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# DIFFERENTIAL ALGEBRA

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## PREFACE

In 1932, the author published *Differential equations from the algebraic standpoint*,<sup>1</sup> a book dealing with differential polynomials and algebraic differential manifolds. In the sixteen years which have passed, the work of a number of mathematicians has given fresh substance and new color to the subject. The complete edition of the book having been exhausted, it has seemed proper to prepare a new exposition.

The title *Differential algebra* was suggested by Dr. Kolchin. The body of algebra deals with the operations of addition and multiplication. We are concerned here with three operations—addition, multiplication and differentiation.

If I am not mistaken, the general nature of the subject here treated is now well enough known among mathematicians to permit me to dispense with a detailed introduction, such as was given in A. D. E. My principal task is to show how much the present book owes to my associates. I am referring to H. W. Raudenbush, W. C. Strodt, E. R. Kolchin, Howard Levi, Eli Gourin and Richard M. Cohn.

Cohn's constructive proof of the theorem of zeros will be found in Chapter V. The theorem on embedded manifolds due to Gourin is contained in Chapter II. Chapter VI contains a discussion of Strodt's work on sequences of manifolds.

In Chapters I, III and IX, there are presented portions of Levi's work on ideals of differential polynomials and on the low power theorem. Of Kolchin's investigation of exponents of differential ideals, I have been able to give only a bare idea. Other work of Kolchin, for instance, proofs for the abstract case of results previously established for the analytic case, is given in Chapter II. His work on the Picard-Vessiot theory, which employs the methods of differential algebra, has just appeared in the *Annals of Mathematics*,<sup>2</sup> and may be permitted to speak for itself.

The contributions of Raudenbush can only be described as fundamental. The basis theorem of Chapter I was, in the analytic case, implicitly contained in A. D. E. It exists there in two parts; the first, the theorem on the completeness of infinite systems; the second, the theorem of zeros. Only casually had I noticed that the two theorems amounted to a basis theorem. I was acquainted with the fact that the theorem on the decomposition of manifolds amounted, in virtue of the theorem of zeros, to a theory of perfect and prime ideals of differential polynomials. In the summer of 1933, I suggested to Raudenbush the problem of constructing a theory of perfect ideals which would be valid in the abstract case. This he accomplished, and, in the course of his work, he brought the basis theorem to its present complete and abstract form. In the proof of

<sup>1</sup> These Colloquium publications, vol. 14. Called below A. D. E.

<sup>2</sup> Kolchin, 14. (See Bibliography, p. 180.)

the basis theorem, the procedure of taking powers is due to Raudenbush. The chains, characteristic sets and methods of reduction existed in the older theorem of completeness.

Raudenbush introduced generic zeros of prime ideals. Here he adapted a method of van der Waerden, which can be traced back to König. Raudenbush gave the first example of a system of differential polynomials with a weak basis. Systems with no strong bases were later produced by Kolchin.

The problems which this book treats are very concrete problems. They deal with situations of the classical theory of differential equations. Seldom would much be lost, as far as the results are concerned, if one limited oneself to the material of classical analysis. The abstract method which we generally employ has, however, a definite utility. It serves to separate algebraic methods from analytic methods. On the whole, it contributes to simplicity, although at times an abstract treatment is less natural than an analytical one. The form in which the results of differential algebra are being presented has thus been deeply influenced by the teachings of Emmy Noether, a prime mover of our period, who, in continuing Julius König's development of Kronecker's ideas, brought mathematicians to know algebra as it was never known before.

In this connection, I should like to say something concerning basis theorems. The basis theorem of Chapter I will be seen to play, in the present theory, the role held by Hilbert's theorem in the theories of polynomial ideals and of algebraic manifolds. When I began to work on algebraic differential equations, early in 1930, van der Waerden's excellent *Moderne Algebra* had not yet appeared. However, Emmy Noether's work of the twenties was available, and there was nothing to prevent one from learning in her papers the value of basis theorems in decomposition problems. Actually, I became acquainted with the basis theorem principle in the writings of Jules Drach<sup>3</sup> on logical integration, writings which date back to 1898. How a basis theorem is employed by him will now be described.

There are two distinct methods for characterizing an irreducible algebraic equation. On the one hand, an equation  $f(x) = 0$  is irreducible if  $f(x)$  cannot be factored. On the other, there is irreducibility if every equation which is satisfied by a single solution of  $f(x) = 0$  is satisfied by all such solutions. The first formulation of irreducibility leads to the notion of irreducible algebraic manifold and to that of irreducible algebraic differential manifold. The second leads to the concept of irreducible system of algebraic differential equations which was employed by Koenigsberger and by Drach. A system of such equations, ordinary or partial, is irreducible if every differential equation which admits a single solution of the system admits all solutions. Drach undertakes to show that, given a system of partial differential equations, the repeated adjunction of new equations will eventually produce an irreducible system. For this he invokes a theorem of Tresse,<sup>4</sup> which states that, in every infinite system

<sup>3</sup> Drach, 4, pp. 292-296.

<sup>4</sup> *Acta Mathematica*, vol. 18 (1894), p. 4.

of partial differential equations, there is a finite subsystem from which the infinite system can be derived by differentiations and eliminations. A study of Tresse's paper will quickly convince one that he claims for his work a generality which it does not have. The statement of his theorem, and his argument, have a definite meaning only for linear systems.

It has not been possible for me to present all of the material which has been developed since the publication of A. D. E. Thus, I have had to pass by most of Kolchin's study of exponents and a good deal of Levi's work on ideals. Of Strodt's paper, only a sketch is given. My own work on general solutions of equations of the second order in one unknown, and of equations of the first order in two unknowns, is also omitted.

I have tried to give, to the present book, the elementary quality which is possessed by A. D. E. Essentially, no previous knowledge of abstract algebra is necessary. As in A. D. E., a treatment is given of Riquier's existence theorem for orthonomic systems of partial differential equations.

New York, N. Y.

January, 1948.



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CHAPTER I  
DIFFERENTIAL POLYNOMIALS AND THEIR IDEALS  
DIFFERENTIAL FIELDS

1. We deal with an algebraic field of characteristic zero, denoting the field by  $\mathfrak{F}$ .  $\mathfrak{F}$ , then, is a collection of elements of one type or another, upon which can be performed the operations of addition, subtraction, multiplication and division, except that division by a certain element 0 of  $\mathfrak{F}$  is excluded. Addition and multiplication are commutative and associative, and multiplication is distributive with respect to addition. Subtraction and division are one-valued operations.  $\mathfrak{F}$  contains a subset which is isomorphic, as regards addition and multiplication, with the system of rational numbers; this subset we consider, as we may, actually to be the system of rational numbers.

We are going to work with fields  $\mathfrak{F}$  of characteristic zero in which an operation of *differentiation* is performable. This operation, which replaces every element  $a$  of  $\mathfrak{F}$  by its *derivative*, an element  $a'$  of  $\mathfrak{F}$ , must be such that, for  $a$  and  $b$  in  $\mathfrak{F}$ ,

$$(1) \qquad (a + b)' = a' + b'$$

and

$$(2) \qquad (ab)' = ba' + ab'.$$

When such an operation of differentiation exists for  $\mathfrak{F}$ , we shall call  $\mathfrak{F}$  a *differential field*.

From (1) with  $b = 0$ , we see that  $0' = 0$ . From (2) with  $b = 1$  and  $a \neq 0$ , it follows that  $1' = 0$ . It is easy to show that the derivative of every rational number is zero. An element with zero for derivative is called a *constant*.

The system of rational numbers, with derivatives taken, as they must be taken, equal to zero, is a differential field. So are the system of real numbers and the system of complex numbers. Further examples are the totality of rational functions of a variable  $x$ , with complex coefficients, and the totality of elliptic functions with a given period parallelogram; in these examples, differentiation is supposed to be performed as in analysis.

If  $\mathfrak{F}$  and  $\mathfrak{F}_1$  are differential fields and if  $\mathfrak{F}_1$  contains  $\mathfrak{F}$ ,  $\mathfrak{F}_1$  is called an *extension* of  $\mathfrak{F}$ . It is understood that, when  $\mathfrak{F}$  is considered by itself, the rational operations and differentiation are performed in it just as when  $\mathfrak{F}$  is regarded as part of  $\mathfrak{F}_1$ .

Hereafter the term *field* will be used as an abbreviation for *differential field*. When an ordinary algebraic field is used, a proper announcement will be made. We repeat that the characteristic will always be zero.

## INDETERMINATES

2. We shall be given frequently a letter such as  $y$ , the first of an infinite sequence of symbols

$$(3) \quad y, y', y'', \dots, y^{(p)}, \dots$$

The symbols in (3) will be used for building polynomials, each polynomial involving, of course, only a finite number of the symbols. We shall call  $y$  a *differential indeterminate* or an *indeterminate*, and  $y^{(p)}$  the  $p$ th *derivative* of  $y$ . Furthermore, for every  $p$ , and for every  $q > 0$ ,  $y^{(p+q)}$  will be called the  $q$ th derivative of  $y^{(p)}$ . It is to be emphasized that only  $y$  in (3) is an indeterminate; the  $y^{(p)}$  are not indeterminates, but derivatives of an indeterminate.

Our problems will deal with any finite number  $n$  of indeterminates  $y_1, \dots, y_n$ . The  $j$ th derivative of  $y_i$  will be written  $y_{ij}$ . We shall call  $y_i$  its own derivative of order zero and shall sometimes write  $y_{i0}$  for  $y_i$ . Where unsubscripted letters  $u, v, \dots, w$  are used for indeterminates, derivatives will be written with subscripts rather than with superscripts. Thus,  $u$  being an indeterminate,  $u_j$  is the  $j$ th derivative of  $u$ .

## DIFFERENTIAL POLYNOMIALS

3. In what follows, we work with an arbitrary field  $\mathfrak{F}$ , which, in every question treated, is assigned in advance. Let there be given indeterminates  $y_1, \dots, y_n$ . By a *differential polynomial* (d.p., singular and plural), we shall mean a polynomial in the  $y_{ij}$  with coefficients in  $\mathfrak{F}$ .

Two d.p. are considered equal if, and only if, their coefficients of like power products in the  $y_{ij}$  are equal. This is part of the basis for calling the  $y$  indeterminates.

In describing a d.p.  $A$ , it is at times desirable to refer to the field  $\mathfrak{F}$ , underlying the discussion, in which  $A$  has its coefficients. This is done by calling  $A$  a d.p. *over*  $\mathfrak{F}$ .

The totality of d.p. in  $y_1, \dots, y_n$  over  $\mathfrak{F}$  will be denoted by  $\mathfrak{F} \{ y_1, \dots, y_n \}$ . At each stage of our work,  $\mathfrak{F}$ , as has been observed, is supposed to be known.  $\mathfrak{F}$  is not changed without an explicit statement being made.

Let  $A$  be a d.p. By the *derivative of*  $A$ , we shall mean the d.p. obtained from  $A$  with the use of (1) and (2) of §1.

Thus, if  $\mathfrak{F}$  is the totality of rational functions of a variable  $x$  and if

$$A = xy_1^2 + x^2y_{21},$$

the derivative of  $A$  is  $y_1^2 + 2xy_1y_{11} + 2xy_{21} + x^2y_{22}$ .

Higher derivatives are defined in the expected way.

Until further notice, capital italics will denote d.p.

By the *class* of  $A$ , if  $A$  is not merely an element of  $\mathfrak{F}$ , we shall mean the greatest  $p$  such that some  $y_{pj}$  is present in a term of  $A$  whose coefficient is distinct from zero. If  $A$  is an element of  $\mathfrak{F}$ ,  $A$  will be said to be of class 0.

We shall at times speak of a d.p.  $A$  as involving a certain  $y_i$  effectively. We shall mean by this that at least one  $y_{ij}$  appears effectively in  $A$ .

By the *order of  $A$  with respect to  $y_i$* , if  $A$  involves  $y_i$  effectively, we shall mean the greatest  $j$  such that  $y_{ij}$  appears effectively in  $A$ . If  $A$  does not involve  $y_i$ , the order of  $A$  in  $y_i$  will be taken as zero.

Let  $A_1$  and  $A_2$  be two d.p. Suppose that some indeterminate  $y_p$  appears effectively in both of them. If  $A_2$  is of higher order in  $y_p$  than  $A_1$ ,  $A_2$  will be said to be of *higher rank than  $A_1$*  and  $A_1$  of *lower rank than  $A_2$* , in  $y_p$ . If  $A_1$  and  $A_2$  are of the same order, say  $q$ , in  $y_p$  and if  $A_2$  is of higher degree than  $A_1$  in  $y_{pq}$ , then, also,  $A_2$  will be said to be of higher rank than  $A_1$  in  $y_p$ . Finally,  $A_2$  will be of higher rank than  $A_1$  in  $y_p$  if  $A_2$  involves  $y_p$  and  $A_1$  does not. Two d.p. for which no difference in rank is established by the foregoing criteria will be said to be of the same rank in  $y_p$ .

If  $A_2$  is of higher class than  $A_1$ ,  $A_2$  will be said to be of *higher rank than  $A_1$* , or to be *higher than  $A_1$* . If  $A_1$  and  $A_2$  are of the same class  $p > 0$ , and if  $A_2$  is of higher rank than  $A_1$  in  $y_p$ , then, again,  $A_2$  will be said to be higher than  $A_1$ . Two d.p. for which no difference in rank is created by what precedes will be said to be of the same rank. Thus, all d.p. of class zero are of the same rank.<sup>1</sup>

If  $A_2$  is higher than  $A_1$  and  $A_3$  higher than  $A_2$ , then  $A_3$  is higher than  $A_1$ .

Where unsubscripted indeterminates  $u, v, \dots, w$  are used, class and relative rank are established by giving to the  $p$ th indeterminate from the left the role of  $y_p$  above.

The following simple fact will be important in our later work.

*Every aggregate of d.p. contains a d.p. which is not higher than any other d.p. of the aggregate.*

If the aggregate contains a d.p. of class zero, any such d.p. answers our requirement. Otherwise, let  $p$  be the least of the classes of the d.p. From the d.p. of class  $p$ , we select those which are of a least order, say  $q$ , in  $y_p$  and from the d.p. just selected we pick one,  $A$ , which is of a lowest degree in  $y_{pq}$ . Then no d.p. in the aggregate is lower than  $A$ .

#### CHAINS

4. If  $A_1$  is of class  $p > 0$ ,  $A_2$  will be said to be *reduced with respect to  $A_1$*  if  $A_2$  is of lower rank than  $A_1$  in  $y_p$ .

The system

$$(4) \quad A_1, A_2, \dots, A_r$$

will be called a *chain* if either

$$(a) \quad r = 1 \quad \text{and} \quad A_1 \neq 0,$$

or

(b)  $r > 1$ ,  $A_1$  is of positive class and, for  $j > i$ ,  $A_j$  is of higher class than  $A_i$  and *reduced with respect to  $A_i$* .

<sup>1</sup> As will be seen in Chapter IX, there are other ways of ordering d.p.

Of course,  $r \leq n$ .

The chain (4) will be said to be of *higher rank* than the chain

$$(5) \quad B_1, B_2, \dots, B_s$$

if either

(a) *there is a  $j$ , exceeding neither  $r$  nor  $s$ , such that  $A_i$  and  $B_i$  are of the same rank for  $i < j$  and that  $A_j$  is higher than  $B_j$*

or

(b)  *$s > r$  and  $A_i$  and  $B_i$  are of the same rank for  $i \leq r$ .*

Two chains for which no difference in rank is created by what precedes will be said to be of the same rank. For such chains,  $r = s$  and  $A_i$  and  $B_i$  are of the same rank for every  $i$ .

Let  $\Phi_1, \Phi_2, \Phi_3$  be chains such that  $\Phi_1$  is higher than  $\Phi_2$  and  $\Phi_2$  higher than  $\Phi_3$ . We write  $\Phi_1 > \Phi_2, \Phi_2 > \Phi_3$ . We shall prove that  $\Phi_1 > \Phi_3$ .

Let  $\Phi_1$  and  $\Phi_2$  be represented by (4) and (5) respectively and let  $\Phi_3$  be

$$C_1, C_2, \dots, C_t$$

Suppose first that  $\Phi_1 > \Phi_2$  for the reason (a) and that  $\Phi_2 > \Phi_3$  for the reason (a). Let  $j$  be the smallest integer such that  $B_j$  is higher than  $C_j$ . Then either  $A_i$  is of the same rank as  $B_i$  for  $i \leq j$  or there is a  $k \leq j$  such that  $A_k$  is higher than  $B_k$ . In either case,  $\Phi_1 > \Phi_3$  by (a). Suppose now that  $\Phi_1 > \Phi_2$  by (b), while  $\Phi_2 > \Phi_3$  by (a). Let  $j$  be taken as above. If  $j > r$ ,  $\Phi_1 > \Phi_3$  by (b). If  $j \leq r$ ,  $\Phi_1 > \Phi_3$  by (a). Now let  $\Phi_1 > \Phi_2$  by (a) while  $\Phi_2 > \Phi_3$  by (b). Let  $j$  be the smallest integer for which  $A_j$  is higher than  $B_j$ . Then  $A_j$  is higher than  $C_j$  and  $A_i$  is of the same rank as  $C_i$  for  $i < j$ . Thus  $\Phi_1 > \Phi_3$  by (a). Finally, if  $\Phi_1 > \Phi_2$  by (b) and  $\Phi_2 > \Phi_3$  by (b), then  $\Phi_1 > \Phi_3$  by (b).

We shall use later the following fact:

*In every aggregate of chains, there is a chain which is not higher than any other chain of the aggregate.*

Let  $\alpha$  be the aggregate. We form a subset  $\alpha_1$  of  $\alpha$ , putting a chain  $\Phi$  into  $\alpha_1$  if the first d.p. in  $\Phi$  is not higher than the first d.p. of any other chain in  $\alpha$  (§3). If the chains in  $\alpha_1$  all consist of one d.p., any chain in  $\alpha_1$  meets our requirements. Suppose that there are chains in  $\alpha_1$  which have more than one d.p. We form the subset  $\alpha_2$  of them whose second d.p. are of a lowest rank. If the chains in  $\alpha_2$  all have just two d.p., any of those chains serves our purpose. If not, we continue, reaching lowest chains in no more than  $n$  steps.

#### CHARACTERISTIC SETS

5. Let  $\Sigma$  be a finite or infinite set of d.p. in  $\mathfrak{F}\{y_1, \dots, y_n\}$  (§3). We do not assume the d.p. in  $\Sigma$  to be distinct from one another.<sup>3</sup> Suppose that the d.p. in  $\Sigma$  are not all zero.

<sup>2</sup> If  $j = 1$ , this is to mean that  $A_1$  is higher than  $B_1$ .

<sup>3</sup> What we are really considering then, is a system of distinct marks, each mark being associated with a d.p. Two marks may be associated with identical d.p.

It is possible to form chains with d.p. in  $\Sigma$ ; for instance, every nonzero d.p. in  $\Sigma$  is a chain. Among all chains in  $\Sigma$ , there are some, by §4, which have a lowest rank. Any such chain will be called a *characteristic set* of  $\Sigma$ .

If  $A_1$  in (4) is of positive class, a d.p.  $F$  will be said to be *reduced with respect to the chain (4)* if  $F$  is reduced with respect to  $A_i$ ,  $i = 1, \dots, r$ .

Let  $A_1$  in (4) be of positive class and let  $\Sigma$  be a system containing (4). We shall prove that, for (4) to be a characteristic set of  $\Sigma$ , it is necessary and sufficient that  $\Sigma$  contain no nonzero d.p. reduced with respect to (4). Suppose that (4) is not a characteristic set of  $\Sigma$ , while (5) is. Suppose that (5) is lower than (4) by (b) of §4. Then  $B_{r+1}$  is reduced with respect to (4). If (5) is lower by (a), there is some  $B_i$  with  $i \leq r$  which is reduced with respect to (4). Suppose now that (4) is a characteristic set and that  $\Sigma$  contains a nonzero d.p.  $F$  which is reduced with respect to (4). If the class of  $F$  is higher than that of  $A_r$ , we get a chain lower than (4) by adjoining  $F$  to (4); otherwise, if the rightmost  $A$  whose class is not exceeded by that of  $F$  is  $A_j$ , the chain  $A_1, \dots, A_{j-1}, F$  is lower than (4).<sup>4</sup>

Let  $\Sigma$  be a system for which (4), with  $A_1$  of positive class, is a characteristic set. We see that, if a nonzero d.p., reduced with respect to (4), is adjoined to  $\Sigma$ , the characteristic sets of the resulting system are lower than (4).

Let  $\Sigma$  be a system of d.p. which are not all zero. The following method for constructing a characteristic set of  $\Sigma$  can actually be carried out when  $\Sigma$  is finite. Of the nonzero d.p. in  $\Sigma$ , let  $A_1$  be one of least rank. If  $A_1$  is of class zero, it is a characteristic set for  $\Sigma$ . Let  $A_1$  be of positive class. If  $\Sigma$  contains no nonzero d.p. reduced with respect to  $A_1$ , then  $A_1$  is a characteristic set. Suppose that such reduced d.p. exist; they are all of higher class than  $A_1$ . Let  $A_2$  be one of them of least rank. If  $\Sigma$  has no nonzero d.p. reduced with respect to  $A_1$  and  $A_2$ , then  $A_1, A_2$  is a characteristic set. If such reduced d.p. exist, let  $A_3$  be one of them of least rank. Continuing, we arrive at a chain (4) which is a characteristic set.

Until further notice, large Greek letters not used as symbols of summation or of multiplication will denote systems of d.p.

## REDUCTION

6. In this section, we deal with a chain (4) with  $A_1$  of positive class.

If a d.p.  $G$  is of class  $p > 0$  and of order  $m$  in  $y_p$ , we shall call  $\partial G / \partial y_{pm}$  the *separant*<sup>5</sup> of  $G$ . The coefficient of the highest power of  $y_{pm}$  in  $G$  will be called the *initial* of  $G$ .<sup>6</sup>

The separant and initial of  $G$  are both lower than  $G$ .

<sup>4</sup> When  $j = 1$ , we use the chain  $F$ .

<sup>5</sup> If  $G$ , arranged as a polynomial in  $y_{pm}$ , is  $\sum_{i=0}^q C_i y_{pm}^i$ , the separant is  $\sum_{i=1}^q i C_i y_{pm}^{i-1}$ . As  $\mathfrak{F}$  has characteristic zero, the separant does not vanish identically.

<sup>6</sup> If the indeterminates are  $u, v, \dots, w$ , then  $w$  will play the role of  $y_p$  above, in the definitions of separant and initial of a d.p. actually involving  $w$ .

In (4), let  $S_i$  and  $I_i$  be respectively the separant and initial of  $A_i$ ,  $i = 1, \dots, r$ . We shall prove the following result.

*Let  $G$  be any d.p. There exist nonnegative integers  $s_i, t_i, i = 1, \dots, r$ , such that, when a suitable linear combination of the  $A_i$ , and of a certain number of their derivatives, with d.p. for coefficients, is subtracted from*

$$S_1^{s_1} \cdots S_r^{s_r} I_1^{t_1} \cdots I_r^{t_r} G,$$

*the remainder,  $R$ , is reduced with respect to (4).*

We limit ourselves, as we may, to the case in which  $G$  is not reduced with respect to (4).

Let  $j$  be the greatest value of  $i$  such that  $G$  is not reduced with respect to  $A_i$ . Let  $A_j$  be of class  $p$ , and of order  $m$  in  $y_p$ . Let  $G$  be of order  $h$  in  $y_p$ .

We suppose first that  $h > m$ . If  $k_1 = h - m$ , then  $A_j^{(k_1)}$ , the  $k_1$ th derivative of  $A_j$ , will be of order  $h$  in  $y_p$ . It will be linear in  $y_{ph}$ , with  $S_j$  for coefficient of  $y_{ph}$ . Using the algorithm of division, we find a nonnegative integer  $v_1$  such that

$$S_j^{v_1} G = C_1 A_j^{(k_1)} + D_1$$

where  $D_1$  is of order less than  $h$  in  $y_p$ . In order to have a unique procedure, we take  $v_1$  as small as possible.

Suppose, for the moment, that  $p < n$ . Let  $a$  be an integer with  $p < a \leq n$ . We shall show that  $D_1$  is not of higher rank than  $G$  in  $y_a$ . We may limit ourselves to the case in which  $D_1 \neq 0$ . Also, since  $S_j$  is free of  $y_a$ , we need only treat the case in which  $y_a$  is present in  $G$ . Let  $G$  be of order  $g$  in  $y_a$ . Then the order of  $D_1$  in  $y_a$  cannot exceed  $g$ . If  $D_1$  were of higher degree than  $G$  in  $y_a$ ,  $C_1$  would have to involve  $y_{ag}$  to the same degree as  $D_1$  and  $C_1 A_j^{(k_1)}$  would contain terms involving  $y_{ag}$  and  $y_{ph}$  which would be balanced neither by  $D_1$  nor by  $S_j^{v_1} G$ . This proves our statement.

If  $D_1$  is of order greater than  $m$  in  $y_p$ , we find a relation

$$S_j^{v_2} D_1 = C_2 A_j^{(k_2)} + D_2$$

with  $D_2$  of lower order than  $D_1$  in  $y_p$  and not of higher rank than  $D_1$  (or  $G$ ) in any  $y_a$  with  $a > p$ . For uniqueness, we take  $v_2$  as small as possible.

We eventually reach a  $D_u$ , of order not greater than  $m$  in  $y_p$ , such that, if

$$s_j = v_1 + \cdots + v_u,$$

we have

$$S_j^{s_j} G = E_1 A_j^{(k_1)} + \cdots + E_u A_j^{(k_u)} + D_u.$$

Furthermore, if  $a > p$ ,  $D_u$  is not of higher rank than  $G$  in  $y_a$ .

If  $D_u$  is of order less than  $m$  in  $y_p$ ,  $D_u$  is reduced with respect to  $A_j$  (as well as to any  $A_i$  with  $i > j$ ). If  $D_u$  is of order  $m$  in  $y_p$ , we find, with the algorithm of division, a relation

$$I_j^h D_u = H A_j + K$$

with  $K$  reduced with respect to  $A_j$ , as well as to  $A_{j+1}, \dots, A_r$ . For uniqueness, we take  $t_j$  as small as possible.

If  $K$  is not reduced with respect to (4), we treat  $K$  as  $G$  was treated. For some  $l < j$ , there are  $s_l, t_l$  such that  $S_l^{s_l} I_l^{t_l} K$  exceeds, by a linear combination of  $A_l$  and its derivatives, a d.p.  $L$  which is reduced with respect to  $A_l, A_{l+1}, \dots, A_r$ . Then

$$S_l^{s_l} S_j^{s_j} I_l^{t_l} I_j^{t_j} G$$

exceeds  $L$  by a linear combination of  $A_l, A_j$  and their derivatives.

Continuing, we reach a d.p.  $R$  as described in the italicized statement.

Our procedure determines a *unique*  $R$ . We call this  $R$  *the remainder of  $G$  with respect to the chain (4)*.

IDEALS OF DIFFERENTIAL POLYNOMIALS

7. Let  $\Sigma$  be a system of d.p. in  $\mathfrak{F}\{y_1, \dots, y_n\}$ , any two d.p. in  $\Sigma$  being distinct from each other. We shall call  $\Sigma$  a *differential ideal of differential polynomials* if  $\Sigma$  satisfies the following two conditions:

(a) If  $A_1, \dots, A_r$  is any finite subset of d.p. in  $\Sigma$ ,

$$C_1 A_1 + \dots + C_r A_r$$

where the  $C$  are any d.p. at all in  $\mathfrak{F}\{y_1, \dots, y_n\}$ , is contained in  $\Sigma$ .

(b) The derivative of every d.p. in  $\Sigma$  is contained in  $\Sigma$ .

Condition (a) makes  $\Sigma$  an algebraic ideal in  $\mathfrak{F}\{y_1, \dots, y_n\}$ . Together, (a) and (b) state that, given any finite subset of d.p. in  $\Sigma$ , every linear combination of the d.p. of the subset, and of their derivatives of any orders, belongs to  $\Sigma$ . The coefficients in the linear combination may be any d.p.

Throughout our work, unless some other indication is made, the term *ideal* will stand for *differential ideal of d.p.*

An ideal contains an infinite number of d.p. unless it consists of the single d.p.  
0. The intersection of any finite or infinite number of ideals is an ideal.

An ideal  $\Sigma$  will be called *perfect* if, whenever a positive integral power of a d.p.  $A$  is contained in  $\Sigma$ ,  $A$  is contained in  $\Sigma$ . The intersection of any finite or infinite number of perfect ideals is a perfect ideal.

An ideal  $\Sigma$  will be called *prime* if, whenever a product  $AB$  is contained in  $\Sigma$ , at least one of  $A$  and  $B$  is in  $\Sigma$ . Every prime ideal is perfect.

Let  $\Lambda$  be any system of (not necessarily distinct) d.p. There exist ideals, for instance  $\mathfrak{F}\{y_1, \dots, y_n\}$ , which contain all d.p. in  $\Lambda$ . The intersection of all ideals containing  $\Lambda$  will be called the *ideal generated by  $\Lambda$*  and will be denoted by  $[\Lambda]$ . A d.p.  $A$  is contained in  $[\Lambda]$  if, and only if,  $A$  is a linear combination of d.p. in  $\Lambda$  and of derivatives, of various orders, of such d.p.<sup>7</sup>

The intersection of all perfect ideals containing  $\Lambda$  will be called the *perfect ideal determined by  $\Lambda$*  and will be denoted by  $\{\Lambda\}$ . One sees that  $\{\Lambda\}$  contains  $[\Lambda]$ .

<sup>7</sup> Unless other indications are given, the coefficients in a linear combination may be any d.p.

8. We represent by  $(\Lambda)$  the totality of linear combinations of d.p.<sup>8</sup> in  $\Lambda$  and, when we wish to express the fact that a difference  $A - B$  is in  $(\Lambda)$ , we shall write

$$A \equiv B, \quad (\Lambda).^9$$

The statements

$$A \equiv B, \quad [\Lambda] \quad ; \quad A \equiv B, \quad \{ \Lambda \},$$

will mean, respectively, that  $A - B$  is contained in  $[\Lambda]$  or in  $\{ \Lambda \}$ .

9. We are going to prove that  $\{ \Lambda \}$  consists of those d.p. which have positive integral powers in  $[\Lambda]$ . We use the following lemma, in which the field is the field of rational numbers.

**LEMMA:** For  $u$  an indeterminate and for every positive integer  $p$ ,

$$(6) \quad u_1^{2p-1} \equiv 0, \quad [u].^{10}$$

Differentiating  $u^p$  and dividing by  $p$ , we have

$$(7) \quad u^{p-1}u_1 \equiv 0, \quad [u^p],$$

which gives (6) if  $p = 1$ . Suppose that  $p > 1$ . By (7),

$$(p-1)u^{p-2}u_1^2 + u^{p-1}u_2 \equiv 0, \quad [u^p].$$

Multiplying by  $u_1$  and using (7), we find that

$$u^{p-2}u_1^3 \equiv 0, \quad [u^p]$$

and we have (6) for  $p = 2$ . Continuing, we find (6) to hold for every  $p$ .

We return now to  $\mathcal{F}\{y_1, \dots, y_n\}$  and to  $\{ \Lambda \}$ . As  $\{ \Lambda \}$  contains  $[\Lambda]$ ,  $\{ \Lambda \}$  contains every d.p. which has a power in  $[\Lambda]$ . If we can show that the totality of such d.p. is an ideal, we shall recognize that totality to be  $\{ \Lambda \}$ . If  $A$  has a power in  $[\Lambda]$ ,  $CA$ , for every  $C$ , has a power in  $[\Lambda]$ . If  $A^p$  and  $B^q$  are in  $[\Lambda]$ ,  $(A+B)^{p+q-1}$  is seen, on being expanded, to be in  $[\Lambda]$ . Thus the set of those d.p. which have powers in  $[\Lambda]$  is closed with respect to linear combination. The lemma above shows that, if  $A^p$  is in  $[\Lambda]$ , then  $(A')^{2p-1}$  with  $A'$  the derivative of  $A$ , is in  $[\Lambda]$ . Our statement is proved.

10. We prove the following lemma, in which the field is that of the rational numbers.

**LEMMA:** If  $u$  and  $v$  are indeterminates and if  $j$  is a nonnegative integer,<sup>11</sup>

$$u^{j+1}v_j \equiv 0, \quad (uv, (uv)_1, \dots, (uv)_j)$$

where  $(uv)_k$  is the  $k$ th derivative of  $uv$ .

<sup>8</sup> Note that we do not use derivatives of d.p. in  $\Lambda$ .

<sup>9</sup> If, for instance,  $\Lambda$  consists of d.p.  $C_1, \dots, C_r$ , we may write

$$A \equiv B, \quad (C_1, \dots, C_r).$$

<sup>10</sup> See remarks on notation in §2.

<sup>11</sup>  $v_0 = v$ .

The statement is true for  $j = 0$ . We make an induction to  $j = r$ , where  $r > 0$ , supposing lower values of  $j$  to have been treated. We have

$$(8) \quad uv_r - 1 \equiv 0, \quad (uv, \dots, (uv)_{r-1}).$$

Then

$$uv_r + ru^{r-1}v_{r-1} \equiv 0, \quad (uv, \dots, (uv)_r).$$

We multiply by  $u$  and use (8). The induction is seen to be accomplished.

Suppose now that,  $\mathfrak{F}$  being any field, we are given a perfect ideal  $\Sigma$  in  $\mathfrak{F}\{y_1, \dots, y_n\}$ . Let  $AB$  belong to  $\Sigma$ . By the lemma, with  $j = 1$ ,<sup>12</sup>  $A^2B'$ , with  $B'$  the derivative of  $B$ , is in  $\Sigma$ . This puts  $AB'$  in  $\Sigma$ . In general, we see that if  $\Sigma$  is a perfect ideal, and if  $AB$  is in  $\Sigma$ , every  $A^{(i)}B^{(j)}$ , superscripts indicating differentiation, is in  $\Sigma$ .

11. We denote the union<sup>13</sup> of systems  $\Sigma_1$  and  $\Sigma_2$  by  $\Sigma_1 + \Sigma_2$ . The intersection of  $\Sigma_1$  and  $\Sigma_2$  will be denoted by  $\Sigma_1 \cap \Sigma_2$ .

Let  $\Sigma$  be any system of d.p. and  $F_1, \dots, F_p$  any finite set of d.p. We shall prove that

$$(9) \quad \{\Sigma + F_1F_2 \dots F_p\} = \{\Sigma + F_1\} \cap \{\Sigma + F_2\} \cap \dots \cap \{\Sigma + F_p\}.$$

It suffices to treat the case of  $p = 2$ . The first member of (9) is easily seen to be contained in each  $\{\Sigma + F_i\}$ . It is enough, then, to consider an  $A$  which is contained in  $\{\Sigma + F_1\}$  and in  $\{\Sigma + F_2\}$  and to prove that  $A$  is in  $\{\Sigma + F_1F_2\}$ . By §9, there is a  $q$  such that

$$(10) \quad A^q = S_1 + G_1; \quad A^q = S_2 + G_2$$

with  $S_1$  and  $S_2$  in  $[\Sigma]$ ,  $G_1$  in  $[F_1]$  and  $G_2$  in  $[F_2]$ . Multiplying the two equations of (10), we have

$$(11) \quad A^{2q} = S_3 + G_1G_2$$

with  $S_3$  in  $[\Sigma]$ . Let

$$(12) \quad \begin{aligned} G_1 &= MF_1 + M_1F_1' + \dots + M_rF_1^{(r)}, \\ G_2 &= NF_2 + N_1F_2' + \dots + N_rF_2^{(r)}, \end{aligned}$$

where superscripts indicate differentiation. Now  $F_1F_2$  is in  $\{F_1F_2\}$ . By §10, every  $F_1^{(i)}F_2^{(j)}$  is in  $\{F_1F_2\}$ . By (12)  $G_1G_2$  is in  $\{F_1F_2\}$ , thus in  $\{\Sigma + F_1F_2\}$ . By (11),  $A^{2q}$ , and therefore also  $A$ , must be in  $\{\Sigma + F_1F_2\}$ .

### BASES

12. Let  $\Sigma$  be an infinite system of (not necessarily distinct) d.p. We shall call a finite subset  $\Phi$  of  $\Sigma$  a *basis* of  $\Sigma$  if  $\{\Phi\}$  contains every d.p. in  $\Sigma$ . Thus, if  $\Phi$  is a basis of  $\Sigma$ , there is, for every  $A$  in  $\Sigma$ , a positive integer  $p$ , depending on  $A$ , such that  $A^p$  is linear in the d.p. in  $\Phi$  and their derivatives of various or-

<sup>12</sup> The complete lemma will be used in Chapter III.

<sup>13</sup> Where other indications are not given, we use d.p. in  $\mathfrak{F}\{y_1, \dots, y_n\}$ , with  $\mathfrak{F}$  any field.

ders. If  $\Phi$  is a basis of  $\Sigma$ , and if  $\Phi_1$  is a finite subset of  $\Sigma$  which contains  $\Phi$ , then  $\Phi_1$  is also a basis of  $\Sigma$ .

We shall prove the

**THEOREM:** *Every infinite system of differential polynomials in  $\mathfrak{F}\{y_1, \dots, y_n\}$  has a basis.*

The fundamental role played by Raudenbush in bringing the basis theorem to its present complete form has been described in the preface.

We assume the existence of infinite systems without bases and work towards a contradiction.

13. **LEMMA:** *Let  $\Sigma$  be an infinite system with no basis. Let d.p.  $F_1, \dots, F_p$  exist such that, when each d.p. in  $\Sigma$  is multiplied by a suitable product of powers of the  $F$ , one secures a system  $\Lambda$  which has a basis. Then at least one of the systems  $\Sigma + F_i$ ,  $i = 1, \dots, p$ , has no basis.*

Let us suppose that each  $\Sigma + F_i$  has a basis. Since a basis may be enlarged (§12), we may assume that we have a finite subset  $\Phi$  of  $\Sigma$  such that  $\Phi + F_i$  is a basis for  $\Sigma + F_i$ ,  $i = 1, \dots, p$ . We suppose, furthermore, enlarging  $\Phi$  if necessary, that when each d.p. in  $\Phi$  is multiplied by a suitable product of powers of the  $F$ , we secure a basis of  $\Lambda$ .

For each  $i$ ,  $\{\Sigma + F_i\}$ , the smallest perfect ideal containing  $\Sigma + F_i$ , is contained in  $\{\Phi + F_i\}$ . Let  $K = F_1 F_2 \dots F_p$ . By §11,  $\{\Sigma + K\}$  is the intersection of the  $\{\Sigma + F_i\}$  and so is contained in every  $\{\Phi + F_i\}$ . As  $\{\Phi + K\}$  is the intersection of the  $\{\Phi + F_i\}$ ,  $\{\Sigma + K\}$  is contained in  $\{\Phi + K\}$ . Thus  $\Phi + K$  is a basis for  $\Sigma + K$ .

It follows that, for every d.p.  $A$  in  $\Sigma$ , there is a relation

$$A^q = G + MK + M_1 K' + \dots + M_r K^{(r)}$$

with  $G$  in  $[\Phi]$ . Then

$$(13) \quad A^{q+1} = GA + MKA + M_1 K'A + \dots + M_r K^{(r)}A.$$

We shall prove that  $KA$  is in  $\{\Phi\}$ . We know that  $\Lambda$  has a basis  $\Psi$ , each d.p. in  $\Psi$  being obtained from one in  $\Phi$  by a multiplication by a power product in the  $F$ . We see immediately that  $\{\Phi\}$  contains  $\{\Psi\}$ ; also that some power of  $KA$  is in  $\{\Psi\}$ . Then  $KA$  is in  $\{\Psi\}$  and hence in  $\{\Phi\}$ .

By §10, each  $K^{(r)}A$  is in  $\{\Phi\}$ . As  $GA$  is in  $\{\Phi\}$ , we see from (13) that  $A^{q+1}$  is in  $\{\Phi\}$ . Thus  $A$  is in  $\{\Phi\}$ , that is,  $\Phi$  is a basis for  $\Sigma$ . This proves the lemma.

14. From among all infinite systems which lack bases, we select one,  $\Sigma$ , whose characteristic sets (§5) are not higher than those of any other system which lacks a basis (§4). Let (4) be a characteristic set of  $\Sigma$ . Then  $A_1$  is not of class zero; otherwise  $A_1$ , an element of  $\mathfrak{F}$ , would be a basis of  $\Sigma$ .

For every d.p. in  $\Sigma$  which is not in (4), let a remainder with respect to (4) be found as in §6. Let  $\Lambda$  be the system composed of the d.p. in (4) and of the

products of the d.p. of  $\Sigma$  not in (4) by the products  $S_1^{a_1} \cdots I_r^{a_r}$  used in their reduction. Let  $\Omega$  be the system composed of (4) and of the remainders of the d.p. of  $\Sigma$  not in (4).

Now  $\Omega$  must have a basis. Otherwise it would certainly have nonzero d.p. not in (4). As such a d.p. would be reduced with respect to (4), (4) could not be a characteristic set of  $\Omega$  (§5). This means that the characteristic sets of  $\Omega$  would be lower than (4), and  $\Sigma$  would not be a system, lacking a basis, of lowest characteristic sets.

We may suppose that  $\Omega$  has a basis  $\Phi$  composed of d.p.

$$A_1, \cdots, A_r; \quad R_1, \cdots, R_s.$$

Let  $H_i$  be the d.p. of  $\Lambda$  which corresponds to  $R_i$ ,  $i = 1, \cdots, s$ . Let  $\Psi$  be the set

$$A_1, \cdots, A_r; \quad H_1, \cdots, H_s.$$

We wish to see that  $\Psi$  is a basis for  $\Lambda$ .

We know that  $\{\Phi\}$  contains  $[A_1, \cdots, A_r]$ .

Because

$$H_i \equiv R_i, \quad [A_1, \cdots, A_r], \quad i = 1, \cdots, s,$$

we have  $H_i \equiv R_i, \{\Phi\}$ . As each  $R_i$  is in  $\{\Phi\}$ , each  $H_i$  is in  $\{\Phi\}$ . Hence  $\{\Phi\}$  contains  $\{\Psi\}$ . Reciprocally,  $\{\Psi\}$  contains  $\{\Phi\}$ , so that  $\{\Phi\}$  and  $\{\Psi\}$  are identical. Now if  $H$  is any d.p. of  $\Lambda$  and  $R$  the corresponding d.p. of  $\Omega$ , we see as above that  $H$ , like  $R$ , is in  $\{\Phi\}$ , therefore in  $\{\Psi\}$ . Thus  $\Psi$  is a basis for  $\Lambda$ .

The lemma of §13 informs us that at least one of the systems  $\Sigma + S_i, \Sigma + I_i$  has no basis. But, for each  $i$ ,  $S_i$  and  $I_i$  are distinct from zero and reduced with respect to (4). Then, by §5, the characteristic sets of  $\Sigma + S_i$  and  $\Sigma + I_i$  are lower than (4). This produces a final contradiction and the truth of the basis theorem of §12 follows.

#### STRONG AND WEAK BASES

15. A basis  $\Phi$  of a system  $\Sigma$  will be called a *strong* basis if there exists a positive integer  $p$  such that the  $p$ th power of every d.p. in  $\Sigma$  is in  $[\Phi]$ . Bases which are not strong may be called *weak*.

We are going to give an example of a system which has no strong basis.<sup>14</sup>

Let  $\Sigma_1$  and  $\Sigma_2$  be ideals. Let  $\Sigma$  be the totality of products  $AB$  where  $A$  is any d.p. of  $\Sigma_1$  and  $B$  any d.p. of  $\Sigma_2$ . We shall call  $[\Sigma]$  the *product* of  $\Sigma_1$  and  $\Sigma_2$  and shall write  $[\Sigma] = \Sigma_1 \cdot \Sigma_2$ . Multiplication as thus defined is commutative and associative.

We use a single indeterminate  $y$  and any field  $\mathfrak{F}$ . We shall prove that the ideal  $[y]^2$  has no strong basis. Suppose that there is a strong basis. Then there is one,  $\Phi$ , made up of d.p.

<sup>14</sup> Raudenbush, 23, and Kolchin, 9. See bibliography on page 180.

$$(14) \quad y_i y_j, \quad 0 \leq i \leq j \leq s,$$

where  $s$  is some integer.

Let  $p$  be such that the  $p$ th power of every d.p. in  $[y]^2$  is in  $[\Phi]$ . We shall prove that the product of any  $p$  d.p. in  $[y]^2$  is in  $[\Phi]$ .

For any  $A$  and  $B$ , and for any positive integer  $r$ , the  $r + 1$  powers  $(A + iB)^r$ ,  $i = 0, \dots, r$ , are linear in the  $r + 1$  products  $A^i B^j$ ,  $i + j = r$ , with a non-vanishing determinant. It follows that  $AB^{r-1}$  is linear in the  $(A + iB)^r$  with rational coefficients. Thus, for instance,  $AB$  is a sum of three terms  $a_i M_i^2$  with rational  $a$  and with  $M$  which are linear, with integral coefficients, in  $A$  and  $B$ . This implies, by what precedes, that for any  $A, B, C$ , the product  $ABC$  is a sum of terms  $a_i M_i^3$  with  $M$  which are linear, with integral coefficients, in  $A, B, C$ .

In this way, we find that the product of any  $p$  d.p. in  $[y]^2$  is linear in  $p$ th powers, and is therefore in  $[\Phi]$ .

Let  $\alpha$  be a positive integer, which will be fixed at a large value later. We consider all d.p.

$$(15) \quad y_{i_1} y_{i_2} \cdots y_{i_p}$$

for which  $i_1 + i_2 + \cdots + i_p = \alpha$ . Each d.p. in (15) is the product of  $p$  d.p. in  $[y]^2$ . If we define the weight of a product of  $y_j$  as the sum of the subscripts, the weight of each d.p. in (15) is  $\alpha$ .

By the nature of (14) and by the homogeneity and isobaricity of the d.p. in (14) and (15), each d.p. in (15) is linear, with coefficients in  $\mathfrak{F}$ , in the d.p.

$$(16) \quad (y_i y_j)_k y_{j_1} \cdots y_{j_{p-2}}$$

where the subscript  $k$  indicates  $k$  differentiations and where one uses all non-negative  $i, j, k, j_q$  for which

$$0 \leq i \leq j \leq s, \quad i + j + k + j_1 + \cdots + j_{2p-2} = \alpha.$$

A set of distinct d.p. (15) is a linearly independent set; we mean by this that there is no nonidentical linear relation, with coefficients in  $\mathfrak{F}$ , among the d.p. Thus the number of distinct d.p. (15) does not exceed the number of distinct d.p. (16).

Let  $\alpha$  be divisible by  $2p$ . Let us consider these d.p. (15) which are obtained by assigning to  $i_1, \dots, i_{2p-1}$  arbitrary values from 0 to  $\alpha/(2p)$  inclusive. When such assignments are made  $i_{2p}$  is determined. The number of times which any one d.p. can be secured from such assignments is at most  $(2p)!$ . Then there are at least

$$(17) \quad \frac{1}{(2p)!} \left( \frac{\alpha}{2p} + 1 \right)^{2p-1}$$

distinct d.p. (15).

In (16) when all subscripts except  $k$  are selected,  $k$  is determined. As the

number of d.p. in (14) is  $(s + 1)(s + 2)/2$ , the number of distinct d.p. (16) is not more than

$$(18) \quad \frac{1}{2}(s + 1)(s + 2)(\alpha + 1)^{2p - 2}.$$

For  $\alpha$  large, the quantity in (17) exceeds that in (18). This furnishes a contradiction which proves that  $[y]^2$  has no strong basis.

DECOMPOSITION OF PERFECT IDEALS

16. We prove the following theorem:<sup>15</sup>

**THEOREM:** *Every perfect ideal of differential polynomials in  $\mathfrak{F}\{y_1, \dots, y_n\}$  has a representation as the intersection of a finite number of prime ideals.*

We suppose that we have a perfect ideal  $\Sigma$  with no such representation. Then  $\Sigma$  is not prime.<sup>16</sup> Let  $A$  and  $B$  be d.p. which are absent from  $\Sigma$  while  $AB$  is contained in  $\Sigma$ . By §11,  $\Sigma$  is the intersection of  $\{\Sigma + A\}$  and  $\{\Sigma + B\}$ . At least one of the latter two ideals must fail to be the intersection of a finite number of prime ideals. Suppose that  $\{\Sigma + A\}$ , which we denote by  $\Sigma_1$ , so fails. Repeating our argument, we find  $\Sigma_1$  to be a proper part of a perfect ideal  $\Sigma_2$  which is not an intersection of a finite number of prime ideals. Continuing, we form, with the help of the axiom of selection, an infinite sequence of perfect ideals

$$(19) \quad \Sigma, \Sigma_1, \dots, \Sigma_p, \dots$$

each a proper part of its successor. Let  $\Omega$  be the union of the ideals in (19) and let  $\Phi$  be a basis for  $\Omega$ .<sup>17</sup> Then  $\Phi$  is contained in some ideal in (19), say in  $\Sigma_q$ . Then  $\Sigma_q$  contains  $\{\Phi\}$ , hence  $\Omega$  and thus  $\Sigma_{q+1}$ . This contradiction proves the theorem.

17. If an ideal  $\Sigma_2$  contains an ideal  $\Sigma_1$ ,  $\Sigma_2$  will be called a *divisor* of  $\Sigma_1$ .

Let a perfect ideal  $\Sigma$  have a representation

$$(20) \quad \Sigma = \Sigma_1 \cap \Sigma_2 \cap \dots \cap \Sigma_p$$

as an intersection of prime ideals.

If  $\Sigma_1$  is a divisor of any other  $\Sigma_i$ , we may suppress  $\Sigma_1$  in (20). On this basis, we suppose that  $\Sigma_1$  is not a divisor of any other  $\Sigma_i$ . By repeated purging, we obtain a representation (20) with no  $\Sigma_i$  a divisor of any  $\Sigma_j$  with  $j \neq i$ .

Let  $\Sigma'$  be any prime divisor of  $\Sigma$ . We shall prove that  $\Sigma'$  is a divisor of some  $\Sigma_i$  in (20). Let this be false, and let  $A_i$ , for each  $i$ , be a d.p. in  $\Sigma_i$  which is not in  $\Sigma'$ . Then  $A_1 A_2 \dots A_p$  is not in  $\Sigma'$ . This contradicts the fact that the product is in  $\Sigma$ .

A prime divisor of  $\Sigma$  which is not a divisor of any other prime divisor of  $\Sigma$

<sup>15</sup> Raudenbush, 21.

<sup>16</sup> We understand the "intersection" of a single aggregate to be that aggregate.

<sup>17</sup> Note that  $\Sigma$ , as it is not prime, does not consist of the single d.p. 0.

will be called an *essential prime divisor* of  $\Sigma$ . The only essential prime divisors of  $\Sigma$  are the  $\Sigma_i$  in (20). Every prime divisor of  $\Sigma$  is a divisor of an essential prime divisor. Our discussion of (20) shows that, in every representation of a perfect ideal  $\Sigma$  as the intersection of a finite number of prime ideals, every essential prime divisor of  $\Sigma$  appears; the other prime divisors are redundant.

We partially summarize what precedes as follows: *Every perfect ideal has a finite number of essential prime divisors, and is the intersection of those divisors.*

#### RELATIVELY PRIME IDEALS

18. Two ideals  $\Sigma_1$  and  $\Sigma_2$  will be said to be *relatively prime* if there are an  $A_1$  in  $\Sigma_1$  and an  $A_2$  in  $\Sigma_2$  such that  $A_1 + A_2 = 1$ .

Let  $\Sigma_1$  and  $\Sigma_2$  be ideals and suppose that  $\{\Sigma_1\}$  and  $\{\Sigma_2\}$  are relatively prime. We shall prove that  $\Sigma_1$  and  $\Sigma_2$  are relatively prime. Let  $A_1 + A_2 = 1$  with  $A_1$  in  $\{\Sigma_1\}$  and  $A_2$  in  $\{\Sigma_2\}$ . Let  $q$  be such that  $A_1^q$  and  $A_2^q$  are in  $\Sigma_1$  and  $\Sigma_2$  respectively. In the expansion of  $(A_1 + A_2)^{2q-1}$ , we let  $B_1$  be the sum of these terms in which the exponent of  $A_1$  is at least  $q$  and  $B_2$  the sum of the remaining terms. Then  $B_1$  is in  $\Sigma_1$ ,  $B_2$  in  $\Sigma_2$  and

$$(21) \quad B_1 + B_2 = 1.$$

We show that the intersection of two relatively prime ideals is their product (§15). The product is in the intersection. Let  $G$  be in the intersection. Then  $G = GB_1 + GB_2$  with  $B_1$  and  $B_2$  as above.  $GB_1$  and  $GB_2$  are in the product; so then is  $G$ .

An ideal which is relatively prime to each of several ideals is easily seen to be relatively prime to their intersection.<sup>18</sup> It follows that, given several ideals, every one of which is relatively prime to every other, the intersection of the ideals is their product.

19. We derive a theorem of decomposition whose significance will be seen after the theory of algebraic differential manifolds is developed in the following chapter.

**THEOREM:** *Let  $\Sigma$  be an ideal. Suppose that  $\{\Sigma\}$  has a representation*

$$(22) \quad \{\Sigma\} = \Omega_1 \cap \Omega_2 \cap \cdots \cap \Omega_p$$

where the  $\Omega$  are perfect ideals,<sup>19</sup> every one of which is relatively prime to every other. Then  $\Sigma$  has a unique representation,

$$(23) \quad \Sigma = \Sigma_1 \cap \Sigma_2 \cap \cdots \cap \Sigma_p,$$

where the  $\Sigma_i$  are ideals such that  $\{\Sigma_i\} = \Omega_i$ .

By §18, the  $\Sigma_i$  are relatively prime in pairs and the intersection in (23) is a product.

<sup>18</sup> One multiplies the equations which express the relative primeness.

<sup>19</sup> Not necessarily prime.

We treat first the case of  $p = 2$ . Let  $A_1 + A_2 = 1$  with  $A_1$  in  $\Omega_1$  and  $A_2$  in  $\Omega_2$ . As  $A_1A_2$  is in  $\{\Sigma\}$ , some  $(A_1A_2)^q$  is in  $\Sigma$ . In  $(A_1 + A_2)^{2^q - 1}$ , let  $B_1$  be the sum of those terms in which the exponent of  $A_1$  is at least  $q$  and  $B_2$  the sum of the remaining terms. Then  $B_1$  is in  $\Omega_1$  and  $B_2$  in  $\Omega_2$ .  $B_1B_2$  is in  $\Sigma$  and  $B_1 + B_2 = 1$ .

We shall prove that  $B_1'$  and  $B_2'$ , accents indicating differentiation, are in  $\Sigma$ . As  $B_1B_2$  is in  $\Sigma$ ,

$$(24) \quad B_1'B_2 + B_1B_2' \equiv 0, \quad (\Sigma).$$

As  $B_1 = 1 - B_2$  and  $B_2' = -B_1'$ , (24) gives  $B_1' \equiv 2B_2B_1'$ ,  $(\Sigma)$ . Then  $B_1' \equiv 4B_2^2B_1'$ ,  $(\Sigma)$ . We know from §10 that  $B_2^2B_1'$  is in  $\Sigma$ . Thus  $B_1'$  is in  $\Sigma$ ; so also is  $B_2'$ .

Let  $\Sigma_i = [\Sigma + B_i]$ ,  $i = 1, 2$ . We prove that  $\{\Sigma_1\} = \Omega_1$ . Because  $B_1$  is in  $\Omega_1$ , that ideal contains  $\{\Sigma_1\}$ . It suffices then to show that if a d.p.  $G$  is in  $\Omega_1$ ,  $G$  is in  $\{\Sigma_1\}$ . Now  $B_2G$  is in  $\{\Sigma\}$  and therefore in  $\{\Sigma_1\}$ . Again,  $B_2G$  equals  $G - B_1G$ . As  $B_1$  is in  $\Sigma_1$ ,  $G$  is in  $\{\Sigma_1\}$ . Similarly  $\{\Sigma_2\} = \Omega_2$ .

We prove now that  $\Sigma = \Sigma_1 \cap \Sigma_2$ . It suffices to show that if  $G$  is any d.p. common to  $\Sigma_1$  and  $\Sigma_2$ ,  $G$  is in  $\Sigma$ . As  $G$  is in  $\Sigma_1$  and  $B_1'$  is in  $\Sigma$ , we have

$$(25) \quad G = C + DB_1$$

with  $C$  in  $\Sigma$ . Because  $G$  is in  $\Sigma_2$ ,  $DB_1$  is in  $\Sigma_2$ . As  $B_1 = 1 - B_2$  and  $B_2$  is in  $\Sigma_2$ ,  $D$  is in  $\Sigma_2$ . Let  $D = E + FB_2$  with  $E$  in  $\Sigma$ . As  $B_1B_2$  is in  $\Sigma$ ,  $DB_1$  is in  $\Sigma$ . This puts  $G$  in  $\Sigma$ .

We have obtained a representation (23). We have to prove uniqueness. Let a second representation be  $\Sigma = \Sigma_1' \cap \Sigma_2'$ . Let  $G$  be any d.p. in  $\Sigma_1'$ . As  $B_2$  is in  $\Omega_2$ , some  $B_2^t$  is in  $\Sigma_2'$ . Then  $GB_2^t$  is in  $\Sigma$ . As  $B_2 = 1 - B_1$  and  $B_1$  is in  $\Sigma_1$ ,  $G$  is in  $\Sigma_1$ . Again, let  $H$  be any d.p. in  $\Sigma_1$ . For some  $t$ ,  $\Sigma_1'$  contains  $B_1^t$  and therefore  $HB_1^t$ . As  $B_1 = 1 - B_2$  and  $HB_2$  is in  $\Sigma$ ,  $\Sigma_1'$  contains  $H$ . We have proved that  $\Sigma_1$  and  $\Sigma_1'$  are identical. So also are  $\Sigma_2$  and  $\Sigma_2'$ . This settles the question of uniqueness.

We now consider any  $p > 2$  and perform an induction. Let  $\Omega'$  be the intersection of the  $\Omega_i$  with  $i > 1$ . By §18,  $\Omega_1$  and  $\Omega'$  are relatively prime. In a unique way,  $\Sigma = \Sigma_1 \cap \Sigma'$  with  $\{\Sigma_1\} = \Omega_1$  and  $\{\Sigma'\} = \Omega'$ . Also  $\Sigma'$  is a unique intersection of ideals  $\Sigma_2, \dots, \Sigma_p$  with  $\{\Sigma_i\} = \Omega_i$  so that we have a representation (23). We have to prove uniqueness. Consider any representation (23) and let  $\Sigma''$  be the intersection of  $\Sigma_2, \dots, \Sigma_p$ . Obviously,  $\{\Sigma''\}$  is in the intersection of  $\Omega_2, \dots, \Omega_p$ . If  $A$  is in that intersection, some  $A^t$  is in each of  $\Sigma_2, \dots, \Sigma_p$ , hence in  $\Sigma''$ . Then  $\{\Sigma''\}$  is identical with  $\Omega'$  as above. It follows that  $\Sigma'' = \Sigma'$  and that (23) is unique.

If the  $\Omega$  are not relatively prime, there may be no decomposition (23) with  $\{\Sigma_i\} = \Omega_i$ . Thus, in  $\mathfrak{F}\{u, v\}$ ,  $\{uv\} = \{u\} \cap \{v\}$ . Levi has shown that  $[uv]$  has no representation  $\Sigma_1 \cap \Sigma_2$  or  $\Sigma_1 \Sigma_2$  where  $\Sigma_1$  and  $\Sigma_2$  are ideals with  $\{\Sigma_1\} = \{u\}$  and  $\{\Sigma_2\} = \{v\}$ .<sup>20</sup>

<sup>20</sup> Levi, 17.

20. Each  $\Sigma_i$  in (23) is of the form  $[\Sigma + B_i]$  with  $B'_i$  in  $\Sigma$ . Suppose now that  $\Sigma$  has a strong basis  $\Phi$  and that the  $q$ th power of every d.p. in  $\Sigma$  is in  $[\Phi]$ . Let  $G$  be any d.p. in  $\Sigma_1$ . Then (25) holds with  $C$  in  $\Sigma$ . It follows that  $G^q$  is in  $[\Phi + B_1]$ . Thus, for every  $i$ ,  $\Phi + B_i$  is a strong basis for  $\Sigma_i$  and the  $q$ th power of every d.p. in  $\Sigma_i$  is in  $[\Phi + B_i]$ .

We would state in conclusion that it is possible to generalize the theory of ideals of d.p. into a theory which applies to algebraic rings of any type in which operations of differentiation exist.<sup>21</sup>

#### THE IDEAL $[y^p]$

21. We work in  $\mathfrak{F}\{y\}$ .<sup>22</sup> The ideal  $[y^p]$ , where  $p$  is any positive integer, has properties which will be useful in Chapters III and VII.

We consider any power product

$$(26) \quad P = y_0^{q_0} y_1^{q_1} \cdots y_r^{q_r}$$

where  $r$  and the  $q$  are any nonnegative integers. The degree and weight of  $P$  will be

$$d = q_0 + \cdots + q_r, \quad w = q_1 + 2q_2 + \cdots + rq_r,$$

respectively. We are going to find a condition on  $d$  and  $w$  which will be sufficient for  $P$  to belong to  $[y^p]$ .

If  $p = 1$ , every  $P$  of positive degree is in  $[y^p]$ . In what follows, we assume that  $p > 1$ .

Let  $d$  be any positive integer. One can express  $d$  in one and only one way in the form

$$(27) \quad d = a(p - 1) + b$$

where  $a$  and  $b$  are integers with  $a \geq 0$ ,  $0 < b \leq p - 1$ . Let

$$f(p, d) = a(a - 1)(p - 1) + 2ab.$$

The function  $f(p, d)$  has the property that, if  $d > p - 1$ ,

$$(28) \quad f(p, d) - f(p, d - p + 1) = 2(d - p + 1).$$

We prove the following theorem, which is due to Levi.<sup>20</sup>

**THEOREM:** Let  $p > 1$  and let  $P$  be a power product in the  $y_i$ , of positive degree  $d$  and of weight  $w$ . If  $w < f(p, d)$ , then  $P \equiv 0$ ,  $[y^p]$ .

22.  $P$  of (26), distinct from<sup>23</sup>

$$Q = y_0^{q'_0} \cdots y_r^{q'_r},$$

<sup>21</sup> Raudenbush 21, and Kolchin, 10, 11.

<sup>22</sup> One understands that  $\mathfrak{F}$  is any field.

<sup>23</sup> When two products have distinct  $r$ , the  $r$  may be made equal by the adjunction of zero powers.

will be said to be lower than  $Q$  if the nonzero difference  $q_i - q'_i$  of greatest  $i$  is negative. This is a transitive relation and, if  $P$  is lower than  $Q$ ,  $RP$ , with  $R$  any power product, is lower than  $RQ$ .

23. Let  $A = y^p$ . We denote the  $j$ th derivative of  $A$  by  $A_j$ . Then  $A_j$  is a sum of terms  $c_i R_i$  where the  $c$  are positive integers and the  $R$  are power products of degree  $p$  and weight  $j$ . Every power product of degree  $p$  and weight  $j$  is an  $R$  in  $A_j$ ; this is easily proved by induction.

Let  $j$ , any nonnegative integer, be written in the form  $rp + s$  where  $r$  and  $s$  are nonnegative integers and  $s < p$ . We shall show that the lowest power product in  $A_j$  is

$$I_{j} = y_r^{p-s} y_{r+1}^s.$$

Suppose that some power product  $P$  in  $A_j$  is not higher than  $L_j$ . Then  $P$  involves no  $y_i$  with  $i > r + 1$  and the degree  $t$  of  $P$  in  $y_{r+1}$  does not exceed  $s$ .  $P$  has  $p - t$  factors  $y_i$  with  $i \leq r$ . Their total weight does not exceed  $r(p - t)$ . Then the weight of  $P$  is no more than  $rp + t$ . Hence  $t = s$  and each factor  $y_i$  of  $P$  with  $i \leq r$  is  $y_r$ . Thus  $P = L$ .

24.  $P$  in (26) will be called a *weak product* if, for  $i = 0, \dots, r - 1$ , one has  $q_i + q_{i+1} < p$ . Thus  $P$  is weak if and only if there is no  $A_j$  whose  $L_j$  is a factor of  $P$ . If  $P$  is not weak, it will be called *strong*.

We prove that if  $F$  is a d.p. with rational coefficients, homogeneous, of degree  $d > 0$ , and isobaric, of weight  $w$ , either  $F \equiv 0, [A]$ , or

$$(29) \quad F \equiv \sum_{k=1}^t a_k Q_k, \quad [A],$$

where  $t$  is a positive integer, the  $a$  rational numbers and the  $Q$  weak products of degree  $d$  and weight  $w$ .

If the power products in  $F$  are all weak, there is nothing to prove. Otherwise, let  $F = gG + R$  where  $G$  is the lowest strong product in  $F$  and  $R$  is free of  $G$ . Let  $G = HL_j$  with  $L_j$  the lowest product in  $A_j$ . Then

$$A_j = rL_j + \sum_{i=1}^m c_i P_i,$$

with  $r$  and  $m$  integers and the  $P$  higher than  $L_j$ . We have

$$F = \frac{g}{r} H (A_j - \sum c_i P_i) + R,$$

so that

$$(30) \quad F \equiv -\frac{g}{r} \sum c_i H P_i + R, \quad [A].$$

The strong products in the second member of (30) are higher than  $G$ . A finite number of repetitions of this process will remove the strong products, so that either  $F \equiv 0, [A]$ , or a relation (29) holds.

25. Now let  $d$  be any positive integer. We shall show that *there exists no weak product of degree  $d$  whose weight is less than  $f(p, d)$ .*

Let  $Q$  be a weak product of degree  $d$ . If  $d \leq p - 1$ ,  $f(p, d) = 0$ , since  $a = 0$  in (27). Thus the weight of  $Q$  is not less than  $f(p, d)$ . Assuming our statement to hold for  $d < s$ , where  $s > p - 1$ , we shall prove it for  $d = s$ . We can write  $Q$ , of degree  $s$ , in the form

$$Q = y_0 y_1 \cdots y_{p-1} Q',$$

with  $Q'$  a power product which involves no derivative which is lower than one or more of the  $y_j$  standing before  $Q'$ . If  $Q'$  involved  $y_0$ ,  $Q$  would be divisible by  $y_0^p$ . If  $Q'$  involved  $y_1$ , the exponents of  $y_0$  and  $y_1$  in  $Q$  would add up to at least  $p$ . Thus  $Q'$  is free of  $y_0$  and  $y_1$ .

Let each  $y_i$  in  $Q'$  be replaced by  $y_{i-2}$ . Then  $Q'$  goes over into a weak product  $Q''$  whose weight is less than that of  $Q'$  by  $2(s - p + 1)$ . By (28), if the weight of  $Q$  were less than  $f(p, s)$ , that of  $Q''$  would be less than  $f(p, s - p + 1)$ . This cannot be, since  $Q''$  is a weak product of degree  $s - p + 1$ .

If now we refer to §24, the theorem of §21 is seen to be established.<sup>24</sup>

Levi showed that for every  $w$  not less than  $f(p, d)$ , there is a power product of degree  $d$  and weight  $w$  which is not in  $[y^p]$ . The proof is too long to be given here.

26. Let  $p$  be any positive integer and  $P$  any power product in the  $y_i$ , of degree  $d$  and weight  $w$ . We prove that if

$$(31) \quad d > \frac{p-1}{2} + \left[ (p-1)w + \frac{(p-1)^2}{4} \right]^{1/2},$$

then  $P \equiv 0, [y^p]$ .

We suppose, as we may, that  $p > 1$ . Let (31) be satisfied. Then

$$(32) \quad (p-1)w < d^2 - d(p-1).$$

Let  $d$  be expressed as in (27). As  $b(p-1-b) \geq 0$ , (32) gives

$$(33) \quad (p-1)w < d^2 - d(p-1) + b(p-1-b).$$

We replace  $d$  in (33) by its expression in (27), finding that  $w < f(p, d)$ . This proves our statement.

#### ADJUNCTION OF INDETERMINATES

27. It is sometimes desirable to enlarge the system of indeterminates  $y_1, \cdots, y_n$ , introducing by their side a new indeterminate  $v$ .

We consider a prime ideal  $\Sigma$  in  $\mathfrak{F}\{y_1, \cdots, y_n\}$ . We can generate with  $\Sigma$  an ideal  $\Sigma_1$  in  $\mathfrak{F}\{y_1, \cdots, y_n; v\}$ . If  $A$  is in  $\Sigma_1$ ,  $A$  is linear in d.p. of  $\Sigma$ , with

<sup>24</sup> In dealing with partial d.p. in Chapter IX, we shall obtain a theorem similar to that of §21 by a simpler method, due to Kolchin. That method does not give the best bound, as the above method does.

d.p. in  $\mathfrak{F}\{y_1, \dots, y_n; v\}$  for coefficients; if  $A$  is arranged as a polynomial in the  $v_i$ , it will have d.p. in  $\Sigma$  for coefficients.

We are going to show that  $\Sigma_1$  is prime.

Let  $A$  and  $B$  be absent from  $\Sigma_1$ , while  $AB$  is contained in it. Let  $A, B$  and  $AB$  be arranged as polynomials in the  $v_i$ . We suppose that no coefficient in  $A$  or  $B$  is in  $\Sigma$ , suppressing those which are.

We order power products as in §22. The coefficient of the first term in  $AB$  is the product of those in  $A$  and  $B$ . This, since  $\Sigma$  is prime and the coefficients in  $A$  and  $B$  are not in  $\Sigma$ , furnishes a contradiction which proves our statement.

FIELD EXTENSIONS

28. Let  $\Sigma$  be an ideal in  $\mathfrak{F}\{y_1, \dots, y_n\}$ . Let  $\mathfrak{F}_1$  be an extension of  $\mathfrak{F}$  (§1) with respect to which the  $y$  are indeterminates.<sup>25</sup>  $\Sigma$  generates an ideal  $\Sigma_1$  in  $\mathfrak{F}_1\{y_1, \dots, y_n\}$ . Each d.p. in  $\Sigma_1$  is linear in d.p. of  $\Sigma$  with coefficients which are d.p. over  $\mathfrak{F}_1$ .

A set of elements  $\gamma_1, \dots, \gamma_r$  of  $\mathfrak{F}_1$  will be said to be *linearly independent with respect to  $\mathfrak{F}$* , or *independent*, if there exists no relation

$$c_1\gamma_1 + \dots + c_r\gamma_r = 0$$

with the  $c$  in  $\mathfrak{F}$  and not all zero.

It is easy to see that every nonzero d.p. over  $\mathfrak{F}_1$  can be written in the form

$$(34) \quad \gamma_1 A_1 + \dots + \gamma_r A_r$$

with  $A$  which are d.p. over  $\mathfrak{F}$ , and with independent  $\gamma$ . Of course,  $r$  is different for different d.p.

We shall prove that if a nonzero d.p.  $G$  in  $\Sigma_1$  is written in the form (34) with the  $A$  d.p. over  $\mathfrak{F}$  and with the  $\gamma$  independent, each  $A$  is in  $\Sigma$ .

$G$  can be written as a linear combination, with coefficients in  $\mathfrak{F}_1$ , of d.p. in  $\Sigma$ . Let

$$(35) \quad \beta_1 B_1 + \dots + \beta_s B_s$$

be such an expression for  $G$ , with  $s$  as small as possible.

We wish to show that each  $\beta$  is linear in the  $\gamma$  in (34), with coefficients in  $\mathfrak{F}$ . Given any power product in the  $y_{ij}$ , we find from (35) that its coefficient in  $G$  is linear in the  $\beta$ . We secure thus a system of linear equations for the  $\beta$ . We say that this system is of rank  $s$ ; that is, it determines the  $\beta$  uniquely. The system is compatible. If it were of rank less than  $s$ , we would be able to replace some of the  $\beta$  by zero and determine the remaining  $\beta$  so as to satisfy the system. Then  $s$  in (35) would not be a minimum.

Thus the  $\beta$  are linear in the  $\gamma$  with coefficients in  $\mathfrak{F}$ . Then, by (35),  $G$  has an expression

$$\gamma_1 C_1 + \dots + \gamma_r C_r$$

---

<sup>25</sup> Thus a d.p. in the  $y$  over  $\mathfrak{F}_1$  (§3) is zero only when all its coefficients are zero.

with the  $C$  in  $\Sigma$ . As

$$\gamma_1(C_1 - A_1) + \cdots + \gamma_r(C_r - A_r) = 0,$$

it follows easily that  $C_i = A_i$ ,  $i = 1, \cdots, p$ . This proves our statement.

#### FIELDS OF CONSTANTS

29. We shall at times wish to assume that  $\mathfrak{F}$  contains at least one element which is not a constant (§1). We establish now a result which will permit us to make this assumption with no real loss of generality.

Suppose that  $\mathfrak{F}$  consists purely of constants. We adjoin to  $\mathfrak{F}$  a quantity  $x$  which we suppose to be *transcendental* with respect to  $\mathfrak{F}$ . By this, we mean that, considering  $x$  as a pure symbol, we form the totality  $\mathfrak{F}_1$  of rational combinations of  $x$  with coefficients in  $\mathfrak{F}$ . Each element of  $\mathfrak{F}_1$  is of the form  $P/Q$  with  $P$  and  $Q$  polynomials in  $x$  with coefficients in  $\mathfrak{F}$ . Two polynomials in  $x$  are considered equal only if coefficients of corresponding powers of  $x$  are equal.<sup>26</sup> Two expressions  $P_1/Q_1$  and  $P_2/Q_2$  are equal if  $P_1Q_2 = P_2Q_1$ . We attribute to  $x$  a derivative equal to unity and differentiate polynomials  $P$ , and quotients  $P/Q$ , using the familiar formulas of the calculus. On this basis,  $\mathfrak{F}_1$  becomes a differential field and, indeed, an extension of  $\mathfrak{F}$ .

Suppose that we have a prime ideal  $\Sigma$  in  $\mathfrak{F}\{y_1, \cdots, y_n\}$ . Let  $\Sigma_1$  be the ideal of d.p. over  $\mathfrak{F}_1$  generated by  $\Sigma$ .

We shall show that  $\Sigma_1$  is *prime*.

Let  $F$  be any d.p. in  $\Sigma_1$  of the type

$$B_0 + B_1x + \cdots + B_rx^r$$

where the  $B$  are d.p. over  $\mathfrak{F}$ . As  $1, x, \cdots, x^r$  are linearly independent with respect to  $\mathfrak{F}$ , each  $B$  is in  $\Sigma$  (§28).

Now let  $AB$ , but neither  $A$  nor  $B$ , be contained in  $\Sigma_1$ . Multiplying  $A, B, AB$  by elements of  $\mathfrak{F}_1$ , we may suppose them to be polynomials in  $x$  with d.p. over  $\mathfrak{F}$  for coefficients. The coefficients in  $AB$  are in  $\Sigma$ . In  $A$  and  $B$ , we suppress all terms with coefficients in  $\Sigma$ . We may suppose  $A$  and  $B$ , when arranged in ascending powers of  $x$ , to start with terms free of  $x$ . Now  $\Sigma$  is prime, the first term of  $AB$  is in  $\Sigma$ , the first terms in  $A$  and  $B$  are not in  $\Sigma$ . This contradiction proves our statement.

<sup>26</sup> This is the basis for calling  $x$  transcendental with respect to  $\mathfrak{F}$ .

## CHAPTER II

### ALGEBRAIC DIFFERENTIAL MANIFOLDS

#### MANIFOLDS AND THEIR DECOMPOSITION

1. Let  $\Sigma$  be any finite or infinite system of d.p. in  $\mathcal{F}\{y_1, \dots, y_n\}$ . Let there be given an extension  $\mathcal{F}_1$  of  $\mathcal{F}$ .<sup>1</sup> Suppose that there exists in  $\mathcal{F}_1$  a set of  $n$  elements  $\eta_1, \dots, \eta_n$  which are such that when each  $y_i$  is replaced by  $\eta_i$  in the d.p. of  $\Sigma$ , those d.p. all reduce to zero. We shall call the set  $\eta_1, \dots, \eta_n$  a *zero* of  $\Sigma$ . Thus a zero of  $\Sigma$  is a *solution* of the system of equations obtained by equating the d.p. in  $\Sigma$  to zero.

If  $\Sigma$  has zeros, the totality of its zeros, for all possible extensions  $\mathcal{F}_1$  of  $\mathcal{F}$ , will be called the *manifold* of  $\Sigma$ , or of the system of equations obtained by equating the d.p. in  $\Sigma$  to zero.<sup>2</sup> A zero of  $\Sigma$  will at times be called a *point* of the manifold of  $\Sigma$ . The manifold of any system will be called an *algebraic differential manifold*, or, more briefly, a *manifold*.

Let  $\mathcal{M}_1$  and  $\mathcal{M}_2$  be respectively the manifolds of systems  $\Sigma_1$  and  $\Sigma_2$ .<sup>3</sup> If  $\mathcal{M}_1$  is contained in  $\mathcal{M}_2$ , we shall say that  $\Sigma_2$  *holds*  $\Sigma_1$ . Also, we shall say that  $\Sigma_2$  *vanishes over* or *holds*  $\mathcal{M}_1$ . If  $\Sigma$  is a system with no zeros, every system will be said to hold  $\Sigma$ .

Let  $\Sigma$  be an infinite system, and  $\Phi$  a basis of  $\Sigma$  (I, §12).<sup>4</sup> Because  $\Sigma$  contains  $\Phi$ ,  $\Phi$  holds  $\Sigma$ . Because every d.p. in  $\Sigma$  has a power in  $[\Phi]$ ,  $\Sigma$  holds  $\Phi$ . Thus, if  $\Sigma$  has zeros,  $\Sigma$  has the same manifold as  $\Phi$ . If  $\Sigma$  has no zeros,  $\Phi$  has no zeros. Thus *the manifold of any infinite system of d.p. is the manifold of some finite subset of the system*.<sup>5</sup>

If  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are manifolds of systems  $\Sigma_1$  and  $\Sigma_2$ , the intersection  $\mathcal{M}_1 \cap \mathcal{M}_2$ , if not vacuous, is the manifold of  $\Sigma_1 + \Sigma_2$ . The union  $\mathcal{M}_1 + \mathcal{M}_2$  is the manifold of the system of all products  $AB$  with  $A$  in  $\Sigma_1$  and  $B$  in  $\Sigma_2$ .

2. A manifold  $\mathcal{M}$  will be said to be *reducible* if it is the union of two manifolds, not necessarily mutually exclusive, which are proper parts of  $\mathcal{M}$ . If  $\mathcal{M}$  is not reducible, it will be called *irreducible*.

A manifold  $\mathcal{M}$  of a system  $\Sigma$  is irreducible if, and only if, whenever a product  $AB$  vanishes over  $\mathcal{M}$ , at least one of  $A$  and  $B$  vanishes over  $\mathcal{M}$ . Suppose first that  $AB$  holds  $\mathcal{M}$  while neither  $A$  nor  $B$  does. The manifolds of  $\Sigma + A$  and

<sup>1</sup> The  $y$  need not be indeterminates with respect to  $\mathcal{F}_1$ .

<sup>2</sup> Unfortunately, the totality of extensions of  $\mathcal{F}$  is an illegitimate totality. At the present time, there is no process of closure for differential fields analogous to the algebraic closure method. One knows, however, that troubles of this sort are not fatal to a theory.

<sup>3</sup> All d.p. have coefficients in  $\mathcal{F}$ , even though extensions are used in connection with zeros.

<sup>4</sup> Chapter I, §12. When no chapter number is given, the chapter is that in which one is reading.

<sup>5</sup> We understand this statement to stand for the two sentences which precede it.

$\Sigma + B$ , whose union is  $\mathfrak{M}$ , will be proper parts of  $\mathfrak{M}$  and  $\mathfrak{M}$  will be reducible. Again, let  $\mathfrak{M}$  be the union of smaller manifolds  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$ , manifolds respectively of systems  $\Sigma_1$  and  $\Sigma_2$ . Let  $A_i, i = 1, 2$ , be a d.p. of  $\Sigma_i$  which does not hold  $\mathfrak{M}$ . The product  $A_1A_2$  holds  $\mathfrak{M}$ .

Let  $\mathfrak{M}$  be the manifold of a system  $\Sigma$ . The totality  $\Omega$  of those d.p. which vanish over  $\mathfrak{M}$  is an ideal, and, indeed, a perfect ideal. We shall call  $\Omega$  the perfect ideal *associated with*  $\mathfrak{M}$ . It will be seen in §7 that  $\Omega$  is  $\{ \Sigma \}$ .  $\mathfrak{M}$  is irreducible if, and only if,  $\Omega$  is prime. When  $\Omega$  is prime, we call it the prime ideal *associated with*  $\mathfrak{M}$ .

3. We prove the following fundamental theorem.

**THEOREM:** *Every manifold is the union of a finite number of irreducible manifolds.*

Let the theorem not hold for the manifold  $\mathfrak{M}$  of some system  $\Sigma$ . Then  $\mathfrak{M}$  is not irreducible. Let  $AB$  hold  $\mathfrak{M}$ , while neither  $A$  nor  $B$  does. Then  $\mathfrak{M}$  is the union of the manifolds of  $\Sigma + A$  and  $\Sigma + B$ . At least one of the latter manifolds must fail to be the union of a finite number of irreducible manifolds. Let such failure occur for the manifold of  $\Sigma + A$ , which system we represent by  $\Sigma_1$ . Continuing, we produce, with the help of the axiom of selection, an infinite sequence

$$(1) \quad \Sigma, \Sigma_1, \dots, \Sigma_p, \dots,$$

each  $\Sigma_p$  containing, while not holding, its predecessor. Let  $\Omega$  be the union of the systems (1) and let  $\Phi$  be a basis of  $\Omega$ . Then  $\Phi$  is contained in some system of (1), say in  $\Sigma_q$ . We see that  $\Phi$  is a basis for  $\Sigma_q$ . By §1,  $\Phi$  and  $\Sigma_q$  have the same manifold. But the same argument shows that  $\Phi$  and  $\Sigma_{q+1}$  have the same manifold. This furnishes the contradiction that  $\Sigma_{q+1}$  holds  $\Sigma_q$ . The theorem is proved.

Let a manifold  $\mathfrak{M}$  have a representation

$$(2) \quad \mathfrak{M} = \mathfrak{M}_1 + \dots + \mathfrak{M}_p$$

as a union of irreducible manifolds  $\mathfrak{M}_i$ . If an  $\mathfrak{M}_i$  contains an  $\mathfrak{M}_j$  with  $j \neq i$ , then  $\mathfrak{M}_j$  may be suppressed in (2). We thus suppose that no  $\mathfrak{M}_i$  contains any  $\mathfrak{M}_j$  with  $j \neq i$ .

Now let  $\Sigma$  be the perfect ideal associated with  $\mathfrak{M}$  and let, for each  $i$ ,  $\Sigma_i$  be the prime ideal associated with  $\mathfrak{M}_i$ . Each  $\Sigma_i$  is a divisor of  $\Sigma$ . If  $A$  is a d.p. common to all  $\Sigma_i$ ,  $A$  holds  $\mathfrak{M}$  and is thus in  $\Sigma$ . Then  $\Sigma$  is the intersection of the  $\Sigma_i$ . If  $j \neq i$ ,  $\Sigma_j$  is not a divisor of  $\Sigma_i$ ; otherwise  $\mathfrak{M}_i$  would contain  $\mathfrak{M}_j$ . Then the  $\Sigma_i$  are the essential prime divisors of  $\Sigma$ .

If  $\mathfrak{M}'$  is an irreducible manifold contained in  $\mathfrak{M}$ , the prime ideal associated with  $\mathfrak{M}'$  is a divisor of some  $\Sigma_i$  (I, §17). Thus  $\mathfrak{M}'$  is contained in some  $\mathfrak{M}_i$ .

An irreducible manifold contained in  $\mathfrak{M}$  which is not part of a larger irreducible manifold contained in  $\mathfrak{M}$  will be called an *essential irreducible component* of  $\mathfrak{M}$

or a component of  $\mathfrak{M}$ .<sup>6</sup> The only components of  $\mathfrak{M}$  are the  $\mathfrak{M}_i$  in (2). Every irreducible manifold contained in  $\mathfrak{M}$  is contained in some component of  $\mathfrak{M}$ . Our discussion shows that in every representation of  $\mathfrak{M}$  as the union of a finite number of irreducible manifolds, every component of  $\mathfrak{M}$  appears; all other irreducible manifolds in the union are redundant.

We partially summarize what precedes. *A manifold  $\mathfrak{M}$  has a finite number of components, and is the union of them. The essential prime divisors of the perfect ideal associated with  $\mathfrak{M}$  are the prime ideals associated with the components of  $\mathfrak{M}$ .*

A component of the manifold of a system  $\Sigma$  will at times be called a component of  $\Sigma$ .

#### ILLUSTRATIONS IN ANALYSIS

4. To illustrate the decomposition of manifolds, we shall employ differential equations of classical analysis.

We use an open region  $\mathbf{A}$  in the plane of the complex variable  $x$ . Our field  $\mathfrak{F}$  will be supposed to consist of functions meromorphic throughout  $\mathbf{A}$ .<sup>7</sup>

Given a system  $\Sigma$ , we consider zeros of it obtained as follows. Let  $\mathbf{B}$  be any open region contained in  $\mathbf{A}$  and let  $y_1(x), \dots, y_n(x)$ , analytic in  $\mathbf{B}$ , annul every d.p. of  $\Sigma$  in  $\mathbf{B}$ . We shall call the entity composed of the  $y_i(x)$  and  $\mathbf{B}$  an *analytic zero*,<sup>8</sup> or a zero, of  $\Sigma$ . Two sets  $y_i(x)$  which are identical from the standpoint of analytic continuation will give different zeros if they are not associated with the same open region. For instance, if we use an open region  $\mathbf{B}_1$  interior to  $\mathbf{B}$ , and use, throughout  $\mathbf{B}_1$ , the  $y_i(x)$  as defined for  $\mathbf{B}$ , we get a different zero of  $\Sigma$ .<sup>9</sup>

The totality of analytic zeros of  $\Sigma$  will be called the *restricted manifold* of  $\Sigma$ . At this point in our work, we have no need to consider other types of zeros of  $\Sigma$  and it will turn out finally that the consideration of the restricted manifold produces a complete theory of the system  $\Sigma$ .

The case in which  $\mathfrak{F}$  consists of meromorphic functions, and in which one uses restricted manifolds, will be called *the analytic case*. All definitions in §§1-3 following that of manifold retain their meaning and all proofs retain their validity, in the analytic case. Thus, in the analytic case, a system  $\Sigma_2$  *holds* a system  $\Sigma_1$  if every analytic zero of  $\Sigma_1$  is a zero of  $\Sigma_2$ .<sup>10</sup> The theorem of §3

<sup>6</sup> No misunderstanding can arise, since the only subsets of manifolds which we employ are essential irreducible components.

<sup>7</sup> It is futile to seek greater generality through the use of functions analytic except for isolated singularities. If  $f(x)$  has an isolated essential singularity for  $x = a$  and if  $c$  is a rational value assumed by  $f(x)$  in every neighborhood of  $a$ , the reciprocal of  $f(x) - c$  has a pole in every neighborhood of  $a$ .

<sup>8</sup> No confusion with the term zero of the theory of functions will arise.

<sup>9</sup> Given an analytic zero, we have to go through the formality of constructing an extension of  $\mathfrak{F}$  in which its  $y(x)$  are contained. This is done by forming all rational combinations of the  $y(x)$  and their derivatives, with coefficients in  $\mathfrak{F}$ . If such a combination coincides in  $\mathbf{B}$  with a function  $f(x)$  in  $\mathfrak{F}$ , we consider the combination to be identical with  $f(x)$ , and thus to be in  $\mathfrak{F}$ .

<sup>10</sup> In §11, it will be seen that, in this case, every zero of  $\Sigma_1$  is a zero of  $\Sigma_2$ . Thus the word *hold* will be established as a word of a single meaning.

becomes: *Every restricted manifold is the union of a finite number of irreducible restricted manifolds.* By a component of the restricted manifold  $\mathfrak{M}$  of a system  $\Sigma$ , we mean an irreducible restricted manifold  $\mathfrak{M}'$  contained in  $\mathfrak{M}$ , which is not part of a larger irreducible restricted manifold contained in  $\mathfrak{M}$ . We may call  $\mathfrak{M}'$  a *restricted component*, or an *analytic component* of  $\Sigma$ . It will be seen in §11 that the perfect ideal associated with the restricted manifold of  $\Sigma$  is identical with the perfect ideal associated with the complete abstract manifold. The essential prime divisors of this perfect ideal furnish both the analytic components of  $\Sigma$  and the full components discussed in §3.

In our present work under the analytic case, the term *manifold* will be understood to mean restricted manifold.

We consider some examples.  $\mathfrak{F}$  will be any field of meromorphic functions.

**Example 1.** Let  $\Sigma$  consist of the single d.p.  $A = y_1^2 - 4y$  in  $\mathfrak{F}\{y\}$ . We call attention to the fact that  $A$ , as a polynomial in  $y$  and  $y_1$ , cannot be factored in any field. The manifold of  $\Sigma$  consists of the functions  $y = (x + c)^2$  with  $c$  constant, and of the function  $y = 0$ .<sup>11</sup> The derivative of  $A$  is  $2y_1(y_2 - 2)$ . Now  $y_2 - 2$  vanishes for every  $(x + c)^2$  but not for  $y = 0$ . Again,  $y_1$  vanishes for  $y = 0$ , but for no  $(x + c)^2$ . Thus  $\mathfrak{M}$  is reducible and is the union of  $\mathfrak{M}_1$ , composed of the functions  $(x + c)^2$ , and of  $\mathfrak{M}_2$ , composed of  $y = 0$ .  $\mathfrak{M}_1$  is the manifold of the system  $A, y_2 - 2$  and  $\mathfrak{M}_2$  is the manifold of  $A, y_1$ . It is obvious that  $\mathfrak{M}_2$  is irreducible. As to  $\mathfrak{M}_1$ , let it be held by  $BC$ . When  $y$  is replaced by  $(x + c)^2$ ,  $B$  and  $C$  become polynomials in  $c$  with coefficients meromorphic in  $\mathfrak{A}$ . If the product of two such polynomials vanishes identically in  $x$  and  $c$ , one of the polynomials does. Thus one of  $B$  and  $C$  holds  $\mathfrak{M}_1$  and  $\mathfrak{M}_1$  is irreducible.

**Example 2.** Let  $\Sigma$  be the d.p.  $A = y_2^2 - y$  in  $\mathfrak{F}\{y\}$ . Differentiating  $A$  successively, we have, over  $\mathfrak{M}$ ,

$$2y_2y_3 - y_1 = 0,$$

$$(3) \quad 2y_2y_4 + 2y_3^2 - y_2 = 0,$$

$$(4) \quad 2y_2y_5 + 6y_3y_4 - y_3 = 0.$$

Multiplying (4) by  $2y_3$  and substituting into the result the expression for  $y_3^2$  found from (3), we have, over  $\mathfrak{M}$ ,

$$y_2(4y_3y_5 - 12y_4^2 + 8y_4 - 1) = 0.$$

Thus  $\mathfrak{M}$  is reducible. It is composed of  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$ , the respective manifolds of

$$A, y_2; \quad A, 4y_3y_5 - 12y_4^2 + 8y_4 - 1.$$

As  $\mathfrak{M}_1$  consists of  $y = 0$ , it is irreducible. We shall see later that  $\mathfrak{M}_2$ , which is the *general solution* of  $A$ , is irreducible.

**Example 3.** The manifold of  $y_1(y_1 - y)$  decomposes into the two irreducible

<sup>11</sup> We shall not encumber our discussions with references to the areas in which the functions in a zero are analytic.

manifolds given by  $y = c$  and  $y = ce^x$ . These two manifolds have  $y = 0$  in common.

**Example 4.** Let  $\Sigma$  consist of  $A = y_1^2 y_2 - y$ . We find with a single differentiation that  $\mathfrak{M}$  is reducible and is made up of the manifolds of

$$A, y_1; \quad A, y_1 y_3 + 2y_2^2 - 1.$$

We call attention to the fact that  $A$  cannot be factored and is of the first degree in  $y_2$ .

**Example 5.** Let  $\Sigma$  be composed of  $A = uy - u_1^2$  in  $\mathfrak{F}\{u, y\}$ . Differentiating, we find over  $\mathfrak{M}$ ,

$$u_1 y + uy_1 - 2u_1 u_2 = 0.$$

Multiplying this equation by  $y$  and using  $A = 0$ , we find

$$(5) \quad u_1 (y^2 + u_1 y_1 - 2u_2 y) = 0.$$

Neither factor in (5) holds  $\mathfrak{M}$ , so that  $\mathfrak{M}$  is reducible. We call attention to the fact that  $A$  cannot be factored and is of zero order in  $y$ .

**Example 6.** In  $\mathfrak{F}\{y, z\}$ , let  $\Sigma$  be

$$y - xy_1 + \frac{y_1 z_1}{4}, \quad z - xz_1 + \frac{y_1 z_1}{4}.$$

We are dealing with a pair of Clairaut equations.  $\mathfrak{M}$  consists of two irreducible manifolds which are, to speak geometrically, the two-parameter family of lines

$$y = ax - \frac{ab}{4}, \quad z = bx - \frac{ab}{4},$$

and their one-parameter family of envelopes

$$y = (x + c)^2, \quad z = (x - c)^2.$$

The above examples might lead one to conjecture that the manifold of any finite system can be decomposed into irreducible manifolds by differentiations and eliminations. We shall see in Chapter V that this is actually so.

#### PRIME IDEALS AND REGULAR ZEROS

5. We return to the use of an abstract field. We shall call  $\mathfrak{F}\{y_1, \dots, y_n\}$  the *unit ideal*. The prime ideal consisting of the d.p. 0 will be called the *zero ideal*. A prime ideal distinct from the unit ideal and the zero ideal will be said to be *nontrivial*.

Let  $\Sigma$  be a nontrivial prime ideal. Let

$$(6) \quad A_1, \dots, A_r$$

be a characteristic set of  $\Sigma$ . The separant and initial of  $A_i$  will be denoted by  $S_i$  and  $I_i$  respectively. As the  $S$  and  $I$  are reduced with respect to (6), they are not in  $\Sigma$  (I, 5).

We prove that, for a d.p.  $G$  to belong to  $\Sigma$ , it is necessary and sufficient that the remainder of  $G$  with respect to (6) be zero. Let  $G$  be in  $\Sigma$ . As the remainder,  $R$ , is in  $\Sigma$  and is reduced with respect to (6), we have  $R = 0$ . Again, let  $R = 0$ . There is a relation

$$(7) \quad S_1^{s_1} \cdots I_r^{i_r} G \equiv 0, \quad (\Sigma).$$

As  $\Sigma$  is prime and the  $S$  and the  $I$  are not in  $\Sigma$ , it must be that  $G$  is in  $\Sigma$ .

A zero of the characteristic set (6) for which every  $S_i$  and every  $I_i$  is distinct from zero will be called a regular zero of (6).<sup>12</sup> We shall prove that every regular zero of a characteristic set of  $\Sigma$  is a zero of  $\Sigma$ . Let  $\eta_1, \cdots, \eta_n$  be a regular zero of (6). Let  $G$  be any d.p. in  $\Sigma$ . In (7), the  $S$  and the  $I$  are not annulled by the  $\eta$ . Then  $G$  is annulled by the  $\eta$ . The  $\eta$  thus constitute a zero of  $\Sigma$ .

#### GENERIC ZEROS OF A PRIME IDEAL

6. Let  $\Sigma$  be a prime ideal distinct from the unit ideal.

Let  $A$  be any d.p., not necessarily contained in  $\Sigma$ . We form a class  $\alpha$  of d.p., putting into  $\alpha$  every d.p.  $G$  such that  $G \equiv A, (\Sigma)$ . We call  $\alpha$  a remainder class, modulo  $\Sigma$ . Thus  $\mathfrak{F}\{y_1, \cdots, y_n\}$  is composed of a set of remainder classes. As  $\Sigma$  contains no element of  $\mathfrak{F}$  except zero, two distinct elements of  $\mathfrak{F}$  belong to distinct remainder classes; there are thus an infinite number of remainder classes.

Let  $\alpha$  and  $\beta$  be two remainder classes. All sums  $A + B$  with  $A$  in  $\alpha$  and  $B$  in  $\beta$  belong to the same remainder class. We call this class  $\alpha + \beta$ . Actually, every d.p. in  $\alpha + \beta$  is the sum of a d.p. in  $\alpha$  and a d.p. in  $\beta$ . We define  $\alpha\beta$  as the remainder class which contains all products  $AB$  with  $A$  in  $\alpha$  and  $B$  in  $\beta$ . Usually  $\alpha\beta$  contains d.p. which are not products  $AB$ . The derivative  $\alpha'$  of  $\alpha$  is defined as the remainder class which contains the derivatives of the d.p. in  $\alpha$ .

The remainder class which contains the d.p. 0 is  $\Sigma$ . We call  $\Sigma$  the zero class. As  $\Sigma$  is prime, a relation  $AB \equiv 0, (\Sigma)$ , implies that either  $A \equiv 0, (\Sigma)$  or  $B \equiv 0, (\Sigma)$ . Thus, if each of two remainder classes is distinct from the zero class, their product is distinct from the zero class.

We now consider pairs  $(\alpha, \beta)$  of remainder classes in which  $\beta$  is not the zero class. Two pairs,  $(\alpha, \beta)$  and  $(\gamma, \delta)$ , will be called equivalent if  $\alpha\delta = \beta\gamma$ . As the equivalence relation is transitive, the totality of pairs of classes separates into sets of equivalent pairs. If  $\mathfrak{A}$  is the set containing  $(\alpha, \beta)$  and  $\mathfrak{B}$  that containing  $(\gamma, \delta)$ , we define  $\mathfrak{A} + \mathfrak{B}$  as the set containing  $(\alpha\delta + \beta\gamma, \beta\delta)$ , and  $\mathfrak{A}\mathfrak{B}$  as the set containing  $(\alpha\gamma, \beta\delta)$ . The operations of subtraction and division are then uniquely determined. In particular,  $\mathfrak{A}/\mathfrak{B}$  can and must be taken as the set containing  $(\alpha\delta, \beta\gamma)$ .<sup>13</sup> The derivative of the set containing  $(\alpha, \beta)$  is defined as the set containing  $(\beta\alpha' - \alpha\beta', \beta^2)$ . With these operations, the sets of pairs of remainder classes become a differential field, which we denote by  $\mathfrak{F}_1$ .

<sup>12</sup> It will be seen in §6 that regular zeros exist.

<sup>13</sup> We attempt division only when  $\gamma$  is not the zero class.

With an element  $a$  of  $\mathfrak{F}$ , we associate the set in  $\mathfrak{F}_1$  containing the pair  $(\alpha, \beta)$  in which  $\alpha$  contains  $a$  and  $\beta$  contains 1. In this way we obtain a subset  $\mathfrak{F}'$  of  $\mathfrak{F}_1$  which is isomorphic with  $\mathfrak{F}$ . We replace each set of  $\mathfrak{F}'$  by the corresponding element of  $\mathfrak{F}$ , and  $\mathfrak{F}_1$  becomes an extension of  $\mathfrak{F}$ .

We are going to find a zero of  $\Sigma$  in  $\mathfrak{F}_1$ . Let  $\omega$  be that one of the remainder classes above which contains unity, and for  $i = 1, \dots, n$ , let  $\alpha_i$  be the class which contains the d.p.  $y_i$ . Let  $\eta_i$  be the set in  $\mathfrak{F}_1$  which contains  $(\alpha_i, \omega)$ .

We shall show that  $\eta_1, \dots, \eta_n$  is a zero of  $\Sigma$ .

Let  $G$  be any d.p. in  $\Sigma$ . The derivative of  $\eta_i$  is the set containing  $(\alpha'_i, \omega)$ , and  $\alpha'_i$  contains  $y_{i1}$ . It follows that when the  $\eta$  are substituted for the  $y$  in  $G$ , we obtain a set containing  $(\beta, \omega)$ , where  $\beta$  is the remainder class containing  $G$ , that is, the zero class. The set just described has 0 as its proxy in  $\mathfrak{F}_1$ . We see that  $\eta_1, \dots, \eta_n$  is a zero of  $\Sigma$ .

We see immediately, in a converse way, that if  $\eta_1, \dots, \eta_n$  annuls a d.p.  $G$ ,  $G$  is contained in  $\Sigma$ .

A zero of  $\Sigma$ , naturally contained in some extension of  $\mathfrak{F}$ , which is such that every d.p. over  $\mathfrak{F}$  which is annulled by the zero is contained in  $\Sigma$ , will be called a *generic zero*<sup>14</sup> of  $\Sigma$ , or a *generic point* of the manifold of  $\Sigma$ . We know that every prime ideal distinct from the unit ideal has a generic zero.

If we take  $\Sigma$  as in §5, we see that a generic zero of  $\Sigma$  is a regular zero of (6).

#### THE THEOREM OF ZEROS

7. We prove the following theorem:

**THEOREM:** *If  $\Sigma$  is a perfect ideal distinct from the unit ideal,  $\Sigma$  has zeros and every differential polynomial which holds  $\Sigma$  is contained in  $\Sigma$ .*<sup>15</sup>

Let  $\Sigma$  be the intersection of essential prime divisors  $\Sigma_i, i = 1, \dots, p$ . No  $\Sigma_i$  is the unit ideal. For each  $\Sigma_i$ , we form a generic zero. Each of these  $p$  zeros is a zero of  $\Sigma$ . Now let  $G$  be a d.p. which holds  $\Sigma$ . As  $G$  is annulled by each of the generic zeros,  $G$  is in each  $\Sigma_i$  and therefore in  $\Sigma$ .

We see, as was stated in §2, that, given a manifold  $\mathfrak{M}$  of a system  $\Sigma$ , the perfect ideal associated with  $\mathfrak{M}$  is  $\{ \Sigma \}$ ; it is the only perfect ideal whose manifold is  $\mathfrak{M}$ .

Modifying slightly the theorem just proved, we obtain the

**THEOREM OF ZEROS:** *Let*

$$F_1, \dots, F_p$$

*be any finite system of differential polynomials and let  $G$  be any differential polynomial which holds that system. Some power of  $G$  is a linear combination of the  $F$  and of their derivatives of various orders, with differential polynomials for coefficients. In particular, if  $F_1, \dots, F_p$  has no zeros, some linear combination of the  $F$  and of their derivatives of various orders equals unity.*

<sup>14</sup> Raudenbush, 20.

<sup>15</sup> A.D.E., Chapter VII, and Raudenbush, 21.

Let  $\Sigma$  be the perfect ideal determined by the  $F$ . If  $\Sigma$  is the unit ideal, unity is a linear combination as described above. Let  $\Sigma$  be distinct from the unit ideal. Then  $G$  is in  $\Sigma$ .

8. Let us reexamine the decomposition theorem of I, §19. Let  $\Sigma$  be an ideal with a manifold  $\mathfrak{M}$  which has a representation

$$\mathfrak{M} = \mathfrak{M}_1 + \cdots + \mathfrak{M}_p$$

where no two  $\mathfrak{M}_i$  have a point in common. If  $\Omega_i$  and  $\Omega_j$  are the perfect ideals associated with  $\mathfrak{M}_i$  and  $\mathfrak{M}_j$ ,  $i \neq j$ , the system  $\Omega_i + \Omega_j$  has no zeros. Then  $\{\Omega_i + \Omega_j\}$  is the unit ideal. This implies a relation  $A + B = 1$  with  $A$  in  $\Omega_i$  and  $B$  in  $\Omega_j$ . Thus  $\Omega_i$  and  $\Omega_j$  are relatively prime. It follows that  $\Sigma$  has a unique representation as the product of ideals whose manifolds are the  $\mathfrak{M}_i$ .

**Example:** We consider, as in Example 1 of §4, the manifold  $\mathfrak{M}$  of  $A = y_1^2 - 4y$ . At the present time, we use the full abstract manifold.  $\mathfrak{M}$  is the union of  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$ , the respective manifolds of  $A, y_2 - 2$  and  $A, y_1$ . As  $y_2 - 2$  and  $y_1$  have no zero in common,  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  have no point in common. As  $y_1(y_2 - 2)$  is in  $[A]$ , we have, by I, §11,

$$(8) \quad \{A\} = \{A, y_2 - 2\} \cap \{A, y_1\}.$$

Of course,  $\{A, y_1\} = \{y\} = [y]$ . As  $[A]$  contains  $y_1(y_2 - 2)$ , it contains  $y_2(y_2 - 2)^2$ . Thus  $[A]$  contains  $BC$  where

$$B = (y_2 - 2)^2, \quad C = 4y_2 - y_2^2.$$

We have  $B + C = 4$ . It follows from (8) and I, §19, that

$$[A] = [A, B][A, C].$$

Let  $\Sigma_1 = [A, C]$ . As  $\Sigma_1$  contains  $y_2B$ , it contains  $y_2(B + C)$  and therefore  $y_2$ . Then, as  $\Sigma_1$  contains  $y_1(y_2 - 2)$ , it contains  $y_1$  and hence  $y$ . Thus  $\Sigma_1 = [y]$  and

$$[y_1^2 - 4y] = [y][y_1^2 - 4y, (y_2 - 2)^2].$$

9. We shall now obtain a theorem of zeros for the analytic case.

With  $\mathfrak{F}$  a field of meromorphic functions, we take  $\Sigma$  as in §5. Let  $G$  be any d.p. not in  $\Sigma$ . We are going to prove the existence of a regular zero of (6), composed of functions  $y_1(x), \cdots, y_n(x)$ , which is not a zero of  $G$ .

The remainder  $R$  of  $G$  with respect to (6) is not zero. Let

$$K = RS_1 \cdots S_r I_1 \cdots I_r$$

where the  $S$  and  $I$  are as in §5. We wish, for a short time, to consider  $K$  and the  $A_i$  not as d.p., but as ordinary polynomials in the  $y_{ij}$ . A letter  $y_{ij}$  enters into our present work only if it appears effectively in some of the  $r + 1$  polynomials. Let  $\sigma$  be the system of polynomials  $A_i$ . By a zero of  $\sigma$ , we shall mean any set of functions  $y_{ij}(x)$ , analytic in some area contained in  $\mathbf{A}$ , which annull every  $A_i$ . We do not ask that  $y_{i, j+1}(x)$  be the derivative of  $y_{ij}(x)$ .

No power of  $K$  is linear in the  $A_i$  with coefficients which are polynomials in the  $\bar{y}_{ij}$ .<sup>16</sup> Otherwise  $K$ , considered as a d.p., would be contained in the prime ideal  $\Sigma$ . We shall now invoke Hilbert's theorem of zeros for polynomials, which is proved in IV, §14. The system  $\sigma$  has at least one zero, composed of functions  $\bar{y}_{ij}(x)$ , which do not annul  $K$ . Let  $a$  be a value of  $x$  at which the  $\bar{y}_{ij}(x)$  and all coefficients in the  $A_i$  and  $G$  are analytic, and at which  $K$ , when the  $\bar{y}_{ij}(x)$  are substituted into it, has a value distinct from zero.

10. We return now to the consideration of  $K$  and the  $A_i$  as d.p. Let  $p_j$  be the class of  $A_j, j = 1, \dots, r$ , and  $m_j$  the order of  $A_j$  in  $y_{p_j}$ . It may be that  $r < n$  in (6), so that there are  $y_i$  which are not among the  $y_{p_j}$ . Every such  $y_i$ , we replace in the  $A$  by a function  $y_i(x)$  analytic at  $a$ , which is chosen with the sole restriction that if some  $y_{ij}$  is a letter used in §9, the  $j$ th derivative of  $y_i$  has at  $a$  the value  $\bar{y}_{ij}(a)$  as in §9. It is a matter of forming convergent series of powers of  $x - a$ , with a finite number of coefficients assigned in advance. For these replacements, each  $A_j$  goes over into an expression  $B_j$  in  $y_{p_1}, \dots, y_{p_r}$  and their derivatives.

We consider the equation  $B_1 = 0$  as an equation determining  $y_{p_1 m_1}$  as a function of  $x, y_{p_1 0}, \dots, y_{p_1, m_1 - 1}$ . We work at  $x = a$ . To every  $y_{p_1 i}$  which is a letter of §9, we assign the value  $\bar{y}_{p_1 i}(a)$ . There may be  $y_{p_1 i}$  with  $i < m_1$  which do not appear in §9. To them we assign arbitrary numerical values. For the values assigned to  $x$  and the  $y_{p_1 i}, B_1$  vanishes. Now  $\partial B_1 / \partial y_{p_1 m_1}$  does not vanish for these values.<sup>17</sup> We can thus solve the equation  $B_1 = 0$  for  $y_{p_1 m_1}$ , finding

$$(9) \quad y_{p_1 m_1} = f_1(x, y_{p_1 0}, \dots, y_{p_1, m_1 - 1})$$

with  $f$  analytic for the assigned values of its arguments and equal to  $\bar{y}_{p_1 m_1}(a)$  for those values.

We now regard (9) as a differential equation of order  $m_1$  for  $y_{p_1}$ . For the initial conditions assigned as above at  $x = a$ , we obtain a solution  $y_{p_1}(x)$  analytic at  $x = a$ . The functions  $y_1(x), \dots, y_{p_1}(x)$  annul  $A_1$  but neither  $S_1$  nor  $I_1$ .

We now substitute  $y_{p_1}(x)$  for  $y_{p_1}$  in  $B_2$  and treat the equation  $B_2 = 0$  as above. Continuing, we construct a regular zero of (6). This zero does not annul  $K$  at  $x = a$ . Thus  $R$ , and also  $G$ , are not annulled by the zero at  $x = a$ .<sup>18</sup>

11. The theorems of §7 now go over to the analytic case. Thus, *if  $\mathfrak{F}$  is a field of meromorphic functions, and if  $\Sigma$  is a perfect ideal distinct from the unit ideal,  $\Sigma$  has a nonvacuous restricted manifold. Every differential polynomial which vanishes over the restricted manifold of  $\Sigma$  is contained in  $\Sigma$ .*

In the theorem of zeros, if  $F_1, \dots, F_p$  has no analytic zeros, unity is contained in the ideal of the  $F$ , so that  $F_1, \dots, F_p$  has no zeros of any type. If there is a

<sup>16</sup> With coefficients in  $\mathfrak{F}$ .

<sup>17</sup> The partial derivative is what  $S_1$  becomes for the replacements made above in the  $A$ . We note that  $K$  does not vanish for the  $\bar{y}_{ij}(a)$ .

<sup>18</sup> The work of §10 shows that a characteristic set of a prime ideal may be regarded as furnishing a system of differential equations, in a standard form, whose solutions more or less make up the manifold of the ideal.

restricted manifold, every d.p. which holds it is in the perfect ideal determined by the  $F$  and is thus annulled by all zeros of  $F_1, \dots, F_p$ .

To sum up, given any system  $\Sigma$  with  $\mathfrak{F}$  as above, and with  $\{\Sigma\}$  distinct from the unit ideal,  $\Sigma$  has a restricted manifold  $\mathfrak{M}$  and an abstract manifold  $\mathfrak{M}'$  which contains  $\mathfrak{M}$ . Both  $\mathfrak{M}$  and  $\mathfrak{M}'$  have  $\{\Sigma\}$  for associated perfect ideal. We shall find on this basis, in dealing with differential equations of analysis, that it suffices generally to work with restricted manifolds.

#### GENERAL SOLUTIONS

12. We use  $\mathfrak{F}\{y_1, \dots, y_n\}$  with  $\mathfrak{F}$  any field. A d.p. of positive class will be said to be *algebraically irreducible* if it is not the product of two d.p. of positive class.

Let  $F$  be of positive class  $p$  and algebraically irreducible. We are going to study the representation of  $\{F\}$  as an intersection of prime ideals.<sup>19</sup>

Denoting the separant of  $F$  by  $S$ , we let  $\Sigma_1$  be the totality of those d.p.  $A$  which are such that

$$(10) \quad SA \equiv 0, \quad \{F\}.$$

By §7,  $A$  is in  $\Sigma_1$  if  $A$  vanishes for every zero of  $F$  which does not annul  $S$ .

Clearly, the sum of two d.p. in  $\Sigma_1$  is in  $\Sigma_1$ , as is also the product of a d.p. in  $\Sigma_1$  by any d.p. From (10) it follows, by I, §10, that  $SA'$ , with  $A'$  the derivative of  $A$ , is in  $\{F\}$ . Then  $A'$  is in  $\Sigma_1$ . Thus  $\Sigma_1$  is an ideal.

We prove now that *the ideal  $\Sigma_1$  is prime*. Let  $AB$  be in  $\Sigma_1$ . Let  $F$  be of order  $m$  in  $y_p$ . The process of reduction used for forming remainders shows the existence of relations

$$(11) \quad S^a A \equiv R, \quad S^b B \equiv T, \quad [F],$$

with  $R$  and  $T$  of order at most  $m$  in  $y_p$ . We shall prove that at least one of  $R$  and  $T$  is divisible by  $F$ . From (11) we have  $SRT \equiv S^{a+b+1}AB$ ,  $[F]$ . As the second member of this congruence is in  $\{F\}$ , the first member is also. Let then

$$(SRT)^c = MF + M_1F' + \dots + M_qF^{(q)},$$

superscripts indicating differentiation. We have

$$F^{(q)} = Sy_{p, m+q} + U$$

where  $U$  is of order less than  $m+q$  in  $y_p$ . We replace  $y_{p, m+q}$  in  $F^{(q)}$  and in the  $M$  by  $-U/S$ . Clearing fractions, we find a relation

$$S^d(RT)^c = NF + N_1F' + \dots + N_{q-1}F^{(q-1)}.$$

Continuing, we find that some  $S^e(RT)^c$  is divisible by  $F$ . As  $F$  is algebraically irreducible, and not a factor of  $S$ ,  $F$  must be a factor of at least one of  $R$  and  $T$ .

<sup>19</sup> Even if  $p < n$ ,  $\{\mathfrak{F}\}$  will contain d.p. of class as high as  $n$ .

Suppose that  $R$  is divisible by  $F$ . By (11),  $SA$  is in  $\{F\}$  so that  $A$  is in  $\Sigma_1$ . Thus  $\Sigma_1$  is prime.

13. We prove now that for a d.p.  $A$  to belong to  $\Sigma_1$ , it is necessary and sufficient that the remainder of  $A$  with respect to  $F$  be zero. In particular, if  $A$  is in  $\Sigma_1$  and if  $A$  has the same order in  $y_p$  as  $F$ ,  $A$  is divisible by  $F$ .

Let  $A$  belong to  $\Sigma_1$ . We have a relation

$$(12) \quad S^a A \equiv B, \quad [F],$$

with  $B$  of order at most  $m$  in  $y_p$ . Now  $SB$  is in  $\{F\}$  so that, as in §12,  $B$  is divisible by  $F$ . This means that the remainder of  $A$  is zero. Conversely, if the remainder is zero, we have (12) with  $B$  divisible by  $F$  so that  $A$  is in  $\Sigma_1$ .

We see, in particular, that  $\Sigma_1$  does not contain  $S$ .

14. We prove that

$$\{F\} = \Sigma_1 \cap \{F, S\}.$$

$\{F\}$  is contained in each ideal in the second member, so that it will suffice to show that the second member is in  $\{F\}$ . Let  $A$  be in  $\{F, S\}$ . For some  $a$ ,

$$(13) \quad A^a = B + C$$

with  $B$  in  $[F]$  and  $C$  in  $[S]$ . Now, let  $A$  also belong to  $\Sigma_1$ . Then  $SA$  is in  $\{F\}$  so that, by I, §10, the product of  $A$  by any derivative of  $S$  is in  $\{F\}$ . Then  $AC$  is in  $\{F\}$  so that  $A^{a+1}$  is in  $\{F\}$ .

15. Let

$$\{F, S\} = \Lambda_1 \cap \cdots \cap \Lambda_r$$

where the  $\Lambda$  are the essential prime divisors of  $\{F, S\}$ . Certain  $\Lambda$  may be divisors of  $\Sigma_1$ . Suppressing these, and using symbols  $\Sigma_i$  with  $i > 1$  for the remaining  $\Lambda$ , we have

$$(14) \quad \{F\} = \Sigma_1 \cap \Sigma_2 \cap \cdots \cap \Sigma_r.$$

Thus,  $\Sigma_1$  is an essential prime divisor of  $\{F\}$  and, in the representation of  $\{F\}$  as an intersection of essential prime divisors, there is precisely one prime ideal, namely  $\Sigma_1$ , which does not contain  $S$ .

16. An interchange of the subscripts of the  $y$  may give  $F$  a new separant. Any such separant involves only derivatives present in  $F$  and is not divisible by  $F$ . Hence, for the original ordering of the  $y$ , such a separant has a remainder with respect to  $F$  which is not zero. Thus, in (14),  $\Sigma_1$  contains no separant of  $F$ , while  $\Sigma_2, \dots, \Sigma_r$  contain every separant.<sup>20</sup>

We shall call the manifold of  $\Sigma_1$  the *general solution* of  $F$ , or of the equation  $F = 0$ .

<sup>20</sup> It is only in our present work that we use several separants for a d.p., one for each indeterminate appearing effectively in the d.p. This matter will not cause confusion elsewhere.

## SINGULAR ZEROS AND SOLUTIONS

17. We take  $F$  as in §12. A zero of  $F$  will be called *nonsingular* if it fails to annul at least one separant of  $F$ , and *singular* if it annuls every separant. Correspondingly, we speak of nonsingular, and of singular, *solutions* of  $F = 0$ .

Every nonsingular zero of  $F$  is contained in the general solution of  $F$ . The other components of  $F$  are made up of singular zeros.

If a d.p.  $G$  vanishes for all nonsingular zeros of  $F$ , then  $G$  is in  $\Sigma_1$ . This is an immediate consequence of the fact that a generic zero of  $\Sigma_1$  is a nonsingular zero of  $F$ . In the analytic case, we get a less trivial result. If  $G$  vanishes for all nonsingular analytic zeros of  $F$ ,  $G$  is contained in  $\Sigma_1$ . This follows from the fact that the product of  $G$  and the separants holds the restricted manifold of  $F$ , therefore the restricted manifold of  $\Sigma_1$ . By the theorem of zeros, the product is in  $\Sigma_1$ , so that  $G$  is in  $\Sigma_1$ .

In the analytic case, we call the restricted manifold of  $\Sigma_1$  the *restricted general solution* of  $F$ , and, as a rule, since misunderstandings do not occur, the *general solution* of  $F$ .

The general solution may contain singular solutions of  $F = 0$ , as well as the nonsingular ones. From what precedes, we see that *a singular solution belongs to the general solution if, and only if, every d.p. which vanishes for all nonsingular solutions vanishes also for the singular solution*. In the analytic case, one uses here only the analytic nonsingular solutions.

18. As  $\Sigma_1$  contains no nonzero d.p. reduced with respect to  $F$ ,  $F$  is a characteristic set for  $\Sigma_1$ . Let  $\Sigma$  be any nontrivial prime ideal (§5) which has a characteristic set consisting of a single d.p.  $G$ . We assume  $G$  to be algebraically irreducible since, if it is not, we can replace it by one of its factors. As  $\Sigma$  consists of those d.p. which have zero remainders with respect to  $G$ , the manifold of  $\Sigma$  is the general solution of  $G$ . The case in which the number  $n$  of indeterminates is unity is of special interest. *For a single indeterminate  $y$ , every irreducible manifold distinct from the manifold of the zero ideal is the general solution of a differential polynomial in  $y$ .*

For  $n > 1$ , this result does not hold. It will be seen, however, in Chapter III, that if  $G$  is any d.p. of positive class, every component of  $G$  is the general solution of some d.p.

19. We consider some examples in the analytic case. In Example 1 of §4, the component  $y = (x + c)^2$  is composed of nonsingular zeros and is the general solution of  $y_1^2 - 4y$ . In Example 2,  $y = 0$  is the only singular zero, so that  $\mathfrak{M}_2$  is the general solution.

In Example 5, a consideration of the two separants shows that the singular zeros are those for which  $u = 0$ . We denote the general solution by  $\mathfrak{M}_1$ . The factor

$$B = y^2 + u_1y_1 - 2u_2y$$

in (5) vanishes, for  $u = 0$ , only if  $y = 0$ . As  $u_1$  in (5) is not divisible by  $A$ ,  $u_1$  does not hold  $\mathfrak{M}_1$ . Thus  $B$  holds  $\mathfrak{M}_1$ , so that the only zero with  $u = 0$  which

can belong to  $\mathfrak{M}_1$  is  $u = 0, y = 0$ . The zeros of  $A$  with  $u = 0$  constitute an irreducible manifold, the manifold  $\mathfrak{M}_2$  of the d.p.  $u$ . Thus  $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$ . We can now see that  $\mathfrak{M}_1$  contains the singular zero  $u = 0, y = 0$ . Let  $G$  be any d.p. in  $\Sigma_1$  and  $\bar{u}, \bar{y}$ , with  $\bar{u} \neq 0$ , a zero of  $A$ . For every constant  $c \neq 0, c\bar{u}, c\bar{y}$  annuls  $A$  and is thus in  $\mathfrak{M}_1$ . Thus  $G$  vanishes for  $c\bar{u}, c\bar{y}$  and hence for  $u = 0, y = 0$ . This puts  $u = 0, y = 0$  in  $\mathfrak{M}_1$ .

For another example of a general solution which contains a singular zero, we consider  $A = y_1^2 - 4y^3$ , whose manifold is  $y = (x + c)^{-2}$  and  $y = 0$ . The only singular zero is  $y = 0$ . We see, letting  $|c|$  increase, that a d.p. which vanishes for every  $(x + c)^{-2}$  vanishes for  $y = 0$ . Thus  $y = 0$  is in the general solution.

20. The above formulation of the concept of the general solution of an algebraic differential equation appears to be the first fully precise one which has ever been given. In the literature in general, the term "general solution" is used in a loose sense. For a differential equation of order  $n$ , an  $n$ -parameter family of solutions is called the "general solution." Some authors are aware that singular solutions should sometimes be considered as belonging to the general solution, but no sharp criterion is given.

It is interesting, however, that a paper on singular solutions published by Lagrange<sup>21</sup> in 1774 shows him to have possessed a really good idea of the nature of a general solution. Dealing with an equation

$$(15) \quad V\left(x, y, \frac{dy}{dx}\right) = 0,$$

he supposes determined for it a one-parameter family of solutions  $y = f(x, a)$ , which he calls the *complete integral*. He seeks conditions for a *particular* (in modern parlance, singular) solution  $y(x)$  to be considered as belonging to the complete integral. He furnishes conditions under which  $y(x)$  satisfies not only (15), but also "all equations of higher orders which can be derived from it." The satisfaction of all such higher equations is given as the condition for  $y(x)$  to belong to the complete integral. How the higher equations are to be determined is left to be guessed. One is apparently supposed to perform differentiations and eliminations, as in the examples treated by Lagrange. It is proper, however, to credit Lagrange with the possession of a heuristic version of the criterion for membership in the general solution given in §17 above, and to regard his work on singular solutions, like that of Laplace and of Poisson which will be considered in Chapter III, as precursive to the present theory.

#### PARAMETRIC INDETERMINATES

21. Let  $\Sigma$  be a nontrivial prime ideal in  $\mathfrak{F}\{y_1, \dots, y_n\}$ .

There may be some  $y$ , say  $y_j$ , such that no nonzero d.p. in  $\Sigma$  involves only  $y_j$ ; that is, every d.p. in which  $y_j$  appears effectively also involves some  $y_i$  with  $i \neq j$ . If there exist such  $y_j$ , let us pick one of them, arbitrarily, and call it  $u_1$ .

There may be a  $y$  distinct from  $u_1$  such that no nonzero d.p. in  $\Sigma$  involves only  $u_1$  and the new  $y$ . If there exist such  $y$ , we pick one of them and call it  $u_2$ .

<sup>21</sup> Lagrange, 15.

Continuing, we find a set  $u_1, \dots, u_q$  ( $q < n$ ), such that no nonzero d.p. of  $\Sigma$  involves the  $u$  alone and such that, given any  $y_j$  not among the  $u$ , there is a nonzero d.p. of  $\Sigma$  in  $y_j$  and the  $u$  alone.<sup>22</sup>

Let the indeterminates distinct from the  $u$ , taken in any order, be represented now by  $y_1, \dots, y_p$  ( $p + q = n$ ).

We now list the indeterminates in the order

$$(16) \quad u_1, \dots, u_q; \quad y_1, \dots, y_p.$$

We shall speak generally as if  $u$  exist. It will be easy to see, in every case, what slight changes of language are necessary when they do not.

Of the nonzero d.p. in  $\Sigma$  involving only  $y_1$  and the  $u$ , let  $A_1$  be one of least rank. There certainly exist d.p. of  $\Sigma$  of class  $q + 2$  which are reduced with respect to  $A_1$ ; for instance, any nonzero d.p. in  $y_2$  and the  $u$  is of this type. Of such d.p., let  $A_2$  be one of least rank.

Continuing, we build a characteristic set of  $\Sigma$ ,

$$(17) \quad A_1, A_2, \dots, A_p.$$

We shall say that  $A_i$  introduces  $y_i$ .

We shall call  $u_1, \dots, u_q$  a *parametric set of indeterminates* for  $\Sigma$ , or for the manifold of  $\Sigma$ .

#### THE RESOLVENT

22. The investigation which we now undertake will show that every irreducible manifold except that of [0] may be regarded as a birational<sup>23</sup> transform of the general solution of some d.p.<sup>24</sup>

Through §23, we shall work with a field  $\mathfrak{F}$  which contains at least one non-constant element.

We present first two lemmas of a special character.

A set of elements  $\eta_1, \dots, \eta_s$  of  $\mathfrak{F}$  will be called *linearly dependent* if there exists a relation

$$(18) \quad c_1\eta_1 + \dots + c_s\eta_s = 0$$

where the  $c$  are constant elements of  $\mathfrak{F}$ , not all zero.

We prove that for  $\eta_1, \dots, \eta_s$  to be linearly dependent, it is necessary and sufficient that

$$(19) \quad \begin{vmatrix} \eta_1 & \dots & \eta_s \\ \eta_1' & \dots & \eta_s' \\ \dots & \dots & \dots \\ \eta_1^{(s-1)} & \dots & \eta_s^{(s-1)} \end{vmatrix} = 0,$$

where superscripts indicate differentiation.

<sup>22</sup> It will be seen in §32 that  $q$  does not depend on the particular manner in which the  $u$  are selected.

<sup>23</sup> The birational transformations which we use will involve derivatives.

<sup>24</sup> A.D.E., Chapter II, and Kolchin, 10. The treatment given here is taken over from Kolchin's paper.

The proof is conducted as in analysis. For the necessity, we differentiate (18)  $s - 1$  times. We secure a set of  $s$  homogeneous equations for the  $c$ . The determinant must vanish, since there is a solution with some  $c$  distinct from zero. For the sufficiency proof, we proceed by induction. For  $s = 1$ , (19) is evidently sufficient. We treat the case of  $s = r$ , supposing earlier cases to have been examined. By (19) the equations

$$(20) \quad c_1 \eta_1^{(j)} + \cdots + c_r \eta_r^{(j)}, \quad j = 0, \cdots, r - 1,$$

are satisfied by elements  $c_1, \cdots, c_r$  of  $\mathfrak{F}$ , not all zero. We may evidently suppose that when the last row and last column are suppressed in (19), the resulting determinant is not zero. Then, in (20),  $c_r \neq 0$ . We may thus take  $c_r$  equal to unity. For  $j \leq r - 2$ , we differentiate (20) and then subtract the equation (20) corresponding to  $j + 1$ .

We find that

$$(21) \quad c'_1 \eta_1^{(j)} + \cdots + c'_{r-1} \eta_{r-1}^{(j)} = 0, \quad j = 0, \cdots, r - 2,$$

where accents indicate differentiation. As the determinant of (21) is not zero, the  $c_i$  with  $i < r$  are constants. This completes the proof.

We prove now that if  $G$  is a nonzero d.p. in  $\mathfrak{F} \{ u_1, \cdots, u_q \}$ , there exist elements  $\mu_1, \cdots, \mu_q$  in  $\mathfrak{F}$  such that  $G$  is not zero for  $u_i = \mu_i, i = 1, \cdots, q$ .

It suffices to treat a d.p.  $G$  in a single indeterminate  $u$ . Let  $\xi$  be a nonconstant element in  $\mathfrak{F}$ . Let  $r$  be any nonnegative integer. We shall prove that if  $G$  is a nonzero d.p. of order not exceeding  $r$ , there is an element

$$(22) \quad c_0 + c_1 \xi + c_2 \xi^2 + \cdots + c_r \xi^r,$$

where the  $c$  are constants in  $\mathfrak{F}$ , which does not annul  $G$ . Let this be false, and let  $H$  be a nonzero d.p. of lowest rank which vanishes for every element (22). Let the order of  $H$  be  $s$ . We know that  $s \leq r$ . It is easy to see that  $s > 0$ .

When  $u$  is replaced in  $H$  by (22) and each  $u_i$  with  $1 \leq i \leq s$  by

$$(23) \quad c_1 \xi^{(i)} + c_2 (\xi^2)^{(i)} + \cdots + c_r (\xi^r)^{(i)},$$

the superscript  $(i)$  denoting  $i$  differentiations,  $H$ , considered as a polynomial in the indeterminates  $c$ , must vanish identically. Its partial derivatives with respect to the  $c$  are thus all zero. We have thus, from  $c_0, \cdots, c_s$ ,

$$(24) \quad \begin{aligned} \frac{\partial H}{\partial u} &= 0, \\ \frac{\partial H}{\partial u} \xi + \frac{\partial H}{\partial u_1} \xi' + \cdots + \frac{\partial H}{\partial u_s} \xi^{(s)} &= 0, \\ \frac{\partial H}{\partial u} \xi^2 + \frac{\partial H}{\partial u_1} (\xi^2)' + \cdots + \frac{\partial H}{\partial u_s} (\xi^2)^{(s)} &= 0, \\ &\dots\dots\dots, \\ \frac{\partial H}{\partial u} \xi^s + \frac{\partial H}{\partial u_1} (\xi^s)' + \cdots + \frac{\partial H}{\partial u_s} (\xi^s)^{(s)} &= 0. \end{aligned}$$

In each  $\partial H/\partial u_i$ ,  $i = 0, \dots, s$ , the substitutions (22), (23) are supposed to be made.

We regard equations (24) as equations for the  $\partial H/\partial u_i$ . As  $\partial H/\partial u_s$  is of lower rank than  $H$ , it does not vanish identically in the  $c$ . The determinant of (24) is therefore zero. This means, by the preceding lemma, that there is a relation

$$a_1\xi' + a_2(\xi^2)' + \dots + a_s(\xi^s)' = 0$$

where the  $a$  are constants in  $\mathfrak{F}$ , not all zero. Then

$$a_1\xi + a_2\xi^2 + \dots + a_s\xi^s = a_0$$

with  $a_0$  a constant. Thus  $\xi$  satisfies an algebraic equation whose coefficients are in  $\mathfrak{F}$  and are not all zero. Let an equation of this type of least degree be

$$f(\xi) = 0.$$

Then  $f'(\xi)\xi' = 0$ . As  $f'(\xi) \neq 0$ , we have  $\xi' = 0$ . We reach the contradiction that  $\xi$  is a constant and the lemma is proved.

23. Working in  $\mathfrak{F}\{u_1, \dots, u_q; y_1, \dots, y_p\}$ ,<sup>25</sup> we consider a nontrivial prime ideal  $\Sigma$  for which the  $u$  are a parametric set (§21). We are going to show the existence in  $\mathfrak{F}$  of elements

$$(25) \quad \mu_1, \dots, \mu_p$$

and the existence of a nonzero d.p.  $G$ , free of the  $y$ , such that either

(a) there exist no two distinct zeros of  $\Sigma$ , contained in a single extension of  $\mathfrak{F}$ ,

$$(26) \quad \begin{array}{ll} \bar{u}_1, \dots, \bar{u}_q; & y'_1, \dots, y'_p, \\ \bar{u}_1, \dots, \bar{u}_q; & y''_1, \dots, y''_p, \end{array}$$

with the same  $u$ , which  $u$  do not annul  $G$ , or

(b) such pairs of zeros exist and, for each pair,

$$(27) \quad \mu_1(y'_1 - y''_1) + \dots + \mu_p(y'_p - y''_p)$$

is not zero.<sup>26</sup>

We consider the system  $\Sigma'$  obtained from  $\Sigma$  by replacing each  $y_i$  by a new indeterminate  $z_i$ . Introducing  $p$  more indeterminates  $\lambda_1, \dots, \lambda_p$ , we consider the perfect ideal  $\Omega$  determined by  $\Sigma, \Sigma'$  and

$$\lambda_1(y_1 - z_1) + \dots + \lambda_p(y_p - z_p).$$

We have thus  $3p + q$  indeterminates, the  $u, y, z, \lambda$ , and we operate in  $\mathfrak{F}\{u; y; z; \lambda\}$ .

Let  $\Lambda$  be any essential prime divisor of  $\Omega$ . Suppose that not every  $y_i - z_i$ ,

<sup>25</sup> We recall that  $\mathfrak{F}$  is supposed to contain nonconstant elements.

<sup>26</sup> If no  $u$  exist, this is to mean that, if  $\Sigma$  has a pair of distinct zeros in a single extension of  $\mathfrak{F}$ , (27) does not vanish for the pair. We take  $G = 1$  in this case.

$i = 1, \dots, p$ , is in  $\Lambda$ . We shall prove that  $\Lambda$  contains a nonzero d.p. which involves no indeterminates other than the  $u$  and  $\lambda$ .

If  $\Lambda$  contains a d.p. in the  $u$  alone,<sup>27</sup> we have our result. Suppose that  $\Lambda$  contains no such d.p.

Since  $\Lambda$  has all d.p. in  $\Sigma$ ,  $\Lambda$  has, for  $j = 1, \dots, p$ , a d.p.  $B_j$  in  $y_j$  and the  $u$  alone. Let  $B_j$  be taken so as to be of as low a rank as possible in  $y_j$ . Then  $S_j$ , the separant of  $B_j$ , is not in  $\Lambda$ .

Similarly let  $C_j$ ,  $j = 1, \dots, p$ , be a d.p. of  $\Lambda$  in  $z_j$  and the  $u$ , of as low a rank as possible in  $z_j$ . Letting  $z_j$  follow the  $u$  in  $C_j$ , we see that the separant  $S'_j$  of  $C_j$  is not in  $\Lambda$ .

To fix our ideas, suppose that  $y_1 - z_1$  is not in  $\Lambda$ . Consider any generic zero of  $\Lambda$ . For it, we have

$$(28) \quad \lambda_1 = - \frac{\lambda_2 (y_2 - z_2) + \dots + \lambda_p (y_p - z_p)}{y_1 - z_1}.$$

From (28) we find, for the  $j$ th derivative of  $\lambda_1$  in the generic zero, an expression

$$(29) \quad \lambda_{1j} = \rho_j (\lambda_2, \dots, \lambda_p; y_1, \dots, y_p; z_1, \dots, z_p),$$

in which  $\rho_j$  is rational in the  $\lambda, y, z$  and their derivatives, with coefficients in  $\mathfrak{F}$ . The denominator in each  $\rho_j$  is a power of  $y_1 - z_1$ .

Let  $B_i$  be of order  $r_i$  in  $y_i$  and  $C_i$  be of order  $s_i$  in  $z_i$ ,  $i = 1, \dots, p$ .

If a  $\rho_j$  involves derivatives of  $y_i$  of order higher than  $r_i$ , we can get rid of those derivatives by using their expressions in the derivatives of  $y_i$  of order  $r_i$  or less, found from  $B_i = 0$ . Similarly, we transform each  $\rho_j$  so as to be of order not exceeding  $s_i$  in  $z_i$ ,  $i = 1, \dots, p$ .

The new expression for each  $\rho_j$ , which will involve the  $u$ , will have a denominator which is a product of powers of  $y_1 - z_1, S_i, S'_i, i = 1, \dots, p$ . Let  $g$  be the maximum of the integers  $r_i, s_i$ . Let

$$h = 2p(g + 1) + 1.$$

Let  $k$  be the total number of letters  $y_{ij}, z_{ij}$  which appear in the relations (29), transformed as indicated. Then  $h > k$ .

We consider the first  $h$  of the relations (29).<sup>28</sup> (That is, we let  $j = 0, 1, \dots, h - 1$ .) Let  $D$ , an appropriate product of powers of  $y_1 - z_1, S_i, S'_i$ , be a common denominator for the second members of these relations. We write

$$(30) \quad \lambda_{1j} = \frac{E_j}{D}, \quad j = 0, \dots, h - 1.$$

Let  $D$  and the  $E_j$  be written as polynomials in the  $k$  letters  $y_{ij}, z_{ij}$  present in them, with coefficients which are d.p. in  $\lambda_2, \dots, \lambda_p$  and the  $u$ . Let  $m$  be the maximum of the degrees of these polynomials (total degrees in the  $y_{ij}, z_{ij}$ ).

<sup>27</sup> At times the term *nonzero* will be omitted. One will always know when it is being tacitly employed.

<sup>28</sup> When  $j = 0$ , (29) is (28).

Let  $\alpha$  represent a positive integer to be fixed later. The total number of distinct power products of degree  $m\alpha$  or less, in  $k$  letters, is<sup>29</sup>

$$(31) \quad \frac{(m\alpha + k) \cdots (m\alpha + 1)}{k!}.$$

Using (30), let us form expressions for all power products of the  $\lambda_{1j}$  in (30) of degree  $\alpha$  or less. Let each expression be written in the form

$$(32) \quad \frac{F}{D^\alpha}.$$

Then  $F$ , as a polynomial in the  $y_{ij}, z_{ij}$ , will be of degree at most  $m\alpha$ .

The number of power products of the  $h$  letters  $\lambda_{1j}$  of degree  $\alpha$  or less is

$$(33) \quad \frac{(\alpha + h) \cdots (\alpha + 1)}{h!}.$$

Now (31) is a polynomial of degree  $k$  in  $\alpha$ , whereas (33) is of degree  $h$  in  $\alpha$ . As  $h > k$  and as  $m, h, k$  are fixed, (33) will exceed (31) if  $\alpha$  is large. Let  $\alpha$  be taken large enough for this to be realized.

If now the  $F$  in (32) are considered as linear expressions in the power products in the  $y_{ij}, z_{ij}$ , we shall have more linear expressions than power products. Hence the linear expressions  $F$  are linearly dependent. That is, some linear combination of the  $F$ , with coefficients which are d.p. in  $\lambda_2, \cdots, \lambda_p$  and the  $u$ , not all zero, vanishes identically.

The same linear combination of the power products of the  $\lambda_{1j}$  will vanish for the generic zero of  $\Lambda$  for which (28) was written. Now this last linear combination is a d.p.  $H$  in the  $u$  and  $\lambda$ .  $H$  is not identically zero, since the power products in the  $\lambda_{1j}$  in  $H$  are all distinct.

As  $H$  vanishes for a generic zero of  $\Lambda$ ,  $H$  is in  $\Lambda$ .

Let  $\Lambda_1, \cdots, \Lambda_r$  be the essential prime divisors of  $\Omega$ . Let  $\Lambda_1, \cdots, \Lambda_s$  each not contain some  $y_i - z_i$  and let  $\Lambda_{s+1}, \cdots, \Lambda_r$  each contain every  $y_i - z_i$ . Let  $H_i$  be a nonzero d.p. in  $\Lambda_i$ ,  $i = 1, \cdots, s$ , involving only the  $u$  and  $\lambda$ . Let  $K = H_1 \cdots H_s$ .

Using the second lemma of §22, we replace each  $\lambda_i$  in  $K$  by an element  $\mu_i$  of  $\mathfrak{F}$ , in such a way that  $K$  reduces to a nonzero d.p.  $G$  in the  $u$ . We shall show that  $G$  and the  $\mu$  serve as in the statement at the head of this section.

The zeros of  $\Omega$  with  $\lambda_j = \mu_j$ ,  $j = 1, \cdots, p$ , will be the zeros of the  $\Lambda_i$  with  $\lambda_j = \mu_j$ . Now the zeros with  $\lambda_j = \mu_j$  of  $\Lambda_1, \cdots, \Lambda_s$  have  $u$  which annul  $G$ . The zeros of  $\Lambda_{s+1}, \cdots, \Lambda_r$ , even with  $\lambda_j = \mu_j$ , have  $y_i = z_i$ ,  $i = 1, \cdots, p$ .

Suppose now that  $\Lambda_1, \cdots, \Lambda_s$  actually exist. Then there exist distinct pairs (26); the  $y'$  can be taken as the  $y$  in a zero of some  $\Lambda_i$ ,  $i \leq s$ , and the  $y''$  as the  $z$ .<sup>30</sup> For any such pair (26), (27) is zero only if the  $u, y', y''$  are in a zero, with

<sup>29</sup> Perron, *Lehrbuch der Algebra*, vol. 1, p. 46.

<sup>30</sup> We are not supposing here that the  $\lambda$  are replaced by the  $\mu$ .

$\lambda_j = \mu_j, j = 1, \dots, p$ , of some  $\Lambda_i$  with  $i \leq s$ . In that case,  $G$  vanishes for the  $u$ .

When every  $\Lambda_j$  contains every  $y_i - z_i$ , we take  $G = 1, \mu_1 = \dots = \mu_p = 0$ .

We have thus produced the required  $G$  and  $\mu$ .<sup>31</sup>

24. We shall now relinquish the condition that  $\mathfrak{F}$  contain a nonconstant element. Let us assume that parametric indeterminates  $u$  exist. We are going to prove the existence of d.p.  $G, M_1, \dots, M_p$ , in the  $u$  alone, with  $G \neq 0$ , such that, for two distinct zeros (26) for which  $G$  does not vanish,

$$(34) \quad M_1(y'_1 - y''_1) + \dots + M_p(y'_p - y''_p)$$

is not zero.

The discussion of §23 holds through the construction of  $K$ . We are going to prove the existence of d.p.  $M_1, \dots, M_p$  in the  $u$  alone, such that when  $\lambda_i$  is replaced by  $M_i$  in  $K$ , the resulting d.p.  $G$  is not identically zero.

Let  $K$  be arranged as a polynomial in the  $\lambda_{1j}$ , with d.p. in  $\lambda_2, \dots, \lambda_p$  and the  $u$  for coefficients. Let  $u_{1k}$  be a derivative of  $u_1$  of order greater than that of any derivative of  $u_1$  which may appear in the coefficients. If  $\lambda_1$  is replaced by  $u_{1k}$ ,  $K$  becomes a d.p.  $K_1$  in  $\lambda_2, \dots, \lambda_p$  and the  $u$  which is not identically zero. Similarly, if we replace  $\lambda_2$  by a sufficiently high derivative of  $u_1$  in  $K_1$ , we obtain a nonzero d.p.  $K_2$  in  $\lambda_3, \dots, \lambda_p$  and the  $u$ . Continuing these replacements, we obtain a nonzero d.p.  $G$  in the  $u$  alone.

Continuing as in §23, we see that the zeros of  $\Omega$  with  $\lambda_j = M_j, j = 1, \dots, p$ , are the zeros of the  $\Lambda_i$  with  $\lambda_j = M_j$ . Now the zeros with  $\lambda_j = M_j$  of  $\Lambda_1, \dots, \Lambda_s$  have  $u$  which annul  $G$ . The zeros of  $\Lambda_{s+1}, \dots, \Lambda_r$ , even with  $\lambda_j = M_j$ , have  $y_i = z_i$  for  $i = 1, \dots, p$ . This proves our statement.

25. The results of §§23, 24 permit us to state that if either

- (a)  $\mathfrak{F}$  does not consist purely of constants, or
- (b) there exist  $u$ ,

triads of d.p.  $G, P, Q$  exist in  $\mathfrak{F}\{u_1, \dots, u_q; y_1, \dots, y_p\}$ , with  $G$  and  $P$  not in  $\Sigma$  and  $G$  free of the  $y$ , such that, for two distinct zeros of  $\Sigma$  in a single extension of  $\mathfrak{F}$ , with the same  $u$ , the zeros annulling neither  $G$  nor  $P$ , the expression  $Q/P$  has two distinct values. For instance, if (a) holds, we can take  $P = 1$  and  $Q = \mu_1 y_1 + \dots + \mu_p y_p$ .

The ideas will be more complete, and even simpler, if we use general d.p.  $P$ . The following is a nontrivial case in which  $P$  is of positive class. Let  $\mathfrak{F}$  be the totality of rational functions of  $x$ . We take  $\Sigma$  as  $\{y_{11}, y_{21}\}$  in  $\mathfrak{F}\{y_1, y_2\}$ . The zeros are  $y_1 = c, y_2 = d$  with  $c$  and  $d$  constant, but otherwise unrestricted. We take  $G = 1$ . If

$$P = y_1 + xy_2, \quad Q = y_1^2 + x^2,$$

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<sup>31</sup> The following example shows that  $\Sigma$  may have many zeros with given  $u$  and that a  $G$  may exist such that, for  $G \neq 0$ , there is only one zero for given  $u$ . Let  $\Sigma$  be the perfect ideal generated by  $u_1 y_1 - u_2$  in  $\mathfrak{F}\{u_1, u_2, y_1\}$ .  $\Sigma$  is prime, since the separant for  $u_2$  is unity. The set  $u_1, u_2$  is parametric. Let  $G = u_1$ . If  $u_1 = u_2 = 0, y_1$  may be taken arbitrarily, but, for given  $u_1, u_2$  with  $G \neq 0$ , there is only one  $y_1$ .

the expression  $Q/P$  assumes distinct values for distinct zeros of  $\Sigma$  with  $P \neq 0$ .<sup>32</sup>

In certain cases in which  $\mathfrak{F}$  consists purely of constants and in which no  $u$  exist, there may exist no pair  $P, Q$  as described above. For instance, let  $\mathfrak{F}$  be the totality of complex numbers. Let  $\Sigma$  be as in the preceding example. For every zero, the  $y_{ij}$  are zero for  $j > 0$ . We therefore lose no generality in seeking a  $P$  and  $Q$  of order zero in  $y_1$  and  $y_2$ . For any such  $P$  and  $Q$ ,  $Q/P$  will yield the same result for infinitely many distinct pairs of constants  $y_1, y_2$ .

In developing the theory of a prime ideal  $\Sigma$  for the case in which  $\mathfrak{F}$  has only constants and in which there are no  $u$ , two courses are open to us. If we adjoin an element  $x$  to  $\mathfrak{F}$ , as in I, §29,  $\Sigma$  will generate, for the enlarged field, a prime ideal whose theory may be expected to be equivalent to that of  $\Sigma$ ; in the analytic case, the ideals have the same restricted manifold. Again, by I, §27, we can introduce a new indeterminate  $u_1$  and  $\Sigma$  will generate a prime ideal in  $\mathfrak{F}\{u_1; y_1, \dots, y_n\}$ . After either type of adjunction, the theory which follows will apply.

26. From this point on, through §30, we work with a nontrivial prime ideal  $\Sigma$ . We assume that either

- (a)  $\mathfrak{F}$  does not consist purely of constants, or
- (b) parametric indeterminates exist.

We take a triad  $G, P, Q$  as in §25. Introducing a new indeterminate,  $w$ , we let  $\Lambda$  represent the ideal  $\{\Sigma, Pw - Q\}$  in  $\mathfrak{F}\{u; y; w\}$ . Let  $\Omega$  be the totality of those d.p.  $G$  in  $\mathfrak{F}\{u; y; w\}$  which have the property that

$$PG \equiv 0, \quad (\Lambda).$$

We see immediately that  $\Omega$  is an ideal. We shall prove that  $\Omega$  is prime.

Let  $B$  and  $C$  be such that  $BC$  is in  $\Omega$ . For  $s$  appropriate,  $P^s B$  minus a linear combination of  $Pw - Q$  and its derivatives is a d.p.  $R$  free of  $w$ . We obtain similarly, from a  $P^t C$ , a d.p.  $S$  free of  $w$ . As  $RS$  is in  $\Omega$ ,  $PRS$  is in  $\Lambda$ . A generic zero of  $\Sigma$  does not annul  $P$ , and thus furnishes a zero of  $\Lambda$ . Thus a generic zero of  $\Sigma$  annuls  $RS$ , so that one of  $R$  and  $S$  is in  $\Sigma$ . If  $R$  is in  $\Sigma$ ,  $P^s B$  is in  $\Lambda$ . Then  $B$  is in  $\Omega$ , so that  $\Omega$  is prime.

We notice that those d.p. of  $\Omega$  which are free of  $w$  are precisely the d.p. of  $\Sigma$ . In particular,  $\Omega$  contains no d.p. in the  $u$  alone.

We are going to show that  $\Omega$  contains a d.p. in  $w$  and the  $u$  alone.

Let  $B_i, i = 1, \dots, p$ , be a d.p. of  $\Sigma$  involving only  $y_i; u_1, \dots, u_q$ , of minimum rank in  $y_i$ . Let  $S_i$  be the separant of  $B_i$ . Consider any generic zero of  $\Omega$ . For it, we have

$$w = \frac{Q}{P}.$$

For the  $j$ th derivative of  $w$ , we have an expression

$$(35) \quad w_j = \frac{Q_j}{P^{j+1}}.$$

<sup>32</sup> As usual, we compare only zeros contained in the same extension.

Using the relations  $B_i = 0$ , we free each  $Q_j$  from those derivatives of each  $y_i$  which are of order higher than the maximum of the orders of  $Q, P$  and  $B_i$  in  $y_i$ . Each  $w_j$  will then be expressed as a quotient of two d.p., the denominator being a product of powers of  $P, S_1, \dots, S_p$ . If we use a sufficient number of the relations (35), as just transformed, we will have more  $w_i$  than there are  $y_{ij}$  in the second members. Using the process of elimination employed in §23, we obtain a d.p.  $K$  in  $w; u_1, \dots, u_q$  which vanishes for a generic zero of  $\Omega$  and is therefore in  $\Omega$ .

27. We now list the indeterminates in the order

$$u_1, \dots, u_q; w; y_1, \dots, y_p$$

and take a characteristic set of  $\Omega$ ,

$$(36) \quad A, A_1, \dots, A_p.$$

Here  $w, y_1, \dots, y_p$  are introduced in succession (§21). The separants for (36) will be represented by  $S, S_1, \dots, S_p$  and the initials by  $I, I_1, \dots, I_p$ .

If  $A$  is not algebraically irreducible, we can replace it by one of its irreducible factors. We assume therefore that  $A$  is algebraically irreducible.

We are going to prove that  $A_1, \dots, A_p$  are of order 0 in  $y_1, \dots, y_p$  respectively and, indeed, that  $A_i$  is of the first degree in  $y_i$ . Thus, since, for  $i > j$ ,  $A_i$  is of lower degree in  $y_j$  than  $A_j$ , each equation  $A_i = 0$  expresses  $y_i$  rationally in terms of  $w; u_1, \dots, u_q$  and their derivatives.

The determination of the manifold of  $\Sigma$  will in this way be made to depend on the determination of the general solution of  $A = 0$  (§16), which equation will be called a *resolvent* of the prime ideal  $\Sigma$ , or of the system of equations obtained by equating the d.p. in  $\Sigma$  to zero.

28. Let us suppose that our claim with respect to the  $A_i$  is false and let  $A_k$  be the  $A_i$  of highest subscript for which it breaks down. Thus the  $A_i$  with  $i > k$ , if they exist, are of zero order in  $y_{k+1}, \dots, y_p$  respectively and are linear in those letters. On the other hand, either  $A_k$  is of positive order in  $y_k$ , or  $A_k$  is of zero order in  $y_k$  and is not linear in  $y_k$ . We shall force a contradiction.

Let  $P_1$  be the remainder with respect to (36) of  $P$  of §26 and let  $U$  be the remainder with respect to (36) of

$$P_1 S_k I_k I_{k+1} \dots I_p.$$

In  $\mathfrak{F}\{u_1, \dots, u_q; w; y_1, \dots, y_k\}$ , let

$$\Xi = (A, A_1, \dots, A_k, U).$$

$U$  is not zero and is reduced with respect to (36).<sup>33</sup> Of all nonzero d.p. in  $\Xi$  which are reduced with respect to (36), let  $B$  be one of a least degree in  $y_{kr}$ , where  $r$  is the order of  $A_k$  in  $y_k$ . We say that  $B$  is free of  $y_{kr}$ .

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<sup>33</sup> The fact that  $A_{k+1}, \dots, A_p$  involve  $y_{k+1}, \dots, y_p$ , which do not figure in  $\Xi$ , need give no concern. Note that  $U$  is free of those indeterminates.

Suppose that this is not so. Let  $C$  be the initial of  $B$ , that is, the coefficient of the highest power of  $y_{kr}$  in  $B$ . For  $m$  appropriate,

$$C^m A_k = DB + E$$

with  $D$  of lower degree than  $A_k$  in  $y_{kr}$ , and  $E$ , if not zero, of lower degree than  $B$  in  $y_{kr}$ .  $E$  is in  $\Xi$ .

We shall prove that  $E$  is in  $\Omega$ . This is certainly true if  $E = 0$ . Suppose that  $E$  is not zero. Let  $F$  be the remainder of  $E$  with respect to  $A, A_1, \dots, A_{k-1}$ .<sup>34</sup> Then  $F$  is in  $\Xi$  and is reduced with respect to (36). If  $F$  were not zero, it would be, like  $E$ , of lower degree than  $B$  in  $y_{kr}$ . Thus  $F = 0$  and  $E$  is in  $\Omega$ .

Thus  $DB$  is in  $\Omega$ .  $B$  is not, since it is reduced with respect to (36). Then  $D$  is in  $\Omega$ . With  $t$  the degree of  $D$  in  $y_{kr}$ , let

$$D = G_0 + G_1 y_{kr} + \dots + G_t y_{kr}^t.$$

As  $E$ , if not zero, is of lower degree than  $A_k$  in  $y_{kr}$ , the initial of  $DB$  is identical with that of  $C^m A_k$ . Now  $C$ , reduced with respect to (36), is not in  $\Omega$ . Thus  $G_t$  is not in  $\Omega$ . It is easy to see that there exist integers  $a, a_1, \dots, a_{k-1}$  such that

$$I^a I_1^{a_1} \dots I_{k-1}^{a_{k-1}} G_i \equiv G'_i, \quad (\Omega), \quad i = 1, \dots, t,$$

where each  $G'_i$  is reduced with respect to  $A, A_1, \dots, A_{k-1}$ . We see that  $G'_i \neq 0$ . Then

$$G'_0 + \dots + G'_t y_{kr}^t$$

is a nonzero d.p. in  $\Omega$  which is reduced with respect to (36). This contradiction proves that  $B$  is free of  $y_{kr}$ .

29. Now let  $\Omega'$  be the totality of those d.p. in  $\Omega$  which are free of  $y_k, \dots, y_p$ . We see immediately that  $\Omega'$  is a prime ideal with  $A, \dots, A_{k-1}$  as a characteristic set.

Let

$$(37) \quad u_i = \tau_i, i = 1, \dots, q; w = \xi; y_i = \eta_i, i = 1, \dots, p,$$

be a generic zero of  $\Omega$ , contained in an extension  $\mathfrak{F}_1$  of  $\mathfrak{F}$ . Then

$$(38) \quad \tau_1, \dots, \tau_q; \xi; \eta_1, \dots, \eta_{k-1}$$

is a generic zero of  $\Omega'$ . We replace  $u_1, \dots, u_q; w; y_1, \dots, y_{k-1}$  in  $A_k$  by the quantities (38). We secure a d.p.  $H_k$  in  $\mathfrak{F}_1 \{y_k\}$ .

We examine  $H_k$ . Let  $A_k$  be arranged as a polynomial in the  $y_{ki}, i = 0, \dots, r$ , with nonzero coefficients. The coefficients are not in  $\Omega$  and hence do not vanish for (38). Thus  $H_k$  has the same degree in  $y_{kr}$  that  $A_k$  has.

Let  $H_k$  be expressed as a product of irreducible factors over  $\mathfrak{F}_1$  and let  $K$  be an irreducible factor which is of order  $r$  in  $y_k$ . Let  $\zeta_k$  be a generic point (§6) in the general solution of  $K$ .

<sup>34</sup> For  $k = 1$ , we take the remainder of  $E$  with respect to  $A$ .

For (38),  $B$  becomes a d.p.  $L$  in  $\mathfrak{F}_1 \{ y_k \}$ . As  $B$  is not in  $\Omega$ ,  $L$  is not identically zero.  $L$  is of order less than  $r$  in  $y_k$  if  $r > 0$  and is an element of  $\mathfrak{F}_1$  if  $r = 0$ . Thus  $L$  cannot vanish for  $y_k = \zeta_k$  (§13). Then  $B$  does not vanish for

$$(39) \quad \tau_1, \dots, \tau_p; \xi; \eta_1, \dots, \eta_{k-1}; \zeta_k.$$

As  $A, A_1, \dots, A_k$  vanish for (39),  $U$  does not. Then (39) annuls none of

$$P_1, S_k, I_k, \dots, I_p.$$

The failure of  $I_{k+1}, \dots, I_p$  to vanish for (39) shows that, when (39) is substituted into an  $A_j$  with  $j > k$ , the equation  $A_j = 0$  determines  $y_j$  as a quantity  $\zeta_j$  in the extension of  $\mathfrak{F}_1$  which contains  $\zeta_k$ . The quantities

$$(40) \quad \tau_1, \dots, \tau_q; \xi; \eta_1, \dots, \eta_{k-1}; \zeta_k, \dots, \zeta_p$$

are seen to constitute a regular zero of (36) which does not annul<sup>35</sup>  $PG$ . Thus (40) is a zero of  $\Omega$  and

$$(41) \quad \tau_1, \dots, \tau_q; \eta_1, \dots, \eta_{k-1}; \zeta_k, \dots, \zeta_p$$

is a zero of  $\Sigma$  which does not annul  $PG$ .

Suppose now that  $r > 0$ . We cannot have  $\eta_k = \zeta_k$ . Otherwise  $y_k - \eta_k$  would be a d.p. in  $\mathfrak{F}_1 \{ y_k \}$  of lower rank than  $K$  which is annulled by  $\zeta_k$ . In (41) and in the generic zero of  $\Sigma$

$$\tau_1, \dots, \tau_q; \eta_1, \dots, \eta_p,$$

we have two zeros of  $\Sigma$  which do not annul  $PG$  and which yield the same value  $\xi$  for  $w$ . This contradicts the nature of  $G, P, Q$ .

We have thus proved that  $A_k$  is of order zero in  $y_k$ .

30. The denial made at the beginning of §28 now becomes a claim that  $A_k$  is not linear in  $y_k$ . We use the material of §29. As  $\zeta_k$  must equal<sup>36</sup>  $\eta_k, y_k - \eta_k$  must be divisible by  $K$ . This means, if  $H_k$  is of degree  $t$  in  $y_k$ , that

$$(42) \quad H_k = \alpha(y_k - \eta_k)^t$$

where  $\alpha$  is the coefficient of  $y_k^t$  in  $H_k$ . Let  $\beta$  be the coefficient of  $y_k^{t-1}$  in  $H_k$ . By (42),

$$(43) \quad t\alpha\eta_k + \beta = 0.$$

In  $\alpha$  and  $\beta$ , we reverse the substitution made to convert  $A_k$  into  $H_k$ . Also, in the first member of (43), we replace  $\eta_k$  by  $y_k$ . We obtain a d.p.

$$My_k + N$$

which is in  $\Omega$ , since, by (43), it is annulled by (37). As  $\alpha \neq 0$ ,  $M$  is not zero.

<sup>35</sup>  $G$ , which is not in  $\Omega$ , cannot vanish for the  $\tau$ .

<sup>36</sup> Note that the theory of the general solution applies to d.p. which do not involve proper derivatives.

Furthermore,  $M$ , which is the product by  $t$  of a coefficient of  $A_k$ , is reduced with respect to  $A, \dots, A_{k-1}$ ; so is  $N$ .

We have a final contradiction of the assumption of falsity made in §28.

Thus, every  $A_i$  is linear in  $y_i$  and, in the manifold of  $\Sigma$ , each  $y_i$  has an expression rational in  $w; u_1, \dots, u_q$  and their derivatives, with coefficients in  $\mathfrak{F}$ .

31. We say that if

$$(44) \quad \bar{u}_1, \dots, \bar{u}_q; \bar{w}; \bar{y}_1, \dots, \bar{y}_p$$

is a zero of  $\Omega$ , then  $\bar{u}_1, \dots, \bar{u}_q; \bar{w}$  belongs to the general solution of  $A$ .<sup>37</sup> Let  $K$  be any d.p. in  $w$  and the  $u$  belonging to the prime ideal whose manifold is the general solution of  $A$ . As the remainder of  $K$  with respect to  $A$  is zero,  $K$  is in  $\Omega$  and therefore vanishes for  $\bar{u}_1, \dots, \bar{u}_q; \bar{w}$ .

32. The introduction of the resolvent accomplishes the following:

(a) It reduces the study of an irreducible manifold  $\mathfrak{M}$  to the study of the general solution  $\mathfrak{M}'$  of some d.p. The correspondence between  $\mathfrak{M}$  and  $\mathfrak{M}'$  may be described as *birational*. Of course, in the expression for  $w$  in terms of the  $y$ , and in those of the  $y$  in terms of  $w$ , derivatives may appear. For zeros in  $\mathfrak{M}$  with  $P = 0$ , there may be no corresponding  $w$ , and for other zeros in  $\mathfrak{M}$  the initial of some  $A_i$  may vanish. For restricted manifolds, we shall gain information on these special zeros in Chapter VI.

(b) It extends into the theory of differential equations a property of systems of algebraic functions of several variables. It is well known that, given a finite system of algebraic functions, we can find a single algebraic function in terms of which, and of the variables, the functions in the system can be expressed rationally.

(c) It furnishes an instrument useful in the treatment of various problems.

#### DIMENSION OF AN IRREDUCIBLE MANIFOLD

33. Let  $\Sigma$  be a nontrivial prime ideal in  $\mathfrak{F}\{y_1, \dots, y_n\}$  with  $\mathfrak{F}$  any field.

We propose to show that, if parametric indeterminates exist, their number,  $q$ , does not depend on the manner in which they are selected; in other words, two sets of parametric indeterminates contain the same number of indeterminates.

Let us suppose that a set  $u_1, \dots, u_q$  has been selected, and that one has, in addition,  $y_1, \dots, y_p$ . It will suffice to show that, given any  $q + 1$  indeterminates among the  $u$  and  $y$

$$z_1, \dots, z_{q+1},$$

there exists a d.p. in  $\Sigma$  which involves only the  $z$ .

We form a resolvent for  $\Sigma$ . As  $u$  exist, this is possible. Let us consider a generic zero of  $\Omega$ . The  $z$  in that zero have expressions rational in  $w$ , the  $u$  and their derivatives. If a  $z_i$  happens to be a  $u$ , say  $u_j$ , the expression for  $z_i$  is simply  $u_j$ . We write

<sup>37</sup> Here we consider  $A$  as a d.p. in  $w$  and the  $u$  alone.

$$(45) \quad z_i = \rho_i(w; u_1, \dots, u_q), \quad i = 1, \dots, q + 1.$$

On differentiating (45) repeatedly, we get expressions for the  $z_{i,j}$  which are rational in the  $w_j$  and  $u_{i,j}$ . Making use of the relation  $A = 0$ , we transform these expressions so as not to contain derivatives of  $w$  of order higher than  $r$ , where  $r$  is the order of  $A$  in  $w$ .

Since there are  $q + 1$  of the  $z$  and only  $q$  of the  $u$ , it follows that if we differentiate (45) often enough (and then transform), the  $z_{i,j}$  will become more numerous than the  $u_{i,j}$  and  $w, w_1, \dots, w_r$ .

It follows as in §23 that there exists a nonzero d.p. in the  $z$  which vanishes for a generic zero of  $\Sigma$ . Such a d.p. is in  $\Sigma$ .

We shall call the number  $q$  the *dimension* of  $\Sigma$ , or of the manifold of  $\Sigma$ . To a nontrivial prime ideal without parametric indeterminates, we attribute the dimension 0. The dimension of  $[0]$  will be defined as  $n$ .

From §18, it follows that, for  $\mathfrak{F}\{y_1, \dots, y_n\}$ , every manifold of dimension  $n - 1$  is the general solution of a differential polynomial.

#### ORDER OF THE RESOLVENT

34. We work with a nontrivial prime ideal  $\Sigma$  of dimension  $q$  in  $\mathfrak{F}\{u_1, \dots, u_q; y_1, \dots, y_p\}$ , the  $u$  being parametric for  $\Sigma$ .<sup>38</sup> We suppose that triads  $G, P, Q$ , and therefore resolvents, exist. Let

$$(46) \quad A_1, \dots, A_p$$

be a characteristic set for  $\Sigma$ , the separant and initial of  $A_i$  being  $S_i$  and  $I_i$ , respectively. We denote the order of  $A_i$  in  $y_i$  by  $r_i$ . Let

$$h = r_1 + \dots + r_p.$$

We shall prove that every resolvent of  $\Sigma$  is of order  $h$  in  $w$ .<sup>39</sup>

We begin by proving that  $\Omega$  contains a d.p. in  $w; u_1, \dots, u_q$  whose order in  $w$  does not exceed  $h$ .

Consider a generic zero of  $\Omega$ ,

$$(47) \quad \bar{u}_1, \dots, \bar{u}_q; \bar{w}; \bar{y}_1, \dots, \bar{y}_p.$$

For it, we have

$$(48) \quad w = \frac{Q}{P}.$$

We shall show the existence of d.p.  $R$  and  $T$ , each of order not exceeding  $r_i$  in  $y_i$ ,  $i = 1, \dots, p$ , such that, for (47),  $T$  is not zero and

$$(49) \quad w = \frac{R}{T}.$$

<sup>38</sup> When  $q = 0$ , there are no  $u$ .

<sup>39</sup> For a brief proof, based on the theory of algebraic fields, see Kolchin, 13.

Let  $Q_1$  and  $P_1$  be the remainders of  $Q$  and  $P$  respectively relative to (46). Let  $Q_1$  be obtained by subtracting a linear combination of the  $A_i$  and their derivatives from

$$S_1^{\sigma_1} \cdots I_p^{\nu} Q$$

and let  $P_1$  be obtained similarly from

$$S_1^{\sigma_1} \cdots I_p^{\nu} P.$$

Then, for (47), we have

$$(50) \quad w = \frac{Q_1 S_1^{\sigma_1} \cdots I_p^{\nu}}{P_1 S_1^{\sigma_1} \cdots I_p^{\nu}}.$$

For  $R$  and  $T$  in (49), we take the numerator and denominator in (50).

We find from (49), for the  $j$ th derivative of  $w$ , an expression

$$(51) \quad w_j = \frac{B_j}{T^{j+1}}.$$

If  $U_j$  is the remainder of  $B_j$  with respect to (46), we can write (51)

$$(52) \quad w_j = \frac{U_j}{W_j}$$

where  $W_j$  is a product of powers of  $T, S_1, \cdots, I_p$ .

Consider (49) and the first  $h$  of the relations (52). Let  $D$  be a common denominator for the second members in these  $h + 1$  relations. We write

$$(53) \quad w_j = \frac{E_j}{D},$$

$j = 0, \cdots, h$ .

Let  $D$ , the  $E$ , and the  $A$  in (46) be written as polynomials in the  $y_{ij}$  with coefficients which are d.p. in the  $u$ . Let  $m$  be the maximum of the degrees of these polynomials.

For convenience, we represent the  $r_i$ th derivative of  $y_i$  by  $z_i$ . Let  $A_i$  be of degree  $v_i$  in  $z_i$ .

Let  $\alpha$  be a positive integer, to be fixed later. In (53), let us form all power products in the  $w_j$  of degree  $\alpha$  or less. Let the expression for each power product be written in the form

$$(54) \quad \frac{F}{D^\alpha}.$$

Then each  $F$  is a polynomial in the  $y_{ij}$ , of degree not exceeding  $m\alpha$ .

Let each expression (54) be written

$$(55) \quad \frac{F I_p^{m\alpha}}{D^\alpha I_p^{m\alpha}}.$$

Consider a particular  $F$ , and let it be written as a polynomial in  $z_p$ . Suppose

that its degree  $d$  in  $z_p$  is not less than  $m$ . Then, as  $A_p = 0$  for (47), we have, letting

$$M = A_p - I_p z_p^v,$$

the relation

$$(56) \quad I_p z_p^d = -M z_p^{d-v}.$$

If

$$F = J_0 + J_1 z_p + \cdots + J_d z_p^d,$$

with the  $J$  free of  $z_p$ , we may write the numerator in (55) in the form

$$(57) \quad (J_0 I_p + \cdots + J_d I_p z_p^d) I_p^{m\alpha-1}.$$

Since  $I_p$  is of degree less than  $m$  in the  $y_{ij}$ , each term in the parentheses in (57) is of degree less than  $m(\alpha + 1)$

We replace  $J_d I_p z_p^d$  by  $-J_d M z_p^{d-v}$  in (57). As  $J_d$  is of degree not exceeding  $m\alpha - d$  in the  $y_{ij}$ , and as  $M$  is of degree at most  $m$ , then  $J_d M z_p^{d-v}$  is of degree less than  $m(\alpha + 1)$  in the  $y_{ij}$ . Thus (55) goes over into

$$\frac{F_1 I_p^{m\alpha-1}}{D^\alpha I_p^{m\alpha}},$$

where  $F_1$  is of degree less than  $m(\alpha + 1)$  in the  $y_{ij}$  and of degree less than  $d$  in  $z_p$ . If the degree of  $F_1$  in  $z_p$  is not less than  $m$ , we repeat the above operation. After  $t \leq m\alpha$  operations, we get an expression

$$(58) \quad \frac{H I_p^{m\alpha-t}}{D^\alpha I_p^{m\alpha}}$$

with  $H$  of degree less than  $m$  in  $z_p$  and of degree less than  $m(\alpha + t)$  in the  $y_{ij}$ . The numerator in (58) is of degree in the  $y_{ij}$  less than

$$m(\alpha + t) + m(m\alpha - t) \leq 2m^2\alpha.$$

Thus, if we let  $D_1 = D I_p^m$ , we can write each power product in the  $w_j$  of degree  $\alpha$  or less in the form

$$(59) \quad \frac{K}{D_1^\alpha}$$

where  $K$  is of degree less than  $2m^2\alpha$  in the  $y_{ij}$  and of degree less than  $m$  in  $z_p$ .

We now write each expression (59) in the form

$$(60) \quad \frac{K I_{p-1}^{2m\alpha}}{D_1^2 I_{p-1}^{2m\alpha}}$$

and employ, with respect to  $z_{p-1}$ , the procedure used above. We find for each expression (60) an equivalent expression

$$(61) \quad \frac{L}{D_2^\alpha}$$

with  $D_2 = D_1 I_{p-1}^{2m^2}$  and with  $L$  of degree less than  $4m^3\alpha$  in the  $y_{ij}$  and of degree less than  $m$  in  $z_p$  and  $z_{p-1}$ . Continuing, we find, for each power product of the  $w_j$ , an expression

$$(62) \quad \frac{W}{D_p^\alpha}$$

where  $W$  is of degree less than  $2^p m^{p+1} \alpha$  in the  $y_{ij}$  and of degree less than  $m$  in  $z_i$ ,  $i = 1, \dots, p$ . Let  $c$  represent  $2^p m^{p+1}$ .

The number of power products in  $z_1, \dots, z_p$  of degree less than  $m$  in each letter is  $m^p$ . Thus, as the  $y_{ij}$  with  $j < r_i$  are  $h$  in number, the number of power products of the  $y_{ij}$  of degree  $c\alpha$  or less, and of degree less than  $m$  in each  $z_i$ , is not more than

$$(63) \quad m^p \frac{(c\alpha + h) \cdots (c\alpha + 1)}{h!}.$$

The number of power products of degree  $\alpha$  or less in the  $h + 1$  letters  $w_j$  is

$$(64) \quad \frac{(\alpha + h + 1) \cdots (\alpha + 1)}{(h + 1)!}.$$

As (64) is of degree  $h + 1$  in  $\alpha$  and (63) is only of degree  $h$ , (64) will exceed (63) for  $\alpha$  large. This, we know from §23, implies the existence of a nonzero d.p. of  $\Omega$  in  $w$  and the  $u$  alone, of order not exceeding  $h$  in  $w$ .

This shows that the order in  $w$  of the resolvent does not exceed  $h$ . Suppose that the order of  $A$  in  $w$  is  $k < h$ . For (47), we have relations

$$(65) \quad y_i = \frac{C_i}{D_i}$$

where the  $C$  and  $D$  are d.p. in  $w$  and the  $u$ , of order not exceeding  $k$  in  $w$ . We obtain from (65) expressions for the  $y_{ij}$ ,  $j = 0, \dots, r_i - 1$ , which are rational in the  $w_j$  and  $u_{ij}$  with powers of the  $D$  for denominators. Using the relation  $A = 0$ , we depress the orders in  $w$  of the numerators until they do not exceed  $k$ . The transformed expressions will have denominators which are power products of  $S$  and the  $D$ .

By an elimination, we obtain a nonzero d.p.  $W$  in the  $u$  and  $y$  which belongs to  $\Omega$ , hence to  $\Sigma$ . This  $W$ , which is of order less than  $r_i$  in each  $y_i$ , is reduced with respect to (46). This is impossible.

We have thus proved that the order in  $w$  of every resolvent is  $h$ .

35. Let  $\Sigma$  be as in §34, except that we waive the condition that resolvents exist.

If we consider any  $h + 1$  of the  $y_{ij}$ , the elimination process of §34 shows that  $\Sigma$  contains a nonzero d.p. which, in addition to those  $y_{ij}$ , involves only the  $u$  and their derivatives.<sup>40</sup> Thus, *if  $\mathfrak{M}$  is the manifold of  $\Sigma$ , there exist  $h$  of the  $y_{ij}$  such that no algebraic relation among those  $y_{ij}$  and any set of  $u_{ij}$  holds throughout*

<sup>40</sup> The statement which follows is an informal one, whose meaning is clear.

$\mathfrak{M}$ , whereas, given any  $h + 1$  of the  $y_{ij}$ , an algebraic relation holds throughout  $\mathfrak{M}$  for those  $y_{ij}$  and certain  $u_{ij}$ .

The quantity  $h$  will be called the *order* of  $\Sigma$  (or of  $\mathfrak{M}$ ) *relative to*  $u_1, \dots, u_q$ . When  $q = 0$ , we call  $h$  the *order* of  $\Sigma$ .

The two numbers  $q$  and  $h$  measure the extensiveness of  $\mathfrak{M}$ . In the analytic case, we may think of  $q$  as the number of arbitrary functions which figure in  $\mathfrak{M}$ , and of  $h$  as the number of arbitrary constants at one's disposal when the arbitrary functions are selected. This can be seen from §10.

The relative order depends, as one would expect, on the choice of the  $u$ . For instance, the manifold of  $y_{11} - y_2$  is irreducible. If we let  $u_1 = y_2$  we have  $h = 1$ . If  $u_1 = y_1$ ,  $h = 0$ .

If  $\mathfrak{F}$  consists purely of constants and if  $\mathfrak{F}_1$  is secured from  $\mathfrak{F}$  by the adjunction of an element  $x$  of derivative unity, the prime ideal  $\Sigma_1$  of d.p. over  $\mathfrak{F}_1$  which  $\Sigma$  generates has the same parametric sets and the same relative orders as  $\Sigma$ . This is because a characteristic set of  $\Sigma$  is also one of  $\Sigma_1$ .<sup>41</sup>

EMBEDDED MANIFOLDS<sup>42</sup>

36. THEOREM: *Let  $\Sigma$  and  $\Sigma'$  be nontrivial prime ideals, with  $\Sigma'$  a proper divisor of  $\Sigma$ , of the respective dimensions  $q$  and  $q'$ . Then  $q \geq q'$ . If  $q = q'$ , every parametric set  $u_1, \dots, u_q$  for  $\Sigma'$  is such a set for  $\Sigma$  and the order of  $\Sigma'$  relative to  $u_1, \dots, u_q$  is less than that of  $\Sigma$ .*<sup>43</sup>

To show that  $q \geq q'$ , we observe that  $\Sigma$ , which is contained in  $\Sigma'$ , can have no d.p. in the  $u_1, \dots, u_{q'}$  of a parametric set for  $\Sigma'$ . Thus we can build a parametric set for  $\Sigma$  starting with  $u_1, \dots, u_{q'}$ .

Suppose now that  $q = q'$ . By the final remark of §35, we may suppose, even if  $q = 0$ , that resolvents exist for  $\Sigma$  and  $\Sigma'$ .

We can build resolvents simultaneously for  $\Sigma$  and  $\Sigma'$ , using a single relation

$$w = \mu_1 y_1 + \dots + \mu_p y_p.$$

The  $\mu$  and  $G$  which serve for  $\Sigma$  will serve also for  $\Sigma'$ , because the manifold of  $\Sigma'$  is part of that of  $\Sigma$ . For  $\Sigma$  we obtain an  $\Omega$ , and for  $\Sigma'$  an  $\Omega'$  which is a proper divisor of  $\Omega$ . Let

$$A, A_1, \dots, A_p; A', A'_1, \dots, A'_p$$

be characteristic sets of  $\Omega$  and  $\Omega'$  respectively. As  $A$  is in  $\Omega'$ ,  $A'$  is not of higher order in  $w$  than  $A$ . Suppose that  $A'$  is of the same order in  $w$  as  $A$ . By §13,  $A$  is divisible by  $A'$ . The algebraic irreducibility of  $A$  and  $A'$  implies that  $A = cA'$  with  $c$  in  $\mathfrak{F}$ . This implies that  $A, A_1, \dots, A_p$  is a characteristic set for  $\Omega'$  as well as for  $\Omega$ . Now a prime ideal is the totality of those d.p. which have zero remainders with respect to one of its characteristic sets. Thus  $\Omega'$

<sup>41</sup> A d.p. in  $\Sigma_1$  which is a polynomial in  $x$  has coefficients in  $\Sigma$ . (I, §29).

<sup>42</sup> Gourin, 5.

<sup>43</sup> If  $q = 0$ ,  $\Sigma'$  is of lower order than  $\Sigma$ .

and  $\Omega$  are identical. This contradiction shows that  $A'$  is of lower order in  $w$  than  $A$ . The theorem is proved.

When  $q = q'$ , not every parametric set for  $\Sigma$  need be such a set for  $\Sigma'$ . Let  $\Sigma = \{y_{10}y_{20} + y_{11}\}$ . Either  $y_1$  or  $y_2$  is a parametric set. If  $\Sigma' = \{y_1\}$ ,  $y_2$  is parametric and  $y_1$  is not.

#### PRIME IDEALS AND FIELD EXTENSIONS

37. Let  $\Sigma$  be a nontrivial prime ideal. Let  $\mathfrak{F}_1$  be an extension of  $\mathfrak{F}$  and  $\Sigma'$  the ideal of d.p. over  $\mathfrak{F}_1$  which  $\Sigma$  generates. We are going to show that  $\Sigma'$  is perfect and we shall discuss the essential prime divisors of  $\Sigma'$ .

Let us suppose first that  $\Sigma$  is of dimension  $q > 0$ , with a parametric set  $u_1, \dots, u_q$ . We build a resolvent for  $\Sigma$ , using a d.p.

$$(66) \quad w - \mu_1 y_1 - \dots - \mu_p y_p.$$

Let

$$(67) \quad A, A_1, \dots, A_p$$

be a characteristic set of  $\Omega$ , with  $A = 0$ , of order  $r$  in  $w$ , a resolvent for  $\Sigma$ .

Suppose now that the irreducible factors of  $A$  over  $\mathfrak{F}_1$  are  $B_1, \dots, B_p$ . Then each  $B_i$  is of order  $r$  in  $w$ . Otherwise, the coefficients of the powers of  $w$ , in  $A$ , having a common factor over  $\mathfrak{F}_1$ , would have one over  $\mathfrak{F}$  and  $A$  would not be algebraically irreducible.

We consider some  $B_j$ . Let its general solution have a generic point

$$(68) \quad \tau_1, \dots, \tau_q; \xi.$$

We now examine any  $A_i$  in (67), denoting its initial by  $I_i$ . It cannot be that  $I_i$  vanishes for (68). Otherwise  $I_i$ , being of order not greater than  $r$  in  $w$ , would be divisible by  $B_j$ . Thus  $A$  and  $I_i$  would have a common factor over  $\mathfrak{F}_1$ , hence one over  $\mathfrak{F}$ . This is impossible because  $I_i$  is of lower rank than  $A$ . Thus the equation  $A_i = 0$ , when  $w$  and the  $u$  are as in (68), determines  $y_i$  as a quantity  $\eta_i$  in the extension of  $\mathfrak{F}_1$  which contains (68).

We consider the quantities

$$(69) \quad \tau_1, \dots, \tau_q; \xi; \eta_1, \dots, \eta_p.$$

The totality  $\Omega_j$  of d.p. over  $\mathfrak{F}_1$  which vanish for (69) is easily seen to be a prime ideal. We shall prove that  $\Omega_j$  contains  $\Omega$ .

To take care of a point which arises later, let us start with any d.p.  $G$  over  $\mathfrak{F}_1$ . Let  $H$  be the remainder of  $G$  with respect to

$$A_1, \dots, A_p.$$

For some  $a$ , if  $S$  is the separant of  $A$ ,

$$S^a H \equiv K, \quad [A],$$

where  $K$  is of order not higher than  $r$  in  $w$ .

Suppose now that  $G$  is in  $\Omega$ . Then  $K$  is divisible by  $A$  and hence by  $B_j$ . Thus  $K$  vanishes for (68). Now  $S$  does not vanish for (68); if it did,  $S$  would be divisible by  $B_j$  and would have a factor over  $\mathfrak{F}$  in common with  $A$ . Then  $G$  vanishes for (69) and is in  $\Omega_j$ . Thus  $\Omega_j$  contains  $\Omega$ .

Let  $\Omega'$  be the ideal of d.p. over  $\mathfrak{F}_1$  which  $\Omega$  generates. Because a d.p. in  $\Omega$  goes over into one in  $\Sigma$  when  $w$  is replaced by  $\mu_1 y_1 + \dots + \mu_p y_p$ , those d.p. of  $\Omega'$  which are free of  $w$  constitute  $\Sigma'$ . Each  $\Omega_j$  contains  $\Omega'$ . Let  $G$  be any nonzero d.p. which is contained in each  $\Omega_j$ . Let  $K$  be found from  $G$  as above. Then  $K$  is divisible by each  $B_j$ , and hence by  $A$ . Thus  $S^a H$  is in  $\Omega'$ . Then some  $JG$ , with  $J = S^a I_1^{a_1} \dots I_p^{a_p}$ , is in  $\Omega'$ . Let  $G$  be written, as in I, §28, in the form

$$(70) \quad \gamma_1 C_1 + \dots + \gamma_m C_m$$

with the  $C$  d.p. over  $\mathfrak{F}$  and the  $\gamma$  linearly independent with respect to  $\mathfrak{F}$ . When we multiply by  $J$  in (70), we get a d.p. in  $\Omega'$ . Hence each  $J C_i$  is in  $\Omega$ . Then each  $C_i$  is in  $\Omega$  and  $G$  is in  $\Omega'$ . Thus  $\Omega'$  is the intersection of the  $\Omega_j$ .

On this basis, if  $\Sigma_j$  is the prime ideal consisting of those d.p. in  $\Omega_j$  which are free of  $w$ ,  $\Sigma'$  is the intersection of  $\Sigma_1, \dots, \Sigma_s$ . Thus  $\Sigma'$  is perfect.

No  $\Omega_j$  contains any  $\Omega_i$  with  $i \neq j$ ; if it did,  $B_i$  would be divisible by  $B_j$ . Now  $\Omega_j$  is the ideal of d.p. over  $\mathfrak{F}_1$  generated by  $\Sigma_j$  and the d.p. in (66). Thus none of the  $\Sigma_i$  contains any other, and the  $\Sigma_i$  are the essential prime divisors of  $\Sigma'$ .

Consider some  $\Sigma_j$ . If it contained a d.p.  $G$  in the  $u$  alone,  $G$  would vanish for the  $\tau$  in (68). Thus each  $\Sigma_j$  is of dimension  $q$ , with the same parametric sets as  $\Sigma$ . One can see now that  $B_j = 0$  is a resolvent for  $\Sigma_j$ . Thus the order of  $\Sigma_j$  relative to any parametric set equals that of  $\Sigma$ .

Suppose now that  $q = 0$ . We adjoin a new indeterminate  $u$ .  $\Sigma$  generates a prime ideal  $\Lambda$  of d.p. in  $u$  and the  $y$  (I, §27).  $\Lambda$  is of dimension unity, with  $u$  as a parametric set and with an order relative to  $u$  equal to the order of  $\Sigma$ .<sup>44</sup> Let  $\Lambda'$  be the ideal of d.p. over  $\mathfrak{F}_1$  generated by  $\Lambda$ . Then  $\Sigma'$  consists of those d.p. in  $\Lambda'$  which are free of  $u$ . Let the essential prime divisors of  $\Lambda'$  be  $\Lambda_1, \dots, \Lambda_s$ . If  $\Sigma_j$  is the prime ideal composed of those d.p. in  $\Lambda_j$  which are free of  $u$ ,  $\Lambda_j$  contains the prime ideal  $\Xi_j$  in  $\mathfrak{F}_1\{u; y_1, \dots, y_n\}$  generated by  $\Sigma_j$ . As  $\Sigma_j$  is a divisor of  $\Sigma'$ ,  $\Xi_j$  is a divisor of  $\Lambda'$ . This means that  $\Lambda_j = \Xi_j$ . Then no  $\Sigma_j$  contains any  $\Sigma_i$  with  $i \neq j$ , and the  $\Sigma_j$  are the essential prime divisors of  $\Sigma'$ . As the order of any  $\Sigma_j$  equals that of  $\Lambda_j$  relative to  $u$ , each  $\Sigma_j$  has the same order as  $\Sigma$ .

We summarize. *Let  $\Sigma$  be a nontrivial prime ideal of dimension  $q$ , and  $\Sigma'$  the ideal of d.p. over  $\mathfrak{F}_1$ , an extension of  $\mathfrak{F}$ , generated by  $\Sigma$ . Then  $\Sigma'$  is perfect and each of its essential prime divisors  $\Sigma_j$ ,  $j = 1, \dots, s$ , is of dimension  $q$ . If  $q > 0$ , every parametric set for  $\Sigma$  is such a set for every  $\Sigma_j$  and the orders of the  $\Sigma_j$  relative to such a set all equal that of  $\Sigma$ . If  $q = 0$ , every  $\Sigma_j$  has the same order as  $\Sigma$ .<sup>45</sup>*

<sup>44</sup> A characteristic set of  $\Sigma$  is one for  $\Lambda$ .

<sup>45</sup> A.D.E., Chapter VI, and Kolchin, 13.

ADJUNCTIONS TO FIELDS<sup>46</sup>

38. Let  $\mathfrak{F}$  be a field and  $\mathfrak{F}_1$  an extension of  $\mathfrak{F}$ . Let  $\sigma$  be any set of elements of  $\mathfrak{F}_1$ . There exist fields which are contained in  $\mathfrak{F}_1$  and contain  $\mathfrak{F}$  and  $\sigma$ . The intersection of all such fields is a field which will be denoted by  $\mathfrak{F}\langle\sigma\rangle$  and will be called the *field obtained by the adjunction of  $\sigma$  to  $\mathfrak{F}$* .  $\mathfrak{F}\langle\sigma\rangle$  consists of all rational combinations of elements of  $\sigma$ , and of derivatives of such elements, with coefficients in  $\mathfrak{F}$ .

A quantity  $\eta$  lying in an extension of  $\mathfrak{F}$  will be said to be *differential with respect to  $\mathfrak{F}$*  if  $\eta$  annuls a nonzero d.p. in one indeterminate over  $\mathfrak{F}$ .

**THEOREM:** *Let  $\mathfrak{F}$  contain a nonconstant element. Let  $\eta_1, \dots, \eta_n$  be elements lying in an extension of  $\mathfrak{F}$ , each differential with respect to  $\mathfrak{F}$ . The field  $\mathfrak{F}\langle\eta_1, \dots, \eta_n\rangle$  contains an element  $\xi$  such that*

$$\mathfrak{F}\langle\eta_1, \dots, \eta_n\rangle = \mathfrak{F}\langle\xi\rangle.$$

Let  $\Sigma$  be the set of those d.p. in  $\mathfrak{F}\{y_1, \dots, y_n\}$  which vanish for  $y_i = \eta_i$ ,  $i = 1, \dots, n$ . Then  $\Sigma$  is a prime ideal of dimension zero. We form a resultant for  $\Sigma$ , using a d.p. as in (66) with  $p = n$ . Let  $\xi = \sum \mu_i \eta_i$ . Consider the initial  $I_i$  of some  $A_i$  in (67). If  $I_i$  vanished for  $\xi$  and the  $\eta$ ,  $I_i$  would go over into a d.p. in  $\Sigma$  when  $w$  is replaced by the sum of the  $\mu_i y_i$ ; thus  $I_i$  would be in  $\Omega$ . It follows that each  $\eta_i$  is contained in  $\mathfrak{F}\langle\xi\rangle$ . This proves the theorem.

## ANALOGUE OF LÜROTH'S THEOREM

39. Let  $\mathfrak{F}$  be any field and  $u$  an indeterminate. The totality of rational combinations of the  $u_i$ , with coefficients in  $\mathfrak{F}$ , is a field which, by §38, it is proper to call  $\mathfrak{F}\langle u \rangle$ .

We prove the following theorem.

**THEOREM:** *Let  $\mathfrak{F}'$  be any extension of  $\mathfrak{F}$  which is contained in  $\mathfrak{F}\langle u \rangle$ . Then  $\mathfrak{F}'$  contains an element  $v$  such that  $\mathfrak{F}\langle v \rangle = \mathfrak{F}'$ .<sup>47</sup>*

This theorem is analogous to a well known theorem on algebraic fields which is equivalent to Lüroth's theorem on the parametrization of unicursal curves.<sup>48</sup>

40. Every element of  $\mathfrak{F}\langle u \rangle$  can be written in various ways in the form  $P/R$  with  $P$  and  $R$  in  $\mathfrak{F}\{u\}$ . We shall write  $F(u)$  for a d.p.  $F$  in  $u$ , irrespective of the number of derivatives of  $u$  which appear in  $F$ .

**LEMMA:** *Let  $P, Q, R$  be in  $\mathfrak{F}\{u\}$ , with  $R$  not zero. Let the relation*

$$(71) \quad \frac{P(\eta)}{R(\eta)} = \frac{P(\tau)}{R(\tau)},$$

where  $\eta$  and  $\tau$  lie in the same extension of  $\mathfrak{F}$  and do not annul  $R$ , imply the relation

<sup>46</sup> Kolchin, 12.

<sup>47</sup> A.D.E., Chapter VIII, and Kolchin, 12.

<sup>48</sup> van der Waerden, *Moderne Algebra*, vol. 1, p. 126.

$$(72) \quad \frac{Q(\eta)}{R(\eta)} = \frac{Q(\tau)}{R(\tau)}$$

Then  $Q(u)/R(u)$  is contained in the field obtained by adjoining  $P(u)/R(u)$  to  $\mathfrak{F}$ .

If  $Q(u)/R(u)$  is an element of  $\mathfrak{F}$ , the conclusion holds in a trivial way. If  $P(u)/R(u)$  is an element of  $\mathfrak{F}$ , (71) holds when  $\eta$  and  $\tau$  are indeterminates. Then (72) holds when  $\eta$  and  $\tau$  are indeterminates and  $Q/R$  is in  $\mathfrak{F}$ . In what follows, we assume that neither  $P/R$  nor  $Q/R$  is in  $\mathfrak{F}$ .

Let  $F$  be a d.p. in  $\mathfrak{F}\{u, y, z\}$ . For the substitution  $y = P(u)/R(u)$ ,  $z = Q(u)/R(u)$ ,  $F$  becomes an element of  $\mathfrak{F}\langle u \rangle$ . There exist d.p. in  $\mathfrak{F}\{u, y, z\}$ , for instance, 0, which vanish for the indicated substitution. The totality  $\Sigma$  of such d.p. is a prime ideal.

We show first that  $\Sigma$  contains no d.p. in  $y$  alone. Let such a d.p.  $G$  exist. Let  $P/R$  be written  $P_1/R_1$  with  $P_1$  and  $R_1$  relatively prime as polynomials in the  $u_i$ . We consider the algebraically irreducible d.p.  $K = P_1(u) - yR_1(u)$ . Let  $u = \tau$ ,  $y = \eta$  be a generic point in the general solution of  $K$ .  $R$  is not annulled by  $\tau$ . Hence  $G(\eta) = 0$ . This contradicts the fact that no nonzero d.p. in  $y$  alone holds the general solution of  $K$ , and our statement is proved. Similarly,  $\Sigma$  contains no nonzero d.p. in  $z$  alone.

A generic zero of  $\Sigma$  satisfies the relations  $y = P(u)/R(u)$ ,  $z = Q(u)/R(u)$ . With an elimination, we find that  $\Sigma$  contains a d.p. in  $y$  and  $z$  alone.

For the order  $y, z, u$ , let  $Z, U$  be a characteristic set of  $\Sigma$  with  $Z$  algebraically irreducible. Here  $y$  is parametric,  $Z$  introduces  $z$  and  $U$  introduces  $u$ . We claim that  $Z$  is of order zero in  $z$  and linear in  $z$ . The justification of this claim will amount to the proof of our lemma.

We denote the order of  $Z$  in  $z$  by  $r$ . Let  $y = \eta$ ,  $z = \zeta$ ,  $u = \tau$  be a generic zero of  $\Sigma$ . In  $Z$ , we replace  $y$  by  $\eta$ , securing a d.p.  $Z_1$  in  $\mathfrak{F}\langle \eta \rangle \{z\}$ , of order  $r$  in  $z$ . Let  $Z_1$  be factored in  $\mathfrak{F}\langle \eta, \zeta, \tau \rangle$ , and let  $Z_2$  be one of those irreducible factors of  $Z_1$  which are of order  $r$  in  $z$ . Let  $\zeta'$  be a generic point in the general solution of  $Z_2$ .

We shall show that  $\eta, \zeta'$ , which annuls  $Z$ , is a generic point in the general solution of  $Z$ . It will suffice to show that  $\eta, \zeta'$  annuls no d.p.  $B$  which is reduced with respect to  $Z$ . On the one hand, this will show that  $\eta, \zeta'$  does not annul the separant of  $Z$ , and is therefore in the general solution. On the other hand, it will prove that a d.p. whose remainder with respect to  $Z$  is not zero cannot vanish for  $\eta, \zeta'$ . We shall know thus that the only d.p. which vanish for  $\eta, \zeta'$  are those which hold the general solution of  $Z$ .

Let  $\eta, \zeta'$  annul a  $B$  as above. By §28, some linear combination  $C$  of  $B$  and  $Z$  is reduced with respect to  $Z$  and free of  $z_r$ . As  $\eta, \zeta'$  cannot annul  $C$ , it cannot annul  $B$ .

Thus, if we substitute  $\eta, \zeta'$  into  $U$ , we obtain a d.p.  $U_1$  in  $u$  whose order in  $u$  is the same as that of  $U$ . Let  $s$  be that common order. We factor  $U_1$  in  $\mathfrak{F}\langle \eta, \zeta, \tau, \zeta' \rangle$ . Let  $U_2$  be one of those irreducible factors of  $U_1$  which are of order  $s$  in  $u$  and let  $\tau'$  be a generic point in the general solution of  $U_2$ .

We say that  $\eta, \zeta', \tau'$  is a generic zero of  $\Sigma$ . For this, it suffices to show that  $\eta, \zeta', \tau'$  annuls no  $C$  which is reduced with respect to  $Z, U$ . Given such a  $C$ , some linear combination of  $Z, U$ , and  $C$  is, by §28, reduced with respect to  $Z, U$  and free of  $u$ . The proof is now easily completed.

We see now that  $\zeta' = \zeta$ . Otherwise  $\tau$  and  $\tau'$ , which do not annul  $R$ , would produce the same  $P/R$  and two distinct  $Q/R$ .

We find as in §§29, 30 that  $r = 0$  and that  $Z$  is linear in  $z$ .

41. There exist d.p. in  $\mathfrak{F}'\{y\}$  which vanish for  $y = u$ . For instance, if  $P(u)/R(u)$  is an element of  $\mathfrak{F}'$ , we can use

$$(73) \quad P(y) - \frac{P(u)}{R(u)}R(y).$$

The totality  $\Sigma$  of such d.p. is a prime ideal. Clearly  $y = u$  is a generic zero of  $\Sigma$ . We shall prove that the manifold of  $\Sigma$  is the general solution of a d.p. of the type (73).

We know that the manifold of  $\Sigma$  is the general solution of some d.p.  $B$  (§18). We suppose each coefficient in  $B$  to be written as the ratio of two d.p. in  $\mathfrak{F}\{u\}$ . Multiplying  $B$  by a suitable element of  $\mathfrak{F}\{u\}$ , we obtain a d.p.  $C$  in  $\mathfrak{F}\{u, y\}$  which is not divisible by any d.p. in  $\mathfrak{F}\{u\}$  actually involving one or more  $u_i$ . If  $C$  is arranged as a polynomial in the  $y_i$ , the ratio of any two of its coefficients will be in  $\mathfrak{F}'$ .

42. There must be a pair of coefficients,  $P(u)$  and  $R(u)$ , in  $C$ , whose ratio is not an element of  $\mathfrak{F}$ . Otherwise, we could secure from  $C$  a d.p. in  $\mathfrak{F}\{y\}$  vanishing for  $y = u$ . Let

$$(74) \quad D = R(u)P(y) - P(u)R(y).$$

We are going to show that  $D$  is the product of  $C$  by an element of  $\mathfrak{F}$ .

Let  $E = D/R(u)$ . We consider  $E$  as a d.p. in  $\mathfrak{F}'\{y\}$ . Then  $E$  vanishes for  $y = u$  and so is in  $\Sigma$ . Hence, if  $S_1$  is the separant of  $B$ , there is a relation

$$(75) \quad S_1^a E \equiv 0, \quad [B].$$

From (75), if we represent the separant of  $C$  for the order  $u, y$  by  $S$ , we secure a relation

$$(76) \quad FS^a D \equiv 0, \quad [C],$$

with  $F$  in  $\mathfrak{F}\{u\}$ . Let

$$(77) \quad C = G_1 \cdots G_p$$

be a resolution of  $C$  into factors which are algebraically irreducible in  $\mathfrak{F}$ . Each  $G$  involves  $y$ . Let  $r$  be the order of  $C$  in  $y$ . We say that each  $G$  is of order  $r$  in  $y$ . Suppose that  $G_1$  is of order  $s < r$  in  $y$ . Then, when  $C$  is arranged as a polynomial in  $y_r$ , each coefficient is divisible by  $G_1$ . Let  $B$  above be arranged as a polynomial in  $y_r$ . Let  $H_1, \dots, H_t$  be the coefficients in  $B$ . The  $H$ , considered

as polynomials in  $y_1, \dots, y_{r-1}$ , are relatively prime. Hence, there is a relation<sup>49</sup>

$$(78) \quad M_1H_1 + \dots + M_rH_r = N$$

where  $N$  and the  $M$  are polynomials in  $y_1, \dots, y_{r-1}$  and where  $N$  is distinct from zero and free of  $y_s$ . We can obtain from (78) a relation which shows that the coefficients of the powers of  $y_r$  in  $C$  are not divisible by a d.p. of order  $s$  in  $y$ .

No two of the  $G$  in (77) have a ratio which is an element of  $\mathfrak{F}$ . Otherwise  $S$  would have a factor in common with  $C$ . As above, we see that this is impossible for the reason that  $S_i$  has no factor in common with  $B$ . By §13, no  $G_i$  holds the general solution of a  $G_j$  with  $j \neq i$ .

We wish to show that  $D$  holds the general solution of each  $G$ . This will follow from (76) if we can show that  $S$  holds no such general solution. For this, we observe first that  $S$  is of order not more than  $r$  in  $y$ . As  $S$  has no factor in common with  $C$ ,  $S$  is not divisible by any  $G$ .

Let  $s$  be the order of  $C$  in  $u$ . By (74) the order  $s'$  of  $D$  in  $u$  does not exceed  $s$ . Let  $G_1, \dots, G_m$  be those  $G$  which are of order  $s$  in  $u$ . As  $s' \leq s$  and as  $D$  holds the general solutions of  $G_1, \dots, G_m$ , it must be that  $s' = s$  and that  $D$  is divisible by each  $G_i$  with  $i \leq m$ .<sup>50</sup> Then let

$$(79) \quad D = KG_1 \dots G_m.$$

The degree of  $G_1 \dots G_m$  in  $u_s$  is that of  $C$ . As, by (74),  $D$  has a degree in  $u_s$  which does not exceed that of  $C$ ,  $K$  is of order less than  $s$  in  $u$ .

Let  $G_{m+1}, \dots, G_{m'}$  be those  $G$  which are of order  $s - 1$  in  $u$ . Their general solutions are held by  $D$  but by no  $G_i$  with  $i \leq m$ . Thus  $K$  holds the general solutions and is divisible by  $G_{m+1} \dots G_{m'}$ .

If  $C$  and  $D$  are arranged as power products in the  $u_i$  and if such power products are ordered as in I, §22, the highest product in  $D$  will not be higher than that in  $C$ . It follows from (77) and (79) that

$$K = LG_{m+1} \dots G_{m'}$$

with  $L$  of order less than  $s - 1$  in  $u$ . Continuing, we find  $D$  to be the product of  $C$  by a d.p.  $M(y)$  in  $\mathfrak{F}\{y\}$ .  $M$  has to be an element of  $\mathfrak{F}$ . Otherwise,  $D$ , by its symmetry, would be divisible by  $M(u)$  and  $M(u)$  would be a factor of  $C$ .<sup>51</sup>

43. Let  $T$  be the d.p. in  $\mathfrak{F}\{y\}$  obtained by dividing  $D$  by  $R(u)$ . Then  $T$  is of type (73) and is the product of  $B$  by an element of  $\mathfrak{F}'$ .

Let  $v = P(u)/R(u)$ . We are going to prove that  $\mathfrak{F}' = \mathfrak{F}\langle v \rangle$ .

Let  $U(u)/V(u)$  be any element of  $\mathfrak{F}'$ . We shall show that  $U(u)/V(u)$  is in  $\mathfrak{F}\langle v \rangle$ . Let

<sup>49</sup> Perron, *Lehrbuch der Algebra*, vol. 1, p. 204.

<sup>50</sup> The remainder of  $D$  with respect to  $G_i$  for the order  $y$ ,  $u$  is zero.

<sup>51</sup> It is easy now to prove that the  $G$  are all of order  $r$  in  $u$ .

$$W = U(y) - \frac{U(u)}{V(u)} V(y).$$

Then  $W$  is in  $\Sigma$ . If  $S_1$  is the separant of  $T$ , there is a relation

$$S_1^a W \equiv 0, \quad [T].$$

This gives

$$XS^a[V(u)U(y) - U(u)V(y)] \equiv 0, \quad [D],$$

with  $X$  in  $\mathfrak{F}\{u\}$  and  $S$  the separant of  $D$ . Let  $u = \tau$ ,  $y = \eta$  be a zero of  $D$ . We suppose that  $X(\tau)R(\tau) \neq 0$ . Let  $\tau, \eta$  annul  $S$ . Then, if

$$Y = P'(y)R(y) - R'(y)P(y),$$

where  $P' = \partial P/\partial y$ , and  $R' = \partial R/\partial y$ ,  $\eta$  annuls  $Y$ . We shall show that  $Y(y)$  is not zero. As  $C$  is not divisible by a d.p. in  $u$  alone,  $D$  is not. Hence  $P$  and  $R$  are relatively prime. If, for instance,  $R'$  is not zero,  $R'$  is not divisible by  $R$ . Thus  $Y(y)$  is not zero. If  $Y(\eta) \neq 0$ ,  $\tau, \eta$  annuls  $V(u)U(y) - U(u)V(y)$ .

We now write  $P(u)/R(u)$  and  $U(u)/V(u)$  with a common denominator  $RVXY$ . Applying the lemma of §40, we find that  $U/V$  is rational in  $P/R$  and its derivatives. This proves the theorem of §39.

44. Let  $w$  be any element of  $\mathfrak{F}'$  such that  $\mathfrak{F}\langle w \rangle = \mathfrak{F}'$ . We seek a relation between  $w$  and the  $v$  found above. The totality of those d.p. in  $\mathfrak{F}\{y, z\}$  which vanish for  $y = v$ ,  $z = w$  is a prime ideal  $\Sigma$  whose manifold is the general solution of a d.p.  $F$  (§33). As  $w$  has an expression rational in  $v$  and its derivatives,  $\Sigma$  contains a d.p. which is of order zero in  $z$  and linear in  $z$ .  $F$  must be such a d.p. Similarly,  $F$  is of zero order in  $y$  and linear in  $y$ . This means that  $w = (\alpha v + \beta)/(\gamma v + \delta)$  where  $\alpha, \beta, \gamma, \delta$  are elements of  $\mathfrak{F}$ .

45. From the theorem of §39, it follows that if  $v$  and  $w$  are elements of  $\mathfrak{F}\langle u \rangle$ , that is, quotients of two d.p. in  $u$ , there exists a quotient  $t$  of two d.p. in  $u$  such that  $v$  and  $w$  are rational in  $t$  and its derivatives while  $t$  is rational in  $v, w$  and their derivatives. This result parallels Lüroth's theorem on unicursal curves.

CHAPTER III  
STRUCTURE OF DIFFERENTIAL POLYNOMIALS

I. Manifold of a Differential Polynomial

THEOREM ON DIMENSION OF COMPONENTS

1. If  $F$  is an algebraically irreducible d.p. in  $\mathfrak{F}\{y_1, \dots, y_n\}$ , the dimension of the general solution of  $F$  is  $n - 1$ . One might inquire as to the dimensions of the other components of  $F$  (II, §3). This question is answered by the following theorem:

**THEOREM:** *Let  $F$  be a differential polynomial<sup>1</sup> of positive class in  $\mathfrak{F}\{y_1, \dots, y_n\}$ . Every component of  $F$  is of dimension  $n - 1$ .*

From II, §33, it follows that *every component of  $F$  is the general solution of a differential polynomial.*

2. Let the essential prime divisors of  $\{F\}$  be  $\Sigma_1, \dots, \Sigma_s$ . We have to show that every  $\Sigma_i$  is of dimension  $n - 1$ . Consider some  $\Sigma_j$  and let  $\eta_1, \dots, \eta_n$  be a generic zero of  $\Sigma_j$ . We shall show that  $\eta_1, \dots, \eta_n$  is a zero of some  $\Sigma_i$  of dimension  $n - 1$ . Any such  $\Sigma_i$  must be contained in  $\Sigma_j$  and must therefore be identical with  $\Sigma_j$ . This will prove that  $\Sigma_j$  is of dimension  $n - 1$ .

3. We use new indeterminates  $z_1, \dots, z_n$ . In  $F$ , we replace each  $y_i$  by  $z_i + \eta_i$ . Then  $F$  goes over into a d.p.  $K$  in  $\mathfrak{F}_0\{z_1, \dots, z_n\}$  where  $\mathfrak{F}_0$  is  $\mathfrak{F}\langle\eta_1, \dots, \eta_n\rangle$ .  $K$  vanishes when each  $z_i$  is replaced by 0.

4. Now let  $W$  be the sum of the terms of lowest degree in  $K$  considered as a polynomial in the  $z_{ij}$ . Let  $V$  be a factor of  $W$ , algebraically irreducible in  $\mathfrak{F}_0$ . Changing subscripts if necessary, we shall assume that  $V$  involves  $z_1$  effectively. Let  $\zeta_1, \dots, \zeta_n$  be a generic point in the general solution of  $V$  and let  $\mathfrak{F}_1$  represent  $\mathfrak{F}_0\langle\zeta_1, \dots, \zeta_n\rangle$ .

ARBITRARY CONSTANTS

5. We shall explain now what is to be meant by the term *arbitrary constant*. At each stage of our work we operate in a definite field; thus far we have met  $\mathfrak{F}, \mathfrak{F}_0, \mathfrak{F}_1$ . A field having been given, we understand by an arbitrary constant with respect to the field, a quantity  $c$  which can be adjoined to the field,<sup>2</sup> which is transcendental with respect to the field (I, §29), and whose derivative is zero.

THE POLYGON PROCESS

6. We are going to show that  $K$  has a zero

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<sup>1</sup> Algebraic irreducibility is not necessary.

<sup>2</sup> That is,  $c$  lies in an extension of the field.

$$(1) \quad \begin{aligned} z_i &= \zeta_i c, & i &= 2, \dots, n, \\ z_1 &= \zeta_1 c + \varphi_2 c^{\rho_2} + \dots + \varphi_i c^{\rho_i} + \dots. \end{aligned}$$

Here the  $\varphi$  are elements of a field  $\mathfrak{F}'$  which contains  $\mathfrak{F}_1$  while  $c$  is an arbitrary constant with respect to  $\mathfrak{F}'$ . The  $\rho$  are rational numbers with a common denominator; they exceed unity and increase with their subscripts.<sup>3</sup>

7. It may be that  $K$  vanishes for  $z_i = \zeta_i c$ ,  $i = 1, \dots, n$ , where  $c$  is an arbitrary constant with respect to  $\mathfrak{F}_1$ . In that case, the  $\zeta_i c$  are suitable expressions (1) with  $\mathfrak{F}' = \mathfrak{F}_1$ .<sup>4</sup> In what follows, we assume that such vanishing does not occur.

We put in  $K$

$$(2) \quad z_i = \zeta_i c, \quad i = 2, \dots, n; \quad z_1 = \zeta_1 c + u_1.$$

Then  $K$  goes over into an expression  $K'$  which is a polynomial in  $c$  and the  $u_i$ . We may write

$$(3) \quad K' = a'(c) + \sum_{i=1}^p b'_i(c) U'_i.$$

Here  $a'(c)$  and the  $b'(c)$  are polynomials in  $c$  with coefficients in  $\mathfrak{F}_1$ ,  $p$  is a positive integer and the  $U'$  are power products, of positive degree, in the  $u_i$ . We know that  $a'(c)$  is not zero. We understand that no  $b'(c)$  is zero.

8. Let  $\sigma'$  be the least exponent of  $c$  in  $a'$  and  $\sigma'_i$  the least exponent of  $c$  in  $b'_i$ . Let  $d_i$  be the total degree of  $U'_i$ . Finally, let

$$(4) \quad \rho_2 = \text{Max} \frac{\sigma' - \sigma'_i}{d_i}.$$

We shall prove that  $\rho_2 > 1$ .

To begin with, if  $d$  is the degree of  $W$  of §4,  $\sigma' > d$  since  $W$  is annulled by the  $\zeta$ . Under (2), the constituent  $W$  of  $K$  contributes to  $K'$  terms which effectively involve one or more  $u_i$ .<sup>5</sup> The total degree of any such term in  $c$  and the  $u_i$  is  $d$ . Thus, for at least one  $i$  in (3), we have  $\sigma'_i + d_i = d$ . As  $\sigma' > d$ , we have then  $\sigma' - \sigma'_i > d_i$ . Thus  $\rho_2 > 1$ .

9. Let  $g'$  be the coefficient of  $c^{\sigma'}$  in  $a'$ . Let  $h'_i$  denote the coefficient of  $c^{\sigma'_i}$ , or denote zero, according as  $(\sigma' - \sigma'_i)/d_i$  equals  $\rho_2$  or is less than  $\rho_2$ . Let

$$(5) \quad L'(u_i) = g' + \sum_{i=1}^p h'_i U'_i.$$

<sup>3</sup> The zero (1) will lie in an extension of  $\mathfrak{F}'$ . How to use formal infinite series, and how the fractional powers of  $c$  are to be regarded, will be obvious.

<sup>4</sup> One sees how to go through the formality of building an extension of  $\mathfrak{F}'$  which contains the  $z_i$  presented.

<sup>5</sup> To see this, it suffices to show that  $W$  does not vanish identically for  $z_2 = \zeta_2, \dots, z_n = \zeta_n$ . Let  $W$  be arranged as a polynomial in the  $z_i$ . The coefficients are d.p. in  $z_2, \dots, z_n$  and thus cannot hold the general solution of  $V$ , which d.p. involves  $z_1$ .

We consider  $L'$  as a d.p. in  $\mathfrak{F}_1\{u_1\}$ . Let  $\{L'\}$  have  $\Omega_1, \dots, \Omega_q$  for essential prime divisors. Each  $\Omega_i$  has a generic zero  $\psi_i$  in an extension (depending on  $i$ ) of  $\mathfrak{F}_1$ . We select one of the  $\psi_i$  in the following manner.

Let  $L'$  be of effective degree  $f$  in the  $u_{1i}$ . Then certain partial derivatives of  $L'$ , of order  $f$ , with respect to the  $u_{1i}$ , are elements of  $\mathfrak{F}_1$  distinct from zero. Of all positive integers  $r$  for which there is a  $\psi_i$  which does not annul every partial derivative of  $L'$  of order  $r$ , let  $f_1$  be the least. We choose a  $\psi_i$  which does not annul every partial derivative of order  $f_1$  and designate it by  $\varphi_2$ . Let  $\mathfrak{F}_2 = \mathfrak{F}_1\langle\varphi_2\rangle$ .

10. From now on we understand that  $c$ , used above, is an arbitrary constant with respect to  $\mathfrak{F}_2$ . It may be that  $\varphi_2 c^{\rho_2}$  causes  $K'$  to vanish when substituted for  $u_1$ . In that case we have suitable expressions (1) with

$$z_1 = \zeta_1 c + \varphi_2 c^{\rho_2}.$$

Let us suppose that the vanishing does not occur.

We make in  $K'$  the substitution

$$(6) \quad u_1 = \varphi_2 c^{\rho_2} + u_2.$$

Then  $K'$  goes over into an expression  $K''$  in  $c$  and  $u_2$  which may be written

$$(7) \quad K'' = a''(c) + \sum b''_i(c) U''_i.$$

Here  $a''$  and the  $b''$  are sums in which each term is the product of a rational power of  $c$  and an element of  $\mathfrak{F}_2$ . We know that  $a'' \neq 0$ , and we assume that no  $b''$  vanishes. The sums  $\sum$  in (3) and in (7) do not necessarily involve the same power products.

Let  $\sigma''$  be the least exponent of  $c$  in  $a''$ ;  $\sigma''_i$  the least exponent in  $b''_i$ ;  $d_i$  the degree of  $U''_i$ . Let

$$(8) \quad \rho_3 = \text{Max} \frac{\sigma'' - \sigma''_i}{d_i}.$$

We are going to prove that  $\rho_3 > \rho_2$ .

Using an indeterminate  $v$ , we replace  $u_1$  in  $K'$  by  $c^{\rho_2}v$ . The  $i$ th term of  $\sum$  in (3) will produce a set of terms, each of the type  $\beta c^{\sigma'} T$ , where  $T$  is a power product in the  $v_i$ ,<sup>6</sup>  $\beta$  an element of  $\mathfrak{F}_1$ , and where

$$(9) \quad q \geq \sigma'_i + \rho_2 d_i.$$

By (4),  $q \geq \sigma'$ . We will have  $q = \sigma'$  only if  $\beta$  is an  $h'$  in (5). On this basis, we may write

$$(10) \quad K'(c^{\rho_2}v) = c^{\sigma'} L'(v) + c^{\tau'} M'(c, v),$$

where, in regard to  $L'$ ,  $M'$ ,  $\tau'$ , the following statements apply.

$L'$  is as in (5) with  $u_1$  replaced by  $v$ .  $M'$  is a polynomial in the  $v_i$  with co-

<sup>6</sup>  $T$  is the same in all terms and is merely  $U''_i$  with  $u_1$  replaced by  $v$ .

efficients which are sums of terms, each the product of a nonnegative rational power of  $c$  by an element of  $\mathfrak{F}_1$ . As to  $\tau'$ , which we understand to be taken as large as possible, it is a rational number greater than  $\sigma'$ .

We now put  $v = \varphi_2 + c^{-\rho_2} u_2$ . Then (10) gives, by (6),

$$(11) \quad K''(u_2) = c^{\sigma'} L'(\varphi_2 + c^{-\rho_2} u_2) + c^{\tau'} M'(c, \varphi_2 + c^{-\rho_2} u_2).$$

Let  $K'$  be of order  $r$  in  $u_1$ . Suppose that, for some set of nonnegative integers  $l_0, \dots, l_r$ ,

$$(12) \quad \frac{\partial^{l_0 + \dots + l_r} L'(u_1)}{\partial^{l_0} u_{10} \dots \partial^{l_r} u_{1r}}$$

does not vanish for  $u_1 = \varphi_2$ . This implies that at least one  $l$  is positive. Let  $Z = u_{20}^{l_0} \dots u_{2r}^{l_r}$ . We shall prove that  $Z$  is present in  $\Sigma$  in (7) and we shall determine the  $\sigma_i''$  associated with that power product.

The coefficient of any  $u_{20}^{l_0} \dots u_{2r}^{l_r}$  in the second member of (11), whether (12) vanishes for it or not, is the quotient by  $l_0! \dots l_r!$  of

$$(13) \quad c^{\sigma' - \rho_2 \lambda} L'_{l_0 \dots l_r}(\varphi_2) + c^{\tau' - \rho_2 \lambda} M'_{l_0 \dots l_r}(c, \varphi_2)$$

where  $\lambda = l_0 + \dots + l_r$ ;  $L'_{l_0 \dots l_r}$  is (12) with  $u_1$  replaced by  $\varphi_2$ ;  $M'_{l_0 \dots l_r}$  is obtained by the same differentiation and substitution from  $M'(c, u_1)$ .

The assumption that (12) does not vanish for  $u_1 = \varphi_2$  implies that  $Z$  is present in (7). The associated  $\sigma_i''$  is given by

$$(14) \quad \sigma_i'' = \sigma' - \rho_2 \lambda = \sigma' - \rho_2 d_i.$$

On the other hand, if  $\varphi_2$  annuls (12) and if  $Z$  is present in (7), we have

$$(15) \quad \sigma_i'' > \sigma' - \rho_2 d_i.$$

We can now study  $\rho_3$  in (8). We have, for every  $i$ ,

$$(16) \quad \frac{\sigma'' - \sigma_i''}{d_i} = \frac{\sigma'' - \sigma'}{d_i} + \frac{\sigma' - \sigma_i''}{d_i}.$$

By (14) and (15) we have

$$(17) \quad \frac{\sigma' - \sigma_i''}{d_i} = \rho_2$$

or

$$(18) \quad \frac{\sigma' - \sigma_i''}{d_i} < \rho_2,$$

according as (12) with suitable  $l$  does not vanish or does vanish. From (17) and (18) we see that  $(\sigma' - \sigma_i'')/d_i$  is a maximum, namely  $\rho_2$ , for those  $i$  for which (12) does not vanish. Such  $i$  exist, as was seen in connection with the stipulation made in regard to  $\varphi_2$  in §9.

From (13) with every  $l$  zero, we see now that  $\sigma'' > \sigma'$ . Turning now to (16), we see that there are  $i$  for which the first member of (16) exceeds  $\rho_2$ .

This proves that  $\rho_3 > \rho_2$ .

11. We now form for  $K''$  a d.p.  $L''$  analogous to (5) and obtain a zero  $\varphi_3$  of  $L''$  in the manner followed for  $\varphi_2$ . In this, we consider  $L''$  as a d.p. in  $\mathfrak{F}_2\{u_2\}$ .

We continue this procedure. It may be that at some stage we reach a  $K^{(k-1)}$  which is annulled by  $\varphi_k c^{\rho_k}$ . In that case

$$(19) \quad z_1 = \zeta_1 c + \varphi_2 c^{\rho_2} + \cdots + \varphi_k c^{\rho_k}$$

is suitable for (1). We suppose in what follows that our procedure does not terminate in a finite number of steps, so that we are led to form an infinite series

$$(20) \quad z_1 = \zeta_1 c + \varphi_2 c^{\rho_2} + \cdots$$

We shall then be working in a field  $\mathfrak{F}'$  which is the union of all  $\mathfrak{F}_i$  and we understand  $c$  to be an arbitrary constant with respect to  $\mathfrak{F}'$ .

We shall prove that the  $\rho_k$  have a common denominator. This will imply that the  $\rho_k$  become infinite with  $k$ . It will be seen also that the  $z_i$  of (1) annull  $K$ .

12. We begin by showing that the degrees of the  $L^{(k)}$  as polynomials in the  $u_{ki}$  do not increase with  $k$ . Let us compare the degree of  $L'$  with that of  $L''$ . Let  $f$ , and  $f_1 \leq f$ , be as in the stipulation of §9 relative to  $\varphi_2$ , with  $f$  the degree of  $L'$ .

In (16),  $(\sigma'' - \sigma')/d_i$  is less for  $d_i > f_1$  than for  $d_i \leq f_1$ . On the other hand,  $(\sigma' - \sigma'_i)/d_i$  obtains its maximum value  $\rho_2$  for some  $d_i$  equal to  $f_1$ . This shows that the first member of (16) cannot be as great as  $\rho_3$  for  $d_i > f_1$ . Referring now to the description of the coefficients in (5), which description is similar to that of the coefficients in  $L''$ , we see that the degree of  $L''$  does not exceed  $f_1$ .

Thus there is a positive integer  $e$  such that, for  $k \geq e$ , the  $L^{(k)}$  are all of the same degree, say  $m$ . Consider any  $k \geq e$ , the corresponding  $L^{(k)}$ , and the partial derivatives of all orders of  $L^{(k)}$  with respect to the  $u_{ki}$ . We shall prove that if  $R$  is any such derivative of order less than  $m$ ,  $R$  is in  $\{L^{(k)}(u_k)\}$ .<sup>7</sup> Let  $\{L^{(k)}\}$  have the essential prime divisors  $\Omega_1, \cdots, \Omega_q$ . If one refers to the stipulation made in regard to the various  $\varphi_i$  (§9), and considers that the degree of  $L^{(k+1)}$  equals that of  $L^{(k)}$ , one sees that  $R$  is annulled by a generic zero of every  $\Omega_i$ . Thus  $R$  is in every  $\Omega_i$  and so in  $\{L^{(k)}\}$ .

Let, now,  $R$  be a partial derivative of  $L^{(k)}$  of order  $m - 1$ , distinct from zero. Then  $R$  is linear in the  $u_{ki}$ . Let  $L^{(k)}$  be decomposed into factors over  $\mathfrak{F}_k$  which are algebraically irreducible. Let  $Z$  be an irreducible factor of the same order in  $u_k$  as  $L^{(k)}$ . Then  $R$  is in  $\{Z\}$  so that the remainder of  $R$  with respect to  $Z$  is zero. As the order of  $R$  in  $u_k$  does not exceed that of  $Z$ ,  $R$  is divisible by  $Z$ . As  $R$  is linear,  $Z$  is the product of  $R$  by an element of  $\mathfrak{F}_k$ . We have thus, for some  $g$ ,

$$(21) \quad L^{(k)} = QR^g$$

where  $Q$ , except perhaps in the case in which it is an element of  $\mathfrak{F}_k$ , has an order which is less than the common order, call it  $h$ , of  $L^{(k)}$  and  $R$  in  $u_k$ . We have

<sup>7</sup> We work in  $\mathfrak{F}_k\{u_k\}$ .

$$\frac{\partial^{\rho} L^{(k)}}{\partial u_k^{\rho}} = \mu Q$$

with  $\mu$  in  $\mathfrak{F}_k$ . As  $\mu Q$  is reduced with respect to  $R$ , it is not in  $\{R\}$  and thus not in  $\{L^{(k)}\}$ . It follows that  $g$  in (21) equals  $m$ , so that

$$(22) \quad L^{(k)} = \lambda R^m$$

with  $\lambda$  in  $\mathfrak{F}_k$ .

In the expression for  $L^{(k)}$  analogous to (5), there is a term like  $g'$  in (5), free of the  $u_{ki}$ . This means that  $R$  has such a term, so that, by (22),  $L^{(k)}$  has terms of the first degree. Thus, in the equation of definition of  $\rho_{k+1}$  analogous to (4), there will be, among those  $i$  which give a maximum, certain  $i$  for which  $d_i = 1$ . In other words, the denominator of  $\rho_{k+1}$  can be taken as the common denominator of  $\sigma^{(k)}$  and the  $\sigma_i^{(k)}$ . For that common denominator, we can use that of  $\rho_2, \dots, \rho_k$ .

This shows that the  $\rho_k$  have a common denominator, so that they approach  $\infty$  with  $k$ .

13. We have to show now that the expressions in (1), obtained as above, annull  $K$ . Because  $a^{(k)}$ , for any  $k > 1$ , is the result of performing in  $K$  the substitutions

$$(23) \quad \begin{aligned} z_i &= \zeta_i c, & i &= 2, \dots, n, \\ z_1 &= \zeta_1 c + \dots + \varphi_k c^{\rho_k}, \end{aligned}$$

it suffices to show that  $\sigma^{(k)}$  approaches  $\infty$  with  $k$ . This is so because the  $\sigma^{(k)}$  increase with  $k$  and have a common denominator.

#### DIMENSIONS OF COMPONENTS

14. We see now that  $F$  of our theorem has a zero

$$(24) \quad \begin{aligned} y_i &= \eta_i + \zeta_i c, & i &= 2, \dots, n, \\ y_1 &= \eta_1 + \zeta_1 c + \varphi_2 c^{\rho_2} + \dots. \end{aligned}$$

Then (24) is a zero of some  $\Sigma_i$  of §2, say of  $\Sigma_k$ . We shall prove that  $\Sigma_k$  is of dimension  $n - 1$ . It will suffice to prove that  $\Sigma_k$  contains no d.p. in  $y_2, \dots, y_n$ . Let  $M$  be such a d.p. in  $\Sigma_k$ . We replace each  $y_i$  in  $M$  by  $z_i + \eta_i$ . Then  $M$  goes over into a d.p.  $N$  in  $\mathfrak{F}_0\{z_2, \dots, z_n\}$  which vanishes for  $z_i = \zeta_i c$ ,  $i = 2, \dots, n$ . Let  $P$  be the sum of the terms of lowest degree in  $N$ . Then  $P$  vanishes for  $z_i = \zeta_i$ ,  $i = 2, \dots, n$ . We have here the contradiction that  $P$ , which is free of  $z_1$ , holds the general solution of  $V$  of §4.

Then  $\Sigma_k$  is of dimension  $n - 1$ . If, in any d.p.  $M$  of  $\Sigma_k$ , we replace each  $y_i$  by its expression in (24), the term free of  $c$  which is obtained is the result of replacing each  $y_i$  in  $M$  by  $\eta_i$ . Thus  $\eta_1, \dots, \eta_n$  is a zero of  $\Sigma_k$ . This, as was seen in §2, implies the truth of our theorem.

DEGREES OF GENERALITY

15. Suppose that  $F$  is algebraically irreducible, and let  $\Sigma_1$ , in §2, be the prime ideal associated with the general solution of  $F$ . Consider any  $\Sigma_i$  with  $i > 1$ . The manifold of  $\Sigma_i$  is the general solution of a d.p.  $A$ . We say that if  $y_k$  is an indeterminate effectively present in  $A$ , the order of  $F$  in  $y_k$  exceeds that of  $A$ . This follows from the fact that  $F$  is in  $\Sigma_i$ , so that, if  $F$  were not of higher order in  $y_k$  than  $A$ ,  $F$  would be divisible by  $A$  and  $\Sigma_i$  would be identical with  $\Sigma_1$ .

II. Low Powers and Singular Solutions

COMPONENTS

16. Let  $F$  be as in §15. We know that the components which  $F$  may have in addition to its general solution are general solutions of d.p.  $A_1, \dots, A_p$ . There arises the problem of determining the  $A$ . More than this, one will desire to know whether the  $A$  are visible in some way in the structure of  $F$ .

There will be developed, in Chapter V, a method for determining a finite set of algebraically irreducible d.p. whose general solutions make up the manifold of  $F$ . However, not all of the general solutions there found need be components of  $F$ ; it may be that some of them are contained in others of them. The problem of selecting the components is identical with that of determining the influence of the components on the structure of  $F$ . It is best formulated as follows, without requiring the algebraic irreducibility of  $F$ . Let  $F$  and  $A$  be d.p. in  $\mathfrak{F} \{ y_1, \dots, y_n \}$  with  $A$  algebraically irreducible. Let  $F$  hold the general solution of  $A$ , that is, let the remainder of  $F$  with respect to  $A$  be zero. It is required to determine whether the general solution of  $A$  is a component of  $F$ . The solution of this problem is contained in the low power theorem presented below.

PREPARATION PROCESS

17. Let  $F$  and  $A$  be any two d.p. of class  $n$ .<sup>8</sup> Let the orders of  $F$  and  $A$  in  $y_n$  be  $m$  and  $l$  respectively. Let  $A_j$  represent the  $j$ th derivative of  $A$ , and  $S$  the separant of  $A$ . We shall show the existence of a nonnegative integer  $t$  and of a positive integer  $r$  such that  $S^r F$  has a representation

$$(25) \quad \sum_{j=1}^r C_j A^{p_j} A_1^{i_{1j}} A_2^{i_{2j}} \dots A_{m-l}^{i_{m-l,j}}$$

with nonnegative  $p_j$  and  $i_{kj}$ , where no two of the  $r$  sets  $i_{1j}, \dots, i_{m-l,j}$  are identical, the  $C_j$  being of orders not exceeding  $l$  in  $y_n$ , and not divisible by  $A$ .

If  $m \leq l$ , we express  $F$  in the form  $CA^p$  with  $C$  not divisible by  $A$  and we understand this expression of  $F$  to be that which is indicated in (25). In what follows, we assume that  $m > l$ .

We let  $z$  represent  $y_n$  and we start with the case of  $m = l + 1$ . Let  $F$  be of degree  $a$  in  $z_{l+1}$ . Then  $S^a F$  can be written as a polynomial in  $Sz_{l+1}$  with coefficients whose orders in  $z$  do not exceed  $l$ . Now

<sup>8</sup> We are not assuming algebraic irreducibility for  $A$ .

$$A_1 = Sz_{l+1} + T$$

with the order of  $T$  in  $z$  at most  $l$ . Thus  $S^a F$  can be written as a polynomial in  $A_1 - T$ , and hence as a polynomial in  $A_1$ , with coefficients whose orders in  $z$  are at most  $l$ . If we write each coefficient in the form  $CA^p$ ,  $p \geq 0$ , with  $C$  not divisible by  $A$ , we have a representation (25) for  $S^a F$ .

Suppose now that (25) can be produced for  $m < s$  where  $s > l + 1$ . We make an induction to  $m = s$ . Let  $F$ , of order  $s$  in  $z$ , be of degree  $a$  in  $z_s$ . We see as above that  $S^a F$  can be written as a polynomial in  $A_{s-l}$  with coefficients whose orders in  $z$  are less than  $s$ . For a sufficiently large positive integer  $b$ , the product of any of these coefficients by  $S^b$  will have a representation (25). Thus  $S^{b+a} F$  has a representation (25).

18. We shall show now that, for any admissible  $t$ , (25) is unique. Let  $S^t F$  have two distinct representations (25). By a subtraction, we get a relation

$$(26) \quad 0 = \sum_{j=1}^v D_j A_1^{t_j} \cdots A_{m-t}^{t_{m-t}'}$$

where the  $v$  sets of exponents are distinct and where the  $D$ , distinct from zero and of order no more than  $l$  in  $z$ , may be divisible by  $A$ .

We have

$$A_{m-t} = Sz_m + T$$

with  $T$  of order less than  $m$  in  $z$ . In (26), let us replace  $z_m$  by  $(u - T)/S$  where  $u$  is an indeterminate in the customary sense of algebra. Then  $A_{m-t}$  is replaced by  $u$  in (26). Continuing, we see that (26) holds if the  $A_j$  are considered as algebraic indeterminates. This contradicts the fact that the  $D$  are not zero.

19. Suppose now that  $A$  is algebraically irreducible. We see, because  $S$  is not divisible by  $A$ , that for two distinct values  $t_1$  and  $t_2$  of  $t$ , with  $t_2 > t_1$ , (25) is the same except that the  $C$  for  $t_2$  are those for  $t_1$  multiplied by  $S^{t_2-t_1}$ .

By taking  $t$  as small as possible, we are led to a unique expression (25). In all that follows, it will be understood that the smallest admissible  $t$  is used.

When  $A$  and  $F$  are both algebraically irreducible, the smallest  $t$  can be found as follows. If  $S$  is an element of  $\mathfrak{F}$ , we take  $t = 0$ . Otherwise, we first secure (25) with any admissible  $t$  and then determine the highest power  $S^q$  of  $S$  which is a factor of every  $C$ . As  $F$  is algebraically irreducible,  $S^t$  must be divisible by  $S^q$ . A division by  $S^q$  will thus give the unique representation sought.

#### THE LOW POWER THEOREM<sup>9</sup>

20. Let  $F$  and  $A$  be of class  $n$ , of the respective orders  $m$  and  $l$  in  $y_n$ , with  $A$  algebraically irreducible. Let  $F$  hold the general solution of  $A$ . Then there is no term in (25) which is free of  $A$  and the  $A_i$ . Otherwise some  $C$  would hold

<sup>9</sup> First proved by the author in paper 31. The analytic sufficiency proof there given is reproduced in Chapter VI. The algebraic sufficiency proof, to be given now, is due to Levi, 17.

the general solution of  $A$ . This is impossible since the  $C$  are of order at most  $l$  in  $y_n$ , and not divisible by  $A$ . We can now state the

**LOW POWER THEOREM:** *For the general solution of  $A$  to be a component of  $F$ , it is necessary and sufficient that (25) contain a term  $C_k A^{p_k}$ , free of proper derivatives of  $A$ , which, if (25) is considered as a polynomial in  $A, A_1, \dots, A_{m-1}$ , is of lower degree than every other term of (25).<sup>10</sup>*

The assumption that  $F$  and  $A$  are of class  $n$  is made only for convenience. Any indeterminate present in  $A$  is present in  $F$  and may be used as  $y_n$ .

The low power theorem is very easily remembered for the case of a single indeterminate  $y$ , with  $A = y$ . It then becomes: *Let  $F$ , in  $\mathfrak{F}\{y\}$ , vanish for  $y = 0$ . For  $y = 0$  to be a component of  $F$ , it is necessary and sufficient that  $F$ , considered as a polynomial in  $y$  and its derivatives, contain a term in  $y$  alone, that is, a term free of derivatives of  $y$ , which is of lower degree than every other term of  $F$ .*

Thus  $y = 0$  is a component of  $y_1 y_2 - y$ , but not of  $y y_3 - y_2$  or of  $y_2 y_3 - y^2$ .

One of the ideas in the sufficiency proof can be seen in a simple example. In  $\mathfrak{F}\{y\}$ , let  $F = y + y_1 y_2, A = y$ . We have

$$y + y_1 y_2 \equiv 0, \quad [F].$$

Differentiating, we find

$$y_1 + y_1 y_3 + y_2^2 \equiv 0, \quad [F],$$

$$y_2 + y_1 y_4 + 3y_2 y_3 \equiv 0, \quad [F].$$

The three congruences may be written

$$(1 + B_{10})y + B_{11} \quad y_1 + B_{12} \quad y_2 \equiv 0, \quad [F],$$

$$B_{20}y + (1 + B_{21}) \quad y_1 + B_{22} \quad y_2 \equiv 0, \quad [F],$$

$$B_{30}y + B_{31} \quad y_1 + (1 + B_{32}) \quad y_2 \equiv 0, \quad [F],$$

where the  $B$  vanish for  $y = 0$ . The determinant  $D$  of the coefficients of  $y, y_1, y_2$ , in the congruences just written, contains unity as a term and is thus not zero. If we solve for  $y$ , we find that

$$yD \equiv 0, \quad [F].$$

Then  $yD$  holds  $F$ . Thus  $D$  holds every component of  $F$  which  $y$  does not. As  $D$  does not vanish for  $y = 0$ , the manifold of  $y$  is not part of any larger irreducible manifold held by  $F$ . This makes  $y = 0$  a component of  $F$ .

The above method can be applied to any d.p.  $F$  of the type  $y + C$  where the terms of  $C$  are of degree at least 2. The  $p$ th derivative of  $F$  contains  $y_p$ . Now, as is easy to see from a consideration of weights, when  $p$  is large each term in the  $p$ th derivative of  $C$  involves a  $y_i$  with  $i < p$ . This leads to a system of congruences of the type met above.

<sup>10</sup> That is, for  $j \neq k, p_k < p_j + i_{1j} + \dots + i_{m-1,j}$ . If  $m = l$ , so that (25) has just one term, and that of the type  $C_k A^{p_k}$ , the condition will be regarded as fulfilled.

For  $F$  of the type  $y^p + C$  with  $p > 1$ , further elements of proof are necessary. These are provided by Levi's theory of power products, considered in Chapter I.

We shall now treat the general case.

#### SUFFICIENCY PROOF

21. Using indeterminates  $w, z, u_1, \dots, u_\sigma$  and the field of rational numbers, we prove the following lemma.

LEMMA: *Let*

$$(27) \quad C = wz^p - \sum_{j=1}^{\sigma} u_j B_j$$

where  $p$  is a positive integer and the  $B$  are power products, of degree  $p + 1$ , in  $z$  and its derivatives. There exists a relation

$$(28) \quad z^d(w^s + D) \equiv 0, \quad [C],$$

with  $d$  and  $s$  positive integers and with each power product in  $D$  of positive degree in the  $z_j$  and of degree  $s$  in the  $w_j, u_{ij}$ .

Let  $r$  be the maximum of the weights of the  $B$ . If  $r = 0$ , each  $B$  is  $z^{p+1}$  and we have immediately a relation (28) with  $d = p$  and  $s = 1$ .<sup>11</sup>

We suppose now that  $r > 0$  and refer to I, §21. Let

$$d = r(p - 1) + 1, \quad t = d(r - 1).$$

We say that every power product in the  $z_j$  of degree  $d$  and weight not more than  $t$  is contained in  $[z^p]$ . If  $p = 1$ , this is a trivial statement. Let  $p > 1$ . In (27) of I, §21, we have, for  $d$  as above,  $a = r, b = 1$ . Then

$$f(p, d) = t + r + 1$$

and the truth of our statement follows.

Let  $E_1, \dots, E_\mu$  be the power products of degree  $d$  and of weight not more than  $t$ . Let  $p_j$  be the weight of  $E_j$ . Let  $G$  represent  $z^p$ . Consider the representation of an  $E_j$  as a linear combination of the derivatives  $G_i$  of  $G$ . Each  $G_i$  is homogeneous, of degree  $p$ , and isobaric, of weight  $i$ . On this basis, we cast out, from the representation of  $E_j$ , all  $G_i$  with  $i > p_j$ ; from the coefficient of a  $G_i$  with  $i \leq p_j$ , we cast out all terms which are not of degree  $d - p$  and of weight  $p_j - i$ . Thus we write, for  $j = 1, \dots, \mu$ ,

$$(29) \quad E_j = \sum_{k=0}^{p_j} H_{jk} G_k$$

where  $H_{jk}$  is either zero or else homogeneous, of degree  $d - p$ , and isobaric, of weight  $p_j - k$ .

By (27),

<sup>11</sup> The case of  $r = 1$  is also trivial.

$$(30) \quad wG \equiv \sum_{j=1}^{\sigma} u_j B_j, \quad [C].$$

Representing by  $(wG)_k$  the  $k$ th derivative of  $wG$ , we have by (30), for  $k = 0, 1, \dots, t$ ,

$$(31) \quad (wG)_k \equiv K_k, \quad [C],$$

where each term in  $K_k$  is of the first degree in the  $u_j$  and of degree  $p + 1$  in the  $z_j$ ; the  $p + 1$  letters  $z_j$  in each term have a total weight no more than  $r + k$ .

By I, §10, we have, for any  $k$ ,

$$(32) \quad w^{k+1}G_k = \sum_{i=0}^k L_{ki}(wG)_i.$$

We may suppose each  $L_{ki}$  to be a d.p. in  $w$  alone, which is homogeneous, of degree  $k$ , and isobaric, of weight  $k - i$ . By (29) and (32), we have for  $j = 1, \dots, \mu$ ,

$$(33) \quad w^{t+1}E_j = \sum_{i=0}^{p_i} M_{ji}(wG)_i.$$

An  $M_{ji}$  which is not zero is homogeneous, of degree  $t$ , in the  $w_k$  and homogeneous, of degree  $d - p$ , in the  $z_k$ ; it is isobaric, of weight  $p_j - i$ , in all of its letters. By (31) and (33),

$$(34) \quad w^{t+1}E_j \equiv \sum_{i=0}^{p_i} M_{ji}K_i, \quad [C].$$

If  $N$  is a term of some  $K_i$ , the total weight of the  $p + 1$  letters  $z_k$  in  $N$  is, as has been noted, no more than  $r + i$ .

We shall now write (34), with prompt explanations, in the form

$$(35) \quad w^{t+1}E_j \equiv \sum c_\nu P_\nu, \quad [C].$$

The sum in (35) depends on  $j$ . Each  $c_\nu$  is a rational number. Each  $P_\nu$  is a power product, which is of the first degree in the  $u_k$ , of degree  $t$  in the  $w_k$  and of degree  $d + 1$  in the  $z_k$ . The total weight of  $P_\nu$  in the  $w_k$  and  $z_k$  is, for some  $i \leq p_j$ , not more than

$$(p_j - i) + (r + i) = p_j + r \leq t + r.$$

Certainly then, the total weight of the  $d + 1$  letters  $z_k$  in  $P_\nu$  is no more than  $t + r$ .

Working with some  $P_\nu$ , let  $Q$  be the product of the  $d + 1$  letters  $z_k$  in  $P_\nu$ . Let  $z_q$  be the highest derivative of  $z$  in  $Q$  and let  $Q = z_q R$ . The weight of  $R$  cannot exceed  $t$ . Otherwise, as  $t = d(r - 1)$ , some derivative in  $R$  would be of order at least  $r$ . We would have  $q \geq r$  and the weight of  $Q$  would exceed  $t + r$ . Then  $R$  is one of the  $E_i$ .

We may now write<sup>7</sup>(35)

$$(36) \quad w^{t+1}E_j \equiv \sum_{i=1}^{\mu} T_{ji}E_i, \quad [C],$$

with each nonzero  $T$  homogeneous, of the first degree, in the  $u_{ik}$ ; homogeneous, of degree  $t$ , in the  $w_k$ ; homogeneous, of the first degree, in the  $z_k$ . We write (36)

$$(37) \quad \begin{aligned} &(T_{11} - w^{t+1})E_1 + T_{12}E_2 + \cdots + T_{1\mu}E_\mu \equiv 0, \quad [C], \\ &\dots\dots\dots \\ &T_{\mu 1}E_1 + T_{\mu 2}E_2 + \cdots + (T_{\mu\mu} - w^{t+1})E_\mu \equiv 0, \quad [C]. \end{aligned}$$

If  $U$  is the determinant of the system (37),

$$UE_j \equiv 0, \quad [C], \quad j = 1, \dots, \mu.$$

We have

$$U = (-1)^\mu w^{\mu(t+1)} + V$$

where each power product in  $V$  is of positive degree in the  $z_j$  and of degree  $\mu(t+1)$  in the  $w_j$  and  $u_{ij}$ . Observing that  $z^d$  is an  $E_j$ , we have (28) with  $s = \mu(t+1)$  and  $D = (-1)^\mu V$ .

22. We now prove a theorem which gives a result somewhat stronger than the sufficiency of the condition in the low power theorem.

**THEOREM:** *If (25) contains a term  $C_k A^{p_k}$  which is the only term in (25) of degree as low as  $p_k$ , every component of  $A$  which is held by  $F$ , but not by  $C_k$ , is a component of  $F$ .*

Let  $\mathfrak{M}$  be a component of  $A$ . If  $m = l$ , (25) becomes  $F = C_k A^{p_k}$  and the theorem holds. Let  $m > l$ . We compare (25) and (27). We let  $A$  in (25) correspond to  $z$  in (27),  $C_k$  to  $w$  and  $p_k$  to  $p$ . In a term of (25) other than  $C_k A^{p_k}$ , we take a power product of degree  $p+1$  in  $A$  and the  $A_i$  and make it a  $B$  as in (27). Corresponding to (28), there is a relation

$$(38) \quad A^d(C_k^s + E) \equiv 0, \quad [S^t F],$$

where  $E$  holds  $A$ .

Suppose now that  $\mathfrak{M}$  is not a component of  $F$  but rather a proper part of a component  $\mathfrak{M}'$  of  $F$ . Then  $A$  does not hold  $\mathfrak{M}'$  so that, by (38),  $C_k^s + E$  holds  $\mathfrak{M}'$ . As  $E$  holds  $A$ ,  $C_k$  holds  $\mathfrak{M}$  and the theorem is proved.

We note that  $C_k$  does not hold the general solution of  $A$ . The question of sufficiency, in the low power theorem, is settled.

23. **THEOREM:** *If (25) contains a term  $C_k A^{p_k}$  which is the only term in (25) of degree as low as  $p_k$ , every zero of  $A$  which is contained in a component of  $F$  which is not held by  $A$  is a zero of  $C_k$ .*

Let the zero  $\eta_1, \dots, \eta_n$  of  $A$  be contained in a component  $\mathfrak{M}$  of  $F$  which is not held by  $A$ . By (38),  $C_k^s + E$  holds  $\mathfrak{M}$ . As  $E$  holds  $A$ ,  $C_k$  is annulled by  $\eta_1, \dots, \eta_n$ .

NECESSITY PROOF

24. Let  $\mathfrak{M}$  be the general solution of  $A$ . The set  $y_1, \dots, y_{n-1}$  is parametric for  $\mathfrak{M}$  and the order of  $\mathfrak{M}$  with respect to  $y_1, \dots, y_{n-1}$  is  $l$ . We shall prove the following theorem, which will settle the question of necessity in the low power theorem.

**THEOREM:** *Let the terms of lowest degree in (25) involve proper derivatives of  $A$  and let  $A_h$  be the highest derivative of  $A$  which appears in the terms of lowest degree. Then  $\mathfrak{M}$  is not a component of  $F$  and  $\mathfrak{M}$  is contained in a component  $\mathfrak{M}_1$  of  $F$  whose order with respect to  $y_1, \dots, y_{n-1}$  is at least  $l + h$ .<sup>12</sup>*

We shall replace  $y_n$  in (25) by  $y_n + u_0$ , where  $u_0$  is an indeterminate, and examine the resulting d.p. in  $u_0$  and the  $y$ . Such a replacement, made in any d.p.  $B$  in the  $y$ , of order  $s$  in  $y_n$  will convert  $B$  into

$$(39) \quad B + B_0 u_{00} + \dots + B_s u_{0s} + \text{terms of higher degree in the } u_{0i},$$

where  $B_i$  is the partial derivative of  $B$  with respect to  $y_{ni}$ .

For  $B = A$ , (39) will contain the term  $Su_{0l}$  and for  $B = A_i$ , (39) will contain  $Su_{0, l+i}$ .

Let  $\eta_1, \dots, \eta_n$  be a generic point of  $\mathfrak{M}$ . In (25), we make the substitution

$$(40) \quad y_i = \eta_i, \quad i = 1, \dots, n - 1; \quad y_n = \eta_n + u_0.$$

Then  $S'F$ , as in (25), goes over into a d.p.  $K$  in  $\mathfrak{F}_0 \{u_0\}$ , where  $\mathfrak{F}_0 = \mathfrak{F} \langle \eta_1, \dots, \eta_n \rangle$ .

Each  $C$  in (25) will produce, under (40), a nonzero term free of the  $u_{0j}$ . As each  $A_i$ ,  $i = 0, \dots, m - l$ , vanishes for the  $\eta$ , while  $S$  does not, the terms of lowest degree in the  $u_{0j}$  produced by  $A_i$  will be of the first degree and will involve  $u_{0, l+i}$ .

From the terms of lowest degree in (25), we select those which are of a highest degree in  $A_h$ . From the terms just taken, we select those which are of a highest degree in  $A_{h-1}$ . We continue through  $A_1$ . Our process isolates a single term of (25)

$$T = C_j A^{p_j} A_1^{h_j} \dots A_h^{h_j}.$$

Under (40),  $T$  produces a term in  $u_{0l}^{p_j} \dots u_{0, l+n}^{h_j}$  which is not cancelled. Thus the sum  $W$  of the terms of lowest degree in  $K$ , which sum is of positive degree, will be of order  $l + h$  in  $u_0$ .

We are going to find for  $K$  a zero

$$(41) \quad u_0 = \zeta c + \varphi_2 c^{p_2} + \dots + \varphi_k c^{p_k} + \dots$$

of the type exhibited in (1).

Let  $V$  be a factor of  $W$ , irreducible in  $\mathfrak{F}_0$ , which is of order  $l + h$  in  $u_0$ . Let  $\zeta$  be a generic point in the general solution of  $V$ . It may be that  $K$  vanishes for

<sup>12</sup> Note that  $y_1, \dots, y_{n-1}$  is parametric for  $\mathfrak{M}_1$  if  $\mathfrak{M}_1$  contains  $\mathfrak{M}$ .

$u_0 = \zeta c$ . If so,  $\zeta c$  is a suitable series (41). In what follows, we assume that the vanishing does not occur.

We make, in  $K$ , the substitution  $u_0 = c\zeta + u_1$ . Then  $K$  goes over into an expression  $K'$ , a polynomial in  $c$  and the  $u_{1i}$ , which may be written as in (3). The lowest exponent of  $c$  in  $a'$  exceeds the degree of  $W$ .  $W$  contributes, to the sum in (3), terms, free of  $c$ , whose degree in the  $u_{1j}$  is the degree of  $W$ . This justifies us in imagining that it is the present  $K'$  which is being used in §§7-13. We secure thus the zero (41) of  $K$ .

25. We have thus a zero of  $S'F$

$$(42) \quad \begin{aligned} y_i &= \eta_i, & i &= 1, \dots, n-1, \\ y_n &= \eta_n + \zeta c + \varphi_2 c^{\rho_2} + \dots \end{aligned}$$

As the  $\eta$  do not annul  $S$ , (42) gives a zero of  $F$ .

Let the components of  $F$  be general solutions of d.p.  $B_1, \dots, B_s$ . Then (42) is in the general solution of some  $B_i$ . To fix our ideas, we suppose that the general solution  $\mathfrak{M}_1$  of  $B_1$  contains (42). Let  $D$  be any d.p. which holds  $\mathfrak{M}_1$ . Then  $D$  must vanish for the  $\eta$ , else it could not vanish for (42). Then  $D$  holds  $\mathfrak{M}$ . Thus  $\mathfrak{M}_1$  contains  $\mathfrak{M}$ .

Under (40), let  $B_1$  go over into a d.p.  $E$  in  $u_0$ . Let  $U$  be the sum of the terms of lowest degree in  $E$ . Then  $U$  is annulled by  $\zeta$ . Hence the order of  $U$  in  $u_0$  is at least that of  $V$ , namely  $l+h$ . Thus  $B_1$  is of order at least  $l+h$  in  $y_n$  so that the order of  $\mathfrak{M}_1$  with respect to  $y_1, \dots, y_{n-1}$  is at least  $l+h$ . This proves our theorem, and, with it, the necessity of the condition in the low power theorem.

#### AN EXAMPLE

26. We consider, in  $\mathfrak{F}\{y\}$ , the d.p.

$$F = B^2 + \prod_{j=1}^m (y_1 - y + jy^2)$$

where  $B = yy_2 + yy_1 - 2y_1^2$  and  $m$  is any positive integer.

We show first that  $F$  is algebraically irreducible. Suppose that  $F$  has a factor  $G$  free of  $y_2$ . Then  $G$  is a factor of  $y^2$ , the coefficient of  $y_2$ . As  $F$  is not divisible by  $y$ , there is no factor free of  $y_2$ . As the equation  $F = 0$  defines  $y_2$  as a function of two branches of  $y$  and  $y_1$ , there are no factors of the first degree in  $y_2$ .

Thus the manifold of  $F$  consists of the general solution,  $\mathfrak{M}$ , and perhaps, of components held by  $S$ , the separant of  $F$ . As  $S = 2yB$ , and as  $B$  holds  $y$ ,  $B$  holds the components other than  $\mathfrak{M}$ . Thus every zero of  $F$  not in  $\mathfrak{M}$  must annul one of the d.p.

$$A_j = y_1 - y + jy^2, \quad j = 1, \dots, m.$$

We have, for each  $j$ , with  $A_j'$  the derivative of  $A$ ,

$$B = yA_j - 2y_1A_j.$$

The low power theorem shows us immediately that the manifold of each  $A_j$  is a component of  $F$ .

FURTHER THEOREMS ON LOW POWERS

27. Levi obtained a very broad theorem, dealing with systems of d.p., which is essentially a generalization of the low power theorem, at least as far as the question of sufficiency in that theorem is concerned. We consider a special case, which involves a single d.p.

Let

$$F = y_1^{p_1} y_2^{p_2} \cdots y_n^{p_n} + D$$

where the  $p$  are nonnegative integers whose sum is positive and where  $D$  is a d.p. of the following description:

- (a) Each of its terms has a degree in the  $y_{ik}$  which exceeds  $p_1 + \cdots + p_n$ .
- (b) Given any of its terms,  $E$ , and any  $y_j$ ,  $E$  is either divisible by  $y_j^{p_j}$  or else of degree higher than  $p_j$  in the  $y_{jk}$ .

It is easy to see, and, in fact, it will be explicitly shown in the course of our work, that, if  $p_i > 0$ , the manifold of  $y_i$  is a component of  $F$ .

We shall prove that *the zero  $y_i = 0, i = 1, \cdots, n$ , of  $F$  is not contained in any component of  $F$  which is not the manifold of some  $y_i$  with  $p_i > 0$ .*

We treat first the case in which only one of the  $p$ , say  $p_n$ , is positive. We collect those terms of  $F$  which are not of degree higher than  $p_n$  in the  $y_{nk}$ ; they are all divisible by  $y_n^{p_n}$ . We write

$$(43) \quad F = G y_n^{p_n} + H$$

where each term of  $H$  is of degree greater than  $p_n$  in the  $y_{nk}$ . We have  $G = 1 + K$  where  $K$  is free of  $y_n$  and vanishes for  $y_i = 0, i = 1, \cdots, n$ .

We can now apply the low power theorem, taking  $A$  as  $y_n$ . The manifold of  $y_n$  is a component of  $F$ . The zero  $y_i = 0, i = 1, \cdots, n$ , of  $y_n$  does not annul  $G$ . By §23, it cannot lie in any component of  $F$  other than the manifold of  $y_n$ .

Suppose now that the proof has been carried through for the case in which no more than  $r$  of the  $p$  are positive, where  $r < n$ . We make an induction to the case in which  $r + 1$  of the  $p$  are positive.

Let  $p_n$  be positive. We use (43).  $H$  satisfies (a) and (b) and each of its terms is of degree greater than  $p_n$  in the  $y_{nk}$ . We have

$$G = y_1^{p_1} \cdots y_{n-1}^{p_{n-1}} + K$$

where  $K$  satisfies the following two conditions:

- (c) Each of its terms is of degree higher than  $p_1 + \cdots + p_{n-1}$  in the  $y_{ik}, i = 1, \cdots, n$ .
- (d) Given any of its terms,  $E$ , and any  $y_j$  with  $j < n$ ,  $E$  is either divisible by  $y_j^{p_j}$  or else of higher degree than  $p_j$  in the  $y_{jk}$ .

We write (43)

$$F = Gy_n^{p_n} + L_1M_1 + \cdots + L_rM_r$$

where the  $M$  are power products of degree  $p + 1$  in the  $y_{nk}$  and where the  $L$ , like  $K$ , satisfy (d) above.

By §21, there exists a relation

$$y_n^d(G^s + N) \equiv 0, \quad [F],$$

where  $N$  is a homogeneous polynomial, of degree  $s$ , in derivatives, proper or improper, of  $G$  and the  $L$ , with coefficients which vanish when  $y_n = 0$ .

Suppose now that the zero  $y_i = 0, i = 1, \dots, n$ , lies in a component  $\mathfrak{M}$  of  $F$  other than the manifold of  $y_n$ . Then  $G^s + N$  holds  $\mathfrak{M}$ . Now  $G^s + N$  is of the form

$$y_1^{sp_1} \cdots y_{n-1}^{sp_{n-1}} + P,$$

where  $P$  satisfies (a) and (b) above if, in those statements,  $p_n$  is taken as zero and each  $p_i$  with  $i < n$  is replaced by  $sp_i$ . By the earlier cases,  $\mathfrak{M}$  is held by some  $y_i$  with  $i < n$  and is thus the manifold of such a  $y_i$ . The result is established.

28. In  $\mathfrak{F}\{y\}$ , let

$$F = y^p y_1^q + D,$$

where  $p > 0, q > 0$ , and each term of  $D$  is of degree greater than  $q$  in proper derivatives. The manifold of  $y_1$  is a component of  $F$ . The only point which this manifold can have in common with other components is  $y = 0$ .

Suppose now that *each term of  $D$  is of degree greater than  $p + q$  in  $y$  and proper derivatives*. We shall show that *the only component of  $F$  which contains  $y = 0$  is the manifold of  $y_1$* .<sup>13</sup>

We find readily that

$$y_1^d(y^{ps} + N) \equiv 0, \quad [F],$$

where each term of  $N$  is of degree greater than  $ps$ .

If  $y = 0$  were in a second component,  $\mathfrak{M}$ , of  $F$ ,  $\mathfrak{M}$  would be held by  $y^{ps} + N$ . That d.p. has  $y = 0$  as a component.

29. In  $\mathfrak{F}\{y\}$ , let

$$(44) \quad F = y^p + D$$

with  $p$  positive and less than the degree of any term of  $D$ . There exists a relation

$$(45) \quad y^d(1 + N) \equiv 0, \quad [F],$$

where  $N$  vanishes for  $y = 0$ . We are interested in the least value of  $d$  for which it is possible to have a relation (45).

It is easy to see that  $d$  cannot be less than  $p$ . If  $F$  is of positive order, the

<sup>13</sup> Levi, 17, where a more general result is secured.

work of §21 gives  $r(p - 1) + 1$ , with  $r$  the greatest of the weights of the terms of  $D$ , as an employable value of  $d$ . If  $p = 1$ , we can thus take  $d$  as unity. It is not possible to take  $d$  as  $p$  for every  $p$ . For instance, let  $F = y^3 + y_1^4$ . Suppose that we have a relation

$$(46) \quad y^3(1 + N) = MF + M_1F' + \cdots + M_sF^{(s)}.$$

For the second member of (46) to have  $y^3$  as one of its terms, it is necessary for  $M$  to have unity as a term. Then  $MF$  has  $y_1^4$  as a term. Equating terms of degree 4 and weight 4 for both sides of (46), we find  $y_1^4 \equiv 0$ ,  $[y^3]$ , which is easily shown to be false.

We now let  $A$  represent  $y^p$  and  $A_j$  the  $j$ th derivative of  $A$ . Suppose that, for some  $m > 0$ ,

$$(47) \quad F = A + \sum_{i=0}^m M_i A_i,$$

where each  $M$  vanishes for  $y = 0$ . We are going to show that  $d$  may be taken as  $p$ .<sup>14</sup>

We assume, as we may, that no  $M$  is zero. We may write, on the basis of (47) with a suitable range for  $i$  and  $j$ ,

$$(48) \quad A \equiv \sum_{i,j} C_{ij} y_i A_j, \quad [F],$$

in which we understand that no  $C$  is zero. If, in the second member of (48), each  $A_j$  is replaced by the  $j$ th derivative of the second member, there results a congruence

$$(49) \quad A \equiv \sum_{i,j,k} C_{ijk} y_i y_j A_k, \quad [F].$$

For each  $C_{ij}$  in (48), we consider  $i + j$ . Let  $r$  be the maximum of these sums. Then, in (49), no  $i + j + k$  can exceed  $2r$ . If the substitution just made is carried out  $s - 1$  times, we find a congruence

$$A \equiv \sum C y_{i_1} y_{i_2} \cdots y_{i_s} A_{i_{s+1}}, \quad [F],$$

$C$  depending on the  $i$ . No sum  $i_1 + \cdots + i_{s+1}$  can exceed  $sr$ . By §21, if  $s = (r + 1)(p - 1) + 1$ , every  $y_{i_1} \cdots y_{i_s}$  will be in  $[A]$ . We have thus a congruence

$$(50) \quad A - \sum D_{ij} A_i A_j \equiv 0, \quad [F].$$

Let  $L$  represent the first member of (50). We know from what precedes that there is a relation

$$A(1 + N) \equiv 0, \quad [L],$$

with  $N \equiv 0$ ,  $[A]$ . Q.E.D.

<sup>14</sup> Levi, 17.

## TERMS OF LOWEST DEGREE

30. We prove the following theorem.

**THEOREM:** Let  $A$  and  $B$  be nonzero d.p. in  $y_1, \dots, y_n$ . Let  $B$  hold  $A$ . Let  $A_1$  be the sum of the terms of lowest degree in  $A$  considered as a polynomial in the  $y_i$  and let  $B_1$  be the corresponding sum for  $B$ . Then  $B_1$  holds  $A_1$ .

A similar result holds for the terms of highest degree.

31. **Remark.** The simplest case is that in which  $B \equiv 0, [A]$ . One might expect to have then  $B_1 \equiv 0, [A_1]$ . We shall show by means of an example that this need not be so. Let, in  $\mathcal{F}\{y\}$ ,

$$A = y_1^2 + y^3; \quad B = 2y_2A - y_1A' = 2y^3y_2 - 3y^2y_1^2.$$

Then  $A_1 = y_1^2, B_1 = B$ . If we had  $B_1 \equiv 0, [A_1]$ , it would follow that  $y^3y_2 \equiv 0, [A_1]$ . The derivatives of  $y_1^2$  have weights which exceed 2. Thus  $y^3y_2$  would have to be a multiple of  $y_1^2$ . This proves our statement. From the expression of  $B$  in terms of  $A$ , one might now conjecture that *some power* of  $B_1$  is linear in  $A_1$  and the *first* derivative of  $A_1$ . In that case, some power of  $y^3y_2$  would be such a linear combination. This is impossible since  $y^3y_2$  is not divisible by  $y_1$ . Actually, the cube of  $B_1$  is linear in  $A_1$  and its first two derivatives.

32. We enter into the proof. If  $A_1$  is free of the  $y$ ,  $B_1$  certainly holds  $A_1$ . In what follows, we assume that the terms of  $A_1$  are of positive degree. Then  $A_1$  vanishes for  $y_i = 0, i = 1, \dots, n$ .

We shall prove the permissibility of assuming that  $A_1$  contains a term involving only the  $y_{1j}$ . Let  $z_2, \dots, z_n$  and  $w_2, \dots, w_n$  be indeterminates. Let  $y_i$ , for  $i > 1$ , be replaced in  $A_1$  by  $z_i + w_i$ . Then  $A_1$  goes over into a d.p.  $C$  in  $y_1, z$  and  $w$ .  $C$  contains terms free of the  $z_{ij}$ ; the sum  $D$  of such terms is found by substituting  $w_i$  for  $y_i$  in  $A_1$  for  $i > 1$ . Let  $t_2$  be an integer which exceeds the order of  $D$  in  $y_1$ . On putting  $w_2 = y_1 t_2$  in  $D$ , we convert  $D$  into a non-zero d.p.  $D_1$  in  $y_1, w_3, \dots, w_n$ . We now replace  $w_3$  in  $D_1$  by  $y_1 t_3$ , where  $t_3$  exceeds the order of  $D_1$  in  $y_1$ . Continuing, we find a substitution

$$(51) \quad y_i = z_i + y_1 t_i, \quad i = 2, \dots, n,$$

which converts  $A_1$  into a d.p.  $E$  in  $y_1$  and the  $z_i, E$  possessing terms free of the  $z_{ij}$ . The terms of  $E$  will have the same degree as those of  $A_1$ .

The substitution (51) may be applied to  $A$  and  $B$  and will give a situation in which  $E$  takes the place of  $A_1$ . This proves the legitimacy of the assumption described above, and, in what follows,  $A_1$  will be understood to have terms involving only the  $y_{1j}$ .

Now let  $\zeta_1, \dots, \zeta_n$  be any zero of  $A_1$ . We wish to show that  $A$  has a zero

$$(52) \quad \begin{aligned} y_i &= \zeta_i c, & i &= 2, \dots, n, \\ y_1 &= \zeta_1 c + \varphi_2 c^{p_2} + \dots, \end{aligned}$$

of the familiar type. If  $A$  vanishes for  $y_i = \zeta_i c, i = 1, \dots, n$ , we have (52).

Otherwise, we replace each  $y_i$  with  $i > 1$  in  $A$  by  $\zeta_i c$  and  $y_1$  by  $\zeta_1 c + u_1$ . Then  $A$  goes over into an expression  $K'$  in  $c$  and  $u_1$  which may be written as in (3). The lowest exponent of  $c$  in  $a'$  exceeds the degree of  $A_1$ . Also, because  $A_1$  has terms in  $y_1$  alone,  $A_1$  contributes to the sum in (3) terms, free of  $c$ , whose degree in the  $u_i$ ; equals the degree of  $A_1$ . The discussion of §§7-13 thus holds for the present  $K'$ , and we have the zero (52) of  $A$ .

As (52) annuls  $B$ , the  $\zeta$  annul  $B_1$ . The theorem is proved.

33. The case of the terms of highest degree, mentioned in §30, is perhaps most conveniently treated as follows. Let  $A_1$  and  $B_1$  be the sums of the terms of highest degree in  $A$  and  $B$  respectively. Using indeterminates  $u; z_1, \dots, z_n$ , we put in  $A$  and  $B$

$$(53) \qquad y_i = z_i/u^2, \qquad i = 1, \dots, n.$$

We have then

$$\begin{aligned} A &= C/u^m, & A_1 &= C_1/u^m, \\ B &= D/u^m, & B_1 &= D_1/u^m, \end{aligned}$$

with  $m$  a positive integer and  $C, C_1, D, D_1$  d.p. in  $u$  and the  $z$ .  $C_1$  and  $D_1$  will be the sums of terms of *least* degree in  $C$  and  $D$ . Because  $B$  holds  $A$ ,  $uD$  holds  $C$ . By what precedes,  $uD_1$  holds  $C_1$ . Because every zero of  $A_1$  yields zeros of  $C_1$  with  $u \neq 0$ ,  $B_1$  holds  $A_1$ .

SINGULAR SOLUTIONS

34. In studying the components of a d.p.  $F$ , and in examining the manner in which they make themselves visible in the structure of  $F$ , we have thus far had no need to assume  $F$  algebraically irreducible. For a closer examination of the components, algebraic irreducibility is important for  $F$ , and accordingly we assume it.

As we saw in Chapter II, the discussion of the manifold of  $F$  is allied to the study of the singular solutions of  $F = 0$ . The general solution of  $F$  contains all nonsingular solutions and sometimes, in addition, some or all singular solutions. If there are other components, they are made up of singular solutions.

The problem of singular solutions has two aspects. On the one hand, one will wish to know how the singular solutions are distributed among the components of  $F$ . On the other, one will, in the analytic case, desire to know how the singular solutions are related analytically to the nonsingular ones. For instance, singular solutions may be envelopes of nonsingular solutions, or may be embedded among them in an interesting way.

35. Let us examine the first question. With what we already know of the components and with what will be developed in Chapter V, we shall be able to produce a set of d.p.

$$(54) \qquad F, A_1, \dots, A_p$$

whose general solutions are the components of  $F$ . The general solution of a d.p.

in (54) contains all nonsingular zeros of the d.p. One will wish to determine the singular zeros which are contained in the general solution. If one has done this, and thus knows the nature of each component of  $F$ , one may be interested in determining the intersection of two or more components; this is a matter of finding the intersection of the general solutions of two or more d.p.

If it were possible effectively to construct bases for the various essential prime divisors of the perfect ideal determined by a given finite system of d.p., the above questions would be answered. For instance, we could get a finite system of d.p. whose manifold is the general solution of  $F$  and, after that, a finite system whose manifold consists of the singular zeros in the general solution.

The problem of determining bases for the prime divisors is at present far from being solved.<sup>15</sup> It is thus a matter, at this time, of treating special differential equations with such methods as one can devise.

For the case of a single indeterminate, the problem of the singular zeros in a general solution, and that of the intersection of components, reduce to the following problem: *Given two algebraically irreducible d.p. in  $y$ ,  $F$  and  $A$ , with  $F$  holding the general solution of  $A$ , to determine whether the general solution of  $A$  is contained in the general solution of  $F$ .*

If  $F$  is of order  $n$  in  $y$ , and  $A$  of order  $n - 1$ , this is merely a matter of deciding whether the general solution of  $A$  is a component of  $F$ . The low power theorem gives the decision. If the order of  $A$  is less than  $n - 1$ , the question becomes complicated. For instance, suppose that  $A$  is of order  $n - 2$ . It may be that  $F$  has certain components  $\mathfrak{M}_1, \dots, \mathfrak{M}_q$  of order  $n - 1$ . The general solution  $\mathfrak{M}$  of  $A$  may be found, when the low power theorem is used, to be contained in some of the  $\mathfrak{M}_i$ . In that case, the question of testing for the presence of  $\mathfrak{M}$  in the general solution of  $F$  is an intricate one, which thus far has been solved only for the case of  $n = 2$ .<sup>16</sup>

For the case of  $n = 2$ , our problem can be reduced to the following: *Let  $F$ , of order 2, vanish for  $y = 0$ . It is required to determine whether  $y = 0$  is contained in the general solution of  $F$ .* This question can always be answered after there are performed a finite number of operations in which one examines polygons, of the Newton type, associated with  $F$ . The discussion is too lengthy to be presented here.

For instance, let  $F$  be the d.p. of §26.  $F$  is annulled by  $y = 0$ . For  $m > 3$ , it follows from §24 that  $y = 0$  is in the general solution of  $F$ . For  $m = 3$ , the methods of the paper cited above show that  $y = 0$  is in the general solution.

We wish, in conclusion, to compare two very simple d.p. According to §28,  $y = 0$  is not in the general solution of  $yy_1 + y_2^3$ . Consider, again,  $yy_1 + y_2^2$ . By §24, its general solution contains  $y = 0$ .

36. The second problem on singular solutions mentioned in §34 belongs to

<sup>15</sup> A theoretical solution of the problem is presented in Chapter V. This solution is incomplete in that one does not know how far the process used in the solution must be carried to be effective.

<sup>16</sup> Ritt, 31.

classical analysis rather than to differential algebra. For instance, Hamburger's work on differential equations of the first order<sup>17</sup> shows that if a singular solution of such an equation is not contained in the general solution, the singular solution is an envelope of nonsingular ones. If the singular solution is contained in the general solution, it is analytically embedded among nonsingular solutions. Another paper of Hamburger's deals with algebraic differential equations of any order  $n$ , supposed to have a component of order  $n - 1$ . Of course, the notions of component, and of general solution, as we have them, did not exist when Hamburger wrote. The component of order  $n - 1$  is shown, speaking geometrically, to consist of envelopes of curves in the general solution. The theory of algebraic differential manifolds throws new light on the analytic theory of singular solutions, and, as one sees in connection with partial differential equations,<sup>18</sup> points the way in analytical investigations.

37. Just as Lagrange dealt, to an extent, with the general solution of a differential equation, so Laplace,<sup>19</sup> in a paper published in 1772, treated questions resembling those of the present chapter. Dealing with a differential equation  $F = 0$  of order  $n$ , in an unknown  $y$ , Laplace uses the term *general integral* to designate a family of solutions depending on  $n$  arbitrary constants. By a *solution* of the given equation, he understands an equation  $A = 0$  of order lower than  $n$  which "satisfies" the given equation. What seems to be meant, in a vague way, is that  $F$  holds the general solution of  $A$ . A *particular integral* is a solution "contained in" the general integral and a *particular solution* is one which is not so contained. Laplace sets the following two problems:

*Being given a differential equation of any order,*

(1) *to determine whether an equation of lower order which satisfies it is contained in the general integral;*

(2) *to determine all of the particular solutions of the given equation.*

The second problem corresponds to that of the determination of the components of a d.p., the problem which is solved by the low power theorem. The first problem corresponds to that of determining whether the general solution of  $A$  is contained in that of  $F$ .

As one would expect, Laplace's treatment of his problems is of a heuristic nature. It does not contain the elements of a sound theory, or even serviceable conjectures. One can have only admiration, however, for his ability to imagine problems which, with the mathematics of his day, could not be soundly formulated, much less solved.

38. A paper published by Poisson in 1806 treats,<sup>20</sup> in a manner somewhat different from that of Laplace, the questions raised by the latter. Poisson's method is most easily understood from his discussion of "algebraic particular solutions." These, which had been considered by Laplace, have for counter-

<sup>17</sup> Hamburger, 6.

<sup>18</sup> Ritt, 41.

<sup>19</sup> Laplace, 16.

<sup>20</sup> Poisson, 19.

parts, in the theory of manifolds, components composed of one point, for instance, the manifold of  $y$  when

$$F = y + \left( \frac{d^2y}{dx^2} \right)^2.$$

Poisson considers that it is proper to call a solution  $y(x)$  of a differential equation an algebraic particular solution if and only if the equation does not have a one-parameter family of solutions  $y(x) + cz$  with  $c$  an arbitrary constant and  $z$  a function of  $x$  and  $c$ . More or less, an algebraic particular solution is, for Poisson, one which cannot be analytically embedded in a one-parameter family of solutions. With this definition, Poisson is able to state, for certain classes of equations, necessary and sufficient conditions for a given solution to be an algebraic particular solution. The results of Poisson may be regarded, as may also those of Laplace, as heuristic equivalents of portions of the low power theorem. For instance, Poisson concludes that  $y = 0$  is a particular solution of

$$\left( \frac{dy}{dx} \right)^m \frac{d^2y}{dx^2} = y^n$$

if and only if  $m \geq n$ . Poisson's treatment of his problem vaguely resembles the necessity proof for the low power theorem.

39. There is an aspect of the theory of singular solutions which is not revealed by our algebraic considerations. The equation

$$(55) \quad y^2 - y \frac{dy}{dx} + \left( \frac{dy}{dx} \right)^3 = 0$$

has  $y = 0$  in its general solution. If we solve for  $y$  in (55) in terms of  $dy/dx$ , we secure two expansions proceeding according to increasing integral powers of  $dy/dx$ . They are

$$(56) \quad y = \frac{dy}{dx} - \left( \frac{dy}{dx} \right)^2 + \cdots,$$

$$(57) \quad y = \left( \frac{dy}{dx} \right)^2 + \cdots.$$

Now the solution  $y = 0$  of (57) can be shown to be an envelope of solutions of (57). Furthermore, the low power theorem can be extended to cover equations like (57), in which infinite series appear. This suggests extending the theory of differential polynomials into one of *differential power series*.<sup>21</sup>

### III. Exponents of Ideals

40. In  $\mathcal{F}\{y_1, \cdots, y_n\}$ , we consider an ideal  $\Sigma$  and, with it,  $\{\Sigma\}$ . Every d.p. in  $\{\Sigma\}$  has a power in  $\Sigma$ . By I, §15, if, for some  $p$ , and for every  $A$  in  $\{\Sigma\}$ ,  $A^p \equiv 0, (\Sigma)$ , the  $p$ th power of  $\{\Sigma\}$  will be contained in  $\Sigma$ . If there is a positive integer  $p$  such that  $\Sigma$  contains  $\{\Sigma\}^p$ , the least such integer is called

<sup>21</sup> Ritt, 33.

the exponent of  $\{ \Sigma \}$  relative to  $\Sigma$ . When no such integer exists, the relative exponent is taken as  $\infty$ .

Relative exponents were investigated by Kolchin.<sup>22</sup> He studied, in particular, for an algebraically irreducible d.p.  $A$  in  $y$ , of the first order, the exponent of  $\{ A \}$  relative to  $[A]$ . The exponent depends on the nature of the singular zeros of  $A$ . We shall content ourselves with the presentation of an example.

41. Let  $A = y_1^2 - 4y$ . (See Example 1 of II, §4.) We shall prove that the exponent of  $\{ A \}$  relative to  $[A]$  is 2. We have, subscripts of  $A$  indicating differentiation,

$$\begin{aligned}
 (58) \quad & A_1 = 2y_1y_2 - 4y_1, \\
 & A_2 = 2y_1y_3 + 2y_2^2 - 4y_2, \\
 & A_3 = 2y_1y_4 + 6y_2y_3 - 4y_3, \\
 & \dots\dots\dots, \\
 & A_r = 2y_1y_{r+1} + \dots + 2ry_2y_r - 4y_r \qquad (r > 2).
 \end{aligned}$$

The unwritten terms in  $A_r$  with  $r > 2$  are of the form  $cy_p y_q$  with  $p + q = r + 2$  and with  $p$  and  $q$  greater than 2 and less than  $r$ . As  $y_1(y_2 - 2)$  is in  $[A]$ ,  $y_2(y_2 - 2)^2$  is in  $[A]$ . If  $r > 2$ ,

$$y_2(y_2 - 2)^2 = P_r(2ry_2 - 4) + c_r$$

with  $c_r$  a constant distinct from zero. We find then, from the last equation of (58),

$$(59) \quad c_r y_r \equiv P_r(2y_1y_{r+1} + \dots), \quad [A],$$

the unwritten terms being as in  $A_r$ .

We shall now prove that, for  $r > 2$ ,

$$(60) \quad y_r(y_2 - 2) \equiv 0, \quad [A].$$

By (59) with  $r = 3$ , we have

$$(61) \quad c_3 y_3 \equiv 2y_1 y_4 P_3, \quad [A].$$

We multiply by  $y_2 - 2$  in (61), noting that  $A_1 = 2y_1(y_2 - 2)$ . We obtain (60) with  $r = 3$ . If we observe that the subscripts in the unwritten terms of (59) exceed 2 and are less than  $r$ , the induction necessary to establish (60) for all  $r$  is accomplished.

Then, for  $r > 2$ ,

$$(62) \quad y_{r+1}(y_2 - 2) + y_r y_3 \equiv 0, \quad [A].$$

As the first term in (62) is in  $[A]$ , we have, for  $r > 2$ ,  $y_r y_3 \equiv 0, [A]$ . Then

$$y_{r+1} y_3 + y_r y_4 \equiv 0, \quad [A],$$

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<sup>22</sup> Kolchin, 10.

so that  $y_r y_4$  is in  $[A]$  for  $r > 2$ . In this way, we see that

$$(63) \quad y_r y_q \equiv 0, \quad [A],$$

for  $p > 2, q > 2$ .

Let  $P$ , any d.p. in  $\{A\}$ , be written

$$P = Q + R,$$

where  $Q$  consists of those terms of  $P$  which involve only  $y, y_1, y_2$ . The  $y_i$  with  $i > 2$  are in  $\{A\}$ . Thus  $Q$  is in  $\{A\}$ . We write

$$Q = M(y_2 - 2) + N$$

with  $N$  free of  $y_2$ . Then  $y_1 N$  is in  $\{A\}$  so that  $N$  is divisible by  $A$ . Thus

$$P \equiv M(y_2 - 2) + R, \quad [A].$$

Then

$$P^2 \equiv M^2(y_2 - 2)^2 + 2MR(y_2 - 2) + R^2, \quad [A].$$

Now  $R^2$  is in  $[A]$  by (63) and  $R(y_2 - 2)$  is in  $[A]$  by (60). Hence

$$P^2 \equiv M^2(y_2 - 2)^2, \quad [A].$$

Each term in  $M$  involves at least one of  $y, y_1, y_2$ . We know that  $y_1(y_2 - 2)$  and  $y_2(y_2 - 2)^2$  are in  $[A]$ . Also, because  $y_1^2 \equiv 4y, (A)$ , we see that  $y(y_2 - 2)$  is in  $[A]$ . Hence  $P^2 \equiv 0, [A]$ .

Thus  $\{A\}^2$  is contained in  $[A]$ .

42. It remains to be proved that  $\{A\}$  is not identical with  $[A]$ . This we show by proving that  $y_3$  is not in  $[A]$ . Suppose that

$$(64) \quad y_3 = CA + C_1 A_1 + \cdots + C_r A_r.$$

In the second member of (64), we put  $y = y_1^2/4, y_2 = 2$ . We find, writing

$$\begin{aligned} B_2 &= y_1 y_3, & B_3 &= y_1 y_4 + 4y_3, \\ B_r &= y_1 y_{r+1} + \cdots + (2r - 2)y_r \end{aligned} \quad (r > 2),$$

that

$$(65) \quad y_3 = D_2 B_2 + \cdots + D_r B_r.$$

We see immediately that  $r > 2$ . The only term in the second member of (65) which can yield a constant times  $y_3$  is  $D_3 B_3$ . Thus  $1/4$  must be a term in  $D_3$  so that  $D_3 B_3$  must contain  $(y_1 y_4)/4$ . The term just mentioned must cancel out in (65). The only  $B$  other than  $B_3$  which contains a term of which  $y_1 y_4$  is a multiple is  $B_4$ , which contains  $6y_4$ . Thus  $D_4$  must contain  $-y_1/24$  so that  $-(y_1^2 y_5)/24$  appears in  $D_4 B_4$ . The only term other than  $B_4$  which contains a factor of  $y_1^2 y_5$  is  $B_5$ , in which  $y_5$  appears. It follows that  $D_5 B_5$  contains a term in  $y_1^3 y_6$ . Continuing, we produce the contradiction that  $r$  in (64) is not exceeded by any integer. This completes the proof.

## CHAPTER IV

### SYSTEMS OF ALGEBRAIC EQUATIONS

1. The preceding chapters contain, of course, a theory of systems of algebraic equations. One has only to suppose oneself working with a system of d.p. which are of order zero in each indeterminate. It is, however, desirable to make a separate examination of algebraic equations.

For instance, the theory of algebraic equations can be developed from the algorithmic standpoint, so that every entity whose existence is established is constructed with a finite number of operations. The results of the algebraic theory will permit us, in Chapter V, to give an algorithmic treatment of various questions connected with finite systems of d.p.

Again, we shall obtain an approximation theorem for systems of algebraic equations (§39) which will be found useful in the study of algebraic differential manifolds in the analytic case.

Our account of algebraic equations differs in certain respects from the classical treatments. On the one hand, it is convenient for us to use the methods of the preceding chapters; on the other, it is necessary for us to develop formal procedures which can be applied later to differential equations.

#### POLYNOMIALS AND THEIR IDEALS

2. In the present chapter, we use an *algebraic* field  $\mathfrak{F}$  of characteristic zero (I, §1), without requiring that an operation of differentiation exist in  $\mathfrak{F}$ . We study polynomials in algebraic indeterminates  $y_1, \dots, y_n$ , with coefficients in  $\mathfrak{F}$ . The totality of such polynomials is represented by  $\mathfrak{F}[y_1, \dots, y_n]$ . Polynomials will be represented by capital italics and systems of polynomials by large Greek letters.

We carry over definitions from Chapter I as follows. Let  $\mathfrak{F}$  be regarded momentarily as a differential field in which all derivatives are zero, and the  $y$  as differential indeterminates. We then define, as in Chapter I, the terms *class*, *separant*, *initial*, *chain*, *characteristic set* and *remainder*.

3. Let  $\Sigma$  be a system of polynomials in  $\mathfrak{F}[y_1, \dots, y_n]$ . We shall call  $\Sigma$  a *polynomial ideal* (p.i.) if, for every finite subset  $A_1, \dots, A_r$  of  $\Sigma$  and for all  $C_1, \dots, C_r$  in  $\mathfrak{F}[y_1, \dots, y_n]$ , the polynomial  $C_1A_1 + \dots + C_rA_r$  is contained in  $\Sigma$ .

If  $\Sigma_1$  and  $\Sigma_2$  are p.i., and if  $\Sigma_2$  contains  $\Sigma_1$ ,  $\Sigma_2$  is called a *divisor* of  $\Sigma_1$ .

Let  $\Lambda$  be any system of polynomials. Let  $(\Lambda)_0$  be the totality of linear combinations of polynomials in  $\Lambda$  with polynomials for coefficients. Then  $(\Lambda)_0$  is a p.i. We call  $(\Lambda)_0$  the *p.i. generated by*  $\Lambda$ .

Let  $\Sigma$  be a p.i. Suppose that, whenever a polynomial  $A$  is such that some

positive integral power of  $A$  is in  $\Sigma$ ,  $A$  itself is in  $\Sigma$ . We shall then call  $\Sigma$  a *perfect p.i.* Let  $\Lambda$  be any system of polynomials. Let  $\{\Lambda\}_0$  be the totality of those polynomials  $A$  for which a positive integer  $p$ , depending on  $A$ , exists such that  $A^p$  is in  $(\Lambda)_0$ . It is easy to see that  $\{\Lambda\}_0$  is a perfect p.i. We call  $\{\Lambda\}_0$  *the perfect p.i. determined by  $\Lambda$ .*

Let  $\Sigma$  be a p.i. We shall say that  $\Sigma$  is prime if, for every pair of polynomials  $A$  and  $B$  with  $AB$  in  $\Sigma$ , at least one of  $A$  and  $B$  is in  $\Sigma$ . Every prime p.i. is perfect.

For p.i., the following theorem holds.

**THEOREM:** *Every perfect p.i. is the intersection of a finite number of prime p.i.*

In §4, we shall show how this theorem can be proved using only material developed in the present book. Let us, however, first found a proof on Hilbert's classic basis theorem for systems of polynomials.

According to Hilbert's theorem, given an infinite system  $\Sigma$  of polynomials,  $\Sigma$  has a finite subset  $\Phi$  such that  $(\Phi)_0$  contains  $\Sigma$ . Now let  $\Sigma$  be a perfect ideal for which our theorem is not true. Then  $\Sigma$  is not prime. Let  $AB$  be in  $\Sigma$  while neither  $A$  nor  $B$  is. We see easily that

$$\{\Sigma + AB\}_0 = \{\Sigma + A\}_0 \cap \{\Sigma + B\}_0$$

and the proof is completed as in I, §16.

4. We may also operate as follows. Let us consider  $\mathfrak{F}$  as a differential field in which all derivatives are zero and let the  $y$  be regarded as differential indeterminates. Let  $\Sigma$  be an infinite system of polynomials, and  $\Phi$  a basis for  $\Sigma$  as in I, §12. We consider any polynomial  $A$  in  $\Sigma$ . Let  $A^p$  be linear in polynomials of  $\Phi$ , and their derivatives, with d.p. for coefficients. The  $j$ th derivative of a polynomial of positive class is isobaric and of weight  $j$ . If, in the linear expression for  $A^p$ , we cast out all terms of positive weight, we have for  $A^p$  an expression linear in polynomials in  $\Phi$ , with polynomials for coefficients. We secure in this way a basis theorem for systems of polynomials which, to be sure, is weaker than Hilbert's theorem, but which is adequate for the purposes of §3.

5. If  $\Sigma$  is a perfect p.i., a prime divisor of  $\Sigma$  which is not a divisor of any other prime divisor of  $\Sigma$  will be called an *essential prime divisor* of  $\Sigma$ . Every perfect p.i. has a finite number of essential prime divisors, and is the intersection of those divisors.

#### ALGEBRAIC MANIFOLDS

6. Let  $\Sigma$  be a system of polynomials  $\mathfrak{F}[y_1, \dots, y_n]$ . Let  $\mathfrak{F}'$  be any extension of  $\mathfrak{F}$ , that is, any algebraic field which contains  $\mathfrak{F}$ . Let there exist in  $\mathfrak{F}'$  a set of elements  $\eta_1, \dots, \eta_n$  which cause every polynomial in  $\Sigma$  to vanish when  $\eta_i$  is substituted for  $y_i$ . The set  $\eta_1, \dots, \eta_n$  will be called a *zero* of  $\Sigma$ . If  $\Sigma$  has zeros, the totality of its zeros, for all extensions  $\mathfrak{F}'$  of  $\mathfrak{F}$ , will be called the *manifold* of  $\Sigma$ . The manifold of a system of polynomials will be called an *algebraic manifold*.

Let an algebraic manifold  $\mathfrak{M}$  be the union of two algebraic manifolds, each a proper part of  $\mathfrak{M}$ . We shall then call  $\mathfrak{M}$  *reducible*. If  $\mathfrak{M}$  is not reducible, it is called *irreducible*.

Let  $\mathfrak{M}$  be an algebraic manifold. The totality  $\Sigma$  of polynomials which vanish over<sup>1</sup>  $\mathfrak{M}$  is a perfect p.i., the *perfect p.i. associated with  $\mathfrak{M}$* .  $\Sigma$  is prime if and only if  $\mathfrak{M}$  is irreducible. If  $\mathfrak{M}$  is irreducible, we call  $\Sigma$  the *prime p.i. associated with  $\mathfrak{M}$* .

We see readily that *every algebraic manifold is the union of a finite number of irreducible algebraic manifolds*.

Let  $\mathfrak{M}$  be the union of irreducible algebraic manifolds  $\mathfrak{M}_1, \dots, \mathfrak{M}_p$ . We suppose that no  $\mathfrak{M}_i$  contains any  $\mathfrak{M}_j$  with  $j \neq i$ . We then call each  $\mathfrak{M}_i$  a *component* of  $\mathfrak{M}$ , or of any system of polynomials whose manifold is  $\mathfrak{M}$ . If  $\Sigma$  is the perfect p.i. associated with  $\mathfrak{M}$  and  $\Sigma_i$  the prime p.i. associated with  $\mathfrak{M}_i$ ,  $\Sigma$  is the intersection of the  $\Sigma_i$  and the  $\Sigma_i$  are the essential prime divisors of  $\Sigma$ .

#### GENERIC ZEROS OF PRIME POLYNOMIAL IDEALS

7. Let  $\Sigma$  be a prime p.i. distinct from the *unit p.i.*,  $(1)_0$ . Let  $A$  be any polynomial, not necessarily contained in  $\Sigma$ . We form a class  $\alpha$  of polynomials, putting into  $\alpha$  every polynomial  $G$  such that  $G - A$  is in  $\Sigma$ . We call  $\alpha$  a *remainder class modulo  $\Sigma$* . If  $\alpha$  and  $\beta$  are remainder classes,  $\alpha + \beta$  is defined as the remainder class which contains every  $A + B$  with  $A$  in  $\alpha$  and  $B$  in  $\beta$ .

We define  $\alpha\beta$  similarly. We call  $\Sigma$ , which is a remainder class, the *zero class*. Because  $\Sigma$  is prime, the product of two nonzero remainder classes is distinct from the zero class.

We now consider pairs  $(\alpha, \beta)$  of remainder classes in which  $\beta$  is not the zero class. Equivalence is defined as in II, §6, and the totality of pairs of classes separates into sets of equivalent pairs. For the sets of equivalent pairs, addition and multiplication are defined, as in II, §6. Subtraction and division are then performable and unique, with the usual reservation in regard to division. The sets of equivalent pairs constitute an algebraic field  $\mathfrak{F}_1$  which, after an adjustment, becomes an extension of  $\mathfrak{F}$ .

Let  $\omega$  be the remainder class which contains 1. Let  $\alpha_i, i = 1, \dots, n$ , be the class which contains  $y_i$ . Let  $\eta_i$  be the set in  $\mathfrak{F}_1$  which contains  $(\alpha_i, \omega)$ . We find that  $\eta_1, \dots, \eta_n$  is a zero of  $\Sigma$ . Every polynomial in  $\mathfrak{F}[y_1, \dots, y_n]$  which vanishes when each  $y_i$  is replaced by  $\eta_i$  is contained in  $\Sigma$ .

Let  $\Sigma$  be as above. Every zero  $\eta_1, \dots, \eta_n$  of  $\Sigma$  which is such that every polynomial in  $\mathfrak{F}[y_1, \dots, y_n]$  which is annulled by the  $\eta$  is in  $\Sigma$  is called a *generic zero* of  $\Sigma$ .

#### RESOLVENTS

8. A prime p.i. distinct from  $(1)_0$  and from  $(0)_0$  will be said to be *nontrivial*. Let  $\Sigma$  be a nontrivial prime p.i. in  $\mathfrak{F}[y_1, \dots, y_n]$ . The  $y$  can be divided

<sup>1</sup> Language as in Chapter II.

into two sets,  $u_1, \dots, u_q$  and  $y_1, \dots, y_p$ ,  $p + q = n$ , such that no nonzero polynomial of  $\Sigma$  is free of the  $y$ , while, for  $j = 1, \dots, p$ , there is a nonzero polynomial in  $\Sigma$  in  $y_j$  and the  $u$  alone. We call the  $u$  a *parametric set*. Let the *indeterminates be listed in the order*

$$u_1, \dots, u_q; \quad y_1, \dots, y_p,$$

and let

$$(1) \quad A_1, \dots, A_p$$

be a characteristic set of  $\Sigma$ .

A *regular zero* of (1) is defined as a zero of (1) which does not annul the initial of any  $A$ . *Every regular zero of (1) is a zero of  $\Sigma$ .*<sup>2</sup>

9. Let  $K$  be any polynomial not contained in  $\Sigma$ . We shall prove that

$$(A_1, \dots, A_p, K)_0,$$

which we represent by  $\Lambda$ , contains a nonzero polynomial in the  $u$  alone.

We start with the observation that the polynomials in  $\Sigma$  which involve no  $y_i$  with  $i > j$ , where  $1 \leq j < p$ , constitute a prime p.i.; we designate this p.i. by  $\Sigma_j$ .

$\Lambda$  contains the remainder of  $K$  with respect to (1). Of all nonzero polynomials in  $\Lambda$  which are reduced with respect to (1), let  $B$  be one which is of a lowest rank. We say that  $B$  is free of the  $y$ .

Suppose that this is not so, and let  $B$  be of class  $q + r$  with  $r > 0$ . The initial  $C$  of  $B$  is not in  $\Sigma$ . There is a relation

$$C^m A_r = DB + E$$

where  $E$ , if not zero, is of lower degree than  $B$  in  $y_r$ . We say that  $E$  is in  $\Sigma$ . Let this be false. If  $r > 1$ , the remainder of  $E$  with respect to  $A_1, \dots, A_{r-1}$  is a nonzero polynomial contained in  $\Lambda$ , which is reduced with respect to (1) and of lower rank than  $B$ . If  $r = 1$ , a similar statement can be made of  $E$  itself. Thus  $E$  is in  $\Sigma$ , so that  $DB$  is in  $\Sigma$ . Then  $D$  is in  $\Sigma$ .  $D$  is of positive degree in  $y_r$ . As the initial of  $DB$  is that of  $C^m A_r$ , the initial  $I$  of  $D$  is not in  $\Sigma$ . If we had  $r = 1$ ,  $D$  would be a nonzero polynomial in  $\Sigma$  which is reduced with respect to (1); this is because  $D$  is of lower degree in  $y_r$  than  $A_r$ . Thus  $r > 1$ . The remainder of  $D$  with respect to  $A_1, \dots, A_{r-1}$  is zero. Thus  $JD$ , with  $J$  some product of powers of the initials of  $A_1, \dots, A_{r-1}$ , is linear in  $A_1, \dots, A_{r-1}$ . If we write  $JD$  as a polynomial in  $y_r$ , its coefficients will be in  $\Sigma_{r-1}$ . Thus  $JI$  is in  $\Sigma_{r-1}$ . This is false because neither  $J$  nor  $I$  is in  $\Sigma_{r-1}$ .

Thus  $B$  is free of the  $y$  and our statement is proved.

<sup>2</sup> We are applying here, to the theory of characteristic sets, an idea due to van der Waerden. See *Mathematische Annalen*, vol. 96 (1927), p. 189; also *Moderne Algebra*, first edition, vol. 2, p. 56.

10. We are going to show the existence of a nonzero polynomial  $G$ , free of the  $y$ , and the existence of a polynomial

$$Q = M_1y_1 + \cdots + M_p y_p$$

where the  $M$  are polynomials free of the  $y$ , such that, for two distinct zeros of  $\Sigma$  with the same  $u$  (if  $u$  exist) lying in the same extension of  $\mathfrak{F}$  and having  $G \neq 0$ ,  $Q$  assumes two distinct values.

We consider the system  $\Sigma'$  obtained from  $\Sigma$  by replacing each  $y_i$  by a new indeterminate  $z_i$ . Using  $p$  more indeterminates  $\lambda_1, \cdots, \lambda_p$  we consider the system  $\Lambda$  composed of  $\Sigma, \Sigma'$  and

$$\lambda_1 (y_1 - z_1) + \cdots + \lambda_p (y_p - z_p).$$

As  $\Lambda$  contains  $\Sigma$ ,  $\Lambda$  has, for  $j = 1, \cdots, p$ , a nonzero polynomial  $B_j$  in  $y_j$  and the  $u$  alone. Similarly, let  $C_j, j = 1, \cdots, p$ , be a nonzero polynomial of  $\Lambda$  in  $z_j$  and the  $u$  alone.

Let  $D$  be the product of the initials<sup>3</sup> of the  $B$  and  $C$ .

Consider a zero of  $\Lambda$  for which  $(y_1 - z_1) D \neq 0$ . For it, we have

$$(2) \quad \lambda_1 = - \frac{\lambda_2 (y_2 - z_2) + \cdots + \lambda_p (y_p - z_p)}{y_1 - z_1}.$$

Let  $m$  be the maximum of the degrees of the  $B_j$  in the  $y_j$  and of the degrees of the  $C_j$  in the  $z_j$ . Let  $\alpha$  be any positive integer. We write, for  $s = 0, \cdots, \alpha$  and for the above zero,

$$\lambda_1^s = \frac{E_s}{(y_1 - z_1)^\alpha},$$

where  $E_s$  is a polynomial. Now it is plain that, using the relations  $B_j = 0, C_j = 0$ , we can depress the degree of  $E_s$  in each  $y$  and in each  $z$  to be less than  $m$ . The new expression for each  $\lambda_1^s$  will be of the form

$$\lambda_1^s = \frac{F_s}{(y_1 - z_1)^\alpha D_s},$$

where  $D_s$  is a product of powers of the initials of the  $B$  and  $C$ . Let  $L$  be the least common multiple of the  $D_s$ . We write

$$(3) \quad \lambda_1^s = \frac{H_s}{(y_1 - z_1)^\alpha L},$$

$s = 0, \cdots, \alpha$ , each  $H_s$  being a polynomial of degree less than  $m$  in each  $y$  and  $z$ . Now the number of power products of the  $y$  and  $z$ , of degree less than  $m$  in each  $y$  and  $z$ , is  $m^{2p}$ . Consequently, if we take  $\alpha \geq m^{2p}$ , we find a nonzero polynomial in  $\lambda_1$ , of degree not greater than  $\alpha$ , whose coefficients are polynomials in  $\lambda_2, \cdots, \lambda_p$  and the  $u$ , which vanishes for every zero of  $\Lambda$  which does not annul  $(y_1 - z_1)D$ . The product  $K_1$  of this polynomial by  $D$  vanishes for every zero of  $\Lambda$  which does not annul  $y_1 - z_1$ .

<sup>3</sup> The initial of  $C_j$  is the coefficient of the highest power of  $z_j$ .

Similarly, for  $i = 2, \dots, p$ , we find a  $K_i$  which vanishes for every zero of  $\Delta$  which does not annul  $y_i - z_i$ .

Let  $M_i, i = 1, \dots, p$ , be polynomials in the  $u$ , which, when substituted for the  $\lambda_i$  in  $K_1 \cdots K_p$ , reduce that polynomial to a nonzero polynomial  $G$  in the  $u$ . Any such set of  $M$  will furnish a  $Q$  as above. The  $M$  may be taken as integers.

11. Introducing a new indeterminate  $w$ , we let  $\Omega$  represent the p.i.  $(\Sigma, w - Q)_0$  in  $\mathfrak{F}[u_1, \dots, u_q; w; y_1, \dots, y_p]$ . It is easy to prove, as in II, §26, that  $\Omega$  is prime. The polynomials of  $\Omega$  which are free of  $w$  are precisely the polynomials of  $\Sigma$ .

As above, we prove that  $\Omega$  has a nonzero polynomial free of the  $y$ .

We arrange the indeterminates in  $\Omega$  in the order

$$(4) \quad u_1, \dots, u_q; w; y_1, \dots, y_p$$

and take a characteristic set for  $\Omega$ ,

$$(5) \quad A, A_1, \dots, A_p.$$

Here  $w, y_1, \dots, y_p$  are introduced in succession.

We take  $A$  irreducible in  $\mathfrak{F}$ .

We are going to prove that each  $A_i$  is linear in  $y_i$ , so that the equation  $A_i = 0$  expresses  $y_i$  rationally in terms of  $w$  and the  $u$ .

12. Let us suppose that our claim is false and let  $A_k$  be the  $A_i$  of highest subscript for which it breaks down. Then every  $A_i$  with  $i > k$  which may exist is linear in  $y_i$ .

Let  $U$  be the remainder with respect to (5) of

$$I_{k+1} \cdots I_p.$$

Of course,  $U$  is free of  $y_{k+1}, \dots, y_p$ . By §9,

$$(A, A_1, \dots, A_k, U)_0$$

in  $\mathfrak{F}[u_1, \dots, u_p; w; y_1, \dots, y_k]$  contains a nonzero polynomial  $B$  in the  $u$  alone. If  $k = p$ , there is no  $B$ .

Let

$$(6) \quad u_i = \tau_i, \quad i = 1, \dots, q; \quad w = \xi; \quad y_i = \eta_i, \quad i = 1, \dots, p,$$

be a generic zero of  $\Omega$ , lying in an extension  $\mathfrak{F}_1$  of  $\mathfrak{F}$ .

We replace the  $u, w$  and  $y_1, \dots, y_{k-1}$  in  $A_k$  by the corresponding quantities in (6). Then  $A_k$  goes over into polynomial  $H_k$  in  $y_k$  over<sup>4</sup>  $\mathfrak{F}_1$ , whose degree in  $y_k$  equals that of  $A_k$ . Let  $K$  be a factor of  $H_k$ , irreducible in  $\mathfrak{F}_1$ . Then  $(K)_0$ , in  $\mathfrak{F}_1[y_k]$ , is a prime p.i. Let  $\zeta_k$  be a generic zero<sup>5</sup> of  $(K)_0$ .

The quantities

<sup>4</sup> That is, with coefficients in  $\mathfrak{F}_1$ .

<sup>5</sup> The irreducibility of  $K$  implies that every zero of  $K$  is a generic zero of  $(K)_0$ .

$$(7) \quad \tau_1, \dots, \tau_q; \xi; \eta_1, \dots, \eta_{k-1}; \zeta_k$$

do not annul  $I_{k+1} \cdots I_p$ . If they did, they would annul  $U$  and therefore  $B$ . Now  $B$ , which is a nonzero polynomial in the  $u$ , cannot vanish for the  $\tau$ . We obtain thus a zero of  $\Omega$ ,

$$(8) \quad \tau_1, \dots, \tau_q; \xi; \eta_1, \dots, \eta_{k-1}; \zeta_k, \dots, \zeta_p$$

lying in an extension of  $\mathfrak{F}_1$ . The zeros (6) and (8) do not annul  $G$  and they have the same  $w$ . They are thus identical. This means that  $\zeta_k = \eta_k$ . The proof that  $A_k$  is linear in  $y_k$  is now completed as in II, §30.

We shall call the equation  $A = 0$  a *resolvent* of  $\Sigma$ .

It is now easy to prove that  $q$ , in §8, is independent of the manner in which the  $u$  are selected. We call  $q$  the *dimension* of  $\Sigma$ .<sup>6</sup> Following II, §36, we can show that if a prime p.i.  $\Sigma'$  is a proper divisor of  $\Sigma$ , the dimension of  $\Sigma'$  is less than that of  $\Sigma$ .

HILBERT'S THEOREM OF ZEROS

13. We prove the following theorem.

**THEOREM:** *If  $\Sigma$  is a perfect p.i. distinct from the unit p.i.,  $\Sigma$  has zeros and every polynomial which holds<sup>7</sup>  $\Sigma$  is contained in  $\Sigma$ .*

Let  $\Sigma_1, \dots, \Sigma_p$  be the essential prime divisors of  $\Sigma$ . No  $\Sigma_i$  is the unit p.i. If  $G$  holds  $\Sigma$ ,  $G$  vanishes for the generic zeros of each  $\Sigma_i$  and is thus in each  $\Sigma_i$ . Then  $G$  is in  $\Sigma$ .

We present now

HILBERT'S THEOREM OF ZEROS: *Let, in  $\mathfrak{F}[y_1, \dots, y_n]$ ,*

$$(9) \quad F_1, \dots, F_s$$

*be any finite system of polynomials, and  $G$  any polynomial which holds that system. Then some power of  $G$  is linear in the  $F$ , with polynomials for coefficients.*

It is a matter of showing that  $G$  is contained in the perfect ideal determined by the  $F$ . If that ideal is the unit p.i.,  $G$  is certainly contained in it. Otherwise, we have merely to apply the theorem which precedes.

14. The analytic case, in which  $\mathfrak{F}$  consists of functions meromorphic in an open region  $\mathbf{A}$ , needs more detailed treatment. We use analytic zeros of (9), the definition being as in Chapter II. Hilbert's theorem then becomes:

*If  $F$  vanishes for every analytic zero of  $F_1, \dots, F_s$ , some power of  $G$  is linear in the  $F$ , with polynomials for coefficients.*

Let  $\Sigma$  be the perfect ideal determined by the  $F$  and suppose that  $G$  is not contained in  $\Sigma$ . Then  $\Sigma$  is not the unit ideal. Let  $\Sigma'$  be an essential prime divisor of  $\Sigma$  in which  $G$  is not contained. It will be seen that we may sup-

<sup>6</sup> The dimension is zero when there are no  $u$ .

<sup>7</sup> As in Chapter II.

pose  $\Sigma'$  to be distinct from the zero ideal. Let the indeterminates be written  $u_1, \dots, u_q; y_1, \dots, y_p$  with the  $u$  parametric for  $\Sigma'$ .

We form a resolvent for  $\Sigma'$ . Let (5) be a characteristic set for the system  $\Omega$ , associated with  $\Sigma'$  as in §11. We shall prove the legitimacy of assuming that the initials of the  $A_i$  in (5) are free of  $w$ . Let

$$A_1 = My_1 + N.$$

As  $A$  and  $M$  are relatively prime polynomials, there is a relation

$$PA + QM = L$$

where  $P$  and  $Q$  are polynomials in  $w$  and the  $u$ , and  $L$  is a nonzero polynomial in the  $u$  alone.<sup>8</sup> Then  $\Omega$  contains  $Ly_1 + QN$ . If  $I$  is the initial of  $A$ , there is a relation

$$I^a QN = CA + R$$

with  $R$  reduced with respect to  $A$ . Then  $I^a Ly_1 + R$  is in  $\Omega$  and may be used in place of  $A_1$  in (5). We treat the other  $A_i$  similarly.

Let  $H$  be the remainder of  $G$  with respect to (5). Some linear combination of  $H$  and  $A$  is a nonzero polynomial  $K$  in the  $u$  alone. Every zero of  $\Sigma'$  which annuls  $G$  annuls  $K$ .

To complete our proof, we have to show that  $\Sigma'$  has a zero which does not annul  $K$ . We fix  $u_1, \dots, u_q$  as analytic functions which annul neither  $K$  nor any initial in (5). We can then find an analytic  $w$  which annuls  $A$  with the selected  $u$ . The equations  $A_i = 0$  then determine  $y_1, \dots, y_p$ .

#### CHARACTERISTIC SETS OF PRIME POLYNOMIAL IDEALS

15. We consider, in  $\mathfrak{F}[u_1, \dots, u_q; y_1, \dots, y_p]$ , a chain

$$(10) \quad A_1, A_2, \dots, A_p,$$

$A_i$  being of class  $q + i$ . We are going to find a condition for (10) to be a characteristic set of a prime p.i.

Since a nontrivial prime p.i. consists of those polynomials which have zero remainders with respect to any characteristic set, (10) cannot be a characteristic set for more than one prime p.i.

16. If  $\mathfrak{F}_1$  is an extension of  $\mathfrak{F}$  and if  $\eta_1, \dots, \eta_r$  is a finite subset of elements of  $\mathfrak{F}_1$ , the totality of rational combinations of  $\eta_1, \dots, \eta_r$  with coefficients in  $\mathfrak{F}$  will be denoted by  $\mathfrak{F}(\eta_1, \dots, \eta_r)$  and will be called the field obtained by the *adjunction of the  $\eta$  to  $\mathfrak{F}$* . Thus, we represent by<sup>9</sup>  $\mathfrak{F}(u_1, \dots, u_q)$  the totality of the rational combinations of the  $u$  with coefficients in  $\mathfrak{F}$ .

17. Considering (10), we suppose first that  $p = 1$ . We shall show that for  $A_1$  to be a characteristic set of a prime p.i. in  $u_1, \dots, u_q; y_1$ , it is necessary and

<sup>8</sup> Chapter II, §42.

<sup>9</sup> Abbreviated below as  $\mathfrak{F}(u)$ .

sufficient that  $A_1$ , considered as a polynomial in  $y_1$ , be irreducible in  $\mathfrak{F}(u)$ . We first prove sufficiency. Let  $A_1$  be irreducible, as indicated. Then  $A_1 = BC$  with  $B$  free of  $y_1$  and  $C$  irreducible in  $\mathfrak{F}$  as a polynomial in  $y_1$  and the  $u$ . Now  $(C)_0$  is a prime p.i. for which  $C$  is a characteristic set. Then  $A_1$  is also a characteristic set for  $(C)_0$ . For the necessity proof, let  $A_1$  be a characteristic set for a prime p.i.  $\Sigma$ . Let  $A_1 = BC$ , where  $B$  and  $C$  are polynomials of positive degree in  $\mathfrak{F}(u)$  [ $y_1$ ]. Clearing fractions, we secure a relation  $GA_1 = HK$  among polynomials in  $\mathfrak{F}[u; y_1]$  with  $H$  and  $K$  of lower degree than  $A_1$  in  $y_1$ . As one of  $H$  and  $K$  is in  $\Sigma$ , we have a contradiction.

18. We understand now that  $p > 1$ . We furnish a necessary and sufficient condition which is of an inductive type. If (10) is a characteristic set of a prime p.i.  $\Sigma_p$ , those polynomials in  $\Sigma_p$  which are free of  $y_p$  constitute a prime p.i. for which  $A_1, \dots, A_{p-1}$  is a characteristic set. Thus, if (10) is a characteristic set of a prime p.i.  $\Sigma_p$ , then

(a)  $A_1, \dots, A_{p-1}$  is a characteristic set of a prime p.i.  $\Sigma_{p-1}$  in  $y_1, \dots, y_{p-1}$ . Let condition (a) be fulfilled. Let

$$(11) \quad \tau_1, \dots, \tau_q; \eta_1, \dots, \eta_{p-1}$$

be any generic zero of  $\Sigma_{p-1}$ . Let  $\mathfrak{F}_{p-1}$  represent the field obtained by adjoining the quantities in (11) to  $\mathfrak{F}$ . The initial of  $A_p$  is not in  $\Sigma_{p-1}$  and thus does not vanish for (11). We shall prove that if (10) is a characteristic set of a prime p.i.,

(b)  $A_p$ , when the indeterminates other than  $y_p$  are replaced by their corresponding quantities in (11), becomes a polynomial in  $\mathfrak{F}_{p-1}[y_p]$  which is irreducible in  $\mathfrak{F}_{p-1}$ .

A few words are necessary to show that our work is not influenced by the choice which is made of a generic zero (11). For (11), let  $A_p$  become a polynomial  $B$  in  $\mathfrak{F}_{p-1}[y_p]$ . Let  $B = CD$  with  $C$  and  $D$  polynomials of positive degree in  $y_p$ , over  $\mathfrak{F}_{p-1}$ . A coefficient in  $C$  or  $D$  may be written in the form  $\varphi/\psi$  where  $\varphi$  is obtained by making the substitution (11) in a polynomial  $P$  in  $u_1, \dots, u_q; y_1, \dots, y_{p-1}$  over  $\mathfrak{F}$ , and where  $\psi$  is obtained similarly from a polynomial  $Q$ . Then  $Q$  is not in  $\Sigma_{p-1}$ . Suppose now that

$$(12) \quad \tau'_1, \dots, \tau'_q; \eta'_1, \dots, \eta'_{p-1}$$

is a second generic zero of  $\Sigma_{p-1}$  and that the adjunction of the quantities (12) to  $\mathfrak{F}$  produces a field  $\mathfrak{F}'_{p-1}$ . If, in the equation  $B = CD$ , we replace the quantities in (11) by those in (12), and bear in mind that an algebraic relation among the quantities in (11), with coefficients in  $\mathfrak{F}$ , holds also for the quantities (12), we secure an equation  $B' = C'D'$  which shows that (12) may be used with the same effect as (11).

It will be proved that the conditions (a) and (b), which are necessary for (10) to be a characteristic set of a prime p.i., are also sufficient.

19. We prove the necessity of condition (b). Let (10) be a characteristic set of a prime p.i.  $\Sigma_p$ . Suppose that there is a relation  $B = CD$  as in §18.

Writing each coefficient in  $C$  and  $D$  in the form  $\varphi/\psi$  as indicated above, we clear fractions. We obtain a relation

$$(13) \quad \delta B = EF$$

where  $\delta$  is a polynomial in the quantities in (11) and  $E$  and  $F$  are polynomials in  $y_p$ , of positive degree, whose coefficients are polynomials in the quantities in (11). We write (13)

$$(14) \quad \delta B - EF = 0.$$

In the first member of (14) we replace each quantity in (11) by the indeterminate which corresponds to it. We obtain a polynomial

$$(15) \quad GA_p - HK.$$

If this polynomial is arranged according to powers of  $y_p$ , its coefficients will vanish for (11) and thus are in  $\Sigma_{p-1}$ . Hence  $HK$  is in  $\Sigma_p$ . Suppose that  $H$  is in  $\Sigma_p$ . The degree of  $H$  in  $y_p$  is less than that of  $A_p$ . As  $G$  and the initial of  $A_p$  are not in  $\Sigma_p$ , the initial of  $H$  is not in  $\Sigma_p$ . Let  $L$  be the remainder of  $H$  with respect to  $A_1, \dots, A_{p-1}$ . Then  $L$  is reduced with respect to (10). Furthermore,  $L$  is not zero (§9). As  $\Sigma_p$  cannot contain a nonzero polynomial reduced with respect to (10), the necessity of (b) is proved.

20. Suppose now that (a) and (b) are satisfied. When (11) is substituted into  $A_p$ ,  $A_p$  becomes a polynomial  $B$ , irreducible in  $\mathfrak{F}_{p-1}$ . Let  $\eta_p$  be a zero of  $B$ . Let  $\Sigma_p$  be the totality of those polynomials in  $\mathfrak{F}[u; y]$  which vanish for

$$\tau_1, \dots, \tau_q; \quad \eta_1, \dots, \eta_p.$$

Then  $\Sigma_p$  is a prime p.i. We shall prove that (10) is a characteristic set of  $\Sigma_p$ . Let the contrary be assumed. Then  $\Sigma_p$  contains a nonzero  $G$  which is reduced with respect to (10). Now  $G$  must be of class  $p$ , else, vanishing for (11), it would be in  $\Sigma_{p-1}$  in spite of being reduced with respect to (10). For (11),  $G$  becomes a polynomial  $H$  in  $\mathfrak{F}_{p-1}[y_p]$  which is annulled by  $\eta_p$  and is of lower degree than  $B$ . The sufficiency of conditions (a) and (b) is thus established.

#### CONSTRUCTION OF RESOLVENTS

21. Before we can give a method for the effective construction of a resolvent for a prime p.i. for which a characteristic set is given, we must have a solution of the following problem.

Let  $\mathfrak{F}_0$  represent  $\mathfrak{F}(u_1, \dots, u_q)$ . Let  $A$  be a polynomial in  $\mathfrak{F}[u_1, \dots, u_q; w]$  irreducible as a polynomial in  $w$  over  $\mathfrak{F}_0$ . Let  $A_1$  be a polynomial in  $\mathfrak{F}[u_1, \dots, u_q; w; y]$ , of positive degree in  $y$ . Let  $w = \eta_1$  be a zero of  $A$  considered as a polynomial in  $\mathfrak{F}_0[w]$ ; of course,  $\eta_1$  lies in an extension of  $\mathfrak{F}_0$ . Let  $\mathfrak{F}_1$  represent  $\mathfrak{F}_0(\eta_1)$ . We assume that the initial of  $A_1$  does not vanish when  $w$  is replaced by  $\eta_1$ . We represent by  $B$  the polynomial in  $\mathfrak{F}_1[y]$  obtained by re-

placing  $w$  by  $\eta_1$  in  $A_1$ . It is required to find the irreducible factors of  $B$  over  $\mathfrak{F}_1$ .<sup>10</sup>

It will be seen that the only knowledge of  $\eta_1$  which we need is that it annuls  $A$ .

Let  $m$  be the degree of  $A$  in  $w$ . We shall show the existence of an extension  $\mathfrak{F}'$  of  $\mathfrak{F}_1$  in which  $A$  has  $m$  distinct zeros  $\eta_1, \dots, \eta_m$ . Let  $C$  be the polynomial in  $\mathfrak{F}_1[w]$ , of degree  $m - 1$ , obtained by dividing  $A$  by  $w - \eta_1$ . Then  $C$  has a zero  $\eta_2$ , lying in an extension  $\mathfrak{F}_2$  of  $\mathfrak{F}_1$ . The irreducibility of  $A$  in  $\mathfrak{F}_0$  implies that  $\eta_1$  and  $\eta_2$  are distinct. Let  $D = C/(w - \eta_2)$ . We secure a zero of  $D$ . Continuing, we obtain a set  $\eta_1, \dots, \eta_m$ .

Let  $z$  be an indeterminate and let  $E_1$  be the polynomial in  $\mathfrak{F}_1[y, z]$  which results on replacing  $y$  in  $B$  by  $y - z\eta_1$ . Let  $E_i, i = 2, \dots, m$ , result from  $E_1$  on replacing  $\eta_1$  by  $\eta_i$ . Let  $G = E_1E_2 \dots E_m$ .

Then  $G$  is a polynomial in  $\mathfrak{F}_0[y, z]$ , the coefficients in  $G$  being capable of determination by the theory of symmetric functions. Let  $G$  be resolved into factors irreducible in  $\mathfrak{F}_0$ . This is possible, provided we are able to factor a polynomial in one indeterminate over  $\mathfrak{F}$ .<sup>11</sup> Let

$$(16) \quad G = H_1 \dots H_r,$$

with each  $H$  a polynomial in  $\mathfrak{F}_0[y, z]$ , irreducible in  $F_0$ .

We wish to show that, for  $j = 1, \dots, r$ ,  $E_1$  and  $H_j$  have a common factor, of positive degree, over  $\mathfrak{F}_1$ . Let this be false for some definite  $j$ . Then there exists a relation

$$(17) \quad U_1E_1 + V_1H_j = W_1$$

with  $U_1, V_1, W_1$  polynomials over  $\mathfrak{F}_1$  and with  $W_1$  free of  $z$  and distinct from zero. In (17), we replace  $\eta_1$  by  $\eta_i$ , where  $1 < i \leq m$ . We secure a relation<sup>12</sup>

$$U_iE_i + V_iH_j = W_i.$$

This shows that  $H_j$  has no common factor over  $\mathfrak{F}'$ , of positive degree in  $z$ , with any  $E_i$ . Similarly  $H_j$  has no common factor over  $\mathfrak{F}'$ , of positive degree in  $y$ , with any  $E_i$ . On the other hand, the factors of  $H_j$  irreducible over  $\mathfrak{F}'$  must be factors of the  $E$ . This proves our statement.

Let  $K_i, i = 1, \dots, r$ , be the highest common factor of  $E_1$  and  $H_i$ , the field being  $\mathfrak{F}_1$ . We determine  $K_i$  by the Euclid algorithm, bearing in mind that a polynomial  $\xi$  in  $\eta_1, u_1, \dots, u_q$  is zero when and only when the polynomial in  $w$  and the  $u$ , obtained by replacing  $\eta_1$  by  $w$  in  $\xi$ , is divisible by  $A$ .

We shall prove that the  $K$  become, for  $z = 0$ , the irreducible factors of  $B$  in  $\mathfrak{F}_1$ .<sup>13</sup> Let

$$B = M_1 \dots M_k$$

<sup>10</sup> Our treatment follows van der Waerden, *Moderne Algebra*, first edition, vol. 1, p. 210.

<sup>11</sup> Perron, *Algebra*, vol. 1, p. 210.

<sup>12</sup> Every  $\eta$  is a generic zero of  $(A)$ , the field being  $\mathfrak{F}_0$ .

<sup>13</sup> We do not establish a one-to-one correspondence between the  $K$  and the irreducible factors. The knowledge of the essentially distinct irreducible factors of  $B$  permits the representation of  $B$  as a product of powers of irreducible factors.

be a resolution of  $B$  into factors irreducible in  $\mathfrak{F}_1$ . Then

$$E_1 = N_1 N_2 \cdots N_k$$

where each  $N_i$  results from  $M_i$  on replacing  $y$  by  $y - z\eta_i$ . It is easy to see that each  $N_i$ , as a polynomial in  $y$  and  $z$ , is irreducible in  $\mathfrak{F}_1$ .

Manifestly each  $N_i$  is a common factor of  $E_1$  and some  $H_j$  in (16). If we can prove that, in this case,  $N_i$  is the *highest common factor* of  $E_1$  and  $H_j$ , we will have our result.

Let  $N_i^{(j)}$ , for  $j = 2, \dots, m$ , be the polynomial obtained from  $N_i$  on replacing  $\eta_1$  by  $\eta_j$ . Let

$$(18) \quad P_i = N_i N_i'' \cdots N_i^{(m)}.$$

Then  $P_i$  is a polynomial in  $\mathfrak{F}_0 [y, z]$  and

$$G = P_1 P_2 \cdots P_k.$$

Each  $H_i$  in (16) is a factor of some  $P_j$ .

Suppose that  $N_1$  is a factor of  $H_1$  and that  $H_1$  is a factor of  $P_1$ . If we can prove that  $N_1$  is the highest common factor of  $E_1$  and  $P_1$ , we will have our result.

Suppose, for instance, that  $P_1$  is divisible by  $N_1 N_2$ . Then by (18),

$$(19) \quad N_1'' \cdots N_1^{(m)} = R(y, z) N_2,$$

where  $R$  is a polynomial in  $\mathfrak{F}_1 [y, z]$ .

The set of terms of highest degree in the first member of (19) is of the form

$$(20) \quad b(y - z\eta_2)^s \cdots (y - z\eta_m)^s$$

with  $b$  a rational combination of the  $u$  and  $\eta$ . The terms of highest degree in the second member give an expression of the type

$$(21) \quad S(y, z) (y - z\eta_1)^t.$$

Now (20) and (21) cannot be equal, since no  $y - z\eta_i$  with  $i > 1$  is divisible by  $y - z\eta_1$ . This completes the proof.

22. We consider a nontrivial prime p.i.  $\Sigma$  in  $\mathfrak{F}[u_1, \dots, u_q; y_1, \dots, y_p]$  for which

$$(22) \quad A_1, \dots, A_p$$

is a characteristic set,  $A_i$  introducing  $y_i$ . In §§24, 25 we show how, when the  $A$  are given, a resolvent can be constructed for  $\Sigma$ .

23. Let  $\lambda_1, \dots, \lambda_p$  be new indeterminates. If  $\Sigma$  is regarded as a system of polynomials in the  $u, \lambda, y$ ,  $\Sigma$  generates a p.i.  $(\Sigma)_0$  which can be seen, as in I, §27, to be prime. Furthermore  $(\Sigma)_0$  contains no nonzero polynomial in the  $u$  and  $\lambda$ .

We see as in §10 that there exists a nonzero  $G$  in the  $u$  and  $\lambda$  such that, for

two distinct zeros of  $(\Sigma)_0$  with the same  $u$  and  $\lambda$ , lying in the same extension of  $\mathfrak{F}$  and not annulling  $G$ ,

$$Q = \lambda_1 y_1 + \cdots + \lambda_p y_p$$

assumes two distinct values.<sup>14</sup>

By §§11, 12, a resolvent exists for  $(\Sigma)_0$  for which  $w = Q$ . Let  $\Omega = (\Sigma, w - Q)_0$  in  $\mathfrak{F}[u; \lambda; w; y]$ . We consider a characteristic set for  $\Omega$

$$(23) \quad R, R_1, \cdots, R_p$$

in which  $w, y_1, \cdots, y_p$  are introduced in succession and in which  $R$  is irreducible in  $\mathfrak{F}$ . Then  $R = 0$  is a resolvent for  $(\Sigma)_0$  and each  $R_i$  is linear in  $y_i$ .

24. We shall show how a characteristic set (23) can actually be constructed.

Using the polynomials in (22), and also  $w - Q$ , we can, by the method of elimination of II, §34, determine, by means of a finite number of rational operations, a nonzero  $U$  in  $w$ , the  $u$ , and  $\lambda$ , which vanishes for every generic zero of  $\Omega$ . It is a matter of considering relations  $w^j = Q^j$  and depressing the degrees of  $Q^j$  in the  $y$  by using the relations  $A_i = 0$ . Then  $U$  is in  $\Omega$ . Now let

$$U = U_1 \cdots U_r$$

with each  $U_i$  irreducible in  $\mathfrak{F}$ . Some  $U_i$  is in  $\Omega$ . The selection of such a  $U_i$  can be made as follows. Consider any  $U_i$  and let  $V$  be the polynomial obtained from it by replacing  $w$  by  $Q$ . For  $U_i$  to be in  $\Omega$ , it is necessary and sufficient that  $V$  be in  $(\Sigma)_0$ . Let  $V$  be arranged as a polynomial in the  $\lambda$ . For  $V$  to be in  $(\Sigma)_0$ , it is necessary and sufficient that every coefficient in the polynomial be in  $\Sigma$ . A coefficient will be in  $\Sigma$  if and only if its remainder with respect to (22) is zero.

A polynomial in  $w$ , the  $u$ , and  $\lambda$  which is in  $\Omega$  is divisible by  $R$ . Thus an irreducible factor of  $U$  which is in  $\Omega$  must be the product of  $R$  in (23) by an element of  $\mathfrak{F}$ .

We have then a method for constructing a resolvent for  $(\Sigma)_0$ . It remains to show how a complete set (23) can be determined.

Let  $W$  be the polynomial which results from  $R$  on replacing  $w$  by  $w + y_1$  and  $\lambda_1$  by  $\lambda_1 + 1$ . Then  $W$  holds  $\Omega$  and is thus in  $\Omega$ . The degree of  $W$  in  $y_1$  is that of  $R$  in  $w$  and the coefficient of the highest power of  $y_1$  in  $W$  is free of  $w$ .

Let  $\mathfrak{F}_0$  represent  $\mathfrak{F}(u_1, \cdots, u_q; \lambda_1, \cdots, \lambda_p)$  and let  $R$  be considered as a polynomial in  $\mathfrak{F}_0[w]$ . Let  $w = \eta$  be any zero of  $R$ . We represent by  $B$  the polynomial in  $y_1$  over  $\mathfrak{F}_0(\eta)$  obtained by replacing  $w$  in  $W$  by  $\eta$ . Let

$$(24) \quad B = B_1 \cdots B_m$$

be a decomposition of  $B$  into factors irreducible in  $\mathfrak{F}_0(\eta)$ , obtained as in §21. The coefficients in the  $B_i$  are rational in  $\eta$ , the  $u$  and  $\lambda$ . Let  $\alpha$  be the product of the denominators of these coefficients. We write

<sup>14</sup> At present we have no way of determining  $G$ .

$$\alpha B = C_1 \cdots C_m.$$

The  $C$  are irreducible in  $\mathfrak{F}_0(\eta)$  and their coefficients are polynomials in  $\eta$ , the  $u$ , and  $\lambda$ . Let  $D$  be the polynomial which results from  $\alpha$  on replacing  $\eta$  by  $w$ . Let  $E_i$  result similarly from  $C_i$ . Let

$$F = DW - E_1 \cdots E_m.$$

Then  $F$  vanishes identically in  $y_1$  if  $w$  is replaced by  $\eta$ . Hence, if  $F$  is arranged as a polynomial in  $y_1$ , its coefficients are divisible by  $R$ . Thus  $F$  is in  $\Omega$ . Then one of the  $E$  is in  $\Omega$ . Suppose that  $E_1$  is found (by test) to be in  $\Omega$ . We say that  $E_1$  is linear in  $y_1$ . If  $I_1$  is the initial of  $R_1$  in (23), we have

$$(25) \quad I_1^u E_1 = HR_1 + K$$

where  $K$  is free of  $y_1$ . Thus, if  $E_1$  were not linear, it would follow that  $C_1$  is reducible in  $\mathfrak{F}_0(\eta)$ .<sup>15</sup>

It is only necessary, then, to take the remainder of  $E_1$  with respect to  $R$  to have a polynomial which will serve as  $R_1$  in (23).

The  $R_i$  with  $i > 1$  are determined in the same way.

It can be arranged, as in §14, so that, for each  $i$ , the initial  $I_i$  of  $R_i$  is free of  $w$ . We suppose this to be done. If two zeros of  $\Omega$  have the same  $u$ ,  $\lambda$ ,  $w$ , they will have the same  $y$  if no  $I_i$  vanishes for their  $u$ ,  $\lambda$ . We may thus take  $G$  as  $I_1 I_2 \cdots I_p$ .

25. It remains to construct a resolvent for  $\Sigma$ . Let  $I$  be the initial of  $R$  in (23). Let  $a_1, \cdots, a_p$  be integers for which  $IG$ , with  $G$  as above, becomes a nonzero polynomial in the  $u$  when each  $\lambda_i$  is replaced by  $a_i$ .

We shall show how (23) yields a resolvent for  $\Sigma$  with

$$(26) \quad w = a_1 y_1 + \cdots + a_p y_p.$$

Let  $\Omega' = (\Sigma, w - a_1 y_1 - \cdots - a_p y_p)_0$  in  $\mathfrak{F}[u; w; y]$ . Then  $\Omega'$  is a prime p.i. For  $\lambda_i = a_i$ ,  $i = 1, \cdots, p$ , (23) becomes a system of polynomials

$$(27) \quad R', R'_1, \cdots, R'_p$$

each of which holds  $\Omega'$  and is therefore in  $\Omega'$ . As  $R$  and  $R'$  have the same degree in  $w$ , (27) is a chain.

We are going to show that  $R'$  is not the product of two polynomials over  $\mathfrak{F}$  which are of positive degree in  $w$ . Thus, if we free  $R'$  of its factors in the  $u$ , we secure a polynomial  $R_0$  which is irreducible in  $\mathfrak{F}$ . The equation  $R_0 = 0$  will be a resolvent for  $\Sigma$ .<sup>16</sup> Also (27) will be a characteristic set of  $\Omega'$ .

If  $R'$  is a product of two polynomials of positive degree in  $w$ ,  $\Omega'$  will have a characteristic set

$$T, T_1, \cdots, T_p,$$

<sup>15</sup> We note that  $I_1$  cannot vanish for  $w = \eta$ .

<sup>16</sup> For  $w$  as in (26), two distinct zeros of  $\Sigma$  with the same  $u$  and  $w$  annul  $G'$ , obtained from  $G$  by putting  $\lambda_i = a_i$ .

with  $T$  of lower degree in  $w$  than  $R'$ . We assume that the initials of the  $T_i$  are free of  $w$ . If  $D$  is the product of those initials, we have, for a generic zero of  $\Sigma$ ,

$$(28) \quad y_i = \frac{E_{i0} + E_{i1}w + \cdots + E_{i, g-1}w^{g-1}}{D},$$

where  $g$  is the degree of  $T$  in  $w$  and the  $E$  are polynomials in the  $u$ . We understand  $w$  to be given by (26).

Let us now consider the prime p.i.

$$\Omega'' = (\Sigma, v - \lambda_1 y_1 - \cdots - \lambda_p y_p)_0$$

in  $\mathfrak{F}[u; \lambda; v; y]$ . We show that  $\Omega''$  contains a nonzero polynomial  $K$ , free of the  $y$ , which is of degree no more than  $g$  in  $v$ . We consider the relations

$$v^j = (\lambda_1 y_1 + \cdots + \lambda_p y_p)^j, \quad j = 0, \cdots, g.$$

We replace the  $y$  by their expressions in (28) and depress the degrees in  $w$  of the second members to less than  $g$ , using the relation  $T = 0$ . By a linear dependence argument, we secure the polynomial  $K$ . This furnishes the contradiction that  $R$  in (23) is of degree at most  $g$  in  $w$ .

Thus  $R_0 = 0$  is a resolvent for  $\Sigma$ .

#### COMPONENTS OF FINITE SYSTEMS

26. Let  $\Phi$  be a finite system of polynomials in  $\mathfrak{F}[y_1, \cdots, y_n]$ , not all zero. We are going to show how to determine characteristic sets of a finite number of prime p.i. whose manifolds make up the manifold<sup>17</sup> of  $\Phi$ . Later, we shall obtain finite systems whose manifolds are the components of  $\Phi$ .

A system  $\Sigma$  of polynomials will be said to be *equivalent* to the set of systems  $\Sigma_1, \cdots, \Sigma_i$  if the manifold of  $\Sigma$  is the union<sup>18</sup> of the manifolds of the  $\Sigma_i$ .

Let

$$(29) \quad A_1, \cdots, A_p$$

be a characteristic set of  $\Phi$ , obtained as in I, §5. If  $A_1$  is of class zero,  $\Phi$  has no zeros. We assume now that  $A_1$  is of positive class. For every polynomial in  $\Phi$ , let the remainder with respect to (29) be determined. If these remainders are adjoined to  $\Phi$ , we get a system  $\Phi'$  equivalent to  $\Phi$ . By I, §5, if some of the remainders are not zero,  $\Phi'$  will have a characteristic set lower than (29). We see, by I, §4, that after a finite number of repetitions of the above operation, we arrive at a finite system  $\Lambda$ , equivalent to  $\Phi$ , with a characteristic set<sup>19</sup> (29) for which either  $A_1$  is of class zero or for which, otherwise, the remainder of every polynomial in  $\Lambda$  is zero.

27. Let us suppose that we are in the latter case. We make a temporary relettering of the  $y$ . If, in the characteristic set (29) of  $\Lambda$ ,  $A_i$  is of class  $j_i$ , we

<sup>17</sup> If  $\Phi$  has no zeros, we obtain (1)<sub>0</sub>.

<sup>18</sup> In this, we understand that if  $\Sigma$  has no zeros, no  $\Sigma_i$  has zeros.

<sup>19</sup> Naturally, (29) is not the same for  $\Lambda$  as for  $\Phi$ .

replace the symbol  $y_j$  by  $y_i$ . The  $q = n - p$  indeterminates not among the  $y_j$  we call, in any order,  $u_1, \dots, u_q$ . We list the indeterminates in the order  $u_1, \dots, u_q; y_1, \dots, y_p$ .

With this change of notation, we proceed to determine, using §§17-19, whether (29) is a characteristic set for a prime p.i.

28. If  $A_1$  is reducible as a polynomial in  $y_1$  over  $\mathfrak{F}(u_1, \dots, u_q)$  and if  $A_1 = MN$  with  $M$  and  $N$  polynomials in  $u_1, \dots, u_q; y_1$ , of positive degree in  $y_1$ , then  $\Lambda$  is equivalent to  $\Lambda + M, \Lambda + N$ . Each of the latter systems, after we revert to the old notation, will have a characteristic set lower than (29).

Suppose now that  $A_1$  is irreducible in  $\mathfrak{F}(u_1, \dots, u_q)$ . We use indeterminates  $\tau_1, \dots, \tau_q$  and the field  $\mathfrak{F}(\tau_1, \dots, \tau_q)$  which we represent by  $\mathfrak{F}_0$ . For  $u_i = \tau_i, i = 1, \dots, q, A_1$  becomes a polynomial  $B_1$  in  $\mathfrak{F}_0 [y_1]$ . Let  $y_1 = \eta_1$  be a zero of  $B_1$ . Let  $B_2$  be the polynomial in  $\mathfrak{F}_0(\eta_1) [y_2]$  which  $A_2$  becomes for  $y_1 = \eta_1, u_i = \tau_i$ . Suppose that  $B_2$  is reducible in  $\mathfrak{F}_0(\eta_1)$ . We have, in analogy to (14),

$$(30) \quad \delta B_2 - EF = 0,$$

where  $\delta$  is a polynomial in  $\eta_1$  and the  $\tau$ .  $E$  and  $F$  are polynomials in  $y_2$ , of positive degree, whose coefficients are polynomials in  $\eta_1$  and the  $\tau$ . When we replace  $\eta_1$  and the  $\tau$  by  $y_1$  and the  $u$ , the first member of (30) becomes a polynomial

$$GA_2 - HK$$

which, when arranged according to powers of  $y_2$ , has coefficients which are divisible by  $A_1$ .

Thus  $GA_2 - HK$  is in  $(A_1)_0$  so that  $HK$  is in<sup>20</sup>  $(A_1, A_2)_0$ . Let  $M$  and  $N$  be, respectively, the remainders of  $H$  and  $K$  with respect to  $A_1$ . Because the initial of  $GA_2$  is not divisible by  $A_1$ , the initials of  $H$  and  $K$  are not so divisible. It follows that  $M$  and  $N$  are not zero (§19). As  $MN$  is in  $(A_1, A_2)_0$ , we see that  $\Lambda$  is equivalent to  $\Lambda + M, \Lambda + N$ , whose characteristic sets, in the old notation, are lower than (29).

29. Suppose that  $B_2$  is irreducible in  $\mathfrak{F}_0(\eta_1)$ . By §18,  $A_1, A_2$  is a characteristic set of a prime p.i.  $\Sigma_2$  in  $y_1, y_2$  and the  $u$ . Let  $\eta_2$  be any zero of  $B_2$ . We shall show that

$$(31) \quad \tau_1, \dots, \tau_q; \quad \eta_1, \eta_2$$

is a generic zero of  $\Sigma_2$ . Let  $G$  be a polynomial in  $\Sigma_2$ . The remainder of  $G$  with respect to  $A_1, A_2$  is zero. As the initials of  $A_1$  and  $A_2$  do not vanish for (31),  $G$  is annulled by (31). Conversely, let  $G$  be a polynomial in  $y_1, y_2$  and the  $u$  which is annulled by (31). The remainder  $R$  of  $G$  with respect to  $A_1, A_2$  also vanishes for (31). Suppose that  $R$  is not zero. If  $R$  is arranged as a polynomial in  $y_2$ , its coefficients will not be divisible by  $A_1$  and thus will not vanish for  $\eta_1$  and the  $\tau$ . Substituting these quantities for  $y_1$  and the  $u$  in  $R$ , we secure

<sup>20</sup> In  $\mathfrak{F}[u; y_1, y_2]$ .

a polynomial in  $y_2$  of lower degree than  $B_2$  which vanishes for  $y_2 = \eta_2$ . This contradiction shows that  $R = 0$ . Then  $G$  is in  $\Sigma_2$ . Thus (31) is a generic zero of  $\Sigma_2$ .

We substitute the quantities (31) into  $A_3$ , securing a polynomial  $B_3$  in  $y_3$  over  $\mathfrak{F}_0(\eta_1, \eta_2)$ . We need a method for finding the irreducible factors of  $B_3$  in  $\mathfrak{F}_0(\eta_1, \eta_2)$ . Let a resolvent be constructed for  $\Sigma_2$  as in §§24, 25, with

$$w - a_1y_1 - a_2y_2 = 0,$$

$a_1$  and  $a_2$  being integers. Now

$$\tau_1, \dots, \tau_q; \quad a_1\eta_1 + a_2\eta_2; \quad \eta_1, \eta_2$$

is a generic zero of the prime p.i. for which (27), with  $p = 2$ , is a characteristic set. Thus  $a_1\eta_1 + a_2\eta_2$  annuls  $R'$ , but not the initials of  $R'_1$  and  $R'_2$ . Hence  $\eta_1$  and  $\eta_2$  are rational in  $a_1\eta_1 + a_2\eta_2$  and the  $\tau$ . Thus, to factor  $B_3$  in  $\mathfrak{F}_0(\eta_1, \eta_2)$  it suffices to factor  $B_3$  in  $\mathfrak{F}_0(a_1\eta_1 + a_2\eta_2)$ . This we know how to do.

Suppose that  $B_3$  is reducible in  $\mathfrak{F}_0(\eta_1, \eta_2)$ . We have, as in (30), a relation

$$\delta B_3 - EF = 0$$

where  $\delta$  is a polynomial in  $\eta_1, \eta_2$  and the  $\tau$ . If, in the first member, we replace  $\eta_1, \eta_2$  and the  $\tau$  by  $y_1, y_2$  and the  $u$ , we secure a polynomial  $GA_3 - HK$  which, when arranged in powers of  $y_3$ , has its coefficients in  $\Sigma_2$ . Let  $L$  be any of these coefficients. Let  $I_i$  represent the initial of  $A_i$  in (29). As the remainder of  $L$  with respect to  $A_1, A_2$  is zero, some  $I_1^a I_2^b L$  is in  $(A_1, A_2)_0$ . Then some

$$I_1^a I_2^b (GA_3 - HK)$$

is linear in  $A_1$  and  $A_2$ , so that  $I_1^a I_2^b HK$  is in  $(A_1, A_2, A_3)_0$ . Let  $M$  and  $N$  be, respectively, the remainders of  $I_1^a I_2^b H$  and  $K$  with respect to  $A_1, A_2$ . Then  $M$  and  $N$  are not zero and  $MN$  is in  $(A_1, A_2, A_3)_0$ . Thus  $\Lambda$  is equivalent to  $\Lambda + M, \Lambda + N$ , each of which, in the old notation, has characteristic sets lower than (29).

30. If  $B$  is irreducible in  $\mathfrak{F}_0(\eta_1, \eta_2)$  then  $A_1, A_2, A_3$  is a characteristic set of a prime p.i.  $\Sigma_3$ , and we continue as above.

All in all, we have a method for testing (29) to determine whether it is a characteristic set for a prime p.i. and for replacing  $\Lambda$  by a pair of systems with characteristic sets lower than (29) when the test is negative.<sup>21</sup>

In developing our method, we have recast the conditions of §§17, 18 and have secured the following theorem.

**THEOREM:** *A chain of polynomials of positive class fails to be a characteristic set of a prime p.i. if and only if there exist two nonzero polynomials, reduced with respect to the chain, whose product is in the p.i. generated by the chain.*

<sup>21</sup> If, when the indeterminates are  $u_1, \dots, u_q; y_1, \dots, y_p$ , (29) is a characteristic set for a prime p.i.  $\Omega$ , then, when we revert to the old notation, (29) will be a characteristic set for the prime p.i. into which  $\Omega$  goes.

31. Using now the old notation for the indeterminates, let us suppose that (29) has been found to be a characteristic set for a prime p.i.  $\Sigma$ . Then  $\Lambda$  is equivalent to <sup>22</sup>

$$(32) \quad \Sigma, \Lambda + I_1, \dots, \Lambda + I_p.$$

Each  $\Lambda + I_i$  has characteristic sets which are lower than (29).

What precedes shows that the system  $\Phi$  of §26 can be resolved into an equivalent set of prime p.i., as far as the determination of characteristic sets for the prime p.i. goes, by a finite number of rational operations and factorizations, if the same can be done for all finite systems whose characteristic sets are lower than those of  $\Phi$ . The final remark of I, §4, gives a quick abstract proof that the resolution is possible for  $\Phi$ . What is more, the processes used above, of reduction, factorization and isolation of prime p.i., give an algorithm for the reduction.

32. It remains to solve the following problem: Given a characteristic set

$$(33) \quad A_1, \dots, A_p$$

of a nontrivial prime p.i.  $\Sigma$  in  $\mathfrak{F}[y_1, \dots, y_n]$ , each  $A_i$  being of class  $q + i$  ( $p + q = n$ ), it is required to find a finite system of polynomials equivalent to  $\Sigma$ .<sup>23</sup>

33. Using indeterminates  $t_{ij}$ , we make the transformation

$$(34) \quad z_i = t_{i1}y_1 + \dots + t_{in}y_n, \quad i = 1, \dots, n.$$

For a zero of  $\Sigma$  in an extension  $\mathfrak{F}_1$  of  $\mathfrak{F}$ , (34) gives quantities  $z$  in the field obtained by adjoining the  $t$  to  $\mathfrak{F}_1$ . Given any  $q + 1$  of the  $z$

$$z_{i_1}, \dots, z_{i_{q+1}}$$

we find, by the method of elimination of II, §34, a nonzero polynomial in them and the  $t$  which vanishes when the  $z$  are replaced by their expressions in (34), with  $y_1, \dots, y_n$  a generic zero of  $\Sigma$ .

Let  $B$  be such a polynomial in  $z_1, \dots, z_{q+1}$  and the  $t$ . Let  $m$  be the degree of  $B$  considered as a polynomial in the  $z$ . We shall show how to obtain a relation  $C = 0$  among  $z_1, \dots, z_{q+1}$  and the <sup>24</sup>  $t$ , where  $C$  is of degree  $m$  as a polynomial in the  $z$  and, in addition, is of degree  $m$  in each  $z$  separately.

We make in  $B$  the transformation

$$(35) \quad z_i = a_{i1}z'_1 + \dots + a_{i, q+1}z'_{q+1}, \quad i = 1, \dots, q+1,$$

where the  $a$  and  $z'$  are indeterminates. Then  $B$  becomes a polynomial  $B'$  in the  $z'$  whose coefficients are polynomials in the  $t$  and the  $a$ . The degree of  $B'$

<sup>22</sup> Note that  $\Lambda$  is contained in  $\Sigma$  because the remainder of every polynomial in  $\Lambda$  with respect to (29) is zero. Every zero of (29) which annuls no initial is a zero of  $\Sigma$ .

<sup>23</sup>  $\Phi$  of §26 leads to several  $\Sigma$ . For each  $\Sigma$ , we reletter the indeterminates appropriately. After finite systems are found, equivalent to the various  $\Sigma$ , we revert to the original lettering.

<sup>24</sup> Satisfied when the  $y$  in (34) are a generic zero of  $\Sigma$ .

in each  $z'_i$  will be effectively  $m$ .<sup>25</sup> Furthermore, we can specialize the  $a$  as integers in such a way that the determinant  $|a_{ij}|$  is not zero and that the coefficient of the  $m$ th power of each  $z'_i$  in  $B'$  becomes a nonzero polynomial in the  $t$ . Let this be done and let  $B''$  be the polynomial in the  $z'$  and  $t$  into which  $B'$  thus goes.

The transformation (34), and (35) with the  $a$  as just fixed, give a transformation

$$(36) \quad z'_i = \tau_{i1}y_1 + \cdots + \tau_{in}y_n, \quad i = 1, \cdots, q + 1,$$

where each  $\tau$  is a linear combination, with rational coefficients, of the  $t_{ij}$  with  $i \leq q + 1$ . From (35), (36), we see that the  $t_{ij}$  with  $i \leq q + 1$  are linear in the  $\tau$  with integral coefficients.

In  $B''$ , we substitute for each  $t$  its expression in terms of the  $\tau$  and we regard the symbols  $\tau$  as indeterminates instead of linear combinations of the  $t$ . Then  $B''$  goes over into a polynomial  $B'''$  in the  $z'_i, \tau_{ij}, i = 1, \cdots, q + 1$ . We see that  $B'''$  vanishes identically in the  $\tau$  if we replace the  $z'$  by their expressions in (36), with the  $y$  a generic zero of  $\Sigma$ . We now replace, in  $B'''$ , each  $\tau_{ij}$  by  $t_{ij}$  and each  $z'_i$  by  $z_i$ . Then  $B'''$  goes over into a polynomial  $C$  in  $z_1, \cdots, z_{q+1}$  and the  $t, C$  being of degree  $m$  as a polynomial in the  $z$  and of degree  $m$  in each  $z$  separately.  $C$  vanishes for the  $z$  as in (34) with the  $y$  a generic zero of  $\Sigma$ .

Evidently the relation  $C = 0$  just described will subsist if we replace  $z_1, \cdots, z_{q+1}$  by any  $q + 1$  of the  $z_i$ , provided that a corresponding substitution is made for the  $t$  in  $C$ .

We now specialize the  $t$  in (34) as integers with a nonvanishing determinant, in such a way that, for every set of  $q + 1$  indeterminates  $z$ , the polynomial over  $\mathfrak{F}$  obtained from  $C$  remains of effective degree  $m$  in each  $z$  appearing in it.

34. We consider the transformation (34) with the  $t$  as just fixed. If the  $y$  are replaced in (33) in terms of the  $z$ , we get a system  $\Phi$  of  $p$  polynomials in the  $z$ . Let characteristic sets be determined for a set of prime p.i. equivalent to  $\Phi$ . Let  $\Sigma_1, \cdots, \Sigma_s$  be those prime p.i. which do not contain the initial of any  $A$  in (33), the  $y$  being replaced in the initials in terms of the  $z$ .<sup>26</sup> There will be one of the  $\Sigma_i$  which holds the remaining  $\Sigma_i$ . This is because, in a resolution of (33) into an equivalent set of prime p.i., none of which is a divisor of any other, there is precisely one p.i. which contains no initial.<sup>27</sup> To determine which  $\Sigma_i$  holds the others, all we need do is to find a  $\Sigma_i$  whose characteristic set holds the other  $\Sigma_i$ . Suppose, for instance, that the characteristic set of  $\Sigma_1$  holds  $\Sigma_2, \cdots, \Sigma_s$ . Then, if  $\Sigma_1$  does not hold  $\Sigma_j$ , the initial of some polynomial in the characteristic set of  $\Sigma_1$  must hold  $\Sigma_j$ . Then surely  $\Sigma_j$  cannot hold  $\Sigma_1$ . Thus, if  $\Sigma_1$  does not hold all  $\Sigma_i$ , no  $\Sigma_j$  can hold all  $\Sigma_i$ . Then  $\Sigma_1$  holds all  $\Sigma_i$ .

$\Sigma_1$  is obtained from  $\Sigma$  of §32 by replacing the  $y$  in terms of the  $z$ . We shall

<sup>25</sup> Perron, *Algebra*, vol. 1, p. 288.

<sup>26</sup> The condition for a polynomial to be contained in a prime p.i. is that its remainder with respect to the characteristic set vanish.

<sup>27</sup> This is seen from (32).

prove that  $\Sigma_1$  has the same dimension as  $\Sigma$ . To begin with, it is easy to see that the polynomials in any  $q + 1$  of the  $z$ , found in §33, belong to  $\Sigma_1$ . On the other hand, if there were fewer than  $q$  indeterminates in a parametric set of  $\Sigma_1$ , we could use a characteristic set of  $\Sigma_1$  to determine a nonzero polynomial in  $\mathfrak{F}[y_1, \dots, y_q]$  belonging to  $\Sigma$ .

Changing the notation if necessary, let  $z_1, \dots, z_q$  be a parametric set for  $\Sigma_1$ . Then  $\Sigma_1$  will have a characteristic set

$$(37) \quad B_1, \dots, B_p$$

in which  $B_i$  introduces  $z_{q+i}$ .

35. We construct a resolvent  $R = 0$  for  $\Sigma_1$ , with

$$(38) \quad w = a_1 z_{q+1} + \dots + a_p z_n,$$

the  $a$  being integers. Let  $R$  be of degree  $g$  in  $w$ .

We shall prove that the initial of  $R$  is an element of  $\mathfrak{F}$ . According to §33, each  $z_i$ ,  $i > q$ , in a zero of  $\Sigma_1$  satisfies with  $z_1, \dots, z_q$  a fixed equation of degree  $m$  in  $z_i$ , the coefficient of  $z_i^m$  being an element of  $\mathfrak{F}$ . The coefficient just mentioned will be assumed to be unity. Then (38) shows that  $w$  satisfies with  $z_1, \dots, z_q$  an equation in which the highest power of  $w$  is unity.<sup>28</sup> This implies that in the irreducible polynomial  $R$ , the coefficient of  $w^g$  is free of  $z_1, \dots, z_q$ . We may and shall assume that coefficient to be unity.

Referring to §25, we see that

$$(39) \quad z_i = \frac{E_{i0} + E_{i1} w + \dots + E_{i, g-1} w^{g-1}}{D},$$

$i = q + 1, \dots, n$ , where  $D$  and the  $E$  are in<sup>29</sup>  $\mathfrak{F}[z_1, \dots, z_q]$ .

36. Let  $t_1, \dots, t_p$ ;  $v$  be new indeterminates and let

$$\Lambda = (\Sigma_1, v - t_1 z_{q+1} - \dots - t_p z_n)_0$$

in  $\mathfrak{F}[z; t; v]$ . Then  $\Lambda$  is a prime p.i. Also  $\Lambda$  contains an irreducible polynomial  $U$  in  $v, z_1, \dots, z_q$  and the  $t$ , the coefficient of whose highest power of  $v$ , say  $v^d$ , is unity.<sup>30</sup>

We shall prove that  $d = g$ . We see first, following §25, that  $d \leq g$ . As  $v$ , in a zero of  $\Lambda$ , equals  $w$  if  $t_i = a_i$ ,  $i = 1, \dots, p$ , we cannot have  $d < g$ .<sup>31</sup>

Let  $v$  be replaced in  $U$  by

$$(40) \quad t_1 z_{q+1} + \dots + t_p z_n.$$

Then  $U$  becomes a polynomial  $V$  in  $z_1, \dots, z_n$  and the  $t$ . Let  $V$  be arranged as a polynomial in the  $t$  with coefficients which are polynomials in the  $z$ .

<sup>28</sup> This is analogous to the fact that the sum of several algebraic integers is an integer. See Landau, *Zahlentheorie*, vol. 3, p. 71.

<sup>29</sup> The relations (39) hold for any zero of  $\Sigma_1$  with  $D \neq 0$ , and for the corresponding  $w$ .

<sup>30</sup> Note that each  $t_i z_{q+i}$  satisfies an equation in which the coefficient of the highest power of  $t_i z_{q+i}$  is unity.

<sup>31</sup> As the coefficient of  $v^d$  in  $U$  is unity,  $U$  cannot vanish identically for  $t_i = a_i$ .

Let  $\Psi$  be the finite system of those coefficients (polynomials in the  $z$ ). We are going to prove, in the following sections, that  $\Psi$  is equivalent to  $\Sigma_1$ . Thus, if the  $z$  are replaced in  $\Psi$  by their expressions (34), we get a finite system of polynomials equivalent to  $\Sigma$ . We shall thus have solved the problem stated in §32.

37. We begin with the observation that for given elements  $z_1, \dots, z_n$  of an extension  $\mathfrak{F}_1$  of  $\mathfrak{F}$  to constitute a zero of  $\Psi$ , it is necessary and sufficient that for  $z_1, \dots, z_n$  as just given,  $V$  vanish for arbitrary  $t$  in<sup>32</sup>  $\mathfrak{F}_1$ . This shows, in particular, that  $\Psi$  holds  $\Sigma_1$ .

Let  $G$  be the discriminant of  $R$  with respect to  $w$  and let

$$H = DG$$

where  $D$  is as in (39). We shall prove that every zero of  $\Psi$  with  $H \neq 0$  is a zero of  $\Sigma_1$ . Let  $\eta_1, \dots, \eta_n$  be such a zero of  $\Psi$ . For  $z_i = \eta_i, i = 1, \dots, q, R$  becomes a polynomial  $T$  in  $w$ . From §21, we see that  $T$  has  $g$  zeros in some extension of  $\mathfrak{F}(\eta_1, \dots, \eta_q)$ . These zeros are distinct, because  $\eta_1, \dots, \eta_q$  do not annul  $G$ . Using each such  $w$  in (39), we get  $g$  distinct zeros,

$$\eta_1, \dots, \eta_q; \quad z_{q+1}^{(j)}, \dots, z_n^{(j)}, \quad j = 1, \dots, g,$$

of  $\Sigma_1$ . Let  $Z$  be the polynomial which  $U$  becomes for  $z_i = \eta_i, i = 1, \dots, q$ . Then<sup>33</sup>

$$(41) \quad Z = \prod_{j=1}^g (v - t_1 z_{q+1}^{(j)} - \dots - t_p z_n^{(j)}).$$

But  $v - t_1 \eta_{q+1} - \dots - t_p \eta_n$  is a factor of  $Z$ . This shows that, for some  $j, z_i^{(j)} = \eta_i, i = q+1, \dots, n$ , and proves our statement.

38. We have to show that a zero  $\eta_1, \dots, \eta_n$  of  $\Psi$  which annuls  $H$  is a zero of  $\Sigma_1$ . Our proof will employ a Newton polygon process, which we can carry out rapidly by using the material of Chapter III.

For  $z_i = \eta_i, i = 1, \dots, q, R$  becomes a polynomial  $J$  in  $w$ . In some extension  $\mathfrak{F}_1$  of  $\mathfrak{F}(\eta_1, \dots, \eta_q), J$  has  $g$  linear factors. We write

$$J = (w - \xi_1) \dots (w - \xi_g).$$

Now let  $b_1, \dots, b_q$  be integers such that

$$H(\eta_1 + b_1, \dots, \eta_q + b_q) \neq 0.$$

Then, if  $c$  is an indeterminate,

$$(42) \quad H(\eta_1 + b_1 c, \dots, \eta_q + b_q c)$$

is a polynomial in  $c$  which is not identically zero. We put in  $R,$

<sup>32</sup> This means that  $V$  vanishes identically in the  $t$ .

<sup>33</sup> Note that  $Z$  is a polynomial in  $v$  and the  $t$  which vanishes for  $v = t_1 z_{q+1}^{(j)} + \dots + t_p z_n^{(j)}$ . We have thus  $g$  distinct factors of  $Z$ . As  $Z$  is of degree  $g$  in  $v$ , with unity for the coefficient of  $v^g$ , it has the expression in (41).

$$z_i = \eta_i + b_i c, \quad i = 1, \dots, g.$$

Then  $R$  goes over into a polynomial  $K$  in  $w$  whose coefficients are polynomials in  $c$ . In  $K$ , we put  $w = \xi_1 + w_1$ . Then  $K$  becomes an expression  $K'$  in  $w_1$  and  $c$  which we write

$$(43) \quad K' = a'(c) + \sum_{i=1}^g b'_i(c) w_1^i.$$

We shall now regard  $\mathfrak{F}_1$  as a differential field in which every derivative is zero. Furthermore, we regard  $w_1$  as a differential indeterminate and  $c$  as an arbitrary constant. We wish to show that  $K'$  in (43) is annulled either by  $w_1 = 0$  or by a series

$$(44) \quad w_1 = \varphi_2 c^{\rho_2} + \dots + \varphi_k c^{\rho_k} + \dots$$

similar to the series employed in Chapter III, with the distinction that  $\rho_2$ , while positive, need not exceed unity.

It may be that  $K'$  is annulled by  $w_1 = 0$ . Let us suppose that this does not happen. Then  $a'(c)$  is not zero. We compare (43) with (3) of III, §7. The role of  $U'_i$  is taken over by  $w_1^i$ . Because  $K'$  vanishes when  $w_1$  and  $c$  are replaced by zero, the lowest exponent of  $c$  in  $a'$  is positive. Again, the only exponent of  $c$  in  $b'_g$  is zero. Thus  $\rho_2$  of III, §7, will be positive. Without further change, the work of Chapter III furnishes the series in (44).

Let

$$\alpha_1 = \xi_1 + \varphi_2 c^{\rho_2} + \dots + \varphi_k c^{\rho_k} + \dots.$$

We have

$$K = (w - \alpha_1) K_1$$

where  $K_1$  is a polynomial in  $w$  of degree  $g - 1$ , whose coefficients are series in  $c$ . The terms free of  $c$  in  $K_1$  are annulled by  $w = \xi_2$ . When  $w$  is replaced by  $\xi_2 + w_1$ ,  $K_1$  goes over into an expression  $K'$  like that in (43) except that the  $a'$  and  $b'$  are infinite series of fractional powers instead of polynomials. We secure a series like (44) which annuls the  $K'$  with which we are now working.

All in all, we have a representation of  $K$

$$K = (w - \alpha_1) \dots (w - \alpha_g)$$

where each  $\alpha_i$  is a series of the type

$$(45) \quad \alpha_i = \xi_i + \varphi_2 c^{\rho_2} + \dots.$$

The  $\rho$  and the  $\varphi$  depend on  $i$ . If we replace  $c$  by a suitable positive integral power  $h^r$  of an indeterminate  $h$ , we have, for  $i = 1, \dots, g$ ,

$$(46) \quad \alpha_i = \xi_i + \psi_{i1} h + \psi_{i2} h^2 + \dots.$$

The  $\psi$  all lie in some extension of  $\mathfrak{F}_1$ .

From this point on, we regard our fields as algebraic fields and  $h$  as an algebraic indeterminate.

The  $\alpha$  are distinct, since  $G$  does not vanish for

$$(47) \quad z_i = \eta_i + b_i h^r, \quad i = 1, \dots, q.$$

We use (39), understanding that (47) holds and that  $w = \alpha_i$ . We secure  $g$  distinct zeros of  $\Sigma_1$ ,

$$\eta_1 + b_1 h^r, \dots, \eta_q + b_q h^r; \quad z_{q+1}^{(j)}, \dots, z_n^{(j)},$$

$j = 1, \dots, g$ . Each  $z_i^{(j)}$  is a series of integral powers of  $h$ . Such a series can contain no negative power of  $h$ . This follows from the fact that  $\Sigma_1$  contains a polynomial in  $z_1, \dots, z_q, z_i$  in which one of the terms of highest degree is a term in  $z_i$  alone (§33).

Let  $\zeta_i^{(j)}$  be the term of  $z_i^{(j)}$  which is of zero degree in  $h$ . Then, for every  $j$ ,

$$\eta_1, \dots, \eta_q; \quad \zeta_{q+1}^{(j)}, \dots, \zeta_n^{(j)}$$

is a zero of  $\Sigma_1$ .

Let  $Z_h$  be the polynomial in  $v$  and  $t_1, \dots, t_p$  which  $U$  of §36 becomes for (47). Then

$$Z_h = \prod_{j=1}^g (v - t_1 z_{q+1}^{(j)} - \dots - t_p z_n^{(j)}).$$

Letting  $Z_0$  represent  $Z_h$  with  $h = 0$ , we have

$$Z_0 = \prod_{j=1}^g (v - t_1 \zeta_{q+1}^{(j)} - \dots - t_p \zeta_n^{(j)}).$$

Now  $v - t_1 \eta_{q+1} - \dots - t_p \eta_n$  is a factor of  $Z_0$ . This shows that  $\eta_{q+1}, \dots, \eta_n$  are the  $\zeta^{(j)}$  for some  $j$ , so that, as we undertook to prove,  $\eta_1, \dots, \eta_n$  is a zero of  $\Sigma_1$ .

We have thus proved that  $\Psi$  is equivalent to  $\Sigma_1$ .

### AN APPROXIMATION THEOREM

39. Working in the analytic case, we prove the following theorem.

**THEOREM:** *Let  $\Sigma$  be a prime p.i. in  $y_1, \dots, y_n$ . Let  $B$  be any polynomial not contained in  $\Sigma$ . Given any zero of  $\Sigma$ , consisting of functions analytic in an open region  $\mathbf{B}$ , there is an open region  $\mathbf{C}$ , contained in  $\mathbf{B}$ , in which the given zero can be approximated uniformly, with arbitrary closeness, by zeros of  $\Sigma$  for which  $B$  is distinct from zero throughout  $\mathbf{C}$ .*

We assume, as we may, that  $\Sigma$  is nontrivial. If the transformation of §33 is effected,  $\Sigma$  may be replaced by  $\Sigma_1$ , while  $B$  goes over into a polynomial  $B_1$  in  $z_1, \dots, z_n$ .

$B_1$  is not in  $\Sigma_1$ . Let  $z_{q+1}, \dots, z_n$  be replaced in  $B_1$  by their expressions (39). We find that, for every zero of  $\Sigma_1$  with  $D \neq 0$ ,

$$(48) \quad B_1 = \frac{M}{D^\mu}$$

where  $M$  is a polynomial in  $w, z_1, \dots, z_q$ . Because  $DB_1$  is not in  $\Sigma_1$ ,  $M$  is not divisible by  $R$  of §35. Thus we have

$$XR + YM = N$$

where  $N$  is a nonzero polynomial in  $z_1, \dots, z_q$ . A zero of  $\Sigma_1$  which annuls  $B_1$  annuls  $N$ .

Let  $\eta_1, \dots, \eta_n$  be a zero of  $\Sigma_1$ , analytic in an open region  $\mathbf{B}$ , which annuls  $N$ .

Shrinking  $\mathbf{B}$  if necessary, we assume that every one of the polynomials in  $w$  and the  $z$  which we meet in what follows has its coefficients analytic throughout  $\mathbf{B}$ .

Let  $H_1 = NH$ . We use constants  $b_i$  such that

$$H_1(\eta_1 + b_1, \dots, \eta_q + b_q)$$

does not vanish for every  $x$ . Then, if  $h$  is a complex variable,

$$(49) \quad H_1(\eta_1 + b_1h, \dots, \eta_q + b_qh)$$

is a polynomial in  $h$  of the type

$$(50) \quad \alpha_r h^r + \dots + \alpha_s h^s$$

where the  $\alpha$  are functions of  $x$  analytic in  $\mathbf{B}$ . As  $H_1$  in (49) vanishes for  $h = 0$ , we have  $r > 0$ . We assume that  $\alpha_r$  is not identically zero.

Let  $\mathbf{B}_1$  be a simply connected open region contained within its boundary in  $\mathbf{B}$ , in which  $\alpha_r$  is bounded away from zero. Let  $h$  be small but distinct from zero. Then (50) cannot be zero at any point of  $\mathbf{B}_1$ . Thus, if

$$(51) \quad z_i = \eta_i + b_i h, \quad i = 1, \dots, q,$$

$R = 0$  will have  $g$  distinct solutions for  $w$ , each analytic in  $\mathbf{B}_1$ . This is because  $H_1$  is divisible by the discriminant of  $R$ .

As  $H_1$  is divisible by  $D$  in (39),  $\Sigma_1$  will have  $g$  distinct zeros with  $z_1, \dots, z_q$  as in (51),

$$z_1, \dots, z_q; \quad z_{q+1}^{(k)}, \dots, z_n^{(k)}, \quad k = 1, \dots, g,$$

each consisting of functions analytic in  $\mathbf{B}_1$ . The  $z^{(k)}$  are given by (39).

Consider a sequence of nonzero values of  $h$  which tend towards zero,

$$(52) \quad h_1, h_2, \dots, h_j, \dots,$$

each  $h_j$  being so small that (50) is distinct from zero throughout  $\mathbf{B}_1$ . For each  $j$ , if

$$(53) \quad z_i = \eta_i + b_i h_j, \quad i = 1, \dots, q,$$

$U$  of §36 will vanish if

$$(54) \quad v = t_1 z_{q+1}^{(k)} + \cdots + t_p z_n^{(k)},$$

$k = 1, \dots, g$ . It is understood, of course, that the  $z^{(k)}$ , which are analytic throughout  $B_1$ , depend on  $h_j$ . For any  $h_j$ , the  $g$  expressions (54) are distinct.

As the equation of degree  $m$  which a  $z_j, j > q$ , satisfies with  $z_1, \dots, z_q$  has unity for the coefficient of  $z_j^m$ , there is a positive number  $d$ , such that, throughout  $B_1$ ,

$$(55) \quad |z_j^{(k)}| < d$$

for  $j = q + 1, \dots, n; k = 1, \dots, g$  and for every  $h$  in (52). This is because the coefficients of  $z_j^{m-1}, \dots, z_j^0$  in the above mentioned equation are bounded quantities.

For each  $h_j$  of (52), let one of the  $g$  expressions (54) be selected, and be designated by  $v^{(i)}$ . We form thus a sequence

$$(56) \quad v', v'', \dots, v^{(i)}, \dots$$

Let  $C$  be any open region which lies with its boundary in  $B_1$ . From (56) we see, using a well known theorem on bounded families of analytic functions,<sup>34</sup> that, for some subsequence of (56), the coefficients of each  $t_i, i = 1, \dots, p$ , converge uniformly throughout  $C$  to an analytic function  $\eta'_i$ . We find thus that if

$$(57) \quad z_i = \eta_i, \quad i = 1, \dots, q,$$

$U$  vanishes for

$$v = t_1 \eta'_{q+1} + \cdots + t_p \eta'_n.$$

Deleting elements of (56) if necessary, we assume that the convergence occurs when the complete sequence (56) is used, rather than one of its subsequences. For each  $h_j$ , there are  $g - 1$  expressions (54) not used in (56). Let one of these be selected for each  $h_j$  and let (56) be used now to represent the sequence thus obtained. As above, we select a subsequence of (56) for which the coefficients of each  $t_i$  converge uniformly in  $C$ . This gives a second expression which causes  $U$  to vanish when (57) holds. Continuing, we find  $g$  expressions

$$(58) \quad v = t_1 \eta_q^{(k)} + \cdots + t_p \eta_n^{(k)}, \quad k = 1, \dots, g,$$

which make  $U$  vanish when (57) holds.

Let  $v_k$  represent the second member of (58). Again, let  $w_k$  represent the second member of (54), it being understood that the subscripts  $k$  are assigned, for each  $h_j$ , in such a way that the coefficient of  $t_i$  in  $w_k$  converges to that in  $v_k$  as  $h_j$  approaches zero.

Then, since the  $g$  expressions  $w_k$  are distinct from one another for each  $h_j$ , we will have, representing by  $Z_j$  the polynomial which  $U$  becomes when (53) holds,

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<sup>34</sup> Montel, *Les familles normales de fonctions analytiques*, p. 21; Dienes, *The Taylor Series*, p. 160.

$$Z_j = (v - w_1) \cdots (v - w_q).$$

By continuity, if we represent  $U$ , when (57) holds, by  $Z$ ,

$$Z = (v - v_1) \cdots (v - v_q).$$

As  $v - t_1\eta_{q+1} - \cdots - t_p\eta_n$  is a factor of  $Z$ , it must be that, for some  $k$ ,

$$\eta_i = \eta_i^{(k)}, \quad i = q + 1, \cdots, n.$$

Thus  $\eta_1, \cdots, \eta_n$  can be approximated uniformly in  $\mathbf{C}$ , with arbitrary closeness, by zeros of  $\Sigma_1$  for which  $B_1$  is distinct from zero throughout  $\mathbf{C}$ . As the  $y$  vary continuously with the  $z$ , we have our theorem.

#### ZEROS AND CHARACTERISTIC SETS

40. We consider a nontrivial prime p.i.  $\Sigma$  in  $\mathfrak{F}[u_1, \cdots, u_q; y_1, \cdots, y_p]$  with the  $u$  a parametric set. Let

$$(59) \quad A_1, \cdots, A_p$$

be a characteristic set for  $\Sigma$ . We know that every zero of (59) for which no initial vanishes is a zero of  $\Sigma$ . We shall prove that *every zero of (59) for which no separant vanishes is a zero of  $\Sigma$* .

Let  $\eta_1, \cdots, \eta_n$  be a zero of (59) which annuls no separant.

In  $A_1$ , we replace  $u_i$  by  $\eta_i + \tau_i$ ,  $i = 1, \cdots, q$ , where the  $\tau$  are indeterminates, and  $y_1$  by  $\eta_{q+1} + y'_1$ . Then  $A_1$  goes over into a polynomial  $B_1$  in  $y'_1$  and the  $\tau$  which vanishes when the indeterminates are all replaced by zero. Because the separant of  $A_1$  does not vanish for the  $\eta$ ,  $B_1$  contains a term  $\alpha y'_1$  with  $\alpha$  in  $\mathfrak{F}(\eta_1, \cdots, \eta_n)$  and distinct from zero. We solve the equation  $B_1 = 0$  for  $y'_1$  in terms of the  $\tau$ , using the formal process of the implicit function theorem for securing a representation of  $y'_1$  as an infinite series of powers of the  $\tau$ . We can do this because of the presence of  $\alpha y'_1$ . Let  $\xi_1$  be the series thus obtained for  $y'_1$ . The terms of  $\xi_1$  are all of positive degree.

The set

$$(60) \quad \eta_1 + \tau_1, \cdots, \eta_q + \tau_q; \quad \eta_{q+1} + \xi_1$$

is a generic zero of the prime p.i. in  $y_1$  and the  $u$  for which  $A_1$  is a characteristic set.

We substitute the quantities (60) into  $A_2$  and replace  $y_2$  by  $\eta_{q+2} + y'_2$ . Then  $A_2$  goes over into a polynomial  $B_2$  in  $y'_2$ . The coefficients in  $B_2$  are series of non-negative powers of the  $\tau$  and the coefficient of  $y'_2$  contains a term free of the  $\tau$ . We can thus solve  $B_2 = 0$  for  $y'_2$ , expressing  $y'_2$  as a series  $\xi_2$  of powers of  $\tau$ , the terms of  $\xi_2$  being of positive degree. By §29,

$$\eta_1 + \tau_1, \cdots, \quad \eta_q + \tau_q; \quad \eta_{q+1} + \xi_1, \quad \eta_{q+2} + \xi_2$$

is a generic zero of the prime p.i.  $\Sigma_2$  for which  $A_1, A_2$  is a characteristic set. It follows that  $\eta_1, \cdots, \eta_{q+2}$  is a zero of  $\Sigma_2$ . Continuing, we find that  $\eta_1, \cdots, \eta_n$  is a zero of  $\Sigma$ .

CHAPTER V  
CONSTRUCTIVE METHODS

CHARACTERISTIC SETS OF PRIME IDEALS

1. We return to differential fields and to differential polynomials. Let

$$(1) \quad A_1, \dots, A_p$$

be a chain in  $\mathfrak{F}\{y_1, \dots, y_n\}$ ,  $A_i$  being of positive class  $j_i$ . We are going to find a necessary and sufficient condition for (1) to be a characteristic set of a prime ideal.

Let the order of  $A_i$  in  $y_{j_i}$  be  $r_i$ . We represent each  $y_{j_i r_i}$  by  $z_i$ . The remaining  $y_{im}$  in (1) we designate now by new symbols<sup>1</sup>  $v_k$ , attributing the subscripts  $k$  in any convenient way. With these replacements, (1) goes over into a chain of polynomials

$$(2) \quad B_1, \dots, B_p$$

in algebraic indeterminates

$$(3) \quad v_1, \dots, v_r; \quad z_1, \dots, z_p.$$

The passage from (1) to (2) is purely formal. Once it is effected, we treat (2) as we would any other set of polynomials in the  $v$  and  $z$ . For the  $B$ , the basic differential field  $\mathfrak{F}$  is regarded as an algebraic field, its operation of differentiation being suppressed. Again, whereas in a zero of (1)  $y_{i, m+1}$  must be the derivative of  $y_{im}$ , any set of  $v, z$  which lie in an algebraic field containing the elements of  $\mathfrak{F}$ , and which annul the  $B$ , is a zero of (2).

We are going to prove that *for (1) to be a characteristic set of a prime ideal, it is necessary and sufficient that (2) be a characteristic set of a prime p.i.<sup>2</sup> in the indeterminates (3).*

2. We prove first the necessity. Suppose that (2) is not a characteristic set of a prime p.i. We refer to IV, §30. There are polynomials  $M$  and  $N$ , reduced with respect to (2), such that  $MN$  is in  $(B_1, \dots, B_p)_0$ . When we replace the  $v$  and  $z$  by the  $y_{im}$ ,  $M$  and  $N$  become, respectively, d.p.  $P$  and  $Q$ , reduced with respect to (1), such that  $PQ$  is in  $(A_1, \dots, A_p)$ . If (1) were a characteristic set of a prime ideal  $\Sigma$ , then  $PQ$ , but neither  $P$  nor  $Q$ , would be in  $\Sigma$ . The necessity is proved.

3. We now prove sufficiency. Let (2) be a characteristic set of a prime p.i. Let  $\Sigma$  be the totality of those d.p.  $G$  for which there exists a power product  $J$  of the separants and initials of the  $A$ , depending on  $G$ , such that

<sup>1</sup> We use only letters effectively present in (1).

<sup>2</sup> As has been indicated, the algebraic field used for (2) is the set of elements of  $\mathfrak{F}$ .

$$JG \equiv 0, \quad [A_1, \dots, A_p].$$

We see from Chapter I that  $\Sigma$  is an ideal. We shall prove that  $\Sigma$  is prime and that (1) is a characteristic set of  $\Sigma$ .

Suppose that  $\Sigma$  contains a d.p.  $PQ$  but neither  $P$  nor  $Q$ . Let  $R$  and  $T$  be, respectively, the remainders of  $P$  and  $Q$  with respect to (1). Then  $\Sigma$  contains  $RT$  but neither  $R$  nor  $T$ .

In what follows, every  $J_i$  will be a power product of the separants and initials of the  $A$ . Some  $J_1RT$  has an expression linear in the  $A$  and their derivatives. Let  $A_p^{(k)}$  be the highest derivative of  $A_p$  in this expression. Suppose that  $k > 0$ . Then

$$A_p^{(k)} = S_p y_{j_p, r_p + k} + U$$

with  $S_p$  the separant of  $A_p$  and  $U$  of order lower than  $r_p + k$  in  $y_{j_p}$ . In the expression for  $J_1RT$ , we replace  $y_{j_p, r_p + k}$  by  $-U/S_p$ . Clearing fractions, we have an expression for some  $J_2RT$  which is free of  $A_p^{(k)}$ . Continuing, we find a  $J_iRT$  which is linear in the  $A$  in (1).

$R$  and  $T$  may contain  $y_{im}$  not effectively present in (1). If so, we adjoin corresponding letters  $v$  to (3). The set (2) will be a characteristic set of a prime p.i. for the enlarged system (3). The prime p.i. just considered will be called  $\Sigma_0$ . Let  $J_i$ ,  $R$ , and  $T$  be regarded as polynomials in the  $v$  and  $z$ . They are not in  $\Sigma_0$ ; neither is their product. Hence the d.p.  $J_iRT$  cannot be linear in the  $A$ . We know thus that  $\Sigma$  is prime.

We have just seen that if each of two nonzero d.p. is reduced with respect to (1), their product is not in  $\Sigma$ . Taking one of the d.p. as unity, we see that  $\Sigma$  contains no nonzero d.p. reduced with respect to (1). Thus (1) is a characteristic set of  $\Sigma$  and the sufficiency proof is completed.

4. We shall prove that *if (1) is a characteristic set of a prime ideal  $\Sigma$ , every zero of (1) for which no separant vanishes is a zero of  $\Sigma$ .*

Let  $G$  be any d.p. in  $\Sigma$ . Proceeding as in I, §6, we can find a power product  $J$  of the separants of the  $A$  such that

$$JG \equiv H, \quad [A_1, \dots, A_p],$$

where  $H$  is of order not more than  $r_i$  in  $y_{j_i}$ ,  $i = 1, \dots, p$ . As  $H$  is in  $\Sigma$ , its remainder with respect to (1) is zero. Hence there is a power product  $J_1$  of the initials of the  $A$  such that

$$J_1H \equiv 0, \quad (A_1, \dots, A_p).$$

We suppose, enlarging (3) if necessary, that every letter in  $H$  has a corresponding letter in (3). Let  $H$  be regarded as a polynomial in the  $z$  and  $v$ . As  $H$  is in the prime p.i. for which (2) is a characteristic set,  $H$  vanishes for all zeros of (2) which annul no separant (IV, §40). Then  $H$  as a d.p. vanishes for all zeros of (1) which annul no separant; the same is true of  $G$ . This proves our statement.

## FINITE SYSTEMS

5. Let  $\Phi$  be any finite system of d.p. in  $\mathcal{F}\{y_1, \dots, y_n\}$ , not all zero. We shall show now how to determine characteristic sets for a finite set of prime ideals equivalent<sup>3</sup> to  $\Phi$ . In §28, we shall give a theoretical process for determining finite sets whose manifolds are the components of  $\Phi$ .

Let (1) be a characteristic set of  $\Phi$ . If  $A_1$  is of class zero,  $\Phi$  is equivalent to the unit ideal. We suppose now that  $A_1$  is of positive class. For every d.p. in  $\Phi$ , let the remainder with respect to (1) be determined. If these remainders are adjoined to  $\Phi$ , we get a system equivalent to  $\Phi$ . If the remainders are not all zero, the new system will have characteristic sets lower than (1). After a finite number of repetitions of the above operation, we arrive at a system  $\Lambda$ , equivalent to  $\Phi$ , with a characteristic set (1) for which either  $A_1$  is of class zero or for which, otherwise, the remainder of every d.p. in  $\Lambda$  is zero.

Let us suppose that we are in the latter case. We determine, by §1, whether (1) is a characteristic set of a prime ideal. If it is not, we see from §1 that  $\Lambda$  is equivalent to  $\Lambda + P$ ,  $\Lambda + Q$ , where  $P$  and  $Q$ , reduced with respect to (1), can be obtained by calculation. Each of  $\Lambda + P$ ,  $\Lambda + Q$  will have characteristic sets lower than (1).

Let us suppose that (1) has been found to be a characteristic set for a prime ideal  $\Sigma$ . Then, by §4,  $\Lambda$  is equivalent to

$$(4) \quad \Sigma, \Lambda + S_1, \dots, \Lambda + S_p$$

where the  $S$  are the separants for (1). Each  $\Lambda + S_i$  has a characteristic set lower than (1).

What precedes shows that the given system  $\Phi$  can be resolved into an equivalent set of prime ideals, as far as the determination of characteristic sets for the ideals goes, by a finite number of rational operations, differentiations and factorizations, provided that the same can be done for all finite systems whose characteristic sets are lower than those of  $\Phi$ . The final remark of I, §4, gives an abstract proof that the resolution is possible for  $\Phi$ . What is more, the processes used above give an algorithm for the resolution.

*In the analytic case, the algorithm obtained above contains a complete elimination theory for systems of algebraic differential equations.* We get all of the zeros of  $\Phi$  by finding, for each characteristic set, those zeros which cause no separant to vanish. A zero of a prime ideal which causes some separant to vanish will be a zero of some system like the  $\Lambda + S_i$  above, and will thus be found among the zeros of some other prime ideal, where it annuls no separant. Thus our algorithm reduces the problem of determining all solutions of a system of algebraic differential equations to a question of applying the implicit function theorem and the existence theorem for systems of differential equations.

One sees, on the basis of the algorithm obtained above, that a system of d.p. in

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<sup>3</sup> We use this term as in Chapter IV.

$\mathfrak{F}\{y_1, \dots, y_n\}$  in which each d.p. is linear in the  $y_i$ ; has a manifold which is irreducible.

6. The work of §§1-5 furnishes a new proof of the fact that the manifold of a finite system of d.p. is composed of a finite number of irreducible manifolds.<sup>4</sup> The new proof does not depend on Zermelo's axiom.

#### TEST FOR A D.P. TO HOLD A FINITE SYSTEM

7. Let  $\Phi$  be any finite system of d.p. Let it be required to determine whether a given d.p.  $G$  holds  $\Phi$ . What one does is to resolve  $\Phi$  into prime ideals as in §5. For  $G$  to hold  $\Phi$ , it is necessary and sufficient that  $G$  hold each prime ideal. The condition for  $G$  to hold one of the prime ideals is that its remainder with respect to the characteristic set of the prime ideal be zero. This gives a test which involves a finite number of steps.

#### CONSTRUCTION OF RESOLVENTS

8. Let

$$(5) \quad A_1, \dots, A_p$$

be given as a characteristic set of a prime ideal  $\Sigma$  in  $\mathfrak{F}\{u_1, \dots, u_q; y_1, \dots, y_p\}$ ,  $A_i$  introducing  $y_i$ . We suppose that either  $\mathfrak{F}$  does not consist purely of constants or  $u$  actually exist.

We shall show how to construct a resolvent for  $\Sigma$ .

We begin by showing how to obtain the d.p.  $G$  of II, §23. Let  $B_i$  be the d.p. obtained from  $A_i$  by replacing each  $y_j$  by a new indeterminate  $z_j$ . We consider the finite system  $\Lambda$  composed of the d.p. in (5), the d.p.

$$B_1, \dots, B_p$$

and also

$$(6) \quad \lambda_1(y_1 - z_1) + \dots + \lambda_p(y_p - z_p)$$

where the  $\lambda$  are indeterminates. We take the indeterminates in the order  $u$ ;  $\lambda$ ;  $y$ ;  $z$ . We apply the process of §5 for resolving  $\Lambda$  into prime ideals, each prime ideal being represented by a characteristic set. The theory of II, §§23, 24, shows that each prime ideal which is not held by every  $y_i - z_i$  has a characteristic set containing a d.p. in the  $u$  and  $\lambda$  alone. We obtain, by a multiplication of such d.p., the d.p.  $K$  of II, §§23, 24.

When  $\mathfrak{F}$  contains a nonconstant element, the determination of  $\mu$  which do not annul  $K$  of II, §23, is an elementary problem whose solution is sufficiently indicated in II, §22. When  $u$  exist, we find the  $M$  of II, §24, by inspection.

Let us limit ourselves now to the case in which  $\mathfrak{F}$  does not consist of constants. Consider the system

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<sup>4</sup> This proof, like that of §§26, 27 below, does not use the basis theorem of I, §12. It is constructive to the extent that it produces characteristic sets for the associated prime ideals.

$$(7) \quad A_1, \dots, A_p, \quad w - (\mu_1 y_1 + \dots + \mu_p y_p)$$

in  $\mathfrak{F} \{ u; w; y \}$ .

The totality of d.p. which vanish for all zeros of (7) which annul no separant is the system  $\Omega$  of II, §26. The manifold of  $\Omega$  is a component of (7) and every other component is held by some separant.

We apply the process of §5 to resolve (7) into prime ideals. We test these prime ideals to see whether they are held by the separant of some  $A_i$ , and pick out those, say  $\Sigma_1, \dots, \Sigma_s$ , which are held by no separant.

As (7) has only one component which is held by no separant, there must be one  $\Sigma_i$  which holds all other  $\Sigma_i$ . To find such a  $\Sigma_i$ , we need only find a  $\Sigma_i$  whose characteristic set holds all other  $\Sigma_i$ . For, let the set for  $\Sigma_1$  hold  $\Sigma_2, \dots, \Sigma_s$ . If  $\Sigma_1$  does not hold  $\Sigma_j$ , the separant of some d.p. in the set for  $\Sigma_1$  must hold  $\Sigma_j$ , so that  $\Sigma_j$  cannot hold  $\Sigma_1$ . Thus, if  $\Sigma_1$  does not hold every  $\Sigma_i$ , no  $\Sigma_j$  can hold every  $\Sigma_i$ .

$\Sigma_1$  is  $\Omega$ .  $\Sigma_1$  has a characteristic set

$$R, R_1, \dots, R_p$$

in which  $R$  is an algebraically irreducible d.p. Then  $R = 0$  is a resolvent of  $\Sigma$  and each  $R_i$  is linear in  $y_i$ .

CONSTRUCTIVE PROOF OF THEOREM OF ZEROS

9. The theorem of zeros states that if  $G$  holds a finite system  $\Phi$ , some power of  $G$  is in  $[\Phi]$ . Richard Cohn<sup>5</sup> has given a proof of the theorem of zeros which provides a method for expressing a power of  $G$  as a linear combination of the d.p. in  $\Phi$  and their derivatives.

First, let  $\Phi$  have no zeros. We shall show constructively that unity is in  $[\Phi]$ . We obtain the system  $\Lambda$  of §5.  $\Lambda$  is contained in  $[\Phi]$ . If  $\Lambda$  contains a nonzero element of class zero, we have the desired expression for unity. Suppose that  $\Lambda$  contains no such element. Let (1) be a characteristic set for  $\Lambda$ . Then (1) is not a characteristic set of a prime ideal. By §1 and by IV, §30, there exist nonzero d.p.  $P$  and  $Q$ , reduced with respect to (1), such that  $PQ \equiv 0 (A_1, \dots, A_p)$ . Neither of the systems  $\Lambda + P$ ,  $\Lambda + Q$  has a zero. Suppose that we are able to obtain relations

$$(8) \quad 1 = M_0 P + M_1 P' + \dots + M_p P^{(p)} + C,$$

$$(9) \quad 1 = N_0 Q + N_1 Q' + \dots + N_p Q^{(p)} + D,$$

where superscripts indicate differentiation and where  $C$  and  $D$  are in  $[\Lambda]$ . If we multiply (8) and (9), we secure a relation

$$(10) \quad 1 = \sum L_i P^{(i)} Q^{(i)} + K$$

with  $K$  in  $[\Lambda]$ . We know from Chapter I how to find a power of any  $P^{(i)} Q^{(i)}$

<sup>5</sup> Cohn, 1.

which is in  $[PQ]$ . Thus, if we raise the second member of (10) to a sufficiently high power, we have a representation of unity as an element of  $[\Phi]$ .

Our problem becomes that of finding expressions for unity in  $[\Lambda + P]$  and  $[\Lambda + Q]$ . This is the familiar situation of systems with characteristic sets lower than that of  $\Phi$ ; one knows how to proceed.

We take now the general case. Let  $z$  be a new indeterminate.<sup>6</sup> The system

$$(11) \quad zG - 1, \quad \Phi$$

has no zeros. Let unity be expressed linearly in the d.p. of (11) and their derivatives. In this expression, let  $z$  be replaced by  $1/G$ . When we clear fractions, we have a power of  $G$  expressed linearly in the d.p. of  $\Phi$  and their derivatives.

#### A SECOND THEORY OF ELIMINATION

10. The theory of elimination for systems of algebraic differential equations given in what precedes is apparently the first accurate such theory ever to have been presented. There exist, in the treatises on differential equations, discussions of the elimination problem for systems of  $n$  equations in  $n$  unknowns, which start from the fact that a general system can be replaced by a system involving only first derivatives.<sup>7</sup> The unsoundness of these discussions is reflected in the reductions to normal form which they claim to effect. The representations at which they arrive are entirely unsuitable for general systems.

We shall develop, in what follows, using the principle of passing to a system involving only first derivatives, a second elimination theory for systems of equations which are algebraic in the unknowns and their derivatives. This treatment of the elimination problem may be regarded, more or less, as a rigorization, for the case of algebraic differential equations, of the discussions in the older literature.

The second elimination theory has the disadvantage, as compared with that given above, of concealing the unknowns present in a given system of equations among new unknowns, which are introduced to reduce the given equations to the first order. The first elimination theory is thus more useful for certain applications.

On the other hand, the second elimination leads, in a natural way, to theorems on the number of arbitrary constants in the solution of a system of algebraic differential equations. This subject, which is really the subject of the order of an irreducible algebraic differential manifold, will be investigated in Chapter VII.

11. We consider<sup>a</sup> a finite system  $\Phi$  of  $\tilde{n}$  nonzero d.p. in  $\mathcal{F}\{y_1, \dots, y_n\}$ , each d.p. being of order not exceeding unity in each  $y$ . We shall show how to obtain

<sup>6</sup> The method used below is that given by Rabinovitch for the proof of Hilbert's theorem of zeros.

<sup>7</sup> See, for instance, Jordan, *Cours d'analyse*, vol. 3, §3, or Forsythe, *Differential Equations*, vol. 2, Chapter I.

the manifold of  $\Phi$ , if  $\Phi$  has zeros, by solving a set of systems of differential equations, each system being essentially in the Jacobi-Weierstrass normal form.<sup>8</sup>

12. Let  $u_1, \dots, u_m$  be those  $y$  whose derivatives are actually present in some of the d.p. in  $\Phi$ . Let  $v_1, \dots, v_r$  be those  $y$  whose derivatives do not appear.<sup>9</sup> Then  $m + r = n$ .

We represent the first derivative of any  $u_i$  by  $u'_i$ .

13. We now consider the  $u, u', v$  as algebraic indeterminates. Then  $\Phi$  becomes a system  $\Psi$  of polynomials in  $\mathcal{F}[u; u'; v]$ ,  $\mathcal{F}$  being regarded as an algebraic field.

Our first step is to find, by the method of Chapter IV, a set of finite systems  $\Lambda_1, \dots, \Lambda_s$  equivalent to  $\Psi$ , each  $\Lambda$  being equivalent to a prime p.i. We assume that no  $\Lambda_i$  holds any  $\Lambda_j$  with  $j \neq i$ . In conducting the decomposition, we order the indeterminates as follows:<sup>10</sup>

$$(12) \quad u_1, \dots, u_m; \quad u'_1, \dots, u'_m; \quad v_1, \dots, v_r.$$

Of course,  $\Phi$  will be equivalent to the systems obtained by regarding the polynomials in each  $\Lambda$  as d.p.

The process which gives the  $\Lambda$  gives, for each  $i$ , a characteristic set of the prime p.i. equivalent to  $\Lambda_i$ .

We consider any  $\Lambda_i$ , calling it, simply,  $\Lambda$ . The prime p.i. equivalent to  $\Lambda$  will be denoted by  $\Omega$ . In what follows, we assume that  $\Omega$  has zeros.

Suppose that the characteristic set of  $\Omega$  contains polynomials in the  $u$  alone, that is, polynomials free of the  $u'$  and  $v$ . Let

$$(13) \quad A_1, \dots, A_p,$$

taken in the order in which they appear in the characteristic set, be those polynomials. Let  $A'_j$  be the polynomial in the  $u$  and  $u'$  obtained, when one regards  $\mathcal{F}$  momentarily as a differential field and  $u$  as a differential indeterminate, by differentiating  $A_j$ . Then, if  $u_k$  appears in  $A_j$ ,  $u'_k$  will appear in  $A'_j$  and, indeed, will appear linearly.

14. The first case which we shall consider is that in which each  $A'_j$  is in  $\Omega$ . We are going to obtain, from the characteristic set of  $\Omega$ , a system of differential equations, in a normal form, whose solutions are zeros of  $\Phi$ .

15. If  $p < m$ , there will be certain  $u$  none of which appears as a  $u$  of highest subscript in any  $A$  in (13). The totality of such  $u$  may be taken as part of a parametric set of  $\Omega$ .<sup>11</sup> We arrange the subscripts of the  $u$  in such a way that the parametric  $u$  above become  $u_1, \dots, u_{m-p}$  and so that the  $u$  of highest subscript in each  $A_j$  in (13) goes over into  $u_{m-p+j}$ .

Now let the subscripts of the  $u'$  in (12) be rearranged in such a way that, for

<sup>8</sup> Forsythe, loc. cit.

<sup>9</sup> The separation of the  $u$  and  $v$  is for the purposes of Chapter VII.

<sup>10</sup> The sequence (12) corresponds to  $y_1, \dots, y_n$  in IV, §26.

<sup>11</sup> To complete the set, we can use those  $u'$  and  $v$  which are not rightmost indeterminates in any polynomial of the characteristic set.

the new ordering of the  $u, u'_j$  may represent the derivative of  $u_j$ .<sup>12</sup> The  $v$  we leave undisturbed.

For the new arrangement of the indeterminates, we write  $B_j$  for  $A_j, B'_j$  for  $A'_j; \Lambda'$  for  $\Lambda, \Omega'$  for  $\Omega$ . Of course,  $\Omega'$  is a prime p.i.

We determine a characteristic set for  $\Omega'$ . This is accomplished by decomposing  $\Lambda'$  into prime p.i. none of which holds any other, each prime p.i. being represented by a characteristic set. Only one prime p.i., namely  $\Omega'$ , will be obtained. It is easy to see that the chain

$$(14) \quad B_1, \dots, B_p$$

can be taken as the first  $p$  polynomials in the characteristic set of  $\Omega'$ . We understand this to be done.

Of course, each  $B'$  is in  $\Omega'$ .

16. We write  $h = m - p$ . We are going to show that

$$(15) \quad u'_{h+1}, \dots, u'_m$$

are not among the parametric indeterminates for  $\Omega'$  as given by the characteristic set<sup>13</sup>  $\Gamma$  of  $\Omega'$ . Let us consider  $B'_1$ . It involves  $u'_{h+1}$  linearly, with  $S$ , the separant of  $B_1$ , for coefficient of  $u'_{h+1}$ . We consider the polynomials of  $\Gamma$  which involve only indeterminates preceding  $u'_{h+1}$ . These polynomials constitute a chain  $\Pi$  which consists of (14) and, perhaps, of polynomials introducing certain  $u'_i$  with  $i \leq h$ . Let  $C_1$  be the remainder of  $B'_1$  with respect to  $\Pi$ . Then  $C_1$ , which is in  $\Omega'$ , involves  $u'_{h+1}$  linearly. The coefficient  $D$  of  $u'_{h+1}$  in  $C_1$  is found by multiplying  $S$  by powers of the initials in  $\Pi$  and subtracting from the result a linear combination of the polynomials in  $\Pi$ . If  $D$  were zero,  $S$  would be in  $\Omega'$ . This shows that  $u'_{h+1}$  is not parametric; if it were,  $C_1$  would be a non-zero polynomial in  $\Omega'$ , reduced with respect to  $\Gamma$ .

As  $C_1$  is only of the first degree in  $u'_{h+1}$ , we may use  $C_1$  as a polynomial in  $\Gamma$  to introduce<sup>14</sup>  $u'_{h+1}$ . We suppose this to be done.

In the same way, the remainder  $C_2$  of  $B'_2$  with respect to  $\Pi + C_1$  involves  $u'_{h+2}$  and can be used in  $\Gamma$ . We continue in this way, showing that the indeterminates in (15) are not parametric, and determining a  $C_i$  which introduces  $u'_{h+i}, i = 1, \dots, p$ .

It is evident that if the  $C$  and the polynomials in  $\Pi$  are considered as d.p. in the  $u$ , each  $C$  holds  $\Pi$ .

17. As given by  $\Gamma$ , the parametric indeterminates are  $u_1, \dots, u_h$ , then perhaps some of the  $u'_i$  with  $i \leq h$  and some of the  $v$ . If the indeterminates are re-ordered so that the parametric ones come first and so that the relative order of the remaining indeterminates is undisturbed,  $\Gamma$  will remain a characteristic set.

<sup>12</sup> We regard the  $u$  momentarily as differential indeterminates.

<sup>13</sup> To be specific, the parametric indeterminates are those none of which is rightmost in any polynomial in  $\Gamma$ . Among them are  $u_1, \dots, u_h$ .

<sup>14</sup> That is, we may replace the polynomial which introduces  $u'_{h+1}$  by  $C_1$ .

We reorder the indeterminates so that the parametric ones appear in the order<sup>15</sup>

$$(16) \quad u_1, \dots, u_k; \quad u'_1, \dots, u'_k; \quad v_1, \dots, v_i; \quad u_{k+1}, \dots, u_h.$$

The remaining indeterminates will appear in the order

$$(17) \quad u_{h+1}, \dots, u_m; \quad u'_{k+1}, \dots, u'_{h+1}, \dots, u'_m; \quad v_{i+1}, \dots, v_r.$$

For this new ordering, we write  $\Lambda''$  for  $\Lambda'$ ,  $\Omega''$  for  $\Omega'$ ,  $\Gamma''$  for  $\Gamma$ ;  $D_i$  for  $B_i$  and  $E_i$  for  $C_i$ ,  $i = 1, \dots, p$ .

18. Those polynomials of  $\Omega''$  which are free of  $u'_{h+1}, \dots, u'_m$  constitute a prime p.i.  $\Delta$ . A characteristic set of  $\Delta$  is found by deleting  $E_1, \dots, E_p$  from  $\Gamma''$ . This is because  $E_i$  involves  $u'_{h+i}$  linearly, so that  $u'_{h+i}$  appears only in  $E_i$  in  $\Gamma''$ .

We build a resolvent  $R = 0$  for  $\Delta$ , using a  $w$  which is a linear combination of

$$(18) \quad u_{h+1}, \dots, u_m; \quad u'_{k+1}, \dots, u'_h; \quad v_{i+1}, \dots, v_r.$$

Each indeterminate in (18) will have an expression which is rational in  $w$  and the indeterminates in (16). If  $R$  is of degree  $g$  in  $w$ , we can, by Chapter IV, write each of these expressions in the form

$$(19) \quad \frac{H_1 + H_2w + \dots + H_gw^{g-1}}{L},$$

where  $L$  and the  $H$  involve only the letters in (16). The  $H$  will depend on the particular indeterminate in (18), but we may, and shall, use the same  $L$  for all of the expressions.

Let  $G$  be the resultant with respect to  $w$  of  $R$  and its separant  $\partial R/\partial w$ . In accordance with IV, §9, let  $M$  be a polynomial in the indeterminates in (16) which vanishes for every zero of  $\Gamma''$  which annuls the product of the initials<sup>16</sup> in  $\Gamma''$ . We may and shall assume that  $L$  is divisible by  $GM$ .

19. We let  $z$  represent any of the indeterminates in (18) and consider, together with  $R = 0$ , the system of equations

$$(20) \quad z = \frac{H_1 + \dots + H_gw^{g-1}}{L},$$

where  $z$  runs through all indeterminates in (18). Speaking in the language of classical analysis, we shall consider these equations as differential equations for  $u_{k+1}, \dots, u_h$  and as algebraic equations for  $u_{h+1}, \dots, u_m; v_{i+1}, \dots, v_r$ .

It is important to explain precisely what we mean by a solution of the system (20). For this, we consider  $\mathfrak{F}$  again as a differential field. Suppose that, in some differential field which is an extension of  $\mathfrak{F}$ , there exist elements  $u_1, \dots, u_m; v_1, \dots, v_r; w$ , with  $L \neq 0$ , which satisfy  $R = 0$  and (20). We shall call the elements  $u$  and  $v$  a solution of (20).

20. We are going to show that (20) has solutions. For the  $j$ th indeterminate from the left in (18), let  $F_j$  represent the polynomial

<sup>15</sup> For  $i \leq k$ ,  $u_i$  and  $u'_i$  are both parametric.

<sup>16</sup> The calculation of  $M$  involves no theoretical difficulty.

$$Lz - (H_1 + \cdots + H_g w^{g-1}).$$

The chain of polynomials

$$(21) \quad R, F_1, \cdots, F_g,$$

where  $g$  is the number of letters in (18), is, by the theory of resolvents, a characteristic set for a nontrivial prime p.i.

We shall now regard the polynomials in (21) as d.p. in  $\mathfrak{F}\{u; v; w\}$ . Let  $\mathfrak{E}$  represent the system of d.p. (21) and  $S$  the separant of  $R$ . We consider those d.p.  $K$  which have the property that

$$(22) \quad LSK \equiv 0, \quad \{\mathfrak{E}\}.$$

The totality  $\Sigma$  of the  $K$  is seen to be an ideal. We shall prove that  $\Sigma$  is prime.

Let  $UV$  belong to  $\Sigma$ . It is easy to see that there are relations

$$L^a S^b U \equiv U_1, \quad L^c S^d V \equiv V_1, \quad [\mathfrak{E}],$$

where  $U_1$  and  $V_1$  are free of  $u_{h+1}, \cdots, u_m; v_{t+1}, \cdots, v_r$ , and are of order at most zero in  $u_{k+1}, \cdots, u_h, w$ . Then  $U_1 V_1$  is in  $\Sigma$ . Thus some power of  $LSU_1 V_1$  is in  $[\mathfrak{E}]$ . Given a relation

$$(LSU_1 V_1)^\alpha = PR + \cdots + QF_g^\alpha,$$

we can proceed as in §3, making substitutions for the  $z$  and their derivatives, and for the derivatives of  $w$ , the substitutions producing a relation

$$L^\beta S^\gamma (U_1 V_1)^\alpha = WR.$$

Thus one of  $U_1$  and  $V_1$  is divisible by  $R$ . Then one of  $U$  and  $V$  is in  $\Sigma$ , and  $\Sigma$  is prime. The procedure just used shows that unity is not in  $\Sigma$ ; neither is  $L$ .

A generic zero of  $\Sigma$  furnishes a solution of (20).

21. In the analytic case, the solutions of (20) are found as follows. Let  $u_1, \cdots, u_k; v_1, \cdots, v_t$  be taken arbitrarily as analytic functions of  $x$ , with the sole restriction that, when they are substituted into  $L$ ,  $L$  becomes a function  $L'$  of  $x, u_{k+1}, \cdots, u_h$  which is not identically zero. Let numerical values be assigned to  $u_{k+1}, \cdots, u_h$ , at some point  $x = a$ , so that  $L' \neq 0$  for these numerical values and for  $x = a$ . Then  $R = 0$  will determine a set of  $g$  functions  $w$  of  $x, u_{k+1}, \cdots, u_h$ , analytic in some neighborhood containing the chosen set of numerical values. This is because  $L$  is divisible by  $G$ . Using any of these analytic functions for  $w$  in (20), we find in those equations in (20) which correspond to  $u'_{k+1}, \cdots, u'_h$ , a set of differential equations which determine  $u_{k+1}, \cdots, u_h$  for the initial conditions. We then use the equations in (20) whose first members are

$$u_{h+1}, \cdots, u_m; \quad v_{t+1}, \cdots, v_r$$

to determine those unknowns.

The system (20) is essentially in the Jacobi-Weierstrass normal form.

22. A solution of (20) consists of quantities

$$(23) \quad u_1, \dots, u_m; \quad v_1, \dots, v_r.$$

If we adjoin to (23) the derivatives of the quantities  $u$ , we get a set of quantities

$$(24) \quad u_1, \dots, u_m; \quad u'_1, \dots, u'_m; \quad v_1, \dots, v_r.$$

We wish to show that the quantities (24) are a zero of  $\Omega''$ .

If we adjoin to (23) the associated quantity  $w$  and also  $u'_1, \dots, u'_n$ , we get a set

$$(25) \quad u_1, \dots, u_m; \quad u'_1, \dots, u'_n; \quad v_1, \dots, v_r; \quad w$$

which annuls every polynomial in (21). We can see, however, that (25) annuls no initial in (21). Firstly,  $L$  is the initial of every  $F$ . Again,  $L$  is divisible by  $G$ , which vanishes when the initial of  $R$  is zero. Thus (25) is a zero of the prime p.i. for which (21) is a characteristic set. If we suppress  $w$  in (25), we get a zero of  $\Delta$ .

We know now that (24) annuls every polynomial in the characteristic set  $\Gamma'$  of  $\Omega''$ , except perhaps the  $E$  of §17. That the  $E$  are annulled follows from the final remark of §16. Now (24) cannot annul any initial in  $\Gamma'$ . This is because  $L$  is divisible by  $M$ . Thus (24) is a zero of  $\Omega''$ . Then (24) is a zero of  $\Lambda''$ .

23. Let  $\Phi_1$  represent  $\Lambda''$  considered as a set of d.p. We have just seen that every solution of (20) is a zero of  $\Phi_1$ . Every zero of  $\Phi_1$  with  $L \neq 0$  satisfies (20) with a suitable  $w$ . Thus, to get the complete manifold of  $\Phi_1$ , we have to add to the solutions of (20) the manifold of  $\Phi_1 + L$ . Now, by IV, §12, every prime p.i. which  $\Lambda'' + L$  holds has a dimension lower than that of  $\Omega''$ .

We keep the facts just adduced in reserve, while we examine again  $\Lambda$  of §13.

24. We suppose now, returning to §§13, 14, that some  $A'_i$ , call it simply  $A'$ , is not in  $\Omega$ . Let  $\Phi_2$  represent  $\Lambda$  considered as a set of d.p. Then  $\Phi_2$  is equivalent to  $\Phi_2 + A'$ . Now any prime p.i. which  $\Lambda + A'$  holds has a dimension lower than that of  $\Omega$ .

25. From §§23, 24, it follows that, if we treat each  $\Lambda_i$  of §13 as  $\Lambda$  was treated, and then begin with the resulting systems  $\Phi_1 + L$  or  $\Phi_2 + A'$  as with  $\Phi$ , we obtain, continuing the process sufficiently, a set of systems in the normal form (20) whose solutions make up the manifold<sup>17</sup> of  $\Phi$ . This completes the investigation undertaken in §11.

26. We consider the prime ideal  $\Sigma$  of §20. The prime p.i. for which (21) is a characteristic set contains the polynomials of  $\Delta$ , all of which are free of  $w$ . Every such polynomial, considered as a d.p., is in  $\Sigma$ . Thus  $\Sigma$  contains d.p. free of  $w$ . The totality of such d.p. is a prime ideal  $\Sigma'$  in  $\mathfrak{F}\{u, v\}$ .

A generic zero of  $\Sigma'$  is a solution of (20) and is thus a zero of  $\Phi_1$  of §23. Then  $\Phi_1$  holds  $\Sigma'$ . We show now that every solution of (20) is a zero of  $\Sigma'$ . When

<sup>17</sup> The  $u$  in the various systems (20) have to be reordered. Of course, if  $\Phi$  has no zeros, we obtain no system (20), but are led to systems of polynomials without zeros.

we adjoin, to a solution of (20), the quantity  $w$  attached to it, we get a zero of  $\Xi$  of §20. As  $L$  is divisible by  $G$ , the solution and  $w$  do not annul  $S$ . It follows from (22) that the solution and  $w$  are a zero of  $\Sigma$ . This proves our statement.

We have thus another proof of the fact that the manifold of the finite system  $\Phi$  of §11 is the union of a finite member of irreducible algebraic differential manifolds.

27. We now consider a finite system  $\Psi$  of nonzero d.p. in  $\mathcal{F} \{ y_1, \dots, y_n \}$ , the  $y$  being involved in  $\Psi$  up to any order. If a  $y_i$  occurs up to the order  $m_i \geq 1$ , we put, with prompt explanations,

$$(26) \quad y_i = u_{i1}, \quad y_{i1} = u_{i2}, \dots, \quad y_{i, m_i - 1} = u_{im_i}, \quad y_{im_i} = u'_{im_i}.$$

The second subscript of a  $y$  indicates an order of differentiation. The  $u_{ij}$  are all distinct differential indeterminates and  $u'_{im_i}$  is the derivative of  $u_{im_i}$ .

If no derivative of  $y_i$  appears in  $\Psi$ , we put

$$(27) \quad y_i = v_i.$$

Making the substitutions (26) and (27) in the d.p. of  $\Psi$  and adjoining to the resulting system the d.p.

$$(28) \quad u'_{ij} - u_{i, j+1},$$

$j = 1, \dots, m_i - 1$ , we obtain a system  $\Phi$  of d.p. in the  $u_{ij}$  and  $v_i$ , each d.p. of order at most unity in each indeterminate. The system  $\Phi$ , aside from the notation in the subscripts, is of the type described in §11.

If  $A$  is a d.p. in the  $u$  and  $v$ , of any orders in its indeterminates, and if  $A$  goes over into a d.p.  $B$  by the substitutions

$$(29) \quad u_{ij}^{(m)} = y_{i, j+m-1}; \quad v_{im} = y_{im},$$

superscripts of  $u$ , and second subscripts of  $v$ , indicating order of differentiation, every zero of  $A$  which annuls each d.p. in (28) gives a zero of  $B$  for which  $y_i = u_{i1}$  for certain  $i$  and  $y_i = v_i$  for the remaining  $i$ .

It follows that if  $\Phi$  is resolved into prime ideals, as in §26, the substitution (29) will produce a set of prime ideals equivalent to  $\Psi$ .

We have thus proved again that the manifold of any finite system of d.p. is the union of a finite number of irreducible manifolds.

We have also secured a second elimination theory for the system  $\Psi$ .

THEORETICAL PROCESS FOR DECOMPOSING THE MANIFOLD OF A FINITE SYSTEM INTO ITS COMPONENTS

28. We deal with any finite system  $\Phi$  in  $\mathcal{F} \{ y_1, \dots, y_n \}$ . Let  $p$  be any positive integer. We denote by  $\Phi^{(p)}$  the system obtained by adjoining to  $\Phi$  the first  $p$  derivatives of each of its d.p.

When the d.p. in  $\Phi^{(p)}$  are regarded as polynomials in the  $y_{ij}$  which they effectively involve,  $\Phi^{(p)}$  goes over into a system  $\Psi^{(p)}$  of polynomials. For algebraic field, we use  $\mathcal{F}$ .

We decompose  $\Psi^{(p)}$ , with a finite number of operations, into finite systems

$$(30) \quad \Lambda_1, \dots, \Lambda_r$$

each of which is equivalent to a prime p.i. and none of which holds any other.

Let the polynomials in the  $\Lambda$  be considered now as d.p. Then each  $\Lambda_i$  goes over into a system  $\Phi_i$  of d.p. Let any  $\Phi_i$  which is held by some  $\Phi_j$  with  $j \neq i$  be suppressed. This can be accomplished with a finite number of operations. There remain, if  $\Phi$  has a manifold, systems

$$(31) \quad \Phi_1, \dots, \Phi_s.$$

We say that, for  $p$  sufficiently great, the manifolds of the  $\Phi_i$  are the components of<sup>18</sup>  $\Phi$ .

29. Let

$$(32) \quad \Sigma_1, \dots, \Sigma_t$$

be finite systems, no two equivalent, whose manifolds are the components of  $\Phi$ . When the d.p. in the  $\Sigma$  are regarded as polynomials in the  $y_{ij}$ , (32) goes over into a set of systems of polynomials

$$(33) \quad \Gamma_1, \dots, \Gamma_t.$$

Let us make any selection of  $t$  d.p., one from each  $\Sigma$ , and take their product. Let the products, for all possible selections, be

$$A_1, \dots, A_g.$$

Then each  $A$  holds  $\Phi$ . By the theorem of zeros, if  $p$  is large, some power of each  $A$  will be linear in the d.p. of  $\Phi^{(p)}$ .

Thus, if each  $A_i$  is considered as a polynomial in its  $y_{jk}$ , and if it is represented then by  $B_i$ , each  $B$  will hold  $\Psi^{(p)}$  if  $p$  is sufficiently large. Let  $p$  be large enough for this.

We shall prove that each  $\Lambda_i$  of (30) is held by some  $\Gamma_j$  of<sup>19</sup> (33). Suppose that  $\Lambda_1$  is not so held. Let  $C_j$  be a polynomial of  $\Gamma_j$ ,  $j = 1, \dots, t$ , which does not hold  $\Lambda_1$ . Then  $C_1 \cdots C_t$ , that is, some  $B$ , does not hold  $\Lambda_1$ . That  $B$  cannot hold  $\Psi^{(p)}$ . This proves our statement.

It follows that each  $\Phi_i$  is held by some  $\Sigma_j$ .

On the other hand, each  $\Sigma_i$  is held by some  $\Phi_j$ . Let this be false. Let  $D_j$  be a d.p. in  $\Phi_j$ ,  $j = 1, \dots, r$  (we restore momentarily the suppressed  $\Phi_i$ ) which does not hold  $\Sigma_1$ . Then  $G = D_1 \cdots D_r$  does not hold  $\Sigma_1$ . Hence  $G$  does not hold  $\Phi$ . Then, if  $G$  is considered as a polynomial in its  $y_{ij}$ , it does not hold  $\Psi^{(p)}$ . This contradicts the fact that  $\Psi^{(p)}$  is equivalent to (30).

Thus, if  $p$  is sufficiently great, the manifolds of the  $\Phi_i$  in (31) are the components of  $\Phi$ .

<sup>18</sup> It is assumed that  $\Phi$  has a manifold.

<sup>19</sup> The indeterminates are those which appear in (30) and (33).

For the above process to become a genuine method of decomposition, it would be necessary to have a method for determining permissible integers  $p$ . In VI, §9 we treat a special case.

**Example 1.** Let  $\Phi$  be  $y_1^2 - 4y$  in  $\mathfrak{F}\{y\}$ . Then  $\Psi^{(p)}$  is equivalent to the system

$$\begin{aligned} y_1^2 - 4y, \quad y_1(y_2 - 2), \quad y_1y_3 + y_2(y_2 - 2), \\ y_1y_4 + 2y_2y_3 + y_3(y_2 - 2), \dots, \\ y_1y_{p+1} + (p-1)y_2y_p + \dots + (p-1)y_{p-1}y_3 + y_p(y_2 - 2). \end{aligned}$$

$\Psi'$  decomposes into the two systems

$$(34) \quad y, y_1,$$

$$(35) \quad y_1^2 - 4y, \quad y_2 - 2,$$

in  $\mathfrak{F}[y, y_1, y_2]$ , each of which is equivalent to a prime p.i. If we adjoin

$$y_1y_3 + y_2(y_2 - 2)$$

to (34), that system decomposes into

$$(36) \quad y, y_1, y_2,$$

$$(37) \quad y, y_1, y_2 - 2$$

in  $\mathfrak{F}[y, y_1, y_2, y_3]$ . The same adjunction to (35) gives (37) and

$$(38) \quad y_1^2 - 4y, \quad y_2 - 2, \quad y_3.$$

Thus (36), (37) and (38) give the decomposition of  $\Psi''$ . Continuing, we find the decomposition of  $\Psi^{(p)}$  to be, for  $p > 2$ ,

$$(39) \quad y, y_1, y_2, \dots, y_p,$$

$$(40) \quad y, y_1, \quad y_2 - 2, \quad y_3, \dots, y_p,$$

$$(41) \quad y_1^2 - 4y, \quad y_2 - 2, \quad y_3, \dots, y_{p+1}.$$

If we regard the last three systems as systems of d.p., (41) gives the general solution of  $y_1^2 - 4y$ , while (39) gives the zero  $y = 0$ , which is a second irreducible manifold. The system (40) of d.p. has no zeros.

We notice that the system of polynomials  $y_1^2 - 4y, y_2 - 2$  holds the system (40) of polynomials. This is in harmony with the fact that every  $\Lambda$  in (30) is held by some  $\Gamma$  in (33).

**Example 2.** Let  $\Phi$  be  $y_1^2 - 4y^3$ , whose manifold was seen in II, §19 to be irreducible. If we let

$$A_1 = 2y_2 - 12y^2$$

and represent the  $r$ th derivative of  $A_1$  by  $A_{r+1}$ , then  $\Psi^{(p)}$  will be

$$\begin{aligned}
 & y_1^2 - 4y^3, \quad y_1A_1, \quad y_2A_1 + y_1A_2, \\
 & \quad y_3A_1 + 2y_2A_2 + y_1A_3, \dots, \\
 & y_pA_1 + (p-1)y_{p-1}A_2 + \dots + (p-1)y_2A_{p-1} + y_1A_p.
 \end{aligned}$$

Then  $\Psi'$  decomposes into

$$(42) \quad y, y_1,$$

$$(43) \quad y_1^2 - 4y^3, A_1.$$

We now examine  $\Psi''$ . The adjunction of  $y_2A_1 + y_1A_2$  to (42) gives the single system

$$(44) \quad y, y_1, y_2.$$

The same adjunction to (43) gives

$$(45) \quad y_1^2 - 4y^3, A_1, A_2,$$

and also (44).

Let us examine  $\Psi'''$ . The adjunction of  $y_3A_1 + 2y_2A_2 + y_1A_3$  to (44) gives the single system

$$(46) \quad y, y_1, y_2$$

in  $\mathfrak{F}[y, \dots, y_4]$ . The same adjunction to (45) gives

$$(47) \quad y_1^2 - 4y^3, A_1, A_2, A_3,$$

as well as the system, held by (46), obtained by adjoining  $y_3$  to (46).

Continuing, it is not difficult to prove that the decomposition of  $\Psi^{(p)}$  is

$$(48) \quad y, y_1, \dots, y_g,$$

where  $g$  is the greatest integer in  $1 + p/2$ , and

$$(49) \quad y_1^2 - 4y^3, \quad A_1, \dots, A_p.$$

The system (49) of d.p. gives the manifold of  $\Phi$ , while (48) (d.p.), whose manifold is  $y = 0$ , is held by (49).

CHAPTER VI  
ANALYTICAL CONSIDERATIONS

NORMAL ZEROS

1. We deal with the analytic case. Let  $\Sigma$  be a nontrivial prime ideal in  $y_1, \dots, y_n$  with a characteristic set

$$(1) \quad A_1, \dots, A_p.$$

A zero of (1) which annuls no separant will be called a *normal* zero of (1). By V, §4, every normal zero of (1) is a zero of  $\Sigma$ . A zero of (1) may annul some separant and still be a zero of  $\Sigma$ . One of our objects, in what follows, is to characterize such zeros of (1).

ADHERENCE

2. Let  $n$  be a fixed positive integer. We consider sets of functions  $y_1(x), \dots, y_n(x)$ , the functions of each set being analytic in some open region which depends on the set. Let  $\mathfrak{A}$  be a family of such sets. Let  $\bar{y}_1(x), \dots, \bar{y}_n(x)$  be a set of functions which are analytic in an open region  $B$ , the set not belonging to  $\mathfrak{A}$ . We shall say that  $\bar{y}_1, \dots, \bar{y}_n$  *adheres* to  $\mathfrak{A}$  if there exists in  $B$  a point  $a$  of the following description. For every positive integer  $m$  and for every  $\epsilon > 0$ , there exists in  $\mathfrak{A}$  a set  $y_1, \dots, y_n$ , the  $y$  being analytic in an open region<sup>1</sup> containing  $a$ , such that

$$(2) \quad |y_{ij}(a) - \bar{y}_{ij}(a)| < \epsilon, \quad i = 1, \dots, n; j = 0, \dots, m.$$

The point  $a$  will be called a point of contact of  $\bar{y}_1, \dots, \bar{y}_n$  with  $\mathfrak{A}$ .

THE THEOREM OF APPROXIMATION

3. Let  $\Sigma$  be as in §1. Let  $B$  be any d.p. which is not in  $\Sigma$ , and  $\mathfrak{A}$  the set of zeros of  $\Sigma$  which do not annul  $B$ . Let  $\bar{y}_1(x), \dots, \bar{y}_n(x)$  be a zero of  $B$  which adheres to  $\mathfrak{A}$ . We shall prove that the  $\bar{y}$  are a zero of  $\Sigma$ .

Let  $G$  be any d.p. in  $\Sigma$ . We have to prove that  $G$  is annulled by the  $\bar{y}$ . Let  $a$  be a point of contact of the  $\bar{y}$  with  $\mathfrak{A}$ . Some of the coefficients in  $G$  may have poles at  $a$ . If so, we divide  $G$  by a power of one of its coefficients and the pole is removed. We thus assume that the coefficients in  $G$  are analytic at  $a$ . If the  $\bar{y}$  are substituted into  $G$ , we secure a  $\gamma(x)$ , analytic at  $a$ . Suppose that, choosing a large  $m$ , and then a small  $\epsilon$ , we find a  $y_1, \dots, y_n$  in  $\mathfrak{A}$  satisfying (2). When the  $y$  are substituted into  $G$ , we obtain a  $\gamma'(x)$  with a Taylor expansion at  $a$  in which the coefficients of the  $(x - a)^i$ ,  $i = 0, \dots, m$ , are very nearly equal to the corresponding coefficients in  $\gamma$ . But  $\gamma' = 0$ . Then  $\gamma = 0$  and the  $\bar{y}$  annul  $G$ .

<sup>1</sup> The  $y$  and their region of analyticity will depend on  $m$  and  $\epsilon$ .

We derive now a converse result.

**THEOREM:** *Every zero of  $B$  which is a zero of  $\Sigma$  adheres to  $\mathfrak{A}$ .*

We reletter the indeterminates so as to have a parametric set  $u_1, \dots, u_q$  and so that  $A_i$  in (1) introduces  $y_i$ . Let

$$(3) \quad \bar{u}_1(x), \dots, \bar{u}_q(x); \quad \bar{y}_1(x), \dots, \bar{y}_p(x),$$

analytic in an open region  $\mathbf{B}$ , be a zero of  $\Sigma$  which annuls  $B$ . We shall find points of contact of (3) with  $\mathfrak{A}$ .

Let  $R$  be the remainder of  $B$  with respect to (1).

Let

$$T = RS_1 \cdots S_p$$

with  $S_i$  the separant of  $A_i$ .

Let  $A_i$  be of order  $r_i$  in  $y_i$ . For every  $y_{is}$  with  $s > r_i$  in a normal zero of (1), we have an expression

$$(4) \quad y_{is} = \frac{E}{F},$$

where  $E$  is a d.p. of class at most  $q + i$  and of order at most  $r_i$  in  $y_j, j = 1, \dots, i$ ;  $F$  is a power product in the  $S$ . The d.p.

$$(5) \quad Fy_{is} - E$$

are in  $\Sigma$ .

Let  $m$  be any integer greater than every  $r_i$ . We adjoin to (1) all d.p. (5) for  $i = 1, \dots, p$ , with  $s \leq m$ . We now consider the  $u_{ij}$  and  $y_{ij}$  as algebraic indeterminates, so that the d.p. in (1) and (5) become a system  $\Phi$  of polynomials. We use here all  $u_{ik}$  and  $y_{ik}$  with  $k \leq m$  and any other  $u_{ik}$  which may occur in (1), in (5) and in  $T$ .

We shall prove that the totality  $\Omega$  of polynomials which vanish for those zeros of  $\Phi$  for which no  $S$  vanishes is a prime p.i. Let  $GH$  vanish for the indicated zeros. By (4) we have, for those zeros,

$$G = \frac{E_1}{F_1}, \quad H = \frac{E_2}{F_2},$$

where  $E_1$  and  $E_2$  involve no  $y_{ij}$  with  $j > r_i$ . Then  $E_1E_2$  vanishes for the above zeros.

By V, §1, (1), regarded as a set of polynomials, is a characteristic set of a prime p.i. The indeterminates which we use at this point are the  $y_{ij}$  with  $j \leq r_i$  and the  $u_{ij}$  in  $\Phi$ . Then either  $E_1$  vanishes for all zeros of the polynomials (1) which annul no  $S$ , or  $E_2$  does. Suppose that  $E_1$  does. Then  $G$  vanishes for all zeros of  $\Omega$  which annul no separant, so that  $\Omega$  is a prime p.i.

We shall prove that, given any zero of  $\Sigma$ , the  $u_{ij}, y_{ij}$  appearing in  $\Phi$ , obtained from the zero, constitute a zero of  $\Omega$ . This is obvious for the normal zeros of (1).

Then if  $G$  is a polynomial in  $\Omega$ ,  $G$ , considered as a d.p., holds  $\Sigma$ . This proves our statement.

We consider the given zero (3) of  $\Sigma$ . It annuls  $T$ . Consider the corresponding zero of  $\Omega$ . By IV, §39, there is, in  $\mathbf{B}$ , an open region  $\mathbf{C}$  of the following description. Given any  $\epsilon > 0$ , we can find a zero  $u_{ik}, y_{jk}$  of  $\Omega$ , analytic in  $\mathbf{C}$ , for which  $T$  is distinct from zero throughout  $\mathbf{C}$ , such that, for every point  $a$  in  $\mathbf{C}$ ,

$$(6) \quad |u_{ik}(a) - \bar{u}_{ik}(a)| < \epsilon, \quad |y_{jk}(a) - \bar{y}_{jk}(a)| < \epsilon, \\ i = 1, \dots, q; j = 1, \dots, p; k = 0, \dots, m.$$

We refer to II, §10. For any point  $a$  in  $\mathbf{C}$ , the  $u_{ik}(a)$  and  $y_{jk}(a)$  in the zero of  $\Omega$  used in (6) furnish initial conditions for a normal zero of (1). It is a matter of constructing functions  $u$  with a certain number of given coefficients in their Taylor expansions at  $a$ , and then using repeatedly the implicit function theorem and the existence theorem for differential equations.<sup>2</sup> Thus, for every  $a$  in  $\mathbf{C}$ , there is a normal zero of (1), analytic at  $a$ , for which  $T$  is not zero at  $a$  and which satisfies (6) with the given zero (3).

We repeat the above operation, using  $2m$  and  $\epsilon/2$  in place of  $m$  and  $\epsilon$ . We find a region  $\mathbf{C}_1$ , in  $\mathbf{C}$ , every point  $a$  of which can be used as above. For convenience we take  $\mathbf{C}_1$  bounded, with its boundary in  $\mathbf{C}$ . Employing  $4m$  and  $\epsilon/4$ , we find a region  $\mathbf{C}_2$  in  $\mathbf{C}_1$ . We continue, determining a sequence of regions  $\mathbf{C}_j$ . There is at least one point  $a$  common to all of these regions. Given any  $m$  and any  $\epsilon$ , there is a normal zero of (1), thus a zero of  $\Sigma$ , analytic at  $a$ , which does not annul  $B$  at  $a$  and for which (6) holds. As  $a$  is a point of contact of the zero (3) with  $\mathfrak{A}$ , our theorem is proved.

4. The foregoing discussion shows that the points of contact of (3) with  $\mathfrak{A}$  are dense in  $\mathbf{B}$ . Thus, if a zero of  $B$  has a point of contact with  $\mathfrak{A}$ , it has a dense set of points of contact.

A point which is not a point of contact of (3) with  $\mathfrak{A}$  will be said to be *exceptional* for (3) relative to  $B$ . That exceptional points may exist is seen from the following example. Let  $\Sigma$  be the prime ideal in  $\mathfrak{F}\{y\}$  whose manifold is  $y = c/x$  with  $c$  an arbitrary constant. Then  $y = 0$  is the only zero which is analytic at  $x = 0$ . Thus, if  $B = y$ , the point  $x = 0$  is an exceptional point.

Strodt has shown<sup>3</sup> that, when  $B$  is given, the exceptional points of all zeros of  $\Sigma$  which annul  $B$  lie on a fixed set which is vacuous, finite or countably infinite. Without proving Strodt's theorem, let us see that, for a particular zero  $\bar{u}, \bar{y}$  which annuls  $\mathbf{B}$ , the set of exceptional points is at most countably infinite.

We refer to IV, §39, inquiring as to the conditions which must be put on a point  $a$  of  $\mathbf{B}$  so that a region  $\mathbf{C}$  containing  $a$  may exist. We find that  $a$  must be distinct from the poles of the coefficients of a finite number of polynomials and must not be a zero of  $\alpha_r$  in (50). Then, the  $z_j$  with  $j > q$  will be bounded in a region containing  $a$  and  $\mathbf{C}$  can be taken so as to contain  $a$ . Thus for each  $m$ ,

<sup>2</sup>  $\mathbf{C}$  is taken so that the coefficients in the  $A$  are analytic throughout  $\mathbf{C}$ .

<sup>3</sup> Strodt, 44.

in §3, we have to avoid, in selecting  $a$ , a set of isolated points. Such a set is finite or countable. The points  $a$  which cannot be used for all  $m$  thus form at most a countable set.

5. If we take  $B$  as the product of the  $S$ , we see that *the manifold of  $\Sigma$  consists of the normal zeros of (1) and of the zeros which adhere to the set of normal zeros.*

We consider now an algebraically irreducible d.p.  $F$  in  $\mathfrak{F}\{y_1, \dots, y_n\}$ . In the analytic case, a singular zero of  $F$  which adheres to the set of nonsingular zeros will be called an adherent singular zero. We see that *the general solution of an algebraically irreducible differential polynomial consists of the nonsingular zeros and of the adherent singular zeros.*

6. Sometimes a sequence of zeros  $u, y$  exists whose Taylor expansions at a point of contact approach those of  $\bar{u}, \bar{y}$  in the manner indicated in (6), without the  $u, y$  converging uniformly to the  $\bar{u}, \bar{y}$  in a neighborhood of  $a$ . We give an example, using  $\mathfrak{F}\{y\}$ . Let

$$A = (yy_2 - y_1^2)^2 - 4yy_1^3.$$

$A$  is algebraically irreducible in the field of all constants because, when equated to zero, it defines  $y_2$  as a two-branched function of  $y$  and  $y_1$ . Equating  $A$  to zero, we find, for  $y \neq 0$ ,

$$\frac{d}{dx} \left( \frac{y_1}{y} \right) = 2 \left( \frac{y_1}{y} \right)^{3/2},$$

the solutions of which are given by

$$(7) \quad y = be^{1/(c-x)}$$

and  $y = b$ , with  $b$  and  $c$  constants. The solution  $y = 0$ , suppressed above, is included among these.

The solutions (7) with  $b \neq 0$  are normal zeros of  $A$  and thus belong to the general solution of  $A$ . Let  $b$  stay fixed in (7) at a value distinct from zero, while  $c$  approaches  $\infty$  through positive values. Then  $y$  approaches  $b$ , uniformly in any bounded domain. It follows that the zeros  $y = b$  with  $b \neq 0$  adhere to the set of normal zeros and are in the general solution. By taking  $c$  as a small negative number, we can make the second member of (7) and an arbitrarily large number of its derivatives small at pleasure at  $x = 0$ . This shows that  $y = 0$  adheres to the normal zeros and is in the general solution.

Of course, if we take  $b$  small, and then  $c$  large and positive, we get a sequence of normal zeros converging uniformly to zero in any preassigned bounded domain. The discussion above, in which essential singularities were used to prove adherence, shows what might conceivably happen in other examples.

All in all, it is not known whether a zero of  $B$  which adheres to  $\mathfrak{A}$  can be approximated uniformly in some area by zeros in  $\mathfrak{A}$ . It is known<sup>4</sup> that such a zero of  $B$  may fail to be embedded analytically among zeros in  $\mathfrak{A}$ .

<sup>4</sup> Ritt, 32.

## ANALYTIC TREATMENT OF LOW POWER THEOREM

7. The low power theorem was first proved for the analytic case.<sup>5</sup> The necessity proof given was essentially that of Chapter III. We shall present here the analytic sufficiency proof, which employs ideas essentially different from those in Levi's algebraic treatment.

We refer to III, §17, and to III, §20. Let (25) of III, §17, which we rewrite

$$(8) \quad \sum_{j=1}^r C_j A^{p_j} A_1^{q_j} \cdots A_m^{r_j - q_j},$$

contain a term  $C_k A^{p_k}$  which is of lower degree than every other term. We suppose, for simplicity, that  $k = 1$ . Let us imagine that the general solution  $\mathfrak{M}$  of  $A$  is not a component of  $F$ , but rather a proper part of some component  $\mathfrak{M}'$  of  $F$ . Then  $A$  does not hold  $\mathfrak{M}'$ .

Let  $\mathfrak{A}$  be the set of points of  $\mathfrak{M}'$  which are not zeros of  $A$ . By §3, every point of  $\mathfrak{M}$  adheres to  $\mathfrak{A}$ .

Let

$$(9) \quad \bar{y}_1(x), \cdots, \bar{y}_n(x)$$

be a point of  $\mathfrak{M}$  which does not annul  $C_1$ . We consider a point of contact  $a$  of (9) with  $\mathfrak{A}$ , produced as in §3, for which the coefficients in  $A$ , and in the  $C_j$  of (8), are analytic. We assume furthermore that  $C_1$  is not annulled at  $a$  by the functions in (9).

A positive integer  $\mu$  and an  $\epsilon > 0$  being taken, let us find, as in §3, a zero  $\tilde{y}_1, \cdots, \tilde{y}_n$  in  $\mathfrak{A}$ , analytic at  $a$ , such that

$$(10) \quad |\tilde{y}_{ij}(a) - \bar{y}_{ij}(a)| < \epsilon, \quad i = 1, \cdots, n; j = 0, \cdots, \mu.$$

We denote by  $c$  a positive real number which later will be made small. We perform, upon the variable  $x$ , the transformation

$$(11) \quad z = \frac{x - a}{c}.$$

For the  $\tilde{y}$  of (10),  $A$  becomes a function  $A(\tilde{y}, x)$  of  $x$ , analytic at  $a$  and distinct from zero at  $a$ . (Note that we use a point of contact of the type produced in §3.)

Let  $s$  be a positive integer which will be fixed later, and let

$$(12) \quad w(z) = c^{-s} A(\tilde{y}, x),$$

where the  $x$  used in  $A$  is related to  $z$  as in (11). Then

$$w_1 = c^{-s+1} A_1(\tilde{y}, x), \cdots, w_{m-l} = c^{-s+m-l} A_{m-l}(\tilde{y}, x)$$

where subscripts of  $w$  indicate differentiation with respect to  $z$ .

Let  $D_j(z) = C_j(\tilde{y}, x)$ . The  $j$ th term in (8) goes over into

<sup>5</sup> Ritt, 31.

$$(13) \quad c^u D_j(z) w^{p_i} w_1^{u_1} \cdots w_m^{u_m - i^{i'}}$$

where

$$u_j = s d_j - e_j,$$

$d_j$  being the degree of the  $j$ th term of (8) in  $A$  and the  $A_i$ , and  $e_j$  its weight.

From the fact that  $d_j > d_1$  when  $j > 1$ , it follows that, if  $s$  is large, every  $u_j$  with  $j > 1$  will exceed  $u_1$ . Let  $s$  be fixed at a value large enough for this to occur.

As (8) vanishes for the  $\tilde{y}$ , we may write

$$(14) \quad D_1(z) w^{p_1} + \sum_{j=2}^r c^{u_j - u_1} D_j(z) w^{p_j} \cdots w_m^{u_m - i^{i'}} = 0.$$

Let the Taylor expansion of  $A(\tilde{y}, x)$  at  $a$  be

$$(15) \quad b_0 + b_1(x - a) + \cdots + b_i(x - a)^i + \cdots.$$

Then  $b_0 \neq 0$ , but, if  $\epsilon$  and  $1/\mu$  are small, a large number of the  $b$ , beginning with  $b_0$ , will be small. This is because the  $\tilde{y}$  approximate to the  $\bar{y}$  and the  $\bar{y}$  annul  $A$ .

We now fix  $c$  in such a way as to make the greatest of the quantities

$$|b_i c^{-s+i}|, \quad i = 0, \cdots, s-1,$$

equal to unity. This is possible because  $b_0 \neq 0$ .

Then  $c$  tends towards zero with  $\epsilon$  and  $1/\mu$ .

For  $|z|$  small, we have, by (11), (12), (15),

$$(16) \quad w(z) = \sum_{i=0}^{\infty} b_i c^{-s+i} z^i.$$

When  $\mu$  and  $1/\epsilon$  increase, the coefficient of  $z^i$  in (16), for a fixed  $i$  exceeding  $s-1$ , will tend towards zero.

It follows that we can select a sequence of approximating zeros  $\tilde{y}$  for which  $w(z)$  tends towards a nonzero polynomial of degree  $s-1$  at most. The convergence occurs in the sense that each coefficient in (16) tends toward the corresponding coefficient in the polynomial.<sup>6</sup> Let  $\gamma(z)$  be such a polynomial.

Let, for  $|x-a|$  small, and for the particular point (9) of  $\mathfrak{M}$ ,

$$C_j(\tilde{y}, x) = \sum_{i=0}^{\infty} h_{ji}(x-a)^i, \quad j = 1, \cdots, r.$$

Then

$$C_j(\tilde{y}, x) = \sum_{i=0}^{\infty} h'_{ji}(x-a)^i,$$

where, for each  $i$ ,  $h'_{ji}$  approaches  $h_{ji}$  as  $\epsilon$  and  $1/\mu$  decrease. We have

<sup>6</sup> In this polynomial, the "coefficient of  $z^i$ " with  $i \geq s$  is understood to be zero.

$$D_j(z) = \sum_{i=0}^{\infty} h'_j c^i z^i.$$

Turning now to (14) and remembering that the  $u_j - u_1$  with  $j > 1$  are positive, we recognize that

$$(17) \quad h_{10} \gamma^n = 0.$$

In short, if the first member of (17) did not have a vanishing expansion in powers of  $z$ , the first member of (14) could not have a vanishing expansion when  $c$  is small and  $w$  approximates to  $\gamma$ . Because  $C_1(\bar{y}, x)$  does not vanish at  $a$ , we have  $h_{10} \neq 0$ . Then, by (17),  $\gamma = 0$ . This contradicts what precedes, so that the sufficiency proof is completed. The theorems of III, §§22, 23 can be obtained by modifying slightly the above procedure.

The transformation performed in (11) and (12) has the form of certain transformations which were discovered by Painlevé and which were applied by him to the study of differential equations whose solutions have fixed critical points.<sup>7</sup>

It might be proposed to treat the sufficiency question by making the substitution  $A = \alpha^h$ , with  $h$  a positive integer, in the relation  $S'F = 0$ . For  $h$  large, the resulting relation could be divided through by  $\alpha^{hn} = A^n$  and we would get a relation which could not be satisfied by a sequence of  $\bar{y}$  for which the coefficients of  $\alpha$  at  $a$  tend to vanish. There is, however, no a priori assurance that such a sequence of  $\bar{y}$  exists. The proof of its existence is complicated and involves the use of the low power theorem.<sup>8</sup>

Poisson, in his study of singular solutions, used a transformation in which the unknown is replaced by a power of itself. A similar transformation was used by Darboux<sup>9</sup> in connection with the singular solutions of partial differential equations.

8. We present another theorem concerning low powers.

**THEOREM:** *Let*

$$(18) \quad y_i^{p_i} + F_i, \quad i = 1, \dots, n,$$

*be d.p. in  $\mathfrak{F} \{ y_1, \dots, y_n \}$  with each  $p_i$  a positive integer and with each  $F_i$  either identically zero or else composed of terms each of which is of total degree greater than  $p_i$  in the  $y_{jk}$ . The zero  $y_i = 0$ ,  $i = 1, \dots, n$ , of (18) is a component of the system (18).*

Let  $y_i = 0$ ,  $i = 1, \dots, n$ , be a proper part of a component  $\mathfrak{M}$  of (18). To fix our ideas, suppose that  $y_1$  does not hold  $\mathfrak{M}$ . We use a point of contact  $a$  of the type of §3. For every  $m$  and for every  $\epsilon > 0$ , there is a point of  $\mathfrak{M}$

$$(19) \quad y_i = \sum_{j=0}^{\infty} b_{ij} (x - a)^j, \quad i = 1, \dots, n,$$

<sup>7</sup> Bulletin de la Société Mathématique de France, vol. 28 (1900), p. 201.

<sup>8</sup> Ritt, 32, and Levi, 17.

<sup>9</sup> See Ritt, 41.

with

$$(20) \quad |b_{ij}| < \epsilon, \quad i = 1, \dots, n; j = 0, \dots, m,$$

and with  $b_{10} \neq 0$ .

We use a positive integer  $s$  and a positive number  $c$ , both of which will be fixed later. Considering a definite point (19), which corresponds to given  $m$ ,  $\epsilon$ , we let

$$(21) \quad w_i(z) = c^{-s}y_i(x), \quad i = 1, \dots, n,$$

where  $z$  is related to  $x$  as in (11).

Each equation  $y_i^{p_i} + F_i = 0$  goes over into an equation

$$(22) \quad w_i^{p_i} + \sum_{j=1}^r c^{\mu_j s - \nu_j} \alpha_j B_j = 0,$$

where the  $B$  are power products in the  $w$  and their derivatives with respect to  $z$ ; the  $\mu$  are positive integers and the  $\nu$  are nonnegative integers. Each  $\alpha_j$  is the coefficient in  $F_i$  of the power product which produces  $B_j$  and we regard the  $\alpha$ , for any  $c$ , as functions of  $z$ . It is unnecessary to express the dependence on  $i$  of  $\sum$  in (22).

Let  $s$  be fixed at a value large enough for every  $\mu s - \nu$  to be positive.

We have, by (19),

$$(23) \quad w_i(z) = \sum_{j=0}^{\infty} c^{-s+j} b_{ij} z^j, \quad i = 1, \dots, n.$$

We now fix  $c$  in such a way that the greatest of the quantities

$$c^{-s+j} |b_{ij}|, \quad i = 1, \dots, n; j = 0, \dots, s-1,$$

equals unity. This is possible because  $b_{10} \neq 0$ . Then, if  $m \geq s-1$  and if  $\epsilon$  is small,  $c$  will be small.

It follows that, by decreasing  $1/m$  and  $\epsilon$ , we can select a sequence of points (19) which yields, for every  $i$ , a sequence of  $w_i$  which tends toward a polynomial which is either identically zero or else of degree  $s-1$  at most. The selection can be made in such a way that, for some  $i$ , the  $w_i$  converge to a polynomial distinct from zero; fixing our ideas, we assume that  $w_1$  tends towards a nonzero polynomial  $\gamma(z)$ .

We now consider (22) with  $i = 1$ . When  $c$  is small and the  $w_i$  are close to their polynomial limits, the expansion of  $\sum$  in (22) will begin with a large number of small coefficients. This contradicts the fact that  $\gamma^{p_1} \neq 0$  and our theorem is established.

#### DIFFERENTIAL POLYNOMIALS IN ONE INDETERMINATE, OF FIRST ORDER

9. Let  $A$ , in  $\mathfrak{F}\{y\}$ , be of the first order in  $y$  and algebraically irreducible. Limiting ourselves to the analytic case, we shall show how to determine, in a

finite number of steps, a finite system of d.p. whose manifold is the general solution of  $A$ .

Let  $A$  be of degree  $m$  in  $y_1$ . We consider the system

$$(24) \quad A, A_1, \dots, A_{m-1}$$

where  $A_j$  is the  $j$ th derivative of  $A$ . Let (24) be considered as a set of polynomials and let it be resolved into finite systems, each equivalent to a prime ideal and none holding any other. There will be precisely one system,  $\Lambda$ , which is not held by  $S$ , the separant of  $A$  (§3). Let the polynomials in  $\Lambda$  be considered now as d.p. and let  $\Sigma$  be the system of d.p. thus obtained.

We shall prove that *the manifold of  $\Sigma$  is the general solution of  $A$* .

10. We know that the general solution  $\mathfrak{M}$  of  $A$  is contained in the manifold of  $\Sigma$ . We have to show that every zero of  $\Sigma$  is in  $\mathfrak{M}$ .

We observe that  $A$  holds  $\Sigma$ . The zeros of  $A$  not in  $\mathfrak{M}$  are zeros of  $S$ . The common zeros of  $A$  and  $S$  are zeros of the resultant of  $A$  and  $S$  with respect to  $y_1$ , which is a nonzero d.p.  $R$  of order zero. It suffices then to show that if a zero  $u$  of  $R$  is a zero of  $\Sigma$ ,  $u$  is contained in  $\mathfrak{M}$ .

Let  $u_j$  be the  $j$ th derivative of  $u$ . Then  $A = 0$  for  $y = u$ ,  $y_1 = u_1$ . There exist an open region  $\mathbf{A}_1$  and an  $h > 0$  such that, for

$$(25) \quad x \text{ in } \mathbf{A}_1 \text{ and } 0 < |y - u| < h,$$

every solution of the algebraic relation  $A = 0$ , for  $y_1$  considered as a function of  $y$  and  $x$ , is given by a series

$$(26) \quad y_1 = u_1 + a_0(y - u)^{q/s} + \dots + a_p(y - u)^{(q+p)/s} + \dots,$$

where the  $a$  are functions of  $x$  analytic in  $\mathbf{A}_1$  and where  $q$  and  $s$  are integers,  $s$  being positive. The particular series used in the second member of (26) depends on the particular solution  $y_1$  used, but, for each series, we have  $s \leq m$ . We suppose that,<sup>10</sup> in each series,  $a_0$  does not vanish for every  $x$ .

The system of functions

$$(27) \quad u, u_1, \dots, u_m$$

is a zero of  $\Lambda$ . In some region  $\mathbf{A}_2$  contained in  $\mathbf{A}_1$ , we can approximate to (27) arbitrarily closely by a zero of  $\Lambda$  with  $R$  distinct from zero throughout  $\mathbf{A}_2$ . We understand that the coefficients in  $A$  are analytic throughout  $\mathbf{A}_2$ .

It follows that, if  $\xi$  is any point in  $\mathbf{A}_2$ , the differential equation  $A = 0$  has solutions analytic at  $\xi$ , with  $R \neq 0$  at  $\xi$ , for which  $y, \dots, y_m$  differ arbitrarily slightly at  $\xi$  from  $u, \dots, u_m$  respectively.<sup>11</sup>

Any such solution satisfies (26), in the neighborhood of  $\xi$ , for an appropriate choice of the series in (26).<sup>12</sup> Hence there must be one of the series for which (26) is satisfied by a zero of  $A$  with  $R \neq 0$  and with  $y, \dots, y_m$  as close as one

<sup>10</sup> We are setting aside the trivial case of  $A = \mu(y_1 - u_1)$  with  $\mu$  in  $\mathfrak{F}$ .

<sup>11</sup> If  $R \neq 0$  at  $\xi$  for a zero of  $\Lambda$ , then  $S \neq 0$  at  $\xi$ .

<sup>12</sup> If  $R \neq 0$  at  $\xi$ ,  $y - u \neq 0$  for a neighborhood of  $\xi$ .

pleases at  $\xi$  to  $u, \dots, u_m$ . In what follows, we deal with such a series and assume  $\xi$  to be taken so that  $a_0 \neq 0$  at  $\xi$ .

We are going to prove that  $q \geq s$ . We assume that  $s > q$  and produce a contradiction.

We see first that  $q > 0$  in (26). Otherwise  $y_1 - u_1$  could not be small at  $\xi$  if  $y - u$  is small at  $\xi$ . Then  $s > 1$ .

We show now that  $2q/s - 1 > 0$ . Differentiating (26), we find

$$(28) \quad y_2 = u_2 + \sum \frac{q+p}{s} a_p (y-u)^{(q+p)/s-1} (y_1 - u_1) \\ + \sum \frac{da_p}{dx} (y-u)^{(q+p)/s}.$$

We replace  $y_1$  in (28) by its expression in (26). As  $q < s$ , (28) becomes

$$(29) \quad y_2 = u_2 + \frac{q}{s} a_0^2 (y-u)^{2q/s-1} + b_1 (y-u)^{(2q+1)/s-1} + \dots,$$

where the  $b$  are analytic in  $A_1$ . If we had  $2q/s - 1 \leq 0$ ,  $y_2 - u_2$  could not be small at  $\xi$  when  $y - u$  is small. Thus

$$\frac{2q}{s} - 1 \geq \frac{1}{s}.$$

Then  $2q \geq s + 1$ . It follows, since  $q < s$ , that  $s > 2$ . If we differentiate (29) and use (26), it follows as above that  $3q/s - 2 > 0$ . We find then that  $3q \geq 2s + 1$  and that  $s > 3$ . Continuing, we find that  $s$  exceeds  $m$ . Then  $q \geq s$ .

We are now able to show that  $u$  belongs to  $\mathfrak{M}$ . In (26), we replace  $y - u$  by  $v^s$ . Then (26) goes over into the differential equation

$$(30) \quad s \frac{dv}{dx} = a_0 v^{q-s+1} + \dots + a_p v^{q+p-s+1} + \dots.$$

Since the second member of (30) is analytic in  $v$  and  $x$  for  $v$  small and  $x$  close to  $\xi$ , then, if we fix  $v$  as a small quantity at  $\xi$ , distinct from 0, (30) will have a solution analytic at  $\xi$ , not identically zero, with any desired finite number of derivatives as small as one pleases<sup>13</sup> at  $\xi$ . Then  $y - u$ , which equals  $v^s$ , while not zero at  $\xi$  will be small at  $\xi$ , together with as great a finite number of its derivatives as one may choose to consider. Zeros of  $A$ , close to  $u$  but distinct from  $u$  at  $\xi$ , cannot annul  $R$ .

Thus if  $u$  annuls  $S$  as well as  $R$ ,  $u$  adheres to the nonsingular zeros of  $A$  and belongs to  $\mathfrak{M}$ . If  $u$  does not annul  $S$ ,  $u$  certainly belongs to  $\mathfrak{M}$ .

#### SEQUENCES OF IRREDUCIBLE MANIFOLDS

11. Let

$$\Sigma_1, \dots, \Sigma_p, \dots$$

<sup>13</sup> Equation (30) is satisfied by  $v = 0$  and its solution is analytic in the constant of integration.

be an infinite sequence of prime ideals in  $\mathfrak{F}\{y_1, \dots, y_n\}$ , each  $\Sigma_i$  a proper divisor of  $\Sigma_{i+1}$ . The intersection of the  $\Sigma_i$  is a prime ideal  $\Sigma$ . Strodt has investigated<sup>14</sup> the relationship of the manifold  $\mathfrak{M}$  of  $\Sigma$  to the manifolds  $\mathfrak{M}_i$  of the  $\Sigma_i$ .

Of course,  $\mathfrak{M}$  contains every  $\mathfrak{M}_i$ . The dimension of  $\mathfrak{M}$  is shown to exceed that of any  $\mathfrak{M}_i$ . A point  $y_1(x), \dots, y_n(x)$  of  $\mathfrak{M}$  which is not in the union  $\mathfrak{N}$  of the  $\mathfrak{M}_i$  adheres to  $\mathfrak{M}$ . For the  $y$  as just given, a point at which the  $y$  are analytic and which is not a point of contact with  $\mathfrak{N}$  is called an exceptional point. The totality of exceptional points, for all zeros of  $\Sigma$  not in  $\mathfrak{N}$ , is at most countably infinite. Such a countably infinite set of exceptional points may exist, and may be dense in  $\mathbf{A}$ . In fact, a given zero of  $\Sigma$  may have a dense set of exceptional points.

#### OPERATIONS UPON MANIFOLDS

12. We use a single indeterminate  $y$ , and deal with the analytic case. Let  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  be manifolds. There exist d.p. which vanish for every sum  $y_1 + y_2$  with  $y_1$  in  $\mathfrak{M}_1$  and  $y_2$  in  $\mathfrak{M}_2$ .<sup>15</sup> The manifold of the totality of such d.p. is called the *sum*<sup>16</sup> of  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  and is denoted by  $\mathfrak{M}_1 + \mathfrak{M}_2$ . The *product* of the two manifolds is defined similarly. Let  $\mathfrak{M}$  be any manifold. Let  $\Sigma$  be the totality of those d.p. which vanish for the derivative of every  $y(x)$  in  $\Sigma$ . The manifold  $\mathfrak{M}'$  of  $\Sigma$  is called the *derivative* of  $\mathfrak{M}$ .

A manifold  $\mathfrak{M}$  is said to be *limited* if either  $\mathfrak{M}$  consists of the single function zero, or  $\mathfrak{M}$  contains nonzero functions and the function zero does not adhere to the set of reciprocals of such functions.  $\mathfrak{M}$  is limited if and only if it is held by a d.p. of the form  $y^p + F$ , where  $F$  either is zero or else consists of terms of degree less than  $p$ .

If  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  are general solutions of d.p. of the first order, and are limited,  $\mathfrak{M}_1 + \mathfrak{M}_2$  and  $\mathfrak{M}_1\mathfrak{M}_2$  are limited. If  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$ , irreducible and of order more than unity, are limited, their limited character may not be communicated to their sum and product. This is seen from examples based on the theory of the elliptic functions. For the case of the product, the result just stated is equivalent to the fact that the product of two manifolds may contain the function zero, even if neither manifold does. The derivative of every limited manifold is limited.

<sup>14</sup> Strodt, 44.

<sup>15</sup> We take  $y_1$  and  $y_2$  with the same domain of analyticity.

<sup>16</sup> Ritt, 37.

## CHAPTER VII

### INTERSECTIONS OF ALGEBRAIC DIFFERENTIAL MANIFOLDS

#### DIMENSIONS OF COMPONENTS OF INTERSECTIONS

1. B.L. van der Waerden has shown<sup>1</sup> that if two irreducible algebraic manifolds in the space of  $y_1, \dots, y_n$  have the respective dimensions  $p$  and  $q$ , every component of their intersection is of dimension at least  $p + q - n$ . For algebraic differential manifolds, there is no such regularity. We shall exhibit, for the case of  $n = 3$ , two irreducible manifolds of dimension 2 whose intersection consists of a single point.

2. Working with  $u, v, y$ , we let

$$F = u^5 - v^5 + y(uv_1 - vu_1)^2.$$

We take  $\mathfrak{F}$  as the field of complex numbers.  $F$  is algebraically irreducible. We shall find its components. A component other than  $\mathfrak{M}$ , the general solution, must be held by the coefficient of  $y$ , therefore by  $u^5 - v^5$ . Let

$$A_j = u - \omega^j v, \quad j = 1, \dots, 5,$$

where  $\omega = e^{2\pi i/5}$ . As

$$uv_1 - vu_1 = v_1 A_j - v A'_j, \quad j = 1, \dots, 5,$$

it follows from the low power theorem that, for each  $j$ , the manifold of  $A_j$  is a component of  $F$ . Thus  $F$  has six components, each of dimension 2.

The manifold  $\mathfrak{M}'$  of  $y$  is two-dimensional. We shall show that  $\mathfrak{M}$  and  $\mathfrak{M}'$  have precisely one point in common, the point  $u = v = y = 0$ .

We show first that  $u = v = y = 0$  is in  $\mathfrak{M}$ . Let  $\bar{u}, \bar{v}, \bar{y}$  be any point of  $\mathfrak{M}$  which does not annul  $u^5 - v^5$ . If  $c$  is an arbitrary constant with respect to  $\mathfrak{F} \langle \bar{u}, \bar{v}, \bar{y} \rangle$ , it follows from the homogeneity of  $F$  and  $u^5 - v^5$  that  $c\bar{u}, c\bar{v}, c\bar{y}$  is in  $\mathfrak{M}$ . Then every d.p. which holds  $\mathfrak{M}$  vanishes for  $u = v = y = 0$  and our statement is proved.

Now let  $\bar{u}, \bar{v}, 0$  be a point of  $\mathfrak{M}$ .

For each  $j$ , we write  $u^5 - v^5 = A_j B_j$ . For every zero of  $F$  with  $y = 0$ , in particular, for every point of  $\mathfrak{M}$  with  $y$  equal to zero,  $u^5 - v^5$ , and therefore some  $A_j$ , vanishes. By III, §23, a zero of an  $A_j$  which lies in  $\mathfrak{M}$  annuls  $B_j$ , and therefore annuls some  $A_k$  with  $k \neq j$ . It follows that  $\bar{u} = \bar{v} = 0$ .

The anomaly which we have just found has nothing to do with "points at infinity." It would be futile to try to remove it by creating a "projective space."

#### ORDERS OF COMPONENTS OF AN INTERSECTION

3. We consider, as in V, §27, a finite system  $\Psi$  of nonzero d.p. in  $y_1, \dots, y_n$ . Let  $y_i$  be involved in  $\Psi$  up to the order  $m_i$ .

<sup>1</sup> *Mathematische Annalen*, vol. 115 (1938), p. 330.

Let  $\mathfrak{M}$ , the manifold of a prime ideal  $\Gamma$ , be a component of  $\Psi$  of dimension  $q$ . We are interested in securing a bound for the order of  $\mathfrak{M}$  when  $q = 0$ , and a bound for the order of  $\mathfrak{M}$  relative to any given parametric set when  $q > 0$  (II, §35). We shall secure a bound for the order, or for the relative order, in terms of the  $m_i$ .

The result which we shall obtain may be regarded as a counterpart, for systems of algebraic differential equations, of Bézout's theorem on the number of solutions of a system of algebraic equations.

4. Suppose first that  $q = 0$ .

We consider the system  $\Phi$  of d.p. in indeterminates  $u_{ij}, v_i$ , obtained from  $\Psi$  as in V, §27. Some prime ideal  $\Sigma'$  as in V, §26, goes over into  $\Gamma$  by the substitution<sup>2</sup> (29).  $\Sigma'$  is contained in a prime ideal  $\Sigma$ , described as in V, §20. We wish to see that  $\Sigma$  is of dimension zero. Suppose that  $\Sigma$  is of positive dimension; it will have a parametric set  $u_1, \dots, u_k; v_1, \dots, v_r$ . Suppose that  $v$  are actually present in this set. Then  $v_1$  corresponds to some  $y_j$  in  $\Psi$ . As  $\Gamma$  has a d.p. in  $y_j$  alone,  $\Sigma$  has a d.p. in  $v_1$  alone. Thus there are no parametric  $v$ .

Let us now consider  $u_1$ . It corresponds to some  $y_j$ .  $\Gamma$  has a nonzero d.p.  $N$  in  $y_j$  alone. We assume  $N$  to be algebraically irreducible. Suppose that  $N$  involves derivatives of  $y_j$  of order less than  $s$ . Let  $y_{j\mu}$  be the lowest such derivative. Then some linear combination of  $N$  and its derivative is free of  $y_{j\mu}$ . Continuing, we find a nonzero d.p.  $P$  in  $\Gamma$ , involving  $y_j$  alone, in which the derivatives of  $y_j$  are of orders at least  $s$ . To  $P$ , there corresponds in  $\Sigma$  a d.p. in  $u_1$  alone.

Thus  $\Sigma$  is of dimension zero. The set (18) thus becomes

$$u_{h+1}, \dots, u_m; \quad u'_1, \dots, u'_h; \quad v_1, \dots, v_r.$$

We consider the system (20) which corresponds to  $\Sigma$ . The second members are expressions in  $u_1, \dots, u_h; w$ .

We seek a bound for  $h$ . If  $y_i$  occurs up to the order  $m_i$  in  $\Psi$ ,  $y_i$  yields  $m_i$  letters  $u$ . Thus

$$h \leq m_1 + \dots + m_n.$$

Let  $\bar{u}_1, \dots, \bar{u}_m; \bar{v}_1, \dots, \bar{v}_r; \bar{w}$  be a generic zero of  $\Sigma$ . Let  $\zeta$  be a derivative of any order of one of the quantities  $\bar{u}, \bar{v}$ . Then  $\zeta$  has an expression which is rational in  $\bar{u}_1, \dots, \bar{u}_h; \bar{w}$ . If  $\zeta$  is one of the quantities just written, the expression is  $\zeta$  itself. Otherwise, we use (20) and  $R = 0$ ; a sufficient number of differentiations and substitutions gives the desired expression for  $\zeta$ . In particular, proper derivatives of  $w$  which appear during the differentiations of the  $z$  are obtained by differentiation from  $R = 0$ .

Let us consider now any  $h + 1$  of the letters  $u_{ij}, v_{ij}$ . They furnish  $h + 1$  quantities  $\zeta$ , with expressions as just described. Using these  $h + 1$  expressions, and the relation  $R = 0$  for  $\bar{u}_1, \dots, \bar{u}_h; \bar{w}$ , we obtain, by an elimination, a non-

<sup>2</sup> We use, at present, equation numbers of Chapter V. The  $u_i$  of  $\Sigma'$  are the  $u_{ij}$  of  $\Phi$ .

zero polynomial in the  $h + 1$  letters  $u_{ij}, v_{ij}$  which is a d.p. in  $\Sigma$ . It follows that, given any  $h + 1$  distinct  $y_{ij}$ ,  $\Gamma$  contains a nonzero d.p. which involves only those  $y_{ij}$ . This means, by II, §35, that the order of  $\Gamma$  cannot exceed  $h$ .

We may thus state the following theorem:

**THEOREM:** *Let  $\Phi$  be a finite system of nonzero d.p. in  $\mathfrak{F}\{y_1, \dots, y_n\}$ . Let  $m_i$  be the maximum of the orders of those derivatives of  $y_i$  which appear in  $\Phi$ . If a component  $\mathfrak{M}$  of  $\Phi$  is of dimension zero, the order of  $\mathfrak{M}$  is at most  $m_1 + \dots + m_n$ .*

5. We now suppose that  $q > 0$ . We write the indeterminates as  $u_1, \dots, u_q; y_1, \dots, y_p$ , with the  $u$  parametric for  $\Gamma$ . Let  $A_1, \dots, A_p$ , with  $A_i$  of order  $r_i$  in  $y_i$ , be a characteristic set for  $\Gamma$ . Let  $B_i, i = 1, \dots, p$ , be a nonzero d.p. in  $\Gamma$  involving only  $y_i$  and the  $u$ . Let  $C$  be a d.p. which is not in  $\Gamma$  and which holds every component of  $\Psi$  other than  $\mathfrak{M}$ .

Let  $\bar{u}_1, \dots, \bar{u}_q; \bar{y}_1, \dots, \bar{y}_p$  be a generic zero of  $\Gamma$ . Let the  $\bar{u}$  be substituted for the  $u$  in  $\Psi$ . Then  $\Psi$  becomes a system  $\Psi'$  in  $y_1, \dots, y_p$  over  $\mathfrak{F}\langle\bar{u}_1, \dots, \bar{u}_q\rangle$ . Each  $B_i$  becomes a nonzero  $B'_i$  and  $C$  a nonzero  $C'$ .

Let the components of  $\Psi$  other than  $\mathfrak{M}$  be manifolds of prime ideals  $\Gamma_1, \dots, \Gamma_s$ . Then  $\Psi'$  is equivalent to  $\Gamma', \Gamma'_1, \dots, \Gamma'_s$ , each accented system resulting from the corresponding unaccented one when the  $u$  are replaced by the  $\bar{u}$ .

The totality of d.p. in  $\mathfrak{F}\langle\bar{u}_1, \dots, \bar{u}_q\rangle\{y_1, \dots, y_p\}$  which vanish for  $\bar{y}_1, \dots, \bar{y}_p$  is a prime ideal  $\Delta$  held by  $\Psi'$ . Each  $\Gamma'_i$  is held by  $C'$ , while  $\Delta$  is not. Thus, the manifold of  $\Delta$  is contained in a component  $\mathfrak{M}'$  of  $\Psi'$  which is held by  $\Gamma'$ .  $\mathfrak{M}'$  must be of dimension zero, since each  $B'$  is in  $\Gamma'$ . Then  $\Delta$  is of dimension zero and, by II, §36, its order  $h$  does not exceed the order of  $\mathfrak{M}'$ .

Now  $\Delta$  contains no d.p. involving only  $y_{ij}$  with  $j < r_i$ ; otherwise, there would be a nonzero d.p. reduced with respect to  $A_1, \dots, A_p$  which vanishes for the generic zero of  $\Gamma$ . By II, §35,  $h \geq r_1 + \dots + r_p$ . By §4, if the highest derivative of  $y_i$  in  $\Psi$  is of order  $m_i$ , the order of  $\mathfrak{M}'$  does not exceed  $m_1 + \dots + m_p$ . Thus

$$r_1 + \dots + r_p \leq m_1 + \dots + m_p.$$

We may thus formulate the following theorem:

**THEOREM:** *Let  $\Phi$  be a finite set of d.p. in  $\mathfrak{F}\{u_1, \dots, u_q; y_1, \dots, y_p\}$ , the  $u$  being a parametric set for a component  $\mathfrak{M}$  of  $\Phi$ . Let  $m_i, i = 1, \dots, p$ , be the maximum of the orders of those derivatives of  $y_i$  which appear in  $\Phi$ . Then the order of  $\mathfrak{M}$  relative to  $u_1, \dots, u_q$  cannot exceed  $m_1 + \dots + m_p$ .*

In what precedes, the condition that  $\mathfrak{M}$  be a component of  $\Phi$  is essential. For instance, taking  $\Phi$  as  $y_{10} + y_{20}$ , the manifold of  $y_{1n} + y_{10}, y_{10} + y_{20}$ , which is of order  $n$ , is, for every  $n$ , held by  $\Phi$ .

6. Jacobi examined, from the heuristic standpoint, the problem of determining the number of arbitrary constants in the solution of a system of  $n$  differential equations in  $n$  unknowns.<sup>3</sup> Taking the system in the form

<sup>3</sup> See Ritt, 29.

$$(1) \quad u_i = 0, \quad i = 1, \dots, n,$$

where each  $u$  involves the unknowns  $y_1, \dots, y_n$ , a certain number of their derivatives and the independent variable  $x$ , Jacobi considers the derivatives of  $y_i$  appearing in  $u_i$  and denotes the maximum of the orders of those derivatives by  $a_{ij}$ . He forms all sums

$$(2) \quad a_{1j_1} + \dots + a_{nj_n}$$

where  $j_1, \dots, j_n$  is a permutation of  $1, \dots, n$ . He arrives at the conclusion that *the number of arbitrary constants in the solution of (1) does not exceed the greatest sum (2)*.

After our study of algebraic differential manifolds, it is unnecessary to insist on the fact that the notion of the number of constants in the solution of a general system never was a notion which was definite in advance. For algebraic systems, the concept is made definite by the theory of orders of irreducible manifolds which has been developed here. It is thus not surprising that Jacobi's work on this question, in spite of its daring and ingenious quality, should not have firm logical structure.

One would be disposed to regard Jacobi's work as conjectural and to expect that his bound would be found valid in a rigorous theory. We shall see later that Jacobi's bound, like weaker ones given before his time, does not have the broad applicability which one might anticipate for it. We shall treat now a situation in which Jacobi's bound is found to hold.

We deal with two nonzero d.p.  $A$  and  $B$  in  $y$  and  $z$ . We represent by  $a$  and  $b$  the respective orders of  $A$  in  $y$  and  $z$ ; by  $c$  and  $d$  the orders of  $B$  in  $y$  and  $z$ . Let

$$h = \text{Max}(a + d, b + c).$$

We prove the following theorem:

**THEOREM:** *If  $\mathfrak{M}$ , of dimension zero, is a component of the system  $A, B$ , the order of  $\mathfrak{M}$  is at most<sup>4</sup>  $h$ .*

We assume that  $\mathfrak{M}$  is of order greater than  $h$  and produce a contradiction. Fixing our ideas, we assume that  $b \geq d$ .

There exist nonzero d.p.  $C$  whose orders in  $y$  and  $z$  do not exceed  $c$  and  $d$  respectively and which hold  $\mathfrak{M}$ , such that the system  $A, C$  has no component of dimension unity containing  $\mathfrak{M}$ .  $B$  is such a d.p. From among all such d.p.  $C$ , we select one which, for the order  $y, z$  of the indeterminates, is of a least rank. The d.p. selected will be denoted by  $D$ .

We are going to prove that  $D$  is free of  $z$ . We assume that  $z$  is present in  $D$  and force a contradiction.

Let  $D$  be of order  $e$  in  $y$  and of order  $f$  in  $z$ . Let  $S$  be the separant of  $D$ . There is a relation

<sup>4</sup> A better bound can be given in the case in which one of  $y$  and  $z$  is absent from one of  $A$  and  $B$ . For instance, if  $B$  is free of  $z$ , it can be shown as below that the order of  $M$  does not exceed  $b + c$ .

$$S'A \equiv E, \quad [D],$$

where  $E$  has an order in  $z$  not exceeding  $f$  and an order in  $y$  not exceeding

$$\text{Max} (a, e + b - f).$$

Suppose that  $S$  does not hold  $\mathfrak{M}$ . Let  $\mathfrak{M}'$  be a component of the system  $D$ ,  $E$  which contains  $\mathfrak{M}$ . Then  $\mathfrak{M}'$  is a component of  $A$ ,  $D$  and is thus of dimension zero. Applying the theorem of §4 to  $D$ ,  $E$  and using II, §36, we find that the order of  $\mathfrak{M}$  does not exceed<sup>5</sup>

$$\text{Max} (a, e + b - f) + f = \text{Max} (a + f, e + b).$$

As  $f \leq d$  and  $e \leq c$ , the order of  $\mathfrak{M}$  cannot exceed  $h$ .

Thus  $S$  holds  $\mathfrak{M}$ . If  $D$  is of degree  $q$  in  $z_f$ , then

$$(3) \quad qD = z_f S + T,$$

where  $T$ , like  $S$ , is of lower rank than  $D$  in  $z$ . Also  $T$  holds  $\mathfrak{M}$ .

By I, §29, we may assume that  $\mathfrak{F}$  has a nonconstant element.<sup>6</sup> We shall prove the existence of an element  $\mu$  in  $\mathfrak{F}$  such that all components of the system

$$(4) \quad A, \quad S + \mu T$$

which contain  $\mathfrak{M}$  are of dimension zero. As  $S + \mu T$  will be of lower rank than  $D$  in  $z$ , our statement that  $D$  is free of  $z$  will be proved.

By (3), the system  $A$ ,  $D$  holds the system

$$(5) \quad A, S, T.$$

Let  $u$  be an indeterminate. We consider the system

$$(6) \quad A, \quad S + uT$$

in  $y, z, u$ . Let the essential prime divisors of the perfect ideal determined by the system (6) be  $\Sigma_1, \dots, \Sigma_r$ . Let  $\Sigma_1, \dots, \Sigma_s$  be those  $\Sigma$  which are not held by (5). We say that each of these ideals contains a nonzero d.p. in  $y$  and  $u$  alone and a nonzero d.p. in  $z$  and  $u$  alone.

Suppose that  $\Sigma_1$  contains no d.p. in  $y$  and  $u$  alone. Then, if the indeterminates are taken in the order  $u, y, z$ ,  $\Sigma_1$  has a characteristic set composed of one d.p., so that the manifold of  $\Sigma_1$  is the general solution of a d.p.  $F$  (II, §§18, 33). Now  $F$  cannot involve  $u$ , for  $F$  will continue to be a characteristic set for  $\Sigma_1$  if the indeterminates are taken in the order  $y, z, u$ , and  $A$ , which is in  $\Sigma_1$ , does not involve  $u$ . We take the remainder of  $S + uT$  with respect to  $F$  for the order  $u, y, z$ . We secure a relation

<sup>5</sup> As  $b \geq d \geq f$ , we have  $e + b - f \geq e$ .

<sup>6</sup> Suppose that  $\mathfrak{F}$  consists purely of constants. The ideal  $\{A, B\}$  has essential prime divisors  $\Sigma_1, \dots, \Sigma_n$ . When an element  $x$ , of derivative unity, is adjoined to  $\mathfrak{F}$ , we secure a larger  $\{A, B\}$ . Its essential prime divisors can easily be shown to be the prime ideals generated by the  $\Sigma$  in  $\mathfrak{F} < x >$ . The new prime ideals have the same characteristic sets as the old ones.

$$J(S + uT) \equiv 0, \quad [F],$$

with  $J$  a power product in the initial and separant of  $F$ . This means, since  $F$  is free of  $u$ , that

$$JS \equiv 0, \quad JT \equiv 0, \quad [F].$$

It follows that each d.p. in (5) is in  $\Sigma_1$ .

This proves that  $\Sigma_i$ , for  $i \leq s$ , contains a d.p.  $H_i$  in  $y$  and  $u$  alone. Similarly each such  $\Sigma_i$  contains a d.p.  $K_i$  in  $z$  and  $u$  alone. Let  $M$  be the product of the d.p.  $H$  and  $N$  the product of the  $K$ . Let  $u$  be fixed as an element  $\mu$  in  $F$  so that  $M$  goes over into a nonzero d.p.  $U$  in  $y$  alone,  $N$  into a nonzero d.p.  $V$  in  $z$  alone.

Then those zeros of

$$(7) \quad A, \quad S + \mu T$$

which are not zeros of (5) must annul  $U$  and  $V$ . A fortiori, all zeros of (7) which are not zeros of  $A, D$  annul  $U$  and  $V$ .

This shows that a component of (7) which is not contained in a component of  $A, D$  is held by  $U$  and  $V$ . Then every component of (7) which contains  $\mathfrak{M}$  is of dimension zero. This proves that  $D$  is free of  $z$ .

$D$  must involve  $y$  effectively, since  $A, D$  has zeros. Denoting still by  $S$  the separant of  $D$ , we secure a relation

$$S^e A \equiv L, \quad [D],$$

where the orders of  $L$  in  $y$  and  $z$  do not exceed  $e$  and  $b$  respectively. We reason with  $D, L$  as with  $D, E$ , above, to show that  $S$  holds  $\mathfrak{M}$ . Then we follow the method above to find a system similar to (4), with  $S + \mu T$  of lower rank than  $D$ . Thus it is not possible to choose a d.p. of least rank among the d.p.  $C$ . This completes the proof that the order of  $\mathfrak{M}$  does not exceed  $h$ .

#### INTERSECTIONS OF GENERAL SOLUTIONS

7. By all the rules of play, the bound  $h$  of §6 should, when  $A$  and  $B$  are algebraically irreducible, apply to the components of dimension zero in the intersection of the general solutions of  $A$  and  $B$ . The general solution of a d.p.  $F$  in  $y$  and  $z$  can be regarded as the solution of the differential equation obtained by solving for the highest derivative of one of  $y$  and  $z$  in the equation  $F = 0$ . To be sure, we would then be dealing with irrational differential equations. However, as Jacobi's considerations are detached from questions of the theory of functions, one would not expect irrationality to have a bearing on the problem. It might be suggested that Jacobi's heuristic work, as well as previous work which yielded bounds like that of §4, was intended to apply to the "general case." If so, the heuristic history of differential equations has been different from that of algebraic equations. Bézout's work of the middle eighteenth century, on the number of solutions of a system of algebraic equations, was entirely heuristic. His conjecture was validated, late in the nineteenth century, not for a "general case" but for all systems.

The actual situation is as follows. If the orders of  $A$  and  $B$  in each of  $y$  and  $z$  do not exceed unity, we get the bound of §6 for a component of the intersection of the general solutions. For higher orders, that bound need not hold. We shall show how to construct, for every  $n > 3$ , a d.p. of order  $n$  in  $y$  and in  $z$  whose general solution intersects the manifold of  $y$  in an irreducible manifold of dimension zero and order  $2n - 3$ .

8. We consider  $A$  and  $B$ , as in §6, assuming them to be algebraically irreducible, with each of  $a, b, c, d$  not greater than unity. We prove the following theorem.

**THEOREM:** *If  $\mathfrak{M}$ , of dimension zero, is a component of the intersection of the general solutions of  $A$  and  $B$ , the order of  $\mathfrak{M}$  does not exceed  $h$ .*

Thus the order of  $\mathfrak{M}$  does not exceed 2.

We represent by  $\mathfrak{N}$  the intersection of the general solutions of  $A$  and  $B$ .

If  $a, b, c, d$  are all zero, the general solutions of  $A$  and  $B$  are their complete manifolds and we have merely to apply the theorem of §4.

Suppose now that  $a = b = 0$  and that at least one of  $c$  and  $d$  is 1. We consider first the intersection  $\mathfrak{M}'$  of the complete manifolds of  $A$  and  $B$ . Every component of  $\mathfrak{M}'$  of dimension zero has an order not exceeding unity. By II, §36, if  $\mathfrak{M}$  is not contained in a component of  $\mathfrak{M}'$  of dimension unity, the order of  $\mathfrak{M}$  does not exceed unity.

We have now to consider the case in which  $\mathfrak{M}$  is contained in a component  $\mathfrak{M}''$  of  $\mathfrak{M}'$  of dimension unity.  $\mathfrak{M}''$  is the general solution of a d.p.  $C$ . Because  $A$  holds  $\mathfrak{M}''$ ,  $C$  must be of order zero in each of  $y$  and  $z$ ; this implies that  $\mathfrak{M}''$  is the manifold of  $A$ . Then  $\mathfrak{M}''$  must be a component of the manifold of  $B$ . Otherwise  $\mathfrak{M}''$  would be contained in the general solution of  $B$  and  $\mathfrak{M}$  would not be a component of  $\mathfrak{N}$ .<sup>7</sup>

We suppose, as we may, that  $A$  involves  $z$  effectively. As  $\mathfrak{M}''$  is a component of  $B$  other than the general solution, we have  $d = 1$  (III, §15). Let  $S$  be the separant of  $A$ . We have, by the low power theorem, a relation

$$S'B = C_0A^p + C_1A^{p_1}A_1^{q_1} + \cdots + C_rA^{p_r}A_1^{q_r}.$$

Here  $A_1$  is the derivative of  $A$  and, for every  $i, p_i + q_i > p$ . The orders of the  $C$  in  $z$  and in  $y$  do not exceed 0 and 1 respectively, and no  $C$  is divisible by  $A$ . By III, §23, as  $\mathfrak{M}$  is in the intersection of  $\mathfrak{M}''$  and the general solution of  $B$ ,  $C_0$  must hold  $\mathfrak{M}$ . The manifold of the system  $C_0, A$  is a proper part of  $\mathfrak{M}''$  and thus, by II, §36, has components which are all of dimension zero. By §6, the order of such a component cannot exceed unity. Then the order of  $\mathfrak{M}$  does not exceed unity; this is what was to be proved.

Suppose now that at least one of  $a$  and  $b$  is unity and that at least one of  $c$  and  $d$  is unity. We take up immediately the case in which  $\mathfrak{M}$  is contained in a component  $\mathfrak{M}''$  of  $\mathfrak{M}'$  of dimension unity; when  $\mathfrak{M}$  is not so contained, it follows

<sup>7</sup> By III, §15, the components of  $B$  other than its general solution are manifolds of d.p. of orders zero in  $y$  and  $z$ .

from §6 that its order does not exceed  $h$ . As  $\mathfrak{M}$  is a component of  $\mathfrak{N}$ ,  $\mathfrak{M}''$  is not part of  $\mathfrak{N}$ . Let, then,  $\mathfrak{M}''$  fail to be contained in the general solution of  $B$ . Then some other component of  $B$  contains  $\mathfrak{M}''$  and is thus identical with  $\mathfrak{M}''$ . By the case which precedes, the components of the intersection of  $\mathfrak{M}''$  with the general solution of  $B$  are of dimension zero and of order at most unity. This completes the proof.

9. We are going to present a d.p.  $F$  in  $y$  and  $z$ , of order 4 in  $y$  and in  $z$ , whose general solution will be shown to intersect the manifold of  $y = 0$  in an irreducible manifold of dimension zero and order 5.

Through §13,  $K_1$  will represent, for any d.p.  $K$ , the derivative of  $K$ . We let

$$(8) \quad A = y_1 - z_3 y^2,$$

$$(9) \quad B = A^4 - y_3^8,$$

$$(10) \quad C = y_3 A_1 - 2y_4 A,$$

$$(11) \quad F = B - y^6 C^2 = A^4 - y_3^8 - y^6 C^2.$$

We use the field of rational numbers. Let us see first that  $F$  is algebraically irreducible. If we consider the equation  $F = 0$  as an algebraic equation for  $y_4$ , we secure a function  $y_4$  of two branches. Thus, if  $F$  were factorable, it would have a factor of positive degree free of  $y_4$ . Such a factor would have to be a factor of  $y^6 A^2$ . As  $F$  is not divisible by  $y$  or by  $A$ ,  $F$  is algebraically irreducible.

Let us now determine the components of  $F$  other than the general solution.

Let  $\mathfrak{N}$  be such a component. As  $\partial F / \partial y_4 = 4y^6 A C$ ,  $\mathfrak{N}$  must be held by  $yC$  or by  $A$ . Suppose that  $A$  holds  $\mathfrak{N}$ . By (10) and (11),  $y_3$  holds  $\mathfrak{N}$ . In every case then,  $B$  holds  $\mathfrak{N}$ .

Now  $B$  is the product of the four d.p.

$$(12) \quad E^{(j)} = y_1 - z_3 y^2 - j y_3^2, \quad j = \pm 1, \quad \pm (-1)^{1/2},$$

each of which is algebraically irreducible. For what follows, it is important to know that the manifold of each  $E$  is irreducible. From the manner in which  $z_3$  figures in (12), one sees that a component of  $E^{(j)}$  other than the general solution is held by  $y$ . Such a component, being of dimension unity, must be the manifold of  $y$ . But the low power theorem shows that the manifold of  $y$  is not a component. This proves the irreducibility of the manifolds of the  $E$ .

We have, for every  $j$ ,

$$C = y_3 E_1^{(j)} - 2y_4 E^{(j)}.$$

Referring to (11), and applying the low power theorem, we see that the manifold of each  $E$  is a component<sup>8</sup> of  $F$ .

It will be proved that the intersection of the general solution of  $F$  with the manifold of  $y = 0$  is the manifold of the system  $y = 0, z_i = 0$ . The latter manifold is of dimension zero and order 5.

<sup>8</sup> For the order  $y, z$  of the indeterminates,  $F$  as it stands is in the form (25) of III, §17.

10. We refer to I, §26. We use any positive integer  $p$  and any power product  $P$  in  $y$  and its derivatives. The degree of  $P$  is denoted by  $d$  and its weight by  $w$ . The second member of (31) of I, §26, will be represented by  $\delta(p, w)$ . Let  $U = y^p$ . We shall prove that  $P$  has a representation as a homogeneous polynomial in  $U$  and derivatives of  $U$ , whose coefficients are homogeneous polynomials<sup>9</sup> in  $y$  and derivatives of  $y$  of a common degree not greater than  $\delta(p, w)$ .

If  $d \leq \delta(p, w)$ ,  $P$  itself is the representation sought. Otherwise, by I, §26,  $P$  is a linear combination of  $U$  and its derivatives, with coefficients all of degree  $d - p$  and none of weight exceeding  $w$ . If  $d - p \leq \delta(p, w)$ , we have the desired representation. Otherwise, the coefficients of  $U$  and its derivatives will be in  $[U]$ . Continuing, we have  $P$  expressed as in our statement.

11. Let  $\Sigma$  be an ideal of d.p. in  $y$  and  $z$ ;  $M$  a d.p. in  $y$  and  $z$ ;  $\alpha$  a nonnegative number. We shall say that  $M$  admits  $\alpha$  as a multiplier with respect to  $\Sigma$  if, for every  $\epsilon > 0$ , there exists an integer  $n_0(\epsilon)$  such that, for every  $n > n_0(\epsilon)$ ,

$$M^n \equiv P, \quad (\Sigma),$$

where  $P$  is a d.p. depending on  $M$  and  $n$  which, arranged as a polynomial<sup>10</sup> in the  $y_i$ , contains no term of degree less than  $n(\alpha - \epsilon)$ .  $P$  may be zero. If  $\alpha$  is a multiplier for  $M$  and if  $0 \leq \gamma < \alpha$ ,  $\gamma$  is also a multiplier.

We prove the following properties of multipliers:

(a) Let  $M$  and  $N$  admit  $\alpha$  and  $\beta$ , respectively, as multipliers with respect to  $\Sigma$ . Let  $\gamma = \text{Min}(\alpha, \beta)$ . Then  $M + N$  admits  $\gamma$  as a multiplier.

(b) For  $M$  and  $N$  as in (a),  $MN$  admits  $\alpha + \beta$  as a multiplier.

(c) Let  $M^p$ , where  $p$  is a positive integer, admit  $\alpha$  as a multiplier. Then  $M$  admits  $\alpha/p$ .

(d) Let  $M$  admit  $\alpha$  as a multiplier. Then  $M_1$ , the derivative of  $M$ , also admits  $\alpha$ .

(e) If  $M \equiv N, (\Sigma)$ ,  $M$  and  $N$  admit the same multipliers.

Proving (a), we take an  $\epsilon > 0$ . Let  $n_0(\epsilon/2)$  serve as above for both  $M$  and  $N$  with respect to  $\epsilon/2$ . We consider  $(M + N)^n$  for any  $n \geq 1$ . Let  $R = M^a N^b$  where  $a + b = n$ . If  $a$  and  $b$  both exceed  $n_0(\epsilon/2)$ , we have  $R \equiv P, (\Sigma)$ , where no term of  $P$  is of degree less than

$$a(\alpha - \epsilon/2) + b(\beta - \epsilon/2),$$

which quantity is not less than  $n(\gamma - \epsilon/2)$ . If  $b \leq n_0(\epsilon/2) < a$ , we have  $R \equiv P, (\Sigma)$ , with no term of  $P$  of degree less than

$$[n - n_0(\epsilon/2)](\alpha - \epsilon/2).$$

The last quantity, if  $n$  is large in comparison with  $n_0(\epsilon/2)$ , exceeds  $n(\alpha - \epsilon)$ . The truth of (a) is now clear.

The proofs of (b), (c), and (e) are trivial.

<sup>9</sup> Over the field of rational numbers.

<sup>10</sup> When  $P$  is thus arranged, its coefficients are d.p. in  $z$ . The definition of multiplier thus gives a special role to  $y$ .

Proving (d), we take an  $\epsilon > 0$  and, relative to  $M$ , an  $n_0(\epsilon/2)$ . Let  $m$  be a fixed integer which exceeds  $n_0(\epsilon/2)$ . We consider an  $n > 0$  and use  $\delta(m, n)$  as in §10. Then  $M_1^n$  is a polynomial in  $M^m$  and its derivatives, with coefficients which are d.p. in  $M$  of degree not greater than  $\delta(m, n)$ . In this expression, every power product in  $M^m$  and its derivatives is of degree not less than

$$(13) \quad q = [n - \delta(m, n)]/m.$$

Now if  $n$  is large,  $\delta(m, n)$ , as one sees from I, §26, is small in comparison with  $n$ , so that  $q$  is only slightly less than  $n/m$ . Each power product in  $M^m$  and its derivatives is congruent to a d.p. whose terms have degrees in the  $y_i$  not less than  $qm(\alpha - \epsilon/2)$ . If  $n$  is large, this quantity exceeds  $n(\alpha - \epsilon)$ , q.e.d.

12. We return to  $F$  of §9, denoting the general solution of  $F$  by  $\mathfrak{M}$ . We show now that a point in  $\mathfrak{M}$  with  $y = 0$  satisfies  $z_5 = 0$ . Later we shall prove that every  $z$  with  $z_5 = 0$  is admissible.

We determine first a d.p.  $G$  which holds  $\mathfrak{M}$ , but no other component of  $F$ .

We have, by (9) and (10),

$$(14) \quad AB_1 - 4A_1B = 4y_3^7C.$$

Thus, by (11) (first representation of  $F$ ), we have when  $F = 0$

$$(15) \quad 4y_3^7B^{1/2} = y^3(AB_1 - 4A_1B).$$

Again, letting  $K = y^3C$ , we have by (11), when  $F = 0$ , the relation  $B^{1/2} = K$ . Thus, for  $F = 0$ ,  $B \neq 0$ ,

$$(16) \quad B^{-1/2}B_1 = 2K_1.$$

Substituting into (15) the expression which (16) furnishes for  $B_1$ , we find, for  $F = 0$ ,  $B \neq 0$ ,

$$(17) \quad 4y_3^{14} + L = 0,$$

where

$$(18) \quad L = -4y^3y_3^7AK_1 + y^6A^2K_1^2 - 4y^6A_1^2B.$$

We designate the first member of (17) by  $G$ . Then  $G$  holds  $\mathfrak{M}$ .

13. In what follows, all multipliers will operate with respect to  $[F, G]$ .

In (11),  $y_3^8$  and  $y^6C^2$  contain no terms of degree less than 8 in the  $y_i$ . Thus  $A^4$  admits 8 as a multiplier so that, by (c) of §11,  $A$  admits 2. Now  $z_3y^2$  admits 2. By (a) of §11,  $y_1$  admits 2. Then, by (d), every  $y_i$  with  $i \geq 1$  admits 2. From (10), using (a), (b), (d), we find that  $C$  admits 4. Referring to (11) and using (e), we see now that  $A^4$  admits 14 so that  $A$  admits 3. By (10), now,  $C$  admits 5 and we find from (11) that  $A$  admits 4. We return to (10) and see that  $C$  admits 6. Also, by (11),  $B$  admits 18. Finally,  $K$  of §12 admits 9.

By (18),  $L$  admits 30. By (17),  $y_3$  admits  $15/7$ . Now  $y_2 - z_4y^2 - 2z_3yy_1$ , which is  $A_1$ , admits 4. As  $y_1$  admits 2,  $y_2 - z_4y^2$  admits 3. Then  $y_3 - z_5y^2 - 2z_4yy_1$  admits 3 so that  $y_3 - z_5y^2$  admits 3. As  $y_3$  admits  $15/7$ ,  $z_5y^2$  admits  $15/7$ .

We infer that  $[F, G]$  contains a d.p. of the type  $(z_5 y^2)^m + M$  where every term of  $M$  is of degree greater than  $2m$  in the  $y_i$ . It follows from III, §23, that a point in  $\mathfrak{M}$  cannot have  $y = 0$  unless  $z_5 = 0$ .

14. Let  $(0, \alpha)$  be a generic point in the manifold of  $y = 0, z_5 = 0$ . We shall prove that  $\mathfrak{M}$  contains  $(0, \alpha)$ . This will imply that  $\mathfrak{M}$  contains the manifold of  $y, z_5$ , and our investigation of  $F$  will be completed.

Representing by  $c$  an arbitrary constant with respect to  $\mathfrak{F} \langle \alpha \rangle$  and by  $v$  a new indeterminate, we make in  $F$  the substitution<sup>11</sup>

$$(19) \quad y = \sum_{j=1}^6 c^j \alpha_2^{j-1} + c^6 v.$$

We represent by  $A', A'_1, B', C', F'$  the expressions into which  $A, A_1, B, C, F$  are transformed when  $z$  is replaced by  $\alpha$  and  $y$  by the second member of (19).

We find from (19)

$$(20) \quad A' = c^6 v_1 + c^7 P,$$

with  $P$  a polynomial in  $\alpha_2, \alpha_3, c, v$ . Then we may write

$$(21) \quad A'_1 = c^6 v_2 + c^7 Q,$$

with  $Q$  a polynomial in  $\alpha_2, \alpha_3, \alpha_4, c, v, v_1$ .

From (19), we have, remembering that  $\alpha_5 = 0$ ,

$$(22) \quad y_3 = 6c^3 \alpha_3 \alpha_4 + \dots; \quad y_4 = 6c^3 \alpha_4^2 + \dots.$$

By (20), (21), (22), we have, putting  $\beta = 6\alpha_3 \alpha_4$  and  $\gamma = 12\alpha_4^2$ ,

$$C' = c^9 (\beta v_2 - \gamma v_1) + c^{10} R,$$

with  $R$  a polynomial in  $\alpha_2, \alpha_3, \alpha_4, c$  and the  $v_j$  with  $j \leq 4$ . We find thus

$$(23) \quad F' = c^{24} [v_1^4 - \beta^3 - (\beta v_2 - \gamma v_1)^2] + c^{25} T,$$

with  $T$  of the type of  $R$ .

Let  $V$  represent the coefficient of  $c^{24}$  in  $F'$ . As  $\beta \neq 0$ , the differential equation  $V = 0$  for  $v$  is effectively of the second order. Let then  $v = \xi$  be a zero (constructed by the abstract method) of  $V$  which does not annul  $v_1^4 - \beta^3$ .

We wish to show that  $F'$  is annulled by a series

$$(24) \quad v = \xi + \varphi_2 c^{\rho_2} + \varphi_3 c^{\rho_3} + \dots$$

of the usual type, with  $\rho_2 > 0$ .

It will suffice to show that  $G' = F'/c^{24}$  is annulled by a series (24). If  $G'$  vanishes for  $v = \xi$ , then  $v = \xi$  is an acceptable series (24). In what follows, we assume that such vanishing does not occur. We put, in  $G'$ ,  $v = \xi + u_1$ . Then  $G'$  goes over into an expression  $K'$  in  $c$  and  $u_1$

$$(25) \quad K' = a'(c) + \sum b'_i(c) u_{10}^{\alpha_i} \dots u_{14}^{\alpha_{14}}.$$

<sup>11</sup> Subscripts of  $\alpha$  indicate differentiation.

Here  $\sum$  contains the terms of  $K'$  which are not free of the  $u_i$ , and  $i$  ranges from unity to some positive integer. As to  $a'$  and the  $b'$ , they are polynomials in  $c$  with coefficients in  $\mathcal{F} \langle \alpha_2, \xi \rangle$ . Because  $\xi$  does not annul  $G$ ,  $a'$  is not zero. On the other hand, because  $G$  vanishes for  $v = \xi, c = 0$ , the lowest power of  $c$  in  $a'$  is positive. Because the bracketed terms in (23) contribute effectively to  $\sum$  in (25), certain of the  $b'$  contain terms of zero power in  $c$ .

Let  $\sigma'$  be the lowest exponent of  $c$  in  $a'$  and  $\sigma'_i$  the lowest exponent of  $c$  in  $b'_i$ . Let

$$\rho_2 = \text{Max} \frac{\sigma' - \sigma'_i}{\alpha_{0i} + \dots + \alpha_{4i}}$$

where  $i$  has the range which it has in  $\sum$ . As  $\sigma' > 0$  and certain  $\sigma'_i$  equal 0,  $\rho_2 > 0$ . We may now suppose ourselves to be working with  $K'$  of III, §7. We obtain the series (24).

We have shown, all in all, that  $F$ , for  $z = \alpha$ , is annulled by a series

$$(26) \quad y = c + c^2\alpha_2 + \dots + c^5\alpha_2^4 + c^6(\alpha_2^5 + \xi) + \dots,$$

where the unwritten terms have rational exponents greater than 6. The series (26) does not annul  $B$  for  $z = \alpha$ . Indeed,

$$B' = c^{24}(v_1^4 - \beta^8) + \dots$$

and the coefficient of  $c^{24}$  does not vanish for  $v = \xi$ .

It follows that every d.p. which holds  $\mathfrak{M}$  vanishes for  $z = \alpha$  and for  $y$  as in (26). This means that  $y = 0, z = \alpha$  is in  $\mathfrak{M}$ .

15. If, in (8) to (11), we replace  $z_3, y_3, y_4$  wherever they occur by  $z_{n-1}, y_{n-1}, y_n$ , with  $n \geq 4$ , we obtain a d.p.  $F$  with a general solution which intersects the manifold of  $y = 0$  in that of  $y = 0, z_{2n-3} = 0$ ; the proofs require only the slightest changes.

In  $F$  of §9, if one replaces  $z_3$  by  $z$ , one obtains a d.p. which is of the first order in  $z$  and whose general solution intersects the manifold of  $y = 0$  in that of  $y = 0, z_2 = 0$ . This, in itself, is sufficiently anomalous. However, if it is desired to secure a d.p.  $F$  whose order in  $z$  cannot be reduced, it suffices to replace  $y_3$  and  $y_4$ , in (9), (10), (11), by  $zy_3$  and its derivative, respectively.

INTERSECTIONS OF COMPONENTS OF A DIFFERENTIAL POLYNOMIAL

16. Dealing with the analytic case, we prove the following theorem:

**THEOREM:** *Let  $F$  be a d.p. in  $y_1, \dots, y_n$ . A zero of  $F$  which is contained in more than one component of  $F$  annuls  $\partial F / \partial y_{ij}$  for  $i = 1, \dots, n$  and for every<sup>12</sup>  $j$ .*

Thus, in particular, if  $F$  vanishes for  $y_i = 0, i = 1, \dots, n$ , and, considered as a polynomial in the  $y_{ij}$ , contains a term of the first degree, the zero  $y_i = 0$  belongs to only one component of  $F$ .

Let

$$(27) \quad \bar{y}_1, \dots, \bar{y}_n$$

<sup>12</sup> The  $j$  for which this result is significant are those for which  $y_{ij}$  appears effectively in  $F$ .

be a zero for which some  $\partial F/\partial y_{ij}$  fails to vanish. We shall prove that (27) is contained in only one component of  $F$ .

We know that systems defining the components can be secured by choosing a sufficiently large positive integer  $p$  and resolving the system of derivatives

$$(28) \quad F, F_1, \dots, F_p,$$

the  $F$  being considered as polynomials in the  $y_{ij}$ , into prime p.i. none of which holds any other. We shall show that, for any  $p \geq 1$ , (28) yields only one prime p.i. whose polynomials vanish when each  $y_{ij}$  in (28) is replaced by  $\bar{y}_{ij}$ ; as determined by (27). This will prove our theorem.

Reassigning the subscripts of the  $y_i$  if necessary, we assume that one or more  $\partial F/\partial y_{ij}$  do not vanish for (27) and let  $m$  be the greatest value of  $j$  for which the vanishing does not occur. Putting the polynomials in (28) equal to zero, we secure a set of equations which we shall regard as equations to be solved for those  $y_{1, m+j}$  for which  $0 \leq j \leq p$ , in terms of  $x$  and the other  $y_{ik}$  in (28).

Let  $\xi$  be a value of  $x$  at which the coefficients in  $F$  and the functions in (27) are analytic, and at which  $\partial F/\partial y_{1m}$  does not vanish for (27). Let  $[\eta]$  represent, collectively, the values at  $\xi$  of the  $\bar{y}_{ij}$  in the zero of (28) derived from (27).

The polynomials in (28) vanish at the point  $\xi, [\eta]$  in the space of  $x$  and the  $y_{ij}$  in (28). We shall examine, at  $\xi, [\eta]$ , the jacobian with respect to  $y_{1m}, \dots, y_{1, m+p}$  of the polynomials in (28). In the first row of this jacobian, which row we understand to consist of partial derivatives of  $F$ , only the first term  $\partial F/\partial y_{1m}$  fails to vanish at  $\xi, [\eta]$ . To treat the other rows, let us imagine the polynomials in (28) to be expanded in powers of the various differences  $y_{ij} - \bar{y}_{ij}$ . The expansion of  $F$  will contain a term  $\alpha(y_{1m} - \bar{y}_{1m})$ , where  $\alpha$  is the function of  $x$  to which  $\partial F/\partial y_{1m}$  reduces for (27). By the nature of  $m, F_1$  must contain the term  $\alpha(y_{1, m+1} - \bar{y}_{1, m+1})$  and can have no term  $\beta(y_{1j} - \bar{y}_{1j})$  with  $j > m+1$ . Thus, in the second row of the jacobian, the value of the second element at  $\xi, [\eta]$  is that of  $\partial F/\partial y_{1m}$ , and the elements which follow have zero values. Continuing, we find the value of the jacobian at  $\xi, [\eta]$  to be the  $(p+1)$ th power of the value of  $\partial F/\partial y_{1m}$ .

Thus, for the neighborhood of  $\xi, [\eta]$ ,  $y_{1m}, \dots, y_{1, m+p}$  are determined by our equations as analytic functions  $f_m, \dots, f_{m+p}$  of  $x$  and the remaining  $y_{ij}$ . By specializing the  $y_{ij}$  in the  $f$  as functions of  $x$ , we can construct zeros of (28). Indeed, we secure in this way all zeros of (28) which, in an area contained in a small neighborhood of  $x = \xi$ , approximate closely to the zero of (28) derived from (27).

Some prime p.i. in the decomposition of (28), call it  $\Sigma$ , is such that all its polynomials vanish when  $y_{1m}, \dots, y_{1, m+p}$  are replaced by their  $f$ . Then  $\Sigma$  must admit the  $\bar{y}_{ij}$  as a zero. If a prime p.i.  $\Sigma'$  which  $\Sigma$  does not hold vanishes for the  $\bar{y}_{ij}$ ,  $\Sigma'$  has, by IV, §39, zeros which are not in the manifold of  $\Sigma$  and which approximate closely to the  $\bar{y}_{ij}$ . Thus, by what precedes,  $\Sigma$  is the only prime p.i. in the decomposition of (28) which has the  $\bar{y}_{ij}$  as a zero. The theorem is proved.

If one allows all the  $\partial F/\partial y_{ij}$  to vanish and requires the nonvanishing of one

or more partial derivatives of the second order, there is no upper bound to the number of components to which a zero of  $F$  may belong. We illustrate this by an example in  $\mathfrak{F}\{y\}$ . Let

$$F = y_2^2 + \prod_{j=1}^m [(x+j)y_1 - y],$$

where  $m$  is any integer greater than unity. Now  $(x+j)y_1 - y$  has  $(x+j)y_2$  as derivative, and therefore has, for every  $j$ , a manifold which is a component of  $F$ . The zero  $y = 0$  belongs to every such component.

#### ANALOGUE OF A THEOREM OF KRONECKER

17. It is a theorem of Kronecker that, given any system of polynomials in  $n$  indeterminates, there exists an equivalent system containing  $n+1$  or fewer polynomials.<sup>13</sup> We present an analogous theorem for d.p.

**THEOREM:** *Let  $\mathfrak{F}$  contain a nonconstant element. Let*

$$(29) \quad F_1, \dots, F_r$$

*be any finite system of d.p. in  $\mathfrak{F}\{y_1, \dots, y_n\}$ . There exists a system composed of  $n+1$  linear combinations of the  $F$ , with coefficients in  $\mathfrak{F}$ , whose manifold is identical with that of (29).*

We introduce  $r(n+1)$  new indeterminates  $u_j^{(i)}$ ,  $i = 1, \dots, n+1$ ;  $j = 1, \dots, r$  and consider the system  $\Lambda$ ,

$$u_1^{(i)}F_1 + \dots + u_r^{(i)}F_r, \quad i = 1, \dots, n+1,$$

in the  $u$  and  $y$ .

Consider a zero of  $\Lambda$  for which  $F_1 \neq 0$ . For it, we have

$$(30) \quad u_1^{(i)} = -\frac{u_2^{(i)}F_2 + \dots + u_r^{(i)}F_r}{F_1},$$

$i = 1, \dots, n+1$ . If we differentiate the relations (30) often enough, the  $u_{ij}^{(i)}$  will be more numerous than the  $y_{ij}$ . By an elimination, we obtain a d.p.  $K_1$  in the  $u$  which is annulled by every zero of  $\Lambda$  for which  $F_1 \neq 0$ . We find, similarly, a  $K_i$  for each  $F_i$  with  $i > 1$ . We fix the  $u_j^{(i)}$  as elements  $\mu_{ij}$  in  $\mathfrak{F}$  which do not annul the product of the  $K$ . Then the manifold of the  $n+1$  d.p.

$$\mu_{i1}F_1 + \dots + \mu_{ir}F_r, \quad i = 1, \dots, n+1,$$

in  $y_1, \dots, y_n$  is identical with that of (29).

The proof just given does not involve the notion of irreducible manifold. It is considerably shorter than the proof given in A.D.E. However, the older proof gives information on the degree to which one can approximate to the representation of a manifold with a system of  $p$  equations with  $1 \leq p \leq n+1$ .

<sup>13</sup> Koenig, *Algebraische Grössen*, p. 234.

CHAPTER VIII  
 RIQUIER'S EXISTENCE THEOREM FOR ORTHONOMIC  
 SYSTEMS

1. In Chapter IX, we shall extend some of the main results of the preceding chapters to systems of partial differential polynomials. In treating the analytic case, we shall use an important existence theorem due to Riquier. This existence theorem will now be developed.

For §§1–19 of Chapter IX, only §2 and §8 of the present chapter are necessary.

MONOMIALS

2. We deal with  $m$  independent variables,  $x_1, \dots, x_m$ . By a *monomial* is meant an expression  $x_1^{i_1} \cdots x_m^{i_m}$ , where the  $i$  are non-negative integers. If  $\alpha = \gamma\beta$ , with  $\alpha, \beta, \gamma$  monomials, then  $\alpha$  is called a *multiple* of  $\beta$ . Given two distinct monomials,

$$x_1^{i_1} \cdots x_m^{i_m}, \quad x_1^{j_1} \cdots x_m^{j_m},$$

the first is said to be *higher* or *lower* than the second according as the first non-zero difference  $i_k - j_k$  is positive or is negative.

The following theorem, due to Riquier, is used only in Chapter IX.

**THEOREM:** *Let*

$$(1) \quad \alpha_1, \alpha_2, \dots, \alpha_a, \dots$$

*be an infinite sequence of monomials. Then there is an  $\alpha_i$  which is a multiple of some  $\alpha_j$  with  $j < i$ .*

Let  $\beta_1$  be one of those  $\alpha$  for which the exponent of  $x_1$  is a minimum. Consider the monomials which come after  $\beta_1$  in (1). Let  $\beta_2$  be a monomial of this class whose degree in  $x_1$  does not exceed that of any other monomial of the class. Of the monomials which follow  $\beta_2$ , let  $\beta_3$  be one of minimum degree in  $x_1$ . Continuing, we form an infinite sequence of monomials

$$(2) \quad \beta_1, \quad \beta_2, \quad \beta_3, \dots$$

whose degrees in  $x_1$  are nondecreasing. We extract similarly, from (2), a sequence in which the degrees in  $x_2$  do not decrease. We arrive finally at an infinite subsequence of (1) in which each monomial is a multiple of all which precede it.

DISSECTION OF A TAYLOR SERIES

3. Let

$$(3) \quad \sum \frac{a_{i_1 \dots i_m}}{i_1! \dots i_m!} x_1^{i_1} \dots x_m^{i_m}$$

be the Taylor expansion at

$$(4) \quad x_i = 0, \quad i = 1, \dots, m,$$

of a function  $u$  of  $x_1, \dots, x_m$  analytic at the point (4). Let  $[\alpha]$  be any given finite and nonvacuous set of distinct monomials. We are going to separate (3), with respect to  $[\alpha]$ , into a set of components.

Let  $a$  be the greatest exponent of  $x_1$  in the set  $[\alpha]$ . We write

$$(5) \quad u = f_0 + x_1 f_1 + \dots + x_1^{a-1} f_{a-1} + x_1^a f_a,$$

where, for  $i < a$ ,  $x_1^i f_i$  contains all terms in (3) in which the exponent of  $x_1$  is precisely  $i$ . As to  $x_1^a f_a$ , it contains all terms divisible by  $x_1^a$ . Then  $f_1, \dots, f_{a-1}$  are series in  $x_2, \dots, x_m$ , while  $f_a$  involves also  $x_1$ .<sup>1</sup>

We define sets of monomials  $[\alpha]_\lambda$ ,  $\lambda = 0, \dots, a$ , as follows. If  $[\alpha]$  contains monomials in which the exponent of  $x_1$  does not exceed  $\lambda$ , then  $[\alpha]_\lambda$  is to consist of all such monomials in  $[\alpha]$ . If there are no such monomials, then  $[\alpha]_\lambda$  is to be unity. Let  $[\beta]_\lambda$  be the set of monomials in  $x_2, \dots, x_m$  obtained by putting  $x_1 = 1$  in  $[\alpha]_\lambda$ . We now give to each  $f_\lambda$ , with respect to  $x_2$ , the treatment accorded to  $u$ , above, with respect to  $x_1$ . For  $\lambda < a$ , we get a representation of the type

$$(6) \quad f_\lambda = f_{\lambda 0} + x_2 f_{\lambda 1} + \dots + x_2^b f_{\lambda b},$$

where  $b$  depends upon  $\lambda$ , the  $f_{\lambda i}$  with  $i < b$  involving  $x_3, \dots, x_m$ , while  $f_{\lambda b}$  involves also  $x_2$ . For  $\lambda = a$ , each  $f_{a i}$  involves  $x_1$ . That is, in the dissection of  $f_a$ , we treat  $x_1$  like  $x_3, \dots, x_m$ .

We now operate on each  $f_{\lambda \mu}$  with respect to  $x_3$ . We use a set of monomials  $[\gamma]_{\lambda \mu}$ , where, if  $[\beta]_\lambda$  has monomials of degree not exceeding  $\mu$  in  $x_2$ ,  $[\gamma]_{\lambda \mu}$  is obtained by putting  $x_2 = 1$  in all such monomials, and where, otherwise,  $[\gamma]_{\lambda \mu}$  is unity.

Continuing, we find an expression for  $u$ ,

$$(7) \quad u = \sum x_1^{i_1} \dots x_m^{i_m} f_{i_1 \dots i_m},$$

the summation extending over a finite number of terms.

**Example:** Let  $u$  be a function of  $x, y, z$ . Let  $[\alpha]$  be

$$xz^2, \quad xy, \quad x^2yz.$$

For  $x$ , we find

$$u = f_0(y, z) + x f_1(y, z) + x^2 f_2(x, y, z).$$

---

<sup>1</sup> We consider every combination  $i_1, \dots, i_m$  to occur in (3), using zero coefficients if necessary.

We now treat each  $f_i$  with respect to  $y$ , the set of monomials being that indicated below:

$$\begin{aligned} f_0(y, z) & \qquad 1; \\ f_1(y, z) & \qquad z^2, y; \\ f_2(x, y, z) & \qquad z^2, y, yz. \end{aligned}$$

Hence

$$\begin{aligned} f_0(y, z) & = f_{00}(y, z), \\ f_1(y, z) & = f_{10}(z) + yf_{11}(y, z), \\ f_2(x, y, z) & = f_{20}(x, z) + yf_{21}(x, y, z). \end{aligned}$$

The final step is

$$\begin{aligned} f_{00}(y, z) & = f_{000}(y, z) & 1; \\ f_{10}(z) & = f_{100} + zf_{101} + z^2f_{102}(z) & z^2; \\ f_{11}(y, z) & = f_{110}(y) + zf_{111}(y) + z^2f_{112}(y, z) & 1, z^2; \\ f_{20}(x, z) & = f_{200}(x) + zf_{201}(x) + z^2f_{202}(x, z) & z^2; \\ f_{21}(x, y, z) & = f_{210}(x, y) + zf_{211}(x, y) + z^2f_{212}(x, y, z) & 1, z, z^2. \end{aligned}$$

Thus the dissection of  $u$  is

$$\begin{aligned} u & = f_{000}(y, z) + xf_{100} + xzf_{101} + xz^2f_{102}(z) \\ & \quad + xyf_{110}(y) + xyzf_{111}(y) + xyz^2f_{112}(y, z) \\ & \quad + x^2f_{200}(x) + x^2zf_{201}(x) + x^2z^2f_{202}(x, z) \\ & \quad + x^2yf_{210}(x, y) + x^2yzf_{211}(x, y) + x^2yz^2f_{212}(x, y, z). \end{aligned}$$

4. Consider any monomial  $\alpha = x_1^{i_1} \cdots x_m^{i_m}$  in  $[\alpha]$  and any monomial  $\beta$  in the expansion of  $u$  which is a multiple of  $\alpha$ . Of course,  $\beta$  appears in one and in only one of the terms in the second member of (7). Let it appear in  $x_1^{i_1} \cdots x_m^{i_m} f_{i_1 \dots i_m}$ . We shall prove that  $x_1^{i_1} \cdots x_m^{i_m}$  is a multiple of  $\alpha$ . For  $m = 1$ , this result certainly holds. Let the result be true for  $m = r - 1$ . We shall prove it for  $m = r$ . We observe first that in the resolution (5) of  $u$ ,  $\beta$  appears in a term  $x_1^{i_1} f_{i_1}$  with  $i_1 \geq j_1$ .

Suppose first that  $i_1 < a$  in (5). Then  $\beta/x_1^{i_1}$  is free of  $x_1$ . Among the monomials used in the dissection of  $f_{i_1}$  will be  $x_2^{j_2} \cdots x_r^{j_r}$  and  $\beta/x_1^{i_1}$  will be a multiple of  $x_2^{j_2} \cdots x_r^{j_r}$ . As there are only  $r - 1$  variables involved now,  $\beta/x_1^{i_1}$  will appear in a term  $\epsilon f_{i_1 i_2 \dots i_r}$  in the dissection (7) of  $f_{i_1}$  with  $\epsilon$  divisible by  $x_2^{j_2} \cdots x_r^{j_r}$ . Thus  $x_1^{i_1} \cdots x_r^{j_r}$  is divisible by  $\alpha$ .

Suppose now that  $i_1 = a$ . Then  $\beta/x_1^a$  is contained in  $f_a$ . Among the monomials used in the dissection of  $f_a$  will be  $x_2^{j_2} \cdots x_r^{j_r}$ . Now the formal scheme in (7) of the dissection of  $f_a$  can be obtained by taking a function  $g$  of  $x_2, \dots, x_r$ ,

dissecting  $g$  with respect to the monomials associated with  $f_a$  and then adjoining  $x_1$  to the variables in the series yielded by  $g$ . That is, the monomials  $x_2^{i_2} \cdots x_r^{i_r}$  in the dissections, analogous to (7), of  $f_a$  and  $g$ , will be the same. Let  $\gamma$  result from  $\beta$  on putting  $x_1 = 1$ . Then  $\gamma$  is found in the dissection of  $g$  with an  $x_2^{i_2} \cdots x_r^{i_r}$  divisible by  $x_2^{j_2} \cdots x_r^{j_r}$ . The same would therefore be true for  $\beta/x^a$  in the dissection of  $f_a$ . This completes the proof.

It follows that every monomial in  $[\alpha]$  is an  $x_1^{i_1} \cdots x_m^{i_m}$  in (7).

5. The set of monomials consisting of all  $x_1^{i_1} \cdots x_m^{i_m}$  in (7) which are multiples of monomials in  $[\alpha]$  will be called the *extended set arising from*  $[\alpha]$ . The set of monomials  $x_1^{i_1} \cdots x_m^{i_m}$  in (7) not in the extended set will be called the set *complementary* to  $[\alpha]$ .

If  $[\alpha]$  is identical with the extended set arising from  $[\alpha]$ , then  $[\alpha]$  will be called *complete*.

Consider a set  $[\alpha]$  which is not complete. We shall prove that it is possible to form a complete set by adjoining to  $[\alpha]$  multiples of monomials in  $[\alpha]$ .

Let  $p$  be the maximum of all exponents in all monomials in  $[\alpha]$ . Then, in (7), no  $i_k$  exceeds  $p$ .

Let  $[\alpha]'$  be the extended set arising from  $[\alpha]$ . Then if  $[\alpha]'$  is not complete, it is a proper subset of its extended set  $[\alpha]''$  (§4). Since we can never get more than  $(p+1)^m$  monomials  $x_1^{i_1} \cdots x_m^{i_m}$  in (7), this process of taking extended sets must bring us eventually to a complete set.

6. In (7), the variables in an  $f_{i_1 \dots i_m}$  will be called the *multipliers* of the corresponding  $x_1^{i_1} \cdots x_m^{i_m}$ , and all other variables will be called *nonmultipliers* of  $x_1^{i_1} \cdots x_m^{i_m}$ . Of course, if  $f_{i_1 \dots i_m}$  is a constant,  $x_1^{i_1} \cdots x_m^{i_m}$  has no multipliers.

Let  $\beta = x_1^{i_1} \cdots x_m^{i_m}$  be a monomial in the extended set arising from  $[\alpha]$ . Let  $x_k$  be a nonmultiplier of  $\beta$ . Then  $\beta x_k$ , as a multiple of some monomial in  $[\alpha]$ , is the product of a monomial  $\gamma$  in the extended set by unity or by multipliers of  $\gamma$  (§4).

We shall prove that  $\gamma$  is higher than  $\beta$ . Let  $\gamma = x_1^{j_1} \cdots x_m^{j_m}$ . If  $j_1 < i_1$ ,  $x_1$  cannot be a multiplier for  $\gamma$  since  $j_1$  is certainly not the maximum of the degrees in  $x_1$  of the monomials in  $[\alpha]$ . Hence  $j_1 \geq i_1$ . It remains to examine the case in which  $j_1 = i_1$ . When we dissect  $f_{i_1}$ , we find that if  $j_2 < i_2$ ,  $x_2$  cannot be a multiplier for  $x_2^{j_2} \cdots x_m^{j_m}$ . Hence  $j_2 \geq i_2$  and we have to study the case in which  $j_2 = i_2$ . Continuing, we see that  $\gamma$  is not lower than  $\beta$  so that, since  $\gamma \neq \beta$ ,  $\gamma$  is higher than  $\beta$ .

7. We associate with every monomial  $x_1^{i_1} \cdots x_m^{i_m}$  the differential operator

$$(8) \quad \frac{\partial^{i_1 + \dots + i_m}}{\partial x_1^{i_1} \cdots \partial x_m^{i_m}}$$

Then the product of two operators corresponds to the product of the corresponding monomials.

Consider any monomial

$$(9) \quad \beta = x_1^{i_1} \cdots x_m^{i_m}$$

in (7). Let the corresponding differentiation be performed upon  $u$ , and after the differentiation, let the nonmultipliers of  $\beta$  be given zero values. Every term in the expansion of  $u$  which is not divisible by  $\beta$  will disappear during the differentiation. Any term divisible by  $\beta$  whose quotient by  $\beta$  contains nonmultipliers of  $\beta$  will disappear when the nonmultipliers are made zero. Hence the above operation gives identical results when applied to  $u$  and to  $\beta f_{i_1 \dots i_m}$ .

MARKS

8. Let  $y_1, \dots, y_n$  be analytic functions of  $x_1, \dots, x_m$ . Riquier effects an ordering of the  $y$  and their partial derivatives in the following way.

Let  $s$  be any positive integer. We associate with each  $x_i$  any ordered set of  $s$  nonnegative integers

$$(10) \quad u_{i_1}, \dots, u_{i_s}$$

With each  $y_i$ , we associate any ordered set of nonnegative integers

$$(11) \quad v_{i_1}, \dots, v_{i_s}$$

taking care that  $y_i$  and  $y_j$  with  $i \neq j$  do not have identical sets (11). The  $j$ th integer in (10) is called the  $j$ th mark of  $x_i$ , and the  $j$ th integer in (11), the  $j$ th mark of  $y_i$ .

If

$$(12) \quad w = \frac{\partial^{k_1 + \dots + k_m}}{\partial x_1^{k_1} \dots \partial x_m^{k_m}} y_i$$

we define the  $j$ th mark of  $w$ ,  $j = 1, \dots, s$ , to be  $v_{i_j} + k_1 u_{i_j} + \dots + k_m u_{i_j}$ .

Consider all of the derivatives<sup>2</sup> of all  $y_i$ . Let  $w_1$  and  $w_2$  be any two of these derivatives. Let the marks of  $w_1$  and  $w_2$  be

$$a_1, \dots, a_s; \quad b_1, \dots, b_s$$

respectively. Suppose that the two sets of marks are not identical. We shall say that  $w_1$  is *higher* than  $w_2$  or is *lower* than  $w_2$  according as the first nonzero difference  $a_i - b_i$  is positive or is negative. If the two sets of marks are identical, no relation of order is established between  $w_1$  and  $w_2$ .

If  $w_1$  is higher than  $w_2$ ,  $\partial w_1 / \partial x_i$  is higher than  $\partial w_2 / \partial x_i$ .

When the marks in (10) and (11) are such that a difference in order exists between any two distinct derivatives, the derivatives of the  $y$  are said to be *completely ordered*.

Suppose that the ordering is not complete. We shall show how to adjoin new marks, after  $u_{i_s}$  and  $v_{i_s}$ , so as to effect a complete ordering. Clearly, the adjunction of such new marks will not disturb any order relationships which may already exist.

Let  $m$  additional marks be assigned, as in the following table:

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<sup>2</sup> Each  $y_i$  will be considered as a derivative of zero order of itself.



been completely ordered by marks. We consider a finite system  $\sigma$  of differential equations,

$$(14) \quad \frac{\partial^{i_1 + \dots + i_m} y_j}{\partial x_1^{i_1} \dots \partial x_m^{i_m}} = g_{i_1 \dots i_m, j}$$

where

(a) in each equation,  $g$  is a function of  $x_1, \dots, x_m$  and of a certain number of derivatives of the  $y_i$ , every derivative in  $g$  being lower than the first member of the equation;

(b) the first members of any two equations are distinct;

(c) if  $w$  is a first member of some equation, no derivative of  $w$  appears in the second member of any equation;

(d) the functions  $g$  are all analytic at some point in the space of the arguments involved in all of them.<sup>3</sup>

We do not assume that every  $y_i$  appears in a first member.

Riquier calls such a system of equations *orthonomic*.

The derivatives of the  $y$  which are derivatives of first members in the orthonomic system are called *principal derivatives*. All other derivatives are called *parametric derivatives*.

11. Given an orthonomic system,  $\sigma$ , we shall show how to obtain an orthonomic system with the same solutions, in which, for each  $y_i$  appearing in the first members, the monomials corresponding as in §7 to those first members which are derivatives of  $y_i$  form a complete set (§5).

Let equations be adjoined to (14), by differentiating the equations in (14), so that, for each  $y$  which occurs in some first member, the monomials corresponding to the enlarged set of first members constitute a complete set. By §5, this can be done. We obtain thus a system  $\sigma_1$  of equations. Certain first members in  $\sigma_1$  may be obtainable from more than one of the first members in  $\sigma$ . In that case, we use any one of the first members in  $\sigma$  which is available.

Consider any one of the equations in  $\sigma$ . Let  $w$  represent its first member, and  $v$  the highest derivative in the second member. If we differentiate the equation with respect to  $x_i$ , the first member becomes  $\partial w / \partial x_i$ . The highest derivative in the new second member will be  $\partial v / \partial x_i$ , which is lower than  $\partial w / \partial x_i$  (§8).

It is clear, on this basis, that  $\sigma_1$  satisfies condition (a).

We attend now to (c). Let  $\mathbf{C}$  be an open region in the space of the arguments in the second members in  $\sigma$  in which the second members are analytic. We consider those solutions of  $\sigma$  for which the indicated arguments lie in  $\mathbf{C}$ .

The second members in  $\sigma_1$  may involve derivatives not in the second members in  $\sigma$ . The second members in  $\sigma_1$  will be polynomials in the new derivatives, with coefficients analytic in  $\mathbf{C}$ .

<sup>3</sup> Thus, in (d), derivatives not effectively present in a  $g$  may be regarded as arguments in that  $g$ . This does not conflict with (a), in which the arguments considered are supposed to be effectively present.

Let  $w$  be the highest derivative present in a second member in  $\sigma_1$  which is a derivative of a first member in  $\sigma_1$ . Then  $w$  is not present in any second member in  $\sigma$ , so that it appears rationally and integrally in the second members in  $\sigma_1$ . Let  $w$  be a derivative of  $v$ , the first member of the equation  $v = g$  in  $\sigma_1$ . Then  $w$  can be replaced, in the second members in  $\sigma_1$ , by its expression obtained on differentiating  $g$ . We obtain thus a system  $\sigma_2$  with the same solutions as  $\sigma_1$  (or  $\sigma$ ) and with the same first members as  $\sigma_1$ . The system  $\sigma_2$  satisfies condition (a). The derivatives higher than  $w$  which appear in the second members in  $\sigma_2$  also appear in the second members in  $\sigma_1$ . Hence, if  $w_1$ , present in the second members in  $\sigma_2$ , is a derivative of a first member in  $\sigma_2$ , then  $w_1$  is lower than  $w$ . We treat  $w_1$  as  $w$  was treated. Since there cannot be an infinite sequence of derivatives each lower than the preceding one, we must arrive, in a finite number of steps, at a system  $\tau$ , with the same solutions as  $\sigma$ , which satisfies (a), (b), (c), and which has complete sets of monomials corresponding to its first members. The second members in  $\tau$  will be polynomials in any derivatives not present in the second members of  $\sigma$ . Hence assumption (d) is satisfied for  $\mathbf{C}$  and for any values of the new derivatives. Thus  $\tau$  is orthonomic and has the same solutions as  $\sigma$ .<sup>4</sup>

Of course, whether we employ  $\sigma$  or  $\tau$ , we get the same set of principal derivatives and the same parametric derivatives.

12. We consider an orthonomic system,  $\sigma$ , whose first members, as in §11, yield complete sets of monomials. We are going to seek solutions of  $\sigma$ , analytic at some point, which, with no loss of generality, may be taken as  $x_i = 0$ ,  $i = 1, \dots, m$ .

Consider any  $y_i$ . Let numerical values be assigned to the parametric derivatives of  $y_i$ , at the origin, with the sole conditions that the second members in  $\sigma$  are analytic for the values given to the derivatives in them and that the series

$$(15) \quad \sum \frac{a_{j_1 \dots j_m}}{j_1! \dots j_m!} x_1^{j_1} \dots x_m^{j_m},$$

where the  $a$  are the values of the parametric derivatives, the subscripts indicating the type of differentiation, converges in a neighborhood of the origin. The series (15) is called the *initial determination* of  $y_i$ . If  $y_i$  does not appear in a first member, (15) is a complete Taylor series.

In what follows, we suppose an initial determination to be given for each  $y_i$ . We shall then develop a process for calculating the values of the principal derivatives at the origin. There will result analytic functions  $y_i$  which satisfy each equation of  $\sigma$  on the spread obtained by equating to zero the nonmultipliers of the monomial corresponding to the first member. Later we shall obtain a condition for the  $y_i$  to give an actual solution of  $\sigma$ .

In the dissection (7) of each  $y_i$  which we shall obtain,<sup>5</sup> those terms whose monomials are multiples of monomials in the complementary set will constitute

<sup>4</sup> With the values of the arguments in the second members in  $\sigma$  lying in  $\mathbf{C}$ .

<sup>5</sup> This dissection is based on the complete set of monomials corresponding to  $y_i$ .

the initial determination of  $y_i$ . Thus the initial determination of each  $y_i$  is a linear combination of a certain number of arbitrary functions, with monomials for coefficients, the variables in the arbitrary functions being specified. This description of the degree of generality of the solution of a system of equations is one of the most important items of Riquier's work.

We replace each  $y_i$  which does not figure in any first member in  $\sigma$  by an arbitrarily selected initial determination. Then  $\sigma$  becomes an orthonomic system in the remaining  $y_i$ , with the same principal derivatives as before for the remaining  $y_i$ . On this basis, we assume, with no loss of generality, that every  $y_i$  figures in a first member.

13. We use the symbol  $\delta$  to represent differential operators. Any principal derivative,  $\delta y_i$ , which is not a first member in  $\sigma$ , can be obtained from one and only one first member in  $\sigma$  by differentiation with respect to multipliers of the monomial corresponding to that first member. This is because the first members yield complete sets. We have thus a unique expression for  $\delta y_i$ ,

$$(16) \quad \delta y_i = g,$$

where the derivatives in  $g$  are lower than  $\delta y_i$ .

The infinite system obtained by adjoining all equations (16) to  $\sigma$  will be called  $\tau$ . Let  $p$  be any nonnegative integer. The system of equations in  $\tau$  whose first members have  $p$  for first mark will be called  $\tau_p$ . Since the first mark of a derivative is the sum of the order of the derivative and of the first mark of the function differentiated, each  $\tau_p$  has only a finite number of equations.

Let  $a$  be the minimum, and  $b$  the maximum, of the first marks in the first members in  $\sigma$ . For the values assigned, in §12, to the parametric derivatives, the equations  $\tau_a, \tau_{a+1}, \dots, \tau_b$  determine uniquely the values at the origin of the principal derivatives whose first mark does not exceed  $b$ . In short, the lowest such derivative has an equation which determines it in terms of parametric derivatives; the principal derivative next in ascending order is determined in terms of parametric derivatives, and, perhaps, the first principal derivative, and so on.

We subject the unknowns  $y_j$  to the transformation

$$(17) \quad y_j = \bar{y}_j + \varphi_j + \sum \frac{c_{j, i_1 \dots i_m}}{i_1! \dots i_m!} x_1^{i_1} \dots x_m^{i_m}$$

where  $\varphi_j$  is the chosen initial determination of  $y_j$  and where the  $c$  are the principal derivatives at the origin of  $y_j$ , of first mark not exceeding  $b$ , found as above.

Then  $\sigma$  goes over into a system  $\sigma'$  in the  $\bar{y}_j$ . In the new system, we transpose the known terms in the first members (these come from the known terms in (17)) to the right. The new system will be orthonomic in the  $\bar{y}_j$ , with the same monomials for its first members as in  $\sigma$ . The second members will be analytic when each  $x_i$  and each parametric derivative is small.

The system  $\tau'$  for  $\sigma'$ , analogous to  $\tau$  for  $\sigma$ , is obtained by executing the transformation (17) on the equations of  $\tau$ .

Thus, if we give to the  $\bar{y}_i$ , in  $\sigma'$ , initial determinations which are identically zero, the principal derivatives at the origin, of first mark not exceeding  $b$ , will be determined as zero by  $\tau'_a, \dots, \tau'_b$ .

On this account, we limit ourselves, without loss of generality, to the search for solutions  $y_1, \dots, y_n$ , of  $\sigma$ , with initial determinations identically zero, assuming that the system  $\tau_a, \dots, \tau_b$  yields zero values at the origin for the principal derivatives whose first marks do not exceed  $b$ .

14. In the second members in  $\tau_{b+1}$ , no derivatives appear whose first marks exceed  $b+1$ . Those derivatives whose first marks are  $b+1$  enter linearly, because they come from the differentiation of derivatives of first mark  $b$  in  $\tau_b$ .

We denote by  $\delta_k y_i$  the second member of (12). Then every equation in  $\tau_{b+1}$  is of the form

$$(18) \quad \delta_i y_\alpha = \sum p_{i\alpha\beta} \delta_j y_\beta + q_{i\alpha},$$

where the  $\delta_j y_\beta$  are of first mark  $b+1$  and where the  $p$  and  $q$  involve the  $x_i$  and derivatives whose first marks are  $b$  or less.

In (18), we consider every derivative of first mark  $b+1$  which is lower than  $\delta_i y_\alpha$  to be present in the second member. If necessary, we take  $p_{i\alpha\beta} = 0$ .

Consider any  $\delta_i y_\alpha$  in (18). Suppose that there is a  $\beta$  such that  $y_\beta$  has derivatives of first mark  $b+1$  which are lower than  $\delta_i y_\alpha$ . For every such  $\beta$ , we let  $r_{i\alpha\beta}$  represent the number of derivatives of  $y_\beta$ , of first mark  $b+1$ , which are lower than  $\delta_i y_\alpha$ . For every other  $\beta$ , we let  $r_{i\alpha\beta} = 1$ , and we suppose that a single derivative of  $y_\beta$  of first mark  $b+1$  appears in the second member of (18), with a zero coefficient. We can thus not continue to say that every derivative in the second member of (18) is lower than the first member, but no difficulty will arise out of this; only a question of language is involved.

Let  $r$  be the maximum of the  $r_{i\alpha\beta}$ .

The  $p$  and  $q$  in (18) are analytic for small values of their arguments. Let the  $p$  and  $q$  be expanded as series of powers of their arguments.

Let  $\epsilon > 0$  be such that each of the above series converges for values of its arguments which all exceed  $\epsilon$  in modulus. Let  $h > 0$  be such that each  $p$  and each  $q$  has a modulus less than  $h$  when the arguments do not exceed  $\epsilon$  in modulus.

Let  $\lambda$  be any positive number less than  $1/n$ .

Following §9, we determine positive numbers  $\xi_i, \zeta_i$ , not less than unity such that, if  $\delta_i y_\alpha$  and  $\delta_j y_\beta$  are of first mark  $b+1$ , with  $\delta_i y_\alpha$  higher than  $\delta_j y_\beta$ , we have

$$(19) \quad \frac{\xi_1^{i_1} \dots \xi_m^{i_m} r_\alpha}{\xi_1^{j_1} \dots \xi_m^{j_m} \zeta_\beta} > \frac{hr}{\lambda}.$$

In what follows, we associate with each  $y_i$  a new unknown function  $u_i$ .

Let

$$\rho = \frac{\xi_1 x_1 + \dots + \xi_m x_m + \sum \delta u}{\epsilon},$$

where  $\sum$  ranges over all derivatives of  $u_1, \dots, u_n$  whose first mark does not exceed  $b$  ( $\delta_i u_\alpha$  is supposed to have the same marks as  $\delta_i y_\alpha$ ).

We consider the system of equations

$$(20) \quad \delta_i u_\alpha = \frac{1}{1 - \rho} \sum \frac{\lambda}{r_{i\alpha\beta}} \frac{\xi_1^{i_1} \dots \xi_m^{i_m} \zeta_\alpha}{\xi_1^{i_1} \dots \xi_m^{i_m} \zeta_\beta} \delta_j u_\beta + \frac{h \xi_1^{i_1} \dots \xi_m^{i_m} \zeta_\alpha}{1 - \rho}$$

which has the general form of (18), with alterations of the form of the  $p$  and  $q$ .

The function

$$\frac{h}{1 - \frac{x_1 + \dots + x_m + \sum \delta u}{\epsilon}}$$

is a majorant for every  $p$  and every  $q$ . As each  $\xi_i$  is at least unity, the same is true of  $h/(1 - \rho)$ .

Thus, in virtue of (19), wherever a  $\delta_j y_\beta$  is lower than  $\delta_i y_\alpha$  in an equation in (18), the coefficient of  $\delta_j u_\beta$  in the corresponding equation of (20) will be a majorant for the coefficient of  $\delta_j y_\beta$ . In the exceptional case where a  $\delta_j y_\beta$  is not lower than  $\delta_i y_\alpha$  and thus has a zero coefficient, the corresponding coefficient in (20) is certainly a majorant. Evidently the terms in (20) which correspond to the  $q$  in (18) are majorants of the  $q$ .

15. We shall show that (20) has a solution in which each  $u_i$  is a function of

$$(21) \quad \xi_1 x_1 + \dots + \xi_m x_m.$$

Consider, in (20), all derivatives of a particular  $u_\alpha$  whose first marks are  $b + 1$ . The first mark of any such derivative is the order (total) of the derivative, plus the first mark of  $u_\alpha$ . Hence all of the derivatives of  $u_\alpha$  which are of first mark  $b + 1$  are of the same order, say  $g_\alpha$ .

Let the  $u_\alpha$ , in what follows, represent functions of (21). Put  $u_\alpha = \zeta_\alpha u'_\alpha$  and let  $u'_{\alpha i}$  be the  $i$ th derivative of  $u'_\alpha$  with respect to (21). Then with  $i = i_1 + \dots + i_m$ ,

$$\frac{\partial^{i_1 + \dots + i_m}}{\partial x_1^{i_1} \dots \partial x_m^{i_m}} u_\alpha = \xi_1^{i_1} \dots \xi_m^{i_m} \zeta_\alpha u'_{\alpha i}.$$

When the  $u_\alpha$  are functions of (21),  $\rho$  becomes a function  $\rho'$  of (21) and of the derivatives of the  $u'_\alpha$  of order less than  $g_\alpha$ ,  $\alpha = 1, \dots, n$ . Equations (20) reduce to

$$(22) \quad u'_{\alpha\theta} = \lambda \sum_{\beta=1}^n \frac{1}{1 - \rho'} u'_{\beta\theta} + \frac{h}{1 - \rho'}.$$

There will be  $n$  equations in (22), one for each  $\alpha$ . All equations in (20) in which a given  $u_\alpha$  appears in the first member yield the same equation (22). We write (22) as

$$(23) \quad u'_{\alpha\theta} = \rho' u'_{\alpha\theta} + \lambda \sum_{\beta=1}^n u'_{\beta\theta} + h.$$



origin. We can ascend, step by step, through all the principal derivatives, because each  $\tau_p$  in §13 has only a finite number of equations.

We obtain thus a complete power series for each  $y_i$ . We are going to prove that these power series converge for small values of the  $x_i$ .

Let  $\delta_i y_\alpha$  be any principal derivative. We shall prove that the modulus of this derivative at the origin does not exceed the value at the origin found for  $\delta_i u_\alpha$  in §16.

For derivatives of first mark less than  $b + 1$ , this is certainly true; those derivatives have zero values. Let the result hold for all derivatives lower than some  $\delta_i y_\gamma$  of first mark greater than  $b$ . The equation in  $\tau$  for  $\delta_i y_\gamma$  is either in (18) or is found by differentiating some equation in (18). Consider the corresponding equation for  $\delta_i u_\gamma$ , which is either in (20), or obtained from (20) by differentiation.

We shall consider the expressions for  $\delta_i y_\gamma$  and  $\delta_i u_\gamma$  as power series in the  $x_i$  and in the derivatives in terms of which  $\delta_i y_\gamma$  and  $\delta_i u_\gamma$  are expressed.

We see that, for every term in the series for  $\delta_i y_\gamma$ , there is a dominating term in the series for  $\delta_i u_\gamma$ . What is more, the series for  $\delta_i u_\gamma$  may have other terms, involving  $\delta_i u_\gamma$  itself, or even higher derivatives. This is because of the exceptional terms in (20), introduced in §14.<sup>7</sup>

Each term in  $\delta_i u_\gamma$  which has a corresponding term in  $\delta_i y_\gamma$  is at least as great as the modulus of that term at the origin, for such terms involve only lower derivatives than  $\delta_i y_\gamma$  or  $\delta_i u_\gamma$ . Terms in  $\delta_i u_\gamma$  which have no corresponding terms in  $\delta_i y_\gamma$  are zero or positive at the origin. They will be positive if they involve no  $x_i$ , and contain only derivatives of first mark at least  $b + 1$  (§16). This proves that the value determined for each  $\delta_i y_\alpha$  by  $\tau$  has a modulus not greater than the value at the origin of  $\delta_i u_\alpha$ .

Thus the series obtained for the  $y_i$  converge in a neighborhood of the origin.

18. We shall now see to what extent the analytic functions  $y_i$ , just obtained, are solutions of  $\sigma$ .

Consider any equation  $\delta y_i = g$  in  $\sigma$ . This equation, and all equations obtained from it by differentiation with respect to multipliers of the monomial corresponding to the first member, are satisfied, at the origin, by the derivatives of  $y_1, \dots, y_n$  at the origin. Hence, if we substitute  $y_1, \dots, y_n$  into  $\delta y_i - g$ , we obtain a function  $k$  of  $x_1, \dots, x_n$  which vanishes at the origin, together with its derivatives with respect to the above multipliers. Thus, in the expansion of  $k$ , only nonmultipliers occur. Then  $k$  vanishes when the nonmultipliers are zero.

Hence  $y_1, \dots, y_n$  satisfy each equation of  $\sigma$  on the spread obtained by equating to zero the nonmultipliers corresponding to the first member of the equation.

19. Let us return now to the most general orthonomic system  $\sigma$  whose first members give complete sets of monomials. We do not suppose that every  $y_i$  appears in some first member.

<sup>7</sup> In our present language, all derivatives in a second member in (18) are lower than the first member.

We consider any point  $x_i = a_i$ ,  $i = 1, \dots, m$ , subject to obvious conditions of analyticity. Let any values be given to the parametric derivatives of the  $y_i$  at  $a_1, \dots, a_m$ , so as to yield convergent initial determinations. Then the principal derivatives are determined uniquely by  $\sigma$  in such a way as to yield analytic functions  $y_1, \dots, y_n$  which satisfy each equation in  $\sigma$  on the spread obtained by equating to  $a_i$  each nonmultiplier  $x_i$  corresponding to the first member of the equation.

This is an immediate consequence of the preceding sections.

#### PASSIVE ORTHONOMIC SYSTEMS

20. Let  $\sigma$  be an orthonomic system, described as in the preceding section. Let the equations in  $\sigma$  be listed so that their first members form an ascending sequence, and let them be written

$$(25) \quad v_i = 0, \quad i = 1, \dots, l.$$

If  $v_i$  is  $\delta y_j - g$ , we attribute to  $v_i$  the  $s$  marks of  $\delta y_j$ . This establishes order relations among the  $v$ , according to the convention of §8. To all of the derivatives of  $v_i$ , we attribute marks as in §8. Thus, the marks of  $\delta v_i$  will be the marks of the highest derivative in  $\delta v_i$ . By the *monomial corresponding to  $v_i$* , we mean the monomial corresponding to  $\delta y_j$ . We shall refer to  $\delta y_j$  as the *first term in  $v_i$* . By the first term of a derivative of  $v_i$ , we shall mean the corresponding derivative of  $\delta y_j$ .

Consider a  $v$  whose corresponding monomial,  $\alpha$ , has nonmultipliers. Let  $x_i$  be such a nonmultiplier. By §6,  $x_i \alpha$  is the product of a  $\beta$ , in the same complete set as  $\alpha$  and higher than  $\alpha$ , by unity or by multipliers of  $\beta$ . Hence, there is a  $v_p$ , higher than  $v$ , such that some  $\delta v_p$  has the same first term as  $\partial v / \partial x_i$ . Then, in the expression

$$(26) \quad \frac{\partial v}{\partial x_i} - \delta v_p$$

all derivatives effectively present are lower than the first term of  $\partial v / \partial x_i$ .

It is clear that (26) is a polynomial in such principal derivatives as it may involve. Let  $w$  be the highest such principal derivative. Then  $w$  is the first term of some expression  $\delta v_q$ , where  $\delta v_q$  is lower than  $\partial v / \partial x_i$ . We choose  $v_q$  so that  $w$  is obtained from it by differentiation with respect to multipliers of the corresponding monomial. This makes  $v_q$  unique. Let then, identically,

$$(27) \quad w = \delta v_q + k,$$

where the derivatives in  $k$  are all lower than  $w$ . We replace  $w$  in (26) by its expression in (27) and find, identically,

$$\frac{\partial v}{\partial x_i} = \delta v_p + h_1(\delta v_q, \dots),$$

where  $h_1$  is a polynomial in  $\delta v_q$  whose coefficients involve no principal derivative

as high as  $w$ . Let  $w_1$  be the highest principal derivative in  $h_1$ . We give it the treatment accorded to  $w$  and find

$$\frac{\partial v}{\partial x_i} = \delta v_p + h_2 (\delta v_q, \delta v_r, \dots),$$

where  $h_2$  is a polynomial in  $\delta v_q, \delta v_r$ . Continuing, we find in a unique manner an identity

$$(28) \quad \frac{\partial v}{\partial x_i} = \delta v_p + h (\delta v_q, \dots, \delta v_z),$$

in which the coefficients in  $h$  involve only parametric derivatives. We now write (28) in the form

$$(29) \quad \frac{\partial v}{\partial x_i} = \delta v_p + \gamma (\delta v_q, \dots, \delta v_z) + \mu,$$

where  $\mu$  is the term of zero degree in  $h$ . Then  $\mu$  is an expression in the parametric derivatives alone. The expression  $\gamma$  vanishes when  $\delta v_q, \dots, \delta v_z$  are replaced by 0.

It is clear that, for any solution of  $\sigma$ , we must have  $\mu = 0$ . The totality of equations  $\mu = 0$ , obtained from all equations of  $\sigma$  for which the monomial corresponding to the first member has nonmultipliers, all nonmultipliers being used, are called the *integrability conditions* for  $\sigma$ .

21. If every expression  $\mu$  is identically zero, the system  $\sigma$  is said to be *passive*.

We shall prove that, if  $\sigma$  is passive, the  $n$  functions  $y_1, \dots, y_n$ , described in §19, which satisfy each equation in  $\sigma$  on a certain spread, constitute an actual solution of  $\sigma$ .

What we have to show is, that for these functions, every  $v_i$  in (25) vanishes identically.

When the  $y_j$  above are substituted into  $v_i$ , we obtain a function  $u_i$  of  $x_1, \dots, x_m$ . If  $v_i$  has no nonmultipliers,  $u_i = 0$ . Otherwise,  $u_i$  vanishes when the nonmultipliers of the monomial corresponding to  $v_i$  are equated to their  $a$ .

If, in (29), where  $\mu$  is now identically zero, the parametric derivatives in  $\gamma$  are replaced by their expressions as functions of the  $x_i$ , found from the  $y_i$ , (29) becomes a system  $\varphi$  of differential equations in the *unknowns*  $v_i$ . Since (29) consisted of identities, before these replacements,  $\varphi$  is satisfied by  $v_i = u_i$ ,  $i = 1, \dots, t$ .

We now attribute to each  $x_i$  an additional mark 0, and to each  $v_i$  an additional mark  $t - i$ . With this change, the derivatives of the  $v_i$  will be completely ordered and the first member in each equation in  $\varphi$  will be higher than every derivative in the second member.

If the second members in  $\varphi$  contain derivatives of the first members, we can get rid of such derivatives, step by step. Then  $\varphi$  goes over into an orthonomic system  $\psi$ , with the same first members as  $\varphi$ .

For our purposes, it is unnecessary to adjoin new equations to  $\psi$  as in §11.

Consider any unknown  $v_i$  which appears in a first member. The derivatives of  $v_i$  in the first members will be taken with respect to certain variables

$$(30) \quad x_a, \dots, x_d.$$

The variables (30), when equated to their  $a_i$ , give a spread on which  $u_i$  vanishes.

The parametric derivatives of  $v_i$  will be the derivatives taken with respect to the variables not in (30). For the corresponding  $u_i$ , each of these parametric derivatives is zero. Now we know that, for given values of the parametric derivatives, there is at most one solution of  $\psi$ . But  $v_i = 0$ ,  $i = 1, \dots, t$ , is a solution of  $\psi$  for which all parametric derivatives vanish. Hence  $u_i = 0$ ,  $i = 1, \dots, t$ .

This proves that, *given a passive orthonomic system, there is one and only one solution of the system for any given initial determinations.*

## CHAPTER IX

### PARTIAL DIFFERENTIAL ALGEBRA

#### PARTIAL DIFFERENTIAL POLYNOMIALS. IDEALS AND MANIFOLDS

1. We use an algebraic field  $\mathcal{F}$  of characteristic zero which admits  $m$  operations of differentiation. Each element  $a$  of  $F$  has  $m$  partial derivatives  $\partial a / \partial x_i$ ,  $i = 1, \dots, m$ . In this, the  $x$  are not necessarily variables. They may merely be symbols which distinguish the derivatives. Each of the  $m$  operations satisfies (1) and (2) of I, §1. In addition,

$$\frac{\partial}{\partial x_j} \left( \frac{\partial a}{\partial x_i} \right) = \frac{\partial}{\partial x_i} \left( \frac{\partial a}{\partial x_j} \right)$$

for every  $i$  and  $j$ . We call  $F$  a *partial differential field*.

In our work below, definitions will usually be as for the case of one operation and will be given, formally, only when there is some necessity for it.

2. We employ indeterminates  $y_1, \dots, y_n$ . With each  $y_i$  are associated symbols

$$(1) \quad \frac{\partial^{i_1 + \dots + i_m} y_i}{\partial x_1^{i_1} \dots \partial x_m^{i_m}},$$

where the  $i_j$  are any nonnegative integers; these are the *partial derivatives*<sup>1</sup> of  $y_i$ .

$\mathcal{F}$  being given, we understand by a *partial differential polynomial* (p.d.p. or d.p.), a polynomial in derivatives of the  $y$  with coefficients in  $\mathcal{F}$ .

3. We understand marks to be attributed to the symbols  $x$  and  $y$  as in VIII, §8, in such a way as to effect a complete ordering.

By the *leader* of a p.d.p.  $A$  which actually involves indeterminates,<sup>2</sup> we shall mean the highest of those derivatives of the  $y$  which are present in  $A$ .

Let  $A_1$  and  $A_2$  be p.d.p. which actually involve indeterminates. If  $A_2$  has a higher leader than  $A_1$ , then  $A_2$  will be said to be of *higher rank* than  $A_1$ . If  $A_1$  and  $A_2$  have the same leader, and if the degree of  $A_2$  in the common leader exceeds that of  $A_1$ , then again  $A_2$  will be said to be of *higher rank* than  $A_1$ . A d.p. which effectively involves indeterminates will be of *higher rank* than one which does not. Two d.p. for which no difference in rank is created by what precedes will be said to be of the same rank.

As in I, §3, we see that every aggregate of p.d.p. contains a d.p. which is not higher than any other d.p. of the aggregate.

4. If  $A_1$  involves indeterminates,  $A_2$  will be said to be *reduced with respect to*

<sup>1</sup> When the  $i_j$  are all zero, (1) represents  $y_i$ .

<sup>2</sup> We mean that  $A$  is not an element of  $\mathcal{F}$ .

$A_1$  if  $A_2$  contains no proper derivative of the leader of  $A_1$  and if  $A_2$  is either zero or of lower degree than  $A_1$  in the leader of  $A_1$ . A set of p.d.p.

$$(2) \quad A_1, \dots, A_r$$

will be called a *chain* if either

(a)  $r = 1$  and  $A_1 \neq 0$ , or

(b)  $r > 1$ ,  $A_1$  involves indeterminates and, for  $j > i$ ,  $A_j$  is of higher rank than  $A_i$  and reduced with respect to  $A_i$ .

When (b) holds, the leader of  $A_j$  is higher than that of  $A_i$  for  $j > i$ .

Relative rank for chains is defined exactly as in I, §4. If  $\Phi_1, \Phi_2, \Phi_3$ , are chains with  $\Phi_1 > \Phi_2$  and  $\Phi_2 > \Phi_3$ , then  $\Phi_1 > \Phi_3$ .

We prove that, in every aggregate of chains, there is a chain which is not higher than any other chain of the aggregate. Let  $\alpha$  be the aggregate. We form a subset  $\alpha_1$  of  $\alpha$ , putting a chain  $\Phi$  into  $\alpha_1$  if the first d.p. of  $\Phi$  is not higher than the first d.p. of any other chain in  $\alpha$ . It may be that the chains in  $\alpha_1$  are merely elements of  $\mathfrak{F}$ ; if so, any of them is a chain of least rank in  $\alpha$ . Let us suppose that the first d.p. in the chains of  $\alpha_1$  actually involve indeterminates. These first d.p. will all have the same leader; we represent that leader by the symbol  $p_1$ . If the chains in  $\alpha_1$  all consist of one d.p., any chain in  $\alpha_1$  meets our requirements. Suppose that there are chains in  $\alpha_1$  which have more than one d.p. We form the subset  $\alpha_2$  of them whose second d.p. are of a lowest rank and indicate the common leaders of these second d.p. by  $p_2$ . Now  $p_2$  is not a proper derivative of  $p_1$ . As we saw above,  $p_2$  is higher than  $p_1$ . If the chains in  $\alpha_2$  all have just two d.p., any of these chains serves our purpose. If not, we continue. Our result will hold unless there is an infinite sequence

$$p_1, p_2, \dots, p_q, \dots$$

of derivatives which increase steadily in rank, no  $p_q$  being a derivative of a  $p_i$  with  $i < q$ . The existence of such a sequence would contradict Riquier's theorem of VIII, §2.

5. Let  $\Sigma$  be a system containing nonzero d.p. We define a *characteristic set* of  $\Sigma$  to be a chain in  $\Sigma$  of least rank.

If  $A_1$  in (2) involves indeterminates, a d.p.  $F$  will be said to be *reduced with respect to (2)* if  $F$  is reduced with respect to  $A_i$ ,  $i = 1, \dots, r$ .

Let  $\Sigma$  be a system for which (2), with  $A_1$  not free of the indeterminates, is a characteristic set. Then no nonzero d.p. in  $\Sigma$  can be reduced with respect to (2). If a nonzero d.p., reduced with respect to (2), is adjoined to  $\Sigma$ , the characteristic sets of the resulting system are lower than (2).

6. In this section we deal with a chain (2) in which  $A_1$  involves indeterminates.

If a d.p.  $G$  has a leader,  $p$ , we shall call the d.p.  $\partial G/\partial p$  the *separant* of  $G$ . The coefficient of the highest power of  $p$  in  $G$  will be called the *initial* of  $G$ .

Let  $S_i$  and  $I_i$  be, respectively, the separant and initial of  $A_i$  in (2).

We prove the following result.

Let  $G$  be any d.p. There exist nonnegative integers  $s_i, t_i, i = 1, \dots, r$ , such that, when a suitable linear combination of the  $A$  and their derivatives is subtracted from

$$S_1^{s_1} \dots S_r^{s_r} I_1^{t_1} \dots I_r^{t_r} G,$$

the remainder,  $R$ , is reduced with respect to (2).

Let  $p_i$  be the leader of  $A_i$ . We limit ourselves, as we may, to the case in which  $G$  involves derivatives, proper or improper, of the  $p$ . Such derivatives will be called  $p$ -derivatives. Let the highest  $p$ -derivative in  $G$  be  $q$  and let  $q$  be a derivative of  $p_j$ . For the sake of uniqueness, if there are several possibilities for  $j$ , we use the largest  $j$  available. To fix our ideas, we assume  $q$  higher than  $p_r$ . Then

$$S_j^q G = CA_j' + B$$

where  $A_j'$  is a derivative of  $A_j$  with  $q$  for leader and where  $B$  is free of  $q$ . Because  $A_j'$  and  $S_j$  involve no derivative higher than  $q$ ,  $B$  involves no  $p$ -derivative which is as high as  $q$ . For uniqueness, we take  $q$  as small as possible.

If  $B$  involves a  $p$ -derivative which is higher than  $p_r$ , we give  $B$  the treatment accorded to  $G$ . After a finite number of steps, we reach a d.p.  $D$  which differs by a linear combination of derivatives of the  $A$  from a d.p.

$$S_1^{s_1} \dots S_r^{s_r} G.$$

$D$  contains no  $p$ -derivative which is higher than  $p_r$ .

We find then a relation

$$I_r^{t_r} D = HA_r + K,$$

where  $K$  is reduced with respect to  $A_r$ .  $K$  may involve  $p_r$ . Aside from  $p_r$ , the only  $p$ -derivatives present in  $K$  are derivatives of  $p_1, \dots, p_{r-1}$ . Such  $p$ -derivatives are lower than  $p_r$ . Let  $q_1$  be the highest of them.

Suppose that  $q_1$  is higher than  $p_{r-1}$ . We give  $K$  the treatment received by  $G$ , obtaining a unique d.p.  $L$  which differs from some

$$S_1^{s_1} \dots S_{r-1}^{s_{r-1}} I_{r-1}^{t_{r-1}} K$$

by a linear combination of  $A_{r-1}$  and proper derivatives of  $A_1, \dots, A_{r-1}$ . The d.p.  $L$  is reduced with respect to  $A_r$  and  $A_{r-1}$ . Aside from  $p_r$  and  $p_{r-1}$ , the  $p$ -derivatives in  $L$  are derivatives of  $p_1, \dots, p_{r-2}$ , and all such  $p$ -derivatives are lower than  $p_{r-1}$ .

Continuing, we determine, in a unique manner, a d.p.  $R$  as described in our statement. We call  $R$  the remainder of  $G$  with respect to (2).

7. Ideals of p.d.p. are defined as in I, §7. In (b) of I, §7, one requires that the  $m$  partial derivatives of any d.p. in  $\Sigma$  belong to  $\Sigma$ .

We define basis as in I, §12. The basis theorem,<sup>3</sup> the decomposition theorem

<sup>3</sup> In dealing with I, §10, one uses the fact that

$$u^2 \frac{\partial v}{\partial x_i} \equiv 0, \quad \left( uw, \frac{\partial w}{\partial x_i} \right).$$

of I, §16, and the theorem on relatively prime ideals of I, §19, go over immediately to the case of several differentiations.

Manifolds are defined as in II, §1. The decomposition theorem of II, §3, then carries over.

The *analytic case* is formulated as follows.  $\mathfrak{F}$  is a set of functions of  $m$  complex variables  $x_1, \dots, x_m$ . There is given an open region  $A$  in the space of the  $x$ . The functions in  $\mathfrak{F}$  are meromorphic at each point of  $A$ . An analytic zero consists of functions which are analytic in an open region contained in  $A$ .

To illustrate the decomposition theorem, we let

$$(3) \quad A = z - (px + qy) + p^2 + q^2,$$

where  $p = \partial z / \partial x$ ,  $q = \partial z / \partial y$ . Putting  $A = 0$ , and differentiating with respect to  $x$ , we find

$$(4) \quad -(rx + sy) + 2(pr + qs) = 0,$$

where  $r = \partial^2 z / \partial x^2$ ,  $s = \partial^2 z / \partial x \partial y$ . Similarly,

$$(5) \quad -(sx + ty) + 2(ps + qt) = 0,$$

where  $t = \partial^2 z / \partial y^2$ . From (4) and (5) we obtain

$$(rt - s^2)(x - 2p) = 0; \quad (rt - s^2)(y - 2q) = 0.$$

Thus, either  $rt - s^2 = 0$  or  $z = (x^2 + y^2)/4$ . The latter zero of  $A$  does not annul  $rt - s^2$ . Thus the manifold of  $A$  is reducible. The zero  $(x^2 + y^2)/4$  is a component of  $A$ . As one will see later, there is one other component, the *general solution* of  $A$ .

8. The question of generic zeros is treated as in II, §6. Given a prime ideal  $\Sigma$  of p.d.p. in  $y_1, \dots, y_n$ , distinct from the unit ideal, one finds a zero  $\eta_1, \dots, \eta_n$  of  $\Sigma$  which annuls no d.p. not contained in  $\Sigma$ . The abstract theorem of zeros of II, §7, then carries over. The analytic case will be treated later.

The theoretical method of V, §28, for resolving a finite system of d.p. into finite systems equivalent to prime ideals is seen to hold for p.d.p.

#### GENERAL SOLUTIONS

9. Let  $F$  be an algebraically irreducible p.d.p. and  $S$  its separant. We see as in II, §12, that the totality  $\Sigma_1$  of those d.p.  $A$  which are such that

$$(6) \quad SA \equiv 0, \quad \{F\},$$

is an ideal. We shall prove that  $\Sigma_1$  is prime. Let  $p$  be the leader of  $F$ . Let  $AB$  be in  $\Sigma_1$ . There exist relations

$$S^a A \equiv R, \quad S^b B \equiv T, \quad [F],$$

where  $R$  and  $T$  involve no proper derivatives of  $p$ . Then  $SRT$  is in  $\{F\}$ . Let then

$$(SRT)^c = MF + M_1F_1 + \dots + M_qF_q,$$

where the  $F_i$  are distinct partial derivatives of  $F$ . The leaders of the  $F_i$  are distinct. We may thus, and shall, assume that the  $F_i$  increase in rank as their subscripts increase. Let  $p'$  be the leader of  $F_q$ . We have

$$F_q = Sp' + U,$$

where the leader of  $U$  is lower than  $p'$ . We replace  $p'$  in  $F_q$  and in the  $M$  by  $-U/S$ . The proof is completed as in II, §12.

As in II, §13, we prove that  $\Sigma_1$  consists of those d.p. which have zero remainders with respect to<sup>4</sup>  $F$ . In particular,  $\Sigma_1$  does not contain  $S$ .

As in II, §§14, 15, we find that  $\{F\}$  has a decomposition into essential prime divisors

$$(7) \quad \{F\} = \Sigma_1 \cap \Sigma_2 \cap \dots \cap \Sigma_s$$

in which  $\Sigma_1$  is the only divisor which does not contain  $S$ .

A change of marks may give  $F$  a new separant. Any such separant involves only derivatives present in  $F$  and is not divisible by  $F$ . Hence, for the original marks, such a separant has a remainder which is not zero. Thus, in (7),  $\Sigma_1$  contains no separant of  $F$ , while  $\Sigma_2, \dots, \Sigma_s$  contain every separant.

We call the manifold of  $\Sigma_1$  the *general solution* of  $F$ .

COMPONENTS OF A PARTIAL DIFFERENTIAL POLYNOMIAL

10. Let  $F$  be a nonzero p.d.p. We shall prove that every essential prime divisor of  $\{F\}$  has a characteristic set consisting of a single d.p. Such a d.p. call it  $A$ , can be taken as algebraically irreducible; the prime divisor consists of those d.p. which have zero remainders with respect to  $A$ . It will follow that *every component of a nonzero p.d.p. is the general solution of some p.d.p.*<sup>5</sup>

11. Let

$$(8) \quad A_1, \dots, A_r$$

be a chain with  $A_1$  not an element of  $\mathfrak{F}$ . Let  $A_i$  have  $S_i$  for separant and  $I_i$  for initial. Let  $G$  be any p.d.p. We shall prove that *there exists a power product  $J$  of the  $S$  and  $I$  such that  $JG$  is a polynomial in the  $A$  and their partial derivatives with coefficients which are d.p. reduced with respect to (8).*

Let  $p_i$  be the leader of  $A_i$ . We limit ourselves, as we may, to the case in which  $G$  involves  $p$ -derivatives. Let the highest  $p$ -derivative in  $G$  be  $q_1$  and let  $q_1$  be a derivative of  $p_j$ . For uniqueness, we use the largest  $j$  available. To fix our ideas, we assume  $q_1$  higher than  $p_r$ . For some partial derivative  $A'_j$  of  $A_j$  we have

$$A'_j = S_jq_1 + T,$$

<sup>4</sup> We obtain  $B$  as in (12) of II, §13, with  $B$  free of proper derivatives of  $p$ .

<sup>5</sup> The case of  $m > 1$  is essentially different from that of  $m = 1$ . For instance, for  $m = 1$  every irreducible manifold in one indeterminate is a general solution. This is not so for p.d.p.

where  $T$  involves no derivative as high as  $q_1$ . Let  $G$  be of degree  $a$  in  $q_1$ . Then  $S^a G$  can be written as a polynomial in  $A'_j - T$ , and hence as a polynomial in  $A'_j$ , with coefficients in which all  $p$ -derivatives are lower than  $q_1$ . Suppose that, among the coefficients just mentioned, there are one or more which involve  $p$ -derivatives higher than  $p_r$ . Let  $q_2$  be the highest such  $p$ -derivative. We give the coefficients which involve  $q_2$ , with respect to  $q_2$ , the treatment accorded above to  $G$  with respect to  $q_1$ . We see now that there is a power product  $J_1$  in one or two of the  $S$  such that  $J_1 G$  is a polynomial in two derivatives of the  $A$ , with coefficients involving no  $p$ -derivative as high as  $q_2$ . We reach ultimately a  $J_u G$ , with  $J_u$  a power product in the  $S$ , in the coefficients of which the  $p$ -derivatives actually present are not higher than  $p_r$ . Some  $I_r^e J_u G$  is a polynomial in  $A_r$  and proper derivatives of the  $A$  with coefficients which are reduced with respect to  $A_r$ . Aside from  $p_r$ , the only  $p$ -derivatives present in the coefficients are derivatives of  $p_1, \dots, p_{r-1}$ . How to complete the proof is now obvious.

Let us examine the expression found for  $JG$ . In our discussion, there appeared a finite sequence of derivatives

$$(9) \quad q_1, q_2, \dots, q_t$$

with  $q_i$  higher than  $q_{i+1}$ ,  $i = 1, \dots, t-1$ , each  $q_i$  being the leader of a derivative  $B_i$ , proper or improper, of some  $A$ .  $JG$  is a polynomial in the  $B$ , with coefficients reduced with respect to (8).

12. Let  $F$  be a nonzero d.p. and let (7) be a decomposition of  $\{F\}$  into essential prime divisors. Suppose that some  $\Sigma_i$  in (7) has a characteristic set consisting of more than one d.p. We let  $\Lambda$  stand for such a  $\Sigma_i$  and consider a characteristic set (8) of  $\Lambda$ .

Treating  $F$  as  $G$  was treated in §11, we obtain a  $J$  as in §11 and let  $H = JF$ . Then  $H$  is a polynomial in the  $A$  and their partial derivatives.

Let  $\eta_1, \dots, \eta_n$  be a generic zero of  $\Lambda$  contained in an extension of  $\mathfrak{F}_0$  of  $\mathfrak{F}$ . We make in  $H$  and in the  $A$  the substitution

$$(10) \quad y_i = \eta_i + z_i, \quad i = 1, \dots, n,$$

using the same marks for  $z_i$  as for  $y_i$ . Each  $A_i$  goes over into a d.p.  $C_i$  over  $\mathfrak{F}_0$ . Let us study  $C_i$  as a polynomial in the  $z$  and their derivatives.  $C_i$  admits the zero  $z_j = 0$ ,  $j = 1, \dots, n$ . We examine the terms of the first degree in  $C_i$ . To  $p_i$ , the leader of  $A_i$ , there corresponds a derivative  $r_i$  of some  $z$ . The coefficient of  $r_i$  in  $C_i$  is what  $S_i$  becomes when the  $\eta$  are substituted into it. Because  $S_i$  is not in  $\Lambda$ ,  $S_i$  does not vanish for the  $\eta$ . Thus  $C_i$  contains effectively terms of the first degree. We represent the sum of these terms by  $D_i$ . The leader of  $D_i$  is  $r_i$ .

We now consider  $H$ . Let  $K$  represent what  $H$  becomes under (10). Our object is to describe the terms of lowest degree in  $K$  considered as a polynomial in the  $z$  and their derivatives.

Referring to the final remarks of §11, we consider  $H$  as a polynomial in the  $B_i$ ,  $i = 1, \dots, t$ . Let  $L$  be the sum of those terms of  $H$  which are of a lowest

total degree in the  $B$ . Then every term of  $L$  is of the form  $MN$  with  $M$  reduced with respect to (8) and  $N$  a power product in the  $B$ . Under (10), let  $M$  and  $N$  go over into  $P$  and  $Q$  respectively. Then  $P$  contains an effective term which is in  $\mathfrak{F}_0$ , while the terms of  $Q$  which are of a lowest total degree in the  $z$  and their derivatives constitute a product of powers of the  $D$  and their derivatives. Let us select, from  $L$ , those terms which are of a highest degree in  $B_1$ . From these latter terms we select those which are of a highest degree in  $B_2$ . Continuing, we are led to a definite term  $MN$  of  $L$  which goes over under (10) into an expression  $PQ$ . Let

$$N = B_a^\alpha B_b^\beta \cdots B_c^\gamma,$$

where  $a < b < \cdots < c$  and  $\alpha, \beta, \cdots, \gamma$  are positive. If  $s_i$  is allowed to represent that derivative of a  $z$  whose marks are those of  $q_i$  in (9), we find that  $PQ$  contains effectively a term in  $s_a^\alpha s_b^\beta \cdots s_c^\gamma$ . This term is one of the terms of lowest degree in  $K$ .

Thus the leader of  $W$ , the sum of the terms of lowest degree in  $K$ , is a derivative, proper or improper, of the leader of some  $D$ .

13. Changing the notation if necessary, we assume that the leader of  $W$  is a derivative of  $z_1$ . We decompose  $W$  into irreducible factors in  $\mathfrak{F}_0$  and consider an irreducible factor  $V$  which effectively involves the leader of  $W$ .

$V$  is a d.p. over  $\mathfrak{F}_0$ . Let  $\zeta_1, \cdots, \zeta_n$  be a generic point in the general solution of  $V$ , contained in an extension  $\mathfrak{F}_1$  of  $\mathfrak{F}_0$ .

Then  $W$  vanishes for the  $\zeta$ . On the other hand, not every  $D_i$  can so vanish. Let us assume that  $D_1$  vanishes. We shall prove that  $D_2$  does not. By the final statement of §12, the leader of  $V$  is not lower than that of  $D_1$ . If  $D_1$  had a lower leader than  $V$ ,  $D_1$  would be reduced with respect to  $V$  and would not vanish for the  $\zeta$ . Thus  $D_1$  has the same leader as  $V$ . Then  $D_1$  is divisible by  $V$ . As  $D_1$  is linear,  $D_1$  is the product of  $V$  by an element of  $\mathfrak{F}_0$ . Thus the general solution of  $V$  is the general solution of  $D_1$ . By §12, the leaders of  $A_i$  and  $D_i$  have the same marks for every  $i$ . Thus the leader of  $D_2$  is not a derivative of that of  $D_1$ . Then the remainder of  $D_2$  with respect to  $D_1$  is not zero so that  $D_2$  does not vanish for the  $\zeta$ .

14. We say that  $K$  is annulled by expressions

$$(11) \quad \begin{aligned} z_i &= \zeta_i c, & i &= 2, \cdots, n, \\ z_1 &= \zeta_1 c + \varphi_2 c^{\rho_2} + \cdots + \varphi_k c^{\rho_k} + \cdots, \end{aligned}$$

with  $\rho_2 > 1$ .

If  $K$  vanishes for  $z_i = \zeta_i c$ ,  $i = 1, \cdots, n$ , we have the desired expressions. Let the vanishing fail to occur. We put in  $K$

$$z_i = \zeta_i c, \quad i = 2, \cdots, n; \quad z_1 = \zeta_1 c + u_1,$$

where  $u_1$  has the same marks as  $z_1$ . The work of III, §§6-13, carries over with very slight changes. Where, in Chapter III, one uses derivatives of an indeterminate up to a certain order, one employs here a set of partial derivatives.

Leaders serve here as derivatives of highest order do in Chapter III. In treating (11) of III, §10, we represent the derivatives of  $u_1$  appearing in  $K'$  by  $v_1, \dots, v_\sigma$  and the corresponding derivatives of  $u_2$  by  $w_1, \dots, w_\rho$ . Assuming that, for certain  $l$ ,

$$\frac{\partial^h + \dots + \iota_l L'(u_1)}{\partial^h v_1 \dots \partial^l v_\sigma}$$

does not vanish for  $u_1 = \varphi_2$ , we prove that  $w_1^h \dots w_\rho^l$  is present in  $K''$ .

15. The series (11) being obtained, we find that  $H$  is annulled by expressions

$$(12) \quad \begin{aligned} y_i &= \eta_i + \zeta_i c, & i &= 2, \dots, n, \\ y_1 &= \eta_1 + \zeta_1 c + \varphi_2 c^{\rho_2} + \dots. \end{aligned}$$

These expressions do not annul  $J$ , since the  $\eta$  do not. Thus  $F$  vanishes for (12). Because the  $D$  of §12 do not all vanish for the  $\zeta$ , the  $C$  do not all vanish for (11), so that the  $A$  in the characteristic set (8) of  $\Lambda$  of §12 do not all vanish under (12). Now some  $\Sigma_i$  in (7) must admit (12) as a zero. Such a  $\Sigma_i$  is necessarily distinct from  $\Lambda$ . On the other hand, such a  $\Sigma_i$  must admit  $\eta_1, \dots, \eta_n$  as a zero, and thus is contained in  $\Lambda$ . As this is impossible, it is established that every prime ideal in the second member of (7) has a characteristic set consisting of one d.p.

16. Suppose now that  $F$  of §10 is algebraically irreducible. Let  $\Sigma_1$  in (7) be the prime ideal associated with the general solution of  $F$ . Consider any  $\Sigma_i$  with  $i > 1$ . Its manifold is the general solution of a d.p.  $A$ . We say that  $F$  effectively involves some proper derivative of the leader of  $A$ .

If this were not true,  $F$  would be divisible by  $A$ , since  $F$  is in  $\Sigma_i$  and the remainder of  $F$  with respect to  $A$  is zero.

Let  $y_j$  be any indeterminate of which some derivative appears effectively in  $A$  and let  $r$  be the maximum of the orders of the derivatives of  $y_j$  in  $A$ . Marks can be chosen for which the leader of  $A$  is a derivative of  $y_j$  of order  $r$ . Thus  $F$  is of higher order than  $A$  in every indeterminate appearing in  $A$ .

THE LOW POWER THEOREM

17. Let  $F$  and  $A$  be two p.d.p. in  $\mathfrak{F}\{y_1, \dots, y_n\}$ , neither an element of  $\mathfrak{F}$ . Let  $S$  be the separant, and  $p$  the leader, of  $A$ . Proceeding as in III, §17, and as in IX, §11, one proves the existence of a nonnegative integer  $t$  such that  $S^t F$  has a representation

$$(13) \quad \sum_{j=1}^t C_j A^{2j} A_1^{4j} \dots A_h^{4j}$$

where the  $A_i$  are distinct proper derivatives of  $A$  and no two sets  $i_{1j}, \dots, i_{hj}$  are identical; the  $C$  involve no proper derivative of  $p$  and are not divisible by  $A$ . If  $F$  involves no proper derivative of  $p$ , there are no  $A_i$  in (13); otherwise the

leader of  $A_k$  is the highest of the derivatives of  $p$  which appear in  $F$ . For a given admissible  $t$ , the representation (13) of  $S^t F$  is unique.

In what follows, we assume  $A$  to be algebraically irreducible and we use the smallest admissible  $t$ .

The low power theorem has the wording of III, §20, except that one uses the representation in (13).

18. We use an indeterminate  $y$  and the field of rational numbers.<sup>6</sup> Let  $p$  be any positive integer. We shall show that every power product of degree  $2p - 1$  in the  $\partial y / \partial x_i$ ,  $i = 1, \dots, m$ , is in  $[y^p]$ .

We may assume that  $p > 1$ . We have, for every  $i$ ,  $y^{p-1} \partial y / \partial x_i \equiv 0$ ,  $[y^p]$ . Thus, for every  $i$  and  $j$ ,

$$(p - 1) y^{p-2} \frac{\partial y}{\partial x_j} \frac{\partial y}{\partial x_i} + y^{p-1} \frac{\partial^2 y}{\partial x_j \partial x_i} \equiv 0, \quad [y^p].$$

We multiply by any  $\partial y / \partial x_k$ . Then, for any  $i, j, k$ ,

$$y^{p-2} \frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} \frac{\partial y}{\partial x_k} \equiv 0, \quad [y^p].$$

Continuing, we verify our statement.

Let  $k$  be any positive integer. We consider the derivatives of  $y$  of order  $k$ , and form power products in those derivatives. We shall show that every such power product which is of degree  $2^k m^{k-1} p$  is in  $[y^p]$ .

For  $k = 1$ , we observe that  $2^k m^{k-1} p = 2p > 2p - 1$ , and use the result proved above. We suppose the proof carried through for  $k < q$ , where  $q > 1$ , and consider the case of  $k = q$ . Among  $2^q m^{q-1} p$  derivatives of order  $q$ , there must be at least  $2^q m^{q-2} p$  which are derivatives of order  $q - 1$  of some one  $\partial y / \partial x_j$ . By the case of  $k = q - 1$ , a product of  $2^q m^{q-2} p$  derivatives as just mentioned is in  $[(\partial y / \partial x_j)^{2p}]$ , thus in  $[y^p]$ .

The weight of a product of powers of derivatives of  $y$  will be understood to be the sum of the orders of the derivatives in the product.

Let  $a$  be a positive integer. Let

$$f(a, p, m) = p(a + 1) \frac{(2m)^{a+1} - 1}{2m - 1}.$$

We shall show that a power product in  $y$  and its derivatives whose degree is  $f(a, p, m)$  and whose weight does not exceed  $af(a, p, m)$  is in  $[y^p]$ .

Let  $P$  be a power product of degree  $f(a, p, m)$  which is not in  $[y^p]$ . For each nonnegative integer  $k$ , the product  $P$ , by what precedes, must involve fewer than  $(2m)^k p$  derivatives of order  $k$ .<sup>7</sup> Thus  $P$  involves fewer than

$$\frac{(2m)^{a+1} - 1}{2m - 1} p = \frac{f(a, p, m)}{a + 1}$$

<sup>6</sup> In §§18, 19, we do not use marks; the order of a partial derivative is the only index of rank which is employed.

<sup>7</sup> We count each derivative of order  $k$  as many times as it appears in  $P$ .

derivatives of order not exceeding  $a$ . Therefore  $P$  has more than  $f - f/(a + 1)$  derivatives of orders exceeding  $a$ . Then the weight of  $P$  exceeds  $af$ .<sup>8</sup>

19. We can now carry over the lemma of III, §21. Let  $r$  be the maximum of the weights of the  $B$ . The cases of  $r = 0$  and  $r = 1$  are trivial. We therefore assume that  $r > 1$  and put

$$d = f(r - 1, p, m), \quad t = d(r - 1).$$

Every power product in  $z$  and its derivatives which is of degree  $d$  and of weight not more than  $t$  is in  $[z^p]$ . The work of III, §21, needs only minor changes. Where one uses there the  $i$ th derivative of a d.p., one employs here appropriate partial derivatives of order  $i$ . The lemma having been extended, one finds the theorems of III, §§22, 23, to hold for p.d.p.

20. The necessity proof can be conducted as follows. We assume that the terms of lowest degree in (13) involve proper derivatives of  $A$ . If we let  $A$  take the place of the chain (8), (13) is an expression for  $S^t F$  like that of  $JG$  in §10, with the difference that the  $C$  are not reduced with respect to  $A$ . For our purposes, it is enough that the  $C$  do not hold the general solution of  $A$ .

We let  $\eta_1, \dots, \eta_n$  be a generic zero in  $\mathfrak{M}$ , the general solution of  $A$ , and make the substitution (10) in  $S^t F$  and in  $A$ . Then  $A$  goes over into a d.p.  $E$  in the  $z$ . To  $p$ , the leader of  $A$ , there corresponds a derivative  $r$  of some  $z$ .  $E$  has terms of the first degree and their sum has  $r$  for leader.

The substitution (10) converts  $S^t F$  into a d.p.  $K$  in the  $z$ . Considering  $K$  as a polynomial in the  $z$  and their derivatives, we let  $W$  be the sum of the terms of lowest degree in  $K$ . The leader of  $W$  is seen to be a proper derivative of  $r$ . We then proceed as in §§13, 14 and find expressions (12) which annul  $F$  but neither  $S$  nor  $A$ . Those expressions furnish a zero in a component  $\mathfrak{M}'$  of  $F$  which is not held by  $A$ . Then  $\eta_1, \dots, \eta_n$  is in  $\mathfrak{M}'$  and  $\mathfrak{M}$  is not a component of  $F$ .

#### CHARACTERISTIC SETS OF PRIME IDEALS

21. Let  $\Sigma$  be a nontrivial prime ideal for which

$$(14) \quad A_1, A_2, \dots, A_r$$

is a characteristic set. One shows, as in V, §1, that when the  $A$  are regarded as ordinary polynomials in the symbols which they involve, (14) is a characteristic set<sup>9</sup> for a prime p.i.  $\Lambda$ . One then proves as in V, §4, that every zero of the p.d.p. (14) which annuls no separant is a zero of  $\Sigma$ .

22. From this point on we limit ourselves to the consideration of the analytic case. Through §25, it will be assumed that the first mark of each  $x$  is unity.

$\mathbf{A}$  being the region in which the functions in  $\mathfrak{F}$  are given, we represent by

<sup>8</sup> The result is due to Kolchin.

<sup>9</sup> As we shall see below, we do not have in this a sufficient condition for (14) to be a characteristic set of a prime ideal.

$\xi_1, \dots, \xi_m$  or, more briefly, by  $\xi$ , a point in  $\mathbf{A}$  at which the coefficients in (14) are analytic. We use the symbol  $[\eta]$  to designate any set of numerical values which one may choose to associate with the derivatives appearing in (14).

We wish to show that there are sets  $\xi, [\eta]$  which annul every  $A$  but none of the separants of the  $A$ . If we consider the  $A$  as ordinary polynomials, Hilbert's theorem of zeros, as derived for the analytic case in IV, §14, holds for them. As no power of the product of the separants is linear in the  $A$ , we can find a system of analytic functions of  $x_1, \dots, x_m$  which annul the  $A$  when substituted for the various derivatives, without annulling any separant. The existence of a set  $\xi, [\eta]$ , described as above, follows. We shall deal with such a set.

Let  $p_i$  be the leader of  $A_i$ . The equation  $A_1 = 0$ , treated as an algebraic equation for  $p_1$ , determines  $p_1$  as a function of the  $x$  and the derivatives lower than  $p_1$  in  $A_1$ , the function being analytic for  $x_i$  close to  $\xi$ ; and for the derivatives lower than  $p_1$  close to their values among the  $[\eta]$ . The value of the function  $p_1$  for the special arguments just mentioned will be the value for  $p_1$  in  $[\eta]$ . Let the expression for  $p_1$  be substituted into  $A_2$ . We can then solve  $A_2 = 0$  for  $p_2$ , expressing  $p_2$  as a function of the  $x$  and of the derivatives other than  $p_1$  and  $p_2$  appearing in  $A_1$  and  $A_2$ . We substitute the expressions for  $p_1$  and  $p_2$  into  $A_3$ , solve  $A_3 = 0$  for  $p_3$ , and continue in this manner for all d.p. in (14).

We find thus a set of expressions for the  $p$ , each  $p$  being given as a function of the  $x$  and of the derivatives other than  $p_1, \dots, p_r$  in (14). We write

$$(15) \quad p_i = g_i, \quad i = 1, \dots, r.$$

If the equations in (15) are considered as differential equations for the  $y$ , they will form an orthonomic system. We shall prove that if (15) is extended into an orthonomic system whose first members give complete systems of monomials (VIII, §11), the extended orthonomic system is passive.

We consider the prime p.i.  $\Lambda$  of §21. The parametric indeterminates in  $\Lambda$  will be those which correspond to the parametric derivatives in (15). We form a resolvent for  $\Lambda$  with

$$(16) \quad w = b_1 p_1 + \dots + b_r p_r,$$

where the  $b$  are integers. Let the resolvent be

$$(17) \quad B_0 w^s + \dots + B_s = 0,$$

and let the expressions for the  $p$  be

$$(18) \quad p_i = \frac{E_{i0} + \dots + E_{i, s-1} w^{s-1}}{D}.$$

Suppose that, in (16), the  $p$  are replaced by the  $g$  of (15). Then  $w$  in (16) becomes a function of the arguments in the  $g$ , analytic at  $\xi, [\eta]$ . We wish to see that the functions  $g_i$  and  $w$  satisfy (17) and (18). We can form a zero of the characteristic set (14) of  $\Lambda$ , in which the leaders of the  $A$  are put equal to the  $g$  and in which the other letters in the  $A$  are represented by the complex variables

of which the  $g$  are functions. This zero of (14) annuls no separant; it is thus a zero of  $\Lambda$ . This is enough to show that the  $g$  and  $w$  satisfy (17) and (18).

We consider each  $g$  in (15) to be expressed by the second member of (18), where  $w$  is a function of the  $x$  and the parametric derivatives, analytic when the arguments are close to their values<sup>10</sup> in  $\xi, [\eta]$ .

Let us show how an orthonomic extension  $\sigma$  of (15), described as in VIII, §11, is formed. We can calculate each  $\partial p_i / \partial x_j$  from (18). In this calculation  $\partial w / \partial x_j$  appears, and can be found from (17). Higher derivatives of the  $p$  are calculated similarly. If principal derivatives appear in an expression for a  $\delta p$ , we can get rid of them step by step. We secure in this way the desired extension  $\sigma$ . Its equations will be of the form (IV, §14)

$$(19) \quad \delta y = \frac{F_0 + \cdots + F_{s-1} w^{s-1}}{T},$$

where  $T$  involves only parametric derivatives. There may be, in the second members of (19), parametric derivatives which do not appear in (15). Such derivatives enter rationally and integrally. We shall allow these derivatives to vary in the neighborhood of any set of numerical values  $[\zeta]$ .

If we refer now to VIII, §20, we see that every  $\mu$  has an expression like the second member of (19). To establish the passivity of  $\sigma$  for the neighborhood of  $\xi, [\eta], [\zeta]$ , we have to show that every  $\mu$ , as a function of the  $x$  and of the parametric derivatives, is identically zero.

Consider some  $\mu$ , say  $\mu_1$ . Let  $Z$  be the numerator in the expression for  $\mu_1$  and let  $P$  represent the first member of (17). Suppose that  $Z$  is not identically zero. Then the resultant  $W$  of  $P$  and  $Z$  with respect to  $w$  is not zero. If we can show that  $W$  is in  $\Sigma$ , we will have a contradiction. Working in the abstract, let us form a generic zero of  $\Sigma$ ; with it is associated a quantity  $w$  as in (16). The generic zero and  $w$  satisfy (19) and thus annul  $Z$ . Then the generic zero annuls  $W$ . Hence  $Z$  is identically zero and  $\sigma$  is passive at  $\xi, [\eta], [\zeta]$ .

23. Let (14), with  $A_1$  not a function in  $\mathfrak{F}$ , be a chain. We shall find necessary and sufficient conditions for (14) to be a characteristic set of a prime ideal.

As a first necessary condition, we have the condition that (14), when regarded as a set of polynomials, be a characteristic set for a prime p.i. This implies the existence of  $r$  functions  $g_i$ , as in (15), which annul the  $A$  when substituted for the  $p$ , without annulling any separant.

Let  $\xi, [\eta]$  be some set of values as above, for which no separant vanishes. A second necessary condition is that the extended system (19) be passive for the neighborhood of  $\xi, [\eta], [\zeta]$ .

We shall prove that, if (14), considered as a set of polynomials, is a characteristic set of a prime p.i., and if, for some set  $\xi, [\eta], [\zeta]$ , (19) is passive, then (14) is a characteristic set of a prime ideal.

Let (14) satisfy the stated conditions. As the expressions for the  $\mu$  vanish

<sup>10</sup> It may be that  $D$  vanishes at  $\xi, [\eta]$ , but this is not a matter for concern.

identically, (19) developed for *any values at all*  $\xi$ ,  $[\eta]$  which annul no separant will be passive.

The passivity of (19) implies that (14) has zeros which annul no separant. We shall prove that the system  $\Sigma$  of d.p. which vanish for all zeros of (14) annulling no separant is a prime ideal for which (14) is a characteristic set.

Let  $GH$  be in  $\Sigma$ . Let  $J_1G \equiv G_1$ ,  $J_2H \equiv H_1$ ,  $[A_1, \dots, A_r]$ , where the  $J$  are power products in the separants and  $G_1, H_1$  involve no proper derivatives of the  $p$ . There may be, in  $G_1$  and  $H_1$ , parametric derivatives not present in (14). But (14), considered as a set of polynomials, will be a characteristic set for a prime p.i.  $\Lambda$ , even after the adjunction of the new parametric derivatives to the indeterminates in (14).

Let us consider any zero of  $\Lambda$  which annuls no separant in (14). By the passivity of (19) for arbitrary sets  $\xi$ ,  $[\eta]$ ,  $[\zeta]$ , the mentioned zero furnishes, at a point free to vary in a region in  $\mathbf{A}$ , initial conditions for a zero  $\eta_1, \dots, \eta_n$  of the d.p. in (14) which annuls no separant.  $\Sigma$  admits  $\eta_1, \dots, \eta_n$  as a zero. It follows that the zero of  $\Lambda$  annuls  $G_1H_1$ . Then  $G_1H_1$ , considered as a polynomial, is in  $\Lambda$ . Suppose then that  $G_1$  is in  $\Lambda$ . Then  $G$  is in  $\Sigma$ . Thus  $\Sigma$ , which we know to be an ideal, is prime. To prove that (14) is a characteristic set for  $\Sigma$ , it suffices to show that  $\Sigma$  contains no nonzero d.p. reduced with respect to (14); such a d.p., by what precedes, would, considered as a polynomial, belong to  $\Lambda$ .

24. Given a set (14) which satisfies the first condition in §23, we can determine, with a finite number of rational operations and differentiations, whether or not (19) is passive. If (19) is not passive, we secure a d.p. involving only parametric derivatives which vanishes for all zeros of (14) which annul no separant.<sup>11</sup>

ALGORITHM FOR DECOMPOSITION

25. Let  $\Phi$  be any finite system of p.d.p., not all zero. As in Chapter V, we can obtain, by a finite number of differentiations, rational operations and factorizations, a set, equivalent to  $\Phi$ , of finite systems  $\Lambda_1, \dots, \Lambda_r$ , which have the following properties:

- (a) The characteristic sets of the  $\Lambda_i$  are not higher than those of  $\Phi$ .
- (b) If the characteristic set of a  $\Lambda_i$  involves indeterminates, the remainder of any d.p. of  $\Lambda_i$  with respect to the characteristic set is zero.
- (c) The characteristic set of a  $\Lambda_i$ , considered as a set of polynomials, is a characteristic set of a prime p.i.

Suppose that  $\Lambda_1$  has a characteristic set (14) with  $A_1$  not in  $\mathfrak{F}$ . If (19) is not passive,  $\Lambda_1$  is equivalent to

$$\Lambda_1 + G, \Lambda_1 + S_1, \dots, \Lambda_1 + S_r,$$

where the  $S$  are the separants, and  $G$ , involving only parametric derivatives,

<sup>11</sup> In (19) and in the analogous expressions for the  $\mu$ , we may use a single  $T$ . If  $Z$  of §22 does not vanish.  $TW$  will serve our purpose.

vanishes for every zero of (14) which annuls no  $S$ . Now all of the systems just obtained have characteristic sets lower than (14). If (19) proves passive,  $\Lambda_1$  is equivalent to

$$\Sigma, \Lambda_1 + S_1, \dots, \Lambda_1 + S_r,$$

where  $\Sigma$  is the prime ideal for which (14) is a characteristic set.

It is clear that, by this process, we arrive in a finite number of steps at a finite number of chains which are characteristic sets of a set of prime ideals equivalent to  $\Phi$ .

The above constitutes an elimination theory for systems of algebraic partial differential equations.

26. The assumption that the first mark of each  $x$  is unity prevents us from using, in the case of one independent variable, the ordering employed in the earlier chapters. Thus, when the first mark is unity, no derivative of  $y_2$  will be higher than every derivative of  $y_1$ . Now, in the case of  $m = 1$ , no two  $p$  in (15) are derivatives of the same  $y$ . Thus, with any marks, when  $m = 1$ , the equations (15) are a set of ordinary differential equations for which the standard existence theorem can be used. We see that, when  $m = 1$ , (14) will be a characteristic set of a prime ideal if it is a characteristic set of a prime p.i.; one may use any marks which effect a complete ordering. In this way, the theory of characteristic sets of prime ideals is so framed as to include, in the case of  $m = 1$ , our earlier considerations.

#### THE THEOREM OF ZEROS

27. We treat the theorem of zeros in the analytic case. Let there be given p.d.p.  $F_1, \dots, F_p$ , and a  $G$  which vanishes for every analytic zero of the  $F$ . We have to show that  $G$  is contained in  $\{F_1, \dots, F_p\}$ . Let  $\Sigma$  be an essential prime divisor of the perfect ideal. Suppose that  $\Sigma$  does not contain  $G$ . Let (14) be a characteristic set for  $\Sigma$ . Let  $R$  be the remainder of  $G$  with respect to (14) and let  $K = RS_1 \dots S_r$ . Then  $K$ , as a polynomial, is not in  $\Lambda$  of §21. A zero of  $\Lambda$  which does not annul  $K$  furnishes initial conditions for a zero of (14) which is a zero of  $\Sigma$  and does not annul  $G$ .

## APPENDIX. QUESTIONS FOR INVESTIGATION

### IDEALS

1. Levi's work shows the nonexistence of a theory of ideals of d.p. possessing the scope of the Lasker-Noether theory of p.i. For d.p., it will be necessary either to use special types of ideals or to use other combinations than intersections and products.

2. Given a finite set of d.p.,  $F_1, \dots, F_r$ , and a d.p.  $G$ , is it possible to determine whether  $G$  is contained in  $[F_1, \dots, F_r]$ ? The methods of Chapter V permit one to decide whether some power of  $G$  is in  $[F_1, \dots, F_r]$ . It is thus a question of determining a smallest admissible exponent.

3. Kolchin's theory of exponents should admit of extension in several directions. The chief problem examined by Kolchin is that of the exponent of  $\{A\}$  relative to  $[A]$ , where  $A$  is a d.p. in  $y$  of the first order. In the theorems obtained by Kolchin, the relative exponents are 1, 2,  $\infty$ . For instance, if  $A = y^2 + y_1^3$ , the exponent is  $\infty$ . Now

$$[A] = [y^p] \cdot \Sigma,$$

with  $p$  a positive integer and  $\Sigma$  an ideal whose manifold is the general solution of  $A$ . One may inquire as to the exponent of  $\{\Sigma\}$  relative to  $\Sigma$ . That exponent may easily be finite. This problem can, of course, be formulated for d.p.  $A$  admitting many singular zeros.

The problem of exponents may be examined for d.p. of order higher than the first and for p.d.p.

4. For  $F = y^p + y_1^q$ , in  $\mathfrak{F}\{y\}$ , with  $q > p$ , what is the smallest integer  $r$  such that

$$y^r G \equiv 0, \quad [F],$$

where  $G$  does not vanish for  $y = 0$ ? This problem can be extended to general classes of d.p.

5. For  $p > 0, i > 0$ , what is the least  $q$  such that  $y_1^q \equiv 0, [y^p]$ ? For  $i = 1$ , it is not hard to show that  $q = 2p - 1$ . In  $\mathfrak{F}\{u, v\}$ , what is the least power of  $u, v$ , which is contained in  $[uv]$ ?

6. The ideals generated by various differential expressions may be examined. One may study the wronskian, the jacobian, the expression  $EG - F^2$  of differential geometry, etc.

7. One may study d.p. over a field of characteristic  $p$ .

### THE DECOMPOSITION PROBLEM

8. The basic problem has been met in Chapter V. It is that of determining the number of times which the d.p. in a finite system  $\Phi$  must be differentiated

before eliminations will produce finite systems whose manifolds are the components of  $\Phi$ . One would hope to secure a bound which depends on the number of d.p. in  $\Phi$ , their orders and degrees.

9. Attached to the decomposition problem is the first problem of Laplace, mentioned in III, §37. Let  $F$  and  $A$  be algebraically irreducible and let  $F$  hold the general solution of  $A$ . It is required to determine whether the general solution of  $A$  is contained in that of  $F$ . The author has shown how to settle this question for d.p.  $F$  of the second order.<sup>1</sup> The methods can perhaps be extended to cover the case in which  $F$ , in  $\mathfrak{F}\{y\}$ , is of order  $n$ , and  $A$  of order  $n - 2$ . One might perhaps undertake to develop a test for the presence of  $y = 0$  in the general solution of a d.p. of the third order. Other problems of this type will readily suggest themselves.

#### INTERSECTIONS

10. One can see from Chapter VII that if there is regularity in the theory of intersections of algebraic differential manifolds, that regularity is not immediately visible. In VII, §1, an anomaly is found in the dimension of the intersection of a general solution with a second irreducible manifold. One might try to use complete manifolds of d.p. rather than general solutions. Thus, let  $F_1, \dots, F_r$  be d.p. in  $\mathfrak{F}\{y_1, \dots, y_n\}$ . Suppose that  $r < n$ . Is every component of the system  $F_1, \dots, F_r$  of dimension at least  $n - r$ ? For  $r = 1$ , we see from III, §1, that the answer is affirmative.

11. One may seek to extend the result of VII, §6, on Jacobi's bound to systems of  $n$  d.p. in  $n$  indeterminates.

The anomaly met in connection with the order of a component of the intersection of two general solutions raises the following problem. Let  $A$  and  $B$  be algebraically irreducible d.p. in  $y$  and  $z$ . Let  $\mathfrak{M}$  be a component of dimension zero in the intersection of the general solutions of  $A$  and  $B$ . It is required to find a bound for the order of  $\mathfrak{M}$  in terms of the orders of  $A$  and  $B$  in  $y$  and  $z$ . It is conceivable, of course, that no bound exists.

12. One may generalize the problem of III, §1, as follows. Let  $\Sigma$  be a non-trivial prime ideal in  $\mathfrak{F}\{u_1, \dots, u_q; y_1, \dots, y_p\}$  with the  $u$  parametric and with

$$(1) \quad A_1, \dots, A_p$$

a characteristic set. Let  $\Sigma_0$  be the prime p.i. for which (1), with the  $A$  considered as polynomials, is a characteristic set. Let  $\Sigma'$  be the system of d.p. obtained from  $\Sigma_0$  when the polynomials in  $\Sigma_0$  are regarded as d.p. What are the dimensions of the components of  $\Sigma'$ ? Does the low power theorem have a generalization for this situation?

#### DIFFERENTIAL POWER SERIES

13. This subject has been mentioned in III, §39. Only one paper has been

<sup>1</sup> Ritt, 31. In connection with §65 of this paper, see the final remarks of §51 of Ritt, 32.

written on it. The entire program awaits development, both for ordinary differential equations and for partial. In the analytic case, the procedure will depend on whether one works in the neighborhood of a point in the space of the independent variables or in the neighborhood of a set of functions constituting a point of a manifold.

BIRATIONAL TRANSFORMATIONS

14. The theory of the resolvent furnishes an instance of the birational equivalence of two irreducible manifolds. The general problem is that of finding conditions for such equivalence. The results of algebraic geometry should be a guide.

In studying birational transformations, one will meet *differential Cremona transformations*. For instance, let

$$Y = y \frac{d}{dx} \left( \frac{z}{y} \right), \quad Z = z \frac{d}{dx} \left( \frac{z}{y} \right).$$

We find

$$y = Y \frac{d}{dx} \left( \frac{Z}{Y} \right), \quad z = Z \frac{d}{dx} \left( \frac{Z}{Y} \right).$$

Is there a theorem on the structure of such transformations of  $y$  and  $z$  similar to M. Noether's theorem on ordinary Cremona transformations?

The analogue of Lüroth's theorem presented in Chapter II may have an extension to fields formed by the adjunction of two indeterminates.

SINGULAR SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

15. For simplicity, we use two independent variables,  $x$  and  $y$ . Let  $F$  be an algebraically irreducible d.p. in  $\mathfrak{F}\{z\}$ , of order  $n$  in  $z$ . Let the components of  $F$  be  $\mathfrak{M}, \mathfrak{M}_1, \dots, \mathfrak{M}_s$ , with  $\mathfrak{M}$  the general solution. Each  $\mathfrak{M}_i$  is the general solution of a d.p.  $F_i$ . Suppose that, for some  $i$ ,  $F_i$  is of order  $n - 1$  in  $z$ . Considering Hamburger's results for ordinary differential equations, one would expect the functions in  $\mathfrak{M}_i$  to be envelopes, with a contact of some natural order, of functions in  $\mathfrak{M}$ . For  $n = 1$ , this question has been studied by the author.<sup>2</sup> For  $n > 1$ , the matter should be more difficult, since there is no theory of characteristics.

DIFFERENCE ALGEBRA

16. This subject has been treated in papers of J. L. Doob, W. C. Strodt, F. Herzog, H. W. Raudenbush, Richard Cohn and the author.<sup>3</sup> The theory is open for cultivation.

<sup>2</sup> Ritt, 41.

<sup>3</sup> See bibliography.

## BIBLIOGRAPHY

1. COHN, R. M. *On the analog for differential equations of the Hilbert-Netto theorem*, Bulletin of the American Mathematical Society, vol. 47 (1941), pp. 268-270.
2. ——— *Manifolds of difference polynomials*, Transactions of the American Mathematical Society, vol. 64 (1948), pp. 133-172.
3. DOOB, J. L., and RITT, J. F. *Systems of algebraic difference equations*, American Journal of Mathematics, vol. 55 (1933), pp. 505-514.
4. DRACH, J. *Essai sur la théorie générale de l'intégration et sur la classification des transcendentes*, Annales de l'Ecole Normale Supérieure, (3), vol. 15 (1898), pp. 245-384.
5. GOURIN, E. *On irreducible systems of algebraic differential equations*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 593-595.
6. HAMBURGER, M. *Ueber die singulären Lösungen der algebraischen Differenzialgleichungen erster Ordnung*, Journal für die reine und angewandte Mathematik, vol. 112 (1893), pp. 205-246. See also *ibid.*, vol. 121 (1899), p. 265, and vol. 122 (1900), p. 322.
7. HERZOG, F. *Systems of mixed difference equations*, Transactions of the American Mathematical Society, vol. 37 (1935), pp. 286-300.
8. KOLCHIN, E. R., and RITT, J. F. *On certain ideals of differential polynomials*, Bulletin of the American Mathematical Society, vol. 45 (1939), pp. 895-898.
9. KOLCHIN, E. R. *On the basis theorem for infinite systems of differential polynomials*, Bulletin of the American Mathematical Society, vol. 45 (1939), pp. 923-926.
10. ——— *On the exponents of differential ideals*, Annals of Mathematics, vol. 42 (1941), pp. 740-777.
11. ——— *On the basis theorem for differential systems*, Transactions of the American Mathematical Society, vol. 52 (1942), pp. 115-127.
12. ——— *Extensions of differential fields*, I, II, Annals of Mathematics, vol. 43 (1942), pp. 724-729; vol. 45 (1945), pp. 358-361.
13. ——— *Extensions of differential fields*, III, Bulletin of the American Mathematical Society, vol. 53 (1947), pp. 397-401.
14. ——— *Algebraic matrix groups and the Picard-Vessiot theory of homogeneous ordinary linear differential equations*, Annals of Mathematics, vol. 49 (1948), pp. 1-42.
15. LAGRANGE, J. L. *Sur les solutions particulières des équations différentielles*, Oeuvres Complètes, vol. 4, pp. 5-108.
16. LAPLACE, P. S. *Mémoire sur les solutions particulières des équations différentielles et sur les inégalités séculaires des planètes*, Oeuvres Complètes, vol. 8, pp. 326-365.
17. LEVI, H. *On the structure of differential polynomials and on their theory of ideals*, Transactions of the American Mathematical Society, vol. 51 (1942), pp. 532-568.
18. ——— *The low power theorem for partial differential polynomials*, Annals of Mathematics, vol. 46 (1945), pp. 113-119.
19. POISSON, S. D. *Sur les solutions particulières des équations différentielles et des équations aux différences*, Journal de l'Ecole Polytechnique, vol. 6, no. 13 (1806), pp. 60-125.
20. RAUDENBUSH, H. W. *Differential fields and ideals of differential forms*, Annals of Mathematics, vol. 34 (1933), pp. 509-517.
21. ——— *Ideal theory and algebraic differential equations*, Transactions of the American Mathematical Society, vol. 36 (1934), pp. 361-368.
22. ——— *Hypertranscendental adjunctions to partial differential fields*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 714-720.
23. ——— *On the analog for differential equations of the Hilbert-Netto theorem*, Bulletin of the American Mathematical Society, vol. 42 (1936), pp. 371-373.

24. RAUDENBUSH, H. W., and RITT, J. F. *Ideal theory and algebraic difference equations*, Transactions of the American Mathematical Society, vol. 46 (1939), pp. 445-453.
25. RITT, J. F. *Manifolds of functions defined by systems of algebraic differential equations*, Transactions of the American Mathematical Society, vol. 32 (1930), pp. 369-398.
26. ———— *Differential equations from the algebraic standpoint*, American Mathematical Society Colloquium Publications, vol. 14, New York, 1932.
27. ———— *Algebraic difference equations*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 303-308.
28. ———— *Systems of algebraic differential equations*, Annals of Mathematics, vol. 36 (1935), pp. 293-302.
29. ———— *Jacobi's problem on the order of a system of differential equations*, Annals of Mathematics, vol. 36 (1935), pp. 303-312.
30. ———— *Indeterminate expressions involving an analytic function and its derivatives*, Monatshefte für Mathematik, vol. 43 (1936), pp. 97-104.
31. ———— *On the singular solutions of algebraic differential equations*, Annals of Mathematics, vol. 37 (1936), pp. 552-617.
32. ———— *On certain points in the theory of algebraic differential equations*, American Journal of Mathematics, vol. 60 (1938), pp. 1-43.
33. ———— *Systems of differential equations, I. Theory of ideals*, American Journal of Mathematics, vol. 60 (1938), pp. 535-548.
34. ———— *On ideals of differential polynomials*, Proceedings of the National Academy of Sciences of the U. S. A., vol. 25 (1939), pp. 90-91.
35. ———— *On the intersections of algebraic differential manifolds*, Proceedings of the National Academy of Sciences of the U. S. A., vol. 25 (1939), pp. 214-215.
36. ———— *On the intersections of irreducible components in the manifold of a differential polynomial*, Proceedings of the National Academy of Sciences of the U. S. A., vol. 26 (1940), pp. 354-356.
37. ———— *On a type of algebraic differential manifold*, Transactions of the American Mathematical Society, vol. 48 (1940), pp. 542-552.
38. ———— *Complete difference ideals*, American Journal of Mathematics, vol. 63 (1941), pp. 681-690.
39. ———— *Bézout's theorem and algebraic differential equations*, Transactions of the American Mathematical Society, vol. 53 (1943), pp. 74-82.
40. ———— *On the manifolds of partial differential polynomials*, Annals of Mathematics, vol. 46 (1945), pp. 102-112.
41. ———— *Analytic theory of singular solutions of partial differential equations of the first order*, Annals of Mathematics, vol. 46 (1945), pp. 120-143.
42. ———— *On the singular solutions of certain differential equations of the second order*, Proceedings of the National Academy of Sciences of the U. S. A., vol. 32 (1946), pp. 255-258.
43. STRODR, W. C. *Systems of algebraic partial difference equations*, Unpublished master's essay, Columbia University, 1937.
44. ———— *Irreducible systems of algebraic differential equations*, Transactions of the American Mathematical Society, vol. 45 (1939), pp. 276-297.



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## COLLOQUIUM PUBLICATIONS

1. H. S. White, *Linear Systems of Curves on Algebraic Surfaces*;  
F. S. Woods, *Forms of Non-Euclidean Space*;  
E. B. Van Vleck, *Selected Topics in the Theory of Divergent Series and  
of Continued Fractions*;  
1905, xii, 187 pp. \$3.00
2. E. H. Moore, *Introduction to a Form of General Analysis*;  
M. Mason, *Selected Topics in the Theory of Boundary Value Problems  
of Differential Equations*;  
E. J. Wilczynski, *Projective Differential Geometry*;  
1910, x, 222 pp. out of print
3. G. A. Bliss, *Fundamental Existence Theorems*, 1913; reprinted, 1934,  
iv, 107 pp. out of print
- 3<sub>2</sub>. E. Kasner, *Differential-Geometric Aspects of Dynamics*, 1913; reprinted,  
1934; reprinted, 1948, iv, 117 pp. 2.50
4. L. E. Dickson, *On Invariants and the Theory of Numbers*;  
W. F. Osgood, *Topics in the Theory of Functions of Several Complex  
Variables*;  
1914, xviii, 230 pp. out of print
- 5<sub>1</sub>. G. C. Evans, *Functionals and their Applications. Selected Topics,  
Including Integral Equations*, 1918, xii, 136 pp. out of print
- 5<sub>2</sub>. O. Veblen, *Analysis Situs*, 1922; second ed., 1931; reprinted, 1946,  
x, 194 pp. 3.35
6. G. C. Evans, *The Logarithmic Potential. Discontinuous Dirichlet  
and Neumann Problems*, 1927, viii, 150 pp. out of print
7. E. T. Bell, *Algebraic Arithmetic*, 1927, iv, 180 pp. out of print
8. L. P. Eisenhart, *Non-Riemannian Geometry*, 1927; reprinted, 1934;  
reprinted, 1949, viii, 184 pp. 2.70
9. G. D. Birkhoff, *Dynamical Systems*, 1927; reprinted, 1948, viii, 295 pp. 4.25
10. A. B. Coble, *Algebraic Geometry and Theta Functions*, 1929; reprinted,  
1948, viii, 282 pp. 4.00
11. D. Jackson, *The Theory of Approximation*, 1930; reprinted, 1946,  
viii, 178 pp. 3.35
12. S. Lefschetz, *Topology*, 1930, x, 410 pp. out of print
13. R. L. Moore, *Foundations of Point Set Theory*, 1932, viii, 486 pp.  
out of print
14. J. F. Ritt, *Differential Equations from the Algebraic Standpoint*, 1932;  
reprinted, 1948, x, 172 pp. 3.00
15. M. H. Stone, *Linear Transformations in Hilbert Space and their Ap-  
plications to Analysis*, 1932; reprinted, 1946; reprinted, 1948, viii,  
622 pp. 8.00

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- |  |      |
|--|------|
| 16. G. A. Bliss, <i>Algebraic Functions</i> , 1933; reprinted, 1948, x, 220 pp.  | 3.75 |
| 17. J. H. M. Wedderburn, <i>Lectures on Matrices</i> , 1934; reprinted, 1944; reprinted, 1946; reprinted, 1949, x, 205 pp.                   | 3.35 |
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| 20. J. L. Walsh, <i>Interpolation and Approximation by Rational Functions in the Complex Domain</i> , 1935, x, 382 pp.                       | 6.00 |
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| 25. G. Birkhoff, <i>Lattice Theory</i> , 1940; enlarged and completely revised ed., 1949, xxvi, 292 pp.                                      | 6.00 |
| 26. N. Levinson, <i>Gap and Density Theorems</i> , 1940, viii, 246 pp.   | 4.00 |
| 27. S. Lefschetz, <i>Algebraic Topology</i> , 1942; reprinted, 1948, vi, 393 pp.   | 6.00 |
| 28. G. T. Whyburn, <i>Analytic Topology</i> , 1942; reprinted, 1948, x, 281 pp.  | 4.75 |
| 29. A. Weil, <i>Foundations of Algebraic Geometry</i> , 1946, xx, 288 pp.  | 5.50 |
| 30. T. Rado, <i>Length and Area</i> , 1948, vi, 572 pp.  | 6.75 |
| 31. E. Hille, <i>Functional Analysis and Semi-Groups</i> , 1948, xii, 528 pp.  | 7.50 |
| 32. R. L. Wilder, <i>Topology of Manifolds</i> , 1949, x, 395 pp.  | 7.00 |
| 33. J. F. Ritt, <i>Differential Algebra</i> , 1950, viii, 184 pp.  | 4.40 |

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