, $y = \dfrac{2bt}{1+t^2}$.

The curve is an ellipse, as can be seen by comparing the equations with those of the circle (8). The freedom equations can also be written

$$x/a : y/b : 1 = 1-t^2 : 2t : 1+t^2,$$

and an advantage of this form can be seen in the fact that the equation of the chord is

$$\begin{vmatrix} x/a & y/b & 1 \\ 1-t_1^2 & 2t_1 & 1+t_1^2 \\ 1-t_2^2 & 2t_2 & 1+t_2^2 \end{vmatrix} = 0,$$

which we obtained as the *third* line of our previous working.

(iv) Tangents

The method illustrated in Example 1, which is useful in obtaining the equation of a chord of any curve given parametrically, can also be adapted to give the equation of a tangent. In order to be perfectly general, let the freedom equations of the curve be

$$x : y : 1 = f(t) : g(t) : h(t) \quad . \quad . \quad . \quad (22)$$

where $f(t)$, $g(t)$, $h(t)$ are any functions of t. Then the equation of the chord joining the points with parameters t, $t+\delta t$ is

$$0 = \begin{vmatrix} x & y & 1 \\ f(t) & g(t) & h(t) \\ f(t+\delta t) & g(t+\delta t) & h(t+\delta t) \end{vmatrix}$$

$$= \begin{vmatrix} x & y & 1 \\ f(t) & g(t) & h(t) \\ f(t+\delta t)-f(t) & g(t+\delta t)-g(t) & h(t+\delta t)-h(t) \end{vmatrix}, \text{ by Property II,}$$

$$= \begin{vmatrix} x & y & 1 \\ f(t) & g(t) & h(t) \\ \dfrac{f(t+\delta t)-f(t)}{\delta t} & \dfrac{g(t+\delta t)-g(t)}{\delta t} & \dfrac{h(t+\delta t)-h(t)}{\delta t} \end{vmatrix}, \text{ by Property I,}$$

since the L.H.S is zero.

Now, as $\delta t \to 0$, the elements of the third row tend to $f'(t)$, $g'(t)$, $h'(t)$, the derivatives of $f(t)$, $g(t)$ and $h(t)$ with respect to t. Hence **the equation of the tangent to the curve (22) at the point with parameter t is**

$$\begin{vmatrix} x & y & 1 \\ f(t) & g(t) & h(t) \\ f'(t) & g'(t) & h'(t) \end{vmatrix} = 0 \quad . \quad . \quad . \quad (23)$$

The functions $f(t)$, $g(t)$, $h(t)$ are not completely determined when the freedom equations are given. The following example makes this clear.

Example 2. Find the equation of the tangent to the curve $x = ct$, $y = c/t$.

Method 1. The freedom equations can be written

$x : y : 1 = t : 1/t : 1/c$; hence the tangent is

$$0 = \begin{vmatrix} x & y & 1 \\ t & 1/t & 1/c \\ 1 & -1/t^2 & 0 \end{vmatrix} = \frac{x}{ct^2} + \frac{y}{c} - \frac{2}{t};$$

i.e. $x + t^2 y = 2ct$.

Method 2. The freedom equations can be written

$$x : y : 1 = ct^2 : c : t; \text{ hence the tangent is}$$

$$0 = \begin{vmatrix} x & y & 1 \\ ct^2 & c & t \\ 2ct & 0 & 1 \end{vmatrix} = cx + ct^2y - 2c^2t,$$

leading to the same result as before.

As a rule it is best to write the freedom equations so that fractions are avoided on the R.H.S. The method can be slightly modified to suit individual cases: the equations in Example 2 would be best written $x : y : c = t^2 : 1 : t$, and the tangent to the ellipse above could be obtained most directly from $x/a : y/b : 1 = 1-t^2 : 2t : 1+t^2$.

(v) Concurrence

Finally we shall prove that, **if the lines** $l_1x+m_1y+n_1 = 0$, $l_2x+m_2y+n_2 = 0$, $l_3x+m_3y+n_3 = 0$ **are concurrent or parallel,** $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$; **conversely, if** $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$ **the three lines are either concurrent or parallel.**

Let us call the lines u_1, u_2, u_3.

We have seen in 5.4 that the equation of any line through the intersection of u_1 and u_2 if they are not parallel, or of any line parallel to them if they are, can be written

$$(l_1+\lambda l_2)x+(m_1+\lambda m_2)y+(n_1+\lambda n_2) = 0 \quad . \quad . \quad (24)$$

when λ has a suitable value. Hence, if the three given lines are concurrent or parallel, the equation of u_3 can be written in the form (24), and consequently

$$\frac{l_3}{l_1+\lambda l_2} = \frac{m_3}{m_1+\lambda m_2} = \frac{n_3}{n_1+\lambda n_2}.$$

If each of these fractions is denoted by k, we have

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = k \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_1+\lambda l_2 & m_1+\lambda m_2 & n_1+\lambda n_2 \end{vmatrix}, \text{ by Property I;}$$

$$= k \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_1 & m_1 & n_1 \end{vmatrix}, \text{ by Property II;}$$

$= 0$, since the first and third rows are alike. This proves the first part of the statement.

Next, if the three lines are neither concurrent nor all parallel, two of them must meet in a point; for the sake of definiteness let us suppose that u_1 and u_3 meet at A. The remaining line u_2 does not go through A, and so the equation of the line through A parallel to u_2 can be written $l_2x+m_2y+n_0 = 0$, where $n_0 \neq n_2$. And since this line is concurrent with u_1 and u_3 we have, in the same way as in the first part of the proof, $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_0 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$.

So $$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} - \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_0 \\ l_3 & m_3 & n_3 \end{vmatrix}$$
$$= (n_0-n_2)\begin{vmatrix} l_1 & m_1 \\ l_3 & m_3 \end{vmatrix},$$ expanding the determinants in terms of the third column. But the first factor of the R.H.S. is not zero; and neither is the second, since u_1 is not parallel to u_3. Hence the determinant on the L.H.S. is not zero.

It follows that this determinant is only zero when the three lines are concurrent or parallel: this proves the second part of the statement.

Example 3. Prove that the altitudes of a triangle are concurrent.

Let the corners of the triangle be P_1, P_2, P_3. Then the equation of the altitude through P_1 is

$$(x_2-x_3)x+(y_2-y_3)y = x_1(x_2-x_3)+y_1(y_2-y_3),$$

and similar expressions give the other two altitudes The determinant formed by the coefficients in these three lines is

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & x_3x_1-x_1x_2+y_3y_1-y_1y_2 \\ x_3-x_1 & y_3-y_1 & x_1x_2-x_2x_3+y_1y_2-y_2y_3 \\ x_1-x_2 & y_1-y_2 & x_2x_3-x_3x_1+y_2y_3-y_3y_1 \end{vmatrix}.$$

Adding the elements of the last two rows to those of the first, each element of the first becomes zero. Hence the determinant is zero and the altitudes are concurrent.

EXERCISE 7F

[Exercise 7d Nos. 1–14 can also be worked here by means of determinants.]

1. Use Exercise 7d, Nos. 23 and 15, to show that

(i) $\Delta\, OP_2P_3 = \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$; (ii) $\Delta\, P_1P_2P_3 = \frac{1}{2}\left(\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} - \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\right)$.

Why is the R.H.S. in (ii) the same as $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$?

2. With the help of Exercise 7d, No. 15, use a geometrical argument to express $\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$ as simply as possible.

In Nos. 3, 4, find whether the given points are collinear.

3. (i) $(-2, 3)$, $(0, 7)$, $(2, 11)$; (ii) $(-1, 17)$, $(4, 8)$, $(13, -8)$.

4. (i) $(-23, -16)$, $(-8, -5)$, $(5, 4)$; (ii) $(-23, -16)$, $(3, -2)$, $(16, 5)$.

In Nos. 5–8, find the equation relating t_1, t_2, t_3 if the points with these parameters are collinear. Reject the solutions $t_2 = t_3$, $t_3 = t_1$, $t_1 = t_2$.

5. $x = t$, $y = t^3$.

6. $x = t^2$, $y = t^3$.

7. $x = t$, $y = t^4+t^3$.

8. $x = t$, $y = t^2+1/t$.

9. The vertices A, B, C, D of a quadrilateral are $(-3, 3)$, $(-2, -5)$, $(7, -11)$, $(9, 6)$. AD meets BC at U, AB meets DC at V. Find the mid-points of AC, BD, and UV, and show that they are collinear.

10. A, B, C, D, E, F are the points $(3, 0)$, $(6, 0)$, $(c, 0)$, $(0, 3)$, $(0, 6)$, $(0, f)$. BF meets CE at L, CD meets AF at M, AE meets BD at N. Show that L, M, N are collinear.

In Nos. 11–14, find the equation of the join of the two points.

11. $(47, -28)$, $(-38, 23)$.

12. $(6, 4)$, $(83, 53)$.

13. $(41, 93)$, $(-3, -6)$.

14. $(-17, 27)$, $(47, -77)$.

In Nos. 15–18, find the equation of the chord joining the points with parameters t_1, t_2 (or θ_1, θ_2).

15. $x : y : a = t^2 : 2t : 1$.

16. $x : y : c = t^2 : 1 : t$.

17. $x = a\cos\theta$, $y = b\sin\theta$.

18. $x = a\sec\theta$, $y = b\tan\theta$.

In Nos. 19–22, find the equation of the tangent to the given curve at the given point.

19. $x = \dfrac{a(1+t^2)}{1-t^2}$, $y = \dfrac{2bt}{1-t^2}$.

20. $x = \dfrac{a(1+t^2)}{2t}$, $y = \dfrac{b(1-t^2)}{2t}$.

21. $x : y : a = \sec^3\theta : \tan^3\theta : 1$.

22. $x : y : a = \cos^3\theta : \sin^3\theta : 1$.

23. Show that the lines $4x-3y+2 = 0$, $5x+7y-12 = 0$, $69x+2y-38 = 0$ satisfy the determinant test for concurrence. Find a value of λ for which $4x-3y+2+\lambda(5x+7y-12) = 0$ represents the third line.

24. Show that the three lines $4x-y+7 = 0$, $2x+3y-1 = 0$, $4x+3y+2 = 0$ are not concurrent. Find a value of λ for which $4x-y+7+\lambda(2x+3y-1) = 0$ can be written $4x+3y+c = 0$, and calculate the value of c.

In Nos. 25, 26, find whether the given lines are concurrent.

25. (i) $5x+4y+5 = 0$, $7x+6y+6 = 0$, $10x+9y+7 = 0$.
(ii) $8x+10y+7 = 0$, $5x-5y-11 = 0$, $13x+5y-4 = 0$.
(iii) $6x-y-9 = 0$, $10x-6y-15 = 0$, $8x+3y-12 = 0$.

26. (i) $x+2y-6 = 0$, $8x-11y+6 = 0$, $5x-7y+4 = 0$.
(ii) $4x+3y+15 = 0$, $3x-3y+8 = 0$, $11x+3y+40 = 0$.
(iii) $8x-5y+5 = 0$, $9x-5y+10 = 0$, $4x-3y-1 = 0$.

27. Show that the three lines $x\cos\alpha+y\sin\alpha = p$, $x\cos\beta+y\sin\beta = q$, $x\cos\gamma+y\sin\gamma = r$ are concurrent if $p\sin(\beta-\gamma)+q\sin(\gamma-\alpha)+r\sin(\alpha-\beta) = 0$.

28. Show that, for all values of λ, the line given by the equation

$$(\lambda^2+3\lambda-5)x+(\lambda^2-2\lambda+1)y-(\lambda^2-7\lambda+7) = 0$$

passes through a fixed point.

[Hint: rewrite the equation in the form

$$\lambda^2(\quad)+\lambda(\quad)+(\quad) = 0.]$$

In Nos. 29, 30, find the relation between t_1, t_2, t_3 if the lines obtained by substituting these values for t in the given equation are concurrent.

29. $(t^3-1)x+ty = at^2$.

30. $tx+y = at(t^2+2)$.

EXERCISE 7M

1. A rectangular hyperbola, with its centre at $(3, -5)$ and asymptotes parallel to the coordinate axes, passes through $A(1, 4)$. Find its equation and write down the equation of the tangent at A.

2. Write down two equations connecting the coordinates of P_1, P_2, P_3, P_4 if P_1P_2 is perpendicular to P_3P_4 and P_1P_3 is perpendicular to P_2P_4. Deduce that P_1P_4 is perpendicular to P_2P_3.

3. The equations $x = \dfrac{7-4\sin t}{2+\sin t}$, $y = \dfrac{17+16\sin t}{2+\sin t}$ represent part of a straight line. Find the constraint equation of the line and the coordinates of the ends of the part represented.

4. X, Y lie on Ox, Oy respectively; P divides XY in the ratio $l : m$. Find the coordinates of X and Y in terms of those of P.

X and Y move on the axes so that $OX+OY = c$. If l, m, c remain constant, find the locus of P.

5. Find the coordinates of the points on the line $x+y = 3$ whose distances from $12x-5y+7 = 0$ are each equal to 1.

6. Without assuming the general formula for the area of a triangle in terms of the coordinates of its vertices, prove that the area of the triangle whose vertices are the points $(0, 0)$, $(1, 2)$, $(-2, 3)$ is 3·5.

The triangle is turned completely over about the line joining the origin to the point $(-2, 3)$. Prove that the vertex originally at the point $(1, 2)$ falls on the line $3x+2y+7 = 0$. [A.L.]

---

7. Find the equations of the tangents to the hyperbola $9x^2-y^2 = 9$ which are parallel to $5x+y = 0$.

8. Show that the conditions of No. 2 are satisfied if

$$P_1P_2^2+P_3P_4^2 = P_1P_3^2+P_2P_4^2 = P_1P_4^2+P_2P_3^2.$$

[Subtract $x_1^2+x_3^2+x_4^2+y_1^2+y_2^2+y_3^2+y_4^2$ throughout.]

9. A, B are two points, one on each axis; C, D are two points on AB such that $\angle COD = 90°$. Prove that $\dfrac{BC.BD}{AC.AD} = -\dfrac{OB^2}{OA^2}$.

10. A variable line through the fixed point (h, k) meets Ox at L, Oy at M. Find the locus of a point which divides LM in the ratio $p : q$.

11. Determine whether the points $(100, 198)$ and $(-100, -199)$ are on the same or on opposite sides of $y = 2x-1$.

Find the incentre of the triangle formed by $x = 0$, $y = 2x-1$, $y+2x = 3$. [A.L.]

12. Find the equation of the circle which circumscribes the triangle formed by the lines $3x+y = 0$, $3x-y = 0$, $2x-1 = 0$.

Find the coordinates of the feet of the perpendiculars from the point $(\frac{5}{2}, \frac{5}{2})$ to the sides of this triangle, and prove that the three feet are collinear. [A.L.]

---

13. If the polar of P, wo a parabola with vertex A, is parallel to AP, show that the locus of P is another parabola.

14. Show that the mid-point of the chord $3x-2y = 1$ of the ellipse $x^2+2y^2 = c^2$ is independent of c.

15. If $P(x, y)$ lies on the line through $A(h, k)$ such that AP makes an angle θ with the positive direction of the x-axis, and if $AP = r$, show with the aid of a

diagram that $\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r$. Hence find the distance of the point (1, 2) on the line $5x-12y+19 = 0$ from the point of intersection of this line with $78x+65y-499 = 0$. [A.L.]

16. P_1, P_2 lie on $lx+my+n = 0$ and $l'x+m'y+n' = 0$ respectively. Show that the lines $lx+my+n+\lambda(l'x+m'y+n') = 0$ and $lx+my+n-\lambda(l'x+m'y+n') = 0$ cut P_1P_2 in points A and B which divide P_1P_2 internally and externally in the same ratio.

If λ is chosen so that A is the mid-point of P_1P_2, what can be said about the lines P_1P_2 and $lx+my+n-\lambda(l'x+m'y+n') = 0$?

17. Prove that the distances of two points from the centre of a circle are in the same ratio as the distances of each from the polar of the other.

18. Find the equation of the image of the line $x-8y+9 = 0$ in the line $2x-3y+5 = 0$.

---

19. If the polar of P, wo a parabola with vertex A, bisects AP, show that the locus of P is another parabola.

20. The equal sides AB, AC of an isosceles triangle are produced to U and V, so that $BU.CV = AB^2$. Prove that UV always passes through a fixed point.

21. Show that the equation of any straight line can be put into the form $\frac{x-a}{\cos\theta} = \frac{y-b}{\sin\theta}$. How do you interpret this equation when θ is equal to 0 or $\frac{1}{2}\pi$?

Prove that the coordinates of the reflexion of the point (a, b) in the line $x\cos\alpha + y\sin\alpha = p$ are $2p\cos\alpha - a\cos 2\alpha - b\sin 2\alpha$, $2p\sin\alpha - a\sin 2\alpha + b\cos 2\alpha$. [A.L.]

22. Write down the coordinates of the point dividing P_1P_2 in the ratio $\lambda : 1$.

Obtain a quadratic equation giving the ratios in which P_1P_2 is divided by the ellipse $x^2/a^2+y^2/b^2 = 1$.

Show that, if P_2 lies on the polar of P_1, the ellipse divides P_1P_2 internally and externally in the same ratio.

23. P, Q are two points equidistant from the axis of a parabola. Prove that the distance of P from the polar of Q is equal to the distance of Q from the polar of P.

24. Prove that the reflection of $px+qy+r = 0$ in $ax+by+c = 0$ is $(a^2+b^2)(px+qy+r) = 2(ap+bq)(ax+by+c)$.

CHAPTER 8

FREEDOM EQUATIONS: CONICS

8.1. Preliminary Notes

(i) Abbreviations

The following abbreviations will be used:

'the point t_1' means 'the point with parameter t_1';

'the tangent at t_1' means 'the tangent at the point t_1';

'the chord t_1t_2' means 'the chord joining the points t_1t_2'.

Similar abbreviations will apply when other letters are used for the parameter.

(ii) How to memorize results

There are a great many results to memorize in this chapter and although some must simply be learnt by heart, the following hints may help you to remember others.

Tangents. The rule of 6.4 can be used. For the parabola and rectangular hyperbola it is also helpful to remember the gradient in terms of the parameter: this enables the equations of both tangent and normal to be written down with no more than mental working.

Chords. The equation of the chord t_1t_2 can be obtained from that of the tangent at t by the following rule.

If, when the equation of the tangent is cleared of fractions, the coefficients are algebraic functions of t and contain no power of t higher than t^2,

replace t^2 by t_1t_2,

$2t$ by t_1+t_2

and leave constants unchanged.

Notice that this rule does *not* apply to trigonometric functions.

Meet of two tangents. The coordinates of the meet of the tangents at t_1 and t_2 can be obtained from the coordinates of the point t by the rule just given, provided the latter are expressed in the form $x : y : k = f(t) : g(t) : h(t)$, where $f(t)$, $g(t)$, $h(t)$ are polynomials in t, of degree 2 or less, and k is a constant. Thus the equations $x = ct$, $y = c/t$ have to be written in the form $x : y : c = t^2 : 1 : t$ before the rule is applied.

Geometrical properties. You will remember these more easily if you draw your own figure to illustrate each of them, and if you compare the different curves, noticing in which properties they agree and in which they differ.

(iii) Warning

As in 6.4, it is important to notice that the hints given in (ii) are merely devices for *remembering* the results, not legitimate means of *obtaining* them. If an examination question asks you to *obtain* or *prove* a result, a method such as those given in the text must be used.

In other cases you may write down and use the given results freely in the solution of any problem, except that it is unwise to assume a result which is very close to what you are asked to prove. This is a matter for common sense.

(iv) Proofs

Many of the more easily obtainable results will be given without proof. In such cases it is always intended that you should prove them yourself as an exercise.

For the trigonometrical work it is important that you should have a ready grasp of the formulæ expressing $\sin A \pm \sin B$, etc., as products, and *vice versa*.

(v) A useful check

The coefficients in our equations will mostly be letters rather than numbers, and mistakes in working can often be detected by giving each letter a **dimension.** The coordinates x and y represent lengths, and so their dimension is 1, while that of x^2, xy and y^2 is 2. If we now give the coefficients dimensions in such a way that the dimensions of all the terms of any one equation are equal, it can be proved that the same will be true of the terms of any derived equation. The number zero cannot be given a dimension, but any other numbers which occur must be given the dimension 0.

Thus, in the equation $ax^2+bx+c = 0$, we are free to give a any dimension we please. Let us choose 1, then the dimension of the first term, ax^2, is 3; and we must then give b and c dimensions 2 and 3 respectively to make the two remaining terms have the same dimension as the first. Consider now the well-known solution of this equation,

$$x = \frac{-b \pm \sqrt{(b^2-4ac)}}{2a}.$$

We notice that b^2 and $4ac$ each have dimension 4, so that the dimension of the square root is 2 and agrees with that of the $-b$ to which it is added; the dimension of the numerator is therefore 2, while that of the denominator is 1; hence the dimension of the R.H.S. is 1, agreeing with the L.H.S. If, in writing down the formula, we had made a slip and omitted one of the letters or changed its degree, the mistake would probably have been shown up by a wrong dimension.

Sometimes the dimensions of the coefficients are already determined by the equation, as in $x^2/a^2+y^2/b^2 = 1$, where the dimension of the R.H.S. is 0, and therefore a and b have dimension 1. Often, as in this case, the equation of a curve is so written that the dimension of each letter is 1, to make checking easier. An exception is the parameter, which usually has to have dimension 0.

8.2. The Parabola

We shall find it convenient to let the constraint equation be

$$y^2 = 4ax \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Although the value of y itself could be used as parameter, and any point of the curve written $(y^2/4a, y)$, it is somewhat neater if we replace y by $2at$, and therefore $y^2/4a$ by at^2, obtaining the freedom equations

$$x = at^2, \quad y = 2at \quad . \quad . \quad . \quad . \quad (2)$$

The equation of **the chord** t_1t_2—that is, the join of $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$—can be obtained in one of the usual ways. It is

$$2x-(t_1+t_2)y+2at_1t_2 = 0 \quad . \quad . \quad . \quad (3)$$

The tangent at t **is**

$$x-ty+at^2 = 0 \quad . \quad . \quad . \quad . \quad (4)$$

Notice that its **gradient is** $1/t$. Equation (4) may be obtained directly or deduced from (3) by letting t_1 and t_2 tend to t.

The tangents at t_1 and t_2 meet at

$$(at_1t_2, a[t_1+t_2]) \quad . \quad . \quad . \quad . \quad (5)$$

This point is called the **pole of the chord** t_1t_2. The chord t_1t_2 is its polar.

The line through the point t at right angles to the tangent there is called the **normal** at t. Its equation is

$$tx+y = at(t^2+2) \quad . \quad . \quad . \quad . \quad (6)$$

The normal at a point of the parabola meets the curve at a second

point whose parameter is sometimes required. Consider the normal at t_1: its equation is

$$t_1x+y = at_1(t_1^2+2),$$

and the point with parameter t lies on it if

$$t_1(at^2)+2at = at_1(t_1^2+2),$$

whence
$$t_1t^2+2t-t_1(t_1^2+2) = 0 \quad . \quad . \quad . \quad (7)$$

Now the normal meets the curve in two points, one of which has parameter t_1, and hence $t = t_1$ is one solution of (7). Using this fact, there is no difficulty in writing (7) as

$$(t-t_1)(t_1t+t_1^2+2) = 0,$$

from which it is clear that the parameter of the other point of intersection is given by

$$t_1t+t_1^2+2 = 0.$$

Hence the normal at t_1 cuts the curve again at t, where

$$t = -\left(t_1+\frac{2}{t_1}\right) \quad . \quad . \quad . \quad . \quad (8)$$

The point S $(a, 0)$ and the line $x+a = 0$ are called the **focus** and **directrix** of the parabola. Any chord through the focus is called a **focal chord**. The chord LSL' parallel to the directrix (see Fig. 49) is called the **latus rectum** (Latin = 'side erected'); L and L' are $(a, \pm 2a)$ and are got by substituting $t = \pm 1$ in (2).

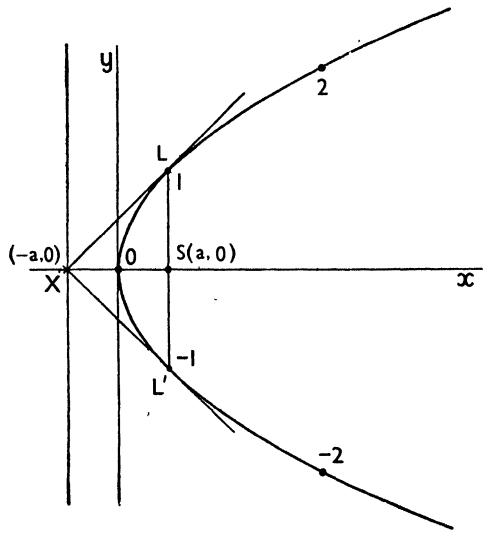


Fig. 49

If it is required to mark the focus and directrix in a sketch of a parabola, a good procedure is:

(i) to draw the axis of symmetry;

(ii) to mark as the focus the point on the axis whose distance from the vertex is one-quarter the width of the curve at that point;

(iii) to mark on the axis the point X at the same distance on the other side of the vertex; the directrix can then be drawn through X at right angles to the axis.

Any line parallel to the axis is called a **diameter** of the parabola. We saw in 6.1 that a parabola can be regarded as a limiting case of an ellipse; as the limit is approached the centre of the ellipse tends to infinity along the major axis, and in the limit any line which passed through the centre becomes parallel to the axis of the parabola.

8.3. Properties of the Parabola

It should hardly be necessary to point out that the properties listed below (with the exception of the last part of III) are properties of *any* parabola, but that for the purpose of proving them it is convenient to place the axes so that the equations are those of 8.2. You should use these equations when making your own proofs, the point P being taken as $(at^2, 2at)$. The properties are illustrated in Figs. 50 and 51.

I. (The focus and directrix property.) Any point P on the curve is the same distance from S as it is from the directrix.

Method 1. Find the constraint equation of the locus of points for which these distances are equal.

Method 2. Write down the distances in terms of t.

II. (The reflector property.) The tangent at any point P on the curve bisects one of the angles between SP and the diameter through P. Hence, rays of light parallel to the axis would be reflected at the interior of a parabolic mirror so as to pass through the focus.

Method 1. As in 7.3, Example 1.

Method 2. Use the formula of 5.2 to find the angles between the lines.

III. The directrix is the polar of the focus, so that tangents at the ends of a focal chord meet on the directrix; these tangents are perpendicular; the line joining the focus to their point of intersection is at right angles to the chord; and if, using (2) of 8.2, the parameters of the ends of the focal chord are t_1, t_2, then $t_1t_2 = -1$.

IV. If the tangent at the vertex and the tangent at P meet at Y, $\angle SYP$ is a right angle.

V. If the tangent at P meets the axis at T, and if N is the foot of the perpendicular from P to the axis, then $TA = AN$, where A is the vertex of the parabola.

VI. If the normal at P meets the axis at G, then $ST = SP = SG$.

VII. (The subnormal property.) NG, known as the subnormal, is constant for all positions of P, being equal to the semi-latus rectum.

VIII. The locus of mid-points of parallel chords is a diameter.

Proof. As the chords are parallel, they all have the same gradient; let it be γ. If t_1, t_2 are the parameters of the ends of a chord,

$$\frac{2}{t_1+t_2} = \gamma, \text{ by (3).}$$

$$\therefore\ t_1+t_2 = \frac{2}{\gamma}.$$

But the ends of the chord are $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$; hence the y-co-ordinate of its mid-point is $a(t_1+t_2)$, which is equal to $\dfrac{2a}{\gamma}$ and is independent of the particular chord chosen.

Hence the locus of mid-points of chords of gradient γ is the diameter $y = \dfrac{2a}{\gamma}$.

An alternative proof can be obtained from the theorem of 6.7.

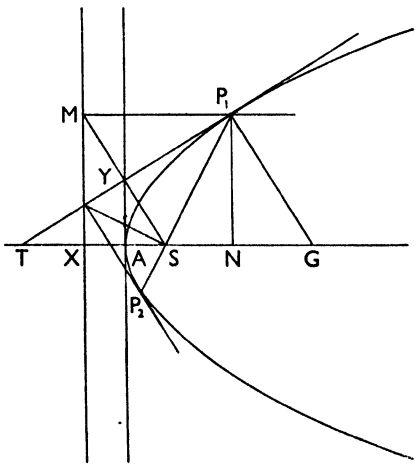


Fig. 50

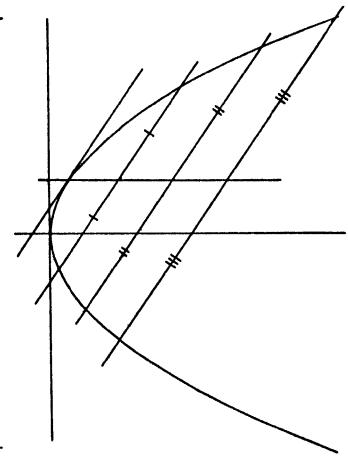


Fig. 51

EXERCISE 8A

Nos. 1–6 are best done by pure geometry.

1. A tangent to a parabola meets the directrix at D and the latus rectum at H; show that D and H are equidistant from the focus.

2. A line intersects a circle; prove that all chords of the circle which are bisected by the line are tangents to a parabola.

3. From any point T on the tangent at P to a parabola perpendiculars TL, TM are drawn to SP and the directrix, S being the focus. Prove that $SL = TM$.

4. Two parabolas have the same focus but their axes are in opposite directions; prove that they cut at right angles.

5. A parabola rolls symmetrically on an equal parabola; find the locus of the focus.

6. A point is marked on a rectangular sheet of paper. If the paper is then folded so that one particular edge always passes through the point, show that the possible positions of the fold are tangents to a parabola.

Chords

7. A chord PQ of a parabola subtends a right angle at the vertex; prove that it cuts the axis in a fixed point.

8. QQ' is a chord of a parabola at right angles to the axis; P is any point on the curve. Prove that PQ, PQ' cut the axis at points equidistant from the vertex.

9. PQ is a focal chord; the line joining P to the vertex cuts the directrix at K; prove that KQ is parallel to the axis.

10. PK, $P'K'$ are the perpendiculars to the directrix from the ends P, P' of a focal chord; prove that PK' and $P'K$ meet at the vertex.

11. P is any point on a parabola with focus S; prove that the tangent at the vertex touches the circle on SP as diameter.

12. The perpendicular bisector of a focal chord PQ cuts the axis at K; prove that $PQ = 2SK$.

Tangents

13. The tangent at P meets the axis at T; show that the tangent at the vertex bisects PT.

14. The tangent at P meets the axis at T; PM is the perpendicular from P to the directrix; prove that MT is parallel to PS, S being the focus.

15. Any line parallel to the y-axis cuts the parabolas $y^2 = 4ax$ and $y^2 = 4bx$ at P and Q respectively; prove that the tangents at P and Q meet on the x-axis.

16. The line joining a point P of a parabola to the vertex meets the directrix at Q; prove that the line joining Q to the focus is parallel to the tangent at P.

17. P, Q, R are three points on a parabola; the chord QR and the tangents at Q, R meet the diameter through P at U, V, W respectively. Prove that $PU^2 = PV.PW$.

18. A, B are two fixed points on the axis of a parabola, equidistant from the focus; prove that the difference of the squares of the distances from A and B to any tangent is constant.

Meets of Tangents

19. The tangents at two points A, B of a parabola meet at C. Show that the line through C parallel to the axis bisects AB.

20. Through the vertex A of a parabola chords AF, AQ are drawn right angles to each other. Show that the tangents at P and Q meet on a ...ed line parallel to the directrix.

21. If the sides of a quadrilateral touch a parabola, show that the join of the mid-points of its diagonals is parallel to the axis.

22. T is the pole of a chord PQ of a parabola, focus S. Prove that $SP.SQ = ST^2$.

23. With the data of No. 22, prove that ST bisects one of the angles between SP and SQ. What does this property become if the chord passes through the focus?

24. With the data of No. 22, prove that the angle which the tangent at P makes with the axis is equal to $\angle STQ$.

25. P, Q are two points on a parabola; the tangents at P and Q meet another tangent at U, V; prove that the mid-point of UV lies on the tangent which is parallel to PQ.

26. P, Q, R are three points on a parabola; the diameters through P, Q meet QR, RP at A, B respectively. Prove that the tangents at P and Q meet at the mid-point of AB.

27. Prove that the product of the distances from the focus S of the parabola $y^2 = 4ax$ to two tangents meeting at P is equal to $a.SP$.

28. Prove that, if two pairs of tangents to a parabola intercept equal lengths on one other tangent, they will intercept equal lengths on every other tangent.

Normals

29. The normal at a point P of a parabola cuts the axis at G; a chord QQ' is drawn through the mid-point of PG parallel to the directrix; prove that $QQ' = 2PG$.

30. From G, the point in which the normal at P meets the axis, a perpendicular GH is drawn to SP, S being the focus. Prove that HP is equal to the semi-latus rectum.

31. Perpendiculars are drawn from the focus to the tangent and normal at any point; show that the join of their feet is parallel to the axis.

32. The tangents at the ends of a focal chord meet at H, the normals meet at K; prove that HK is parallel to the axis.

33. PQ is a chord of a parabola and is concurrent with the axis and directrix; the tangents at P and Q meet at H, the normals meet at K. Prove that HK is bisected by the axis.

34. A chord QR of a parabola subtends a right angle at a point P of the curve; prove that QR meets the normal at P in a fixed point (the **Frégier point** of P).

Normal Chords

35. The normal at P to a parabola meets the curve again at Q; the tangents at P and Q meet at T. Prove that TP is bisected by the directrix.

36. With the data of No. 35, prove that the line through the focus S, perpendicular to ST, bisects TQ.

37. PQ is a focal chord; the normal at P cuts the parabola again at R; prove that the diameter through Q bisects PR.

38. PP' is a focal chord; the normals at P and P' meet the parabola again at Q, Q'. Prove that $Q'Q = 3PP'$ and that the two chords are parallel.

8.4. The Rectangular Hyperbola

We take as the constraint equation

$$xy = c^2 \quad . \quad . \quad . \quad . \quad (9)$$

and as freedom equations

$$x = ct, \quad y = \frac{c}{t}, \quad \text{or } x : y : c = t^2 : 1 : t \quad . \quad . \quad (10)$$

Since the parameter is proportional to x, all points of the curve can be obtained by a suitable choice of t. We cannot use the value $t = 0$, however, as it does not give a finite value to y.

The chord t_1t_2 is

$$x + t_1t_2y - c(t_1 + t_2) = 0 \quad . \quad . \quad . \quad (11)$$

A chord through the centre of a conic is a **diameter**; here the centre is O and so the chord t_1t_2 is a diameter if

$$t_1 + t_2 = 0 \quad . \quad . \quad . \quad . \quad (12)$$

The tangent at t is

$$x + t^2y - 2ct = 0 \quad . \quad . \quad . \quad . \quad (13)$$

and this equation is easily written down if you remember that the **gradient is $-1/t^2$**. The gradient of every tangent is thus negative, as can be seen from a sketch of the curve.

The tangents at t_1 and t_2 meet where

$$x : y : c = 2t_1t_2 : 2 : t_1 + t_2 \quad . \quad . \quad . \quad (14)$$

i.e. at $\left(\dfrac{2ct_1t_2}{t_1 + t_2}, \dfrac{2c}{t_1 + t_2}\right)$.

The normal at t is

$$t^3x - ty = c(t^4 - 1) \quad . \quad . \quad . \quad . \quad (15)$$

and the normal at t_1 meets the curve again at t_2, where

$$t_2 = -\frac{1}{t_1^3} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The transverse axis is $x = y$, the conjugate axis $x + y = 0$. The vertices are (c, c), $(-c, -c)$.

8.5. Properties of the Rectangular Hyperbola

Besides having all the properties of the general hyperbola (see 8.9), the rectangular hyperbola has several properties of its own. You should draw figures to illustrate them.

I. (The orthocentric property.) If the vertices of a triangle lie on a rectangular hyperbola, the altitudes also meet at a point of the curve.

Let the vertices be P_1, P_2, P_3 and have parameters t_1, t_2, t_3; then the gradient of P_2P_3 is $-\dfrac{1}{t_2t_3}$, by (11). Let the altitude through P_1 meet the curve again at P_4, with parameter t_4. The gradient of P_1P_4 is $-\dfrac{1}{t_1t_4}$, and, since P_1P_4 is perpendicular to P_2P_3,

$$\left(-\frac{1}{t_1t_4}\right)\left(-\frac{1}{t_2t_3}\right) = -1,$$

from which $$t_1t_2t_3t_4 = -1 \quad . \quad . \quad . \quad . \quad (17)$$

This equation also shows that P_2P_4 is perpendicular to P_3P_1 and that P_3P_4 is perpendicular to P_1P_2. Hence P_1P_4, P_2P_4, and P_3P_4 are the altitudes of $\triangle P_1P_2P_3$ and meet at P_4 on the curve.

In the same way, the altitudes of $\triangle P_2P_3P_4$ meet at P_1, and so on, so that each of the four points is the orthocentre of the triangle formed by the other three.

II. A chord of a rectangular hyperbola subtends equal or supplementary angles at the ends of (*a*) any perpendicular chord, (*b*) any diameter.

III. If O is the centre of a rectangular hyperbola and P a point on it, OP and the tangent at P are equally inclined to an asymptote.

IV. The lines joining any point of a rectangular hyperbola to the ends of a diameter are equally inclined to an asymptote.

Property III is a limiting case of this, as one end of the diameter tends to the given point on the curve.

EXERCISE 8B

In this exercise, r.h. = rectangular hyperbola.

Chords

1. Q and R are two fixed points of a r.h.; P is a variable point of the curve. If PQ, PR meet an asymptote at U, V, prove that UV is of constant length.

2. M is the mid-point of a chord PQ, O is the centre; show that PQ and OM are equally inclined to an asymptote.

3. P is any point on a r.h., QQ' is a diameter. If PQ' cuts the asymptotes at X, Y, show that $XY = PQ$.

4. With the data of No. 2, if OM cuts the curve at R, show that $PM^2 = OM^2 - OR^2$.

5. P, Q are two points of a r.h., centre O; the line through O at right angles to OP is met by the lines through Q parallel to the asymptotes in the points L, M. Show that LPM is a right angle.

6. N is the foot of the perpendicular to the line $x = y$ from a point P on the r.h. $xy = c^2$. Prove that the power of N wo the circle $x^2+y^2 = 2c^2$ is equal to PN^2.

7. A chord PQ of a r.h. meets the asymptotes at X, Y; R is a point of the curve such that OR is perpendicular to PQ, O being the centre. Prove that $XP.PY = OR^2$.

8. The vertices of a triangle lie on a r.h.; prove that the perpendiculars to its sides, at the points where they meet one of the asymptotes, are concurrent.

Tangents

9. The tangents at P, Q to a r.h. meet one asymptote at L, M. Prove that PQ meets the asymptote at the mid-point of LM.

10. PQ is a chord parallel to the conjugate axis; O is the centre of the curve; prove that OQ is perpendicular to the tangent at P.

11. P, Q, R are points of the curve such that $\angle QPR = 90°$; prove that QR is perpendicular to the tangent at P.

12. P, Q are two points on the r.h. $xy = c^2$, such that the tangent at P is perpendicular to OQ and meets it at T. Prove that $OT.OQ = 2c^2$.

13. If PP' is a diameter of the r.h. $xy = c^2$ and the tangent at P meets the asymptotes at Q and R, prove that $P'Q$ and $P'R$ are tangents to the r.h. $3xy+c^2 = 0$. [A.L.]

14. Two lines are drawn through the centre of a r.h. making equal angles with an asymptote; if H, K are the feet of the perpendiculars to them from a point P of the curve, prove that HK is perpendicular to the tangent at P.

Meets of Tangents

15. The tangents at two points P, Q of a r.h. meet at R. Show that the diameter through R bisects PQ.

16. Through a vertex C of a r.h. chords CP, CQ are drawn at right angles to each other; show that the tangents at P and Q meet on the conjugate axis.

17. The tangents at P, Q meet at K; M is the mid-point of PQ. If a rectangle $KLMN$ is drawn with sides parallel to the asymptotes, prove that L and N lie on the curve.

18. The sides of a quadrilateral touch a r.h.; show that the join of the mid-points of its diagonals passes through the centre of the curve.

19. If the tangents at the ends of a chord of a r.h. meet at H, and K is the foot of the perpendicular from the centre O to the chord, prove that $OH.OK$ is constant for all positions of the chord.

20. If the tangents to a r.h. at P and Q cut at T, and O is the centre, prove that $\angle TOP$ and $\angle TPQ$ are either equal or supplementary.

Normals

21. If the normal at a point P of a r.h. meets the axes *of the hyperbola* at A, B, show that P is the mid-point of AB.

22. C, C' are the vertices of a r.h., P is any point of the curve; the lines through P perpendicular to CP, $C'P$ meet CC' at H, H'; prove that the normal at P bisects HH'.

23. The normal at P cuts the curve again at Q; if O is the centre and M the mid-point of PQ, prove that OM is perpendicular to OP.

24. If the normal at P cuts the curve again at Q, and O is the centre, prove that $PQ^2 = 3OP^2+OQ^2$.

8.6. The Circle and Ellipse

We saw in 7.1 that the circle

$$x^2+y^2 = a^2 \quad . \quad . \quad . \quad . \quad (18)$$

can have freedom equations

$$x = \frac{a(1-t^2)}{1+t^2}, \quad y = \frac{2at}{1+t^2}$$

or

$$\mathbf{x : y : a = 1-t^2 : 2t : 1+t^2} \quad . \quad . \quad . \quad (19)$$

Alternative freedom equations are got by taking the angle AOP

as the parameter of a point P, A being $(a, 0)$. If $\angle AOP = \theta$, then, since $OP = a$, the coordinates of P are

$$x = a\cos\theta, \quad y = a\sin\theta \quad . \quad . \quad . \quad (20)$$

In many respects this is the simpler representation, but it is worth bearing in mind that trigonometric functions are often less easy to manipulate than algebraic.

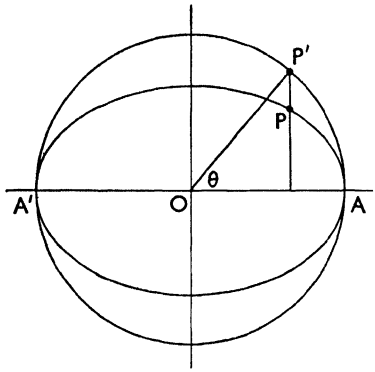


Fig. 52

There is a simple relationship between (19) and (20). $\triangle A^1OP$ is isosceles and hence

$$\angle POA = 2\angle PA'A.$$

It follows that $t = \tan\frac{1}{2}\theta$, and (19) contains the well-known expressions for $\sin\theta$ and $\cos\theta$ in terms of the tangent of the half-angle.

The relationship between the circle (18) and the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad . \quad (21)$$

has already been mentioned in 4.4: the y-coordinates of P and P' in Fig. 52 are in the ratio $b : a$ and hence the freedom equations of the ellipse are

$$x = \frac{a(1-t^2)}{1+t^2}, \quad y = \frac{2bt}{1+t^2}$$

which can be written

$$x/a : y/b : 1 = 1-t^2 : 2t : 1+t^2 \quad . \quad . \quad (22)$$

or

$$x = a\cos\theta, \quad y = b\sin\theta \quad . \quad . \quad . \quad (23)$$

Notice that here t does not represent the gradient of $A'P$ but that of $A'P'$, where P' is the **corresponding point** on the circle (18), the **auxiliary circle** of the ellipse; in the same way, θ measures the angle $P'OA$, not POA. The angle θ is called the **eccentric angle** of P: the reason for this name will be seen when we consider the properties of the curve.

The equation of the chord t_1t_2 is

$$(1-t_1t_2)\frac{x}{a} + (t_1+t_2)\frac{y}{b} = 1+t_1t_2 \quad . \quad . \quad . \quad (24)$$

and of the tangent at t

$$(1-t^2)\frac{x}{a} + 2t\frac{y}{b} = 1+t^2 \quad . \quad . \quad . \quad (25)$$

If $$t_1t_2 = -1 \quad . \quad . \quad . \quad . \quad (26)$$
the chord goes through the centre and is a diameter.

The tangents at t_1 and t_2 meet where
$$x/a : y/b : 1 = 1-t_1t_2 : t_1+t_2 : 1+t_1t_2 \quad . \quad . \quad (27)$$

Using the freedom equations (23), the gradient of the chord $\theta_1\theta_2$ is
$$\frac{b(\sin\theta_1 - \sin\theta_2)}{a(\cos\theta_1 - \cos\theta_2)} = \frac{2b\cos\frac{\theta_1+\theta_2}{2}\sin\frac{\theta_1-\theta_2}{2}}{-2a\sin\frac{\theta_1+\theta_2}{2}\sin\frac{\theta_1-\theta_2}{2}} = -\frac{b\cos\frac{\theta_1+\theta_2}{2}}{a\sin\frac{\theta_1+\theta_2}{2}},$$
and so the equation of the chord is
$$\frac{x}{a}\cos\frac{\theta_1+\theta_2}{2} + \frac{y}{b}\sin\frac{\theta_1+\theta_2}{2} = \cos\frac{\theta_1+\theta_2}{2}\cos\theta_2 + \sin\frac{\theta_1+\theta_2}{2}\sin\theta_2,$$
which simplifies to
$$\frac{x}{a}\cos\frac{\theta_1+\theta_2}{2} + \frac{y}{b}\sin\frac{\theta_1+\theta_2}{2} = \cos\frac{\theta_1-\theta_2}{2} \quad . \quad . \quad (28)$$

Notice that the R.H.S. is zero if $\theta_1-\theta_2 = \pi$, so that points whose parameters differ by π lie at the ends of a diameter.

The tangent at θ can be deduced from (28) by letting θ_1 and θ_2 tend to θ, and is
$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \quad . \quad . \quad . \quad (29)$$

Equation (29) can be written down at sight from (21) and (23) by the rule of 6.4. The rule of 8.1 for writing down the equation of a chord does not apply here, but if it were blindly carried out it would yield
$$\frac{x}{a}\cos\frac{\theta_1+\theta_2}{2} + \frac{y}{b}\sin\frac{\theta_1+\theta_2}{2} = 1,$$
which has the L.H.S. of (28) but not the R.H.S. It represents a *tangent* to the ellipse, namely at the point $\frac{\theta_1+\theta_2}{2}$; a chord parallel to this must be nearer to the origin and the R.H.S. of its equation must therefore be smaller. This fact may help you to remember the R.H.S. of (28); notice also that the equation is not altered by interchanging θ_1 and θ_2.

The tangents at θ_1 and θ_2 meet where
$$x/a : y/b : 1 = \cos\frac{\theta_1+\theta_2}{2} : \sin\frac{\theta_1+\theta_2}{2} : \cos\frac{\theta_1-\theta_2}{2} \quad . \quad (30)$$
Notice that the coordinates are proportional to those of $\left(a\cos\frac{\theta_1+\theta_2}{2}, b\sin\frac{\theta_1+\theta_2}{2}\right)$, which would lie on the ellipse, but are

greater than them; this is as it should be, since the tangents meet at a point outside the curve.

The normal at θ is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \quad . \quad . \quad . \quad (31)$$

If, in what has been said above about an ellipse, the value of b is made equal to a, the corresponding facts about a circle are obtained. The extent to which an ellipse deviates from a circular shape depends upon the ratio $b : a$; in the case in which $b < a$ it is usually measured by $\sqrt{\left(1 - \frac{b^2}{a^2}\right)}$ which is called the **eccentricity** of the curve. If the eccentricity is denoted by e, then

$$a^2 - b^2 = a^2e^2 \quad . \quad . \quad . \quad . \quad (32)$$

This equation rhymes and is easily remembered. Notice that the eccentricity of an ellipse is always less than 1.

The two points $(\pm ae, 0)$ are called the **foci** of the ellipse and the two lines $ex \pm a = 0$ its **directrices.** As in the case of the parabola, any chord through a focus is called a **focal chord,** and each focal chord perpendicular to the major axis a **latus rectum.**

Substituting $x = ae$ in the equation of the ellipse we find that

$$y^2 = b^2(1 - e^2) = b^4/a^2,$$

and hence the semi-latus rectum is $\frac{b^2}{a}$.

To locate the foci in a sketch of an ellipse, notice that (32) means that their distance from an end of the minor axis is equal to the major semi-axis: $SB = OA$ in Fig. 53.

If $b > a$, the major axis lies along Oy. The eccentricity is then $\sqrt{(1 - a^2/b^2)}$, the foci are $(0, \pm be)$, and the directrices and latera recta are parallel to Ox.

The equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point $(a\cos\theta, a\sin\theta)$, obtained by putting $b = a$ in (29), is

$$x\cos\theta + y\sin\theta = a \quad . \quad . \quad . \quad (33)$$

This is sometimes called the **perpendicular form** of the equation of a line, since it brings in the length a of the perpendicular from the origin to the line. It offers one way of finding the distance of a point from a line (see Exercise 7c, No. 38). An answer to a common examination question about it is given in the following example.

Example. Show that the equation $lx+my+n = 0$ can be reduced to the form $x \cos \theta + y \sin \theta = p$, where p is positive, whatever the signs of l, m, and n; and interpret p and θ.

Drop a perpendicular ON from the origin to the line $lx+my+n = 0$ and let its length be p (so that p is positive); then N lies on the circle $x^2+y^2 = p^2$. Let ON make an angle θ with Ox, measured counter-clockwise; then, by the definitions of $\sin \theta$ and $\cos \theta$, the coordinates of N are $(p \cos \theta, p \sin \theta)$. The given line, being perpendicular to ON, is the tangent at N to the circle, and hence its equation can be written $x \cos \theta + y \sin \theta = p$.

8.7 Properties of the Ellipse

I. (The focus and directrix property). The distances of any point of the curve from a focus and from the corresponding directrix are in the ratio $e : 1$.

Method 1. Let the point be $(a \cos \theta, b \sin \theta)$ and find its distances from $(-ae, 0)$ and $ex+a = 0$. Using the fact that $b^2 = a^2(1-e^2)$, from (32), notice that

$$a^2(\cos \theta + e)^2 + b^2 \sin^2 \theta = a^2(e \cos \theta + 1)^2.$$

Method 2. Let the point be (x, y) and find its locus if the squares of its distances from $(-ae, 0)$ and $ex+a = 0$ are in the ratio $e^2 : 1$.

This property is true of the parabola, where $e = 1$, and we shall see that it is also shared by the hyperbola. If a cone is cut by a plane, and two spheres are drawn touching both cone and plane (Dandelin's construction), the points at which the spheres touch the plane are the foci of the curve in which the plane cuts the cone. The spheres touch the cone in two circles, and the planes of these circles cut the first plane in the directrices. The focus and directrix property can then be established by pure geometry, using the fact that all the tangents from a point to a sphere are equal.

II. (The bifocal property.) The sum of the distances of any point of the curve from the two foci is constant and equal to the major axis of the ellipse.

A particular case of this property has already been met in Exercise 4d, No. 13, where it was also used to give a rough method of drawing an ellipse. It can be proved by either of the methods used for I, or it can be deduced from I by showing that the sum of the focal distances is equal to the product of the eccentricity and the distance between the directrices.

III. (The reflector property.) If P is any point on an ellipse whose foci are S and S', the tangent at P is the exterior bisector of the angle $S'PS$. Hence, if a reflector is of the shape formed by rotating an ellipse about its major axis, rays of light from one focus will be reflected so as to pass through the other. The name *focus* (= hearth) was introduced by Kepler in 1604 because of this property.

Method 1. Let the tangent at P meet the major axis at T; show that $SP/S'P = ST/S'T$ and use the converse of a theorem about the bisectors of an angle of a triangle.

Method 2. Let p, p' denote the distances of S and S' from the tangent at P, and let q, q' be their distances from the normal (the signs of q and q' being opposite because the normal lies between the foci); prove that

$$\frac{p}{p'} = -\frac{q}{q'}$$

and deduce the result.

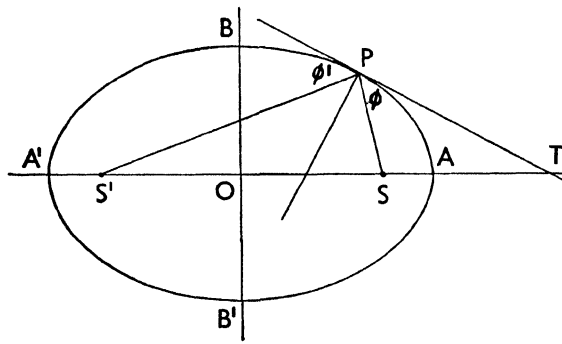


Fig. 53

The property may also be deduced from II by a kinematical argument. Imagine a particle moving counter-clockwise round the ellipse with velocity v, and let SP, $S'P$ make angles ϕ, ϕ' with the tangent as in Fig. 53. If P is the position of the particle, the direction of its velocity will be along the tangent there and hence the resolved parts of its velocity along SP and PS' will be $v\cos\phi$ and $v\cos\phi'$. These measure respectively the rates at which SP is increasing and $S'P$ decreasing. But since $SP+S'P$ is constant, these rates are equal. Hence $v\cos\phi = v\cos\phi'$, and $\phi = \phi'$.

IV. Each directrix is the polar of the corresponding focus; the

tangents at the ends of a focal chord meet on the corresponding directrix.

From the definition of a polar it follows that the two halves of this statement are equivalent, and only one of them need be proved.

V. The product of the distances of the foci from any tangent is equal to the square of the minor semi-axis.

VI. The foot of the perpendicular from a focus to a tangent lies on the auxiliary circle.

Let $P_0(x_0, y_0)$ be the point at which the perpendicular from $(ae, 0)$ meets the tangent at $(a\cos\theta, b\sin\theta)$; then since P_0 lies on the tangent,

$$bx_0\cos\theta + ay_0\sin\theta = ab;$$

and since it lies on the perpendicular,

$$ax_0\sin\theta - by_0\cos\theta = a^2e\sin\theta.$$

Squaring and adding these equations we find

$$(b^2\cos^2\theta + a^2\sin^2\theta)(x_0^2+y_0^2) = a^2(b^2 + a^2e^2\sin^2\theta).$$

Now $b^2\cos^2\theta + a^2\sin^2\theta = b^2 + (a^2-b^2)\sin^2\theta = b^2 + a^2e^2\sin^2\theta$; this expression is never zero and we can therefore divide through by it. Hence $x_0^2+y_0^2 = a^2$ and P_0 lies on the auxiliary circle $x^2+y^2 = a^2$.

We have here solved a locus problem by a new method: the conditions which give rise to the locus were expressed not only in terms of the coordinates of a point on the locus but also in terms of another unknown, in this case θ. Such methods are often necessary and will be considered further in Chapter 9.

VII. (The diameter property.) If two diameters of an ellipse are such that the tangents at the ends of the first are parallel to the second, then each of the diameters is the locus of mid-points of chords parallel to the other.

Proof. Let the first diameter go through the point α, so that its equation is

$$bx\sin\alpha - ay\cos\alpha = 0 \quad . \quad . \quad . \quad \text{(i)}$$

Now the gradient of the tangent at α is $-\frac{b}{a}\cot\alpha$, while that of the chord $\theta_1\theta_2$ is $-\frac{b}{a}\cot\left(\frac{\theta_1+\theta_2}{2}\right)$, and these two are equal if $\theta_1+\theta_2 = 2\alpha$; that is to say, if $\alpha-\theta_1 = \theta_2-\alpha$. Let each of these equal ϵ, then $\theta_1 = \alpha-\epsilon$, $\theta_2 = \alpha+\epsilon$, and any chord parallel to the tangent at α joins the points $\alpha-\epsilon$ and $\alpha+\epsilon$ for a suitable value of ϵ.

Amongst these parallel chords is that for which $\epsilon = \pi/2$, and as the parameters of its ends differ by π it is in fact the required second

diameter. Notice that if $\pi/2$ is added to or subtracted from the parameter of one end of the second diameter, the result is the parameter of one or other end of the first, since multiples of 2π can be ignored. The relationship between the two diameters is therefore symmetrical.

The mid-point of one of the parallel chords is $\left(\frac{a}{2}[\cos\overline{\alpha-\epsilon}+\cos\overline{\alpha+\epsilon}], \frac{b}{2}[\sin\overline{\alpha-\epsilon}+\sin\overline{\alpha+\epsilon}]\right)$, which simplifies to $(a\cos\alpha\cos\epsilon, b\sin\alpha\cos\epsilon)$; as this satisfies (i), the first diameter is the locus of mid-points of chords parallel to the second. Because of the symmetry of the relationship between the diameters, the second is also the locus of mid-points of chords parallel to the first.

Two such diameters are called **conjugate,** and each diameter is said to be conjugate to the chords parallel to the other.

The gradients of the two diameters are $\frac{b}{a}\tan\alpha$ and $-\frac{b}{a}\cot\alpha$, so that **the product of the gradients is $-b^2/a^2$.**

An alternative method of proving this property will be found in 8.9, where it is used to prove the same result in the case of a hyperbola.

VIII. (The trammel property.) If a rod YXP moves so that Y remains on the y-axis, X on the x-axis, then P traces out an ellipse with semi-axes equal to YP and XP.

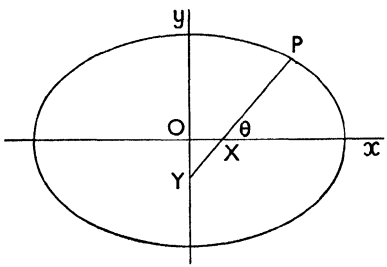


Fig. 54

Let $YP = a$, $XP = b$, and let the rod make an angle θ with the x-axis as in Fig. 54. Then the coordinates of P are $(a\cos\theta, b\sin\theta)$ and it therefore traces out the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

As the rod does not pass through the centre but is eccentric, θ is called the eccentric angle of P. An instrument based on this property is actually used for drawing ellipses: X and Y are two pins whose positions on the rod can be adjusted to give the required values of a and b; the rod carries a pencil at P and the pins run in two perpendicular slots in a base-plate. The device is known as a trammel.

None of these eight properties yields anything of interest in the case in which $b = a$ and the ellipse becomes a circle. The eccentricity is

then zero, both foci are at the centre, and the directrices have vanished, having receded to an infinite distance.

EXERCISE 8C (N.B.—Problems about the foci are often best done by pure geometry, using Properties I–VI)

Find the eccentricity of the ellipse, the coordinates of its foci and the equations of its directrices.

1. $x^2/25+y^2/16 = 1$.
2. $x^2/25+y^2/169 = 1$.
3. $x^2/4+y^2/9 = 1$.
4. $x^2/4+y^2 = 1$.
5. $2x^2+3y^2 = 6$.
6. $2x^2+y^2 = 8$.

In Nos. 7–10, find the equation of the ellipse and the coordinates of its centre.

7. Focus (4, 0), directrix $x = 0$, $e = \frac{1}{2}$.
8. Focus O, directrix $y = 4$, $e = 1/\sqrt{2}$.
9. Foci O and (0, 4); $x = 3$ a tangent.
10. Foci (0, 2) and (6, 2); Ox a tangent.
11. Give a geometrical interpretation of the equation

$$(2x-5)^2 = (x+1)^2+(y-2)^2.$$

12. P is any point on the ellipse $3x^2+4y^2 = 2x+1$. If PN is the perpendicular from P to Oy, show that $2OP = PN+1$.

What does this tell you about a focus, a directrix, and the eccentricity of the ellipse?

13. In Fig. 53, show that $AS.SA' = b^2$.
14. Find the eccentricity of the ellipse $ax^2+by^2 = 1$.
15. A chord of an ellipse, centre O, drawn in a fixed direction, meets the major axis at H; the perpendicular bisector of the chord meets the major axis at K. Prove that OH/OK is constant.
16. If the chord $\theta\phi$ of the ellipse $x^2/a^2+y^2/b^2 = 1$ subtends a right angle at $(a, 0)$, show that

$$(a^2-b^2)\cos\frac{\theta+\phi}{2} = (a^2+b^2)\cos\frac{\theta-\phi}{2}$$

and that the chord passes through a fixed point on the major axis Ox.

In Nos. 17–42, AA', BB' are the major and minor axes of an ellipse with centre O, their lengths being $2a$ and $2b$; S, S' are the foci; P is any point on the curve, P' the corresponding point on the auxiliary circle; PN is the perpendicular from P to AA'; the tangent and normal at P meet the major axis at T, G, the minor axis at T', G'.

Tangents

17. Prove that $ON.OT = a^2$.
18. Prove that $\dfrac{a^2}{OT^2}+\dfrac{b^2}{OT'^2} = 1$.

19. The circle on SS' as diameter meets BB' at Y, Y'; prove that the sum of the squares of the perpendiculars from Y, Y' to any tangent to the ellipse is constant.
20. Show that the circle with centre S and radius SP touches the tangent at P' to the auxiliary circle.
21. H is any point on the tangent at B; show that the other tangent from H to the ellipse is also a tangent to the circle HSS'.
22. The eccentric angles of P, Q, R are α, β, γ. If $2\alpha = \beta+\gamma+2n\pi$, where n is a whole number, show that the chord QR is parallel to the tangent at P.

If the tangent at Q is also parallel to RP, show that the tangent at R is parallel

to PQ. Verify that these conditions are all fulfilled if $\beta = \alpha + \frac{2\pi}{3}$, $\gamma = \alpha - \frac{2\pi}{3}$.

Meets of Tangents

23. The tangents at A, A' meet any other tangent at H, H'. Prove that $AH.A'H' = b^2$.

24. The tangents at A and P meet at H; prove that OH is parallel to $A'P$.

25. If U is the mid-point of a chord and V is the pole of the chord, prove that U and V are collinear with O. If OUV meets the curve at P, prove that $OU.OV = OP^2$.

26. The tangents at the ends of a chord meet at a point D on a directrix; prove that the product of the distances of O and D from the chord is equal to $-b^2$.

27. Q is the point of the curve such that $S'Q$ is parallel to SP, and PQ is not a diameter; prove that the tangents at P and Q meet on the auxiliary circle.

28. Four tangents to an ellipse form a parallelogram; if two of its vertices lie on the directrices, show that the other two lie on the auxiliary circle.

Normals

In Nos. 29–32, prove the given statement.

29. $OG = e^2ON$.

30. $PG : PG' = b^2 : a^2$.

31. $SG = eSP$.

32. $OG.OT = a^2e^2$.

33. Justify the following construction for the normal at P. Let C be the centre of the circle PBB'; take G on OA so that C is the mid-point of OG; then PG is the normal.

34. The normal at P meets $P'O$ at R; prove that $OR = a+b$ and deduce a geometrical construction for a normal to an ellipse.

Conjugate Diameters

In Nos. 35–42, OQ is conjugate to OP.

35. Prove that $OP^2+OQ^2 = a^2+b^2$.

36. Prove that AP and $A'P$ are parallel to conjugate diameters.

37. Prove that the tangents at P and Q meet on the ellipse $x^2/a^2+y^2/b^2 = 2$.

38. If the parallelogram $OPRQ$ is completed, show that its area is ab.

39. A parallelogram is inscribed in an ellipse; prove that its sides are parallel to conjugate diameters.

40. A parallelogram is circumscribed about an ellipse; show that its diagonals lie along conjugate diameters.

41. If OM is the perpendicular from O to the tangent at P, prove that $OM.OQ = ab$.

42. Prove that $PG : OQ = b : a$.

Miscellaneous

43. If one circle lies within another, prove that the locus of centres of circles touching both is either of two ellipses.

44. Ellipses are described with a given focus and corresponding directrix; show that the locus of the ends of their minor axes is a parabola.

45. Obtain the locus of points from which tangents to the ellipse $x^2/a^2+y^2/b^2 = 1$ are at right angles thus: let $P(x, y)$ be on the locus, let the tangents from P be $lx+my = -n$ and $mx-ly = -n'$, so that the coordinates of P satisfy both these equations; square and add the sides of both equations, use the fact that, if $lx+my+n = 0$ touches the ellipse, $n^2 = a^2l^2+b^2m^2$, and so prove that x and y satisfy $x^2+y^2 = a^2+b^2$.

This is therefore the equation of the locus, which is known as the **director circle** of the ellipse.

What length in Fig. 53 is equal to its radius?

46. If P is any point on the director circle (see No. 45), prove that the product of the distances of P and O from the polar of P wo the ellipse is constant.

47. An ellipse rolls on a fixed equal ellipse; if to begin with both major axes are in line, find the loci of the foci of the moving curve.

Deduce that it would be possible to have two elliptical gear wheels, each pivoted at a focus, so that the rate of rotation of one would fluctuate when that of the other was constant.

48. A piece of paper is cut in the shape of a circle, and a point is marked on it. If the paper is folded so that the circumference passes through the point, show that the fold is always a tangent to a certain ellipse.

(Consider the locus of mid-points of lines joining the point to the circumference, and use Property VI.)

8.8. The Hyperbola

Consider first the rectangular hyperbola

$$x^2-y^2 = a^2 \quad . \quad . \quad (34)$$

and let A' be the vertex $(-a, 0)$. Any line through A', unless its gradient is ± 1 or infinite, will meet the curve in one other point P; we can express the coordinates of P in terms of the gradient t of $A'P$ as the parameter, and any point of the hyperbola can be obtained in this way by giving t a suitable value.

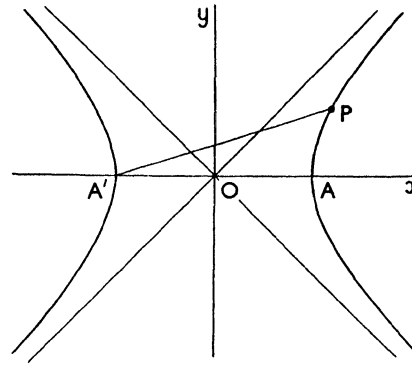


Fig. 55

To find where the line through A' with gradient t meets the curve again, let us express the line itself parametrically, as in 7.2. Thus any point on $A'P$ is given by

$$x = u-a, \quad y = ut \quad . \quad . \quad . \quad (i)$$

where u is the parameter controlling the position of the point on the line. Substituting these values in (34) we find that, if the point (i) lies on the hyperbola,

$$(u-a)^2-u^2t^2 = a^2;$$

$$\therefore \quad (1-t^2)u^2-2au = 0;$$

$$\therefore \quad u = 0 \text{ or } \frac{2a}{1-t^2}.$$

The value $u = 0$ merely gives the point A'; hence P is given by the other value of u. Substituting this value in (i) we obtain as the coordinates of P,

$$x = \frac{a(1+t^2)}{1-t^2}, \quad y = \frac{2at}{1-t^2} \quad . \quad . \quad . \quad (35)$$

and these are the freedom equations of the rectangular hyperbola.

The more general hyperbola whose equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad (36)$$

is obtained from the curve (34) by altering all the y-coordinates in a constant ratio, the new coordinate being obtained from the old by multiplying by b/a—the same process as that which turns a circle into an ellipse. Hence a suitable parametric form of (36) is

$$x = \frac{a(1+t^2)}{1-t^2}, \quad y = \frac{2bt}{1-t^2}$$

or $$\mathbf{x/a : y/b : 1 = 1+t^2 : 2t : 1-t^2} \quad . \quad . \quad (37)$$

but you should notice that the gradient of $A'P$ is now no longer t but bt/a.

The algebraic equations (37) can be replaced by simple trigonometric ones if we write $t = \tan \frac{1}{2}\theta$, making

$$\mathbf{x = a \sec \theta, \quad y = b \tan \theta} \quad . \quad . \quad . \quad (38)$$

or $$\mathbf{x/a : y/b : 1 = 1 : \sin \theta : \cos \theta} \quad . \quad . \quad (39)$$

It can be seen at once that these values of x and y do satisfy (36), but the "angle" θ does not represent anything worth looking at in the figure. The values of θ at different points on the curve are shown in Fig. 56. If the sum of two parameters is π, the corresponding points lie at opposite ends of a diameter, since their coordinates are numerically equal but opposite in sign.

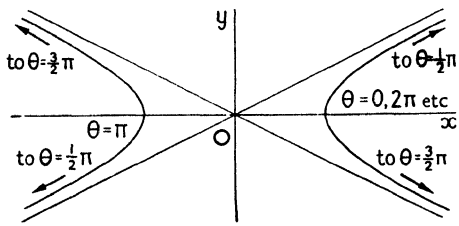


Fig. 56

If the equation of the hyperbola is written $1+y^2/b^2 = x^2/a^2$ and you compare this, (37) and (39) with the corresponding equations for the ellipse, namely (21), (22) and $x/a : y/b : 1 = \cos\theta : \sin\theta : 1$, you will notice that the equations for the hyperbola can be got from

those of the ellipse by exchanging x/a and 1. This makes it easy to write down at sight the equations for the hyperbola corresponding to (24)–(30). These we now give, but you should verify them for yourself by direct means in order to be able to "obtain" them when required.

The chord t_1t_2 is $(1+t_1t_2)\frac{x}{a}-(t_1+t_2)\frac{y}{b}=1-t_1t_2$ . . (40)

The tangent at t is $(1+t^2)\frac{x}{a}-2t\frac{y}{b}=1-t^2$ . . . (41)

The pole of t_1t_2 is given by

$$x/a : y/b : 1 = 1+t_1t_2 : t_1+t_2 : 1-t_1t_2 \quad . \quad . \quad (42)$$

The chord $\theta_1\theta_2$ is

$$\frac{x}{a}\cos\frac{\theta_1-\theta_2}{2}-\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1+\theta_2}{2} \quad . \quad . \quad (43)$$

The tangent at θ is $\frac{x}{a}-\frac{y}{b}\sin\theta=\cos\theta$ . . . . (44)

The pole of $\theta_1\theta_2$ is given by

$$x/a : y/b : 1 = \cos\frac{\theta_1-\theta_2}{2} : \sin\frac{\theta_1+\theta_2}{2} : \cos\frac{\theta_1+\theta_2}{2} \quad . \quad (45)$$

Other considerations arise in finding the equation of a normal, and it cannot be written down by this simple exchange rule. The normal at θ is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2 \quad . \quad . \quad . \quad (46)$$

Since the equation of the hyperbola can also be got from that of the ellipse by changing the sign of b^2, this same change is made in defining the eccentricity of a hyperbola. Thus

$$a^2+b^2=a^2e^2 \quad . \quad . \quad . \quad . \quad (47)$$

and e is always greater than 1. The eccentricity of a hyperbola cannot reasonably be regarded as measuring its deviation from a circular shape but is given a meaning by the focus and directrix property (Property I below). For a rectangular hyperbola $e=\sqrt{2}$, since then $a=b$. Other terms are defined exactly as in the case of the ellipse: thus the foci are $(\pm ae, 0)$, the directrices $ex\pm a=0$ and the auxiliary circle $x^2+y^2=a^2$. Exercise 8d, Nos. 13, 14, will help you to sketch their relative positions.

8.9. Properties of the Hyperbola

The first seven properties given below correspond with the first seven properties of the ellipse. You should prove them yourself, using the same methods as for the ellipse. Notice where there are slight differences between the two cases.

I. (The focus and directrix property.) The distances of any point of the curve from a focus and from the corresponding directrix are in the ratio $e : 1$.

II. (The bifocal property.) The difference between the distances of any point of the curve from the two foci is constant and equal to the transverse axis.

III. (The reflector property.) If P is any point on a hyperbola whose foci are S and S', the tangent at P is the interior bisector of the angle $S'PS$.

Hence rays of light from one focus would be reflected by a hyperbolic mirror so as to appear to have come from the other focus.

IV. A directrix is the polar of the corresponding focus; tangents at the ends of a focal chord meet on the corresponding directrix.

V. The product of the distances of the foci from any tangent is equal to b^2.

Notice, however, that the foci lie on opposite sides of the tangent, whereas in the ellipse they lay on the same side.

VI. The foot of the perpendicular from a focus to a tangent lies on the auxiliary circle.

VII. (The diameter property.) If two diameters of a hyperbola are such that the tangents at the ends of the first are parallel to the second, then each of the diameters is the locus of mid-points of chords parallel to the other.

This may be proved by the same method as was used for the ellipse, but the result can be obtained more easily without freedom equations.

Proof. Let P_1 and P_2 be two points on the curve. Then

$$b^2x_1^2-a^2y_1^2 = a^2b^2$$

and

$$b^2x_2^2-a^2y_2^2 = a^2b^2.$$

$$\therefore \quad b^2(x_1^2-x_2^2)-a^2(y_1^2-y_2^2) = 0;$$

$$\therefore \quad b^2(x_1+x_2) = a^2(y_1+y_2)\frac{y_1-y_2}{x_1-x_2} \quad . \quad . \quad . \quad \text{(i)}$$

Now let γ denote the gradient of the chord P_1P_2 and (x, y) its mid-point; then

$$\gamma = \frac{y_1-y_2}{x_1-x_2}, \quad x = \frac{x_1+x_2}{2}, \quad y = \frac{y_1+y_2}{2}.$$

So, from (i),

$$b^2x = a^2\gamma y \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

and this is the locus of mid-points of chords which have a given gradient γ. As a tangent is a limiting case of a chord, (ii) is also satisfied by the points of contact of tangents with gradient γ.

Now (ii) represents a line through the origin, i.e. a diameter. Let its gradient be γ'. Then

$$\gamma\gamma' = b^2/a^2 \quad . \quad . \quad . \quad . \quad (48)$$

The symmetry of this result shows that the locus of mid-points of chords with gradient γ' is the diameter with gradient γ, and this completes the proof of the property.

Notice, however, that only one of the two conjugate diameters cuts the hyperbola, since from (48) the gradient of an asymptote

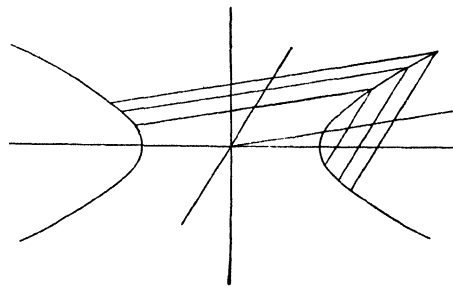


Fig. 57

$(\pm b/a)$ lies between the gradients of the two diameters. This is illustrated in Fig. 57.

VIII. If a chord P_1P_2 of a hyperbola cuts the asymptotes at Q_1 and Q_2, P_1P_2 and Q_1Q_2 have the same mid-point. Thus $P_1Q_1 = P_2Q_2$.

(Prove this by applying VII to both the curve and its asymptotes.)

IX. The point of contact of a tangent to a hyperbola lies midway between the points in which the tangent meets the asymptotes.

This is a limiting case of VIII, but may also be proved directly.

X. A variable tangent to a hyperbola makes with the asymptotes a triangle of constant area.

In 8.5 we established five properties of a rectangular hyperbola. These are true for any such curve and hence for the rectangular hyperbola (34).

8.10. Confocal Conics

Conics having the same foci are called **confocal.** To find the equations of conics having the same foci as the ellipse (21), consider first the equation

$$\frac{x^2}{p}+\frac{y^2}{q}=1 \quad . \quad . \quad . \quad . \quad (49)$$

It represents an ellipse if p and q are both positive, a hyperbola if they differ in sign; and the cases in which we are interested are those in which p is positive and greater than q.

If (49) represents an ellipse, we can write $a^2=p$, $b^2=q$, and the x-coordinates of the foci are $\pm\sqrt{(a^2-b^2)}=\pm\sqrt{(p-q)}$. If it represents a hyperbola, we can put $a^2=p$, $b^2=-q$, and the x-coordinates of the foci are $\pm\sqrt{(a^2+b^2)}=\pm\sqrt{(p-q)}$. In either case, therefore, the foci are $(\pm\sqrt{(p-q)}, 0)$.

It follows that the conics represented by (49) for different values of p and q are confocal if $p-q$ is constant.

In particular, the conics represented by

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}=1 \quad . \quad . \quad . \quad . \quad (50)$$

when λ takes different values, are confocal, and we say that (50) represents a system of confocal conics. If $a>b$, the conics of the system are ellipses when $\lambda<b^2$ and hyperbolas when $b^2<\lambda<a^2$.

Two properties of confocals deserve mention.

I. Two confocals of a system pass through any point, one an ellipse and the other a hyperbola.

Let P_0 be the given point. The conics of the system (50) pass through it if

$$x_0^2(b^2-\lambda)+y_0^2(a^2-\lambda)=(a^2-\lambda)(b^2-\lambda),$$

i.e. $$\lambda^2+(x_0^2+y_0^2-a^2-b^2)\lambda-(b^2x_0^2+a^2y_0^2-a^2b^2)=0.$$

When $\lambda=a^2$, the L.H.S. of this equation becomes $(a^2-b^2)x_0^2$ and is therefore positive.

When $\lambda=b^2$, it becomes $-(a^2-b^2)y_0^2$ and is negative.

As $\lambda\to-\infty$, it is positive since the coefficient of λ^2 is positive.

Hence it is zero for two values of λ, one between a^2 and b^2 (when the curve is a hyperbola), and one less than b^2 (when the curve is an ellipse).

II. If two confocals cut, they cut at right angles.

Let λ_1, λ_2 be the two values of λ for which (50) goes through P_0, so that

$$\frac{x_0^2}{a^2-\lambda_1}+\frac{y_0^2}{b^2-\lambda_1}=1 \quad \text{and} \quad \frac{x_0^2}{a^2-\lambda_2}+\frac{y_0^2}{b^2-\lambda_2}=1.$$

Subtract and divide through by $\lambda_2-\lambda_1$, which is not zero; then

$$\frac{x_0^2}{(a^2-\lambda_1)(a^2-\lambda_2)}+\frac{y_0^2}{(b^2-\lambda_1)(b^2-\lambda_2)}=0.$$

But this is the condition that the tangents to the two curves at P_0, namely

$$\frac{x_0x}{a^2-\lambda_1}+\frac{y_0y}{b^2-\lambda_1}=1 \quad \text{and} \quad \frac{x_0x}{a^2-\lambda_2}+\frac{y_0y}{b^2-\lambda_2}=1,$$

should be perpendicular. The curves therefore cut orthogonally.

Property II also follows from the reflector properties of the ellipse and hyperbola, since by these the tangents at P_0 to the two curves are respectively the exterior and interior bisectors of $\angle S'P_0S$.

Fig. 58 shows the general appearance of a system of confocals.

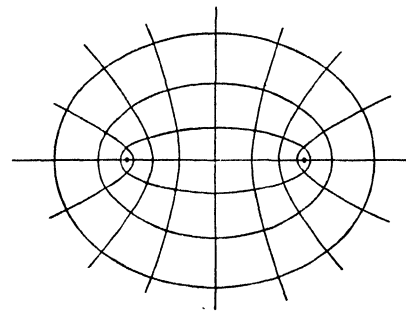


Fig. 58

EXERCISE 8D

In Nos. 1–6, find the eccentricity of the hyperbola, the coordinates of its foci, and the equations of its directrices.

1. $\frac{x^2}{9}-\frac{y^2}{16}=1.$
2. $\frac{x^2}{9}-\frac{y^2}{16}+1=0.$
3. $\frac{x^2}{9}-\frac{y^2}{4}+1=0.$
4. $\frac{x^2}{16}-\frac{y^2}{4}=1.$
5. $5x^2-4y^2=20.$
6. $x^2-3y^2=4.$

Nos. 7–10 are best done by pure geometry.

7. Prove that the locus of the centre of a circle which touches two given non-concentric circles with unequal radii is an ellipse or a hyperbola.

Draw sketches distinguishing between the two cases and showing how the two branches of the hyperbola arise.

8. The tangent and normal at a point P of a hyperbola meet the conjugate axis at T', G'; if S, S' are the foci, show that the circle SPS' passes through T' and G'.

9. A circular hole is cut in a sheet of paper and a point S is marked on the sheet. If the paper is folded so that S lies on the edge of the hole, show that all positions of the fold are tangents to a hyperbola.

(Consider the locus of points midway between S and points of the circle. Use Property VI.)

10. Draw two intersecting lines and mark a point not on them. Use Property

VIII to construct other points on the hyperbola which passes through the marked point and has the lines as asymptotes. Hence sketch the curve.

In Nos. 11–34, the following notation is used. $P'P$ is any diameter of a hyperbola with centre O, eccentricity e, transverse axis $A'A$, and foci S', S. N is the foot of the perpendicular from P to $A'A$; the tangent at P meets the asymptotes at H, H', the transverse axis at T, and the conjugate axis at T'. The normal at P meets these axes at G, G'. The vertices of the conjugate hyperbola are B, B'.

Points

11. A parallelogram is drawn with its sides parallel to the asymptotes of a hyperbola. If the ends of one diagonal lie on the curve, show that the other diagonal passes through the centre.

12. H, K are two points, one on each asymptote, and the parallelogram $OHJK$ is drawn. If HJ, JK cut the curve at L, M, prove that LM is parallel to HK.

13. Show that the directrices meet the asymptotes on the auxiliary circle.

14. Prove that the foot of the perpendicular from a focus to an asymptote lies on the corresponding directrix.

Tangents

15. Prove that $ON.OT = OA^2$.

16. Prove that $OH.OH' = OS^2$.

17. AP, $A'P$ and the tangent at P meet any line parallel to BB' at L, N, M respectively; prove that M is the mid-point of LN.

18. The line through H parallel to OH' cuts the curve at Q; show that $OH' = 4HQ$.

Meets of Tangents

19. The tangents at P and Q meet at M; prove that OM bisects PQ.

20. The tangents at P and Q meet at M; prove that $\triangle OPM$ and $\triangle OMQ$ are equal in area.

21. If the sides of a quadrilateral touch a hyperbola, show that the line joining the mid-points of its diagonals passes through the centre.

22. If the tangents at P and Q meet at M on the auxiliary circle, prove that the lines through P and Q parallel to OM pass through the foci.

Normals

23. Prove that $OG.OT = OS^2$.

24. Prove that $OG = e^2ON$.

25. If the normal at P is parallel to an asymptote, prove that $OG = SG'$.

26. Justify the following construction for the normal at P. The line through P parallel to BB' meets an asymptote at K; the line through K at right angles to OK cuts AA' at G; then PG is the required normal.

27. Prove that $SG = eSP$, and hence that the normal at P bisects $\angle S'PS$.

28. If OF is the perpendicular from O to the tangent at P, prove that $OF.PG = OB^2$.

Conjugate Diameters

29. A parallelogram is drawn with its sides parallel to the asymptotes; prove that its diagonals are parallel to conjugate diameters.

30. Two parallel tangents cut another tangent at K, K'; prove that OK, OK' lie along conjugate diameters.

31. The diameter conjugate to OP meets the conjugate hyperbola at Q; prove that the triangles OPQ and OAB are equal in area.

32. With the data of No. 31, prove that PQ is parallel to one asymptote and bisected by the other.

33. With the data of No. 31, prove that $OP^2-OQ^2 = OA^2-OB^2$.

Are OP and OQ conjugate with respect to the conjugate hyperbola?

34. If a diameter cuts a directrix at K, and S is the corresponding focus, prove that SK is perpendicular to the conjugate diameter.

Confocals [Let $\lambda = 0$ in (50) where possible]

35. P is a point on a hyperbola; prove that its power wo the auxiliary circle is equal to the square of the minor semi-axis of the confocal ellipse through P.

36. The eccentricities of two confocals are e, e'; if $ee' = 1$, prove that the curves meet at the ends of their latera recta.

37. An ellipse with major semi-axis a_1, and a confocal hyperbola with transverse semi-axis a_2, meet at P. If S, S' are the foci, prove that $SP \cdot S'P = a_1^2-a_2^2$.

38. Two confocal ellipses are given. P_1, P_2 are points, one on each, having equal eccentric angles; Q_1, Q_2 are two more such points; prove that $P_1Q_2 = P_2Q_1$.

EXERCISE 8M

1. Find the area of the triangle with sides $x = 3y$, $x+y = 8$, $5x = 3y$.

2. A, B are two fixed points; P is any other point. Points Q, R are taken so that $AQ = PA$, $BR = PB$, $\angle PAQ = \angle RBP = 90°$, Q is to the right of PA, and R is to the left of PB. Prove that the mid-point of QR is independent of the position of P.

3. Prove that the circle on a focal chord of a parabola as diameter touches the directrix.

4. Find the common tangents of $xy = 2$ and $x^2-y^2 = 3$.

5. If, with the notation of Exercise 8c, Nos. 17–42, a point X is taken on OA such that $AX = SP$, prove that $BX = OP$.

6. Prove that the equation of a straight line may be written in the form $x \cos \alpha + y \sin \alpha = p$. State the geometrical interpretation of p and α.

Find the coordinates of the points on the axis of x whose distances from the point $(-a, a)$ are equal respectively to their distances from the line $x+y = 2a$. [A.L.]

---

7. Prove that the points with coordinates $(3, 7)$, $(-5, 22)$, $(-20, 14)$, $(-12, -1)$ are at the corners of a square.

Find the area of the part of this square which lies in the positive quadrant. [A.L.]

8. P, Q, O, R, S are five points in order on a line, O being the mid-point of PS and QR. A line MA is drawn through O, such that $\angle MSP = \angle MAP = 90°$. H, K are two points on PA, equidistant from P; and points U, V are taken on MS, such that $\angle UQH = \angle VRK = 90°$. Prove that $UM = MV$.

[Take O as origin, OA as x-axis, and let P, Q be (a, at), (b, bt).]

9. A focal chord PQ of a parabola meets the directrix at D; if the tangents at P and Q meet at E, prove that $DE^2 = DP \cdot DQ$.

10. Find the equations of the common tangents to the hyperbolas $x^2-2y^2 = 2$ and $2x^2-y^2+2 = 0$.

11. If H is the foot of the perpendicular from a focus S of an ellipse to the tangent at P, and O is the centre, prove that OH bisects SP.

12. The eccentric angles of two points A, B of an ellipse are $\theta \pm \delta$. If the tangents at A and B meet at T, and S is a focus, prove that $SA \cdot SB = ST^2 \cos^2 \delta$.

---

13. Through a point P on the diagonal BD of a square $ABCD$, lines are drawn parallel to the sides and meet AB in Q, AD in R. Show that BR, CP and DQ are concurrent.

14. The line $lx+my+n = 0$ meets the sides BC, CA, AB of a triangle ABC in D, E, F. If A, B, C are (x_1, y_1), (x_2, y_2), (x_3, y_3), show that $\dfrac{BD}{DC} = -\dfrac{lx_2+my_2+n}{lx_3+my_3+n}$, and deduce Menelaus' Theorem, that $\dfrac{BD \cdot CE \cdot AF}{DC \cdot EA \cdot FB} = -1$.

15. Show that the area of the triangle formed by the three points t_1, t_2, t_3 on the parabola $x = at^2$, $y = 2at$ is $a^2(t_2-t_3)(t_3-t_1)(t_1-t_2)$, and that this is double the area of the triangle formed by the tangents at these points.

16. The eccentricity of a hyperbola is $\sec \alpha$, α being an acute angle; show that the eccentricity of the conjugate hyperbola is $\operatorname{cosec} \alpha$.

17. With the notation of Exercise 8c, Nos. 17–42, show that the power of a point P of an ellipse wo the circle on BB' as diameter is equal to $ON \cdot OG$.

18. P, P' are the ends of a diameter of an ellipse, Q is any other point on the curve; if the tangent at P' meets PQ at K, prove that the tangent at Q bisects $P'K$.

---

19. Find the equations of the tangent and normal at any point on the curve $x = a \cos^3 t$, $y = a \sin^3 t$, where t is a variable parameter.

Show that the axes intercept a length a on the tangent and a length $2a \cot 2t$ on the normal. [A.L.]

20. Show that the line OP_1 divides P_2P_3 in the ratio $x_1y_2-x_2y_1 : x_3y_1-x_1y_3$.

Hence prove Ceva's Theorem, that if points D, E, F on the sides BC, CA, AB of a triangle ABC are such that AD, BE, CF are concurrent, $\dfrac{BD \cdot CE \cdot AF}{DC \cdot EA \cdot FB} = +1$.

21. Prove that the orthocentre of the triangle formed by any three tangents to a parabola lies on the directrix.

22. Two rectangular hyperbolas are such that the axes of either are the asymptotes of the other; prove that they cut orthogonally.

23. TP, TQ are tangents to an ellipse, centre O; prove that the triangles OPT and OTQ are equal in area.

24. The normal at P to an ellipse cuts the curve again at Q; if θ, ϕ are the eccentric angles of P, Q, prove that

$$a^2 \sin\theta \sin\frac{\theta+\phi}{2} + b^2 \cos\theta \cos\frac{\theta+\phi}{2} = 0.$$

---

25. Show that the coordinates of the image of P_1 in $x-2y = 0$ are $\frac{1}{5}(3x_1+4y_1)$, $\frac{1}{5}(4x_1-3y_1)$.

Hence show that the curve $11x^2-4xy+14y^2 = 30$ is symmetrical about this line.

26. A, B are the points $(am, -al)$, $(bq, -bp)$ on the lines $lx+my=0$, $px+qy=0$. If the line $lx+my+\lambda(px+qy)=0$ cuts AB at C, find an expression for the ratio $AC:CB$.

If the line $lx+my+\mu(px+qy)=0$ cuts AB at D, show that the value of $\frac{AC.DB}{CB.AD}$ is independent of a and b.

27. PK is the perpendicular from a point P to the directrix of a parabola with focus S; the polar of P cuts the axis at H. Prove that (i) $PK=SH$, (ii) SK is perpendicular to the polar of P.

28. If $(c\tan\theta, c\cot\theta)$ and $(c\tan\phi, c\cot\phi)$ are two points on the hyperbola $xy=c^2$, and if $\theta+\phi$ is constant, show that the join of the points passes through a fixed point on the conjugate axis of the hyperbola.

29. The tangent to an ellipse at a fixed point P on it is cut by a variable pair of conjugate diameters in L, M; show that $LP.PM$ is constant.

30. With the notation of Exercise 8c, Nos. 17–42, if K is the foot of the perpendicular from G' to SP, show that $PK=a$.

---

31. The sides of a parallelogram are $lx+my+n=0$, $l'x+m'y+n'=0$, $lx+my+p=0$, $l'x+m'y+p'=0$. Prove that its area is

$$\pm\frac{(n-p)(n'-p')}{lm'-l'm}.$$

32. Write down the coordinates of the centroid of the triangle $P_1P_2P_3$.

Points Q_1, Q_2, Q_3 are taken, outside the triangle, such that the triangles $P_2P_3Q_1, P_3P_1Q_2, P_1P_2Q_3$ are similar; prove that the triangles $P_1P_2P_3$, $Q_1Q_2Q_3$ have the same centroid.

[Let N_1, the foot of the perpendicular from Q_1 to P_2P_3, divide P_2P_3 in the ratio $\lambda:1$; let $N_1Q_1=\mu P_2P_3$. Express the coordinates of Q_1 in terms of λ, μ, and those of P_2 and P_3.]

33. Find the equations of the two parabolas which have the x-axis as axis, the point $(a, 0)$ as focus, and pass through $(at_0^2, 2at_0)$. Prove that the two curves cut at right angles at this point.

34. Q, R are two fixed points on a hyperbola, P a variable point on the curve; if PQ, PR meet an asymptote at H, K, prove that HK is constant.

35. Prove that the product of the distances of a point of a circle from any two tangents is equal to the square of its distance from their chord of contact.

36. The normal at a point P of an ellipse, of which S is a focus, meets the ellipse again in Q, and the normal at Q meets the major axis in L. A line through L parallel to PQ meets SP in R. Prove that $SQ=SR$. [C.S.]

[This is best proved by pure geometry, using 8.7.]

CHAPTER 9

THREE NEW METHODS

9.1. Loci and Elimination

The method of solving locus problems, given in 3.3, was

(i) to call any point on the locus (x_0, y_0);

(ii) to express in terms of x_0 and y_0 the conditions leading to the locus, so obtaining an equation which these coordinates satisfy, and

(iii) to drop the suffixes from x_0 and y_0.

But in some cases the second operation is difficult or impossible to carry out directly.

Example 1. The perpendicular from the centre of the ellipse $b^2x^2+a^2y^2 = a^2b^2$ to the tangent at any point P meets the line SP at Q, S being the focus $(ae, 0)$. Find the locus of Q.

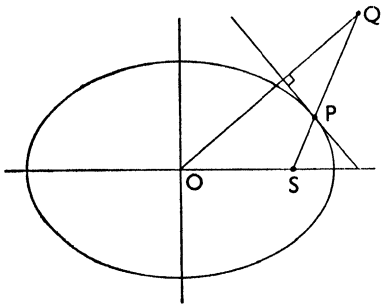


Fig. 59

If we attempt the direct method, we can let Q be (x_0, y_0), and write down the equation of SQ. But here a difficulty arises, for SQ meets the ellipse not only at P but at another point as well. The difficulty is a double one: in the first place an attempt to find where the line meets the conic would lead to very heavy algebra, and in the second place, even if the points were found we should not know at which of them the tangent would be perpendicular to OQ.

In such a case it is best to modify the usual method by introducing another unknown as well as naming the coordinates of a point on the locus.

Let P be $(a \cos \theta, b \sin \theta)$, then SP is

$$bx \sin \theta + a(e-\cos \theta)y = abe \sin \theta.$$

The tangent is $bx \cos \theta + ay \sin \theta = ab$, and so OQ is

$ax \sin\theta - by\cos\theta = 0$. As Q lies on both of these lines, we have

$$bx_0 \sin\theta + a(e-\cos\theta)y_0 = abe\sin\theta \quad . \quad . \quad \text{(i)}$$

and

$$ax_0 \sin\theta - by_0 \cos\theta = 0 \quad . \quad . \quad . \quad \text{(ii)}$$

At this stage let us pause to consider what we are trying to do. So far, we have proceeded as though we were constructing a point on a locus in pure geometry—we have, so to speak, joined SP, drawn the tangent at P, and dropped a perpendicular to it from O—and the natural step to take next would be to produce the perpendicular to meet SP; that is, to find x_0 and y_0 from (i) and (ii). This would in fact give a pair of freedom equations for the locus, expressing the co-ordinates of any point of it in terms of the parameter θ.

But a locus problem is not usually regarded as solved when the freedom equations of the locus are found, since we may not be able to recognize what curve they represent. It is the constraint equation which we require, and before it can be obtained we must get rid of the unknown, θ, which we introduced at the beginning. Our business with (i) and (ii), therefore, is *not* to find x_0 and y_0, but to **eliminate θ,** and this is best done **by setting aside for a time any interest in x_0 and y_0, and concentrating attention entirely upon the parameter.**

Thus (i) and (ii) can be written

$$ay_0\cos\theta + b(ae-x_0)\sin\theta = aey_0$$

and

$$by_0\cos\theta - ax_0\sin\theta = 0,$$

from which we find

$$\cos\theta = \frac{a^2ex_0}{a^2x_0+b^2(ae-x_0)}, \quad \sin\theta = \frac{abey_0}{a^2x_0+b^2(ae-x_0)}.$$

We can now use the identity $\cos^2\theta + \sin^2\theta = 1$ and obtain

$$\begin{aligned} a^2e^2(a^2x_0^2+b^2y_0^2) &= \{(a^2-b^2)x_0+ab^2e\}^2 \\ &= (a^2e^2x_0+ab^2e)^2; \\ \therefore\ a^2x_0^2+b^2y_0^2 &= a^2e^2x_0^2+2ab^2ex_0+b^4; \\ \therefore\ x_0^2+y_0^2-2aex_0-b^2 &= 0. \end{aligned}$$

This equation is satisfied by the coordinates (x_0, y_0) of any point on the locus. The locus equation is therefore got by dropping the suffixes. It represents a circle, centre S, passing through the ends of the minor axis.

Example 1 illustrates the elimination of trigonometric functions by means of the identity $\cos^2\theta+\sin^2\theta = 1$; this and its equivalent forms are often useful. It may not always be necessary to obtain explicit expressions for $\sin\theta$ and $\cos\theta$ as in the example above: in 8.7, Property VI, for instance, a short-cut was found by squaring and

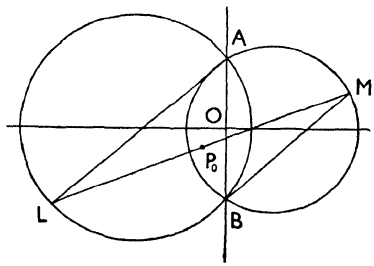


Fig. 60

adding the sides of the two equations.

The next two examples show how an algebraic parameter may be eliminated.

Example 2. The circles

$$x^2+y^2+2px-c^2=0$$

and

$$x^2+y^2+2qx-c^2=0$$

meet at A and B. A variable line through A cuts the first circle again at L; a parallel line through B cuts the second circle again at M. Find the locus of the mid-point of LM.

It is easily seen that the circles meet at the points $(0, \pm c)$. Which are we to call A and which B? Will the choice make any difference to the locus? We do not know at the outset and so our only course is to make an arbitrary choice—we let A be $(0, c)$—and to consider at the end of the problem whether our choice has affected the answer.

Let P_0 be any point on the locus. If we are to employ the direct method we have now to obtain information about L, M and the three variable lines in terms of x_0, y_0, p, q, c. It is not at once clear how to do this, so we had better introduce another unknown and eliminate it later. The unknown should preferably serve as a parameter of both L and M, and their common gradient γ seems the obvious choice.

To find the coordinates of L and M in terms of γ we first express the line AL parametrically. Any point on it is $(t, c+\gamma t)$, and this lies on the first circle if

$$t^2+(c+\gamma t)^2+2pt-c^2 = 0,$$

from which

$$t = 0 \quad \text{or} \quad -\frac{2(p+c\gamma)}{1+\gamma^2}.$$

Hence L is

$$\left(-\frac{2(p+c\gamma)}{1+\gamma^2}, \frac{c-2p\gamma-c\gamma^2}{1+\gamma^2}\right).$$

It is not necessary to go through similar working to obtain M. The second circle is got from the first by replacing p by q, and if the sign of c is changed we obtain BM instead of AL. So if both these changes are made in the coordinates of L we shall have the coordinates of M. They are

$$\left(-\frac{2(q-c\gamma)}{1+\gamma^2}, -\frac{c+2q\gamma-c\gamma^2}{1+\gamma^2}\right).$$

Hence the coordinates of P_0 are given by

$$x_0 = -\frac{p+q}{1+\gamma^2}, \quad y_0 = -\frac{\gamma(p+q)}{1+\gamma^2} \qquad \text{(iii)}$$

Having obtained the coordinates of P_0 in terms of γ, we have now to eliminate γ, and must therefore focus our attention not upon x_0 and y_0 but upon γ itself. It is readily seen from (iii) that $\gamma = y_0/x_0$, and if we substitute this value in the first equation we obtain, after simplifying,

$$x_0^2+y_0^2+(p+q)x_0 = 0.$$

The equation of the locus is therefore

$$x^2+y^2+(p+q)x = 0$$

and represents a circle whose centre is midway between the centres of the two given circles and which passes through the origin.

Did it matter which was A and which was B? No, since the locus equation is not altered by changing the sign of c.

Notice that, since $\gamma = y_0/x_0$, OP_0 is parallel to AL and BM. This could have been seen from pure geometry, since O and P_0 are the mid-points of AB and LM, and if we had noticed it at the beginning we could have found L and M in terms of x_0 and y_0 and solved the problem by the direct method. There is little to choose between the two methods in this instance.

Example 3. Find the locus of mid-points of chords of the parabola $y^2 = 4ax$ which subtend a right angle at the vertex.

If we call a chord UV and its mid-point P_0, it is natural to consider the gradients of OU and OV. Their product is -1, so that if the gradient of OU is γ, that of OV is $-1/\gamma$. U and V are then found to be $\left(\frac{4a}{\gamma^2}, \frac{4a}{\gamma}\right)$ and $(4a\gamma^2, -4a\gamma)$, so that

$$x_0 = 2a\left(\gamma^2+\frac{1}{\gamma^2}\right),$$

$$y_0 = -2a\left(\gamma-\frac{1}{\gamma}\right),$$

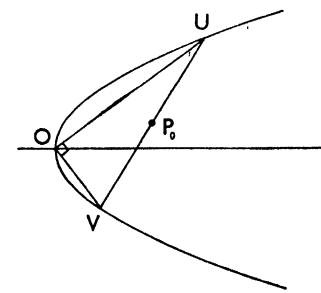


Fig. 61

and we have to eliminate γ between these two equations. Here it is not convenient to express γ in terms of x_0 and y_0; instead, we must use the fact that

$$\left(\gamma-\frac{1}{\gamma}\right)^2 = \gamma^2+\frac{1}{\gamma^2}-2,$$

and this shows that the locus is

$$y^2 = 2a(x-4a),$$

a parabola, vertex $(4a, 0)$, and latus rectum half that of the original curve.

It may occasionally be necessary to introduce more than one extra unknown in order to solve a locus problem. If you look back at the proof of 8.9, Property VII, you will see that, in order to find the locus of mid-points of chords with gradient γ, no fewer than four extra unknowns—x_1, y_1, x_2, y_2—were introduced and successfully eliminated. That, however, was a freak case; the difficulty of elimination usually mounts with distressing rapidity as further unknowns are introduced, and as a general rule you should introduce as few as possible.

Exercise 9A

1. Find the locus of mid-points of focal chords of the parabola $y^2 = 4ax$.
2. The normal at a variable point P of the parabola $y^2 = 4ax$ meets the x-axis at G. Find the locus of the mid-point of PG.
3. P is a variable point on the parabola $y^2 = 4ax$. The line through the focus parallel to the tangent at P meets the diameter through P at Q. Find the locus of Q.
4. P is a variable point on the parabola $y^2 = 4ax$; if the perpendicular from (x_0, y_0) to the tangent at P meets the diameter through P at Q, find the locus of Q.
5. PM is the perpendicular from $P(at^2, 2at)$ to Oy. Find the locus of Q, the foot of the perpendicular from M to OP.
6. C is the point (c, c); P a variable point on the hyperbola $xy = c^2$. If CP meets Ox at X, find the locus of a point Q on CX, such that $CP = XQ$.
7. A variable line through A (h, k) meets Ox and Oy at X, Y. On XY is taken a point P such that $YP = XA$, these being regarded as directed lengths. Find the locus of P, and use geometrical reasoning to show that OA is the tangent to the locus at A.
8. A variable line through the fixed point (h, k) meets Ox at X, Oy at Y. Find the locus of the point which divides XY in the ratio $p : q$.
9. The base of a triangle is fixed; find the locus of the vertex if the ratio of the tangents of half the base angles is constant.
10. A 45° set-square ABC moves so that the ends of its hypotenuse AB remain on Ox and Oy. Find the locus of C if $AC = BC = a$.
11. The normal at P to an ellipse meets the major axis at G. Prove that the locus of the mid-point of PG is a concentric ellipse.
12. The normal at a variable point P of the ellipse $b^2x^2+a^2y^2 = a^2b^2$ meets

Ox at G; the line through G parallel to Oy meets OP at Q; show that the locus of Q is a concentric and similar ellipse.

13. P, Q are the ends of a pair of conjugate diameters of the ellipse $b^2x^2+a^2y^2 = a^2b^2$. Find the locus of the meet of the tangents at P and Q.

14. With the data of No. 13, find the locus of the mid-point of PQ.

15. P is any point on the ellipse $b^2x^2+a^2y^2 = a^2b^2$, and Q is the corresponding point on the auxiliary circle. The normal at P to the ellipse meets OQ at R. Find the locus of R.

16. PP' is a variable chord of an ellipse, centre O, and is parallel to the minor axis. Show that the locus of the meet of OP' and the normal at P is an ellipse.

17. The normal at a point P of the hyperbola $b^2x^2-a^2y^2 = a^2b^2$ meets the transverse axis at G, and GQ is drawn at right angles to this axis and equal to GP. Find the locus of Q.

18. P is a variable point on the hyperbola $b^2x^2-a^2y^2 = a^2b^2$, P' its image in the transverse axis, A and A' the vertices. If AP meets $A'P'$ at Q, prove that the locus of Q is $b^2x^2+a^2y^2 = a^2b^2$.

19. P, Q are variable points on the circles $x^2+y^2-ax = 0$, $x^2+y^2-by = 0$, such that PQ passes through O. Find the locus of the mid-point of PQ.

20. A variable line through $(b, 0)$ meets the circle $x^2+y^2 = a^2$ at P, P'. A, A' are the points $(0, a)$, $(0, -a)$. Find the locus of the meet of AP and $A'P'$.

21. A point moves so that the tangents from it to a parabola enclose an angle of 45°; show that its locus is a rectangular hyperbola and explain the significance of the two branches of the locus.

22. Two normals to the parabola $y^2 = 4ax$ are perpendicular; find the locus of their point of intersection.

9.2. Envelope Equations

A curve may be regarded as being traced out by a moving point, or as being continually touched (*enveloped*) by a moving line. In the first case it is called a locus, and the condition that the point (x, y) should lie on it is called its **locus equation;** in the second case it is called an **envelope,** and the condition that the line $lx+my+n = 0$ should be a tangent to it is called its **envelope equation,** or sometimes its *tangential equation*, or *line-equation.* The envelope equations of four well-known curves are given below, together with their locus equations which are, of course, merely the constraint equations which we have already met.

Locus equation	Envelope equation	
$y^2 = 4ax$	$am^2 = ln$	(1)
$xy = c^2$	$4c^2lm = n^2$	(2)
$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$	$a^2l^2+b^2m^2 = n^2$	(3)
$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$	$a^2l^2-b^2m^2 = n^2$	(4)

We give proofs of (1) and (4); you should prove the other two yourself in a similar way.

The tangent to $y^2 = 4ax$ at $(at^2, 2at)$ is $x - ty + at^2 = 0$ and this is the same line as $lx + my + n = 0$ if

$$\frac{l}{1} = -\frac{m}{t} = \frac{n}{at^2} \quad . \quad . \quad . \quad . \quad (5)$$

Here we have introduced t and must eliminate it again. To do so we notice that, since the three fractions of (5) are equal, the product of the first and last is equal to the square of the second. Hence the line $lx + my + n = 0$ touches the parabola if $ln/a = m^2$, i.e. $am^2 = ln$.

Other properties of equal fractions are proved in algebra books and are often useful in elimination. Thus the tangent to $x^2/a^2 - y^2/b^2 = 1$ at $(a \sec \theta, b \tan \theta)$ is

$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta - 1 = 0,$$

and this is the same line as $lx + my + n = 0$ if

$$\frac{al}{\sec \theta} = \frac{-bm}{\tan \theta} = \frac{-n}{1}:$$

Hence

$$\frac{a^2l^2 - b^2m^2}{\sec^2 \theta - \tan^2 \theta} = \frac{n^2}{1},$$

from which (4) is obtained immediately.

It will help you to remember the envelope equations given above if you notice (*a*) that in each case the x and y of the locus equation are replaced by l and m respectively in the envelope equation;

(*b*) that the coefficients in the locus equation are replaced by their reciprocals, and

(*c*) that each term in the envelope equation is of the second degree in l, m, and n jointly. The only things you have then to take note of are the disappearance of the 4 in (1) and its appearance in (2).

These rules do not extend readily to equations containing other terms, and we shall not consider the general case here. It is worth noticing, however, that the envelope equation of any **circle** can be written down from the fact that the distance from the centre to any tangent is equal to the radius.

A **point** also has an envelope equation, namely the condition that the line $lx + my + n = 0$ should pass through it. The line passes through the point (h, k) if $lh + mk + n = 0$; hence the envelope equation of (h, k) is $hl + km + n = 0$. What comes to the same thing,

however, and is more important, is that, **if l, m, n vary but satisfy a first-degree equation with constant coefficients, the variable line $lx+my+n = 0$ passes through a fixed point.**

For example, if $3l-4m+5n = 0$, this can be written $l(\frac{3}{5})+m(-\frac{4}{5})+n = 0$ and shows that the line $lx+my+n = 0$ passes through the point $(\frac{3}{5}, -\frac{4}{5})$.

We shall illustrate the use of envelope equations of conics by means of three examples. The envelope equation of a point will be used in Example 3 of 9.3.

Example 1. A variable tangent to the hyperbola $b^2x^2-a^2y^2 = a^2b^2$ meets the ellipse $b^2x^2+a^2y^2 = a^2b^2$ at L and M; the tangents to the ellipse at L and M meet at P. Find the locus of P.

Let P be (x_0, y_0); then since LM is the polar of P wo the ellipse, its equation is $b^2x_0x+a^2y_0y = a^2b^2$.

As this is a tangent to the hyperbola,

$$a^2(b^2x_0)^2-b^2(a^2y_0)^2 = (a^2b^2)^2;$$
$$\therefore \quad b^2x_0^2-a^2y_0^2 = a^2b^2.$$

The locus of P is therefore the hyperbola itself.

Example 2. (8.7, Property VI.) Find the locus of the foot of the perpendicular from $S(ae, 0)$ to a tangent to the ellipse $b^2x^2+a^2y^2 = a^2b^2$.

Let P_0 be any point on the locus. The gradient of SP_0 is $y_0/(x_0-ae)$; hence the equation of the line through P_0 at right angles to SP_0 is

$$(x_0-ae)x+y_0y-(x_0^2+y_0^2-aex_0) = 0,$$

and since this is a tangent to the ellipse,

$$a^2(x_0-ae)^2+b^2y_0^2 = (x_0^2+y_0^2-aex_0)^2.$$

The locus equation is therefore

$$a^2(x-ae)^2+b^2y^2 = (x^2+y^2-aex)^2 \quad . \quad . \quad (i)$$

This is an equation of the fourth degree, whereas it should have been $x^2+y^2 = a^2$. Can something have gone wrong? If we re-write (i) we get

$$(x^2+y^2)^2-(2aex+b^2)(x^2+y^2)+a^2(2aex+b^2-a^2) = 0,$$

and this can be factorized, giving

$$(x^2+y^2-a^2)(x^2+y^2-2aex+a^2-b^2) = 0.$$

Hence a point on the locus satisfies either or both of the equations

$$x^2+y^2 = a^2$$

and

$$x^2+y^2-2aex+a^2e^2 = 0.$$

The first of these represents the auxiliary circle; what of the second? It can be written

$$(x-ae)^2+y^2 = 0$$

and so represents a point circle at the focus S. As it is not possible at this stage to explain in what sense the point S can claim to belong to the locus, we disregard it.

Comparing the solution just given with that in 8.7, it is tempting to conclude that the direct method of solving a locus problem may put into the answer extra points which would not be given by the method of elimination. But such a conclusion is unjust. Actually the solution given in 8.7 was one of those 'polished products of omniscient cunning' mentioned in the preface, and it enabled us to divide through by the offending factor $b^2\cos^2\theta + a^2\sin^2\theta$ without letting it obscure the answer.

Let us go back to that solution and see what would have happened if we had made a more pedestrian approach to the elimination. We had

$$bx_0\cos\theta + ay_0\sin\theta = ab$$

and

$$ax_0\sin\theta - by_0\cos\theta = a^2e\sin\theta.$$

The second of these could have been rewritten as

$$by_0\cos\theta - a(x_0 - ae)\sin\theta = 0$$

and we could then have obtained

$$\cos\theta = \frac{a(x_0-ae)}{x_0^2+y_0^2-aex_0}, \quad \sin\theta = \frac{by_0}{x_0^2+y_0^2-aex_0}.$$

On eliminating θ we should once again have been led to equation (i).

The only fair conclusion, therefore, is that the solution of a locus problem by *either* method may contain unwanted factors, and we have simply to get rid of them by any means available.

Example 3. A line moves so that the algebraic sum of its distances from (1, 2), (−2, −1), (1, −1) is equal to 6; find its envelope.

Let $lx+my+n = 0$ be any position of the line Then.

$$\frac{(l+2m+n)+(-2l-m+n)+(l-m+n)}{\sqrt{(l^2+m^2)}} = \pm 6;$$

$$\therefore \quad (3n)^2 = 36(l^2+m^2);$$

$$\therefore \quad 4l^2+4m^2 = n^2,$$

and this, by (3), is the envelope equation of the circle $x^2+y^2 = 4$. The envelope is therefore a circle with centre at O and radius 2.

EXERCISE 9B

Loci

1. A variable tangent to the parabola $y^2 = 4ax$ meets $y^2 = 8ax$ at L and M. Find the locus of the mid-point of LM.

2. The tangent at any point of the parabola $y^2 = 4ax$ meets the axes at L and M. Find the locus of the mid-point of LM.

3. Find the locus of a point whose polar wo $y^2+2bx = 0$ touches the parabola $y^2 = 4ax$.

4. If the polar of P wo the parabola $y^2 = 4ax$ touches the parabola $x^2 = 4by$, show that the locus of P is $xy+2ab = 0$ and that the polar of P wo the second parabola touches the first.

5. A variable tangent to $2xy = c^2$ cuts Ox, Oy at X, Y. The lines joining $(c, -c)$ to X, and $(-c, c)$ to Y, meet at P. Find the locus of P.

6. A variable tangent to $b^2x^2+a^2y^2 = a^2b^2$ meets the parabola $y^2 = 2bx$ at L, M; the tangents at L and M meet at P. Find the locus of P.

7. The tangent to the circle $x^2+y^2 = a^2$ at a variable point L on it meets Ox at M; OL meets the line $x = b$ at N. Find the locus of P, if PM and PN are perpendicular to Ox and Oy.

8. A triangle has two of its sides along the coordinate axes and its third side is any tangent to the circle $x^2+y^2 = a^2$. Prove that the locus of the circumcentre of the triangle is $a^2(x^2+y^2) = 4x^2y^2$. [A.L.]

Envelopes

9. A line moves so that, if it meets Ox, Oy at X, Y, $OX.OY$ is constant. Find its envelope.

10. A variable line meets the lines $x \pm y+a = 0$ at L, M. Find its envelope if the mid-point of LM lies on Oy.

11. Find the envelope of a line which moves so that the product of its distances from the points $(\pm 5, 0)$ is equal to -16.

12. Find the envelope of a line if the product of its distances from (c, c) and $(-c, -c)$ is equal to $-c^2$.

13. O is a fixed point midway between two parallel lines on which points P, Q move so that $\angle POQ = 90°$. Find the envelope of PQ.

14. A tangent to the hyperbola $b^2x^2-a^2y^2 = a^2b^2$ cuts the lines $x = \pm a$ at H, K. If K' is the image of K in Ox, show that HK' is a tangent to the ellipse $b^2x^2+a^2y^2 = a^2b^2$.

15. A, A' are the vertices of an ellipse, P a variable point on the curve; if the tangents at A, A' meet $A'P$, AP at Q, Q' respectively, show that QQ' envelopes an ellipse.

16. Prove that a chord which joins the ends of a pair of conjugate diameters of an ellipse always touches a similar ellipse.

Common Tangents

17. Find the common tangents of the parabola $y^2 = 4x$ and the hyperbola $3x^2-4y^2 = 12$.

18. Find the common tangents of the circles $x^2+y^2 = 1$, $(x-10)^2+y^2 = 49$.

19. Prove that, if $a^2l^2+b^2m^2-n^2 = 0$, the line $lx+my+n = 0$ touches the ellipse $x^2/a^2+y^2/b^2 = 1$ at the point $(-a^2l/n, -b^2m/n)$.

Find the common tangents of the two ellipses $x^2/a^2+y^2/b^2 = 1$ and $x^2/c^2+y^2/d^2 = 1$, and show that their points of contact lie on the ellipse $(b^2+d^2)x^2+(a^2+c^2)y^2 = a^2d^2+b^2c^2$. [A.L.]

20. Find the equations of the common tangents to the ellipses whose envelope equations are $a^2l^2+b^2m^2 = n^2$ and $c^2l^2+d^2m^2 = n^2$, and show that they also touch the ellipse whose envelope equation is $a^2l^2+b^2m^2-n^2+k(c^2l^2+d^2m^2-n^2) = 0$.

9.3. Twins

When a line cuts a conic, it may happen (as in 6.6 or 7.1) that the coordinates of one point of intersection are known and that we want to find those of the other. In such a case, although a quadratic equation may appear temporarily, this equation can be solved by factors, and the coordinates of the required point are then obtained from a *linear* equation.

But if neither of the points is known beforehand, any attempt to find one of them will lead to a quadratic equation giving both (see 7.2, Examples 4 and 6). This is a feature, not only of the points in which a line cuts a conic, but of pairs of points or lines in various other cases—for example, the pair of tangents from a given point to a conic (see 7.2, Example 3) or the pair of points in which these tangents meet a given line. It is a feature which occurs, in fact, whenever two elements can be found from a quadratic equation, if there is no immediate way of distinguishing between them.

Any two such points or lines will be called **twins,** and just as it is easiest to refer to human twins by their surname, so in dealing with twin points it is best not to treat them individually but to use our knowledge of the sum and product of the roots of the quadratic equation from which they spring, since this preserves the symmetry between the points, and the roots themselves will generally be such clumsy expressions as to be practically useless for anything but numerical calculation.

These ideas, which we have already used in 6.7 to find the mid-point of a chord, are best developed by means of examples.

Example 1. Prove that on the axis of a parabola there is a point K having the property that, if a chord PQ passes through it, the value of $\frac{1}{PK^2} + \frac{1}{QK^2}$ is constant for all positions of the chord.

We begin by letting the parabola be $y^2 = 4ax$ and taking a point $K(k, 0)$ on the axis. We have then to show that a value of k can be found giving K the required property.

If we draw a chord through K, there is no way of telling which end is P and which is Q: P and Q are twins and we must obtain information about them from a quadratic equation. There are several possible ways of doing this. In the first place we could write down the equation of any line through K and, by eliminating x between it and the equation of the parabola, obtain a quadratic whose roots

would be the y-coordinates of P and Q. Or we could find the equation giving the parameters of P and Q by letting $(at^2, 2at)$ lie on the line through K. Either method could be used successfully, but as we are concerned with the distances PK and QK it will be better if we can introduce them more directly into the equation. This we can do by expressing the freedom equations of a line through K in terms of the distance from K as in 7.2.

Consider the point $(k + r\cos\theta, r\sin\theta)$. Its distance from K is r, and it lies on the line through K with gradient $\tan\theta$. It lies also on the parabola if

$$r^2\sin^2\theta = 4ak+4ar\cos\theta,$$

i.e.
$$r^2\sin^2\theta - 4ar\cos\theta - 4ak = 0 \quad . \quad . \quad . \quad \text{(i)}$$

and this is the equation giving the values of r which measure KP and KQ.

Let the roots of (i) be r_1 and r_2, so that

$$r_1+r_2 = \frac{4a\cos\theta}{\sin^2\theta}, \quad r_1r_2 = -\frac{4ak}{\sin^2\theta}.$$

Then
$$\frac{1}{PK^2} + \frac{1}{QK^2} = \frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{r_1^2 + r_2^2}{r_1^2r_2^2} = \frac{(r_1+r_2)^2 - 2r_1r_2}{(r_1r_2)^2}$$
$$= \frac{16a^2\cos^2\theta + 8ak\sin^2\theta}{16a^2k^2} = \frac{k+(2a-k)\cos^2\theta}{2ak^2}.$$

This is constant, as θ varies, if $2a-k = 0$. Hence the point $K(2a, 0)$ has the required property.

What would have happened if we had not treated P and Q as twins? We could have let PQ be the line $x+hy = k$, found the actual coordinates of P and Q, and thereafter wallowed in square roots until reaching the answer. Or we could have let P be $(at^2, 2at)$, found the equation of PK, found that it met the parabola again at $Q\left(\frac{k^2}{at^2}, -\frac{2k}{t}\right)$, and obtained the distances PK and QK in terms of t. This would have led to the answer, though somewhat clumsily. Or we might have admitted half-heartedly that P and Q were twins by giving them parameters t_1 and t_2, showing that the chord t_1t_2 passes through K if $t_1t_2 = -\frac{k}{a}$, and trying to make use of this fact in the calculation of $\frac{1}{PK^2} + \frac{1}{QK^2}$. This is a very awkward method unless one parameter is eliminated.

Example 2. Find the equation of the chord joining the points $(4a^2, 3a)$, $(4\beta^2, 3\beta)$ on the parabola $4y^2 = 9x$.

Prove that the condition for this chord to touch the circle $x^2+y^2-5x+4 = 0$ is $(\alpha+\beta)^2 = (4\alpha\beta+1)(\alpha\beta+1)$.

The triangle PQR inscribed in the parabola is such that its sides PQ, PR touch the circle. Deduce from your previous result that the side QR also touches the circle. [A.L.]

We will leave out the first two parts, as they present no special difficulty and do not concern our present discussion.

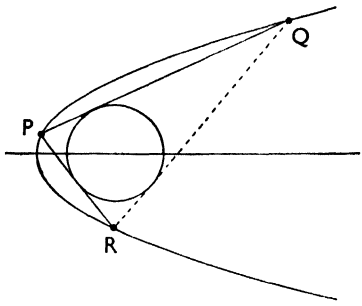


Fig. 62

It is clear which point of the figure should be called P, but we cannot discriminate between Q and R: they are twins, and we must obtain their parameters as the roots of a quadratic equation. This leads to a difficulty of language: the letter used as the unknown in an equation cannot also denote one of the roots, and so it is better not to mention the parameters of Q and R until we have obtained the equation whose roots they are. A slightly indirect approach is needed.

Let P be $(4a^2, 3a)$, and let $(4t^2, 3t)$ be any point on the parabola. The chord joining it to P touches the circle if

$$(a+t)^2 = (4at+1)(at+1),$$

i.e. if $$(4a^2-1)t^2+3at-(a^2-1) = 0.$$

The parameters of Q and R are the roots of this equation. Let them be β, γ; then

$$\beta+\gamma = \frac{-3a}{4a^2-1}, \quad \beta\gamma = -\frac{a^2-1}{4a^2-1}.$$

Now QR touches the circle if $(\beta+\gamma)^2 = (4\beta\gamma+1)(\beta\gamma+1)$.

But $$(\beta+\gamma)^2 = \frac{9a^2}{(4a^2-1)^2},$$

and $$(4\beta\gamma+1)(\beta\gamma+1) = \left(1-\frac{4a^2-4}{4a^2-1}\right)\left(1-\frac{a^2-1}{4a^2-1}\right)$$

$$= \left(\frac{3}{4a^2-1}\right)\left(\frac{3a^2}{4a^2-1}\right) = \frac{9a^2}{(4a^2-1)^2}.$$

Hence QR touches the circle.

Example 3. A chord of the parabola $y^2 = 4ax$ subtends a right angle at the point $P_0(at_0^2, 2at_0)$; prove that it passes through a fixed point (the Frégier point of P_0).

Let the equation of the chord be $lx+my+n = 0$. We can find where it cuts the curve by putting $x = at^2$, $y = 2at$; this gives the equation $alt^2+2amt+n = 0$. If the roots are t_1, t_2, we have

$$t_1+t_2 = -\frac{2m}{l}, \quad t_1t_2 = \frac{n}{al} \quad . \quad . \quad . \quad \text{(i)}$$

Now the chords t_0t_1, t_0t_2 are at right angles, hence

$$\frac{2}{t_0+t_1} \cdot \frac{2}{t_0+t_2} = -1;$$

i.e. $$t_1t_2+t_0(t_1+t_2)+t_0^2+4 = 0.$$

Substituting the results of (i) in this, we find

$$\frac{n}{al} - \frac{2mt_0}{l} + t_0^2+4 = 0;$$

i.e. $$(t_0^2+4)al-2at_0m+n = 0.$$

Hence the chord always goes through the point $[a(t_0^2+4), \ -2at_0]$.

The ideas of this section can also be used to find the equation of a chord of a conic with algebraic freedom equations, or of a tangent to such a conic.

Example 4. Find the equation of the chord t_1t_2 of the hyperbola $x/a : y/b : 1 = 1+t^2 : 2t : 1-t^2$.

Let the chord be $lx+my+n = 0$. Then it meets the conic where $al(1+t^2)+2bmt+n(1-t^2) = 0$,

i.e. $$(al-n)t^2+2bmt+(al+n) = 0 \quad . \quad . \quad . \quad \text{(i)}$$

The roots of this equation are t_1, t_2; but the equation with these roots can be written

$$t^2-(t_1+t_2)t+t_1t_2 = 0 \quad . \quad . \quad . \quad \text{(ii)}$$

We therefore have

$$\frac{al-n}{1} = \frac{-2bm}{t_1+t_2} = \frac{al+n}{t_1t_2} = \frac{2al}{1+t_1t_2} = \frac{-2n}{1-t_1t_2} \quad . \quad \text{(iii)}$$

where the first three fractions are obtained by comparing (i) and (ii), and the last two are derived from them by using the properties of equal fractions.

From the second, fourth, and fifth we see that

$$l : m : n = \frac{1+t_1t_2}{a} : -\frac{(t_1+t_2)}{b} : -(1-t_1t_2).$$

The equation of the chord is therefore

$$(1+t_1t_2)\frac{x}{a} - (t_1+t_2)\frac{y}{b} = 1-t_1t_2.$$

EXERCISE 9C

Unless the contrary is stated, the constraint and freedom equations of the curves are those of Chapter 8.

Find the equation of the chord t_1t_2 of:

1. $y^2 = 4ax$. 2. $xy = c^2$.

Find the equation of the tangent at t to:

3. The hyperbola. 4. The ellipse.

In Nos. 5–8, write down the equation giving the values of t for which the tangent to the given curve passes through P_0, and deduce the coordinates of the meet of the tangents at t_1 and t_2.

5. $xy = c^2$. 6. $y^2 = 4ax$.

7. The ellipse. 8. The hyperbola.

Find the coordinates of the mid-point of the chord of contact of tangents from P_0 to:

9. $y^2 = 4ax$. 10. $xy = c^2$.

11. The line $lx+my+n = 0$ cuts $xy = c^2$ at P, Q; C is (c, c). Find the orthocentre of $\triangle CPQ$.

12. The line $lx+my+n = 0$ cuts $y^2 = 4ax$ at P, Q; find the orthocentre of $\triangle OPQ$.

13. With the data of No. 12, show that $\tan POQ = \dfrac{4\sqrt{a(am^2-ln)}}{n+4al}$. Explain the occurrence of the expression am^2-ln on the top and $n+4al$ on the bottom of the fraction.

14. With the data of No. 11, show that $\tan PCQ = \dfrac{\sqrt{(n^2-4c^2lm)}}{c(l+m)}$. Explain the occurrence of n^2-4c^2lm in the numerator and $l+m$ in the denominator.

15. Prove that triangles can be drawn whose vertices lie on $xy = c^2$ and whose sides touch $y^2 = 4ax$.

16. Prove that triangles can be drawn whose vertices lie on $y^2 = 4ax$ and whose sides touch $xy = c^2$.

17. The line $lx+my+n = 0$ meets the ellipse at P, Q. If the tangents at P, Q are perpendicular, prove that PQ touches the ellipse $\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} = \dfrac{1}{a^2+b^2}$.

18. The line $lx+my+n = 0$ cuts the circle $x^2+y^2 = a^2$ at P, Q; A is $(-a, 0)$. Show that $\tan PAQ = \dfrac{1}{n}\sqrt{(a^2l^2+a^2m^2-n^2)}$, and deduce that, if $\angle PAQ$ remains constant, the line envelopes a concentric circle.

19. A variable line meets the parabola at P, Q; the tangents at P, Q meet Oy at R, S. If RS subtends a right angle at $(b, 0)$, prove that the line goes through a fixed point, and find its coordinates.

20. A variable line meets $xy = c^2$ at P, Q; if the mid-point of PQ lies on the rectangular hyperbola $xy-hx-ky = 0$, show that the line goes through a fixed point, and find its coordinates.

21. If a chord PQ of the ellipse subtends a right angle at $(-a, 0)$, show that it goes through a fixed point, and find the coordinates of the point.

22. Prove that the Frégier point of t_0 on the ellipse is given by

$$\frac{x}{a} : \frac{y}{b} : 1 = (a^2-b^2)(1-t_0^2) : -2(a^2-b^2)t_0 : (a^2+b^2)(1+t_0^2).$$

23. Show that the gradients of the tangents from P_0 to $xy = c^2$ satisfy the equation $x_0^2\gamma^2-2(x_0y_0-2c^2)\gamma+y_0^2 = 0$.

Two parallel tangents meet another tangent at P, Q; prove that the asymptotes are the bisectors of $\angle POQ$.

24. Show that the gradients of the tangents from P_0 to $y^2 = 4ax$ satisfy the equation $x_0\gamma^2-y_0\gamma+a = 0$, and deduce

(i) that perpendicular tangents meet on the directrix;

(ii) that tangents equally inclined to $x = y$ meet on the latus rectum.

25. Find the locus of points from which the product of the gradients of the two tangents to $x^2+y^2 = a^2$ is k^2.

26. If the tangents from P_0 to the ellipse make an angle θ with each other, show that

$$\tan\theta = \pm\frac{2\sqrt{(b^2x_0^2+a^2y_0^2-a^2b^2)}}{x_0^2+y_0^2-a^2-b^2}.$$

Hence find the locus of P if the tangents from it are perpendicular.

27. The tangents from P to $x^2+y^2 = a^2$ cut Oy at Q, R. If the mid-point of QR is $(0, b)$, show that the locus of P is the parabola $bx^2+a^2y = a^2b$.

28. The tangents from P to $x^2+y^2 = a^2$ cut Oy at Q, R; if QR subtends a right angle at $(c, 0)$, $c>a$, show that the locus of P is the hyperbola $(c^2-a^2)x^2-a^2y^2 = a^2c^2$.

29. The tangents from P to $x^2+y^2 = a^2$ cut $x+a = 0$ at Q, R. If $QR = 2b$, show that the locus of P is $(a^2-b^2)x^2+a^2y^2+2ab^2x = a^2(a^2+b^2)$, and distinguish the different cases.

30. With the data of No. 29, show that $\angle QOR$ is never 90°.

31. Show that, on the major axis of the ellipse, there are two positions of a point C having the property that, if any chord through C cuts the ellipse at P, Q, $CP^{-2}+CQ^{-2}$ is constant. Find the coordinates of C and the value of the constant.

32. Show that, on the conjugate axis of the hyperbola when $a>b$, there are two positions of a point C having the same property as in No. 31. Find the coordinates of C and the value of the constant.

33. Two perpendicular chords PQ, RS of the ellipse meet at a focus S. Prove that $\dfrac{1}{PQ}+\dfrac{1}{RS} = \dfrac{a^2+b^2}{2ab^2}$.

34. Two perpendicular chords PQ, RS of a rectangular hyperbola meet at any point C. Prove that $CP.CQ+CR.CS = 0$.

35. The point $A(4, 1)$ lies inside the circle $x^2+y^2 = 25$. A chord through A meets the circle at points P, Q such that $PA = 2AQ$. Prove that there are two possible directions of the chord, one of which is parallel to the axis Oy, and find the tangent of the acute angle between the two directions. [A.L.]

36. Two perpendicular chords of the ellipse are trisected at P; show that the locus of P is

$$b^2(a^2+9b^2)x^2+a^2(9a^2+b^2)y^2 = a^2b^2(a^2+b^2).$$

9.4. Triplets and Larger Groups

Not only may twin points occur: we may also have triplets or larger groups of points whose coordinates are obtained from the roots of an equation of the third or higher degree. If the equation is

$$a_0t^n+a_1t^{n-1}+a_2t^{n-2}+\ldots\ldots+a_{n-1}t+a_n = 0,$$

it is shown in algebra books that

the sum of the roots is $-\frac{a_1}{a_0}$,

the sum of products of pairs of roots is $\frac{a_2}{a_0}$,

the sum of products of roots taken three at a time is $-\frac{a_3}{a_0}$,

.....

the product of the roots is $(-1)^n \frac{a_n}{a_0}$.

The geometrical properties of the points concerned can be deduced from these relationships.

Triplets may occur in connection with concurrent normals to a parabola.

Example 1. The normals at the points P, Q, R of a parabola are concurrent; prove that the centroid of $\triangle PQR$ lies on the axis of the curve.

The normal at t to the parabola $x = at^2$, $y = 2at$ goes through (x_0, y_0) if

$$at^3+(2a-x_0)t-y_0 = 0 \quad . \quad . \quad . \quad \text{(i)}$$

and the roots of this equation are the parameters of P, Q, R.

Let them be t_1, t_2, t_3, so that P, Q, R are $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, $(at_3^2, 2at_3)$. As the coefficient of t^2 in (i) is zero, $t_1+t_2+t_3 = 0$. But the y-coordinate of the centroid of $\triangle PQR$ is $\frac{2a}{3}(t_1+t_2+t_3)$ and hence is zero. The centroid therefore lies on the axis.

If the coordinates of one point of a triplet are known, the other two points can be regarded as twins and their coordinates obtained from a quadratic equation.

Example 2. If the normals to a parabola at P and Q meet on the curve, prove that PQ meets the axis in a fixed point.

With the same equations as in Example 1, the normal at t goes through $(at_0^2, 2at_0)$ if

$$t^3+(2-t_0^2)t-2t_0 = 0.$$

This is satisfied by $t = t_0$, and hence $t-t_0$ is a factor of the L.H.S. Dividing by this factor, we obtain

$$t^2+t_0t+2 = 0 \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

as the equation whose roots are the parameters of P and Q. If these parameters are t_1 and t_2, the chord PQ is

$$2x-(t_1+t_2)y+2at_1t_2 = 0$$

and meets the axis where $x = -at_1t_2$. But from (ii), $t_1t_2 = 2$, and so PQ meets the axis at the fixed point $(-2a, 0)$.

The points in which a line meets a cubic curve form a triplet, and this fact can often be used to find the relation between the parameters of collinear points on such a curve. In simple cases the method is easier than that of 7.6.

Example 3. Diocles' Cissoid (see 12.2) is given by the freedom equations $x : y : a = t^2 : t^3 : 1+t^2$.

(*a*) Find the relation between t_1, t_2, t_3 if the points with these parameters are collinear.

(*b*) If the tangent at P meets the curve again at Q, find the relation between the parameters of these two points.

Any line $lx+my+n = 0$ meets the curve where

$$amt^3+(al+n)t^2+n = 0.$$

Let t_1, t_2, t_3 be the roots of this, then, since the coefficient of t is zero,

$$t_2t_3+t_3t_1+t_1t_2 = 0 \quad . \quad . \quad . \quad . \quad \text{(i)}$$

and this is the required relation.

If the line moves to that $t_3 \to t_1$, in its limiting position it will be the tangent at t_1. But as $t_3 \to t_1$, (i) becomes

$$t_2t_1+t_1^2+t_1t_2 = 0,$$

i.e.

$$t_1+2t_2 = 0 \quad . \quad . \quad . \quad \text{(ii)}$$

provided $t_1 \neq 0$. Hence (ii) is the relation between the parameters t_1, t_2 of P and Q respectively.

The point Q, at which the tangent at P to a cubic curve meets the curve again, is called the **tangential** of P.

Sets of four points can occur when two conics intersect.

Example 4. What can be said about the parameters of four points of the parabola $x = at^2$, $y = 2at$ if a circle can be drawn through them? Is the converse true?

Find the locus of the centre of the circle if three points of intersection are coincident.

Let the equation of the circle be $x^2+y^2+2gx+2fy+c = 0$; it meets the parabola where $a^2t^4+2a(2a+g)t^2+4aft+c = 0$.
Let the roots of this equation be t_1, t_2, t_3, t_4, then

$$t_1+t_2+t_3+t_4=0 \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$t_2t_3+t_3t_1+t_1t_2+t_1t_4+t_2t_4+t_3t_4=\frac{2(2a+g)}{a} \quad . \quad \text{(ii)}$$

$$t_2t_3t_4+t_1t_3t_4+t_1t_2t_4+t_1t_2t_3=-\frac{4f}{a} \quad . \quad . \quad \text{(iii)}$$

$$t_1t_2t_3t_4=\frac{c}{a^2} \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

The first of these is independent of the particular circle chosen, and is therefore the relation between the parameters of any four concyclic points on the parabola. Since a circle can be drawn through any three non-collinear points, only one condition is needed to make four points of the curve concyclic, and so the converse is true, namely that, if $t_1+t_2+t_3+t_4=0$, the four points with these parameters are concyclic. This can be seen in greater detail from the fact that, if t_1, t_2, t_3, t_4 are any four numbers satisfying (i), values of g, f, c can be obtained from (ii), (iii), and (iv); and the circle whose equation contains these values then cuts the parabola in points having the chosen parameters.

If three points of intersection are coincident, we can put t_2 and t_3 equal to t_1, and (i) then shows that $t_4=-3t_1$. Substituting these values in (ii) and (iii), and at the same time putting $x=-g$, $y=-f$, so that the centre of the circle is (x, y), we find

$$-6t_1^2=\frac{2(2a-x)}{a}, \quad \text{and} \quad -8t_1^3=\frac{4y}{a}.$$

Hence $4(x-2a)^3=27ay^2$, and this is the equation of the locus. It represents a **semicubic parabola** having a **cusp** at $(2a, 0)$, i.e. the two branches come to a point there and stop. The curve is shown in Fig. 63.

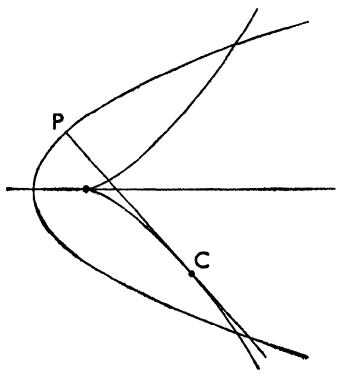


Fig. 63

A circle meeting a conic in three coincident points is called a **circle of curvature** and the locus of its centre is called the **evolute** of the conic. It is shown in calculus books that the normals to the curve are tangents to the evolute, and that if a thread of suitable length were wrapped round the evolute and unwound, the locus of its end would

be the original curve. If the normal to the curve at P touches th evolute at C, C is called the **centre of curvature** of the curve at I and CP the **radius of curvature.** The **curvature** itself is the reciproc of this. C is the centre of the circle which meets the curve in thre coincident points at P; it can also be found as the limiting positio of the meet of the normals at P and Q, as Q tends to P along th curve.

In the case of the parabola in Fig. 63, three normals can be draw to the curve from any point to the right of the evolute (since thre tangents can be drawn to the evolute), but only one from any poir to its left.

EXERCISE 9D (The freedom equations are those of Chapter 8)

The Parabola $y^2 = 4ax$

1. If the normals at P, Q, R are concurrent, prove that PQ and OR are equal inclined to the axis.

2. Find the orthocentre of the triangle formed by the feet of the normals whic pass through (x_0, y_0).

3. A variable chord through the focus meets the parabola at P_1 and P_2. Th normals at P_1 and P_2 meet at N. Prove that the equation of the locus of N $y^2 = a(x-3a)$. [A.L.]

4. The normal at P meets the axis at G, and PG is produced to H so th $GH = \frac{1}{2}PG$. Show that two perpendicular normals pass through H.

5. Three normals pass through P; if two of the normals are coincident, sho that the locus of P is the evolute $27ay^2 = 4(x-2a)^3$.

6. A variable chord passes through $(c, 0)$; if the normals at its ends meet P, find the locus of P.

7. Prove that the circle through the feet of three concurrent normals pass also through the vertex.

8. Two chords PQ, RS make equal angles with the axis but are not paralle show that a circle can be drawn through their ends.

9. P is a variable point on the parabola; if the circle on OP as diameter mee the curve again at Q, R, show that QR passes through a fixed point and find i coordinates.

10. With the data of No. 9, find the locus of the mid-point of QR.

In Nos. 11–15, a triangle is formed by the tangents at t_1, t_2, t_3, the roots c $t^3+pt^2+qt+r = 0$.

11. Show that the centroid G of the triangle is $(aq/3, -2ap/3)$.

12. Show that the orthocentre H is $[-a, -a(p+r)]$.

13. Show the circumcentre C is $[\frac{1}{2}a(q+1), \frac{1}{2}a(r-p)]$.

14. Show that C, G, H are collinear, and that $GH = 2CG$. The line CGH called the **Euler Line** of the triangle.

15. If the gradient of the Euler line is $1/t_4$, show that

$$3+\Sigma t_1t_2+3t_1t_2t_3t_4 = 0.$$

What is the geometrical meaning of the symmetry of this result?

16. Show that the circumcircle of a triangle of tangents goes through the focu

The Rectangular Hyperbola $xy = c^2$.

17. If the normals at points with parameters t_1, t_2, t_3, t_4 are concurrent, show that $\Sigma t_1 t_2 = 0$ and that $t_1t_2t_3t_4 = -1$. Deduce that the four points are orthocentric and that the sum of the gradients of their joins is zero.

18. The normals at P_1, P_2, P_3, P_4 are concurrent at P_0.
Prove that (i) $x_1x_2x_3x_4 = y_1y_2y_3y_4$;
(ii) $x_1+x_2+x_3+x_4 = x_0$, $y_1+y_2+y_3+y_4 = y_0$.

19. With the data of No. 18, prove that
$$OP_1^2+OP_2^2+OP_3^2+OP_4^2 = OP_0^2.$$

20. With the data of No. 18, prove that OG_1 is parallel to the normal at P_1, G_1 being the centroid of the triangle $P_2P_3P_4$.

21. A circle cuts the hyperbola at points with parameters t_1, t_2, t_3, t_4; show that $t_1t_2t_3t_4 = 1$. If the points are P, Q, R, S, deduce that PQ and RS make equal angles with an axis of the hyperbola.

22. If a circle cuts a rectangular hyperbola in four points, show that the centre of the hyperbola lies midway between any one of the points and the orthocentre of the triangle formed by the other three.

23. If the perpendicular from the centre of a rectangular hyperbola to a chord QR cuts the curve at P, show that the circle PQR touches the curve at P.

24. If a circle cuts a rectangular hyperbola at P, Q, R, S, prove that the diameter of the hyperbola perpendicular to PQ bisects RS.

25. Show that the evolute has freedom equations
$$x = \tfrac{1}{2}c\left(3t + \frac{1}{t^3}\right), \quad y = \tfrac{1}{2}c\left(\frac{3}{t} + t^3\right),$$
where t is the parameter of point at which the circle of curvature touches the hyperbola.

26. A circle of radius a cuts the hyperbola at P, Q, R, S; prove that $OP^2+OQ^2+OR^2+OS^2 = 4a^2$.

The Ellipse $x^2/a^2+y^2/b^2 = 1$

27. If the points with parameters t_1, t_2, t_3, t_4 are concyclic, prove that $\Sigma t_1 = \Sigma t_2t_3t_4$.
If the eccentric angles of the points are θ_1, θ_2, θ_3, θ_4, deduce that $\tan \frac{1}{2}(\theta_1+\theta_2) + \tan \frac{1}{2}(\theta_3+\theta_4) = 0$, and hence that $\theta_1+\theta_2+\theta_3+\theta_4 = 2n\pi$.

28. P and Q are the ends of conjugate diameters of an ellipse. The circles of curvature at P and Q meet the ellipse again at P' and Q' respectively. Prove that PQ' and $P'Q$ are parallel, and find the envelope of $P'Q'$. [S.L.]

29. If the normals at t_1, t_2, t_3, t_4 are concurrent, show that $\Sigma t_1 t_2 = 0$, $t_1t_2t_3t_4 = -1$.
If the corresponding eccentric angles are θ_1, θ_2, θ_3, θ_4, deduce that $\theta_1+\theta_2+\theta_3+\theta_4 = (2n+1)\pi$.
If a circle is drawn through three of the points, where does it cut the ellipse again?

30. If the normals at P, Q, R, S are concurrent, and the equation of PQ is $\frac{lx}{a} + \frac{my}{b} + n = 0$, show that the equation of RS is $\frac{x}{al} + \frac{y}{bm} - n = 0$.

Cubic Curves

In Nos. 31, 32, find the relation between t_1, t_2, t_3 if the points with these parameters are collinear.

31. (i) $x : y : a = t^3 : 1 : t$; (ii) $x : y : a = (1-t^2) : t(1-t^2) : 1+t^2$.

32. (i) $x : y : a = t^3 : t^2 : 1$; (ii) $x/a : y/b : 1 = t : t^3 : 1+t^2$.

In Nos. 33 and 34 the curve is a cubic on which three points are collinear if $t_1+t_2+t_3 = 0$.

33. *ABC*, *DEF*, *GHJ*, *ADG*, *BEH* are five lines meeting the curve at the lettered points; prove that *C*, *F*, *J* are collinear.

34. Points *A*, *B*, *C*, *D* are taken on the curve, such that each is the tangential of the preceding one; if *AC*, *BD* meet the curve at *P*, *Q*, show that *PQ* touches the curve at *P*.

Exercise 9M

1. *P* is a point on one of two equal parabolas which touch at their vertices; prove that the polar of *P* wo the second parabola touches the first at *P'*, the image of *P* in the axis.

2. Find the locus of centroids of equilateral triangles whose vertices lie on $xy = c^2$.

3. The normal at *P* to an ellipse cuts the axes at *X*, *Y*. Show that the locus of the mid-point of *XY* is an ellipse similar to the original curve with axes interchanged.

4. If *PP'* is a diameter of the rectangular hyperbola $xy = c^2$, and the tangent at *P* meets the asymptotes at *Q* and *R*, prove that *P'Q* and *P'R* are tangents to the rectangular hyperbola $3xy+c^2 = 0$. [A.L.]

5. The line $lx+my+n = 0$ meets the parabola $y^2 = 4ax$ at *P* and *Q*, and the lines joining *P*, *Q* to $(c, 0)$ meet the curve again at *R*, *S*. Show that the equation of *RS* is $nx-mcy+lc^2 = 0$, and prove that, if *PQ* passes through a fixed point, then *RS* also passes through a fixed point. [A.L.]

6. A circle is drawn on the chord *AB* of the parabola $y^2 = 4ax$ as diameter. It meets the parabola again at *P*, *Q*; *P'*, *Q'* are the reflections of *P*, *Q* in the axis of the parabola; *P'F* and *Q'E* are drawn parallel to the axis of the parabola and meet *AB* at *F*, *E*. Prove that $P'F = Q'E = 4a$. [A.L.]

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7. A point moves so that its distance from the first of two given lines is proportional to the square of its distance from the second. Show that its locus is a parabola with axis parallel to the second line, and that the first line touches it where it meets the second.

8. Show that two conjugate diameters of a rectangular hyperbola are equally inclined to an asymptote.

9. Two circles touch at *O*; if *P*, *Q* are variable points, one on each circle, such that $\angle POQ = 90°$, find the locus of the mid-point of *PQ*.

10. Points *P*, *F*, *G* have coordinates $(at^2, 2at)$, $(a, 0)$, $(3a, 0)$ respectively; and *PF*, *PG* meet the parabola $y^2 = 4ax$ again at *Q* and *R*. Find the equation of the line *QR*. Prove that *QR* touches the parabola $y^2 = 3ax$. [A.L.]

11. A variable line meets the circle $x^2+y^2 = a^2$ at *P*, *Q*; *A* is $(-a, 0)$; *AP*, *AQ* meet *Oy* at *R*, *S*; if the mid-point of *RS* is $(0, b)$, prove that the line goes through a fixed point, and find its coordinates.

12. *P*, *Q* are two points on opposite branches of the rectangular hyperbola $xy = c^2$. Prove that the circle on *PQ* as diameter meets the hyperbola again at two (real) points, and that those points are the ends of a diameter of the hyperbola [A.L.]

---

13. *N* is the foot of the perpendicular from any point *P* to its polar wo a parabola, focus *S*. Prove that *SN* is equal to the distance of *P* from the directrix.

Which of the listed properties of the parabola does this become when P is on (i) the curve, (ii) the directrix?

14. M is the mid-point of a chord of a rectangular hyperbola which subtends a right angle at a point P on the curve; O is the centre of the hyperbola. Prove that $\angle POM = 90°$.

15. An ellipse slides along its major axis and is cut in P and Q by a fixed line perpendicular to that axis; prove that the normals to the ellipse at P and Q remain normal to a similar fixed ellipse.

16. A line moves so that the product of its distances from two fixed points on the same side of it is constant. Prove that it envelopes an ellipse.

17. PQ is a chord of a parabola drawn in a fixed direction. Prove that the locus of the point of intersection of the normals at P and Q is a straight line, which is itself a normal to the parabola. [A.L.]

18. A circle passes through the centre O of a rectangular hyperbola and cuts the curve at A, B, C, D. If G is the centroid of the triangle ABC, prove that $\angle DOG$ is a right angle.

---

19. The tangent at P to a rectangular hyperbola meets an asymptote at R; the line through R parallel to the other asymptote cuts the curve at Q; PQ cuts the asymptotes at H, K; prove that P and Q are the points of trisection of HK.

20. L, M are the points (at, bt), $(a/t, -b/t)$. Show that

(i) they lie on the asymptotes of the hyperbola $x^2/a^2 - y^2/b^2 = 1$;

(ii) the line LM touches the hyperbola at its mid-point;

(iii) $t = OL/OS = OS/OM$, S being a focus.

21. Find the locus of the feet of perpendiculars from the focus of a parabola to normals to the curve.

[It is easiest to take the focus as origin. Reject the factor representing a point-circle at the focus.]

22. L, M are two points on the circle $x^2+y^2 = a^2+b^2$, such that OL, OM lie along conjugate diameters of the ellipse $b^2x^2+a^2y^2 = a^2b^2$. Prove that LM is a tangent to the ellipse and deduce that the circle is the director circle of the ellipse (the locus of meets of perpendicular tangents).

23. If PQ is any chord through the focus S of the ellipse $b^2x^2+a^2y^2 = a^2b^2$, show that $\dfrac{1}{PS} + \dfrac{1}{SQ} = \dfrac{2a}{b^2}$.

[Pay attention to the signs of PS and SQ.]

24. K is the centre of curvature of a rectangular hyperbola at P_0; the line parallel to the normal at P_0, and midway between it and the centre, cuts the curve at Q_1, Q_2; if Q_1P_1, Q_2P_2 are diameters, prove that the normals at P_1 and P_2 pass through K.

---

25. A line moves so that the product of the perpendiculars to it from two fixed points is constant. Prove that the locus of the feet of these perpendiculars is a circle, the same for each.

26. The ends of one axis of a variable central conic are fixed; prove that the locus of the points of contact of tangents to it from a fixed point on the other axis is a parabola.

27. If the eccentric angles of the ends of a variable chord of an ellipse differ by a constant, prove that the chord envelopes a similar concentric ellipse.

28. If the tangents from P to $y^2 = 4ax$ cut off a constant length b on Oy, show that the locus of P is $y^2 - 4ax = b^2$,

29. Prove that, if the centroid and orthocentre of a triangle coincide, it is equilateral.

Prove that the three points on the rectangular hyperbola described by the point (t, t^{-1}), whose parameters are the roots of the equation in θ $\lambda^3\theta^3+3\lambda^3\theta^2-3\theta-\lambda = 0$, are the vertices of an equilateral triangle, and that λ is the parameter of the point at which the circumcircle of the triangle meets the hyperbola again. [S.L.]

30. The normals to the parabola $y^2 = 4ax$ at points P, Q, R on it meet at (x_0, y_0), and the tangents at P, Q, R are drawn. Show that the perpendiculars to these tangents at the points where they meet the axis are concurrent, and find the point of concurrence.

---

31. A line moves so that the algebraic sum of the perpendiculars to it from a number of fixed points is zero; prove that it always passes through a fixed point.

32. Prove that the equation of the normal at any point of the curve given by $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$ is $x\cos\frac{3}{2}t+y\sin\frac{3}{2}t = \cos\frac{1}{2}t$.

Prove that the normals at the points t and $t+\pi$ intersect at right angles, and find the locus of their point of intersection. [A.L.]

33. Two confocals are given; if p, q are the distances of the centre from a tangent to one and a parallel tangent to the other, prove that p^2-q^2 is constant for all such pairs of tangents.

34. If the polars wo the ellipse $b^2x^2+a^2y^2 = a^2b^2$ of two points on the auxiliary circle are perpendicular, show that the locus of their meet is $a^2(x^2+y^2) = a^4+b^4$.

35. PP' is a diameter of a rectangular hyperbola; the circle with centre P, passing through P', cuts the curve again at L, M, N; prove that the triangle LMN is equilateral.

36. The normals at P, Q, R to the parabola $y^2 = 4ax$ meet at (x_0, y_0); show that the perpendiculars to the sides of the triangle PQR at the points where they meet the axis are concurrent, and find the point of concurrence.

CHAPTER 10

THE GENERAL CONIC

10.1 Rotation about the Origin

In Chapter 2 we saw how the equation of a curve is altered when the curve is shifted without rotation. Let us now see what happens when it is rotated about O through an angle θ.

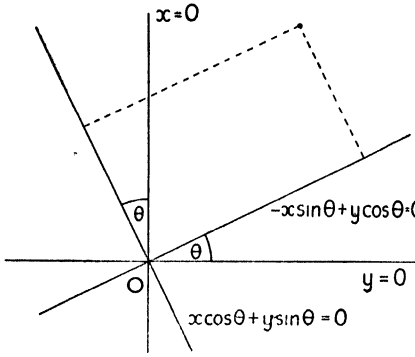


Fig. 64

If a line lying along Ox is so rotated, its gradient becomes $\tan\theta$ and its equation can be written

$$-x\sin\theta + y\cos\theta = 0 \quad . \quad . \quad . \quad (1)$$

this form having been chosen because the L.H.S. measures the distance of (x, y) from the line and reduces to y when $\theta = 0$. In the same way, the equation of a line rotated from Oy through an angle θ can be written

$$x\cos\theta + y\sin\theta = 0 \quad . \quad . \quad . \quad (2)$$

Now the letters x, y in the equation of a curve measure the distances of any point on it from the lines Oy, Ox. Hence, if the curve is rotated about O through an angle θ, the same equation holds between the distances of any point on it from the lines (2) and (1). The equation of the new curve can thus be got from the old by substituting

$$\left.\begin{array}{rl} x\cos\theta + y\sin\theta & \text{for } x \\ -x\sin\theta + y\cos\theta & \text{for } y \end{array}\right\} \quad . \quad . \quad . \quad (3)$$

Notice that, if $\theta = 180°$, the substitutions amount to putting $-x$ for x, $-y$ for y, and this we have already done in 2.4 to reflect a point in the origin. Also the distance of a point from O is unaltered by the substitution (3) since

$$(x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2 = x^2 + y^2.$$

If we do not wish to bring in the angle of rotation explicitly, we can let $x = 0$, $y = 0$ rotate into

$$lx+my = 0$$

and

$$-mx+ly = 0$$

The rotation is then effected by substituting

$$\left.\begin{aligned} &\frac{lx+my}{\sqrt{(l^2+m^2)}} \text{ for } x,\\ &\frac{-mx+ly}{\sqrt{(l^2+m^2)}} \text{ for } y, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

Example 1. Rotate the curve

$$ax^2+by^2+c = 0 \quad . \quad . \quad . \quad . \quad (5)$$

through an angle θ about the origin.

The new equation is

$$a(x\cos\theta + y\sin\theta)^2+b(-x\sin\theta + y\cos\theta)^2+c = 0,$$

i.e. $(a\cos^2\theta + b\sin^2\theta)x^2+2(a-b)\cos\theta\sin\theta\, xy$

$$+(b\cos^2\theta + a\sin^2\theta)y^2+c = 0 \quad . \quad (6)$$

We notice that it contains a term in xy as well as terms in x^2 and y^2.

Example 2. Trace the curve

$$(4x-3y)^2 = 10(3x+4y).$$

The equation can be written

$$\frac{(4x-3y)^2}{4^2+3^2} = \frac{2(3x+4y)}{\sqrt{(3^2+4^2)}}$$

and represents the parabola $y^2 = 2x$ rotated about O so that the axis and the tangent at the vertex are respectively $4x-3y = 0$ and $3x+4y = 0$. To see how the parabola lies we notice that the given equation can only be satisfied if $3x+4y$ is positive. The curve is therefore as shown in Fig. 65.

Notice that the rotation of a curve, effected by the substitutions (3) or (4), does not alter the degree of its equation. We have also seen that the degree is unaltered if a curve is shifted. When cartesian coordinates are used, therefore, it is possible to say that a curve is of a certain degree without needing to specify the particular axes to which its equation is referred.

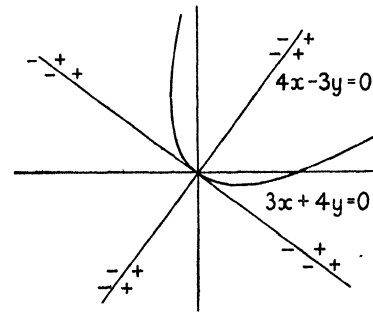


Fig. 65

10.2. The General Equation of the Second Degree

We shall show that the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad . \quad . \quad (7)$$

represents a conic, provided that at least one real point can be found which satisfies it.

Let us begin by shifting the curve

$$ax^2+2hxy+by^2+k=0 \quad . \quad . \quad . \quad (8)$$

so that its centre is at (x_0, y_0). Its equation will then be

$$a(x-x_0)^2+2h(x-x_0)(y-y_0)+b(y-y_0)^2+k=0$$

and can be written

$$ax^2+2hxy+by^2-2(ax_0+hy_0)x-2(hx_0+by_0)y \\ +ax_0^2+2hx_0y_0+by_0^2+k=0.$$

This is the same as (7) if

$$ax_0+hy_0+g=0 \quad . \quad . \quad . \quad . \quad (9)$$

$$hx_0+by_0+f=0 \quad . \quad . \quad . \quad . \quad (10)$$

and

$$ax_0^2+2hx_0y_0+by_0^2-c+k=0 \quad . \quad . \quad (11)$$

Provided $ab-h^2 \neq 0$, (9) and (10) can be solved to give

$$x_0=\frac{hf-bg}{ab-h^2}, \quad y_0=\frac{gh-af}{ab-h^2},$$

and a value of k can be found by substituting these results in (11).

To simplify the working, we shall use A, B, C, F, G, H to denote the cofactors of the corresponding small letters in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

and the determinant itself will be denoted by δ.

Thus $A=bc-f^2, \quad B=ca-g^2, \quad C=ab-h^2,$

$F=gh-af, \quad G=hf-bg, \quad H=fg-ch,$

and $\delta=abc+2fgh-af^2-bg^2-ch^2$.

With this notation, the solutions of (9) and (10) are

$$x_0=\frac{G}{C}, \quad y_0=\frac{F}{C} \quad . \quad . \quad . \quad . \quad (12)$$

To find the value of k, we notice that (11) can be written

$$x_0(ax_0+hy_0)+y_0(hx_0+by_0)-c+k=0.$$

Now, from (9) and (10), the contents of the two brackets are respectively equal to $-g$, $-f$, so that

$$k=gx_0+fy_0+c=\frac{gG+fF+cC}{C}.$$

But $gG+fF+cC$ is the expansion of the determinant in terms of its third row, and is therefore equal to δ. Hence $k = \delta/C$, and we can say that, **if $C \neq 0$, (7) represents the curve**

$$ax^2+2hxy+by^2+\frac{\delta}{C} = 0 \quad . \quad . \quad . \quad (13)$$

shifted so that its centre is at $\left(\frac{G}{C}, \frac{F}{C}\right)$. Alternatively (13) may be regarded as the equation of the given curve referred to parallel axes through its centre.

Example 1 of 10.1 shows that (13) represents the curve

$$a'x^2 + b'y^2 + \frac{\delta}{C} = 0 \quad . \quad . \quad . \quad (14)$$

rotated about O through an angle θ, if

$$a'\cos^2\theta + b'\sin^2\theta = a \quad . \quad . \quad . \quad . \quad (15)$$

$$b'\cos^2\theta + a'\sin^2\theta = b \quad . \quad . \quad . \quad . \quad (16)$$

and

$$(a'-b')\sin 2\theta = 2h \quad . \quad . \quad . \quad . \quad (17)$$

By subtraction we obtain, from (15) and (16),

$$(a'-b')\cos 2\theta = a-b \quad . \quad . \quad . \quad . \quad (18)$$

and this, together with (17), gives

$$\tan 2\theta = \frac{2h}{a-b} \quad . \quad . \quad . \quad . \quad (19)$$

Values of θ can always be found satisfying this equation.

Squaring and adding (17) and (18), we obtain

$$(a'-b')^2 = (a-b)^2+4h^2 \quad . \quad . \quad . \quad (20)$$

in which the R.H.S. is never negative. Hence a real value of $a'-b'$ can be found.

Adding (15) and (16), we have

$$a'+b' = a+b; \quad . \quad . \quad . \quad . \quad (21)$$

squaring this and subtracting (20),

$$a'b' = ab-h^2 = C \quad . \quad . \quad . \quad (22)$$

Values of a' and b' can always be found from (20) and (21), and hence (13) can always be regarded as a rotated version of (14), the angle of rotation being given by (19). Or (14) can be regarded as the equation of the given curve referred to its principal axes.

To trace (7), it only remains to see what (14) represents, and this will depend upon the signs of a', b' and δ/C.

Case 1: a' and b' have like sign.

(14), and therefore (7), represents an ellipse if the sign of δ/C is opposite to that of a' and b'; the ellipse degenerates to a point if

$\delta = 0$; and if a', b' and δ/C all have the same sign, no real points satisfy the equation.

Case 1 arises when $a'b' > 0$, and (22) shows that this condition is equivalent to $C > 0$. The sign of a' and b' is, by (21), that of $a+b$. It follows that (7) **represents a proper ellipse* if $ab-h^2 > 0$ and $\frac{\delta}{a+b} < 0$.**

Case 2: a' and b' have opposite signs, i.e. $a'b' < 0$.

(14) then represents a hyperbola. Its asymptotes are $a'x^2+b'y^2 = 0$, and when, after rotation, (14) becomes (13), the equation of the asymptotes becomes

$$ax^2+2hxy+by^2 = 0 \quad . \quad . \quad . \quad (23)$$

More will be said of this equation in 10.4.

The hyperbola (14) is rectangular if $a'+b' = 0$. By (21), this condition is the same as $a+b = 0$. The hyperbola degenerates into a line-pair if $\delta = 0$.

Hence, **if $ab-h^2<0$, (7) represents a hyperbola whose asymptotes are parallel to the lines obtained by equating to zero the terms of highest degree. The hyperbola is rectangular if $a+b = 0$ and degenerates to a line-pair if $\delta = 0$.**

Notice that $a+b = 0$ is a sufficient condition for (7) to represent a rectangular hyperbola (possibly degenerate), for then $ab-h^2 \leqslant ab < 0$, since a and b have opposite signs.

There remains the case in which $ab-h^2 = 0$. Multiplying through by a, and replacing ab by h^2, (7) can be written

$$(ax+hy)^2+a(2gx+2fy+c) = 0 \quad . \quad . \quad (24)$$

This equation shows that the square of the distance of any point of the curve from the line $ax+hy = 0$ is proportional to its distance from $2gx+2fy+c = 0$, and if these two lines were at right angles it would follow that the curve was a parabola. In general they are not, but (24) can be written

$$(ax+hy+\lambda)^2+2a(g-\lambda)x+2(af-h\lambda)y+(ac-\lambda^2) = 0 \quad . \quad . \quad (25)$$

and the lines $ax+hy+\lambda = 0$, $2a(g-\lambda)x+2(af-h\lambda)y+(ac-\lambda^2) = 0$ are perpendicular if $a^2(g-\lambda)+h(af-h\lambda) = 0$,

i.e.
$$\lambda = \frac{a(ag+hf)}{a^2+h^2} \quad . \quad . \quad . \quad . \quad (26)$$

With this value of λ, (25) can be written in the form

$$(ax+hy+\lambda)^2 = k(hx-ay+\mu)$$

and hence, as in 10.1, Example 2, it represents a parabola.

* The ellipse may, in a special case, be a circle. The condition for (7) to represent a circle was found in 4.2.

But this last step cannot be taken if $\delta = 0$. For if $\delta = 0$ and $ab = h^2$,

$$\begin{aligned} 0 = a\delta &= a^2bc+2afgh-a^2f^2-abg^2-ach^2 \\ &= ac(ab-h^2)+2afgh-a^2f^2-g^2h^2 \\ &= -(gh-af)^2. \end{aligned}$$

Hence $gh = af$, and (26) can be written

$$\lambda = \frac{a^2g+af.h}{a^2+h^2} = \frac{a^2g+gh^2}{a^2+h^2} = g = \frac{af}{h}.$$

When these last two expressions are substituted for λ, (25) becomes

$$(ax+hy+g)^2+ca-g^2 = 0,$$

and represents the two parallel lines

$$ax+hy+g = \pm\sqrt{(g^2-ca)}$$

provided $ca-g^2 < 0$.

Thus, **when $ab-h^2 = 0$, (7) represents a parabola with axis parallel to $ax+hy = 0$; if $\delta = 0$ the parabola degenerates into two parallel lines; these lines are distinct if $ca-g^2 < 0$, coincident if $ca-g^2 = 0$, but no points satisfy the equation if $ca-g^2 > 0$.**

10.3. Abstract Geometry

In developing coordinate geometry we have used algebra as a tool to measure geometrical figures and discover their properties, and algebraic results have been of interest only in so far as they have fulfilled this purpose: thus, if a locus problem leads to the equation $x^2+y^2+4 = 0$, we have regarded this as meaning *merely* that no points satisfy the locus-condition, and no interest has attached to the fact that the equation was $x^2+y^2+4 = 0$ instead of, say, $x^2+y^2+1 = 0$ or $(x-1)^2+(y-2)^2+4 = 0$. But it is possible to take another viewpoint: to regard the algebra as fundamental and to use geometrical terms to describe it, whether these terms are appropriate to the real world or not. The geometry is then abstract: a 'line' is not a figure in space but a first-degree equation, a 'point' is merely a pair of values of x and y. The variables may, if we wish, take on complex as well as real values, and then $x^2+y^2+4 = 0$ will be called a 'circle', its 'radius' will be $2i$ and its 'centre' (0, 0). As a reminder that the equation cannot be illustrated by a plane figure, it is usual to qualify the word "circle" by saying that it is 'imaginary'.

From this point of view the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 0$ can be described in two ways. It can be called a 'degenerate ellipse', as hitherto, or, since it can be written $\frac{x}{a} = \pm\frac{iy}{b}$, it can be called an 'imaginary line-

pair'. A 'conic' is thus degenerate when it splits up into two 'lines', 'real', 'coincident', or 'imaginary'.

To put these ideas on a logical footing would be beyond our present scope, and we shall therefore not attempt to deduce anything from them but merely borrow their language whenever it enables us to express more concisely results obtained by the geometry of real space. No confusion will be caused if we omit the inverted commas: a statement refers to real space if it makes sense when read in that way, but if it does not (through mentioning imaginary elements), it is to be regarded as an abstract geometrical description of the algebra concerned. We shall still often say that an equation *represents* a certain curve, even if the geometry is abstract.

With this convention, the following table shows the curves represented by the equation

$$ax^2+2hxy+by^2+2gx+2fy+c = 0$$

under certain conditions, and summarizes the results of 10.2.

Condition	Curve
$\delta \neq 0$: $ab-h^2 > 0$	Ellipse, real if $\dfrac{\delta}{a+b} < 0$
$ab-h^2 = 0$	Parabola
$ab-h^2 < 0$	Hyperbola
$a+b = 0$	Rectangular hyperbola
$\delta = 0$	As above, but degenerating to a line-pair

Since the value of δ discriminates between the cases in which the conic degenerates and those in which it is proper, δ is called the **discriminant** of equation (7). Notice that we can also say that the curve is a hyperbola, parabola, or ellipse according as the factors of the second-degree terms are real and distinct, repeated, or complex.

Example 1. Trace the curve

$$33x^2+20xy+54y^2+66x+20y+62 = 0,$$

finding the coordinates of its centre, the equations of its principal axes, and its equation referred to them.

Here $a = 33$, $b = 54$, $c = 62$, $f = 10$, $g = 33$, $h = 10$, so that $G = 10.10-33.54 = -1682$, $F = 33.10-33.10 = 0$,

$C = 33.54-10.10 = 1682$, and $\delta = gG+fF+cC = 29.1682$. The centre is therefore $(-1, 0)$ and the equation of the curve referred to parallel axes through the centre is

$$33x^2+20xy+54y^2+29 = 0.$$

By (19), the angle made by a principal axis with Ox is θ, where $\tan 2\theta = -20/21$. Let $\tan\theta = t$, then

$$\frac{2t}{1-t^2} = -\frac{20}{21},$$
$$\therefore \quad 10t^2-21t-10 = 0,$$
$$\therefore \quad t = -\frac{2}{5} \text{ or } \frac{5}{2}.$$

Let us choose one of these values, say $\tan\theta = -\frac{2}{5}$, and a particular value of θ to satisfy it, say a negative acute angle having $\sin\theta = -\frac{2}{\sqrt{29}}$, $\cos\theta = \frac{5}{\sqrt{29}}$. Then by (15) and (16), the equation of the curve referred to its principal axes is $a'x^2+b'y^2+29 = 0$, where

$$25a'+4b' = 29.33,$$
$$4a'+25b' = 29.54,$$

from which $a' = 29$, $b' = 29.2$. Hence the equation is

$$x^2+2y^2+1 = 0.$$

The original equation therefore represents an imaginary ellipse with semi-axes i and $i/\sqrt{2}$. The 'major' axis is the new x-axis, and this is the line through $(-1, 0)$ with gradient $-\frac{2}{5}$, namely $2x+5y+2 = 0$; similarly the 'minor' axis is $5x-2y+5 = 0$.

Example 2. Trace the parabola

$$49x^2+14xy+y^2-38x+66y+14 = 0,$$

finding the semi-latus rectum, the coordinates of the vertex and the equation of the axis. Sketch the curve.

We confirm that the curve is a parabola by noticing that the second-degree terms are $(7x+y)^2$; to trace the curve we must therefore write the equation in the form

$$(7x+y+a)^2+b(x-7y+c) = 0.$$

Comparing the coefficients of x and y, and the constant terms, we see that the two equations are the same if

$$14a+b = -38, \quad 2a-7b = 66, \quad a^2+bc = 14.$$

Hence $a = -2$, $b = -10$, $c = -1$, and the parabola is

$$(7x+y-2)^2 = 10(x-7y-1) \quad . \quad . \quad . \quad (i)$$

The axis and tangent at the vertex are thus respectively $7x+y-2=0$ and $x-7y-1=0$, meeting at $(0{\cdot}3, -0{\cdot}1)$, the vertex. To find the semi-latus rectum we write (i) in the form

$$\frac{(7x+y-2)^2}{7^2+1^2} = \sqrt{2}\,\frac{x-7y-1}{\sqrt{(7^2+1^2)}},$$

as in 10.1, Example 2. Comparing this with $y^2 = 4ax$, the semi-latus rectum is seen to be $\frac{1}{2}\sqrt{2}$.

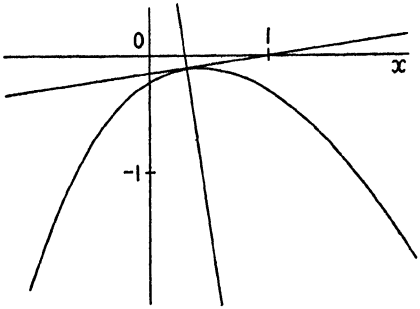


Fig. 66

At all points of the curve, $x-7y-1$ is positive, by (i). The parabola is therefore as in Fig. 66, the points of intersection with Oy being found from the equation

$$y^2+66y+14=0.$$

Exercise 10A

Warning. In numerical examples on the general equation do not forget that f, g, h are the coefficients of $2y$, $2x$, $2xy$.

Find the equation when the given curve is rotated about O through the given acute angle.

1. $xy = 8$, $30°$.
2. $xy+y^2 = 2$, $45°$.
3. $x^2+y^2-5x = 0$, $\tan^{-1}\frac{3}{4}$.
4. $x^2+4y^2 = 4$, $\tan^{-1}\frac{1}{2}$.

In Nos. 5–10, what curve does the equation represent? In each case give the equation referred to the principal axes, asymptotes, or axis and tangent at the vertex, etc., as suggested by the form of the given equation, and state the angle through which the curve represented by your equation must be rotated to make it coincide with the given curve.

5. $(2x+3y+4)^2-(3x-2y-1)^2 = 52$.
6. $(12x-5y+1)^2+52(5x+12y+3) = 0$.
7. $4(2x-y)^2+9(x+2y)^2 = 180$.
8. $(x+7y)^2+(7x-y)^2 = 50$.
9. $(3x-y+6)^2 = 40$.
10. $(3x-4y+3)(4x+3y-4) = 100$.

In Nos. 11, 12, find the values of A, B, C, F, G, H, δ.

11. $2x^2-6xy+4y^2-4x+6y+1 = 0$.
12. $5x^2+6xy+2y^2+2x+1 = 0$.

In Nos. 13–18, work through 10.2 using the given equation in place of (7).

13. $6x^2+4xy+9y^2-4x+32y-6 = 0$.
14. $7x^2-6xy-y^2+40x-8y+56 = 0$.
15. $16x^2+24xy+9y^2-132x+26y-159 = 0$.
16. $4x^2+12xy+9y^2+4x+6y-3 = 0$.
17. $12x^2-11xy-5y^2+24x-11y+12 = 0$.
18. $13x^2+14xy+13y^2-66x-54y+123 = 0$.

In Nos. 19–24, trace the curve as in 10.3, Examples 1 and 2.

19. $4xy+3y^2+4y = 4$.

20. $x^2+xy+y^2-6x-6y+9=0$.
21. $17x^2+12xy+22y^2-78x+52y+169=0$.
22. $2x^2-3xy-2y^2-x+7y-3=0$.
23. $9x^2-12xy+4y^2-36x+24y+36=0$.
24. $x^2-4xy+4y^2+25y=0$.
25. If a conic passes through the origin, and the axes are the tangent and normal to it there, show that its equation can be written $ax^2+2hxy+by^2+2gx=0$.
26. A conic touches both coordinate axes; show that its equation can be written $a^2x^2+2hxy+b^2y^2+2acx+2bcy+c^2=0$.

10.4. Line-pairs

The equation $3x^2+5xy-2y^2=0$ can be written

$$(3x-y)(x+2y)=0$$

and is satisfied if $3x-y=0$ or $x+2y=0$. It therefore represents these two lines.

In the same way,

$$ax^2+2hxy+by^2=0 \quad . \quad . \quad . \quad (27)$$

represents two lines which may be real and distinct, coincident, or imaginary, because if complex numbers are allowed we can always factorize the L.H.S. and write

$$ax^2+2hxy+by^2=(lx+my)(l'x+m'y) \quad . \quad . \quad (28)$$

so that (27) represents the two lines

$$lx+my=0, \quad l'x+m'y=0.$$

These two lines are called **elements** of the pair (27).

From (28),

$$ll'=a, \quad lm'+l'm=2h, \quad mm'=b \quad . \quad . \quad (29)$$

and you should use these equations to prove the following two results.

(A) One angle between the elements of (27) is $\tan^{-1}\dfrac{2\sqrt{(h^2-ab)}}{a+b}$.

$$\left[\text{Begin with } \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.\right]$$

(B) The two lines through O, perpendicular to the elements of (27), are given by the equation $bx^2-2hxy+ay^2=0$.

[The required lines are $mx-ly=0$ and $m'x-l'y=0$.]

It follows from (A) that the elements of (27) are perpendicular if $a+b=0$, coincident if $ab-h^2=0$, and imaginary if $ab-h^2>0$. This agrees with what we found in 10.2.

In 10.2 we saw that when

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad . \quad . \quad (7)$$

represents a hyperbola, its asymptotes are parallel to the elements of (27), and that when it represents a parabola, its axis is parallel to the double line which (27) then represents. Considering the degenerate cases of these curves, it follows that, **when (7) represents a line-pair, its elements are parallel to those of (27), and are perpendicular if $a+b=0$, parallel or coincident if $ab-h^2=0$.**

We conclude this section with a direct proof that (7) represents a line-pair if, and only if, $\delta = 0$.

To solve for x, we write (7) in the form

$$ax^2+2(hy+g)x+(by^2+2fy+c) = 0.$$

Hence, if $a \neq 0$,

$$ax = -(hy+g) \pm \sqrt{\{(hy+g)^2-a(by^2+2fy+c)\}}$$

and this will represent a curve unless the square root reduces to a linear function of y, when it will represent two lines. The square root reduces if, and only if, the expression in curly brackets, namely

$$(h^2-ab)y^2+2(gh-af)y+(g^2-ac),$$

is a perfect square, and the condition for this is that

$$(gh-af)^2-(h^2-ab)(g^2-ac) = 0.$$

When multiplied out, the condition becomes $a\delta = 0$ and, as we have supposed $a \neq 0$, is equivalent to $\delta = 0$.

If $a = 0$ but $b \neq 0$, the result can be reached by solving for y.

If $a = b = 0$, and $h \neq 0$, (7) is

$$2hxy+2gx+2fy+c = 0,$$

i.e.
$$2(hx+f)(hy+g)+ch-2fg = 0,$$

and this represents a line-pair if, and only if, $2fg-ch = 0$. But this is precisely what the equation $\delta = 0$ reduces to when $a = b = 0$ and $h \neq 0$.

If $a = b = h = 0$, (7) is not of the second degree and the result is not intended to apply.

Thus, provided (7) is of the second degree, it represents a line-pair if, and only if, $\delta = 0$.

Example. Show that $2x^2+xy-6y^2-5x+11y-3 = 0$ represents a line-pair, and find its elements.

[*Note.*—Since finding the elements will prove that the equation represents a line-pair, we do not need to evaluate δ.]

Method 1. The equation can be written

$$2x^2+(y-5)x-(6y^2-11y+3) = 0.$$

$$\therefore \quad 4x = -(y-5) \pm \sqrt{\{(y-5)^2+8(6y^2-11y+3)\}}$$
$$= -y+5 \pm \sqrt{(49y^2-98y+49)}$$
$$= -y+5 \pm (7y-7)$$
$$= 6y-2 \quad \text{or} \quad -8y+12.$$
$$\therefore \quad 2x-3y+1 = 0 \quad \text{or} \quad x+2y-3 = 0.$$

The equation thus represents a line-pair and these are its elements.

Method 2. Ignoring the terms in y, the given L.H.S. can be factorized as $(2x \quad +1)(x \quad -3)$.

Ignoring the terms in x, it can be factorized as

$$(\quad -3y+1)(\quad 2y-3).$$

[Notice that the signs have been chosen to make this as much like the first version as possible.]

Lastly, ignoring all but the second-degree terms, the L.H.S. is $(2x-3y \quad)(x+2y \quad)$.

From these three results we can write the equation in the form $(2x-3y+1)(x+2y-3) = 0$ and conclude that it represents a line-pair, the elements being as before.

10.5. Line-pair joining O to the Meets of a Line and Conic

We saw in 6.6 that the coordinates of two points in which a line cuts a conic are obtained by solving the two equations simultaneously. If the equations are used to eliminate y, the result is an equation in x giving the x-coordinates of the two points. For example, if we obtain $x^2-5x+4 = 0$, we conclude that the x-coordinates are 1 and 4. But geometrically the equation $x^2-5x+4 = 0$ represents the two *lines* $x = 1$ and $x = 4$, one through each point of intersection, and it is clear that, whenever we eliminate y between the equations of a line and a conic, we obtain the lines parallel to Oy through the points in which they meet. A similar result follows if x is eliminated.

In the same way, if we use the two equations to eliminate all terms of less than the second degree, the equation we obtain represents two lines through O, and these lines also pass through the two points of intersection because the coordinates of either point satisfy the original two equations and hence any derived from them. The method of elimination is best shown by an example.

Example. Find the equation of the lines joining O to the meets of $4x-3y = 10$ and $x^2+y^2+3x-6y-20 = 0$ and show that they are perpendicular.

The equation of the given line can be written $1 = \frac{4x-3y}{10}$, and we can obtain a **homogeneous** equation of the second degree (i.e. one in which *every* term except the zero term is of the second degree) by multiplying the first-degree terms of the equation of the circle by $\frac{4x-3y}{10}$, the constant term by $\left(\frac{4x-3y}{10}\right)^2$, thus:

$$x^2+y^2+(3x-6y)\left(\frac{4x-3y}{10}\right)-20\left(\frac{4x-3y}{10}\right)^2 = 0.$$

This reduces to $2x^2-3xy-2y^2 = 0$, the equation of the required line-pair. Its elements are perpendicular because the sum of the coefficients of x^2 and y^2 is zero; the two lines are in fact $2x+y = 0$ and $x-2y = 0$.

Exercise 10B

In Nos. 1, 2, find the equations of the elements of the given line-pairs.

1. (i) $2x^2+xy-y^2-3x+3y-2 = 0$; (ii) $2x^2+8xy+8y^2+x+2y-1 = 0$.
2. (i) $2x^2-5xy+2y^2+8x-7y+6 = 0$;
   (ii) $9x^2-12xy+4y^2-24x+16y+16 = 0$.

In Nos. 3, 4, find one of the angles between the two lines.

3. (i) $x^2-16xy-11y^2 = 0$; (ii) $x^2+xy-6y^2 = 0$;
   (iii) $x^2+2xy-3y^2 = 0$; (iv) $3x^2+4xy+y^2 = 0$;
   (v) $x^2-2xy\sec\theta+y^2 = 0$.
4. (i) $x^2-4xy+y^2 = 0$; (ii) $x^2-4xy+4y^2 = 0$;
   (iii) $x^2-4xy-y^2 = 0$; (iv) $x^2-4xy = 0$;
   (v) $(1+\sqrt{2}\cos\theta)x^2+2\sqrt{2}\,xy\sin\theta+(1-\sqrt{2}\cos\theta)y^2 = 0$.

In Nos. 5, 6, write down the equations of the line-pairs whose elements pass through O and are perpendicular to those of the given line-pairs.

5. (i) $6x^2-xy-y^2 = 0$; (ii) $2x^2+3xy+y^2 = 0$.
6. (i) $2x^2-xy-3y^2 = 0$; (ii) $3x^2-2xy-3y^2 = 0$.

In Nos. 7, 8, find the equation of the line-pair whose elements pass through the given point and are parallel to the given lines.

7. (1, 2), $x^2+xy-6y^2 = 0$. 8. (1, −1), $x^2-16xy-11y^2 = 0$.

In Nos. 9–12, find the values of λ for which the given equation represents a line-pair.

9. $2x^2+xy-6y^2-3x+8y+\lambda = 0$.
10. $x^2+xy-6y^2+\lambda x+\lambda y+6 = 0$.
11. $x^2+2(1+2\lambda)xy+(5+4\lambda)y^2+2(1+\lambda)x+2(5+\lambda)y+5+2\lambda = 0$.
12. $15x^2-2xy-y^2-70x+26y-25+\lambda(x^2+2xy+y^2-18x-2y+1) = 0$.

In Nos. 13–16, find the equation of the line-pair joining O to the points of intersection of the given line and conic.

13. $x+y = 2$, $6x^2+6y^2-5x-7y = 2$. Find also the equations of the elements.
14. $x+3y+1 = 0$, $2xy-2x-10y = 3$. Show that the two lines are equally inclined to Ox.
15. $2x-y+3 = 0$, $2x^2-y^2+8x+7 = 0$. What does the result tell you about the line and curve?

16. $2x-3y=4$, $13y^2+6x+15y+4=0$. What does the result tell you about the line and curve?

In Nos. 17, 18, find the equations of the tangents from O to the given curve, i.e. the lines joining O to the points of intersection of the curve with the polar of O.

17. $6xy-7y^2-8x-4y-4=0$. 18. $x^3+4y^2-4x-56y+100=0$.

19. Prove that the cosine of one of the angles between the lines of $ax^2+2hxy+by^2=0$ is $\dfrac{a+b}{\sqrt{\{(a-b)^2+4h^2\}}}$.

20. If $lx+my+n=0$ is parallel to one of the lines of $ax^2+2hxy+by^2=0$, prove that $am^2-2hlm+bl^2=0$.

21. Find the equation of the line-pair bisecting the angles between the elements of $ax^2+2hxy+by^2=0$ as follows.

Show that freedom equations of the line through any point P_0 at right angles to OP_0 can be written

$$x=x_0-ty_0, \quad y=y_0+tx_0.$$

Find the equation giving the parameters of the points in which this line meets the given line-pair.

If P_0 lies on an angle-bisector, what can be said about the roots of this equation? This condition being satisfied, find the locus of P_0.

22. In No. 21, what can be said about the roots of the equation in t if OP_0 is perpendicular to an element of the given line-pair?

Hence prove statement (B) of 10.4.

23. Find the equation of the angle-bisectors of $ax^2+2hxy+by^2=0$ by writing $\tan\theta=y/x$ in (19) of 10.2.

24. Find the equation of the angle-bisectors of $ax^2+2hxy+by^2=0$ as follows.

From (28) of 10.4, the elements of the line-pair make angles α, β with Ox where $\tan\alpha=-l/m$, $\tan\beta=-l'/m'$. Let (x, y) be a point on one of the angle-bisectors; then if this bisector makes an angle θ with Ox, show that $\tan\theta=y/x$, that $\theta=\dfrac{\alpha+\beta}{2}+\dfrac{n\pi}{2}$, and hence that $\tan 2\theta=\tan(\alpha+\beta)$.

Using (29), eliminate α, β, θ, l, m, l', m' and obtain the required result.

25. If the lines joining O to the points of intersection of a variable line $lx+my+n=0$ with the conic $ax^2+2hxy+by^2+2gx+2fy=0$ are perpendicular, show that $lx+my+n=0$ passes through a fixed point on the normal to the conic at O (the Frégier point of O wo the conic) and find its coordinates.

26. If the lines joining O to the points of intersection of a variable line with the conic $ax^2+2hxy+by^2+2gx+2fy=0$ make equal angles with Ox, show that the variable line passes through a fixed point, and find its coordinates.

27. Obtain the envelope equations of (i) $xy=c^2$, (ii) $b^2x^2+a^2y^2=a^2b^2$ by writing down the condition that the lines joining O to the points of intersection of $lx+my+n=0$ with the curve shall be coincident.

28. Using the method of No. 27, show that the envelope equation of $ax^2+2hxy+by^2+2gx+2fy+c=0$ can be written

$$(clm+hn^2-fln-gmn)^2=(cl^2+an^2-2gln)(cm^2+bn^2-2fmn).$$

29. Find the locus of P if the lines joining O to the points of contact of the tangents from P to the circle $x^2+y^2-2x=3$ are perpendicular.

30. Two perpendicular lines through the centre meet an ellipse at U, V; the tangents at U, V meet at P. Show that the locus of P is a concentric ellipse.

31. Prove that the locus of mid-points of chords of the circle $x^2+y^2-2ax+2b^2=0$ which subtend a right angle at the origin is the circle

$x^2+y^2-ax+b^2=0$. [If your work leads to a locus of the fourth degree, reject the solution $(x-a)^2+y^2=0$.] [A.L., adapted.]

32. Find the equation of the pair of lines joining O to the points of intersection of the conic $17x^2-12xy+8y^2=20$ and the circle $x^2+y^2=r^2$. By writing down the condition that the lines shall be coincident, find the lengths of the semi-axes of the conic.

33. Prove that the length intercepted by $ax^2+2hxy+by^2=0$ on the line $lx+my+n=0$ is $\pm\dfrac{2n\sqrt{\{(l^2+m^2)(h^2-ab)\}}}{am^2-2hlm+bl^2}$.

[Write down freedom equations for $lx+my+n=0$ based on the point P_0 which is supposed to lie on it.]

34. Prove that the area of the triangle formed by $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ is $\pm\dfrac{n^2\sqrt{(h^2-ab)}}{am^2-2hlm+bl^2}$.

10.6. The Common Points of Two Conics

It will simplify our discussion if we use the letter s to represent a second-degree polynomial in x and y, so that $s=0$ represents the equation of a conic. In the absence of any other indication, s will denote the general expression $ax^2+2hxy+by^2+2gx+2fy+c$, and s' the same expression with dashed coefficients, but we shall also use s and the Greek letter σ (sigma) to represent particular second-degree expressions, introducing them by a phrase such as "Let the parabola be $s\equiv y^2-4ax=0$".

The equation $s+\lambda s'=0$, where λ is a number, then represents a curve through all the points common to the two conics $s=0$ and $s'=0$, since, when the coordinates of any of these points are substituted, both s and s' become zero. This is still true if the coordinates are not real but complex.

If the first conic is not degenerate, the axes can be chosen so that the coordinates of any point on it are expressed in terms of a parameter t by one of the pairs of equations of Chapter 8, and by substituting these expressions in the equation of the second conic an equation is obtained whose roots are the parameters of the points of intersection. In general, this equation is of the fourth degree, and hence two conics which are not both degenerate can meet in four points but not more.

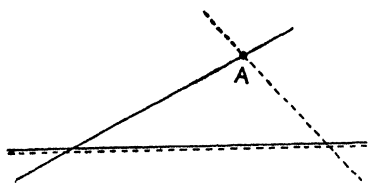


Fig. 67

If both conics are line-pairs, it is clear from a figure that they too cannot meet in more than

four points unless they have a line in common, as in Fig. 67, where the full lines represent one line-pair and the dotted lines the other. The common points are then A, together with all points of the common line.

Conversely, if three of four given points are collinear, any conic through all four must be a line-pair with the line of collinearity as one of its elements, since a line cannot meet a non-degenerate conic in three points.

In what follows, we shall suppose that no three of the points are collinear, unless the contrary is expressly stated.

As an illustration of two conics meeting in four points we can let $s \equiv y^2-x$, $s' \equiv x^2+y^2-6x+4$. It is easily verified that the parabola $s = 0$ and the circle $s' = 0$ both go through the points $(1, \pm 1)$, $(4, \pm 2)$. Hence the conic

$$y^2-x+\lambda(x^2+y^2-6x+4) = 0 \quad . \quad . \quad . \quad (30)$$

goes through these points whatever the value of λ. By a suitable choice of λ it can also be made to go through any chosen fifth point not on $s' = 0$. Thus if the fifth point is (2, 0), (30) goes through it if $-2+\lambda(4-12+4) = 0$, whence $\lambda = -\frac{1}{2}$; and if we substitute this value of λ in (30), the resulting equation

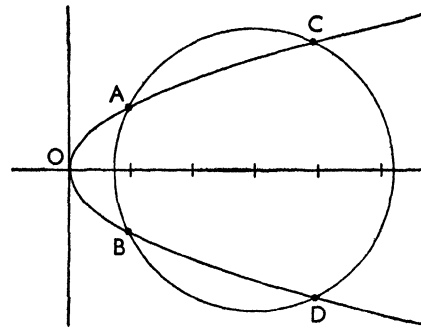


Fig. 68

$$y^2-x-\tfrac{1}{2}(x^2+y^2-6x+4) = 0 \quad . \quad . \quad . \quad (31)$$

represents a conic through all five points.

Now since two conics cannot meet in more than four points, it follows that not more than one conic can be drawn through five points. From this we deduce that (30) can be made to represent *any* conic through the four points of intersection, except the circle itself. For, given any conic through the four points, we can choose any other point P on it, choose λ so that (30) goes through P, and it then follows that (30) represents the given conic.

Although we have used a particular case by way of illustration, the reasoning has been general, and we can say that **if $s = 0$, $s' = 0$ meet in four distinct points, the system**

$$s+\lambda s' = 0 \quad . \quad . \quad . \quad (32)$$

(together with $s' = 0$) comprises all the conics through these four points. The conics of the system are said to form a **pencil.**

When written in full, (32) is

$$(a+\lambda a')x^2+2(h+\lambda h')xy+(b+\lambda b')y^2 \\ +2(g+\lambda g')x+2(f+\lambda f')y+c+\lambda c' = 0.$$

By the tests of 10.3, it represents a rectangular hyperbola if

$$a+\lambda a'+b+\lambda b' = 0,$$

a parabola if

$$(a+\lambda a')(b+\lambda b')-(h+\lambda h')^2 = 0,$$

and a line-pair if

$$\begin{vmatrix} a+\lambda a' & h+\lambda h' & g+\lambda g' \\ h+\lambda h' & b+\lambda b' & f+\lambda f' \\ g+\lambda g' & f+\lambda f' & c+\lambda c' \end{vmatrix} = 0.$$

As these equations are respectively of the first, second, and third degree in λ, the pencil generally contains one rectangular hyperbola, two parabolas, and three line-pairs. If, however, $s = 0$ and $s' = 0$ are both rectangular hyperbolas, $a+b = a'+b' = 0$ and hence $a+\lambda a'+b+\lambda b' = 0$ for all values of λ, so that all the curves are rectangular hyperbolas. Since an equation has to satisfy *two* conditions in order to represent a circle, the pencil (32) only contains a circle if the four points happen to be concyclic.

These results can be illustrated by the pencil (30). The rectangular hyperbola of the system, given by $\lambda = -\frac{1}{2}$, is (31), which can be written $x^2-y^2-4x+4 = 0$. It is degenerate, being the two lines $x-2\pm y = 0$, i.e. AD and BC in Fig. 68. The parabolas are given by $\lambda(\lambda+1) = 0$, whence $\lambda = 0$ or -1. The first value gives $y^2 = x$, the second gives $x^2-5x+4 = 0$, representing the pair of parallel lines AB and CD. The line-pairs are given by

$$\begin{vmatrix} 2\lambda & 0 & -1-6\lambda \\ 0 & 2+2\lambda & 0 \\ -1-6\lambda & 0 & 8\lambda \end{vmatrix} = 0$$

[notice that (30) has to be multiplied by 2 to avoid fractions in the coefficient of $2x$!], which can be written

$$(10\lambda+1)(2\lambda+1)(\lambda+1) = 0.$$

We have already found the line-pairs corresponding to $\lambda = -\frac{1}{2}$ and $\lambda = -1$; $\lambda = -\frac{1}{10}$ gives the equation of AC and BD.

The line-pairs of a pencil are often important and provide an easy way of obtaining the equation of the pencil through four given points.

Example 1. Find the equation of the rectangular hyperbola through $A(-1, 0)$, $B(2, -1)$, $C(1, 2)$, $D(0, 2)$.

The equations of the joins of the points are

$$AD\ 2x-y+2=0, \quad BC\ 3x+y-5=0,$$
$$CD\ \ y-2=0, \quad AB\ x+3y+1=0.$$

Hence two line-pairs of the pencil through the four points are $(2x-y+2)(3x+y-5)=0$ and $(y-2)(x+3y+1)=0$, so that any conic of the pencil is

$$(2x-y+2)(3x+y-5)+\lambda(y-2)(x+3y+1)=0.$$

This represents a rectangular hyperbola if $5+3\lambda=0$; hence the required equation is

$$3(2x-y+2)(3x+y-5)-5(y-2)(x+3y+1)=0,$$

which reduces to

$$9x^2-4xy-9y^2-x+23y-10=0.$$

Example 2. If a circle meets a conic at A, B, C, D, show that AB and CD make supplementary angles with an axis of the conic.

Let the coordinate axes be parallel to the axes of the conic; then its equation $s=0$ contains no term in xy.

Let the equations of AB and CD be $lx+my+n=0$, $l'x+m'y+n'=0$. Then the equation

$$(lx+my+n)(l'x+m'y+n')+\lambda s=0 \quad . \quad . \quad \text{(i)}$$

represents any conic through A, B, C, D, and hence, for some value of λ, represents the given circle. The coefficient of xy in (i) is therefore zero, so that $lm'+l'm=0$. As this condition can be written $l/m+l'/m'=0$, the sum of the gradients of AB and CD is zero and the two lines make supplementary angles with either axis.

EXERCISE 10C [Simple cases of conics through four points were also given in 6b Nos. 27–30]

1. Find the conic of the system $x^2+(\lambda-1)xy+\lambda y^2+2x-(\lambda-1)y=0$ which passes through (2, 2). What type of curve is it?

2. Find the conic of the system of No. 1 which touches the y-axis. What type of curve is it? Is there a member of the system which touches the x-axis?

3. Show that any conic through $(0, -6)$, $(8, 2)$, $(5, 4)$, $(6, 3)$ is a rectangular hyperbola, and find its equation if its asymptotes are parallel to the coordinate axes. What can be said about the four points?

4. Find the equation of the rectangular hyperbola which passes through $(17, -7)$, $(-7, -7)$, $(-4, 2)$, $(-4, 0)$ and has axes parallel to the coordinate axes.

5. Find the equation of the conic through $(\pm 1, 0)$, $(2, \pm 1)$, $(-2, 2)$, and show that it does not cut the y-axis. Sketch it.

6. Find the equation of the conic through $(0, 0)$, $(0, 2)$, $(-1, -2)$, $(1, \pm 1)$. Investigate its intersections with $y=0$ and with $2x=1$. Hence sketch it.

7. Find the parabolas and rectangular hyperbola of the pencil

$$x^2+14xy-5y^2-10x+2y+1+\lambda(x^2+20xy-8y^2-13x+2y+4) = 0.$$

Find also the member whose axes are parallel to the coordinate axes.

8. Find the parabolas and rectangular hyperbola of the pencil

$$5x^2-2(1-2\lambda)xy+5(2-\lambda)y^2+(3-\lambda)x-(7+\lambda)y = 0,$$

and show that the asymptotes of the rectangular hyperbola are parallel to the lines bisecting the angles between the axes of the parabolas.

In Nos. 9, 10, find the parabolas, circle and rectangular hyperbola through the given points, and sketch them.

9. (0, 0), (−8, 4), (27, 9), (7, −1).

10. (0, 0), (1, −2), (9, −6), (16, 8).

11. Find the relation between λ and μ if the equations

$$x^2+y^2-25+\lambda(9x^2-16y^2) = 0 \text{ and } x^2+y^2-25+\mu(x^2-16) = 0$$

represent the same curve. Show by means of a sketch why the two pencils are the same.

12. Explain, in the light of 10.6, why $(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$ represents the circle on P_1P_2 as diameter.

13. If $s \equiv ax^2+by^2-1$, $s' \equiv ab(x^2+y^2)-(a+b)$, show that the directrices of $s = 0$ are a line-pair of $s+\lambda s' = 0$.

14. Two parabolas with their axes at right angles cut in four points; show that the axes of every conic through the four points are parallel to the axes of the parabolas.

15. A variable conic passes through the points $(-4a, 0)$, $(2a, \pm 2a\sqrt{3})$, and $(0, 0)$. Find the locus of its centre.

16. If $ax^2+2hxy+by^2+2gx+2fy+1 = 0$ represents a line-pair, prove that the third pair of lines through the points where it meets the axes is $ax^2+2(2fg-h)xy+by^2+2gx+2fy+1 = 0$.

17. The equation of one pair of sides of a parallelogram is $ax^2+2hxy+by^2 = 0$ and the opposite corner is P_0. Show that the equations of the diagonals are $xy_0-x_0y = 0$ and $2x(ax_0+hy_0)+2y(hx_0+by_0) = ax_0^2+2hx_0y_0+by_0^2$.

By writing down the condition for the diagonals to be perpendicular, find the equation of the angle-bisectors of $ax^2+2hxy+by^2 = 0$.

18. The equation of the diagonals of a parallelogram is $ax^2+2hxy+by^2 = 0$, and of one pair of opposite sides is $lx+my \pm n = 0$. Find the equation of the other pair of sides.

19. The equation $ax^2+2hxy+by^2 = 0$ represents two lines AOA', BOB'. The perpendicular from a point P_0 to AOA' cuts BOB' at L, and the perpendicular from P_0 to BOB' cuts AOA' at M. Prove that the equation of the circle on LM as diameter is

$$(a+b)(x^2+y^2)-2(bx_0-hy_0)x+2(hx_0-ay_0)y+(bx_0^2-2hx_0y_0+ay_0^2) = 0.$$

20. The line $px+qy = 1$ meets the conic $ax^2+2hxy+by^2+2gx+2fy+c = 0$ at A and B; and from the origin O lines OA, OB are drawn to meet the conic again at C and D. Find the equation of the pair of lines OA, OB, and hence, or otherwise, show that the equation of the line CD is

$$c(px+qy+1)+2(gx+fy) = 0. \text{ [A.L.]}$$

21. Line-pairs parallel to $a_1x^2+2h_1xy+b_1y^2 = 0$ and $a_2x^2+2h_2xy+b_2y^2 = 0$ are drawn respectively through P_1 and P_2. Show that they form a cyclic quadrilateral if and only if $h_1(a_2-b_2) = h_2(a_1-b_1)$.

What can you deduce about their angle-bisectors? [Use the answer to Exercise 10b, No. 23.]

22. A, B, C, D are four points on a circle; AB meets CD at P_1, AD meets BC at P_2. Prove that the sum of the powers of P_1 and P_2 wo the circle is equal to $P_1P_2^2$.

23. P is a variable point of a parabola with vertex O; if the circle on OP as diameter meets the parabola again at Q, R, show that QR passes through a fixed point.

24. A pencil of conics passes through four points A, B, C, D; if AB and CD meet at O, show that the polar of O is the same wo every conic of the system.

10.7. Coaxal Circles

The equation $s = 0$ represents a circle if $a = b$ and $h = 0$. Hence if $s = 0$, $s' = 0$ are both circles, the equation

$$s+\lambda s' = 0 \quad . \quad . \quad . \quad . \quad (32)$$

will also represent a circle unless $\lambda = -a/a'$, when the second-degree terms vanish and it represents a line.

Since two circles meet in two points, real or imaginary, and only one circle can be drawn through three points, an argument like that of 10.6 shows that (32) represents a circle or line through these two points of intersection, and can be made to represent all such circles (except $s' = 0$) by varying λ.

In what follows, we shall suppose that the coefficients of x^2 and y^2 are 1 in both the expressions s and s', so that $s \equiv x^2+y^2+2gx+2fy+c$, $s' \equiv x^2+y^2+2g'x+2f'y+c'$. As we saw in 4.2, the values of these expressions when the coordinates of a given point are substituted are then the powers of that point wo the two circles $s = 0$, $s' = 0$. Also the line of the system is $s-s' = 0$, and, since its equation can be written $s = s'$, it is the locus of points whose powers wo the two circles are equal. It is called the **radical axis** of the two circles.

The coefficient of x^2 (or y^2) in the expression $s+\lambda s'$ is $1+\lambda$, and so the power of a point wo the circle (32) is obtained by substituting the coordinates of the point in the expression $(s+\lambda s')/(1+\lambda)$. Hence the radical axis of the circles $s+\lambda_1 s' = 0$, $s+\lambda_2 s' = 0$ is

$$\frac{s+\lambda_1 s'}{1+\lambda_1} = \frac{s+\lambda_2 s'}{1+\lambda_2}.$$

Provided $\lambda_1 \neq \lambda_2$ this reduces to $s-s' = 0$, the same line as before. **Thus all pairs of circles of the system $s+\lambda s' = 0$ have the same radical axis.** For this reason, the circles are said to be **coaxal.**

The equation of any coaxal system can be written

$$x^2+y^2+2\lambda x+c = 0 \quad . \quad . \quad . \quad (33)$$

For let the x-axis be the join of the centres of two circles of the system; then their equations can be written

$$s \equiv x^2+y^2+2gx+c = 0 \quad . \quad . \quad . \quad (34)$$

$$s' \equiv x^2+y^2+2g'x+c' = 0 \quad . \quad . \quad . \quad (35)$$

and their radical axis is $2(g-g')x+c-c' = 0$. If we now choose the radical axis as y-axis, $c = c'$. The coaxal system $s+ks' = 0$, of which (34) and (35) are members, is then

$$(1+k)(x^2+y^2)+2(g+kg')x+(1+k)c = 0,$$

and this is the same as (33) if we write $\lambda = (g+kg')/(1+k)$.

Three different types of coaxal system arise corresponding to different values of c in (33).

Case 1. $c < 0$. The radical axis meets (33) where $y^2+c = 0$, i.e. at $(0, \pm\sqrt{-c})$. The coaxal system is thus of an *intersecting* type, consisting of all the circles through these two points.

Case 2. $c > 0$. The radical axis meets any circle of the system at the two imaginary points $(0, \pm\sqrt{-c})$. Since these are also the points of intersection of every pair of circles of the system, the circles do not meet in real points but form a *non-intersecting* coaxal system. As (33) can be written $(x+\lambda)^2+y^2 = \lambda^2-c$, the system contains two point-circles with centres $(\pm\sqrt{c}, 0)$, obtained by putting $\lambda = \mp\sqrt{c}$. They are called the **limiting points** of the system.

Case 3. $c = 0$. The system then consists of every circle which touches Oy at O. The points of intersection and the limiting points all coincide at O, and the system is of an intermediate type.

Two circles are said to be **orthogonal** if the tangents to them at a point of intersection are at right angles. This is equivalent to saying that the radii drawn to a point of intersection are at right angles. Since the centre of

$$x^2+y^2+2gx+2fy+c = 0 \quad . \quad . \quad . \quad (36)$$

is $(-g, -f)$ and the square of its radius is

$$g^2+f^2-c,$$

the condition for (36) and

$$x^2+y^2+2g'x+2f'y+c' = 0 \quad . \quad . \quad . \quad (37)$$

to be orthogonal is

$$(g-g')^2+(f-f')^2 = g^2+f^2-c+g'^2+f'^2-c',$$

i.e. $$\mathbf{2gg'+2ff'-c-c' = 0.}$$

This condition is easily remembered by comparing it with the expression for the square of the radius of (36).

The condition for (37) and (33) to be orthogonal is $2\lambda g'-c-c' = 0$. If they are orthogonal when λ takes two different values, then

$g' = 0$, $c' = -c$, and they are orthogonal for all values of λ. Hence, writing μ instead of f', we can say that the circles

$$x^2+y^2+2\lambda x+c = 0$$

and

$$x^2+y^2+2\mu y-c = 0$$

are orthogonal for all values of λ and μ. We have here two coaxal systems such that the common points of one are the limiting points of the other, and the two systems are orthogonal—that is to say, every circle of one is orthogonal to every circle of the other. They are shown in Fig. 69.

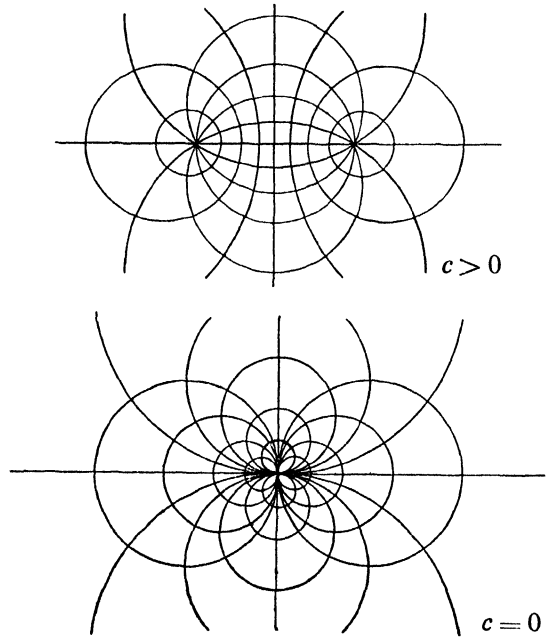


Fig. 69

EXERCISE 10D

1. Prove that the three common chords of the circles $x^2+y^2-6y-1 = 0$, $x^2+y^2-2x-2y+1 = 0$, $x^2+y^2+6x-4y-21 = 0$ taken in pairs are concurrent and find the coordinates of the point of intersection.

Find the equation of the circle which cuts each of the given circles orthogonally. [A.L.]

2. Find the equation of the circle cutting orthogonally the circles $x^2+y^2+2x+4y = 0$, $x^2+y^2+4x+6y-4 = 0$, $x^2+y^2-4x+6y+4 = 0$.

3. Find the equation of the radical axis and the coordinates of the limiting points of the coaxal system of which $x^2+y^2+2x-8y+7 = 0$ and $x^2+y^2-4x+4y-2 = 0$ are members.

4. Find the limiting points of the system $x^2+y^2+(\lambda-3)x-2(\lambda-2)y+2\lambda = 0$.

5. Show that the locus of a point whose powers wo two given circles are in a given ratio is a coaxal circle.

6. Show that the difference between the powers of a point wo two fixed circles is proportional to its distance from their radical axis.

7. Prove that the polars of a point wo the circles of a coaxal system are concurrent or parallel.

8. Show that a limiting point has the same polar wo every circle of a coaxal system.

9. Show that the radical axes of three circles taken in pairs are concurrent or parallel. (The point of concurrence is called the **radical centre.**)

10. Circles of a coaxal system meet a fixed circle not belonging to the system; show that the common chords are concurrent or parallel.

11. If a circle is orthogonal to two circles, show that its centre lies on their radical axis and that it passes through the limiting points of their coaxal system.

12. If $s = 0$, $s' = 0$, $s'' = 0$ are three circles whose centres are not collinear, show that a circle (real or imaginary) can be found cutting the circle $\lambda s+\mu s'+\nu s'' = 0$ orthogonally for all values of λ, μ, ν.

13. A variable circle cuts a conic in two fixed points A, B and two variable points P, Q; show that the direction of PQ is fixed.

14. O is a point on the radical axis of a coaxal system. If the x-axis meets any circle of the system in P, Q, prove that $OP \cdot OQ$ is constant.

15. If the common chord of two circles, one from each of two given orthogonal coaxal systems, passes through a fixed point, prove that the locus of the mid-point of the join of their centres is a straight line.

16. Two systems of coaxal circles are orthogonal; show that the common chords of equal circles (one of each system) touch a rectangular hyperbola.

17. If the polars of P wo two circles meet at Q, prove that the mid-point of PQ lies on the radical axis.

18. Show that the locus of the foot of the perpendicular from a fixed point P to its polar wo a variable circle of a coaxal system is a circle whose centre is on the radical axis.

19. Circles are drawn on a series of parallel chords of a rectangular hyperbola as diameters. Prove that they form a coaxal system, and show that the orthogonal system of circles is that corresponding to the chords in the perpendicular direction. [S.L.]

20. If the common chord of two circles, one belonging to the system $x^2+y^2+2\lambda x = c^2$, the other to $x^2+y^2+2\mu y+c^2 = 0$, passes through the fixed point P_0, show that the locus of the points of intersection of the circles is in general a third-degree curve, but reduces to one of the circles if P_0 lies on one of the axes.

10.8. Conics meeting in Coincident Points

In this section we shall show that the statement of 10.6, that the equation $s+\lambda s' = 0$ represents *any* conic through the four points of intersection of $s = 0$ and $s' = 0$, is still true when these points are not all distinct, and we shall consider the various cases which arise. As the proof is rather long it will be given only in outline, but hints for filling in the details will be given in square brackets.

To begin with, suppose that $s = 0$ is non-degenerate; the coordin-

ates of any point on it can then be expressed in terms of a parameter t, and the parameters of its intersections with $s' = 0$ are given by a quartic equation in t. The statement that $s = 0$ and $s' = 0$ meet in four points some of which are coincident means neither more nor less than that some of the roots of this equation are equal.

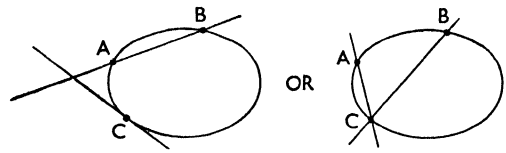
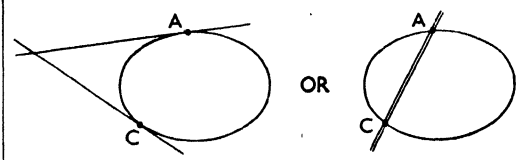
I. If $s = 0$ is met by $s' = 0$ in four points with parameters t_1, t_2, t_3, t_4 (some of which may be equal), then it is met by $s+\lambda s' = 0$ in the same four points. [For let the freedom equations of $s = 0$ be $x = F(t), y = G(t)$; then, when these functions are substituted in the equation $s+\lambda s' = 0$, the expression s vanishes identically.]

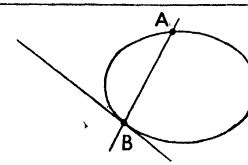
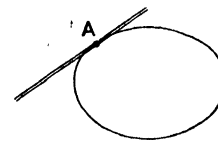
II. If $\sigma = 0$, $\sigma' = 0$ are two conics of the pencil $s+\lambda s' = 0$, then any member of the pencil is also a member of the pencil $\sigma+\mu\sigma' = 0$, and *vice versa* (except the conics $\sigma' = 0$ and $s' = 0$ respectively, but these exceptions vanish if we agree to include $s' = 0$ in the pencil $s+\lambda s' = 0$). Hence the pencil can be generated by any two of its members. [Let $\sigma \equiv \alpha(s+\lambda_1 s')$, $\sigma' \equiv \beta(s+\lambda_2 s')$, then $\sigma+\mu\sigma' = 0$ is the same as $s+\lambda s' = 0$ if $\lambda = (\alpha\lambda_1+\mu\beta\lambda_2)/(\alpha+\mu\beta)$, etc.]

III. The pencil $s+\lambda s' = 0$ contains line-pairs. [The values of λ are found by equating the discriminant to zero.]

IV. Hence any conic $\sigma = 0$ (different from $s = 0$) belongs to the pencil $s+\lambda s' = 0$ if, and only if, the pencils $s+\lambda s' = 0$, $s+\mu\sigma = 0$ have a line-pair in common.

V. The ways in which $s = 0$ can be met by a line-pair in coincident points are as follows.

Points of intersection	Line-pairs meeting $s = 0$
ABCC	
AACC	

Points of intersection	Line-pairs meeting $s = 0$
ABBB	
AAAA	

(The pairs of close parallel lines indicate a line-pair consisting of two coincident lines.)

Hence in all cases the line-pairs are completely determined when the points of intersection and the tangents to $s = 0$ at any coincident points are given.

VI. Hence, by IV, the pencil $s+\lambda s' = 0$ includes every conic which meets $s = 0$ in the same four points as $s' = 0$.

VII. Also, by II, **if the pencil contains two distinct line-pairs, it can be generated by them,** i,e. we can let $s = 0$ be degenerate.

We will now take, one by one, the four ways in which coincidences can occur amongst the four points of intersection.

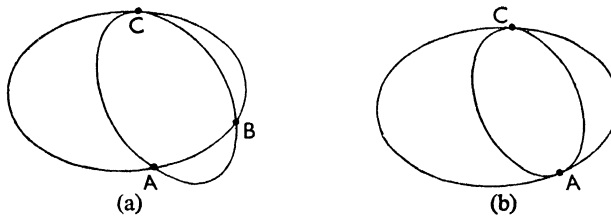


Fig. 70

ABCC. All non-degenerate conics of the pencil have a common tangent at *C* and are said to have ordinary or **two-point contact** there.

Example 1. Find the equations of the two parabolas which pass through $A(10, 0)$, $B(5, 0)$, $C(0, 1)$, and touch the line $x+y-1 = 0$ at *C*.

The line AC is $x+10y-10=0$, BC $x+5y-5=0$, AB $y=0$. Hence, referring to the figures in the table above, we see that the line-pairs of the pencil of conics through A, B, C, touching $x+y-1=0$ at C are

$$y(x+y-1)=0 \text{ and } (x+10y-10)(x+5y-5)=0,$$

so that any conic of the system is

$$x^2+15xy+50y^2-15x-100y+50+\lambda(xy+y^2-y)=0.$$

This is a parabola if

$$4(50+\lambda)=(15+\lambda)^2,$$

whence $\lambda=-1$ or -25.

Substituting these values of λ, we find the equations of the two parabolas to be

$$(x+7y)^2-15x-99y+50=0 \text{ and } (x-5y)^2-15x-75y+50=0.$$

Example 2. Find the equation of the circle which passes through the origin and the two points of intersection of $ax^2+by^2=0$ and $lx+my+n=0$.

If we call the two points of intersection A and B, and the origin C, the circle meets the given line-pair at A, B, and twice at C. It is therefore a member of the pencil

$$(lx+my+n)(l'x+m'y)+\lambda(ax^2+by^2)=0 \quad . \quad . \quad \text{(i)},$$

where $l'x+m'y=0$ is the (as yet undetermined) tangent to it at C.

Now (i) represents a circle if

$$ll'-mm'+\lambda(a-b)=0$$

and

$$lm'+l'm=0.$$

The second of these is satisfied if $l'=l$, $m'=-m$, and to satisfy the first we must then put $\lambda=-(l^2+m^2)/(a-b)$. Hence the required circle is

$$(a-b)(lx+my+n)(lx-my)-(l^2+m^2)(ax^2+by^2)=0,$$

which reduces to

$$(am^2+bl^2)(x^2+y^2)-n(a-b)(lx-my)=0.$$

AACC. All non-degenerate conics of the pencil have ordinary contact with $s=0$ at A and C. They are said to have **double contact** along AC. One parabola of the pencil degenerates into the line AC taken twice.

Example 3. Find the equation of the pair of tangents from (3, 2) to the ellipse $x^2+4y^2=4$.

The pair of tangents has double contact with the ellipse along the polar of (3, 2), namely $3x+8y-4=0$, and is therefore a member of

the pencil

$$(3x+8y-4)^2+\lambda(x^2+4y^2-4) = 0.$$

Since (3, 2) itself lies on the pair of tangents, we have

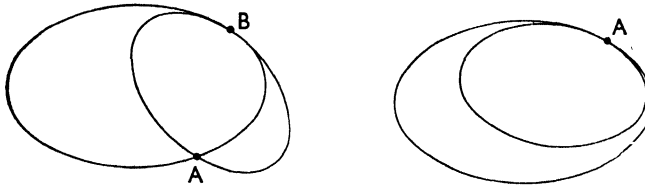
$$21^2+\lambda.21 = 0,$$

whence $\lambda = -21$ and the required equation is

$$(3x+8y-4)^2-21(x^2+4y^2-4) = 0,$$

or, on simplification,

$$3x^2-12xy+5y^2+6x+16y-25 = 0.$$



(a) (b)

Fig. 71

ABBB. The non-degenerate conics have **three-point contact** with $s = 0$ at B, and all have the same curvature there since the circle of curvature has contact of this order.

Example 4. Find the equation of the circle of curvature of the parabola $y^2 = 4ax$ at $(at^2, 2at)$.

The circle is a member of the pencil

$$(x-ty+at^2)(\qquad\qquad)+\lambda(y^2-4ax) = 0,$$

where the contents of the second bracket, if equated to zero, would represent a line through $(at^2, 2at)$. Since the equation of the pencil can contain no term in xy if it is to represent a circle, the second bracket can be filled in as $(x+ty-3at^2)$, and the circle is obtained when $\lambda = t^2+1$. It is

$$x^2+y^2-2a(3t^2+2)x+4at^3y-3a^2t^4 = 0.$$

AAAA. The non-degenerate conics have four-point contact and are said to **osculate** (Latin *osculare* = kiss), as this is the highest order of contact which two conics can have without being identical. One parabola of the pencil degenerates into the tangent at A taken twice.

10.9. The Apollonius Hyperbola

If a conic and a point P_0 are given, the locus of a point P, such that PP_0 is perpendicular to the polar of P wo the conic, is a rectangular

hyperbola called the Apollonius hyperbola corresponding to P_0 and the conic.

Case 1: *Central Conic.* Let the conic be $ax^2+by^2+c=0$. The polar of P_1 is $ax_1x+by_1y+c=0$,
the line P_1P_0 is $(y_1-y_0)(x-x_0)-(x_1-x_0)(y-y_0)=0$,
and these are perpendicular if $ax_1(y_1-y_0)-by_1(x_1-x_0)=0$. The locus is therefore

$$(a-b)xy-ay_0x+bx_0y=0.$$

Case 2: *Parabola.* Let the conic be $y^2-4ax=0$. The polar of P_1 is $2ax-y_1y+2ax_1=0$ and is perpendicular to P_1P_0 if

$$2a(y_1-y_0)+y_1(x_1-x_0)=0.$$

The locus is therefore

$$xy+(2a-x_0)y-2ay_0=0.$$

The Apollonius hyperbola has the following properties:

I. It passes through P_0 and the centre of the conic.

II. Its asymptotes are parallel to the axes of the conic.

(Properties I and II need slight modification if the conic is a parabola.)

III. It meets the conic at the feet of the normals from P_0 to the conic.

[For if it meets the conic at P, PP_0 is perpendicular to the tangent at P; conversely, if P is the foot of a normal from P_0 to the conic, P lies on the hyperbola.]

IV. If P_0 is the centre of the conic, the hyperbola degenerates into the axes of the conic.

Property III gives a useful method of dealing with concurrent normals, alternative to that of Chapter 9. Property IV enables us to write down the axes of a conic when we know its centre.

Example. Find the meet of the normals to the conic $ax^2+by^2+c=0$ at the points where it is cut by the line $lx+my+n=0$.

Let the point be (x_0, y_0), so that the Apollonius hyperbola corresponding to it is $(a-b)xy-ay_0x+bx_0y=0$. This is a member of the pencil of conics of which two members are $ax^2+by^2+c=0$ and $lx+my+n=0$ together with some other line. The equation of the Apollonius hyperbola can therefore be written

$$ax^2+by^2+c+(lx+my+n)(l'x+m'y+n')=0,$$

there being no need to introduce a factor λ since we have the choice of the values of l', m', n'. Comparing coefficients we have

$$a+ll'=0, \quad b+mm'=0, \quad c+nn'=0,$$

$$\frac{a-b}{lm'+ml'}=\frac{-ay_0}{ln'+nl'}=\frac{bx_0}{mn'+nm'}.$$

Hence $$\frac{lm(a-b)}{bl^2+am^2} = \frac{-anly_0}{cl^2+an^2} = \frac{bmnx_0}{cm^2+bn^2},$$

and the point of intersection is

$$\left(\frac{(a-b)l}{bn} \cdot \frac{cm^2+bn^2}{bl^2+am^2}, -\frac{(a-b)m}{an} \cdot \frac{cl^2+an^2}{bl^2+am^2}\right).$$

Exercise 10E

ABCC

1. Find the equation of the rectangular hyperbola through $(-1, -7)$ and $(7, -1)$ which touches the circle $x^2+y^2 = 50$ at $(5, 5)$.

2. An ellipse, with axes parallel to Ox and Oy, cuts the parabola $y^2 = 4x$ at $(1, -2)$ and touches it at $(4, 4)$. Find the other point in which the two curves cut.

3. A circle passes through the vertex of a parabola and touches it at a variable point; find the locus of its centre.

4. Find the circumcentre of the triangle formed by the line-pair $ax^2+2hxy+by^2 = 0$ and the line $lx+my+n = 0$.

5. A circle touches a parabola at P, cuts it at Q and R, and passes through the focus S. Prove that QR passes through a fixed point T, such that the directrix is the perpendicular bisector of ST.

6. Conics are drawn through three given points to touch a fixed line at one of the points; show that the locus of their centres is a conic, and that, if the direction of the given tangent is varied, the centre-loci so obtained form another pencil of conics.

AACC

7. Find the equation of the conic through O which has double contact with the rectangular hyperbola $xy = 6$ along the line $3x-2y = 12$, and show that it is a parabola.

8. Find the equation of the ellipse which passes through the centre of the circle $x^2+y^2-4x-4y+4 = 0$ and touches it where it touches the axes.

9. A circle has double contact with a parabola; show that the chord of contact is perpendicular to the axis.

10. The polar of P, wo a circle having double contact with the parabola $y^2 = 4ax$, is $y = c$; as the circle varies, show that the locus of P is $4ax-cy = 4a^2$.

11. If the tangents from P to the ellipse $b^2x^2+a^2y^2 = a^2b^2$ are perpendicular, show that the locus of P is $x^2+y^2 = a^2+b^2$, the **director circle.**

12. A parabola touches a hyperbola at A and B, and passes through the centre. Show that the tangents at A and B meet on the conjugate hyperbola.

13. A variable conic touches the axes at fixed points X, Y. Prove that the locus of its centre is the line joining O to the mid-point of XY.

14. A variable rectangular hyperbola touches a fixed conic at U, V, and passes through a fixed point O. Prove that UV touches a circle with centre O.

ABBB

15. A parabola with axis parallel to Ox has three-point contact with the rectangular hyperbola $xy = c^2$; find the locus of its vertex.

16. A variable rectangular hyperbola has three-point contact at O with the circle $x^2+y^2-ax = 0$. Show that it passes through a fixed point other than O, and find its coordinates.

17. A variable rectangular hyperbola has three-point contact with a circle at a fixed point. Show that the locus of its centre is a circle.

18. The conics of a pencil have three-point contact at A and pass through B. Show that the centre-locus is a conic touching all members of the pencil at A and having its own centre at C, the point on AB such that $AB = 4AC$.

19. Show that the circle having three-point contact with the ellipse $b^2x^2+a^2y^2 = a^2b^2$ at $(a\cos\theta, b\sin\theta)$ is

$$x^2+y^2-2(a^2-b^2)\left(\frac{x}{a}\cos^3\theta - \frac{y}{b}\sin^3\theta\right)+a^2+b^2-3(a^2\sin^2\theta + b^2\cos^2\theta) = 0,$$

and hence that the evolute of the ellipse is given by

$$ax : by : a^2-b^2 = \cos^3\theta : -\sin^3\theta : 1.$$

20. Show that the equation of any parabola having three-point contact with the circle $x^2+y^2-4ax = 0$ at O can be written $(x\tan\theta-y)^2 = 4ax$. By writing this equation in the form $(x-x_0)^2+(y-y_0)^2 = (x\cos\theta+y\sin\theta-p)^2$, show that the focus of the parabola always lies on a certain circle and that the directrix passes through a fixed point. Find the equation of the circle and the coordinates of the point.

AAAA

21. Find the equation of the parabola which has four-point contact with the ellipse $b^2x^2+a^2y^2 = a^2b^2$ at $(-a, 0)$.

22. Show that a circle can only have four-point contact with an ellipse at the end of an axis.

23. Show that the polar of $(a, 0)$ wo any rectangular hyperbola osculating the parabola $y^2 = 4ax$ touches the parabola $y^2 = 4a(3x+2a)$.

24. Prove that the centres of a pencil of osculating conics lie on a straight line.

Apollonius Hyperbola

25. Obtain the axes of the conic $ax^2+2hxy+by^2+c = 0$ by writing down the equation of the Apollonius hyperbola of the centre.

What follows by letting $c\to 0$?

26. If the Apollonius hyperbola touches the conic, show that the generating point is on the evolute of the conic.

27. Show that all conics through the feet of the four normals from P_0 to $x^2-y^2 = a^2$ are rectangular hyperbolas, and that their centres lie on a circle.

28. The normals at four points A, B, C, D of the ellipse $b^2x^2+a^2y^2 = a^2b^2$ are concurrent; if the equation of AB is $\frac{lx}{a}+\frac{my}{b}+n = 0$, prove that the equation of CD is $\frac{x}{al}+\frac{y}{bm}-n = 0$. These two lines are called **co-normal chords.**

29. Show that the feet of the normals from the point (f, g) to the rectangular hyperbola $xy = c^2$ are the intersections of the hyperbola with the rectangular hyperbola $x^2-y^2-fx+gy = 0$. Show that, if $fg = 4c^2$, the same four points are the feet of the normals to the second hyperbola from the point $(-\frac{1}{2}f, -\frac{1}{2}g)$. [C.S.]

30. A pencil of conics passes through four points on a circle, centre C; show that the centres of the system all lie on the Apollonius hyperbola corresponding to C and any member of the system.

EXERCISE 10M

1. A tangent to a rectangular hyperbola cuts the asymptotes at X, Y; the perpendiculars to it at X and Y cut the asymptotes at Y', X'. Show that $X'Y'$ is also a tangent to the curve.

2. The coordinates of points A and B are (α, β) and (α', β'). Prove that the circle on AB as diameter is given by the equation $(x-\alpha)(x-\alpha')+(y-\beta)(y-\beta') = 0$.

The line $lx+my = 1$ meets the conic $ax^2+by^2 = 1$ at the points A and B. Find the equation of the circle on AB as diameter. Prove that, if this circle passes through the origin, the given line touches a certain fixed circle. [A.L.]

3. A chord PQ of the conic $ax^2+by^2+c = 0$ subtends a right angle at O. A second conic, which meets the coordinate axes in the same points as the first, is met by PQ at R and S. Prove that RS also subtends a right angle at O.

4. Find the locus of the meet of the tangents at two points of the parabola $y^2 = 4ax$ if the normals at the two points meet on the line $x = h$.

5. Two independent systems of coaxal circles are such that the radical axis of either is the line of centres of the other. Show that the product of the radii of any two circles, one of each system, that touch each other, is constant. [A.L.]

6. Prove that the product of the distances of a variable point of a conic from two opposite sides of an inscribed quadrilateral is proportional to the product of its distances from the other two sides.

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7. Show that any chord of the parabola $y^2 = 4ax$, which is trisected by the axis, touches the parabola $y^2+32ax = 0$.

8. S is a focus of an ellipse and P is a point on the curve; prove that the circle on SP as diameter touches the auxiliary circle.

9. Write down the equation of the line P_0P_1. By using the envelope equation of the ellipse $b^2x^2+a^2y^2 = a^2b^2$, show that the equation of a pair of tangents from P_0 to the curve can be written $a^2(y-y_0)^2+b^2(x-x_0)^2 = (xy_0-x_0y)^2$. Hence find the locus of P if the tangents from P are at right angles (the director circle).

10. Prove that four normals can be drawn to the rectangular hyperbola $xy = a^2$ from any point $H(h, k)$.

If K, L, M, N are the feet of these normals and if the circle through K, L, M has its centre at C and meets the hyperbola again at N', prove that the origin O bisects NN' and that the middle point of CN divides OH in the ratio 1 : 3. [S.L.]

11. If the rectangular hyperbola $ax^2+2hxy-ay^2+2gx+2fy+c = 0$ cuts Ox at X_1, X_2 and Oy at Y_1, Y_2, prove that the radical axis of the circles on X_1X_2 and Y_1Y_2 as diameters is $gx+fy+c = 0$.

12. By considering the cases of 10.8 as limits of those in which the four common points of two conics are distinct, show that, of the three line-pairs of the pencil $s+\lambda s' = 0$,

two are identical in the $ABCC$ and $AACC$ cases, and

all three are identical in the $ABBB$ and $AAAA$ cases.

It follows that in the first two cases the discriminant equation has two equal roots, and that in the second two it has three equal roots.

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13. P is a variable point on the circle $x^2+y^2 = 9$; X is (6, 0). If the bisector of $\angle POX$ meets PX at Q, find the locus of Q.

14. If the line $lx+my+n = 0$ meets the curve $xy = c^2$ at P and Q, show that $PQ^2 = (l^2+m^2)(n^2-4c^2lm)/l^2m^2$.

15. Show that the equation of the parabola touching Ox and Oy and having its own axis along the line $x = y$ can be written $\sqrt{x} \pm \sqrt{y} \pm \sqrt{c} = 0$. Draw a sketch, showing the different signs which must be taken in the equation to obtain the different parts of the curve.

16. A circle, centre (α, β), meets the parabola $y^2 = 4ax$ at P, Q, R, S. Prove that the four points in which the circles of curvature at P, Q, R, S meet the parabola again lie on a circle whose centre is $(9\alpha - 16a, -27\beta)$. [S.L.]

17. If the tangent at P to a hyperbola, centre O, cuts the auxiliary circle at L, M, show that the circle OLM goes through the foot of the perpendicular from P to the transverse axis.

18. Prove that the polars of a point wo two conics having double contact meet on the chord of contact.

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19. P is a variable point on the circle $x^2+y^2 = a^2$, M the foot of the perpendicular from P to the x-axis. Find the locus of the points of intersection of OP and CM, if C is $(0, a)$.

20. Show that the circle $x^2+y^2+2g'x+2f'y+c' = 0$ passes through the ends of a diameter of the circle $x^2+y^2+2gx+2fy+c = 0$ if

$$2(g^2+f^2-c) = 2gg'+2ff'+c+c'.$$

If a third circle is drawn, concentric with the first and cutting the second orthogonally, show that the area of the second circle is half that of the region between the first and third.

21. An equilateral triangle has its centre at the origin and one of its sides is $x+y = 1$; find the equations of the other sides and show that its altitudes are represented by $x^3+3x^2y-3xy^2-y^3 = 0$.

22. C_n denotes the centre of the circle through the feet of the three normals from P_n to a parabola. Prove that, if the points P_1, P_2, P_3 are collinear, then the points C_1, C_2, C_3 are also collinear. [S.L., adapted.]

23. In general, how many circles of the coaxal system through two points of a parabola touch the curve? Are there any exceptions?

24. The rectangular hyperbola through the four points common to the circle $x^2+y^2+2gx+2fy+c = 0$ and the two straight lines $ax^2+2hxy+by^2 = 0$ meets the x-axis at A, B and the y-axis at C, D. Prove that the equation of the line joining the mid-points of AB and CD is $\dfrac{x}{g} - \dfrac{y}{f} = \dfrac{a+b}{a-b}$. [A.L.]

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25. A variable tangent to a parabola cuts the tangents at the ends of the latus rectum at P, Q. Show that $\angle PSQ = 90°$, S being the focus.

26. If the line $lx+my+n = 0$ cuts the parabola $y^2 = 4ax$ at P and Q, find the circumcentre of the triangle OPQ.

27. Show that the equation $ax^2+2hxy+by^2+2gx+2fy+c = 0$ represents two parallel lines if $ab-h^2 = bg^2-af^2 = 0$, and that the distance between them is then $2\sqrt{\left(\dfrac{g^2-ac}{a(a+b)}\right)}$.

28. The points P', Q', R', S' on the auxiliary circle of the ellipse $b^2x^2+a^2y^2 = a^2b^2$ correspond to the points P, Q, R, S on the ellipse. If the normals at P, Q, R, S all pass through (x_0, y_0), show that the lines through P, Q, R, S parallel to OP', OQ', OR', OS' are concurrent, and find the point of concurrence.

29. X and Y lie on Ox and Oy; two conics have a common tangent at O and touch XY at X, Y respectively; if they meet again at A and B, show that the axes bisect the angles between OA and OB.

30. The conic $x^2-2xy-8y^2-6x+24y = 0$ meets any conic through O (0, 0), $P(-6, 6)$, $Q(-2, 0)$, $R(0, 2)$ at O, P, and two other points L and M. Show that LM passes through a fixed point and find its coordinates.

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31. If P' is the point on the auxiliary circle corresponding to a point P on the ellipse $b^2x^2+a^2y^2 = a^2b^2$, of which S is the focus $(ae, 0)$, find the locus of the meet of OP' and SP.

32. Show that the circles $x^2+y^2+2gx+2fy+c = 0$ and
$$x^2+y^2+2g'x+2f'y+c' = 0 \text{ touch if}$$
$$(2gg'+2ff'-c-c')^2 = 4(g^2+f^2-c)(g'^2+f'^2-c').$$

33. P, Q are variable points on a conic through a fixed point O. If OP, OQ make equal angles with the tangent at O, show that PQ passes through a fixed point on this tangent.

34. The circle on a variable chord PQ of a parabola as diameter touches the parabola at another point. Prove that the locus of the centre C of the circle is an equal parabola and that PQ touches this parabola at C.

35. A variable rectangular hyperbola has three-point contact with a given conic at a given point C. Show that it passes through a fixed point on the normal at C.

36. Find the orthocentre of the triangle formed by the lines $lx+my+n = 0$ and $ax^2+2hxy+by^2 = 0$ as follows. Write down the coordinates of a variable point P on the altitude from the origin; write down the equation of the line-pair whose centre is P and whose elements are perpendicular to those of $ax^2+2hxy+by^2 = 0$; finally find the condition that this new line-pair should be a member of the pencil of conics which has, as its remaining line-pairs, $ax^2+2hxy+by^2 = 0$ and $lx+my+n = 0$ taken together with some other line.

CHAPTER 11

COORDINATE SYSTEMS

11.1. Introduction

Although Apollonius in his treatise on conic sections (about 250 B.C.) had based his calculations on the distances of a point from two perpendicular lines and had in effect obtained the equations of 6.1, these equations were expressed in geometric rather than algebraic terms—for instance, the square of a number had to be represented by the area of an actual square—and it was not until nearly two thousand years later that René Descartes (1595–1650) realized not only that a curve could be represented by an equation relating the coordinates of a moving point which traced it out, but also that from this equation *every* property of the curve could be obtained. Pursuing his idea he brought out in 1637 the first treatise on coordinate geometry, and the coordinate system used is called Cartesian in his honour.

In using the cartesian system we can imagine a **coordinate network** of perpendicular lines, as on a sheet of squared paper but of infinitely fine mesh, each line representing a particular value of one of the coordinates; and the point (4, 7), for example, can be regarded as the meet of the lines $x = 4$ and $y = 7$.

But why should the network be rectangular? Or why should it consist of straight lines? All that is needed is a network consisting of two families of lines or curves, each member of one family representing a different value of the first coordinate, and each member of the other a different value of the second; a point is then located by the intersection of one member of each family, and an equation connecting the coordinates still represents a curve whose properties can be found from its equation when the properties of the curves of the network are known. Three systems will be considered in the course of this chapter, but first we will look at a means whereby one of them can be obtained, and which is of interest in its own right.

11.2. Orthogonal Projection

If, from all points of a figure in a plane π, perpendiculars are drawn to another plane π', they meet π' in the points of a figure called the **orthogonal projection** of the first figure. The process by which the second figure is obtained is also called orthogonal projection.

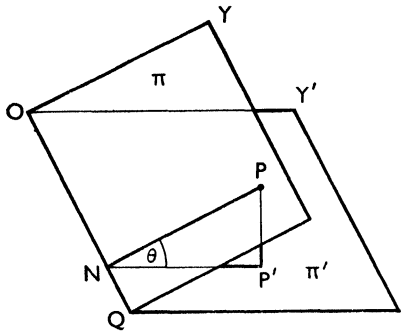

Fig. 72

To find the relationship between the two figures, take a point O on the common line OQ of π and π', and let the plane through O, perpendicular to OQ, meet π in OY, π' in OY'. Let any point P in π have coordinates (x, y) referred to axes OQ, OY, and let its orthogonal projection P' have coordinates (x', y') referred to OQ, OY'. Then from Fig. 72, if θ is the acute angle between the planes,

$$x' = x, \quad y' = y\cos\theta \quad . \quad . \quad . \quad (1)$$

and the result of orthogonal projection is that distances in one direction are unaltered, while distances in the perpendicular direction are multiplied by a constant factor $\cos\theta$.

Since the second equation of (1) can be written $y = y'\sec\theta$, the equation of the orthogonal projection of a given curve can be obtained by substituting x' for x, $y'\sec\theta$ for y, in its equation. From this it follows that **the degree of a curve is unaltered by orthogonal projection.** . . . . . (A)

A line thus projects into a line.

If these substitutions are made in the general equation of the second degree, we obtain

$$ax'^2+2shx'y'+s^2by'^2+2gx'+2sfy'+c = 0,$$

where s has been written for $\sec\theta$, and this represents an ellipse, parabola or hyperbola according as $a(s^2b)-(sh)^2 \gtreqless 0$. This criterion reduces to $ab-h^2 \gtreqless 0$; hence **an ellipse, parabola or hyperbola projects into a conic of the same type** . . . . . (B)

The circle $x^2+y^2 = a^2$ and the rectangular hyperbolas $x^2-y^2 = \pm a^2$, however, project into the ellipse $x'^2+y'^2\sec^2\theta = a^2$ and the hyperbolas $x'^2-y'^2\sec^2\theta = \pm a^2$. It is customary to omit the

dashes from the coordinates of the projected points; if we do this and write

$$b = a\cos\theta \quad . \quad . \quad . \quad . \quad (2)$$

the equations of the projected curves become the familiar $x^2/a^2+y^2/b^2 = 1$ and $x^2/a^2 - y^2/b^2 = \pm 1$. Given any positive quantities a and b, with $a>b$, we can always find an acute angle θ satisfying (2); hence **any ellipse or hyperbola can be regarded as the orthogonal projection of a circle or rectangular hyperbola** . . . (C)

In the same way, by taking the common line of the planes parallel to the other axis, **any ellipse or hyperbola can be projected into a circle or rectangular hyperbola** . . . . . (D)

the radius of the circle, for example, being equal to the minor semi-axis of the ellipse.

By considering intersections, it can be seen that **coincident points project into coincident points** . . . . . (E)

so that a tangent to a curve projects into a tangent to the projected curve.

In general, lengths and angles are altered by projection, but **parallel lines project into parallel lines** . . . . . (F)

and **the ratio of lengths of parallel lines is unaltered by orthogonal projection** . . . . . (G)

The first of these follows by considering the projections of the lines $lx+my+n = 0$ and $lx+my+n' = 0$. To prove the second, let P_1, T, P_2, U be the points (x_1, y_1), (x_1+ht, y_1+kt), (x_2, y_2), (x_2+hu, y_2+ku), so that P_1T, P_2U are parallel and of lengths $t\sqrt{(h^2+k^2)}$ and $u\sqrt{(h^2+k^2)}$. If we write c for $\cos\theta$, they project into (x_1, cy_1), (x_1+ht, cy_1+ckt), (x_2, cy_2), (x_2+hu, cy_2+cku), and the corresponding lengths, $t\sqrt{(h^2+c^2k^2)}$ and $u\sqrt{(h^2+c^2k^2)}$, are in the same ratio. In particular, **mid-points project into mid-points, and the centre of a curve into the centre of the projected curve** . . (H)

A rectangle whose sides are of length p, q, and are parallel to OQ and OY respectively, projects into a rectangle with sides p, cq, parallel to OQ and OY'. The projected area is therefore c times the old. As the area of any figure can be regarded as the limit of the sum of areas of many small rectangles, this relation between the areas of the original and projected figures is always true, and **the ratio of coplanar areas is unaltered by orthogonal projection.** . . (I)

Problems to be solved by orthogonal projection are usually worded in terms of pure geometry, and the solution then depends on the statements (A)—(I), which are similarly worded.

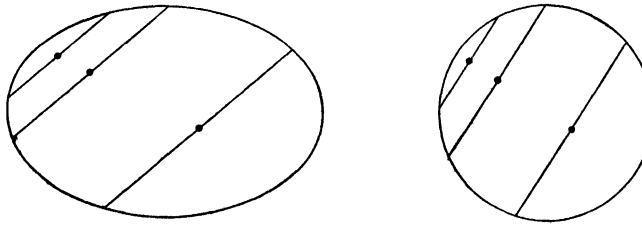

Fig. 73

Example 1. Prove that the locus of mid-points of parallel chords of an ellipse is a diameter.

Project the ellipse into a circle; then the parallel chords and their mid-points project into parallel chords of the circle and their mid-points. But the mid-points of the chords of the circle lie on the perpendicular diameter. As this is a line through the centre of the circle it is the projection of a diameter of the ellipse, and on this the given mid-points lie. (The diameter of the ellipse is not, however, in general perpendicular to the given chords, since angles are usually altered by projection.)

Example 2. P, Q are two points on an ellipse, centre O; the tangents at P and Q meet at T; OT meets PQ at R and the ellipse at S. Prove that $OR.OT = OS^2$.

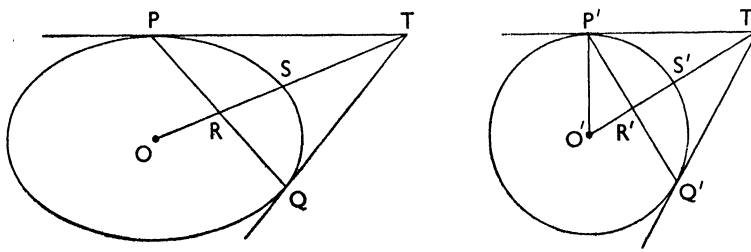

Fig. 74

Project the ellipse into a circle and denote corresponding points by means of dashes, as in Fig. 74. Then $P'Q'$ is perpendicular to $O'T'$ and hence the triangles $O'P'R'$, $O'T'P'$ are similar, having a right angle each and a common angle at O'. Hence

$$\frac{O'R'}{O'P'} = \frac{O'P'}{O'T'},$$

$$\therefore \quad \frac{O'R'}{O'S'} = \frac{O'S'}{O'T'}.$$

But these are ratios of lengths along a line, and were therefore unaltered by projection.

Hence $\dfrac{OR}{OS} = \dfrac{OS}{OT}$, and $OR.OT = OS^2$.

EXERCISE 11A

1. If an ellipse is projected orthogonally into a circle, what do conjugate diameters become?

2. Prove that the area of the ellipse $x^2/a^2+y^2/b^2 = 1$ is πab.

3. A parallelogram is inscribed in an ellipse; prove that its sides are parallel to conjugate diameters.

4. A parallelogram is circumscribed about an ellipse; prove that its diagonals lie along conjugate diameters.

5. Prove that all parallelograms which are inscribed in an ellipse and have conjugate diameters as diagonals are equal in area.

6. Prove that, of all parallelograms circumscribing an ellipse, those having sides parallel to conjugate diameters are of least area.

7. ABC is a variable triangle inscribed in a given ellipse, and is such that the tangent at B to the ellipse is parallel to AC, and the tangent at C to the ellipse is parallel to AB. These two tangents intersect at a point P. Show that the locus of P is an ellipse whose axes lie along those of the given ellipse. [A.L.]

8. Show how to inscribe in an ellipse a triangle of maximum area which has one corner at a given point of the curve.

9. A tangent to an ellipse meets two conjugate diameters at U, V; prove that the other tangents from U and V are parallel.

10. A triangle ABC is inscribed in an ellipse. If a line is drawn through each vertex, parallel to the diameter which is conjugate to that parallel to the opposite side, show that these lines are concurrent.

11. AB is any diameter of an ellipse, and Q is any point on the tangent at A. If BQ meets the ellipse again at R, prove that the tangent at R bisects AQ. [A.L.]

12. The centre of an ellipse is C, the major axis is AA', and P is any point on the line of the minor axis BB'. The line AP meets the ellipse again at Q, and RR' is the diameter of the ellipse parallel to AP. Prove that $AP.AQ = 2CR^2$. [A.L.]

13. AA' is a fixed diameter of an ellipse, centre C. P is any point on the ellipse, and CQ, CQ' are semi-diameters parallel to AP, $A'P$ respectively. Prove that $AP^2/CQ^2+A'P^2/CQ'^2$ is constant. [A.L.]

14. Chords PQ, RS of an ellipse meet at T. Prove that the ratio of $TP.TQ$ to $TR.TS$ is equal to the square of the ratio of the lengths of diameters parallel to PQ and RS.

15. "A chord of a fixed circle, which moves so that its mid-point always lies on a fixed line, envelopes a parabola." What does this statement become when the figure is projected orthogonally?

16. "The locus of mid-points of chords of a circle which pass through a fixed point is a circle." Generalize this statement by means of orthogonal projection.

17. Prove that any two intersecting lines can be projected orthogonally into perpendicular lines, and deduce that an ellipse can be drawn to touch each side of a given parallelogram at its mid-point.

18. A, A' are the vertices of an obtuse-angled hyperbola, and P a variable point of the curve. Prove that it is possible to project the figure orthogonally so that if P projects into P', A and A' remaining unchanged, $\angle AP'A' = \angle APA'$ for all positions of P.

11.3. Oblique Cartesian Coordinates

In the oblique cartesian system the axes are usually graduated with equal scales as before, but are not at right angles. The equation $x = h$ represents a line through the graduation h on Ox, parallel to Oy, and $y = k$ represents a line similarly defined; the point (h, k) is their meet, as in Fig. 75. To discover the properties of this coordinate system we shall first show that its network is the orthogonal projection of a rectangular cartesian network.

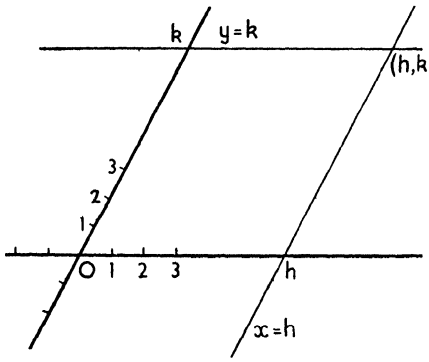

Fig. 75

Let a rectangular coordinate network N lie in a plane π, and let π' be a plane meeting π in the line $x = y$, which we will call OQ (see Fig. 76a). When the figure is projected on to π', lines parallel to OQ are unaltered in length, while lengths at right angles to this are multiplied by $\cos\theta$, and the result (Fig. 76b) is an oblique network N' in which the axes make equal angles with OQ and the scales on the two axes are equal, though not the same as before projection. If we denote the angle xOy in N' by ω (omega), $\tan\frac{1}{2}\omega = \cos\theta$; hence a value of θ can be found to make ω equal to

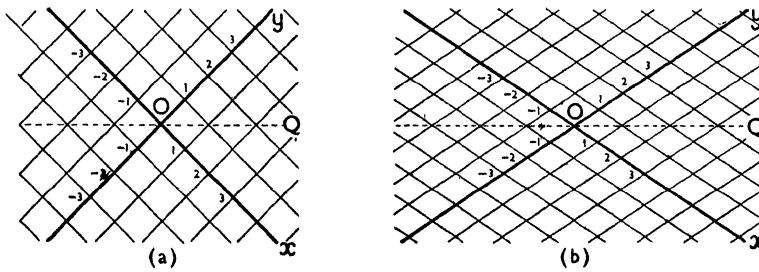

Fig. 76

any given acute angle. If an obtuse value of ω is required, it can be obtained by letting OQ be the line $x+y = 0$ in N.

Any oblique network N' is thus the orthogonal projection of a rectangular network N. Further, since every line of N projects into the corresponding line of N', every point of N' is the projection of the point *with the same coordinates* in N, and every curve in N' is the projection of the curve *with the same equation* in N. This is not what we had in 11.2, where the two coordinate systems were alike, and hence the algebra of that section does not apply here. But the statements (A)—(I) are independent of coordinate systems and therefore still true. We note some of their consequences.

From (A), a first-degree equation represents a line, and the equation of the line P_1P_2 is the same as before.

From (F), the tests for parallel lines still hold good, but not those for perpendicularity, nor the formulæ for angle and distance.

From (G) and (H), the point dividing P_1P_2 in a given ratio has the same coordinates as before, and so has the centre of a given conic.

From (A) and (B), any second-degree equation represents a conic, and the tests for an ellipse, parabola, and hyperbola, as well as that for degeneracy, still apply. But the equation of a circle is no longer what it was.

From (C), by a suitable choice of axes, the equation of any ellipse can be written $x^2+y^2 = a^2$. The axes are conjugate diameters, since each is the locus of mid-points of chords parallel to the other; evidently they are the two *equal* conjugate diameters.

The equation of any hyperbola referred to its asymptotes as axes is $xy = c^2$. The equation $x^2-y^2 = a^2$, however, still represents a rectangular hyperbola since its asymptotes, $x\pm y = 0$, are the bisectors of $\angle xOy$ and therefore perpendicular.

From (E), the equations of tangents and polars can still be used.

Most of these results can be summed up by saying that **the formulæ for rectangular axes still hold good unless they concern distance or angle.**

EXERCISE 11B

1. Show that the equation $x^2/a^2-y^2/b^2 = 1$ represents a hyperbola and that the coordinate axes lie along conjugate diameters.
2. Show that the equation $y^2 = 4ax$ represents a parabola whose axis is parallel to Ox.
3. Sketch the curves $y = x^2$, $xy = 1$ when $\omega = 60°$.
4. Sketch the curves $x^2+y^2 = 1$, $x^2-y^2 = 1$ when $\omega = 60°$.

5. What are the equations of the axes of $x^2+y^2 = a^2$?

6. What does the equation $x(x+y) = a^2$ represent? Sketch the curve when $\omega = 60°$, and state the equation of one axis of symmetry.

7. Prove Ceva's Theorem, that if L, M, N are points on the sides BC, CA, AB of a triangle, such that AL, BM, CN are concurrent, then

$$\frac{BL}{LC}\cdot\frac{CM}{MA}\cdot\frac{AN}{NB} = +1.$$

8. Prove Menelaus' Theorem that, if a line meets the sides BC, CA, AB of a triangle at L, M, N, then

$$\frac{BL}{LC}\cdot\frac{CM}{MA}\cdot\frac{AN}{BN} = -1.$$

9. On the sides AC, BC of a triangle ABC, variable points P, Q are taken so that $a/CP+b/CQ$ is constant, where a, b are constant numbers. Prove that PQ passes through a fixed point.

10. Through a point P on the diagonal BD of a parallelogram $ABCD$, lines are drawn parallel to the sides and meet AB at Q, AD at R. Show that BR, CP, DQ are concurrent.

11. Q is a fixed point on a parabola, M is any point on the diameter through Q, and PP' the chord of the parabola bisected at M. Show that PP'^2/QM is constant for all positions of M.

12. The mid-point M of a chord PP' of an ellipse, centre O, lies on a diameter AA'; BB' is the diameter parallel to PP'. Prove that $OM^2/OA^2+MP^2/OB^2 = 1$.

13. P is a variable point of a hyperbola, centre O; lines are drawn through P parallel to the asymptotes so that each meets the other asymptote, the points of intersection being Q, R. Prove that $OQ.OR$ is constant.

14. PQ is a variable chord of a rectangular hyperbola and has its mid-point on a fixed diameter AA' of the curve; prove that the power of the centre of the hyperbola wo the circle on PQ as diameter is constant.

What can be deduced about this circle and the circle on AA' as diameter?

15. P, Q are two points on a parabola; the tangent at P meets the diameter through Q at T; show that the tangent at Q bisects PT.

16. Two parabolas with parallel axes touch at O; any line parallel to the tangent at O cuts them at P, Q respectively; prove that the tangents at P, Q meet on the diameter through O.

17. Q, R are two fixed points of a hyperbola, and P is a variable point of the curve. If PQ, PR meet an asymptote at U, V, prove that UV is of constant length.

18. The tangents at P, Q to a hyperbola meet one asymptote at L, M. Prove that PQ passes through the mid-point of LM.

19. Prove that a variable tangent to a hyperbola makes with the axes a triangle of constant area.

20. If the sides of a quadrilateral touch a hyperbola, show that the join of the mid-points of its diagonals passes through the centre of the curve.

21. H, K are two points, one on each asymptote of a hyperbola, centre O, and the parallelogram $OHJK$ is drawn. If HJ, JK cut the curve at P, Q, prove that PQ is parallel to HK.

22. A parallelogram is drawn so that the ends of one diagonal lie on a hyperbola whose asymptotes are parallel to the sides. Prove that the other diagonal passes through the centre of the curve.

23. Use the cosine formula to find the length of P_1P_2 in terms of $x_1, y_1, x_2, y_2, \omega$.

24. Find the equation of the circle, centre O, with radius a.

25. What does the equation $(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$ represent?

26. Find the equations of the lines through O at right angles to the axes.

11.4. Polar Coordinates

In polar coordinates the axes are replaced by a point O (the **pole**) and an **initial line** OX passing through it; the position of the point P with coordinates (r, θ) is then defined as follows. Let a directed line OR first lie along OX, the senses of the two lines being the same, and let OR then be rotated counter-clockwise about O through the directed angle θ, which we shall suppose to be measured in radians. With OR in its new position, P is the point on it such that the directed length of OP (the **radius vector**) is equal to r.

If r and θ are positive, everything is straightforward: if in Fig. 77 the length of OP is 1 unit and the acute angle POX is $\pi/6$, P is the point $(1, \pi/6)$. But the point $(-1, -5\pi/6)$ is also P, because these coordinates instruct us first to rotate OR *clockwise* from OX through $5\pi/6$ radians (making it lie along OQ in the figure), and then to draw a radius vector 1 unit long in the direction *opposite* to OQ. In the same way the coordinates $(1, \pi/6 \pm 2n\pi)$ and $(-1, -5\pi/6 \pm 2n\pi)$ represent the same point P for all integral values of n.

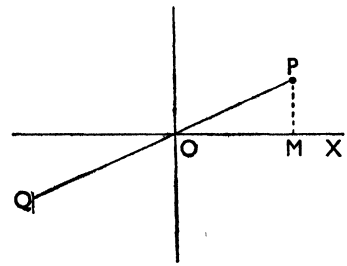

Fig. 77

To change from polar to cartesian coordinates, or *vice versa*, suppose that in Fig. 77 the point P has polar coordinates (r, θ), and has cartesian coordinates (x, y) when O is the origin and OX the x-axis. Let PM be the perpendicular from P to OX; then on the one hand $OP = r$ and $\angle MOP = \theta$, while on the other, $OM = x$ and $MP = y$. Hence

$$x = r\cos\theta, \quad y = r\sin\theta \quad . \quad . \quad . \quad (3)$$

and

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x \quad . \quad . \quad . \quad (4)$$

If we are given the equation of a curve in terms of one set of coordinates, (3) and (4) enable us to express it in terms of the other; or if r is given in terms of θ, (3) provides cartesian freedom equations with θ as parameter. When both kinds of coordinates are used together there is of course no necessity for the origin to be the pole and the x-axis the initial line, but we shall always suppose this to be the case unless the contrary is stated; in other cases (3) and (4) will need modification.

Example 1. Express the equation $x^2+y^2-2ax = 0$ in polar coordinates.

It is $r^2-2ar\cos\theta = 0$, or, dropping the factor r, $r = 2a\cos\theta$.

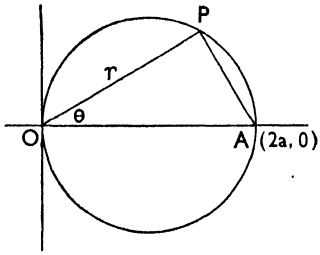

Fig. 78

This could have been seen geometrically. If A is the point $(2a, 0)$, the given equation represents the circle on OA as diameter. Since the angle in a semicircle is a right angle, $r = OP = OA\cos\theta = 2a\cos\theta$. The polar equations of other circles through O may be obtained in a similar way.

Notice that, as θ increases from 0 to $\pi/2$, P travels from A to O along the upper half of the circle. As soon as θ becomes obtuse, however, $\cos\theta$ becomes negative, so that the tip of the radius vector, instead of swinging over into the second cartesian quadrant, moves to the opposite side of the pole and enters the fourth quadrant with r negative. As θ then increases to π, P moves round the lower half of the circle to A. Further increase in θ causes the process to repeat itself, and the circle has been described a second time when θ reaches 2π.

Example 2. Find the cartesian equation of $r\cos(\theta-\alpha) = a$.

Expanding the cosine and using (3) we obtain as the required equation $x\cos\alpha + y\sin\alpha = a$.

Once again the result is clear from a figure. If P is any point on the locus and M the point with polar coordinates (a, α), the given equation shows that $OP\cos MOP = OM$, so that OMP is a right angle. Hence the locus is the line through the cartesian point $M(a\cos\alpha, a\sin\alpha)$, perpendicular to OM, and its equation follows accordingly. As θ increases from 0 to 2π the line is traced out twice.

Example 3. Show that the equation of a conic with a focus at the pole and major axis along the initial line may be written

$$\frac{l}{r} = 1 + e\cos\theta,$$

where l is its semi-latus rectum and e its eccentricity.

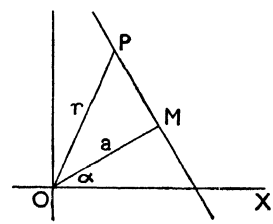

Fig. 79

Let the directrix corresponding to the

focus O cut the positive part of the initial line; let P be any point on the curve and L one end of the latus rectum; let PN, LM be the perpendiculars from these points to the directrix.

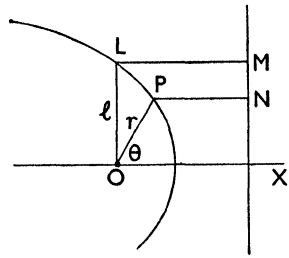

Fig. 80

Then $OL = l$, $OP = r$, and by the focus-and-directrix property $l = eLM$, $r = ePN$. Also $LM = PN + r\cos\theta$. Hence, multiplying this last equation by e, $l = r(1 + e\cos\theta)$ and the result follows.

11.5. Polar Curve-tracing

Symmetry

Any line through O is obtained by giving θ a constant value. We shall find the image of a point in the line OA with equation $\theta = \alpha$ and so obtain a test for symmetry about this line.

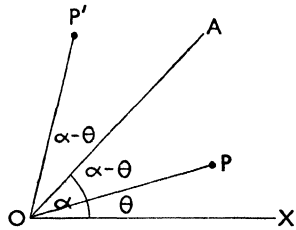

Fig. 81

In Fig. 81, let P' be the image of $P(r, \theta)$ in OA. Since $\angle POA = \alpha - \theta$, $\angle AOP'$ has the same value and hence

$$\angle XOP' = \angle XOA + \angle AOP' = 2\alpha - \theta.$$

The image of (r, θ) in $\theta = \alpha$ is thus $(r, 2\alpha - \theta)$. Hence **a curve is symmetrical about $\theta = \alpha$ if its equation is unaltered when $2\alpha - \theta$ is written for θ.**

Also, since the image of P in O is $(r, \theta+\pi)$ or $(-r, \theta)$, **a curve is symmetrical about O if its equation is unaltered when θ is replaced by $\theta+\pi$, or when r is replaced by $-r$.**

Unfortunately these tests are incomplete: a curve may be symmetrical and yet not satisfy them because more than one pair of coordinates can represent the same point.

Example 1. Trace the curve $r = a\theta$.

Clearly r is zero when $\theta = 0$ and increases continually with θ, never becoming infinite. The curve is therefore a spiral. Negative values of θ give negative values of r and again produce a spiral. The

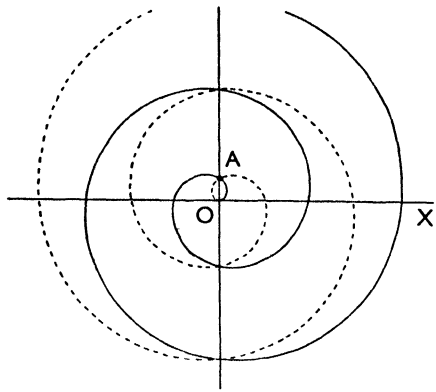

Fig. 82

curve, which is known as **Archimedes' Spiral,** is shown in Fig. 82, where the part for which θ is negative is shown dotted. The radial distance between consecutive "turns" of the spiral is everywhere $2\pi a$, and variation in the value of a merely alters the scale of the curve: all Archimedean spirals are therefore similar. The curve has mechanical applications because a cam of this shape, uniformly rotated, will lift a rod at a uniform rate: a sewing machine has something of the sort for winding the bobbin.

Three facts about polar equations are illustrated by this curve. In the first place it is symmetrical about $\theta = \pi/2$ and yet its equation does not satisfy the usual test for symmetry. How this comes about may be seen by considering the point A in the figure: its coordinates are $(\pi a/2, \pi/2)$ and $(-\pi a/2, -\pi/2)$ on the two halves of the spiral.

In the second place, the equation $r = a(\theta+2n\pi)$ represents this

same curve for all integral values of n, since the lines $\theta = \alpha$ and $\theta+2n\pi = \alpha$ are identical. A given curve, therefore, may have more than one polar equation referred to a given pole and initial line.

Thirdly, to obtain the cartesian equation of the curve we write $\tan\theta = \tan(r/a)$, which becomes

$$y = x\tan\left\{\frac{\sqrt{(x^2+y^2)}}{a}\right\},$$

and this cannot be simplified to an algebraic equation because a line meets the curve in an infinite number of points, whereas it can only meet an nth-degree curve in n points. The polar and cartesian equations of a curve are not as a rule both algebraic, owing to the nature of equations (3) and (4).

Tangents

The angle ϕ between the radius vector and a tangent to a curve

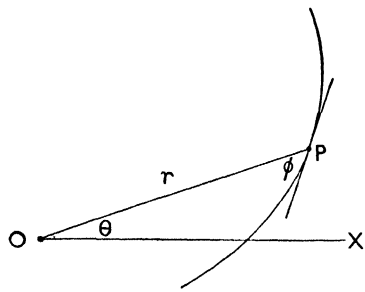

Fig. 83

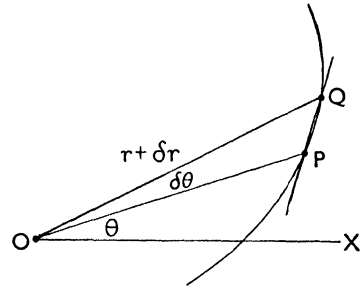

Fig. 84

(measured as in Fig. 83) is given by

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} \quad . \quad . \quad . \quad . \quad (5)$$

Proof. Let P be the point of contact and Q a nearby point of the curve, as in Fig. 84, so that $OP = r$, $OQ = r+\delta r$, $\angle XOP = \theta$, $\angle XOQ = \theta+\delta\theta$. Then in the triangle OPQ, by a well-known trigonometric formula,

$$\tan\tfrac{1}{2}(P-Q) = \frac{OQ-OP}{OQ+OP}\cot\tfrac{1}{2}POQ$$

$$= \frac{\delta r}{(2r+\delta r)\tan\frac{1}{2}\delta\theta}.$$

Now let $Q\to P$ along the curve. The limit of the L.H.S. is

$\tan\frac{1}{2}\{(\pi-\phi)-\phi\}$, reducing to $\cot\phi$, while that of the R.H.S. is $\frac{1}{r}\frac{dr}{d\theta}$.

If the curve passes through the pole, it is easy to find the equation of the tangent there. For let the curve pass through O when $\theta = \theta_0$, and let $P(r_1, \theta_1)$ be any other point of the curve. The equation of the chord OP is $\theta = \theta_1$; as $P \to O$ it tends to the limiting form $\theta = \theta_0$ and represents the tangent at O. Thus **if $r \to 0$ as $\theta \to \theta_0$, the line $\theta = \theta_0$ is a tangent to the curve at O.**

In Example 1, $r \to 0$ as $\theta \to 0$, and the initial line is the tangent at O, as in Fig. 82.

If P is a point on a curve, O the pole, and if the tangent at P and the line through O at right angles to OP meet at T, then OT is called a **polar subtangent** to the curve (Fig. 85). A **polar subnormal** (OG in Fig. 85) is similarly obtained from the normal.

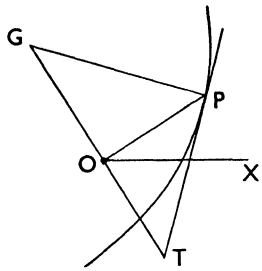

Fig. 85

Asymptotes

The asymptotes of a curve are bound up with the values of θ for which r becomes infinite. If $r \to \infty$ as $\theta \to \alpha$, there may be an asymptote parallel to the line $\theta = \alpha$, as in Fig. 86. Let this line be OA, and $P(r, \theta)$ any point of the curve, then $\angle POA = \alpha - \theta$, and the distance of P from OA is $r \sin(\alpha - \theta)$. If, as $\theta \to \alpha$, this expression tends to a limit h, a line drawn parallel to $\theta = \alpha$, at a distance h from it on the appropriate side, will be an asymptote because the distance of the curve from it tends to zero. The equation of the line is $r \sin(\alpha - \theta) = h$, since every point of it is at a distance h from OA. Thus **if, as $\theta \to \alpha$, $r \to \infty$ but $r \sin(\alpha - \theta)$ tends to a limit h, the line $r \sin(\alpha - \theta) = h$ is an asymptote.**

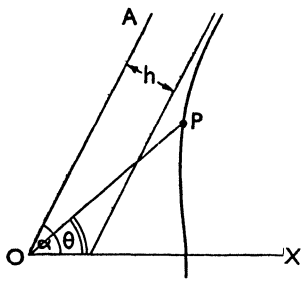

Fig. 86

It may happen, however, that $r \sin(\alpha - \theta)$ does not tend to a finite limit, and in that case there is no asymptote parallel to $\theta = \alpha$.

Example 2. Trace the curve $r = a \sec^2\theta$.

As $\sec^2\theta = \sec^2(-\theta) = \sec^2(\pi - \theta)$, the curve is symmetrical about

the lines $\theta = 0$ and $\theta = \pi/2$. When $\theta = 0$, $r = a$; as θ increases to $\pi/2$, r increases without limit; there may therefore be an asymptote parallel to $\theta = \pi/2$. Now the equation of the curve can be written $r \sin(\pi/2 - \theta) = a \sec\theta$, and the R.H.S. tends to infinity as $\theta \to \pi/2$; hence there is no asymptote and the curve recedes infinitely far from $\theta = \pi/2$.

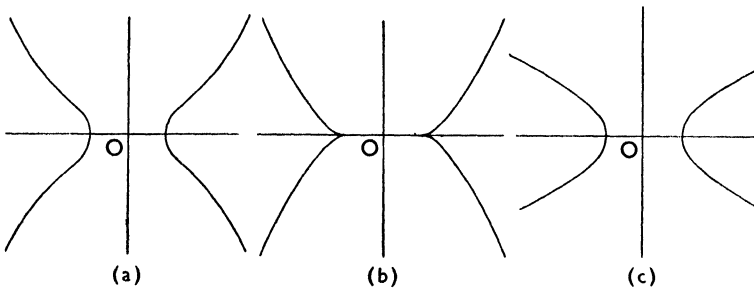

Fig. 87

It follows that the general form of the curve is as in Fig. 87 (a) or (b); it cannot be like (c) because there the curve has no points for which θ is in the neighbourhood of $\pm\frac{\pi}{2}$. To decide between (a) and (b) it will be sufficient to find the value of ϕ when $\theta = 0$. We have $\cot\phi = \frac{1}{r}.2\sec\theta\tan\theta = 2\sin\theta$, so that $\phi = \pi/2$. The curve is therefore as in Fig. 87 (a).

Example 3. Find the asymptotes of the curve $\frac{l}{r} = 1 + e\cos\theta$ when $e > 1$.

We already know from 11.4, Example 3, that the curve is a hyperbola, and it is clear that $r \to \infty$ as θ approaches an obtuse angle α for which $\cos\alpha = -1/e$ and therefore $\sin\alpha = \sqrt{(e^2-1)}/e$.

Now $r\sin(\alpha-\theta) = \dfrac{l\sin(\alpha-\theta)}{1+e\cos\theta}$, which takes the meaningless form $\frac{0}{0}$ if we put $\theta = \alpha$. There is a theorem in Calculus to deal with this situation: the limit as $\theta \to \alpha$ is unaltered if we differentiate the top and bottom of the fraction. This gives $l\cos(\alpha-\theta)/e\sin\theta$, which tends to $l/e\sin\alpha$ as $\theta \to \alpha$. Hence the asymptote is $r\sin(\alpha-\theta) = l/\sqrt{(e^2-1)}$. The other asymptote is its image in the initial line, namely $r\sin(\alpha+\theta) = l/\sqrt{(e^2-1)}$.

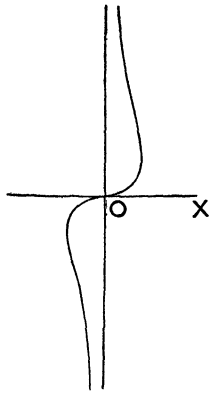

Fig. 88

Example 4. Trace the curve $r^2 = a^2 \tan\theta$.

When $\theta = 0$, $r = 0$; as θ increases to $\pi/2$, r increases numerically without limit and can be either positive or negative; between $\theta = \pi/2$ and $\theta = \pi$ no value of r is possible; and the sequence repeats when $\theta = \pi$. The curve therefore passes through the pole, has symmetry about it, and may have an asymptote parallel to $\theta = \pi/2$. The tangent at O is $\theta = 0$ (since $r \to 0$ as $\theta \to 0$), and to locate the possible asymptote we have to find the limit of $r\cos\theta$ as $\theta \to \pi/2$. Now $r\cos\theta = \pm a\sqrt{(\sin\theta\cos\theta)}$, and this tends to zero as $\theta \to \pi/2$; hence the line $\theta = \pi/2$ is itself the asymptote. The curve is therefore as shown in Fig. 88.

Exercise 11c

State the polar coordinates of the points with the given cartesian coordinates. In each case arrange that r is positive and that $-\pi < \theta \leqslant \pi$.

1. $(4, 0)$, $(1, -\sqrt{3})$, $(-2\sqrt{2}, 2\sqrt{2})$, $(-3, 0)$.
2. $(2, 2)$, $(0, -2)$, $(-\sqrt{3}, 3)$, $(0, 0)$.

Rewrite the polar coordinates of the following points so that $r > 0$ and $-\pi < \theta \leqslant \pi$.

3. $(2, 5\pi/4)$, $(-1, -3\pi/2)$, $(-1, 0)$.
4. $(3, -7\pi/3)$, $(-2, 5\pi/6)$, $(-5, 5\pi)$.

Find the polar equation of the curve with the given cartesian equation.

5. $lx+my+n = 0$.
6. $y^2 = 4a(x+a)$.
7. $(x^2+y^2)^2 = a^2(x^2-y^2)$.
8. $x(x^2+y^2) = ay^2$.

Find the cartesian equation of the curve with the given polar equation.

9. $r\sin(\theta-\alpha) = a$.
10. $r = a\sec\theta$.
11. $r^2\sin 2\theta = a^2$.
12. $r^2 = a^2\sec 2\theta$.
13. $l/r = 1+e\cos\theta$.
14. $r = a\tan\theta$.

In Nos. 15–26, trace the curve, finding the polar equations of the tangents at O, the cartesian equations of the asymptotes, if any, and noting any symmetry.

15. $r = a(1+\cos\theta)$.
16. $r = a(1+\sqrt{2}\cos\theta)$.
17. $r\sin 2\theta = 2a$.
18. $r = a\tan\theta$.
19. $r\theta = a$.
20. $r^2\theta = a^2$.
21. $r = a\cos^2\dfrac{3\theta}{2}$.
22. $r = a\cos 3\theta$.
23. $r\cos\theta + a\cos 2\theta = 0$.
24. $r^2 = a^2\cot 2\theta$.
25. $(x^2+y^2)^2 = 2axy^2$.
26. $x^2(x^2+y^2) = 4(x^2+y^2-ay)^2$.
27. A cam consists of the portion of the Archimedean spiral $r = a\theta+b$ from $\theta = 2n\pi$ to $\theta = (2n+1)\pi$, and its image in the initial line. Show that all chords of the cam which pass through O are of equal length.
28. Show that all chords of the *cardioid* $r = a(1+\cos\theta)$ which pass through O are of equal length.

29. Write down the square of the distance between the points whose polar coordinates are $(1, \alpha)$, $(1, \beta)$: (i) by using the cosine formula, (ii) by writing down their cartesian coordinates and using the formula of 3.1. Hence, find an expression for $\cos(\alpha-\beta)$.

30. Show that (i) $OP_1 . OP_2 \cos P_1OP_2 = x_1x_2+y_1y_2$;

(ii) $OP_1 . OP_2 \sin P_1OP_2 = \pm(x_1y_2-x_2y_1)$.

31. Show that the curve $r\theta = a$ has a constant polar subtangent. What does this tell you about its asymptote?

32. Show that the Archimedean spiral $r = a\theta$ has a constant polar subnormal. What can be deduced for large values of θ?

33. Show that the equation $r = ae^{\theta \cot \alpha}$, where e is the base of natural logarithms, represents a spiral for which $\phi = \alpha$. It is known as an **equiangular spiral.**

34. Show that the polar subtangent and subnormal of an equiangular spiral are both proportional to r.

35. Show that the polar subtangent to Archimedes' spiral $r = a\theta$ is equal to $r\theta$. If P is the nearest point to O at which the spiral meets the positive part of the initial line, deduce that OT, the polar subtangent corresponding to P, is equal to the circumference of a circle of radius OP.

This result, which does not entail the numerical measurement of either a or θ, was Archimedes' contribution towards the squaring of the circle.

36. Show that, for the *lituus* $r^2\theta = a^2$, $\triangle POT$ has constant area.

37. A variable chord PQ of the conic $l/r = 1+e\cos\theta$ passes through O; prove that $1/OP+1/OQ$ is constant. Draw a diagram to show what this means in the case of a hyperbola and of an ellipse.

38. POP' is a focal chord of the parabola $2a/r = 1-\cos\theta$; show that the locus of the mid-point of PP' is $2a/r = \sec\theta-\cos\theta$, and find its cartesian equation.

39. Two focal chords of a rectangular hyperbola are at right angles; prove that their lengths are equal.

40. PP', QQ' are variable perpendicular focal chords of an ellipse; prove that $1/PP'+1/QQ'$ is constant.

41. An ellipse has semi-axes a and b, and O its *centre*. If P, Q are two points on the curve such that PQ subtends a right angle at O, prove that

$$1/OP^2+1/OQ^2 = 1/a^2+1/b^2.$$

42. Two perpendicular lines through any point O cut a conic at P_1 and P_2, Q_1 and Q_2 respectively. Prove that

$$\left(\frac{1}{OP_1}-\frac{1}{OP_2}\right)^2+\left(\frac{1}{OQ_1}-\frac{1}{OQ_2}\right)^2$$

is constant for all positions of the lines, due regard being had to sign.

43. Find in its simplest form the equation of the tangent at the point t on the curve $x = a\cos^3 t$, $y = a\sin^3 t$.

Show that perpendicular tangents meet on the curve whose polar equation is $2r^2 = a^2\cos^2 2\theta$.

Sketch both curves.

44. A, B are the points with polar coordinates $(c, 0)$, (c, π); find the polar equation of the locus of P if $PA . PB = c^2$, and sketch the curve.

45. Prove that the equation of the tangent to the curve $1/r = f(\theta)$ at the point where $\theta = \alpha$ is $1/r = p\cos\theta + q\sin\theta$, where $p = f(\alpha)\cos\alpha-f'(\alpha)\sin\alpha$, $q = f(\alpha)\sin\alpha+f'(\alpha)\cos\alpha$, and the dashes denote differentiation wo θ.

46. Show that the polar equation of a conic referred to a focus as pole may be put in the form $l/r = 1+e\cos\theta$, and find the equation of the tangent at the point $\theta = \alpha$.

A conic has foci S and H, and the tangents from a general point R touch the conic at P and Q. Prove that RS and RH bisect the angles PSQ and PHQ respectively. [C.S.]

47. Trace the curve $r = a(\cos\theta+\cos 2\theta)$, showing that it crosses itself at the points $(a/\sqrt{2}, \pm\pi/4)$.

48. Sketch the curve $r = a(\sin 2\theta+\sin\theta)$, noting any symmetry.

49. Trace the curve $r^3\sin 4\theta = \sin(\theta+\alpha)$, (a) when $0<\alpha<\frac{1}{4}\pi$ and (b) when $\alpha = 0$. Show how the shape of the curve passes over into its limiting form as α tends to zero. [C.S.]

50. Sketch the curve $r(1-2\cos\theta) = 3a\cos 2\theta$, and find the equations of its asymptotes. [C.S.]

11.6. Bipolar Coordinates

In these, the position of a point is specified by means of its distances r_1, r_2 from two fixed points A_1, A_2, the poles. The equations $r_1 = a$, $r_2 = b$ then represent circles, and from the bifocal property of conics it follows that the equations $r_1+r_2 = c$ and $r_1-r_2 = d$ represent an ellipse and one branch of a hyperbola. Even so slight a generalization as $ar_1+br_2 = c$, however, produces a curve (the cartesian oval: see 12.3, Nos. 42–48) whose properties are much more difficult to investigate.

Notice that in bipolar coordinates each pair of coordinates represents *two* points. They are therefore only suited to the treatment of curves which are symmetrical about the join of the poles.

Exercise 11D

In this exercise, $A_1A_2 = 2c$, and O is the mid-point of A_1A_2. What do the following equations represent?

1. $r_1 = r_2$.
2. $r_1+r_2 = 2c$.
3. $r_2-r_2 = 2c$.
4. $r_1^2+r_2^2 = 4c^2$.

5. Plot **Cassini's Ovals**, $r_1r_2 = a^2$, when a^2 has the value (i) $24c^2/25$, (ii) c^2, (iii) $5c^2/4$.

Cassini's ovals are a section of the equipotential surfaces formed when two long thin parallel wires of equal radius are given equal electrical charges of like sign.

6. Show that $r_1 = \lambda r_2$ represents a system of coaxal circles having A_1, A_2 as limiting points. (These replace the curves of No. 5 if the charges are of opposite sign.)

7. Show that $r_1^2+r_2^2 = a^2$ represents a circle, centre O, and find its radius.

8. Show that $r_1^2-r_2^2 = a^2$ represents a line perpendicular to A_1A_2.

9. Use Hero's formula for the area of a triangle to show that the equation of the pair of lines parallel to A_1A_2 and distant a from it is

$$(r_1^2-r_2^2)^2-8c^2(r_1^2+r_2^2)+16c^2(a^2+c^2) = 0.$$

10. Show that the pair of lines through O, making equal angles α with A_1A_2, have an equation of the form $(r_1^2-r_2^2)^2 = 8a^2(r_1^2+r_2^2-2c^2)$, and find α in terms of a and c.

11.7. Coordinates in Three Dimensions

To apply rectangular cartesian coordinates to three-dimensional figures, an axis Oz is drawn at right angles to both Ox and Oy, and is graduated on the same scale as the other two.

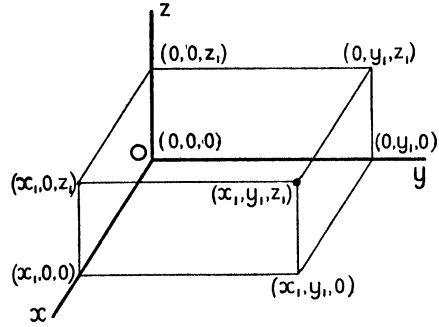

Fig. 89

The equations $x = 0$, $y = 0$, $z = 0$ represent the **coordinate planes** yOz, zOx, xOy; $x = x_1$, $y = y_1$, $z = z_1$ represent planes parallel to these, the point in which they meet being (x_1, y_1, z_1). The six planes together form a rectangular parallelepiped, as in Fig. 89, where the coordinates of all the corners are shown.

The six planes $x = x_1$, $y = y_1$, $z = z_1$, $x = x_2$, $y = y_2$, $z = z_2$ form a rectangular parallelepiped with edges $|x_1-x_2|$, $|y_1-y_2|$, $|z_1-z_2|$, having P_1 and P_2 at opposite ends of a diagonal. Hence

$$P_1P_2^2 = (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2 \quad . \quad . \quad (6)$$

The two lines P_0P_1, P_0P_2 are perpendicular if $P_0P_1^2+P_0P_2^2 = P_1P_2^2$. Using (6) and simplifying the result, we can say that **P_0P_1 is perpendicular to P_0P_2 if**

$$(x_0-x_1)(x_0-x_2)+(y_0-y_1)(y_0-y_2)+(z_0-z_1)(z_0-z_2) = 0 \quad . \quad (7)$$

By considering orthogonal projections on the coordinate planes it can be seen that the coordinates of the **point dividing P_1P_2 in the ratio $\lambda : 1$** are given by

$$x = \frac{x_1+\lambda x_2}{1+\lambda}, \quad y = \frac{y_1+\lambda y_2}{1+\lambda}, \quad z = \frac{z_1+\lambda z_2}{1+\lambda} \quad . \quad . \quad (8)$$

and that these can serve as freedom equations for the line P_1P_2.

From (6), the equation of the **sphere** with centre O and radius a is

$$x^2+y^2+z^2 = a^2 \quad . \quad . \quad . \quad . \quad (9)$$

The **tangent plane** at the point P_1 is

$$x_1x+y_1y+z_1z = a^2 \quad . \quad . \quad . \quad (10)$$

For if P is any point on the tangent plane, P_1P is perpendicular to OP; hence $x_1(x-x_1)+y_1(y-y_1)+z_1(z-z_1) = 0$, by (7). The equation of the plane is therefore

$$x_1x+y_1y+z_1z = x_1^2+y_1^2+z_1^2 \quad . \quad . \quad . \quad (11)$$

and the R.H.S. is equal to a^2 since P_1 lies on the sphere.

Any first-degree equation represents a plane. For let the equation be $lx+my+nz = p$; (11) reduces to this if

$$\frac{x_1}{l} = \frac{y_1}{m} = \frac{z_1}{n} = \frac{p}{l^2+m^2+n^2}.$$

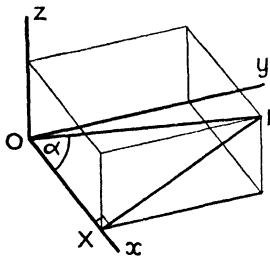

Fig. 90

The **direction of a line** OP is conveniently measured by the angles α, β, γ which it makes with Ox, Oy, Oz. From the right-angled triangle OXP in Fig. 90,

$$\cos\alpha = x/OP,$$

and similarly

$$\cos\beta = y/OP, \cos\gamma = z/OP,$$

so that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1.$$

The cosines of the three angles are called the **direction cosines** of OP.

If the direction cosines of a line AB are λ, μ, ν, and those of another line AC are λ', μ', ν', then

$$\cos BAC = \lambda\lambda'+\mu\mu'+\nu\nu' \quad . \quad . \quad . \quad (12)$$

It is sufficient to prove (12) for the case in which A is the origin. For the method see Exercise 11e, No. 7.

The **angle between two planes** is equal to the angle between normals to them. A normal to (10) has direction cosines proportional to x_1, y_1, z_1; hence the direction cosines of a normal to $lx+my+nz = p$ are l/r, m/r, n/r, where $r^2 = l^2+m^2+n^2$. Thus (12) can be used to find the angle between two planes whose equations are given.

Just as (9) represents a sphere, so any equation relating x, y, z represents a surface, and the equation of a tangent plane to a surface of the second degree is obtained by an obvious extension of the rules used in two dimensions.

Of particular interest is a **surface of revolution.** If the axis of revolution is Oz, any section perpendicular to this (obtained by keeping z constant) is a circle with its centre on Oz. Since the square of the radius is measured by x^2+y^2, the value of this expression must depend on z only, and the equation of the surface can be written $x^2+y^2 = f(z)$.

In particular, the equation

$$x^2+y^2-k^2z^2 = 0 \quad . \quad . \quad . \quad . \quad (13)$$

represents a **circular cone,** since it has the required form and reduces to $x = \pm kz$ when $y = 0$. Any equation in which every term is of the same degree represents some kind of cone, since if (x, y, z) satisfies the equation, so does (tx, ty, tz), and the surface is therefore generated by lines through O.

Two surfaces usually meet in a **curve,** and a line or curve has to be represented either by means of the equations of two surfaces, or by means of freedom equations. Thus $x^2+y^2 = a^2$ represents a cylinder, not a circle; a circle could be represented by the two equations $x^2+y^2 = a^2$, $z = c$, or by the freedom equations $x = a\cos\theta$, $y = a\sin\theta$, $z = c$. The x-axis could be represented by the equations $y = 0$, $z = 0$, or by the freedom equations $x = t$, $y = 0$, $z = 0$.

Exercise 11E

1. Prove that the lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent at a point three-quarters of the way from each vertex to the opposite face. This point is called the **centroid** of the tetrahedron.

2. Prove that the lines joining mid-points of opposite edges of a tetrahedron pass through the centroid and are bisected there.

3. If the opposite edges of a tetrahedron are equal, prove that the joins of mid-points of opposite edges are perpendicular to those edges.

4. In No. 3, show that the sum of squares of edges is equal to four times the sum of squares of joins of mid-points of opposite edges.

5. If two pairs of opposite edges of a tetrahedron are perpendicular, prove that the third pair are also perpendicular.

6. In No. 5, show that the sum of squares of each pair of opposite edges is the same.

7. B, C are the points (l, m, n), (l', m', n'); $OB = r$, $OC = r'$. Write down expressions for (i) r, r' in terms of the other letters; (ii) BC^2, using (6); (iii) $\cos BOC$, by applying the cosine formula to $\triangle BOC$.

What are the direction cosines of OB and OC? Hence obtain (12).

8. Show that (7) is a special case of (12).

What are represented by the equations in Nos. 9–12?

9. $y^2+z^2 = a^2$. 10. $y^2+z^2 = 2ay$.

11. $y^2+z^2 = 2yz$. 12. $y^2+z^2 = x^2$.

13. Prove that the sphere on P_1P_2 as diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2) = 0.$$

14. What is represented by the equation

$$(x-a)^2+(y-b)^2+(z-c)^2 = d^2?$$

15. By considering a sphere with centre O, and a plane parallel to a coordinate plane, prove that a plane section of a sphere is a circle.

16. Prove that two spheres meet in a plane curve.

17. Prove that the equation of the right bisector plane of P_1P_2 is
$$2(x_1-x_2)x+2(y_1-y_2)y+2(z_1-z_2)z = x_1^2+y_1^2+z_1^2-(x_2^2+y_2^2+z_2^2).$$

18. Show that the plane $lx+my = p$ is parallel to Oz.

19. Show that the line $x = 2-3t$, $y = 3+2t$, $z = 1+t$ is parallel to the plane $5x+6y+3z = 11$. What is the equation of the parallel plane in which it lies?

20. Find the equation of the plane through the lines $x = 1+2t$, $y = 3-t$, $z = 2+t$ and $x = 1-3t$, $y = 3-2t$, $z = 2+3t$.

21. Show that the planes $3x+y+5z = 0$ and $7x+4y-5z = 0$ are perpendicular, and find the equation of the plane through O perpendicular to both.

22. Find the ratio of the direction cosines of the line of intersection of the two planes given in No. 21.

23. Prove that the distance of P_1 from the plane $lx+my+nz = p$ is $(lx_1+my_1+nz_1-p)/\sqrt{(l^2+m^2+n^2)}$.

24. What is represented by the equation
$$18(3x+y+5z)^2+7(7x+4y-5z)^2 = 630?$$

25. Prove that lines with direction cosines in the ratios $l : m : n$, $m : n : l$, $n : l : m$ are mutually perpendicular if a plane perpendicular to one of them meets the axes in points the sum of whose coordinates is zero.

26. A plane passes through the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$; find the ratio of the direction cosines of the perpendicular to it from O. Find also the length of this perpendicular.

27. Find the equation of the surface obtained by rotating the hyperbola $x^2/a^2-y^2/b^2 = 1$ about Ox. Show that the plane $x = c$ meets it in a circle if $c^2>a^2$, but not at all if $c^2<a^2$. The surface is thus in two parts and is called a **hyperboloid of two sheets.**

Show that any plane parallel to $z = 0$ meets the surface in a hyperbola, and that all such hyperbolas are similar.

28. Find the equation of the surface obtained by rotating the hyperbola $x^2/a^2-y^2/b^2 = 1$ about Oy, and show that every plane perpendicular to Oy meets it in a circle and every plane parallel to Oy meets it in a hyperbola. It is called a **hyperboloid of one sheet.**

29. Find the equation of the **paraboloid of revolution** obtained by rotating the parabola $y^2 = 4az$ about Oz; show that any plane parallel to $x = 0$ cuts it in a parabola, and that all such parabolas have equal latera recta.

30. Show that the **hyperbolic paraboloid** $x^2-y^2+4az = 0$ is met by any plane parallel to $x = 0$ or $y = 0$ in a parabola, and by any plane parallel to $z = 0$ in a hyperbola. Its general appearance is thus that of a pass between two mountains.

31. Show that the points P, P' given by $x = \pm 2at$, $y = 2a(t+u)$, $z = au(2t+u)$ move along straight lines as t varies. Deduce that an infinite number of pairs of lines lie entirely on the surface of the hyperbolic paraboloid $x^2-y^2+4az = 0$, which is therefore a **ruled surface.**

32. P_1, P_2 are the points given by $x_1 = a\cos(\theta+\alpha)$, $y_1 = c$, $z_1 = a\sin(\theta+\alpha)$, $x_2 = a\cos(\theta-\alpha)$, $y_2 = -c$, $z_2 = a\sin(\theta-\alpha)$. What are the loci of P_1 and P_2 as θ varies?

By writing down the coordinates (x, y, z) of any point on P_1P_2, show that
$$cx = a(c\cos\alpha\cos\theta - y\sin\alpha\sin\theta),$$
$$cz = a(y\sin\alpha\cos\theta + c\cos\alpha\sin\theta),$$
and deduce that, as θ varies, the line P_1P_2 sweeps out the one-sheeted hyperboloid
$$c^2(x^2+z^2) = a^2(y^2\sin^2\alpha+c^2\cos^2\alpha),$$
which is therefore also a ruled surface.

33. A variable point of a curve is given by the freedom equations $x : y : z : 1 = q_1(t) : q_2(t) : q_3(t) : q_4(t)$, where $q_1(t) = a_1t^2+b_1t+c_1$, etc. Show that a plane can be found in which the point lies for all values of t. The locus is thus a plane curve, and is in fact a conic.

34. A curve is given by the freedom equations $x = at$, $y = bt^2$, $z = ct^3$. Show that a plane meets the curve in not more than three points, and hence that no four points of the curve are coplanar. Deduce that no three points are collinear.

The curve is known as a **twisted cubic.** Verify that it and the line $x = y = 0$ are the intersections of the hyperbolic cone $b^2xz = acy^2$ and the parabolic cylinder $a^2y = bx^2$.

CHAPTER 12

SOME INTERESTING CURVES

12.1. Introduction

We have seen that curves of the second degree can all be regarded as sections of a cone; they can be classified as ellipses, parabolas, and hyperbolas, with circles and rectangular hyperbolas as special cases, all these different types of curve being readily distinguishable from one another and all having many simple properties; and it might be expected that the curves of any higher degree would also have some common source and could be classified into clearly defined species each having a number of characteristic geometrical properties. Unfortunately, this is not so. The number of species into which, for example, cubic curves have been divided by different writers ranges from 5 to 219, and although geometrical properties exist, they are as a rule not sufficiently elementary to be included here. Most of the curves mentioned in this chapter, therefore, should be regarded as isolated specimens rather than as illustrations of a type: some are here because they happen to have one or two simple properties, others because they are historically famous.

Some of the earliest studies of curves were made in an attempt to solve two problems which engaged the attention of Greek geometers for several centuries. The first was the problem of trisecting any given angle. This cannot be done with ruler and compasses alone; expressed algebraically, it requires the solution of an equation of the third degree, and several cubic curves effect the trisection quite easily. Apparently, however, the first practical method (about 180 B.C.) used a fourth-degree curve, Nicomedes' Conchoid, which can be described mechanically. Much later, at the end of the third century A.D., Pappus of Alexandria solved the problem by means of conics (see Exercise 5M, No. 36).

The other problem was the duplication of the cube. It is said that in the time of Plato the Delians were told by their oracle that, to get rid of a plague from which they were suffering, they must construct a cubical altar double the size of the existing one. This necessitated

finding a geometrical construction for the cube root of 2. The problem was generalized to that of finding two mean proportionals between two given quantities, i.e. if a and b are two given lengths, the problem was to find lengths x, y, such that $\frac{a}{x} = \frac{x}{y} = \frac{y}{b}$, for then $\left(\frac{a}{x}\right)^3 = \frac{a}{b}$. Menæchmus soon solved it by means of the intersection of two conics: in our modern notation, any two of the equations $x^2 = ay$, $y^2 = bx$, $xy = ab$ lead to the required result. But it continued to attract attention: Nicomedes showed that it could be solved by his conchoid (of which he was inordinately proud), and later, about 100 B.C., Diocles gave a simpler solution using his cissoid, a cubic curve.

In doing the examples of this chapter, you should bear in mind that the methods of pure geometry, by which some of the curves were originally investigated, will often shorten the working. For example, many of the curves are defined geometrically and it may be that other facts besides the equation can be deduced from the definition.

12.2. Cubic Curves

The Curves $x^3 = a^2y$, $x^3 = ay^2$, $x^2y = a^3$

Suitable freedom equations are respectively

$$x : y : a = t : t^3 : 1,$$
$$x : y : a = t^2 : t^3 : 1,$$
$$x : y : a = t^3 : 1 : t^2.$$

1. Show that the points t_1, t_2, t_3 on the first curve are collinear if $t_1+t_2+t_3 = 0$, and that points with the same parameters on the second or third are collinear if $1/t_1+1/t_2+1/t_3 = 0$. [Use the method of 9.4.]

If the tangent at P (the point with parameter t) meets the curve again at Q, show that the parameter of Q is $-2t$, $-t/2$, $-t/2$ in the three cases respectively.

If PQ meets the x-axis at A, the y-axis at B, show that for each curve $PA : AB : BQ$ is constant.

2. Two lines meet one of the curves at ABC, DEF; AD, BE, CF meet the curves again at L, M, N respectively; prove that L, M, N are collinear.

By making the two original lines approach coincidence with each other, deduce a property of the tangentials of three collinear points.

3. With the data of No. 1, if K is a point dividing PQ in a fixed ratio (say $k : 1$), show that as P moves round the curve the locus of K is similar curve.

4. If PR is a variable chord of the third curve subtending a right angle at the origin, prove that the locus of the meet of the tangents at P and R is an ellipse

touching the x-axis at O and touching the curve at the points where $t^2 = \frac{1}{2}$. [The elimination is made easier by putting $t - \frac{1}{t} = u$.]

5. If PR is a variable chord of the curve $x^3 = ay^2$ subtending a right angle at the cusp, prove that the locus of the meet of the tangents at P and R is a parabola touching the curve at the points where $t^2 = \frac{1}{2}$. [Put $t - 1/t = u$.]

6. Show that the normal to the parabola $y^2 = 4a(x+2a)$ at the point $(at^2-2a, 2at)$ is the tangent to the semi-cubic parabola $4x^3 = 27ay^2$ at the point $(3at^2, -2at^3)$. The semi-cubic parabola is thus the evolute of the parabola.

Diocles' Cissoid

A tangent is drawn at A to a circle on OA as diameter; a variable line through O cuts the circle at R and the tangent at S; P is the point on OS such that $OP = RS$. Then the locus of P is the cissoid (= ivy-leaf-shaped), perhaps so called because the edge of an ivy leaf has a clearly marked cusp where it joins the stem, and the cissoid has a similar cusp at the origin.

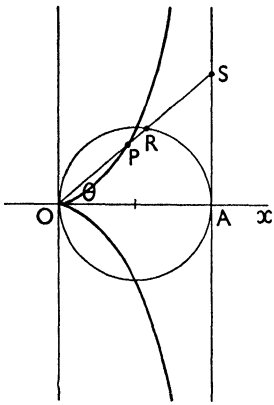

Fig. 91

If we let O and A be the origin and $(a, 0)$ and let OS make an angle θ with OA, then $OS = a \sec\theta$, $OR = a\cos\theta$, and hence the polar equation of the cissoid is

$$r = a(\sec\theta - \cos\theta), \text{ i.e.}$$

$$\boldsymbol{r = a\sin\theta\tan\theta.}$$

From this, $x = a\sin^2\theta$, $y = a\sin^2\theta\tan\theta$, or, putting $t = \tan\theta$,

$$\boldsymbol{x : y : a = t^2 : t^3 : 1+t^2.}$$

Hence $a - x = \dfrac{a}{1+t^2}$, and $x^3 = y^2(a-x)$ is the constraint equation of the curve. The line $x = a$ is an asymptote.

7. **Solution of the Delian Problem.** Let B be $(0, b)$, and let OS move round until P lies on AB. Prove that then $(y/x)^3 = b/a$. Hence $(AS/OA)^3 = b/a$ and the problem is solved.

8. Find the equation of the tangent at P in terms of t, the parameter of P; find the parameter of the point U in which this tangent meets the curve again.

Hence prove that

(i) if OU meets the asymptote at V, $SA = 2AV$;

(ii) if the tangent at P meets the asymptote at T, $AS = 2ST$.

9. If PQ is a variable chord of the cissoid, subtending a right angle at the cusp, find the parameter of Q in terms of that of P and prove that

(i) $\angle PAQ$ is also a right angle;

(ii) PL and RQ are both parallel to the asymptote, L being the point in which OQ cuts the circle;

(iii) if the normal at P and the perpendicular at S to the asymptote meet QO produced at M, N, then $LO = MN$;

(iv) the locus of the mid-point of PQ is a line parallel to the asymptote;

(v) the locus of the point of intersection of the normals at P and Q is a line parallel to the asymptote;

(vi) the locus of the points of intersection of the tangents at P and Q is a circle on AC as diameter, C being $\left(\frac{a}{4}, 0\right)$.

[Find the coordinates of the point in terms of t, the parameter of P; simplify the expressions by putting $u = t - 1/t$; obtain u^2 in terms of x, then eliminate u^2 by substituting in the expression for y^2.]

10. **Newton's Construction.** A rod LMN, right-angled at M and having $MN = a$, moves so that LM passes through $(-a/2, 0)$ and N lies on $x = a/2$; show that the mid-point of MN always lies on the cissoid.

See also 12.5, Nos. 79, 91.

The Strophoid

If a variable circle of fixed radius a moves so that its centre remains on a fixed line, the locus of the point of contact of a tangent to it from a fixed point A distant a from the line is called a logocyclic curve or strophoid (Greek *strepho*, I turn). The curve was investigated by G. P. de Roberval (1602–75).

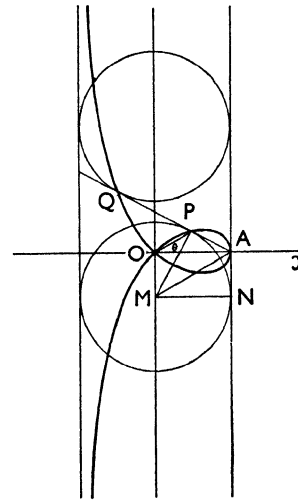

Fig. 92

To find its equation, let the line be the y-axis and A $(a, 0)$. Let the centre of the circle be at M, and let P be the point of contact of the tangent from A. Complete the rectangle $AOMN$. Then since the tangents to a circle are equal, $AP = AN$. Also

$$\angle APM = \angle ANM = 90°.$$

$\therefore$ $\triangle APM \equiv \triangle ANM$;

$\therefore$ $\triangle APM \equiv \triangle MOA$ (the other half of the rectangle);

$\therefore$ $OP // MA$.

Now let $\angle POA = \theta$. Then $\angle PMA = \angle AMN = \theta$.

$\therefore$ $x = MP \cos PMN = a \cos 2\theta$,

and $y = x \tan \theta = a \cos 2\theta \tan \theta$.

Putting $t = \tan \theta$ we obtain alternative freedom equations

$$x : y : a = 1 - t^2 : t(1 - t^2) : 1 + t^2.$$

The curve, shown in Fig. 92, has a **double-point** or **node** at O (i.e.

a point at which it crosses itself), given by $t = \pm 1$, and has $x+a = 0$ as its only asymptote.

11. Show that the polar equation is $r = a(2\cos\theta - \sec\theta)$ and the cartesian constraint equation $x(x^2+y^2) = a(x^2-y^2)$.
Find the equation of the tangent at t.

12. Find the gradient of AP; if it cuts the curve again at Q, show that

(i) $\angle POQ = 90°$;

(ii) the mid-point Y of PQ lies on the y-axis, and $YP = YQ = YO$;

(iii) $AP.AQ = a^2$;

(iv) if M, L are the centres of the circles touching APQ at P and Q, then LM subtends a right angle at A;

(v) the tangents at P and Q make equal angles with APQ; if this angle is ϕ, then $\tan\phi = \cos PAO$;

(vi) if these tangents meet the asymptote at U, V, then the mid-point of UV lies on APQ produced;

(vii) if these tangents intersect at T, then OT is parallel to APQ;

(viii) the locus of T is the cissoid $x(x^2+y^2)+ay^2 = 0$;

(ix) if the normal at P and the line through A perpendicular to AP meet at H, the locus of H is a parabola having A as focus and $x = 0$ as directrix.

[Parts (vii) and (viii) can be awkward unless carefully handled. For (vii), find the equation of the line joining O to the point of intersection as in 5.4. For (viii), use (ii) and (v) to show that the point lies on the perpendicular bisector of PQ; find the equation of this bisector; in it and the equation of the line found in (vii) put $u = \dfrac{t^2-1}{2t}$; finally eliminate u between these two equations.]

13. A and W are two points on a circle, centre O, A being fixed and W variable. The bisector of $\angle WOA$ and the perpendicular from W to OA meet at P. Show that the locus of P is a strophoid.

See also 12.4, No. 56; 12.5, Nos. 79, 86, 87, 92, 93.

Maclaurin's Trisectrix

A is the point $(-4a, 0)$; a variable line through O meets the circle on OA as diameter at R and the line $x+a = 0$ at S. If a point P is taken on the variable line so that $RS = OP$, sense being taken into account, the locus of P is known as the Trisectrix of Colin Maclaurin (1698–1746).

14. Show that the equation of the trisectrix is $x(x^2+y^2) = a(3x^2-y^2)$, that its appearance is similar to that of the strophoid but with a different size and shape of loop, that the gradient at the origin is $\pm\sqrt{3}$, and that $x+a = 0$ is its asymptote.

15. A point P moves so that the angles made with the positive direction of the x-axis by OP and QP, where Q is $(2a, 0)$, are in the ratio $1 : 3$. Show that the polar equation of the locus of P is $r = a(4\cos\theta - \sec\theta)$ and that its cartesian equation is as given above. The curve can therefore be used to trisect angles; hence its name.

[Most of the Greek attempts to trisect an angle depended on Fig. 93, which explains itself and from which the required polar equation is easily obtained.]

See also 12.5, Nos. 79, 86.

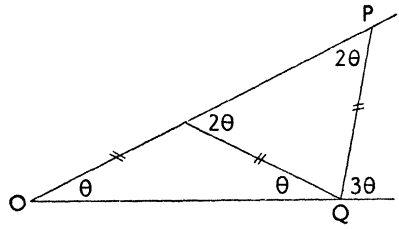

Fig. 93

Agnesi's Witch

Let UV be a line parallel to the tangent to a circle at a given point O on it; let a variable line through O meet UV at L and the circle at M; let the lines through L, M, respectively perpendicular and parallel to UV, intersect at P; then the locus of P is known as Agnesi's Witch after Maria Gaetana Agnesi, who was Professor of Mathematics at Bologna in the mid-eighteenth century and one of the few famous woman mathematicians of former times. She does not appear to have invented the curve, but refers to it in her book *Istituzione Analytiche* (1748) as 'the curve commonly known as the Witch'.

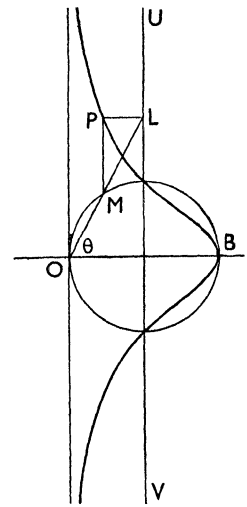

Fig. 94

To find the equation of the curve let O be the origin, let the equation of UV be $x = a$, and let the circle have diameter b. Then if OM makes an angle θ with the x-axis, the coordinates of P are given by

$$x = b\cos^2\theta, \quad y = a\tan\theta,$$

or, putting $\tan\theta = t$,

$$x = \frac{b}{1+t^2}, \quad y = at.$$

Eliminating t, we find that

$$x(y^2+a^2) = a^2b.$$

Agnesi considered the case in which $b = a$, but many later writers have put $b = 2a$. The results given in the following examples are true for all cases.

16. (i) Find the condition that the three points with parameters t_1, t_2, t_3 shall be collinear and show that the corresponding values of θ satisfy $\theta_1+\theta_2+\theta_3 = \pm\frac{1}{2}\pi$.

(ii) By letting two of the values of θ tend to the same limit, find the relation between α and β if the tangent at $(b\cos^2\alpha, a\tan\alpha)$ meets the curve again at $(b\cos^2\beta, a\tan\beta)$; hence give a geometrical construction for the tangent.

(iii) What values of θ give a point of inflexion on the curve?

17. P is the point with parameter t; B is $(b, 0)$; BP meets the curve at R, and Q is the reflection of R in the x-axis.

(i) What are the parameters of Q and R?

(ii) If OLM is called the line generating P, what can be said about the lines generating P and Q?

18. With the data of No. 17, show that

(i) the mid-point of PR lies on $2x = b$;

(ii) the tangents at P and Q meet on the curve;

(iii) $BP.BQ = BC^2$, C being the point in which BP meets $y = a$

See also 12.5, No. 88.

12.3. Quartic Curves

Nicomedes' Conchoid

Given a fixed line and a fixed point O, distant a from it, let a variable line through O meet the fixed line at N, and along ON mark NP equal to a fixed length b; then the locus of P is Nicomedes' Conchoid. It can be described mechanically with the aid of two pieces of wood or metal. A base plate has a peg and a slot to represent the fixed point and line, while a movable pointer has a slot to fit over the peg, a peg to slide in the fixed slot, and a hole for a pencil at a distance b beyond the peg, as in Fig. 95.

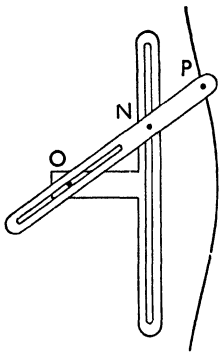

Fig. 95

Although this definition of a conchoid gives only a single branch, its polar or cartesian equation represents a curve with two branches, the second being the locus obtained when NP is measured in the opposite sense to ON.

To find the equation, let the fixed line be $x = a$, meeting the x-axis at A; let O be the origin, and let ON make an angle θ with OA. Then $ON = a \sec \theta$ and so the polar equation of the curve is

$$r = a \sec \theta + b.$$

When θ increases by a further 180°, $\sec \theta$ changes sign and N is again the point at which $r = a \sec \theta$, but this time the point at which $r = a \sec \theta + b$ is on the same side of N as O. Thus the branch further from O is obtained when θ is in the first or fourth quadrants, while the other two quadrants give rise to the other branch.

The cartesian equation may be obtained by writing the polar equation $r \cos \theta - a = b \cos \theta$. Then

$$x - a = b \cos \theta;$$

$$\therefore \quad r(x-a) = bx;$$

$$\therefore \quad (x^2+y^2)(x-a)^2 = b^2x^2.$$

The line $x = a$ is the curve's only asymptote. The branch nearer O has three possible forms: if $b>a$ it has a double-point at O and a loop, as in Fig. 96; if $b = a$ it has a cusp at O and resembles a cissoid; while if $b<a$ it looks somewhat similar to the other branch and explains the name conchoid, which means *mussel-shaped.*

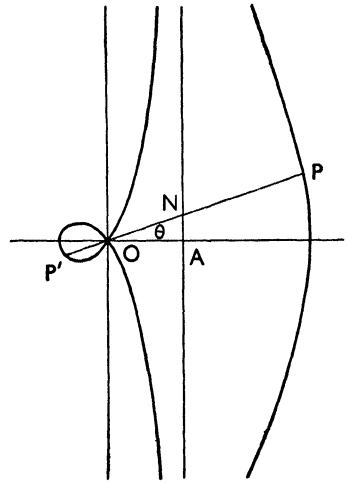

Fig. 96

The last case also shows a curious feature: the cartesian equation gives an **isolated point** at O which has no mechanical significance and does not satisfy the polar equation.

19. (i) Obtain freedom equations for the curve in terms of the parameter θ.
(ii) Find the equation of the tangent at the point θ.
(iii) If PO meets the curve again at P', and if the tangents at P and P' meet the line $x = 2a$ at T and T', show that $TM = MT'$, where M is the point of intersection of PP' and $x = 2a$.

20. The tangent at P meets the asymptote at E; V is the foot of the perpendicular from E to OP; prove that $PV = ON$.

21. The circle $x^2+y^2 = b^2$ meets the asymptote at B, B'; prove that OB, OB' are the tangents at O to the conchoid.

22. The circle $x^2+y^2 = b^2$ meets OP at R, such that OR, PP' are in the same sense; the tangent to the circle at R meets the asymptote at D; show that (with the notation of Nos. 19, 20)
(i) $DN = NE$;
(ii) D, T', P are collinear.

23. **Solution of the Delian Problem.** Show that, if a line through A making an angle of 60° with the x-axis meets a cuspidal conchoid (i.e. $b = a$) at P and Q, P and O being on opposite sides of the asymptote, then $AQ = a$, $ON = 2\frac{1}{2}a$, and $AP = 2^{\frac{1}{2}}a$.

The more general result is as follows. Given any conchoid with $b > a$, let Q be a point at which the curve meets $2x = a$ and let one of the tangents at O meet the asymptote at B; produce QB to meet the other branch of the curve at P. Then, defining N as before, if $QB = c$, $ON = u$, $BP = v$, it can be shown that $\frac{c}{u} = \frac{u}{v} = \frac{v}{2b}$. To prove this, however, it is best to ignore the equation of the curve and to proceed thus: first produce OQ to meet the asymptote at C and show that $OQ = QC = OB = NP = b$; apply Menelaus' Theorem to $\triangle POQ$ with transversal NBC, and also express in two ways the square of the distance from O to PBQ; you will then have two simultaneous equations leading to the required result.

24. **Trisection of an angle.** If in the equation of the conchoid, $b = 2a\sec\alpha$ where α is an acute angle, show that the line $y = a\tan\alpha$ meets the curve in four points and that if these points are referred to polar coordinates in the usual way, the four values of θ are $\frac{\alpha}{3}$, $\frac{\alpha+2\pi}{3}$, $\frac{\alpha+4\pi}{3}$ and $\pi-\alpha$.

[This can be done by using the polar equations of line and curve, or else by pure geometry with the help of Fig. 93.]

Pascal's Limaçon

O is a fixed point on a given circle of radius a; on a variable chord OM two points P, P' are taken such that $P'M$ and MP are both equal to a fixed length b; then the locus of P and P' is known as a limaçon. If O is taken as pole and the diameter OA of the circle as initial

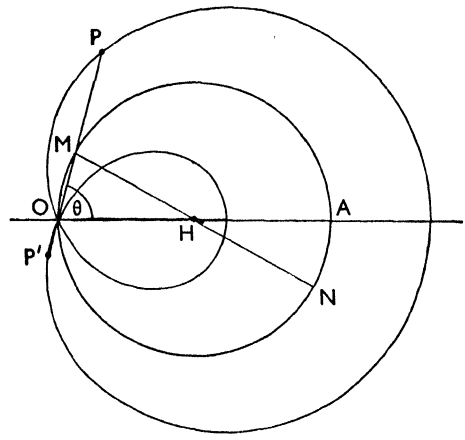

Fig. 97

line, the polar equation of the curve is

$$r = 2a\cos\theta + b,$$

since $OM = 2a\cos\theta$. If the value of r given by the equation is that of P in the figure, then when θ is increased by π the corresponding point P' will be represented, since $\cos(\theta+\pi) = -\cos\theta$.

When $b>2a$ the curve consists of a single loop, since r is always positive; when $b = 2a$ it has a cusp at O and is known as a **cardioid** (see 12.4); when $b<2a$ it has the shape shown in the figure and the part of the curve from $\theta = 0$ to $\theta = \pi$ gives it the name of limaçon (= snail).

From the polar equation we obtain at once the freedom equations $x = (2a\cos\theta + b)\cos\theta$, $y = (2a\cos\theta + b)\sin\theta$. Writing the polar equation in the form $(r^2-2ar\cos\theta)^2 = b^2r^2$ we see that the cartesian equation is

$$(x^2+y^2-2ax)^2 = b^2(x^2+y^2),$$

showing that it is a curve of the fourth degree.

25. If S is $\left(\frac{4a^2-b^2}{4a}, 0\right)$, show that the bipolar equation of the curve referred to O and S is

$$4ar_2 \pm 2br_1 = 4a^2-b^2,$$

where the upper sign gives the inner loop, the lower sign the outer.

[If you first get $\pm 4ar_2 = 2br_1+4a^2-b^2$, show that r_1 has been counted as negative on the inner loop and that the sign of the whole R.H.S. is everywhere positive.]

26. If the origin is moved to H, the mid-point of OA, show that

(i) the curve has freedom equations

$$x = a\cos 2\theta + b\cos\theta, \quad y = a\sin 2\theta + b\sin\theta;$$

(ii) the equation of the normal at P is

$$x(2a\sin 2\theta + b\sin\theta) - y(2a\cos 2\theta + b\cos\theta) = ab\sin\theta;$$

(iii) if MN is a diameter of the given circle, N is the point

$$(-a\cos 2\theta, -a\sin 2\theta);$$

(iv) the normals at P and P' pass through N.

27. Using the polar equation of the curve with O as pole, prove that $\cot\phi = -\frac{ON}{OP}$; hence prove the last part of No. 26.

28. If, with the origin as in No. 26, S is the point $(-b^2/4a, 0)$, prove that $\angle PSA = 2\phi - \pi$.

[Show first that $\tan PSA = \tan 2\phi$ and then consider the initial values of the angles.]

29. A circle is drawn with centre $(a, 0)$ and radius $\sqrt{(\frac{1}{2}b^2+a^2)}$. A point P then moves so that

$$PT^4 = 2ab^2.PL$$

where PT is the tangent from P to the circle and PL is the perpendicular from P to the line $8ax+b^2 = 0$. Show that the locus of P is the same limaçon as before.

30. Two circular discs, each of diameter b, lie touching each other on a horizontal plane. One circle is fixed to the plane; to the other is fixed a radial wire on which a point P is marked at a distance a from the centre. Initially the wire, if produced, would pass through the centre of the fixed circle, and the centre of the movable circle lies between this centre and P. The movable circle then rolls without slipping around the fixed circle. Show that when the line joining the centres makes an angle θ with its initial direction, the coordinates of P, referred to the centre of the fixed circle as origin and the initial position of the line as x-axis, are the same as in No. 26 (i), and hence that the locus of P is a limaçon.

31. Two perpendicular tangents are drawn, one to each of two fixed circles; show that, as the tangents vary, the locus of their point of intersection is either of two limaçons.

[Choose the origin so that the join of the centres of the circles subtends a right angle there and so that the ratio of the distances of the centres from the origin is equal to the ratio of the radii; choose the direction of the x-axis so that it bisects the join of centres; use as few unknowns as possible in writing down the centres and radii of the circles and express the coordinates of the points of contact in terms of a single parameter.]

32. The **Trisectrix** is the name given to the limaçon when $a = b$, so that H lies on the curve. In this case show that

(i) $\angle OPH = \frac{1}{3} \angle PHA$;

(ii) if PH meets the curve again at Q and R, then $HP+HQ+HR = 0$, sense being taken into account.

[The first part is best done by pure geometry, using Fig. 93; for the second, express HP in terms of θ and consider the angles which must replace θ to obtain Q and R. Alternatively, the second part may be done entirely by cartesian methods, taking the origin at H and using the theory of equations.]

33. The bipolar equation of a curve is $r_2^2 = 4cr_1$; show that the curve is a limaçon. [Let the two poles have cartesian coordinates $(c, 0)$, $(2c, 0)$ respectively.]

See also 12.3, No. 46; 12.5, Nos. 80, 82, 89, 94, 95.

Lemniscates

A lemniscate (the original Greek word means *looped knot*) is any curve in the form of a figure eight, but the name is chiefly applied to a few well-known curves of this type which we shall briefly consider. One of the earliest references to a figure-eight curve is in the writings of Eudoxus (408–355 B.C.), who thought that a planet traced out a locus of this kind on the surface of a celestial sphere, the curve being given by the intersection of the sphere with a cylinder. Although this is a three-dimensional curve it is closely related to a plane curve, Gerono's Lemniscate (see No. 40).

The other lemniscates to be included here are conveniently introduced by considering Watt's *Parallel Motion*. James Watt (1736–1819), wishing to keep the motion of a piston rod straight without the use of sliding parts, devised a linkage consisting of three rods $F'A'$, $A'A$, AF pivoted to one another at A' and A and to the chassis at F' and F, the axes of the four pivots being parallel; if a point P was taken

in $A'A$ so that $F'A.A'P = PA.AF$, he found that the locus of P was very nearly straight in part of its course. This linkage he called his Parallel Motion.

To find the equation of the locus we shall consider only the symmetrical case in which $F'A' = AF$ and P is the mid-point of $A'A$. Let F, F' be the points $(c, 0)$, $(-c, 0)$, let $F'A' = AF = a$,

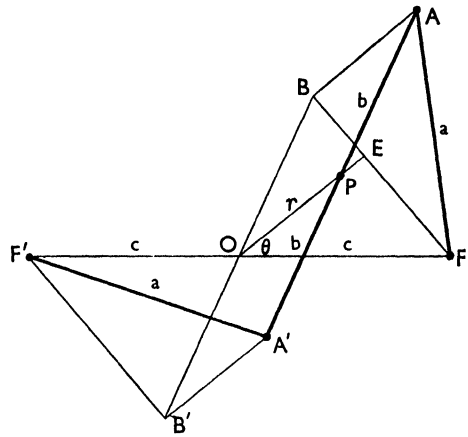

Fig. 98

$A'P = PA = b$. Through O draw a line $B'OB$ parallel to $A'A$ and having $B'O = OB = b$, so that $B'A'PO$ and $OPAB$ are parallelograms. Join $F'B'$ and FB. Then $\triangle F'OB' \equiv \triangle FOB$; $\therefore F'B' /\!/ = FB$; $\therefore \triangle F'B'A' \equiv \triangle FBA$, since $B'A' = BA$; $\therefore$ $\angle F'B'A' = \angle FBA$. But also, since $F'B' /\!/ BF$ and $B'A' /\!/ BA$, we have $\angle F'B'A' + \angle FBA = 180°$; hence the two angles are right angles.

Now let P have polar coordinates (r, θ) referred to O and OF, and let OP (produced) meet BF at E; then $BA = r$, and hence $BF^2 = a^2 - r^2$. Applying the cosine formula to $\triangle FOB$, we have

$$OB^2 = OF^2 + BF^2 - 2OF.BF\cos EFO;$$

$$\therefore \quad b^2 = c^2 + a^2 - r^2 - 2c(a^2 - r^2)^{\frac{1}{2}}\sin\theta;$$

$$\therefore \quad (r^2 - a^2 + b^2 - c^2)^2 = 4c^2(a^2 - r^2)\sin^2\theta.$$

Multiplying through by r^2, we find the cartesian equation to be

$$(x^2+y^2)(x^2+y^2-a^2+b^2-c^2)^2 + 4c^2y^2(x^2+y^2-a^2) = 0.$$

This sixth-degree curve, which may be called **Watt's Lemniscate,** can assume a variety of forms for different values of a, b, c (see 12.4, Nos. 76, 77); in particular, when $b = c$ its equation becomes

$$(x^2+y^2-a^2)\{(x^2+y^2)(x^2+y^2-a^2) + 4c^2y^2\} = 0,$$

giving either $x^2+y^2-a^2 = 0$ or $(x^2+y^2)(x^2+y^2-a^2)+4c^2y^2 = 0$. The first of these represents a circle, while the second gives **Booth's Lemniscate,**

$$(x^2+y^2)^2 = a^2x^2-(4c^2-a^2)y^2.$$

The circle occurs because $F'A'AF$ can form a parallelogram with $OP = a$; the lemniscate occurs when A' and F are on the same side of $F'A$, as in Fig. 98.

Finally, if in Booth's Lemniscate we put $2c^2 = a^2$, so that it is possible for $F'A'FA$ to form a square with the rods making up two sides and a diagonal, we obtain **Bernoulli's Lemniscate,** named after James Bernoulli (1654–1705), which has for its equation

$$(x^2+y^2)^2 = a^2(x^2-y^2),$$

or

$$r^2 = a^2\cos 2\theta.$$

The shape of the curve is shown in Fig. 99, the points F' and F are called its foci.

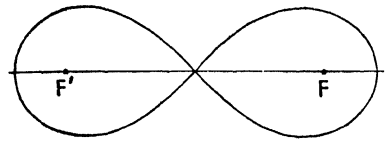

Fig. 99

Nos. 34–38 refer to Bernoulli's lemniscate.

34. Show that (i) $\phi = 2\theta+\frac{1}{2}\pi$;
(ii) the normal at θ makes an angle 3θ with the initial line;
(iii) if FP, $F'P$ make angles α, α' with the initial line, then $\alpha+\alpha' = 4\theta$;
(iv) the angle between FP and the normal at P is equal to that between $F'P$ and the radius vector to P;
(v) if T is a point at which the tangent is parallel to Ox, then $OT = OF$ and OT makes an angle of 30° with Ox.

35. Show that the bipolar equation of Bernoulli's lemniscate referred to the foci is $2r_1r_2 = a^2$. The curve is therefore an Oval of Cassini.

36. If p is the length of the perpendicular from O to the tangent at P, show that $a^2p = r^3$. This is known as the **pedal equation** of the curve.

37. If the tangent at P meets the curve at Q and R, and M is the mid-point of QR, show that $OM = OP$ and that the angles which OM and OP make with the x-axis are in the ratio 5 : 1. Hence write down the polar equation of the locus of M and sketch the curve which it represents.

[Show that the cartesian coordinates of any point on the tangent are given by $x = r\cos\theta + s\sin 3\theta$, $y = r\sin\theta - s\cos 3\theta$, where (r, θ) are the polar coordinates of P and s is the distance of the point from P; substitute these coordinates in the equation of the curve to obtain an equation in s; reject the zero roots and find half the sum of the others.]

38. (i) Find the point of intersection of the lemniscate with the circle $x^2+y^2 = at(x-y)$; hence find freedom equations for the lemniscate.

(ii) Find the condition that the points with parameters t_1, t_2, t_3, t_4 should be concyclic. [Cf. 9.4].

(iii) A circle meets the lemniscate at A, B, C, D; a circle through A and B meets the curve again at G, H; a circle through C and D meets the curve again at L, M; show that G, H, L, M are concyclic.

See also 12.5, Nos. 81, 83, 97.

39. **Booth's Lemniscate.** Show that

(i) if $a<2c$, the general appearance of the curve is similar to Bernoulli's, except that the gradient of the tangents at O is different;

(ii) if $a = 2c$, the curve consists of two circles;

(iii) if $a>2c$, the curve (apart from an isolated point at O) consists of a single loop with no double-point.

(iv) Explain how the circles arise in case (ii).

[Use the polar equation for (i) and (iii).]

See also 12.5, Nos. 81 and 97.

40. **Gerono's Lemniscate**

Take a third coordinate axis at right angles to the plane of the paper, and let z represent the height of a point above the paper. Then the distance of a point from O is $\sqrt{(x^2+y^2+z^2)}$ and hence the equation of a sphere with centre O and radius a is $x^2+y^2+z^2 = a^2$; and if the sphere is raised until it touches the x-y plane its new equation will be

$$x^2+y^2+(z-a)^2 = a^2. \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

The distance of a point from the x-axis is $\sqrt{(y^2+z^2)}$ and hence the equation of a cylinder with radius b and axis along the x-axis is $y^2+z^2 = b^2$. If this cylinder is raised until it too touches the x-y plane, its new equation will be

$$y^2+(z-b)^2 = b^2 \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

The intersection of these two surfaces is Eudoxus' Hippopede; if perpendiculars are dropped from every point of it to the x-y plane the equation of the locus of their feet is got by eliminating z between (i) and (ii). Show that it is

$$4(a-b)^2y^2 = x^2\{4b(a-b)-x^2\},$$

and that it represents a figure-eight curve provided $a>b$.

In Gerono's lemniscate, $b = a/2$, so that its equation is $a^2y^2 = x^2(a^2-x^2)$. Show that the tangents at O are the same as for Bernoulli's, and that it cuts the x-axis at the same points.

41. A is a variable point on the circle $x^2+y^2 = a^2$; B is the foot of the perpendicular from A to the line $x = a$; the line through A parallel to Oy cuts OB at P; show that the locus of P is Gerono's lemniscate.

Cartesian Ovals

Any curve with a bipolar equation of the form

$$mr_1+lr_2 = nc \quad . \quad . \quad . \quad . \quad \text{(i)}$$

where l, m, n are positive or negative constants and c is the distance between the two poles, is known as a cartesian oval, and the two poles are called its foci. Clearly there is no loss of generality if we suppose $m^2 \geqslant l^2$, and we can set aside the cases in which $m = \pm l$ since the curve is then an ellipse or hyperbola defined by its bifocal property. We are thus left with the cases in which $m^2 > l^2$.

To find the cartesian equation of the curve it will be convenient to choose axes so that the two foci are respectively $A\left(\frac{cl^2}{m^2-l^2}, 0\right)$ and $B\left(\frac{cm^2}{m^2-l^2}, 0\right)$, and to take numbers α, β, γ such that $\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n} = \sqrt{\left(\frac{c}{m^2-l^2}\right)}$. The bipolar equation can then be written $\beta r_1 = \gamma(\beta^2-\alpha^2)-\alpha r_2$, the foci A, B are $(\alpha^2, 0)$, $(\beta^2, 0)$, and hence the cartesian equation is

$$\beta\{(x-\alpha^2)^2+y^2\}^{\frac{1}{2}} = \gamma(\beta^2-\alpha^2)-\alpha\{(x-\beta^2)^2+y^2\}^{\frac{1}{2}}.$$

On squaring, rearranging and squaring again (as in 3.3, Example 2), this gives

$$(x^2+y^2-\beta^2\gamma^2-\gamma^2\alpha^2-\alpha^2\beta^2)^2+4\alpha^2\beta^2\gamma^2(2x-\alpha^2-\beta^2-\gamma^2) = 0 \quad . \quad (1)$$

42. Show that the same equation is reached by starting with any of the bipolar equations

$$mr_1+lr_2 = nc \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$
$$mr_1-lr_2 = nc \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$
$$-mr_1+lr_2 = nc \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

If l, m, n are positive, with $m>l$, and if negative values of r_1, r_2 are not allowed, show that the value of mr_1+lr_2 is least when $r_1 = 0$, and hence that there are real points satisfying (i) provided $n>l$. [Notice that $r_1+r_2 = c$ at any point between and collinear with A and B, and $r_1+r_2>c$ elsewhere.] By considering the values of mr_1-lr_2 along the line AB, show that (ii) gives a real curve for all positive values of n, but (iii) only when $n<l$.

Hence the cartesian equation always represents *two* ovals provided that l, m, n are positive, that $l \neq m$, and that n is not equal to the lesser of l and m. The exceptional case here excluded is considered in No. 46.

43. If C $(\gamma^2, 0)$ is a third pole from which a radius vector r_3 is measured, show that the bipolar equations

$$\pm mr_1 \pm lr_2 = nAB,$$
$$\pm nr_1 \pm lr_3 = mAC,$$
$$\pm nr_2 \pm mr_3 = lBC$$

all lead to the same cartesian equation, namely (1). The curve thus has *three* foci, A, B, C.

44. Show that the polar equation of the curve referred to A as pole and AB as initial line is

$$r^2-2r(\beta\gamma-\alpha^2\cos\theta)+(\gamma^2-\alpha^2)(\beta^2-\alpha^2) = 0 \quad . \quad . \quad (2)$$

[Put $r_1 = r$ and eliminate r_2 between (i) and $r_2^2 = r^2+c^2-2cr\cos\theta$.]

45. A variable line meets the curve in four points P, Q, R, S; show that the sum of the distances of these points from any focus is the same for all positions of the line.

[The polar equation of any line is $r(\mathrm{A}\cos\theta+\mathrm{B}\sin\theta) = \mathrm{C}$. Obtain a quartic equation in r by eliminating θ between this and (2); and show that the sum of the roots is $4\beta\gamma$.]

46. Show that the bipolar equation $\pm mr_1 \pm lr_2 = lc$ (where c is the distance AB between the poles), represents a limaçon, that if $m<l$ the limaçon has a node at A, and that if $m>l$ the limaçon has no node but the bipolar equation also

gives an isolated point at A. A limaçon is thus a particular case of a cartesian oval.

[Put $\alpha = \gamma$ in (2).]

47. A point moves so that the fourth power of the length of the tangent from it to a given circle is proportional to its distance from a given line; prove that its locus is a pair of cartesian ovals.

[Use equation (1).]

48. **Mechanical construction of cartesian oval**

A, B, X are three collinear points; a variable triangle PQR moves so that PQ passes through A, PR through B, QR through X, while Q and R lie on given circles with centres at A and B; prove that the locus of P is a cartesian oval with A and B as foci. A mechanism of pins and slotted rods can easily be devised to satisfy the conditions.

[This is best proved by pure geometry: apply Menelaus' Theorem to $\triangle PAB$.]

See also 12.5, No. 96.

12.4. Other Mechanical Loci

Epicycloids and Hypocycloids

If a circle rolls without slipping on the outside of the circumference of a fixed circle (the **pitch circle**), the locus of a given point on the

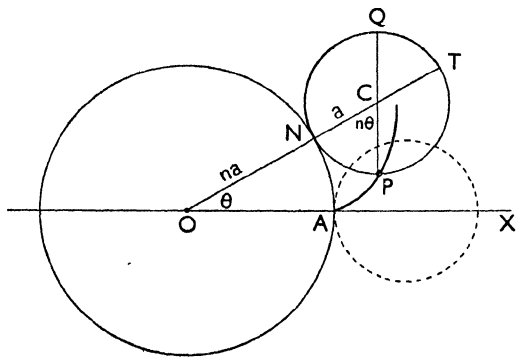

Fig. 100

circumference of the rolling circle is called an **epicycloid**; if the moving circle rolls on the inside, the locus is called a **hypocycloid.**

To find freedom equations for an epicycloid, let the pitch circle have radius na, let O be its centre and A a point at which it is met by the locus. Take O as origin, OA as x-axis let the rolling circle have radius a, let C be its centre, P the point on its circumference which traces out the locus, and N its point of contact with the pitch circle. Then the tracing point moves from A to P as the moving circle rolls to its present position from the position in which its centre was

on the x-axis, and, since it does not slip, the arcs AN and PN are equal.

Hence, if $\angle COA = \theta$, $\angle OCP = n\theta$ and PC makes an angle $(n+1)\theta$ with the x-axis. The coordinates of P are therefore given by

$$x = OC\cos\theta - CP\cos(n+1)\theta,$$
$$y = OC\sin\theta - CP\sin(n+1)\theta,$$

and the **freedom equations of an epicycloid** can be written

$$\boldsymbol{x = (n+1)a\cos\theta - a\cos(n+1)\theta,}$$
$$\boldsymbol{y = (n+1)a\sin\theta - a\sin(n+1)\theta.}$$

The gradient of the curve at P is

$$\frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{\cos\theta - \cos(n+1)\theta}{\sin(n+1)\theta - \sin\theta} = \tan(\tfrac{1}{2}n+1)\theta.$$

But this is also the gradient of PT, where NT is a diameter of the rolling circle, since $\angle CTP = \frac{1}{2}\angle NCP$. Hence ***PT* and *PN* are the tangent and normal** to the curve at P. This also follows from the fact that N is the instantaneous centre of rotation of the rolling circle.

In particular, AX is the tangent at A and the curve therefore has a cusp there. Since the circumference of the pitch circle is n times that of the rolling circle, the curve will meet the pitch circle at n points and so have n cusps.

If PQ is a diameter of the rolling circle, the locus of Q is the n-cusped epicycloid obtained from the original curve by rotation about O through $1/2n$ of a revolution. Its freedom equations are readily seen to be

$$x = (n+1)a\cos\theta + a\cos(n+1)\theta,$$
$$y = (n+1)a\sin\theta + a\sin(n+1)\theta.$$

49. By drawing a figure for the case in which the rolling circle is inside the pitch circle, show that the **freedom equations of an *n*-cusped hypocycloid** can be written

$$\boldsymbol{x = (n-1)a\cos\theta + a\cos(n-1)\theta,}$$
$$\boldsymbol{y = (n-1)a\sin\theta - a\sin(n-1)\theta,}$$

and that the tangent and normal can be constructed in the same way as before.

50. **Construction of tangents to an epicycloid or hypocycloid**

Show that QT is the tangent to the locus of Q, and that, if it meets the circle with centre O and radius $(n+2)a$ again at U, $\angle UOA = (n+1)\angle TOA$.

Hence tangents to an n-cusped epicycloid can be constructed as follows: choose a multiple of $n+1$ and mark that number of equally spaced points on the circumference of a circle, numbering them consecutively 0, 1, 2, 3, . . ; draw the tangent to the circle at O, join 1 and $n+1$, 2 and $2(n+1)$, and so on. If a sufficiently large number of points is taken, the tangents will themselves give a good approximation to the curve.

Tangents to an n-cusped hypocycloid may be similarly constructed by taking a multiple of $n-1$ points marked $\ldots -3, -2, -1, 0, 1, 2, \ldots$ and joining 1 and $-(n-1)$, 2 and $-2(n-1)$, etc., but greater accuracy is needed in the drawing.

51. If a circle of radius $2a$ rolls round the outside of a fixed circle of radius na, show that a diameter of the rolling circle is always tangential to an n-cusped epicycloid and that a similar fact is true of a hypocycloid.

[Use No. 50 and the fact that $\angle CTP = \frac{1}{2}\angle NCP$.]

52. Use No. 50 to show that normals to an n-cusped epicycloid touch another n-cusped epicycloid, the pitch radii of the two curves being in the ratio $n+2 : n$; and that a similar fact is true of hypocycloids, the ratio being $n-2 : n$. An epicycloid or hypocycloid is thus the evolute of a curve of the same type.

53. If a circle rolls round a *smaller* fixed circle which touches it internally, the locus of a point on the rolling circle is sometimes called a **pericycloid.**

If the fixed and rolling circles have centres O and K, radii b and kb, obtain freedom equations for the curve in the form,

$$x = kb\cos\frac{k-1}{k}\phi - (k-1)b\cos\phi,$$

$$y = kb\sin\frac{k-1}{k}\phi - (k-1)b\sin\phi,$$

where ϕ is the angle which OK makes with the negative part of the x-axis.

By writing $k = 1+\frac{1}{n}$, $b = na$, and $\phi = (n+1)\theta$, show that the curve is in fact an epicycloid.

Some particular cases are given special names. If $n = 0$, the fixed circle is a mere point, the epicycloid is a circle, and no hypocycloid is possible. If $n = 1$, the epicycloid is known as a **cardioid** (= heart-shaped) and there is again no hypocycloid. If $n = 2$, the epicycloid is called a **nephroid** (= kidney-shaped—but it looks more like two kidneys), and the equation of the hypocycloid shows that it is merely a segment of a straight line, a diameter of the fixed circle. The hypocycloid with three cusps is called a **deltoid,** that with four an **astroid** (= star-shaped).

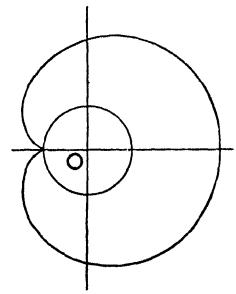

Fig. 101

The Cardioid

The locus of Q when $n = 1$ is given by

$$\left.\begin{aligned} x &= 2a\cos\theta + a\cos 2\theta, \\ y &= 2a\sin\theta + a\sin 2\theta. \end{aligned}\right\} \quad \ldots \quad \text{(i)}$$

If, however, the curve is shifted through a distance a in the direction OX, the equations become

$$\begin{aligned} x &= 2a\cos\theta + a(1+\cos 2\theta) = 2a\cos\theta\,(1+\cos\theta), \\ y &= 2a\sin\theta + a\sin 2\theta \qquad\quad = 2a\sin\theta\,(1+\cos\theta), \end{aligned}$$

and hence the polar equation of the curve is

$$r = 2a(1+\cos\theta).$$

The curve is therefore a particular case of a limaçon, as mentioned in 12.3, and has all the properties found in Nos. 26–30, though possibly in a degenerate form.

54. Show that a chord of the cardioid (i) which passes through the cusp $(-a, 0)$ is of constant length $4a$, that the tangents at its ends meet at right angles on the circle $x^2+y^2 = 9a^2$, and that the normals at its ends meet on the circle $x^2+y^2 = a^2$.

55. If r is the distance from the cusp to a point P of the curve, and p the length of the perpendicular from the cusp to the tangent at P, prove that $r^3 = 4ap^2$.

56. The tangent at any point P of a strophoid (see Nos. 11, 12) meets the line through the vertex A at right angles to AP at Z; prove that the locus of Z is a cardioid.

57. Rays of light from $(3a, 0)$ are reflected by the circle $x^2+y^2 = 9a^2$; use No. 50 to show that after one reflection the rays become tangents to the cardioid given by (i). A curve to which rays of light are tangential is called a **caustic** curve.

See also 12.5, Nos. 80, 82, 94.

The Nephroid

58. Rays of light parallel to the x-axis are reflected by the circle $x^2+y^2 = 16a^2$; use No. 50 to show that the caustic curve is the nephroid $x = 3a\cos\theta - a\cos 3\theta$, $y = 3a\sin\theta - a\sin 3\theta$. Draw it.

In practice half of the caustic is, in optical language, virtual; the other half can be seen in a cup of tea if the sun shines on it.

The Deltoid

From No. 49, a 3-cusped hypocycloid is given by

$$x = a(2\cos\theta + \cos 2\theta),\quad y = a(2\sin\theta - \sin 2\theta).$$

It has cusps where $\theta = 0$ and $\pm\frac{2\pi}{3}$, i.e. at the points $A(3a, 0)$ $B\left(-\frac{3a}{2}, \frac{3a\sqrt{3}}{2}\right)$, $C\left(-\frac{3a}{2}, -\frac{3a\sqrt{3}}{2}\right)$. The curve lies within the equilateral triangle ABC and its method of generation shows that it is nearest the origin when θ is midway between the values which give its cusps. Thus when $\theta = \pi$, $x = -a$, $y = 0$, and the curve touches the circle $x^2+y^2 = a^2$ at this and two other points.

The constraint equation,

$$(x^2+y^2)^2 - 8ax(x^2-3y^2) + 18a^2(x^2+y^2) = 27a^4,$$

can be obtained from No. 61. The curve is thus of the fourth degree.

59. Show that the freedom equations of the curve can be written

$$x+a : y : a = 4(1-t^2) : 8t^3 : (1+t^2)^2$$

where $t = \tan\frac{1}{2}\theta$, and that the tangent at t is $t(x+a)+y = \frac{4at}{1+t^2}$.

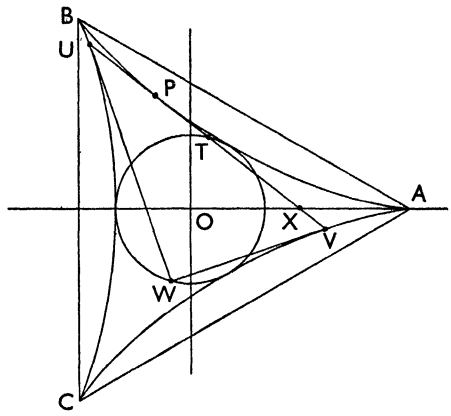

Fig. 102

60. If the tangent at P meets the curve again at U and V, UV meets the x-axis at X and the line $x+a=0$ at Y, and the tangents at U and V meet at W (see Fig. 102), prove that

(a) $\angle UWV = 90°$;
(b) the locus of W is the circle $x^2+y^2=a^2$;
(c) T, the mid-point of UV, lies on this same circle;
(d) O is the mid-point of TW;
(e) $UV = 4a$;
(f) $UX = VY$;
(g) the angles which UW, VW make with Ox are half those which UV makes with Ox.

[Let t_0 be the parameter of P; find the equation giving the parameters of the points in which the curve is met by the tangent at P; this is of the fourth degree but contains the factor $(t-t_0)^2$; divide by this factor and show that the parameters of U and V are the roots t_1, t_2 of $t_0t^2-2t-t_0=0$; use the fact that $t_1t_2=-1$ to prove (a). Find the point in which the tangents at t_1, t_2 meet; use the known values of t_1+t_2 and t_1t_2 to show that $(1+t_1^2)(1+t_2^2)=4(1+1/t_0^2)$ and hence that W is $(-a\cos\theta_0, -a\sin\theta_0)$; hence prove ($b$). Similarly show that the mid-point of UV is $(a\cos\theta_0, a\sin\theta_0)$, proving ($c$) and ($d$). Deduce ($e$) by pure geometry. Find X and Y; prove (f) by showing that T is also the mid-point of XY. Prove (g) by rewriting the equation $t_0t^2-2t-t_0=0$ to give t_0 in terms of t.]

61. Show that the freedom equations can be written

$$3a+2x : 3a-x-\sqrt{3}y : 3a-x+\sqrt{3}y = \frac{1}{4t^2} : \frac{1}{(t+\sqrt{3})^2} : \frac{1}{(t-\sqrt{3})^2}.$$

Hence show that, if α, β, γ are the distances of a point of the curve from BC, CA, AB,

$$\alpha^{-\frac{1}{2}} \pm \beta^{-\frac{1}{2}} \pm \gamma^{-\frac{1}{2}} = 0.$$

62. A variable line is cut at Y by a tangent to a circle and at X by the diameter through the point of contact; if the line moves so that the mid-point of XY lies on the circle, prove that it remains tangential to a deltoid.

[Use No. 60 (c) and (f).]

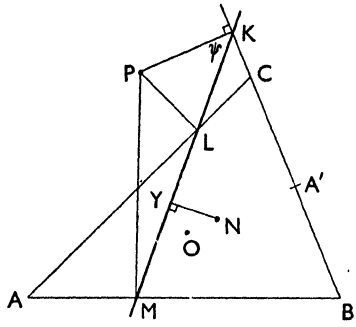

Fig. 103

63. A tangent to the given deltoid makes an angle ψ with the x-axis, and the perpendicular from the origin to the tangent is of length p. Prove that, if suitable conventions are adopted for the signs of p and ψ, $p = a \sin 3\psi$.

64. A **pedal line** of a triangle is the line joining the feet of the perpendiculars to the sides from a point P on the circumcircle: it can easily be proved by pure geometry that the three feet are collinear. Complete the following proof, due to Jacob Steiner (1796–1863), that as the point P moves round the circumcircle the pedal line remains tangential to a deltoid.

K, L, M are the feet of the perpendiculars to the sides from a point P on the circumcircle of $\triangle ABC$; O, N are the circumcentre and nine-point centre of the triangle; A' is the mid-point of BC; the pedal line KLM makes an angle ψ with PK; NY, the perpendicular from N to KLM, is of length p; $OC = R$, the circumradius of the triangle.

(*a*) Prove that $p = A'K\cos\psi - NA'\sin(\psi + NA'O)$
$= R\sin OPK\cos\psi - \frac{1}{2}R\sin(\psi - B + C)$.

[Draw $A'Z \perp KLM$, and $NW \perp A'Z$; remember that $NA' \| AO$.]

(*b*) Prove that $OPK = OPC + CPK = OCP + CPK$
$= OCL + PKL + KLC$
$= 2\psi - B + C$.

(*c*) Hence prove that $p = \frac{1}{2}R\{2\sin(2\psi - B + C)\cos\psi - \sin(\psi - B + C)\}$
$= \frac{1}{2}R\sin(3\psi - B + C)$.

(*d*) Use No. 63 to show that KLM is tangential to a deltoid which touches the nine-point circle of the triangle at three points.

The Astroid

Writing $n = 4$ in No. 49 we obtain $x = 3a\cos\theta + a\cos 3\theta = 4a\cos^3\theta$, $y = 3a\sin\theta - a\sin 3\theta = 4a\sin^3\theta$ as the freedom equation of an astroid.

Since the factor 4 is not particularly useful we shall drop it, making the linear dimensions of the curve one-quarter of their previous value. Any point P on the curve is then given by the freedom equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta,$$

and the constraint equation can, with a little labour, be shown to be

$$(x^2 + y^2 - a^2)^3 + 27a^2x^2y^2 = 0,$$

so that the curve is of the sixth degree.

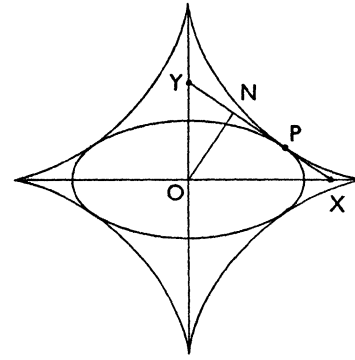

Fig. 104

65. Show that (*a*) the equations of the tangent and normal at P are
$$x\sin\theta+y\cos\theta = a\sin\theta\cos\theta,$$
$$x\cos\theta-y\sin\theta = a\cos 2\theta;$$
(*b*) if the tangent meets the axes at X, Y, XY is of constant length;
(*c*) if N is the foot of the perpendicular from O to the tangent, $XP = NY$;
(*d*) the lines $x\pm y = 0$ intercept a constant length $2a$ on the normal.

66. If an ellipse touches the curve at P and at the three other points obtained by reflection in the axes, show that the semi-axes of the ellipse are equal to YP and PX, so that their sum is constant.

67. Show that there are two ways in which a square can be drawn so that its corners are on the astroid and the two sides which meet at each corner are the tangent and normal to the curve at that point, the parameters of the corners being given by $\tan 2\theta = \pm 2$.

68. With the notation of No. 65, show that the equation of the locus of N is $(x^2+y^2)^3 = a^2x^2y^2$.
Find the polar equation and sketch the locus.

69. Show that perpendicular tangents of the astroid meet on the curve $2(x^2+y^2)^3 = a^2(x^2-y^2)^2$.
Find the polar equation and sketch the curve. What is the relationship between this curve and the locus of No. 68?

The Cycloid

A cycloid is the locus of a point on the circumference of a circle as the circle rolls without slipping along a fixed straight line; it is thus the limiting case of an epicycloid or hypocycloid as the radius of the fixed circle tends to infinity.

Freedom equations are easily got from a figure. Let OX be the fixed line, P the tracing point on the circumference of a rolling circle

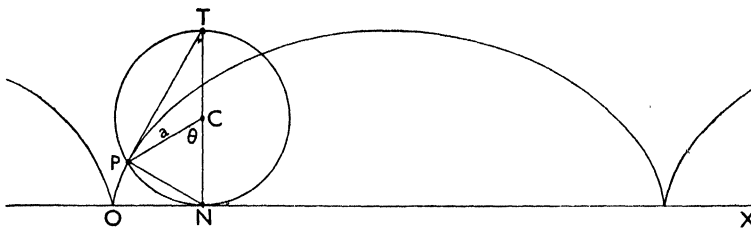

Fig. 105

with centre C and radius a, and N the point of contact of this circle with the fixed line. Let O be one of the points at which the curve meets the fixed line, so that ON is equal to the arc PN which has rolled over it, and both are equal to $a\theta$, where $\theta = \angle PCN$.

The x-coordinate of P is equal to $ON-PC\sin\theta = a\theta-a\sin\theta$; the y-coordinate of P is equal to $NC-PC\cos\theta = a-a\cos\theta$. Thus the required freedom equations are

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$$

The curve consists of a series of **arches**; consecutive arches join to form cusps, these cusps being given by $\theta = 0, 2\pi, 4\pi$ etc.

It is sometimes convenient to take axes so that the x-axis is the fixed line as before, but the origin comes midway between two cusps and the y-axis passes through the highest point of the curve. Such a point was formerly given by $\theta = \pi$, so to find the new equations we have only to replace θ by $\theta + \pi$ and subtract $a\pi$ from the value of x, getting

$$x = a(\theta + \sin\theta), \quad y = a(1 + \cos\theta).$$

If this curve is moved parallel to the y-axis until the x-axis is tangential to it, the new equations will be

$$x = a(\theta + \sin\theta), \quad y = -a(1 - \cos\theta).$$

This and the curve in its original position with a cusp at O have an interesting property discovered by Christian Huyghens (1629–1695) (see No. 71).

When the equation of a cycloid is given in a problem it may not be immediately clear where the cusps are or which way up the curve is. The best way to begin is to notice first whether the values of y are restricted while those of x can increase without limit: if this is so, the fixed line of the cycloid is parallel to the x-axis and the cusps are given by $dx/d\theta = 0$.

70. Show that (i) the tangent and normal to the curve at P are PT and PN, where N has the same meaning as before and NT is a diameter of the rolling circle;
(ii) consecutive perpendicular tangents cut off a constant length on the common tangent of the arches, while consecutive perpendicular normals cut off the same length on the line of cusps.

71. P is given by $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$; Q by $x = a(\theta + \sin\theta)$, $y = -a(1 - \cos\theta)$. Sketch the loci of P and Q, and show that PQ is a normal to the first and a tangent to the second. Hence if a thread of length $2a$ has one end attached to a cusp of the second curve, is held taut, and wraps itself round the second curve, the locus of its other end is the first curve.

Watt's Lemniscate

We have already seen in 12.3 that this curve is the locus of the mid-point of a rod of length $2b$ whose ends are joined by rods of length a to two fixed points $2c$ apart; we have also seen that, if these points are $(\pm c, 0)$, the equation of the curve is

$$(x^2+y^2)(x^2+y^2-a^2+b^2-c^2)^2+4c^2y^2(x^2+y^2-a^2) = 0,$$

and we have considered in some detail the cases in which $b = c$.

The mechanical part of the locus (i.e. all but possible isolated points on the x-axis) may be readily constructed by using strips of postcard as the 'rods'. The points about which the strips are to pivot can be marked with a compass-point; drawing-pins can serve as pivots, those at the ends of the 'rod' of length $2b$ being used point upwards; a sharp pencil then traces the curve through a pinhole at the mid-point of this 'rod'. If, during part of the motion the 'rods' of length a get in the way of the pencil, it may be possible to avoid the trouble by cutting away part of the connecting rods or by using curved strips instead of straight; any short lengths of arc still missing may be filled in freehand. The complete curve is always symmetrical about both axes, but to obtain the whole of it the strips may have to be uncoupled at times and recoupled in a new position.

72. (i) Show algebraically that the curve has two real points on the x-axis (other than O) provided $a^2 > b^2 - c^2$.

(ii) From purely geometrical considerations show that (*a*) the locus is physically impossible if $|b-c| > a$; (*b*) if the curve meets the x-axis at points other than O, then $b \geqslant c$ and $a^2 > b^2 - c^2$. [Notice that the two rods of length a must be equally inclined to the x-axis.]

If $b < c$, it will be seen that the equation still gives two (isolated) points on the x-axis. The origin satisfies the equation under all conditions.

73. Show that the curve meets the y-axis (apart from O) in

(i) 2 points if $b+c > a > |b-c|$;

(ii) 4 points if $a > b+c$.

74. Show that (i) the curve has real tangents at O provided $b+c \geqslant a > |b-c|$ [they may be found by equating the second-degree terms to zero];

(ii) these tangents coincide with the y-axis if $b+c = a$, and with the x-axis if $b^2-c^2 = a^2$;

(iii) the tangents at O cut the curve again where

$$4a^4c^2x^2 = \{a^4-(b^2-c^2)^2\}\{a^2-(b-c)^2\}\{(b+c)^2-a^2\},$$

and hence that the intersections are real when $a^2 > |b^2-c^2|$;

(iv) these tangents are perpendicular when $(a^2-b^2)^2 = c^2(2b^2-c^2)$.

A tangent at O meets the branch to which it is tangential in at least 3 coincident points; (iii) shows that, if $a^2 = |b^2-c^2|$, it meets it in 5 coincident points. The curve is then remarkably straight in the neighbourhood of the origin. See 76 (ii) and 77 (ii).

75. If $a > b-c$, so that the curve meets the y-axis in 4 points, the inner pair of which are where $y^2 = a^2-(b-c)^2$, show that each of the lines

$$y = \pm\sqrt{\{a^2-(b-c)^2\}}$$

meets the curve in

(i) 4 coincident points if $a^2b = (b+c)^3$;

(ii) 6 coincident points if also $(a^2-b^2-6bc-3c^2)^2 = 4bc(3b^2+2c^2)$.

(iii) Show that, if both these conditions are fulfilled,

$$12b^5-9b^4c+8b^3c^2+6b^2c^3-c^5 = 0,$$

and verify that this equation is satisfied approximately when $b : c = 1 : 2{\cdot}864$. The curve is then as straight as possible near its inner intersections with the y-axis. See 77 (viii).

76. $b>c$. Construct the curve when the ratio $a:b:c$ has each of the given values. In each case show that a, b, c satisfy the stated conditions and notice the geometrical features of the curve which follow from these conditions as predicted by Nos. 72–75.

(i) $\frac{1}{2}\sqrt{2}(\sqrt{7}-1)[\triangleq 1{\cdot}164] : 2 : 1$; $b^2-c^2>a^2>(b-c)^2$; $(a^2-b^2)^2 = c^2(2b^2-c^2)$.

(ii) $\sqrt{3} : 2 : 1$; $a^2 = b^2-c^2$.

(iii) $\frac{1}{2}\sqrt{2}(\sqrt{7}+1)[\triangleq 2{\cdot}578] : 2 : 1$; $(b+c)^2>a^2>b^2-c^2$; $(a^2-b^2)^2 = c^2(2b^2-c^2)$.

(iv) $3 : 2 : 1$; $a = b+c$.

(v) $3{\cdot}2 : 2 : 1$; $(b+c)^3/b>a^2>(b+c)^2$.

(vi) $\frac{3}{2}\sqrt{6}[\triangleq 3{\cdot}674] : 2 : 1$; $a^2 = (b+c)^3/b$.

(vii) $4{\cdot}4 : 2 : 1$; $a^2>(b+c)^3/b$.

[If all these are to be constructed on one sheet of paper, it is quickest to mark all the required values of a on the two connecting rods at the beginning and to construct the curves in the reverse order. As parts of the strips become superfluous, they can then be cut off.]

77. $b<c$. Proceed as in No. 76. If the curves are drawn on separate sheets of paper, the isolated points on the x-axis should also be marked in.

(i) $0{\cdot}149 : 1 : 1{\cdot}1$; $c^2-b^2>a^2>(c-b)^2$; $(a^2-b^2)^2 = c^2(2b^2-c^2)$. If b is taken as 5 in., the lemniscate is about an inch high.

(ii) $1 : 1 : \sqrt{2}$; $a^2 = c^2-b^2$; $(a^2-b^2)^2 = c^2(2b^2-c^2)$.

(iii) $1{\cdot}407 : 1 : 1{\cdot}1$; $(b+c)^2>a^2>c^2-b^2$; $(a^2-b^2)^2 = c^2(2b^2-c^2)$.

(iv) $2{\cdot}1 : 1 : 1{\cdot}1$, $16 : 7 : 9$, $3 : 1 : 2$; $a = b+c$. In the second of these cases it can be shown that the line $y = a/\sqrt{3}$ meets the curve in two sets of three coincident points.

(v) $2{\cdot}6 : 1 : 1{\cdot}1$; $(b+c)^3/b>a^2>(b+c)^2$.

(vi) $3{\cdot}043 : 1 : 1{\cdot}1$; $a^2 = (b+c)^3/b$.

(vii) $4 : 1 : 1{\cdot}1$; $a^2>(b+c)^3/b$.

(viii) $7{\cdot}598 : 1 : 2{\cdot}864$; $a^2 = (b+c)^3/b$; $(a^2-b^2-6bc-3c^2)^2 = 4bc(3b^2+2c^2)$. Values of a somewhat less and greater than this can also be taken for comparison with (v), (vi), (vii).

12.5. Derived Curves

In 12.3 we defined a limaçon in terms of a circle, a point on it, and a length: given a fixed point O and a variable point M on a circle, a limaçon can be defined as the locus of the points on OM at a fixed distance from M. The method by which the circle is used to obtain the limaçon is a simple one, and it is natural to enquire what curves will result if we carry out the same operation on a curve other than a circle, and what properties of the original curve will still be found in the derived curve. Further, the same question can be asked about other operations by which one curve is derived from another. Three such ways of derivation will be considered, chosen less for their intrinsic importance than for the light they throw on the relationship between some of the curves we have already met. There are other transformations, including a more important one which needs to be

treated at greater length and can only be mentioned here. We saw in 6.1 that, if lines are drawn from a suitable point in space to all the points of a circle, they meet any other plane in a circle, ellipse, or hyperbola. Each of these is known as a **conical projection** of the original circle. This same transformation can turn any non-degenerate conic section into any other, leaving important properties of the original figure unaltered; any cubic curve can be projected into another having symmetry about a point; and by projection many other interesting and elegant results can be obtained.

Pedal

The locus of the foot of the perpendicular from a fixed point O to a variable tangent to a given curve (the *base-curve*) is known as the **pedal** of the base-curve with respect to O. Pedal curves were first investigated by Maclaurin.

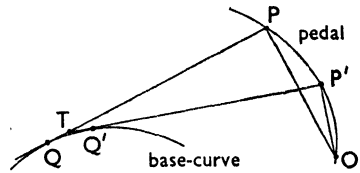

Fig. 106

If the perpendiculars from O to the tangents at Q, Q' to the base-curve meet them at P, P', and if the two tangents intersect at T, then T, P, P', O are concyclic and hence $\angle OPP' = \angle OTP'$. Now let Q' tend to Q along the curve, so that P' tends to P along the pedal, and the limiting position of PP' is the tangent at P to the pedal. Then

(i) **the angle between OQ and the tangent at Q to the base-curve is equal to the angle between OP and the tangent at P to the pedal;**

(ii) the tangent at P to the pedal is also the tangent at P to the circle on OQ as diameter hence **the circle on OQ as diameter touches the pedal at P, and the pedal is the envelope of such circles.**

78. What are the pedals of a parabola, ellipse, and hyperbola with respect to a focus?

79. Show that Diocles' cissoid, the strophoid, and Maclaurin's trisectrix (see 12.2) are the pedals of a parabola with respect to its vertex, the point of intersection of its axis and directrix, and a point on the axis three times as far from the vertex.

80. Show that the pedal of a circle with respect to a point O is a limaçon (see 12.3) unless O is on the circle, when it is a cardioid.

If a point Q on the circle gives rise to a point P on the pedal, show that the normal at P passes through the mid-point of OQ; with the notation of Nos. 25–29, locate the centre of the circle and hence prove No. 26 (iv).

81. Show that Booth's lemniscate (see 12.3) is the pedal of an ellipse or hyperbola with respect to its centre, and, in particular, that Bernoulli's lemniscate is the pedal of a rectangular hyperbola.

82. Construct a limaçon, using No. 80 and property (ii) above, as follows. As Q moves round the circle, the mid-point of OQ moves round a circle of half the radius with its centre at half the distance from O. Draw *this* circle, not the original one, and then draw a large number of circles having their centres on this circle and passing through O. The limaçon is the envelope of all these circles, and an approximation to it will be visible without further drawing. A cardioid may be similarly constructed.

83. Repeat No. 82, starting with a rectangular hyperbola, and obtain Bernoulli's lemniscate.

Cissoid

If a variable line through a fixed point O cuts two given curves at Q, R respectively, and a point P is taken on the line so that the directed lengths OP, RQ are equal, the locus of P is called the **cissoid**

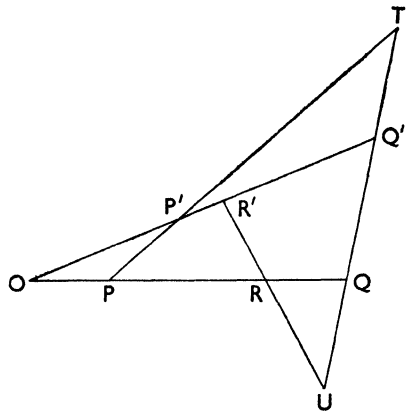

Fig. 107

of the two given curves for the pole O. Thus Diocles' cissoid is the cissoid of a line and a circle touching it, the pole being the other end of the diameter through the point of contact.

Notice that the two given curves must be mentioned in a particular order, for if the order is reversed the new cissoid is the reflection of the old one in the pole. The cissoid is most easily treated in polar coordinates: if $OQ = r_1$, $OR = r_2$, $OP = r$, then $r = r_1 - r_2$, the values of θ for the three points either being the same or differing by multiples of 2π.

Let P, R, Q be defined as above, and let P', R', Q' be new positions of these points; then if PP' and QQ' meet at T, QQ' and RR' at U, we shall show that $TQ' = QU$.

For, applying Menelaus' Theorem to the triangle OQQ' and the two transversals $PP'T$ and $R'RU$, we have

$$\frac{OP.QT.Q'P'}{PQ.TQ'.P'O} = -1 = \frac{RQ.UQ'.R'O}{OR.QU.Q'R'}.$$

But $OP = RQ$, $Q'P' = R'O$, $PQ = OR$, and $P'O = Q'R'$,

$$\therefore \frac{QT}{TQ'} = \frac{UQ'}{QU}, \quad \therefore \frac{QT}{TQ'} + 1 = \frac{UQ'}{QU} + 1, \quad \therefore \frac{QQ'}{TQ'} = \frac{QQ'}{QU},$$

$$\therefore \quad TQ' = QU.$$

Now let Q' tend to Q along the first given curve, and we have the following result; **let tangents be drawn to the two given curves and the cissoid at Q, R, P respectively, let the first of these meet the third at T and the second at U, then $TQ = QU$.**

84. Prove the property given above by using $r = r_1 - r_2$, and $r \cot \phi = \frac{dr}{d\theta}$.

Take particular care of signs.

85. Use this property to prove No. 8 (ii).

86. Two perpendicular diameters of a circle are drawn. Show that the cissoid of one diameter and the circle, the pole being one end of the other diameter, is a strophoid (see 12.2). Show further that, if the first diameter is replaced by a parallel line midway between it and the pole, the new cissoid is Maclaurin's trisectrix.

87. With the notation of 12.2, show that a strophoid is the cissoid of its asymptote and itself, the pole being A. [Use No. 12 (ii).]

Hence prove No. 12 (vi).

88. With the notation of 12.2, show that Agnesi's witch is the cissoid of its asymptote and itself, the pole being B. [Use No. 18 (i).]

89. If the second given curve is a circle with the pole as centre, the cissoid is called a **conchoid** of the first given curve. Nicomedes' curve (see 12.3) is thus the conchoid of a straight line.

(i) If the tangents at the corresponding points Q, P if the given curve and its conchoid meet at T, and V is the foot of the perpendicular from T to OQP, show that $OQ = PV$.

(ii) Hence prove No. 20.

(iii) With the notation of Fig. 97 of 12.3, if the tangent at M to the circle and the tangent at P to the limaçon meet at T, and TV is the perpendicular from T to OP, show that $OM = PV$. Hence, by considering the intersections of VT and ON with the circle on TN as diameter, prove No. 26 (iv).

Inverse

If O is fixed and P, Q are points collinear with O, such that $OP.OQ = k^2$, k being constant, the locus of P and the locus of Q are said to be **inverses** of each other with respect to a circle with

centre O and radius k, or, more simply, inverses with respect to O. If the polar equation of the locus of Q is $r = f(\theta)$, that of the locus of P is $r = k^2/f(\theta)$.

The chief property of inverse curves is that the tangents to them at P and Q make supplementary angles with the radius vector OPQ. For, if P' and Q' are new positions of P and Q, $OP.OQ = OP'.OQ'$; hence P, Q, Q', P' are concyclic; hence $\angle OPP' = \angle OQ'Q$, and the result follows as Q' tends to Q.

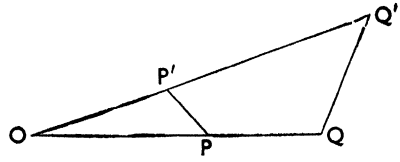

Fig. 108

90. Show that (i) if the cartesian equation of a curve is given, the equation of the inverse with respect to O can be obtained by substituting $k^2x/(x^2+y^2)$ for x and $k^2y/(x^2+y^2)$ for y.

(ii) the inverse of the circle $x^2+y^2+2gx+2fy+c = 0$ is

$$k^4+2k^2(gx+fy)+c(x^2+y^2) = 0,$$

and that this represents a straight line or a circle according as the original circle does or does not pass through O.

91. Show that the inverse of a parabola with respect to its vertex is Diocles' cissoid.

92. Show that the inverse of a rectangular hyperbola with respect to a vertex is a strophoid.

93. Use No. 12 (iii) to show that a strophoid is its own inverse with respect to its vertex A; hence deduce No. 12 (v).

94. (i) Show that the inverse of a conic with respect to a focus is a limaçon and that, with the notation of 12.3, the cases $b>2a$, $b = 2a$, $b<2a$ occur when the conic is an ellipse, parabola, and hyperbola respectively.

(ii) Show that the second focus inverts into the point S of No. 25, and deduce that the angles which the tangents at P to the limaçon and the circle OPS make with OP, are in the ratio 1 : 2. Hence prove No. 28.

95. Show that the bipolar equation of a limaçon, given in No. 25, can be written $\{4ar_2-(4a^2-b^2)\}^2 = 4b^2r_1^2$, and hence, by applying the cosine formula to triangle OSP, that the polar equation of the curve referred to S as pole and SA as initial line is

$$16a^2r^2-8ar(4a^2+b^2\cos\theta)+(4a^2-b^2) = 0.$$

Deduce that the limaçon is its own inverse with respect to the circle having S as centre and passing through O.

96. Use No. 44 to show that, using a suitable radius, a cartesian oval is its own inverse with respect to a focus.

97. Show (i) that the inverse of a conic with respect to its centre is Booth's lemniscate;

(ii) that cases (i), (ii), (iii) of No. 39 occur respectively when the conic is a hyperbola, pair of parallel lines, and ellipse;

(iii) that Bernoulli's lemniscate is the inverse of a rectangular hyperbola.

(iv) Use the property of concyclic points on a rectangular hyperbola (see 9d, No. 21) to show that if a straight line or circle cuts Bernoulli's lemniscate in four points P, Q, R, S, and if T is the other end of the diameter through P, then the circle through the centre O and any two of the points Q, R, S, T cuts orthogonally the circle through O and the other two points.

ANSWERS

The following abbreviations are used throughout:
C = circle, E = ellipse, H = hyperbola, LP = line-pair, P = parabola, Pt = point, RH = rectangular hyperbola; C = centre, V = vertex, r = radius.

PART I

Exercise 1A, *p.* 15.

1. (*b*) (4, −1); (*c*) (i) *EBF*; (ii) *CF*, *ED*; (iii) *BD* perp. to *CF* and *ED*, *CD* perp. to *EBF*, *BC* perp. to *DF*.
2. (*b*) (15, 16); (*c*) (i) *STR*; (ii) *PSUQ*; (iii) *PSVR*, *QUVR*.

Exercise 2A, *p.* 18.

1. i perp. to ii. **2.** i//ii. **3.** i//ii. **4.** ii perp. to iii.
5. All go through (3, 4). **6.** All go through (12, −10).
7. $y = 0$, $x = 0$. **8.** $x-y = 0$, $x+y = 0$.

Exercise 2B, *p.* 19.

1. A, C, D; 4; (16, 10); they meet at (16, 10). **2.** E, H; 2; they meet at (4, −2).
3. $k = h^2$. **4.** $h+k+1 = 0$. **5.** $hk = 6$. **6.** $k = 2h^2$. **7.** $3h+4k = 0$.
8. $k^2 = 5h$. **9.** $k = 2h^2$. **10.** $hk = 3$. **11.** 3, 6. **12.** 9.

Exercise 2C, *p.* 23.

In Nos. 1–22 the equation of the image in Ox is given first. The letters x, y, O after the equations mean that the curve is symmetrical about Ox, Oy, O.

1. $y = -x$, $y = -x$; O.
2. $x = 0$, $x = 0$; x, y, O.
3. $y = -x^2$, $y = x^2$; y.
4. $y = -x^4$, $y = x^4$; y.
5. $y = -x^3$, $y = -x^3$; O.
6. $y^2 = x^3$, $y^2 = -x^3$; x.
7. $y^2 = x$, $y^2 = -x$; x.
8. $y^3 = -x$, $y^3 = -x$; O.
9. $y^2 = x^2$, $y^2 = x^2$; x, y, O.
10. $y^4 = x^4$, $y^4 = x^4$; x, y, O.
11. $y = -(x+1)^2$, $y = (x-1)^2$.
12. $y^2 = (x+1)^2$, $y^2 = (x-1)^2$; x.
13. $y = -x^2-1$, $y = x^2+1$; y.
14. $y^2 = x-1$, $y^2 = -x-1$; x.
15. $y = -1/x$, $y = -1/x$; O.
16. $y = -1/x^3$, $y = -1/x^3$; O.
17. $y = -1/x^2$, $y = 1/x^2$; y.
18. $y^2 = 1/x$, $y^2 = -1/x$; x.
19. $y^2 = x^2+1$, $y^2 = x^2+1$; x, y, O.
20. $y^2 = 4-x^2$, $y^2 = 4-x^2$; x, y, O.
21. $y = \pm\sqrt{(1+x)}$, $y = \pm\sqrt{(1-x)}$; x.
22. $y = \pm\sqrt{x}$, $y = \pm\sqrt{(-x)}$; x.

In Nos. 23–32 the letters A, B mean that the curve is symmetrical about $x = y$ and $x+y = 0$ respectively.

23. A. **24.** B. **25.** Neither. **26.** Neither. **27.** B. **28.** B.
29. AB. **30.** AB. **31.** AB. **32.** Neither.

EXERCISE 2D, *p.* 30.

The letter O means that the curve passes through the origin.

1. RH $xy = 6$, $C(2, 3)$, O. **2.** RH $xy+6 = 0$, $C(0, -1)$.
3. P $y^2 = x$, $V(2, 1)$. **4.** P $y = 2x^2$, $V(-3, -10)$.
5. P $y^2+4x = 0$, $V(4, 4)$, O. **6.** P $y^2 = x$, $V(-9, -3)$, O.
7. RH $xy+8 = 0$, $C(3, -1)$. **8.** RH $xy+18 = 0$, $C(-6, 3)$, O.
9. P $y = x^2$, $V(3, 4)$; 4. **10.** RH $xy = 8$, $C(-2, 1)$; between 1 and 5.

EXERCISE 3A, *p.* 34.

1. $OP = 4\sqrt{17}$, $OQ = 10$, $OR = \sqrt{41}$, $OS = \sqrt{17}$, $PQ = 2\sqrt{29}$, $PR = 21$, $PS = 17$, $QR = \sqrt{137}$, $QS = 13$, $RS = 10$. **2.** $AB = \sqrt{178}$, $AC = 15$, $AD = 13$, $AO = 10$, $BC = \sqrt{241}$, $BD = 17$, $BO = \sqrt{106}$, $CD = 4$, $CO = \sqrt{37}$, $DO = 3\sqrt{5}$. **15.** Kite. **16.** Isosceles trapezium. **17.** O; O; R; R.
18. A; R; O; A. **19.** 5. **20.** 13.
21. $(1\pm19, 1\pm17)$, $(1\pm17, 1\pm19)$, $(1\pm25, 1\pm5)$, $(1\pm5, 1\pm25)$, $(1\pm23, 1\pm11)$, $(1\pm11, 1\pm23)$.
22. $(3\pm17, \pm6)$, $(3\pm6, \pm17)$, $(3\pm15, \pm10)$, $(3\pm10, \pm15)$, $(3\pm18, \pm1)$, $(3\pm1, \pm18)$.
23. Perp. bisector of AB. **24.** Perp. bisector of PQ.

EXERCISE 3B, *p.* 37.

5. $(-7, 12)$. **6.** $(-3, -14)$. **7.** $(-8, 2)$. **8.** $(10, -8)$.
9. Vertex Q of parallelogram OP_1QP_2.

EXERCISE 3C, *p.* 42.

1. $3x-2y+7 = 0$. **2.** $3x-y = 0$. **3.** $2x+3y = 0$. **4.** $y = 3$.
5. $x-5y = 11$. **6.** $2y = 3$. **7.** $x-5y = 13$. **8.** $y = 0$.
9. True for all points, by A's theorem.
10. For all points, $PA^2+PB^2 = 2PC^2+50$, by A's theorem.
11. Part of RH $xy-2x-2y = 4$, like $xy = 8$, $C(2, 2)$.
12. Part of RH $xy-4x = 8$, like $xy = 8$, $C(0, 4)$.
13. Part of P $y^2-4x+4 = 0$, like $y^2 = 4x$, $V(1, 0)$.
14. Part of P $x^2-8y+16 = 0$, like $8y = x^2$, $V(0, 2)$. **15.** RH $xy+8 = 0$.
16. Branch of RH $xy = 2$.
17. Part of RH $xy-6x-6y+18 = 0$, like $xy = 18$, $C(6, 6)$.
18. As No. 16. **19.** $x^2+y^2 = 25$; C, centre O.
20. $x^2+y^2 = 6x$; C, centre $(3, 0)$.
21. As No. 19; 100; right angle; circle on AB as diam.
24. As No. 20; 3. **25.** P $y^2-4x = 4$, like $y^2 = 4x$, $V(-1, 0)$.
26. $9x^2 = 16y^2$; the lines $3x = \pm4y$. **27.** $y = 3$. **28.** $x = 1$.
29. $y = 12$. **30.** $x = 4$. **31.** $3x+4y = 12$. **32.** $x-2y = 0$.
33. P $y = 2(x-3)^2$, like $y = 2x^2$, $V(3, 0)$. **34.** RH $xy = 1$. **35.** RH $xy = 24$.
36. P $y^2 = 4x$. **39.** RH with the fixed lines as asymptotes.
40. P with vertex A and axis AB.

EXERCISE 4A, *p.* 49.

1. (i) $x^2+y^2=9$; (ii) $x^2+y^2+2x+4y=4$; (iii) $x^2+y^2-6x-6y+9=0$; (iv) $x^2+y^2-6x+8y=0$. **2.** (i) $x^2+y^2=4$; (ii) $x^2+y^2-10x+4y+25=0$; (iii) $x^2+y^2+2x-2y=14$; (iv) $x^2+y^2+8x=0$. **3.** (i) (0, 0), 4; (ii) (−2, 5), 1; (iii) (−4, 0), 4; (iv) (5, −12), 13. **4.** (i) (0, 0), 3; (ii) (−8, 15), 17; (iii) (−1, 5), 6; (iv) (6, 0), 4. **5.** (i) $4x^2+4y^2=1$; (ii) $2x^2+2y^2-6x-10y=9$; (iii) $3x^2+3y^2-3x+4y=0$; (iv) $x^2+y^2-3x-2y+1=0$.
6. (i) $9x^2+9y^2=4$; (ii) $x^2+y^2-4x+5y+8=0$; (iii) $2x^2+2y^2+5x-12y=0$; (iv) $5x^2+5y^2-2x-4y=4$.
7. (i) (1, 2), 3; (ii) $(-\frac{5}{2}, 6)$, $6\frac{1}{2}$; (iii) $(\frac{3}{2}, 2)$, 3; (iv) $(\frac{2}{5}, -\frac{4}{5})$, 1.
8. (i) (−2, −3), 5; (ii) $(\frac{1}{2}, -1)$, $\frac{3}{2}$; (iii) $(-\frac{2}{3}, 1)$, 1; (iv) $(\frac{1}{2}, \frac{1}{5})$, $\frac{7}{10}$.
9. 2, (1, 2), 1. **10.** 3, $(-\frac{4}{7}, \frac{3}{7})$, $\frac{2}{7}$. **11.** (i) Ext., (ii) no, (iii) int., (iv) ext.
12. (i) nearly ext., (ii) int., (iii) ext., (iv) nearly int.
13. $x^2+y^2-2x+4y=$ (*a*) 20, (*b*) 44, (*c*) 4.
14. $x^2+y^2+4x-4y=113$, $x^2+y^2+4x-4y+7=0$; both internal.
15. Out, on, in, on. *A*, *B*, *C*, *D* lie in that order on chord *BD*.
16. On, out, in, on. **17.** $\frac{1}{2}$, $\frac{3}{2}$. **18.** (0, 6), (0, −4).

EXERCISE 4B, *p.* 52.

1. $x^2+y^2-x-7y=0$. **2.** $x^2+y^2-8y=9$. **3.** $(8\frac{1}{2}, 5)$. **4.** (9, 7).
5. Different constant terms; $x^2+y^2-2x+3y=89$.
6. Different coefficients of y; $x^2+y^2-10x-9y+17=0$.
7. $x^2+y^2-6x-8y=0$. **8.** $x^2+y^2-8x-6y+9=0$.
9. $2x^2+2y^2-18x-15y+28=0$. **10.** $x^2+y^2-5x-7y+6=0$.
11. $x^2+y^2-25x=100$; $(0, \pm 10)$.
12. $x^2+y^2-25x+25y+100=0$; (5, 0), (20, 0).
13. y_1; $x^2+y^2-2x_1x-2y_1y+x_1^2=0$; $x^2+y^2-8x-4y+16=0$; $x^2+y^2+8x-20y+16=0$.
14. x_1; $x^2+y^2-2x_1x-2y_1y+y_1^2=0$; $x^2+y^2-20x-20y+100=0$; $x^2+y^2-40x+20y+100=0$.

EXERCISE 4C, *p.* 55.

1. $(\pm 5, 0)$, $(0, \pm 3)$. **2.** $(0, \pm 6)$, $(\pm 4, 0)$. **3.** $(\pm 4, 0)$, $(0, \pm 2)$.
4. $(0, \pm 10)$, $(\pm 4, 0)$. **5.** $(0, \pm\frac{1}{2})$, $(\pm\frac{1}{3}, 0)$. **6.** $(\pm 1, 0)$, $(0, \pm\frac{1}{3})$.
7. (2, −1); 2, 1; *Ox*. **8.** (2, 1); $\frac{5}{2}$, $\frac{5}{3}$; *Ox*. **9.** $(-\frac{1}{2}, 2)$; 2, $1/\sqrt{2}$; *Oy*.
10. $(\frac{1}{3}, -\frac{3}{2})$; $\sqrt{2}$, $\frac{2}{3}$; *Oy*. **11.** $x^2+2y^2=18$. **12.** $3x^2+2y^2=36$.
13. $36x^2+4y^2=9$. **14.** $9x^2+16y^2=4$. **15.** $9x^2+16y^2-18x+64y=71$.
16. $9x^2+4y^2+54x-8y+49=0$. **17.** $36x^2+64y^2-36x-64y=119$.
18. $81x^2+64y^2+54x-64y=119$.

EXERCISE 4D, *p.* 57.

The coordinates are those of the centre, and are followed by the radius or the semi-axes, the first semi-axis being parallel to *Ox*.

1. $x^2+y^2-4x+2y+1=0$; (2, −1), 2. **2.** $x^2+y^2-4x=12$; (2, 0), 4.
3. $x^2+y^2=25$; (0, 0), 5. **4.** $x^2+y^2=9$; (0, 0), 3.

5. $9x^2+25y^2 = 3600$; (0, 0), 20, 12. **6.** $25x^2+16y^2 = 3600$; (0, 0), 12, 15.
7. $x^2+y^2-26x+144 = 0$; (13, 0), 5. **8.** $x^2+y^2-10y+9 = 0$; (0, 5), 4.
9. $x^2+y^2+10x-4y = 92$; (−5, 2), 11.
10. $x^2+y^2-32x-10y+56 = 0$; (16, 5), 15.
11. $3x^2+4y^2 = 16$; (0, 0), $4/\sqrt{3}$, 2. **12.** $x^2+4y^2 = 4$; (0, 0), 2, 1.
13. $9x^2+25y^2 = 225$; (0, 0), 5, 3. **14.** $64x^2+289y^2 = 18{,}496$; (0, 0), 17, 8.
15. $x^2+y^2-8x-6y = 75$; (4, 3), 10. **16.** $x^2+y^2 = 225$; (0, 0), 15.
17. $x^2+y^2+12x+12y+16 = 0$; (−6, −6), $2\sqrt{14}$.
18. $x^2+y^2-24x-24y+20 = 0$; (12, 12), $2\sqrt{67}$.
19. Line $9x+9y = 1$. **20.** $\alpha+\beta = \gamma$. **21.** $x^2+y^2 = 0$; (0, 0) only.
22. $5(x-\frac{2}{5})^2+5y^2+19\frac{1}{5}$ cannot be zero. **23.** $x^2+9y^2 = 9$; (0, 0), 3, 1.
24. $x^2+9y^2 = 36$; (0, 0), 6, 2. **25.** $4x^2+9y^2 = 36$; (0, 0), 3, 2.
26. $x^2+y^2-10x-4y+20 = 0$; (5, 2), 3. **27.** Circle with mid-point as centre.
28. Circle, centre on line joining points. **30.** Circle, centre on join of centres.
32. Ellipse.

EXERCISE 4M, *p.* 59.

To save space, the equations of loci are so written that the curves can easily be traced.

2. 0·55, 5·45. **3.** $x^2+y^2-10x-10y+25 = 0$, $x^2+y^2-34x-34y+289 = 0$; (9, 2). **4.** Line $6x = 13$. **5.** P $y^2 = 4(x+1)$ or $y^2+4(x-1) = 0$. **6.** $[0, \pm\sqrt{(b^2-a^2)}]$; $2\{b+\sqrt{(b^2-a^2)}\}$.
8. Equidistant from O and (−1, 3). **9.** (8, 9). **10.** C $x^2+y^2 = \frac{1}{4}c^2$.
11. P $y^2+12(x-3\frac{1}{3}) = 0$. **12.** $25x^2+16y^2-150x-96y = 31$.
13. Both. **15.** P $y^2 = 8(x-2)$. **16.** Line $x+y-c = 0$.
17. Circle, centre at centre of rectangle. **20.** $2x^2+2y^2-9x-6y+12 = 0$.
21. RH $xy = \frac{1}{4}c^2$. **22.** $x^2+y^2-(a+b)x\pm(a-b)y+\frac{1}{4}(a+b)^2 = 0$.
24. $3x^2+4y^2-48x+144 = 0$; (4, 0), (12, 0), (6, ±3), (10, ±3).
25. (±2, ±3), (±1, ±4), (±5, ±2).
27. Circle, centre the centre of the square. **29.** Parabola.
30. Ellipse, centre A, axes bisecting BAC.
32. $x^2+y^2\pm6x-8y+24 = 0$, $x^2+y^2\pm2x = 24$.
33. (±3, ±4), (±4, ±3); (±3, ±2·4), (±4, ±1·8).
34. Line $3x-2y = 18$; all go through (6, 0) and $(2\frac{4}{13}, 5\frac{7}{13})$. **35.** Circle.
36. Rect. hyp., centre A, asymptotes bisecting BAC.

EXERCISE 5A, *p.* 68.

1. (i) −2; (ii) $\frac{3}{4}$; (iii) y_1/x_1. **2.** (i) $-\frac{3}{2}$; (ii) 0; (iii) $-b/a$.
3. (i) $x-y+2 = 0$; (ii) $x-6y-7 = 0$; (iii) $x+3 = 0$.
4. (i) $x+3y-19 = 0$; (ii) $8x-3y-28 = 0$; (iii) $y+1 = 0$.
5. $3x-y-6 = 0$, $x-y+2 = 0$, $x+2y+5 = 0$; 3, 7, $\frac{1}{2}$.
6. $2x+y+3 = 0$, $x+3y+4 = 0$, $x+y = 0$; $\frac{1}{2}$, $\frac{1}{3}$, −1.
13. (i) $3x-2y-14 = 0$; (ii) $4x+5y-22 = 0$; (iii) $y-3 = 0$.
14. (i) $2x-y-14 = 0$; (ii) $x+3y+11 = 0$; (iii) $x-3 = 0$.
15. (i) $x^2+y^2-5x+7y = 0$; (ii) and (iii) $x^2+y^2-8x-2y = 33$.
16. (i) $x^2+y^2 = p^2+q^2$; (ii) $x^2+y^2-(p-q)x-(p+q)y = 0$;
(iii) $x^2+y^2-9x-2y = 0$.

17. $3x+2y-32=0$, $2x-y-5=0$, $x+3y-27=0$; (6, 7).
18. $y-1=0$, $x+1=0$, $x+3y-2=0$; (−1, 1).
19. (i) $2x-3y+6=0$; (ii) $x+2=0$. **20.** (i) $5x+7y-6=0$; (ii) $y-3=0$.
21. $x+3y+1=0$, $7x+y-13=0$, $x-2y-4=0$; (2, −1).
22. $y+4=0$, $x-y-9=0$, $17x-7y-113=0$; (5, −4).
23. $4x-y-14=0$, $3x+2y-16=0$. **24.** $x-y-3=0$, $x+y-3=0$.
27. $3x-2y-12=0$, $x+2y-12=0$, $x+y-9=0$; (6, 3).
28. $2x-y+2=0$, $2x+5y+14=0$, $x+y+4=0$; (−2, −2). **29.** $xy+6=0$.
30. $xy=9$. **33.** E $x^2/16+y^2/4=1$. **34.** E $x^2/4+y^2/9=1$.
35. RH $xy=6$. **36.** RH $(x-5)(y-4)+20=0$.
42. (p sec α, 0), (0, p cosec α). **43.** (ii) γ, γ'; (iii) $\gamma\gamma'=-1$.
44. $1+\gamma^2$, $1+\gamma'^2$, $(\gamma-\gamma')^2$; $\gamma\gamma'=-1$.

EXERCISE 5B, *p.* 77.

1. $-\frac{1}{2}$, $\frac{8}{7}$, $\frac{3}{4}$, 2, 0. **2.** $\frac{2}{3}$, -3, $\frac{1}{3}$, -3, 0. **3.** (i) $2x-3y+9=0$;
(ii) $x-3=0$; (iii) $y-5=0$. **4.** (i) $10x+2y+23=0$; (ii) $y-1=0$;
(iii) $2x+5=0$. **5.** (i) $5x+2y+3=0$; (ii) $y+4=0$; (iii) $x-1=0$.
6. (i) $9x-3y+4=0$; (ii) $x=0$; (iii) $3y-4=0$. **7.** $tx+y-at(t^2+2)=0$.
8. $t^3x-ty-c(t^4-1)=0$. **9.** (5, −1), (5, −6). **10.** (−27, 7), (2, 7).
11. (i) 36°52′; (ii) 60°. **12.** (i) 135°; (ii) 150°. **15.** 3, 2. **16.** −3, 5.
19. $-\frac{2}{3}$. **20.** −4. **21.** $7x+9y=0$. **22.** $x-9y+17=0$.
23. $6x+5y-5=0$. **24.** $2x-y-21=0$. **25.** (−2, 1). **26.** (2, −1).
27. $(\frac{2}{3}, -\frac{4}{3})$. **28.** (4, 2). **29.** (1, 2), $(1-4\lambda, 2-3\lambda)$.
30. (4, 2), $(4-10/\lambda, 2+24/\lambda)$. **31.** $(\gamma-3)/(1-2\gamma)$. **32.** $(6+2a)/(2-a)$.
33. (3, −4). **34.** (−2, 3). **35.** (±1, −4). **36.** (0, 4). **37.** (a, 0).
38. (1, 2). **39.** No. **40.** $x+8y-7=0$. **41.** The common chord.
42. $x^2+y^2-x-y-8=0$.

EXERCISE 5M, *p.* 78.

1. $lx\pm my\pm n=0$. **2.** 30°, 90°, 120°. **3.** C $x^2+y^2=c^2$.
4. $28x-29y=0$. **5.** $5x+y=0$, $x-5y=0$. **6.** (st, st).
7. $lx+my-n=0$, $mx+ly\pm n=0$. **11.** $x+(2-\sqrt{3})y=4(3-\sqrt{3})$.
13. Half parabola, vertex A, axis AB. **14.** Common chord. **15.** 1, −1.
16. (9, 4), (6, 0); (1, 10), (−2, 6). **17.** $[c, \frac{1}{2}(c^2+1)]$.
20. $(-a, -b)$; $(1, \pm\sqrt{3})$. **21.** $\frac{7}{2}$, $-\frac{1}{2}$. **22.** (6, 3), (3, 10).
23. Circle on AB as diameter.
25. Change both signs and exchange coords. *ad lib.*; $(\pm 10, \pm 10)$, $(\pm 5, \mp 5)$.
27. $x^2+y^2-6x-5y+9=0$. **31.** For $2x^3y^4$ read x^2y^4.
33. Part of parabola $x^2+4(y-1)=0$. Circles touch Ox and other semicircle.

EXERCISE 6A, *p.* 89.

1. (i) $7x^2+16y^2-2400x=0$, $(171\frac{3}{7}, 0)$; (ii) $y^2=144x$;
(iii) $56x^2-169y^2+21{,}840x=0$, (−195, 0).
2. (i) $24x^2+25y^2-240x=0$, (5, 0); (ii) $y^2=12x$;
(iii) $24x^2-25y^2+240x=0$, (−5, 0). **4.** >45°; 45°. **5.** $y^2=4x^2$; in the two generators $y=\pm 2x$. **6.** $x^2+2y^2=0$; a point at the origin.

11. $b^2x^2+a^2y^2-2ab^2x = 0$; P $y^2 = 4cx$.
12. $b^2x^2-a^2y^2+2ab^2x = 0$; P $y^2 = 4cx$. **15.** at; $\frac{1}{2}a(t+1/t)$; it is an asymptote.
21. H $x^2/16-y^2/9 = 1$. **22.** H $x^2/16-y^2/9+1 = 0$.
23. One branch of H of No. 21; the other branch. **24.** H $x^2/9-y^2/16 = 1$.
26. RH with vertices at ends of perpendicular diameter.

Exercise 6B, *p.* 93.

In Nos. 1–14, 17–20, 23–24, the equation is that of the curve referred to its centre or vertex.

1. RH $2xy = 5$, $C(-1, \frac{1}{2})$. **2.** C, $r = 3$, $C(0, 1)$. **3.** P $x^2 = 2y$, $V(-2, 3)$.
4. E $4x^2+y^2 = 25$, $C(-\frac{1}{2}, 0)$. **5.** PtC at $(-1, 3)$.
6. LP $x = \pm y$, $C(-\frac{3}{2}, -\frac{3}{2})$. **7.** P $y^2 = x$, $V(-\frac{9}{4}, -\frac{3}{2})$.
8. RH $xy = 8$, $C(-2, 0)$. **9.** C, $r = \frac{5}{2}$, $C(\frac{3}{2}, -2)$. **10.** P $4x^2 = y$, $V(\frac{5}{2}, -25)$. **11.** RH $x^2-y^2 = 9$, $C(5, -4)$. **12.** PtC at $(5, 4)$.
13. E $x^2+2y^2 = 18$, $C(0, 3)$. **14.** P $y^2+x = 0$, $V(9, -1)$.
15. Lines $y = 6$, $y = -1$. **16.** Lines $x = 1$, $y = -2$.
17. LP $2x = \pm y$, $C(-\frac{1}{2}, 1)$. **18.** PtE at $(-\frac{1}{2}, -1)$.
19. H $x^2-4y^2+4 = 0$, $C(1, -1)$. **20.** E $x^2+4y^2 = 25$, $C(-3, 2)$.
21. Lines $x = -2$, $y = 3$. **22.** Line $x = 2$ twice. **23.** E $9x^2+16y^2 = 25$, $C(1, 1)$. **24.** H $9x^2-25y^2+16 = 0$, $C(1, -1)$. **26.** P $x^2+4y = 0$.
27. LP, RH, P, C, E; all curves touch at $(0, 2)$ and go through $(\pm 4, 0)$.
28. E, C, P, RH, LP; all curves touch at $(0, -8)$ and go through $(\pm 12, 0)$.
29. C, E, P, H, LP, RH; all curves touch at $(0, \pm 3)$.
30. LP, P, E, C, E, P, H; all curves touch at $(-6, 0)$ and go through $(2, -4)$ and $(18, 12)$.

Exercise 6C, *p.* 101.

1. (i) $x+y+1 = 0$; (ii) $x-2y+9 = 0$; (iii) $x+2 = 0$.
2. (i) $x+1 = 0$; (ii) $2x-y-3 = 0$; (iii) $x-y-4 = 0$.
3. (i) $x-3y+6 = 0$; (ii) $8x+y+8 = 0$; (iii) $5x-2y = 4$; (iv) $3x+2y = 25$; (v) $5x-2y = 5$. **4.** (i) $8x+3y+48 = 0$; (ii) $19x-y = 90$; (iii) $9x-25y+60 = 0$; (iv) $2x-5y+17 = 0$; (v) $x-4y = 7$.
5. $x-2y = 0$; member of LP. **6.** $x+3 = 0$; member of LP.
7. LP crossing at $(2, -3)$. **8.** PtE at $(1, -2)$. **9.** $(4, 3)$, $(2, 1)$.
10. $(3, 3)$, $(3, -2)$. **11.** $(6, 9)$, $(0, 1)$. **12.** $(4, -2)$, $(-1, 0)$.
13. $x+5 = 0$, $4x-3y = 25$, $3x+4y = 25$; $(-5, 60)$, $(13, 9)$, $(19, -8)$.
17. $3x+2y = 12$. **18.** $x+1 = 0$, $x-4y+17 = 0$; $4x+y = 0$.
19. $3x+5y = 30$. **20.** $5x-8y = 9$.
21. $(t^2+1)x-(t^2-1)y=4t$. **23.** $16x+9y = 25$; -11, 23.
24. $x+4y = 4$; -5, 8. **25.** $3x+5y = 25$; $(75/17, 40/17)$.
28. $(ax_1+hy_1+g)x+(hx_1+by_1+f)y+(gx_1+fy_1+c) = 0$; the polar of P_1.
32. $(-x_1, 0)$. **34.** All the circles go through $(7, 1)$.
37. $(a^2-b^2)xy+b^2y_0x-a^2x_0y = 0$. **38.** $(a^2+b^2)xy-b^2y_0x-a^2x_0y = 0$.
39. $x^2-y^2-x_0x+y_0y = 0$. **40.** $xy+(2a-x_0)y-2ay_0 = 0$.

EXERCISE 6D, *p.* 108.

1. (i) (2, 1), (−1, 3); chord; (ii) (4, 3) twice; tangent; (iii) (1, −1); line // asymptote; (iv) line does not cut. **2.** (i) Line does not cut; (ii) (0, 1), (3, 2); chord; (iii) (1, −2) twice; tangent; (iv) (−1, 2); line // asymptote. **7.** $(at^2, 2at)$. **8.** $(ct, c/t)$. **9.** $[a(1-t^2)/(1+t^2), 2at/(1+t^2)]$. **10.** $[\frac{1}{2}a(t+1/t), \frac{1}{2}a(t-1/t)]$. **11.** $(a\cos\theta, b\sin\theta)$. **12.** $(a\sec\theta, b\tan\theta)$. **13.** 2; 2. **14.** $\frac{7}{2}, -\frac{1}{2}$. **15.** $(-\frac{1}{2}, 1)$. **16.** $(-\frac{8}{3}, \frac{3}{2})$. **17.** $3x-4y+25=0$, $4x+3y=25$; 90°. **18.** $x\pm y+2=0$. **19.** $x=0$; $2x+3y=0$ asymptote. **20.** (36/13, 15/13). **21.** ± 2; $x\pm 2y+4a=0$. **22.** $4x-2y+1=0$. **23.** $am^2=ln$. **24.** $4c^2lm=n^2$. **25.** $a^2(l^2+m^2)=n^2$. **26.** $a^2l^2-b^2m^2=n^2$. **29.** (−1, −3). **30.** (14, −5). **35.** C $x(x-h)+y(y-k)=0$. **36.** P $(y-\frac{1}{2}k)^2=2a(x-h+k^2/8a)$. **37.** H $(x-\frac{1}{2}h)^2/a^2-(y-\frac{1}{2}k)^2/b^2=\frac{1}{4}(h^2/a^2-k^2/b^2)$. **38.** RH $(x-\frac{1}{2}h)(y-\frac{1}{2}k)=\frac{1}{4}hk$. **39.** P $y^2=x$. **40.** P $4y^2=x$. **41.** a^2/b. **42.** a^2/b. **43.** $x+2y=0$; (2, −1). **44.** 1, 9; (−2, −2), (−6, −6).

EXERCISE 6M, *p.* 110.

1. $x^2+y^2-8x\pm 6y+16=0$, $x^2+y^2+16x\pm 30y+64=0$. **2.** $(-4\frac{1}{2}, -7\frac{1}{2})$, (6, −11), $(9\frac{1}{2}, -\frac{1}{2})$. **4.** RH $(x-2)(y-5)=10$. **5.** $(6\frac{1}{2}, 2)$; $4x-y+1=0$. **7.** $4\frac{4}{5}$. **8.** (8, −9), (8, 5), (−6, 5), (−6, −9); $x^2+y^2-2x+4y=44$. **9.** $(1, a+b+c+abc)$. **10.** RH $xy=c^2$. **11.** $(4a/\gamma^2, 4a/\gamma)$, $(4a\gamma^2, -4a\gamma)$. **12.** Branch of RH $x^2-y^2=a^2$; circle on LM as diameter. **14.** $6x+y=15$ or -22; $x-6y=-16$ or 21. **15.** C $x(x-c)+y(y-c)=0$. **17.** 90°, 34°42′. **19.** $25a/8$, $5a/2$. **20.** −5, −3, −1. **21.** $x^2+y^2=c^2$. **22.** (4, 7), (−4, 5), (−2, −3). **24.** $[0, ay_0/(a-x_0)]$, $[0, ay_0/(a+x_0)]$; $(0, -b^2/y_0)$. **26.** $x-y=0$, $x-3y+7=0$. **27.** Parallel and midway between it and their meet. **30.** RH $x^2-y^2=2a^2$. **31.** C $x(x-2)+y^2=0$. **32.** $4x+7y=12$. **33.** $bx=ay$, and AB itself. **37.** $x+2y+5a=0$. **38.** $(\frac{3}{2}, \frac{1}{2})$, $(\frac{10}{6}, \frac{4}{3})$, $(\frac{5}{2}, 1)$; $2x-4y=1$. **39.** $(l^2+m^2)(x^2+y^2-a^2)+2n(lx+my+n)=0$. **42.** AB and $x+y=a+b$. **43.** $x-y+5=0$; $x\pm y\pm 5=0$. **44.** $(-7\frac{1}{2}, -5\frac{1}{2})$; $(-7, -6\frac{1}{2})$. **46.** $y_0x+x_0y=x_0y_0$. **47.** RH $x^2-y^2=\frac{1}{4}(a^2-b^2)$. **48.** AB and $x=y$. **49.** C $x(x-1)+y^2=0$. **50.** $(r\cos\theta, r\sin\theta)$; $x\cos\alpha+y\sin\alpha=p$. **51.** $x^2-2(h+kt)x+h^2+k^2=0$. **52.** $-b^2x_0/a^2y_0$; $b^2x_0x+a^2y_0y=b^2x_0^2+a^2y_0^2$. **55.** $[k^2x_1/(x_1^2+y_1^2), k^2y_1/(x_1^2+y_1^2)]$. **57.** (i) $2x(x_1-x_2)+2y(y_1-y_2)=x_1^2-x_2^2+y_1^2-y_2^2$;

(ii) $\left(-\dfrac{(l^2-m^2)x_1+2lmy_1+2ln}{l^2+m^2}, -\dfrac{2lmx_1-(l^2-m^2)y_1+2mn}{l^2+m^2}\right)$.

60. Branch of H $3x^2-y^2+2cx=c^2$.

PART II

EXERCISE 7A, *p.* 119.

1. $x = 3t/(1+t^3)$, $y = 3t^2/(1+t^3)$;

t	-10	-2	$-1{\cdot}2$	-1	$-0{\cdot}8$	$-0{\cdot}5$	$-0{\cdot}1$	0	$0{\cdot}1$	$0{\cdot}5$	1	2	10
x	$0{\cdot}03$	$0{\cdot}86$	$4{\cdot}95$	$\pm\infty$	$-4{\cdot}92$	$-1{\cdot}71$	$-0{\cdot}30$	0	$0{\cdot}30$	$1{\cdot}33$	$1{\cdot}50$	$0{\cdot}67$	$0{\cdot}03$
y	$-0{\cdot}30$	$-1{\cdot}71$	$-5{\cdot}93$	$\mp\infty$	$3{\cdot}93$	$0{\cdot}86$	$0{\cdot}03$	0	$0{\cdot}03$	$0{\cdot}67$	$1{\cdot}50$	$1{\cdot}33$	$0{\cdot}30$

2. $x = (3-t^2)/(1+t^2)$, $y = t(3-t^2)/(1+t^2)$;

$\pm t$	0	$\frac{1}{7}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	$\sqrt{3}$	2	3
x	3	$2{\cdot}92$	$2{\cdot}20$	1	$0{\cdot}23$	0	$-0{\cdot}20$	$-0{\cdot}60$
$\pm y$	0	$0{\cdot}42$	$1{\cdot}10$	1	$0{\cdot}35$	0	$-0{\cdot}40$	$-1{\cdot}80$

3. $x = 2at/(t^2+1)$, $y = a(t^2-1)/(t^2+1)$.

4. $x = a(1+t^2)/2t$, $y = a(1-t^2)/2t$.

13. $3x-4y = 3$; the second gives only positive values of y.

14. $y^2 = x+4$; the second gives only $-4 \leqslant x \leqslant 0$. **16.** $x-ty+at^2 = 0$.

19. $x+t^2y-2ct = 0$; $(2ct, 0)$, $(0, 2c/t)$. **21.** $2a$. **22.** $\pm 2a$.

23. $alt^2+2amt+n = 0$; $am^2 = ln$. **24.** $clt^2+nt+cm = 0$; $4c^2lm = n^2$.

25. $(al-n)t^2-2amt-(al+n) = 0$; $a^2(l^2+m^2) = n^2$.

26. $a(l-m)t^2+2nt+a(l+m) = 0$; $a^2(l^2-m^2) = n^2$.

EXERCISE 7B, *p.* 126.

1. $(\frac{11}{8}, \frac{7}{5})$, $(\frac{1}{4}, \frac{3}{4})$, $(\frac{4}{3}, -\frac{1}{3})$.

2. $(\frac{5}{11}, \frac{2}{11})$, $(\frac{2}{5}, \frac{1}{5})$, $(\frac{3}{7}, \frac{1}{7})$.

3. $(8, -7)$, $(-1, -3)$, $(-2, 1)$.

4. $(1, -2)$, $(6, 7)$, $(-3, -10)$.

5. $-24{:}11$. **6.** $3{:}2$.

7. $-1{:}3$; parallel; A; B.

8. AB; $1{:}2$; B; $-7{:}4$.

9. $[\frac{1}{3}(x_1+x_2+x_3), \frac{1}{3}(y_1+y_2+y_3)]$.

10. $(-3, 2)$, $(0, 1)$, $(1, 0)$.

11. $\frac{5}{2}\sqrt{29}$. **12.** $\frac{4}{5}\sqrt{13}$.

13. $2x-5y+18 = 0$, $2x+y = 6$.

14. $x+4y = 0$, $3x+4y = 8$.

15. $x = 2+4t$, $y = 1+3t$; 5; all lines concurrent.

16. $x = 3+12t$, $y = -2-5t$; $13(9+2\lambda)/(3-\lambda)$; 3; 39.

EXERCISE 7C, *p.* 130.

1. 2, 0, 1, $3\frac{1}{8}$, $\pm(x_1 \cos\alpha+y_1 \sin\alpha-p)$.

2. $\sqrt{41}$, 3, $3\frac{1}{4}$, 0, $\pm a\sqrt{(1+t^2)}$.

3. B, E.

4. (c).

5. $c<-14$ or $c>-1$.

6. $(3, 3)$.

7. No.

8. $12-2x+y$, $4-y$, $x+y-5$.

9. $2x+y+3 = 0$, $x-2y = 1$; the second.

10. $10x+2y = 11$, $2x-10y = 3$; the second.

11. $(3, 2)$.

12. $(-13, -2)$.

13. Ecentre.

14. Centre $(1, -2)$.

15. $(1, -1)$; $(2, 2)$, $(-3, -3)$, $(6, -6)$.

16. $x^2+y^2-10x-3y+25 = 0$.

17. 3.

18. $4x+3y+9 = 0$, $4x+3y = 11$.

19. $(6, 7)$.

20. $5x+12y = 29$.

22. 24.

24. $x = 0$ (twice), $3x \pm 4y = 8$.

25. $8x^2-12xy+17y^2-40x-20y = 0$.

26. $44x^2+600xy-551y^2 = 169$.

27. RH like $xy = 1$, asymptotes $3x-4y = 12$, $4x+3y = 12$.

28. P like $y^2 = 4x$, axis $3x-4y = 12$, tangent at vertex $4x+3y = 12$.

29. E like $x^2/4+y^2 = 1$, major axis $4x-2y = 1$, minor axis $x+2y = 2$.

30. H like $x^2/4-y^2/9 = 1$, trans. axis $6x-2y+3 = 0$, conj. axis $3x+9y+2 = 0$.

31. $(5, 2)$.

32. 4.

38. The distance from P_1 to the original line; opposite.

EXERCISE 7D, *p.* 134.

1. 27. **2.** -32. **3.** -12. **4.** $17\frac{1}{2}$. **5.** 0. **6.** 0.
7. $\frac{1}{2}(x_1y_2-x_2y_1)$. **8.** $a(x-y)$. **9.** $-\frac{1}{2}(b-c)(c-a)(a-b)$.
10. $\frac{1}{2}(b-c)(c-a)(a-b)$. **11.** 4:1. **13.** $2x+3y=9$.
14. RH $(x-a)(y+a)+a^2=0$. **17.** $24\frac{1}{2}$. **18.** -17. **19.** 9. **20.** 0.

EXERCISE 7E, *p.* 138.

1. -18. **2.** 8. **3.** 0. **4.** -5. **5.** $1+a^2+b^2+c^2$. **6.** 0.
7. $x_2y_3-x_3y_2+x_3y_1-x_1y_3+x_1y_2-x_2y_1$. **8.** $abc+2fgh-af^2-bg^2-ch^2$.
9. 0. **10.** 0. **11.** -43. **12.** 1. **13.** $2abc$. **14.** $4abc$.
15. 0. **16.** 0. **17.** $a(b-c)(a-b)$. **18.** $b^2(a-c)(a+c-2)$.
19, 20. $(y-z)(z-x)(x-y)$. **21.** a, b. **22.** 0 (twice), $a+b+c$.

EXERCISE 7F, *p.* 143.

1. Expand in terms of last column. **2.** $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.
3. Yes; no. **4.** No; yes. **5.** $t_1+t_2+t_3=0$. **6.** $t_2t_3+t_3t_1+t_1t_2=0$.
7. $t_1^2+t_2^2+t_3^2+t_2t_3+t_3t_1+t_1t_2+t_1+t_2+t_3=0$. **8.** $t_1t_2t_3=1$.
9. $(2, -4)$, $(3\frac{1}{2}, \frac{1}{2})$, $(-4, -22)$. **11.** $3x+5y=1$. **12.** $7x-11y+2=0$.
13. $9x-4y+3=0$. **14.** $13x+8y+5=0$. **15.** $2x-(t_1+t_2)y+2at_1t_2=0$.
16. $x+t_1t_2y=c(t_1+t_2)$.
17. $(x/a)\cos\frac{1}{2}(\theta_1+\theta_2)+(y/b)\sin\frac{1}{2}(\theta_1+\theta_2)=\cos\frac{1}{2}(\theta_1-\theta_2)$.
18. $(x/a)\cos\frac{1}{2}(\theta_1-\theta_2)-(y/b)\sin\frac{1}{2}(\theta_2+\theta_2)=\cos\frac{1}{2}(\theta_1+\theta_2)$.
19. $(1+t^2)x/a-2ty/b=1-t^2$. **20.** $(1+t^2)x/a-(1-t^2)y/b=2t$.
21. $x\cos\theta-y\cot\theta=a$. **22.** $x/\cos\theta+y/\sin\theta=a$. **23.** $\frac{5}{11}$.
24. $\frac{8}{3}$; $\frac{18}{7}$. **25.** No; yes; yes. **26.** Yes; no; yes. **29.** $t_1t_2t_3=1$.
30. $t_1+t_2+t_3=0$.

EXERCISE 7M, *p.* 145.

1. $xy+5x-3y+3=0$; $9x-2y=1$.
2. $x_1x_3+x_2x_4+y_1y_3+y_2y_4=x_1x_4+x_2x_3+y_1y_4+y_2y_3=x_1x_2+x_3x_4+y_1y_2+y_3y_4$.
3. $x+y=12$; (1, 11), (11, 1).
4. $[(l+m)x/m, 0]$, $[0, (l+m)y/l]$; $(l+m)(lx+my)=lmc$.
5. (21/17, 30/17), (−5/17, 56/17). **7.** $5x+y\pm4=0$.
10. RH $(p+q)xy=pkx+qhy$. **11.** Opposite; $(2\sqrt{5}-4, 1)$.
12. $x^2+y^2=5x$; $(-\frac{1}{2}, \frac{3}{2})$, (1, 3), $(\frac{1}{2}, \frac{5}{2})$. **15.** 3. **16.** Parallel.
18. $7x-4y+11=0$.
22. $(x_2^2/a^2+y_2^2/b^2-1)\lambda^2+2(x_1x_2/a^2+y_1y_2/b^2-1)\lambda+(x_1^2/a^2+y_1^2/b^2-1)=0$.

EXERCISE 8A, *p.* 153.

5. Directrix of the fixed parabola. **23.** Part of Property III.

EXERCISE 8C, *p.* 165.

1. $\frac{3}{5}$, $(\pm 3, 0)$, $3x \pm 25 = 0$. **2.** $\frac{12}{13}$, $(0, \pm 12)$, $12y \pm 169 = 0$.
3. $\frac{1}{3}\sqrt{5}$, $(0, \pm\sqrt{5})$, $y\sqrt{5} \pm 9 = 0$. **4.** $\frac{1}{2}\sqrt{3}$, $(\pm\sqrt{3}, 0)$, $x\sqrt{3} \pm 4 = 0$.
5. $\frac{1}{3}\sqrt{3}$, $(\pm 1, 0)$, $x \pm 3 = 0$. **6.** $\frac{1}{2}\sqrt{2}$, $(0, \pm 2)$, $y \pm 4 = 0$.
7. $3x^2+4y^2-32x+64 = 0$; $(5\frac{1}{3}, 0)$. **8.** $2x^2+y^2+8y = 16$; $(0, -4)$.
9. $13x^2+9y^2-36y = 81$; $(0, 2)$. **10.** $4x^2+13y^2-24x-52y+36 = 0$; $(3, 2)$.
11. Locus of point whose distance from $(-1, 2)$ is twice its distance from $2x = 5$.
12. 0; $x+1 = 0$; $\frac{1}{2}$. **14.** $\sqrt{\left(\frac{a-b}{a}\right)}$ or $\sqrt{\left(\frac{b-a}{b}\right)}$.
47. Circles with centres at the fixed foci and radii equal to the major axis.

EXERCISE 8D, *p.* 173.

1. $\frac{5}{3}$, $(\pm 5, 0)$, $5x \pm 9 = 0$. **2.** $\frac{5}{4}$, $(0, \pm 5)$, $5y \pm 16 = 0$.
3. $\frac{1}{2}\sqrt{13}$, $(0, \pm\sqrt{13})$, $y\sqrt{13} \pm 4 = 0$. **4.** $\frac{1}{2}\sqrt{5}$, $(\pm 2\sqrt{5}, 0)$, $x\sqrt{5} \pm 8 = 0$.
5. $\frac{3}{2}$, $(\pm 3, 0)$, $3x \pm 4 = 0$. **6.** $\frac{2}{3}\sqrt{3}$, $(\pm\frac{4}{3}\sqrt{3}, 0)$, $x \pm \sqrt{3} = 0$. **33.** Yes.

EXERCISE 8M, *p.* 175.

1. 12. **4.** $3x+y \pm 2\sqrt{6} = 0$. **6.** $(0, 0)$, $(-8a, 0)$. **7.** $10\frac{6}{8}\frac{7}{0}$.
10. $x \pm y \pm 1 = 0$. **19.** $x \sin t + y \cos t = a \sin t \cos t$;
$x \cos t - y \sin t = a \cos 2t$. **26.** $\lambda a : b$, λ/μ.
32. $\frac{1}{3}(x_1+x_2+x_3)$, $\frac{1}{3}(y_1+y_2+y_3)$; $\frac{x_2+\lambda x_3}{1+\lambda} - \mu(y_2-y_3)$, $\frac{y_2+\lambda y_3}{1+\lambda} + \mu(x_2-x_3)$.
33. $y^2 = 4ax$, $y^2 = 4at_0^2\{a(1+t_0^2)-x\}$.

EXERCISE 9A, *p.* 182.

1. P $y^2 = 2a(x-a)$. **2.** P $y^2 = a(x-a)$. **3.** P $y^2 = 2a(x-a)$.
4. RH $y(x-x_0+2a) = 2ay_0$. **5.** C $x(x-4a)+y^2 = 0$. **6.** RH $x(y+c) = 2c^2$.
7. RH $x(y-2k)+hk = 0$. **8.** RH $(p+q)xy-pkx-qhy = 0$.
9. Hyperbola with foci at ends of base. **10.** Lines $x \pm y = 0$.
13. E $b^2x^2+a^2y^2 = 2a^2b^2$. **14.** E $b^2x^2+a^2y^2 = \frac{1}{2}a^2b^2$.
15. C $x^2+y^2 = (a+b)^2$. **17.** H $x^2/(a^2+b^2)-y^2/b^2 = 1$.
19. C $x(x-\frac{1}{2}a)+y(y-\frac{1}{2}b) = 0$. **20.** RH $b(x^2-y^2)-2a^2x+a^2b = 0$.
21. Angle containing parabola is 45° or 135°. **22.** P $y^2 = a(x-3a)$.

EXERCISE 9B, *p.* 186.

1. P $3y^2 = 16ax$. **2.** P $2y^2+ax = 0$. **3.** P $ay^2 = b^2x$.
5. C $(x-c)^2+(y-c)^2 = 4c^2$. **6.** RH $x^2-y^2 = a^2$. **7.** H $x^2/a^2-y^2/b^2 = 1$.
9. RH $xy = \text{const.}$ **10.** P $y^2 = 4ax$. **11.** H $x^2/9-y^2/16 = 1$.
12. RH $2xy = c^2$. **13.** Circle, centre O, touching lines.
17. $x \pm y + 1 = 0$. **18.** $4x \pm 3y = 5$, $3x \pm 4y + 5 = 0$.
19. 20. $x\sqrt{(d^2-b^2)} \pm y\sqrt{(a^2-c^2)} \pm \sqrt{(a^2d^2-b^2c^2)} = 0$.

EXERCISE 9C, *p.* 192.

1. $2x-(t_1+t_2)y+2at_1t_2=0$. **2.** $x+t_1t_2y=c(t_1+t_2)$.
3. $(1+t^2)x/a-2ty/b=1-t^2$. **4.** $(1-t^2)x/a+2ty/b=1+t^2$.
5. $y_0t^2-2ct+x_0=0$; $x:y:c=2t_1t_2:2:t_1+t_2$.
6. $at^2-y_0t+x_0=0$; $x:y:a=t_1t_2:t_1+t_2:1$.
7. $b(x_0+a)t^2-2ay_0t-b(x_0-a)=0$; $x/a:y/b:1=1-t_1t_2:t_1+t_2:1+t_1t_2$.
8. $b(x_0+a)t^2-2ay_0t+b(x_0-a)=0$; $x/a:y/b:1=1+t_1t_2:t_1+t_2:1-t_1t_2$.
9. $\left(\frac{y_0^2-2ax_0}{2a},\ y_0\right)$. **10.** $(c^2/y_0,\ c^2/x_0)$. **11.** $(-cl/m,\ -cm/l)$.
12. $\left(-\frac{4al+n}{l},\ -\frac{m(4al+n)}{l^2}\right)$.
13. Angle is zero if line touches curve; $(4a,\ 0)$ is Frégier point of O.
14. Angle is zero if line touches curve, 90° if gradient is 1. **19.** $(b^2/a,\ 0)$.
20. $(2k,\ 2h)$—or else the origin. **21.** $\left(a\frac{b^2-a^2}{b^2+a^2},\ 0\right)$.
25. H $k^2x^2-y^2=(k^2-1)a^2$. **26.** C $x^2+y^2=a^2+b^2$.
29. $b<a$, E; $b=a$, P; $b>a$, H. **31.** $\left(\pm a\sqrt{\frac{a^2-b^2}{a^2+b^2}},\ 0\right)$, $(a^2+b^2)/a^4$.
32. $\left(0,\ \pm b\sqrt{\frac{a^2+b^2}{a^2-b^2}}\right)$, $(a^2-b^2)/a^4$. **35.** $\frac{8}{15}$.

EXERCISE 9D, *p.* 197.

2. $(x_0-6a,\ -\frac{1}{2}y_0)$. **6.** P $ay^2=c^2(x-2a-c)$.
9. $(-4a,\ 0)$. **10.** P $y^2=2a(x+4a)$.
15. Euler line of triangle of tangents at any three of t_1, t_2, t_3, t_4 is parallel to tangent at fourth.
28. $x^2/a^2+y^2/b^2=\frac{1}{2}$. **29.** Diametrically opposite the fourth.
31. (i) $t_1+t_2+t_3=0$; (ii) $\Sigma t_2t_3+1=0$. **32.** (i) $\Sigma t_2t_3=0$; (ii) $\Sigma t_1=t_1t_2t_3$.

EXERCISE 9M, *p.* 199.

2. $xy=c^2$. **9.** Circle having centres of given circles at ends of diameter.
10. $x+2ty+3at^2=0$. **11.** $(-a,\ a^2/b)$. **13.** I; part of III.
21. Parabola with $\frac{1}{4}$ of given latus rectum.
30. $(2a-x_0,\ -y_0)$. **32.** $x^2+y^2=1$. **36.** $(0,\ \frac{1}{2}y_0)$.

EXERCISE 10A, *p.* 210.

1. $x^2\sqrt{3}-2xy-y^2\sqrt{3}+32=0$. **2.** $xy-y^2+2=0$.
3. $x^2+y^2-4x-3y=0$. **4.** $8x^2-12xy+17y^2=20$.
5. RH $x^2-y^2=4$, $\tan^{-1}\frac{3}{2}$. **6.** P $y^2+4x=0$, $\tan^{-1}\frac{12}{5}$.
7. E $x^2/9+y^2/4=1$, $-\tan^{-1}\frac{1}{2}$. **8.** C $x^2+y^2=1$, $\tan^{-1}7$ or anything.
9. LP $y=\pm2$, $\tan^{-1}3$. **10.** RH $xy=4$, $-\tan^{-1}\frac{4}{3}$.
11. -5, -2, -1, 0, -1, -3, 1. **12.** 2, 4, 1, 3, -2, -3, -1.
19. $C(-1,0)$; H $4x^2-y^2=4$; trans. axis $2x-y+2=0$, conj. axis $x+2y+1=0$.
20. $C(2,2)$; E $x^2+3y^2=6$; major axis $x+y=4$, minor axis $x=y$.
21. $C(3,-2)$; PtE $x^2+2y^2=0$; 'major' axis $2x+3y=0$,

'minor' axis $3x-2y=13$. **22.** $C(1, 1)$; LP $x^2=y^2$; axes $3x-y=2, x+3y=4$.
23. $3x-2y=6$ twice.
24. P $y^2+x\sqrt{5}=0$; axis $x-2y=5$, tangent at vertex $2x+y=5$; $V(3, -1)$.

EXERCISE 10B, *p.* 214.

1. $2x-y+1=0$, $x+y=2$; $x+2y+1=0$, $2x+4y=1$.
2. $2x-y+2=0$, $x-2y+3=0$; $3x-2y=4$ twice.
3. 60°, 45°, $\tan^{-1}2$, $\tan^{-1}\frac{1}{2}$, θ. **4.** 60°, 0°, 90°, $\tan^{-1}4$, 45°.
5. $x^2-xy-6y^2=0$; $x^2-3xy+2y^2=0$.
6. $3x^2-xy-2y^2=0$; $3x^2-2xy-3y^2=0$.
7. $x^2+xy-6y^2-4x+23y=21$. **8.** $x^2-16xy-11y^2-18x-6y+6=0$.
9. -2. **10.** ± 5. **11.** $-2, 0, 1$. **12.** $-8, -3, 0$.
13. $(3x-y)(x-2y)=0$. **14.** $x^2=3y^2$. **15.** $(x+y)^2=0$; they touch.
16. $x^2+y^2=0$; they do not meet in real points. **17.** $x+3y=0$, $2x-y=0$.
18. $2x+3y=0$, $3x-8y=0$.
21. $(bx_0^2-2hx_0y_0+ay_0^2)t^2+2\{h(x_0^2-y_0^2)-(a-b)x_0y_0\}t+(ax_0^2+2hx_0y_0+by_0^2)=0$; sum is zero; $h(x^2-y^2)=(a-b)xy$. **22.** One root is infinite.
23. $(x^2-y^2)/(a-b)=xy/h$. **25.** $[-2g/(a+b), -2f/(a+b)]$.
26. $(-f/h, -g/h)$. **29.** $3x^2+3y^2-14x=21$.
32. $r^2(17x^2-12xy+8y^2)=20(x^2+y^2)$; 1, 2.

EXERCISE 10C, *p.* 219.

1. $x^2-2xy-y^2+2x+2y=0$; RH. **2.** $x^2+y^2+2x=0$; circle; no.
3. $xy-2y=12$; they are orthocentric. **4.** $x^2-y^2-10x+2y=56$.
5. $8x^2-9xy-24y^2+18y=8$. **6.** $13x^2-4xy-2y^2-11x+4y=0$;
$(\frac{11}{13}, 0)$, none.
7. $x^2-4xy+4y^2-x+2y=8$, $x^2+2xy+y^2-4x+2y=5$;
$x^2+6xy-y^2-6x+2y=3$; $x^2+2y^2-3x+2y=6$.
8. $(x-3y)^2+2x=0$, $(2x+y)^2+x-7y=0$; $x^2+2xy-y^2=2y$.
9. $y^2=x+6y$, $x^2=13x+42y$; $x^2+y^2=14x+48y$; $x^2-y^2=12x+36y$.
10. $x^2=13x+6y$, $y^2=4x$; $x^2+y^2=17x+6y$; $x^2-y^2=9x+6y$.
11. $16\lambda\mu+25\lambda-\mu=0$. **15.** $x^2+y^2=4a^2$. **17.** $(x^2-y^2)/(a-b)=xy/h$.
18. $(am-hl)x+(hm-bl)y\pm n\sqrt{(h^2-ab)}=0$. **21.** They form a rectangle.

EXERCISE 10D, *p.* 223.

1. (3, 1); $x^2+y^2-6x-2y+7=0$. **2.** $x^2+y^2-2x-2y=6$.
3. $2x-4y+3=0$; (1, 0), (0, 2). **4.** (1, −1), (−1, 3).

EXERCISE 10E, *p.* 230.

1. $7x^2-2xy-7y^2-110x+30y+450=0$. **2.** (9, −6).
3. Semi-cubic parabola.
4. $x:y:n=(a-b)l+2hm:2hl-(a-b)m:2(bl^2-2hlm+am^2)$.
7. $(3x+2y)^2=24(3x-2y)$. **8.** $x^2+xy+y^2-4x-4y+4=0$.
15. $8xy=9c^2$. **16.** $(-a, 0)$. **20.** $x^2+y^2=ax$, $(-a, 0)$.
21. $ay^2=2b^2(x+a)$.
25. $h(x^2-y^2)=(a-b)xy$; represents angle-bisectors of line-pair.

EXERCISE 10M, *p.* 232.

2. $(am^2+bl^2)(x^2+y^2)-2blx-2amy = l^2+m^2-a-b$. **4.** $y^2 = a(x+h-2a)$.
9. $x^2+y^2 = a^2+b^2$. **13.** $x^2+y^2 = 4x$. **19.** $x^2 = a(a-2y)$.
21. $(\sqrt{3}+1)x-(\sqrt{3}-1)y+2 = 0$, $(\sqrt{3}-1)x-(\sqrt{3}+1)y = 2$.
23. Three; 2 if points symmetrical about axis; 1 if circle of curvature at one point goes through other.
26. $[(4al^2+4am^2-ln)/2l^2, mn/2l^2]$. **28.** $[ax_0/(a+b), by_0/(a+b)]$.
30. $(-1, 1)$. **31.** $b(2x-ae)^2+2(a+b)y^2 = b(a+b)^2$.
36. $x/l = y/m = -n(a+b)/(am^2-2hlm+bl^2)$.

EXERCISE 11A, *p.* 239.

1. Perpendicular diameters. **15.** For 'circle' read 'ellipse'.
16. For 'circle', 'circle', read 'ellipse', 'similar ellipse'.

EXERCISE 11B, *p.* 241.

5. $x \pm y = 0$. **6.** Hyperbola with asymptotes $x = 0$, $x+y = 0$; $y = 0$.
14. They are orthogonal.
23. $\sqrt{\{(x_1-x_2)^2+(y_1-y_2)^2+2(x_1-x_2)(y_1-y_2)\cos\omega\}}$.
24. $x^2+y^2+2xy\cos\omega = a^2$.
25. Ellipse on P_1P_2 as diameter, having equal conjugate diameters parallel to $x = 0$, $y = 0$.
26. $x\cos\omega+y = 0$, $x+y\cos\omega = 0$.

EXERCISE 11C, *p.* 250.

1. $(4, 0)$, $(2, -\frac{1}{3}\pi)$, $(4, \frac{3}{4}\pi)$, $(3, \pi)$.
2. $(2\sqrt{2}, \frac{1}{4}\pi)$, $(2, -\frac{1}{2}\pi)$, $(2\sqrt{3}, \frac{2}{3}\pi)$, $(0, \theta)$.
3. $(2, -\frac{3}{4}\pi)$, $(1, -\frac{1}{2}\pi)$, $(1, \pi)$. **4.** $(3, -\frac{1}{3}\pi)$, $(2, -\frac{1}{6}\pi)$, $(5, 0)$.
5. $r(l\cos\theta+m\sin\theta)+n = 0$. **6.** $r(1-\cos\theta) = 2a$. **7.** $r^2 = a^2\cos 2\theta$.
8. $r = a(\sec\theta-\cos\theta)$. **9.** $x\sin\alpha-y\cos\alpha+a = 0$. **10.** $x = a$.
11. $2xy = a^2$. **12.** $x^2-y^2 = a^2$. **13.** $(1-e^2)x^2+y^2+2elx = l^2$.
14. $x^2(x^2+y^2) = a^2y^2$. **15.** $\theta = \pi$; about $\theta = 0$.
16. $\theta = \pm\frac{3}{4}\pi$; about $\theta = 0$. **17.** $x = \pm a$, $y = \pm a$; about $\theta = 0, \frac{1}{2}\pi, \pm\frac{1}{4}\pi$.
18. $\theta = 0$; $x = \pm a$; about $\theta = 0, \frac{1}{2}\pi$. **19.** $y = a$; about $\theta = \frac{1}{2}\pi$.
20. $y = 0$; about O. **21.** $\theta = \pi, \pm\frac{1}{3}\pi$; about $\theta = 0, \pm\frac{1}{3}\pi$.
22. $\theta = \frac{1}{2}\pi, \pm\frac{1}{6}\pi$; about $\theta = 0, \pm\frac{1}{3}\pi$. **23.** $\theta = \pm\frac{1}{4}\pi$; $x = a$; about $\theta = 0$.
24. $\theta = \pm\frac{1}{4}\pi$; $x = 0$, $y = 0$; about O. **25.** $\theta = 0, \frac{1}{2}\pi$; about $\theta = 0$.
26. $\theta = 0$; about $\theta = \frac{1}{2}\pi$.
29. $2-2\cos(\alpha-\beta)$; $(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2$; $\cos\alpha\cos\beta+\sin\alpha\sin\beta$.
31. It is the same distance from O. **32.** Normal tends to tangent to circle $r = a$.
38. $y^2 = 2ax$. **43.** $x\sin t+y\cos t = a\sin t\cos t$.
44. $r^2 = 2c^2\cos 2\theta$. **46.** $l/r = (e+\cos\alpha)\cos\theta+\sin\alpha\sin\theta$.
48. About $\theta = \frac{1}{2}\pi$. **50.** $r\sin(\frac{1}{8}\pi\pm\theta) = \frac{3}{2}a$.

EXERCISE 11D, *p.* 252.

1. Right bisector of A_1A_2. **2.** A_1A_2. **3.** A_1A_2 produced both ways.
4. Circle on A_1A_2 as diameter. **7.** $\sqrt{(\frac{1}{2}a^2-c^2)}$. **10.** $\cos\alpha = a/c$.

EXERCISE 11E, *p.* 255.

7. $\cos BOC = (ll'+mm'+nn')/rr'$. **9.** Cylinder with axis Ox.
10. Cylinder touching zOx along Ox. **11.** Plane $y = z$ twice.
12. Cone, vertex O, axis Ox. **14.** Sphere, centre (a, b, c), radius d.
19. $5x+6y+3z = 31$. **20.** $x+9y+7z = 42$. **21.** $5x-10y-z = 0$.
22. $-5:10:1$. **24.** Circular cylinder, radius 1, axis the line of No. 22.
26. $bc:ca:ab$; $abc/\sqrt{(b^2c^2+c^2a^2+a^2b^2)}$. **27.** $x^2/a^2-(y^2+z^2)/b^2 = 1$.
28. $(x^2+z^2)/a^2-y^2/b^2 = 1$. **29.** $x^2+y^2 = 4az$. **32.** Circles of radius a.

CHAPTER 12, *p.* 259.

2. They are collinear. **8.** $t(3+t^2)x-2y = at^3$; $-\frac{1}{2}t$.
9. $t_2 = -1/t_1$. **11.** $(1-4t^2-t^4)x+4ty = a(1-t^2)^2$.
16. $t_2t_3+t_3t_1+t_1t_2 = 1$; $\beta = \pm\frac{1}{2}\pi-2\alpha$; $\pm\frac{1}{6}\pi$.
17. $-1/t$, $1/t$; perpendicular. **19.** $x = a+b\cos\theta$, $y = a\tan\theta+b\sin\theta$; $x(a\sec^3\theta+b)+by\tan\theta = \sec\theta(a\sec\theta+b)^2$. **37.** $r^2 = a^2\cos\frac{2}{3}\theta$.
38. $x:y:a = t(t^2+1):t(t^2-1):t^4+1$; $t_1t_2t_3t_4 = 1$. **68.** $r = \frac{1}{2}a\sin 2\theta$.
69. $r\sqrt{2} = a\cos 2\theta$; No. 68 with linear dimensions multiplied by $\sqrt{2}$, and rotated through $\frac{1}{4}\pi$.
78. Tangent at vertex; auxiliary circle; auxiliary circle. **80.** A.

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