UNIVERSAL LIBRARY

The Dryden Press
Mathematics Publications

GENERAL EDITOR

EDWIN R. SMITH

THE IOWA STATE COLLEGE

PLANE AND SPHERICAL Trigonometry

FRANK A. RICKEY
Associate Professor of Mathematics
Louisiana State University

and

J. P. COLE
Major, U.S.A., Retired
Professor of Mathematics
Louisiana State University

THE DRYDEN PRESS
New York

Designed by Burnshaw
Manufactured in the U.S.A.
Second Printing: March 1943

PREFACE

This book is intended as a text for a forty-eight hour course in college trigonometry. The three fundamental purposes guiding the preparation were to make the book teachable, the material useful, and the essentials so organized as to become a permanent part of the student's knowledge. The subject matter is organized around two fundamentals: the definitions of the trigonometric functions and the addition formulas. The emphasis is placed upon *method* rather than upon rule, upon *remembering* formulas as consequences of the fundamentals rather than upon assigned memory work. The practice exercises are largely based on this idea.

While the authors have made no attempt at novelty, they have not hesitated to introduce new material or treatments which they feel are fundamentally sound and which have stood the test of classroom practice while the work was in its preliminary stages at the Louisiana State University.

Plotting in polar coordinates has been introduced as part of the chapter on the functions of the general angle with gratifying results. With the definitions and values of the functions of the general angle fresh in their minds, students find the idea of polar coordinates simple and interesting. Marked improvement in their subsequent work in analytic geometry has been observed. Reciprocally, the plotting in polar coordinates gives the students a tangible basis for considering angles not found in triangles. Purposeful drill on evaluating the functions is made possible.

The development of the formulas in analytic trigonometry is accomplished with complete generality without use of special cases.

Applications of trigonometry of interest to the armed services have received considerable attention. The mil as a unit of angle measure has been fully treated. Vector representation of velocities, accelerations, and forces is stressed. Exercises with a military turn are numerous. The chapter on spherical trigonometry, though intentionally brief, provides the student with the knowledge necessary to solve the usual applied problems in that subject.

Applications are given early in the course and are made a major item throughout. Verification of identities is regarded as a process of simplification rather than puzzle solving. Only the simplest forms of identities and other abstract applications are introduced in the first half of the book to encourage confidence.

It is accepted as good practice to include in college algebra texts review work on essentials of high school algebra. Proceeding on the basis that a background of geometric material is equally helpful to trigonometry students, the authors have introduced a review of basic topics in plane geometry and a brief treatment of solid mensuration. Many colleges do not find it practicable to offer formal courses in solid geometry, yet are handicapped by the inadequate background in three dimensional concepts on the part of their students. The geometric subject matter occupies the space usually devoted to five place tables in trigonometry texts.

For those who wish to teach the use of common logarithms as part of the trigonometry course, a short treatment of common logarithms is included in the appendix. It is hoped that this develops computation with common logarithms as the simple process that it really is.

The four-place tables are arranged in essentially the same fashion as the usual five-place tables and the instructions for interpolating are designed to carry over to use in five-place tables, should the student need them later.

Answers to the odd numbered exercises are conveniently placed in the text.

The sincere thanks of the authors are offered those who so generously aided in the preparation; to their colleagues for many helpful criticisms and suggestions, particularly to N. E. Rutt, who critically read the first manuscript, and to H. L. Smith for ideas of the general development of the formulas of analytic trigonometry; to E. R. Smith, Iowa State College, for his valuable suggestions and cooperation; and to J. A. Cooley and G. M. Ewing who read the galleys and contributed to the accuracy of presentation.

The cordial, fine spirit of the staff of The Dryden Press has made the task of production a very pleasant one.

F. A. R.
J. P. C.

C O N T E N T S

PART ONE

Plane Trigonometry

Chapter I. USEFUL FACTS ABOUT PLANE FIGURES

1. Introduction 2
2. Definitions and nomenclature 2
3. Similar plane figures 5
4. The Pythagorean relation 7
5. Limitations of the methods of plane geometry 9

Chapter II. THE TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES

6. Definition 11
7. Functions of a variable 11
8. Trigonometric functions 12
9. Versed sine and coversed sine 13
10. Trigonometric functions of 30°, 45°, and 60° 14
11. Trigonometric functions of complementary angles. Trigonometric co-functions 17
12. Fundamental trigonometric relations 18
13. Values of trigonometric ratios 20

Chapter III. THE RIGHT TRIANGLE

14. Solving the right triangle 24
15. Logarithms of trigonometric functions 25
16. Use of logarithms in solving right triangles 26
17. Angles of elevation and depression. Problems involving right triangles 28
18. Vectors, components, and resultants 31

Chapter IV. DIRECTED ANGLES AND UNITS OF MEASUREMENT

19. Angles as measures of rotation 36
20. Units of measurement 37
21. Relations between the units 37
22. The central angle and the intercepted arc 40
23. Areas of the circular sector and circular segment 42
24. Linear speed and angular velocity of rotation 44
25. Relations of θ, $\sin\theta$, and $\tan\theta$ for small values of θ 45

Chapter V. TRIGONOMETRIC FUNCTIONS OF THE GENERAL ANGLE

26. Introduction 49
27. A rectangular coordinate system 49
28. Distance between two points 50
29. Standard position of an angle 51
30. The trigonometric functions of any angle . . 52
31. Some observations regarding functions of the general angle 54
32. Finding the values of the functions of any angle. Reduction to functions of an acute angle . . . 56
33. Value of the functions of the quadrantal angles 58
34. Functions of negative angles . . 60
35. Loci in polar coordinates 61

Chapter VI. VARIATION AND GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

36. The sine and the cosine 64
37. The tangent and cotangent . . . 69
38. Composite functions. Addition of ordinates . . . 70

Chapter VII. FUNCTIONS INVOLVING MORE THAN ONE ANGLE

39. Introduction . . 74
40. The cosine addition formulas . 75
41. Generalization of the relations $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ and $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ 77
42. The sine addition formulas 77
43. The tangent addition formulas 79
44. A note on the periodicity of the functions $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ 80
45. The double angle formulas 81
46. The half angle formulas . . . 82
47. The sum and difference of sines and cosines of two angles 83

Chapter VIII. SOLUTION OF OBLIQUE TRIANGLES

48. Oblique triangles 88
49. The law of sines 88
50. Problems involving the law of sines 90
51. The law of cosines 94
52. Problems involving the law of cosines 96

Chapter VIII. SOLUTION OF OBLIQUE TRIANGLES (Continued)

53. The law of tangents ... 98
54. Applying the law of tangents ... 99
55. Relation of the half angle to the three sides ... 100
56. Areas of triangles ... 102
57. Cyclical permutations in the triangle formulas ... 104

Chapter IX. INVERSE TRIGONOMETRIC FUNCTIONS. TRIGONOMETRIC EQUATIONS

58. Inverse functions ... 107
59. Inverse trigonometric functions ... 107
60. Trigonometric equations ... 111
61. Solutions of the equations $\sin\theta = a$, $\cos\theta = a$, $\tan\theta = a$ 112
62. Solution of other forms of trigonometric equations ... 114

Chapter X. COMPLEX NUMBERS AND DE MOIVRE'S THEOREM

63. Complex numbers ... 118
64. Graphical representation of complex numbers ... 119
65. Addition and subtraction of complex numbers ... 120
66. Multiplication and division of complex numbers 121
67. The product of two or more complex numbers in the polar form De Moivre's theorem ... 122
68. The quotient of two complex numbers in the polar form 124
69. Review exercises in plane trigonometry ... 125

PART TWO

Spherical Trigonometry

Chapter XI. USEFUL FACTS ABOUT SOLID FIGURES

70. Fundamental definitions and relations ... 130
71. Similar figures ... 132
72. Prisms. Parallelepipeds ... 134
73. Cylindrical surfaces. Cylinders ... 136
74. Pyramids ... 138
75. Conical surfaces. Cones ... 139
76. The frustrum ... 141
77. The prismatoid and its mensuration ... 143
78. The theorem of Pappus ... 144
79. The sphere. Figures on the sphere ... 146
80. Spherical areas and volumes ... 150

Chapter XII. SPHERICAL TRIANGLES

81. The spherical triangle 154
82. Special geometric properties 155
83. The law of sines 155
84. The law of cosines 157
85. Spherical right triangles 158
86. Napier's rules 158
87. Cases that are solvable in a right spherical triangle 159
88. The area of a spherical triangle 161
89. Solution of the oblique spherical triangle 162
90. The terrestrial sphere 171
91. The celestial sphere 175

APPENDIX

I. Accuracy of computed results 180
II. Common logarithms 182

TABLES

I. Squares, square roots, cubes, and cube roots 193
II. Logarithms of numbers 194
III. Natural functions 196
IV. Logarithms of sines, cosines, tangents, and cotangents . . . 202

INDEX, 208

Part One

PLANE TRIGONOMETRY

Chapter I

USEFUL FACTS ABOUT PLANE FIGURES

1. INTRODUCTION

Geometry may be defined as that branch of mathematics which investigates the relations and properties of solids, surfaces, lines, and angles. *Trigonometry* is based on the study of the relations between the sides and angles of triangles. It is clear that trigonometry and geometry must be closely related. Indeed, a knowledge of the essential definitions and facts of plane geometry forms a foundation for the study of trigonometry. In turn, many problems whose solutions cannot be found by the methods of ordinary geometry are readily solved with the aid of trigonometry.

This study will begin with a review of a few essentials of plane geometry.

2. DEFINITIONS AND NOMENCLATURE

Every straight line is considered as extending indefinitely in both directions. The word **line** will be used in this work to mean a **straight** line. A **line segment** is a portion of a straight line joining two given points on the line. A **half line,** or **ray,** is a portion of a line extending indefinitely in one direction from a given point on the line.

An **angle** is the figure formed by two rays drawn from a point which is called the **vertex** of the angle. An angle can be measured in **degrees,** in **radians,** or in **mils.** An angle of one degree is

$\frac{1}{360}$ of the angular space about a point. If an angle of one degree is divided into sixty equal parts, each angular part is called one **minute.** Again, one minute contains sixty **seconds.** Measurement of angles in seconds can be accurately made only with extremely precise instruments. For example, at a distance of one mile from the vertex of an angle of one second, the sides are separated by a distance approximately equal to the thickness of a lead pencil. In this book angles will be measured to the nearest minute. The symbols for degrees, minutes, and seconds are °, ′, and ″, respectively.

A **polygon** is a closed plane figure formed by line segments. A **regular polygon** is a polygon having all sides and all angles equal.

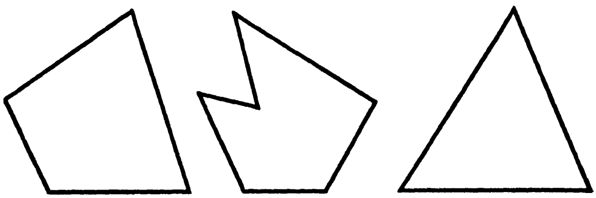


FIGURE 1.

A **triangle** is a polygon of three sides. Classified as to its sides, a triangle is called **scalene, isosceles,** or **equilateral** according as it has no sides equal, two sides equal, or all three sides equal. A **right triangle** has one right angle. Other triangles are called **oblique triangles.**

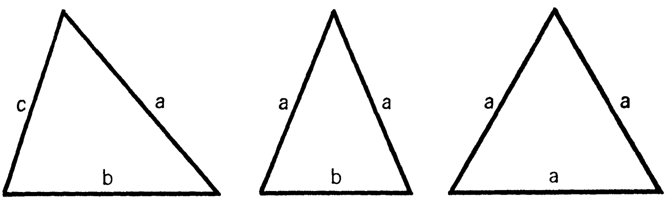


FIGURE 2.

The line segment from a vertex of a triangle perpendicular to the opposite side is called an **altitude** of the triangle. The side to which the altitude is drawn is called the **base** of the triangle. Obviously, any of the three sides of a triangle can be taken as a base. The sides of a right triangle which form the right angle are known as **legs,** or simply the **sides,** of the right triangle. The longest side, opposite the right angle, is the **hypotenuse.** It is clear that either of the legs of a right triangle may be taken as the altitude of the triangle and the other leg as the base. A **median** of any triangle is a line segment joining a vertex of the triangle to the midpoint of the opposite side.

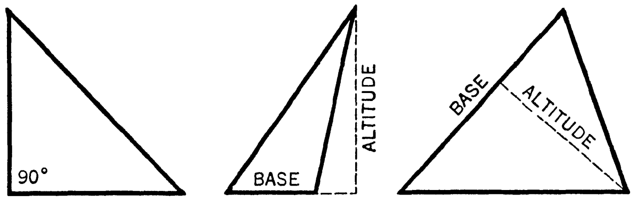


FIGURE 3.

A four-sided polygon is called a **quadrilateral.** A **parallelogram** is a quadrilateral whose opposite sides are parallel. A **rectangle** is a parallelogram all the angles of which are right angles. A **square** is an equilateral rectangle. A **trapezoid** is a quadrilateral having

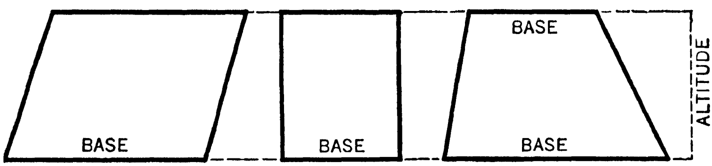


FIGURE 4.

only two sides parallel. The parallel sides of a trapezoid are called its **bases.** The altitude of a trapezoid or of a parallelogram is the perpendicular distance between its bases. Many polygons are named according to the number of sides. A polygon of five, six, seven, eight, or ten sides is called a **pentagon, hexagon, heptagon, octagon,** or **decagon,** respectively.

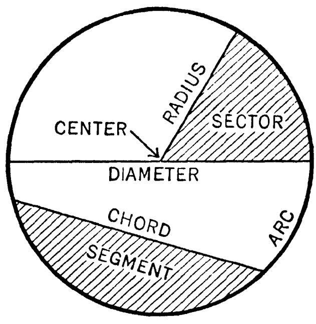


FIGURE 5.

A **circle** is a plane closed curve every point of which is equidistant from a point within called the **center.** The various terms used in connection with the circle are illustrated in Figure 5. The **circumference** of the circle is the perimeter or **length** of the curve. The ratio of the circumference of any circle to its **diameter** is denoted by the symbol π. This number cannot be represented exactly in a decimal form but is approximately 3.1416.

A circle can be constructed so as to be tangent to each of the sides of a regular polygon and such a circle is called the **inscribed circle.** The center of the inscribed circle is also the center of the circle passing through the vertices of the polygon. The latter circle is known as the **circumscribed circle.** The radii of the inscribed and circumscribed circles are respectively called the **apothem** and **radius** of the regular polygon. Their common center is the **center of the polygon.**

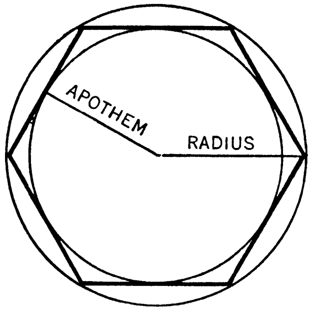


FIGURE 6.

3. SIMILAR PLANE FIGURES

Two polygons are **similar** if they have the same number of sides, and if their corresponding sides are proportional and their corresponding angles are equal.

In the case of two triangles, it is known that if their corresponding angles are equal their corresponding sides *must* be proportional, and *vice versa.* Hence two triangles are known to be similar either if their corresponding angles are equal *or* if their corresponding sides are proportional.

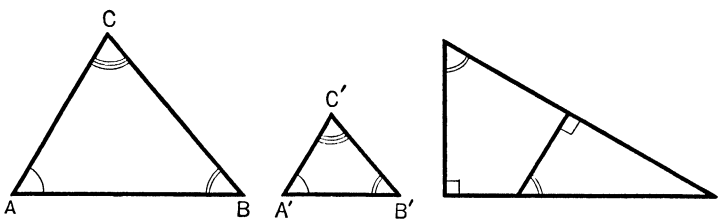


FIGURE 7.

Two important and useful facts about similar figures are: (**a**) any pairs of corresponding lines of similar figures are proportional, and (**b**) the areas of any similar figures are proportional to the *squares* of any corresponding lines. For example, the perimeters of two similar triangles are to each other as two corresponding medians, altitudes, or sides; the areas of two similar polygons are to each other as the squares of their perimeters, etc. (NOTE: the same proportions hold for corresponding lengths and areas of any two circles.)

The process of finding unknown magnitudes by proportion is often facilitated by the use of logarithms. The method is illustrated in example 2 below. (See Appendix for review of logarithms.)

EXAMPLE 1. A triangle interior to a given triangle is formed by joining the consecutive midpoints of the sides of the given triangle. Compare the perimeters of the two triangles; their areas.

SOLUTION: Since a line segment joining the midpoints of two sides of a triangle is known to be parallel to the third side and equal to one-half of it in length, each side of the small triangle is half the corresponding side of the large one. Hence the triangles are similar, having their corresponding sides proportional. Let p, p' and A, A' be the perimeters and areas of the large and small triangles, respectively. Then

$$\frac{p}{p'} = \frac{2}{1} = 2,$$

and

$$\frac{A}{A'} = \frac{2^2}{1^2} = 4.$$

Hence the large triangle has twice the perimeter and four times the area of the small triangle.

EXAMPLE 2. The area of a certain regular polygon whose radius is 3.84 inches is 425.2 square inches. Find the area of a similar polygon whose radius is 7.44 inches.

SOLUTION: Let x = area of the second polygon. Then

$$\begin{aligned} \frac{x}{425.2} &= \frac{(7.44)^2}{(3.84)^2}. \\ \log x - \log 425.2 &= 2(\log 7.44 - \log 3.84). \\ \log x &= 2.6286 + 2(0.8716 - 0.5843) \\ &= 3.2032. \\ x &= 1597 \text{ sq. in.} \end{aligned}$$

EXERCISES

1. A flag pole standing on level ground casts a shadow 52 feet long at the same time that a yard stick held vertically casts a shadow 25 inches long. How tall is the flag pole? ANS. 74.9 ft.
2. The distance from home plate to second base on a standard baseball diamond 90 feet square is 127.3 feet. How far is it from home plate to second base on a junior size diamond 60 feet square?
3. If it costs \$120 to fence in a certain plot of ground, how much will it cost to fence in a plot of similar shape but of twice the area? ANS. \$169.70

4. A street light is 24 feet above the ground. How long is the shadow of a 6 foot man who stands 21 feet from the point directly under the light?
5. A leather belt connects a pulley to a flywheel and is crossed to obtain a reversed direction of rotation. The flywheel is 6 feet in diameter and the pulley is 1 foot in diameter. If the belt touches 13 feet of the circumference of the flywheel and has no sag, how much of the circumference of the pulley does it touch? ANS. 2.17 ft.
6. On a map two towns are $2\frac{3}{8}$ inches apart. The map is drawn to the scale 1″ = 10 miles. How far apart are the towns?
7. Construct any acute angle BAC. Along AC select three random points P, P', P''. From each of these points drop perpendiculars to AB meeting AB at Q, Q', Q'', respectively. By actual measurement, verify that
$$\frac{PQ}{QA} = \frac{P'Q'}{Q'A} = \frac{P''Q''}{Q''A}.$$
8. One side of a polygon whose area is 8,465 square feet is 147 feet long. Find the length of the corresponding side of a similar polygon whose area is 5,029 square feet. ANS. 113.3 ft.
9. The diameter of the earth is about $3\frac{2}{3}$ times that of the moon. Compare the intensity of the "full-moon-light" on the earth with that of the "full-earth-light" on the moon. (Assume that the areas of their apparent circles of light are proportional to the intensities of their light.)
ANS. Earth-light is $13\frac{4}{9}$ times as bright as moon-light.
10. The legs of a certain right triangle are 11 inches and 25 inches, respectively. From a point on the hypotenuse perpendiculars are let fall to the legs of the triangle forming a rectangle whose length is 15 inches. Find the width of the rectangle.
11. A military observer wished to approximate the distance to a foot soldier. He had a ruler and string. He measured off a foot of string and tied this to one end of the ruler. He held this end of the ruler 12 inches from his eye and on a horizontal line with the head of the soldier. He then observed the feet of the soldier, and the distance on the ruler between the two observations was $\frac{3}{4}$ inches. Find the distance from the observer to the soldier assuming the latter to be 6 feet in height. ANS. 96 ft.

4. THE PYTHAGOREAN RELATION

One of the most important and widely applied laws of plane geometry is the celebrated right triangle relation, the first proof of which is attributed to the Greek mathematician and philosopher Pythagoras (582–507 B.C.). It is the familiar truth that *the square upon the hypotenuse of any right triangle is equal to the sum of the squares upon the two legs.* Or, if c, a, and b denote the hypotenuse and two legs respectively, then

$$c^2 = a^2 + b^2.$$

EXAMPLE 1. A ship sails 15 miles eastward and then 12 miles northward. How far is it then from the starting point?

SOLUTION: The unknown distance is the length of the hypotenuse of a right triangle whose legs are 12 and 15. Hence by the Pythagorean law,

$$\begin{aligned} d^2 &= 12^2 + 15^2 \\ &= 369. \\ d &= \sqrt{369} = 19.2 \text{ mi.} \end{aligned}$$

EXAMPLE 2. Find the radius of a regular octagon, each side of which is 10 inches.

SOLUTION: By extending two perpendicular sides of the octagon an isosceles right triangle will be formed (see figure). Let x equal the length of each leg, so that

$$x^2 + x^2 = 10^2,$$

from which

$$x^2 = 50$$

and $x = 5\sqrt{2}$ units.

Now if M denotes the midpoint of AB,

$$OM = x + 5 = 5(1 + \sqrt{2}).$$

Since

$$\begin{aligned}\overline{OB}^2 &= \overline{OM}^2 + \overline{MB}^2 \\ &= 25(1 + \sqrt{2})^2 + 25 \\ &= 25(1 + 2\sqrt{2} + 2 + 1) \\ &= 50(2 + \sqrt{2}),\end{aligned}$$

it follows that

$$OB = 5\sqrt{4 + 2\sqrt{2}} \text{ units.}$$

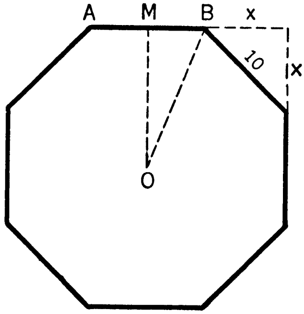


FIGURE 8.

EXERCISES

1. Find the altitude of an equilateral triangle each side of which is 10 inches. ANS. $5\sqrt{3}$ in.
2. Find the altitude of an isosceles triangle whose base is 10 inches and whose perimeter is 36 inches.

If a, b, and c are the sides of a triangle and respectively have the lengths given below, verify that they satisfy the Pythagorean relation and hence the triangles are right triangles:

3. 8, 15, 17.
4. 7, 24, 25.
5. 35, 12, 37.
6. $2n$, $n^2 - 1$, $n^2 + 1$.
7. 1, $\sqrt{3}$, 2.
8. $a^2 - b^2$, $2ab$, $a^2 + b^2$.
9. $2\sqrt{mn}$, $m - n$, $m + n$.
10. $\dfrac{2xy}{x - y}$, $x + y$, $\dfrac{x^2 + y^2}{x - y}$.
11. In a circle of radius 12 inches a chord of length 8 inches is drawn. Find the distance from the center of the circle to the chord. ANS. $8\sqrt{2}$ in.
12. The distance that an observer in an elevated position can see in level country or at sea may be defined as the length of a tangent drawn from the observer to the earth's surface. Prove that this distance d, in miles, for an observer elevated to a height h, in feet, is given approximately for small values of h by the formula

$$d = \sqrt{\frac{3h}{2}}.$$

(*Suggestion:* The radius of the earth is 3960 miles. h feet $= \dfrac{h}{5280}$ miles. Apply the Pythagorean rule and in the resulting expression omit the term $\left(\dfrac{h}{5280}\right)^2$ which would be negligible for small values of h.)

13. Apply the formula of Ex. 12 to determine how far an observer 100 feet above the surface of the ocean can see an object at water level.

ANS. 12.2 mi.

14. What is the greatest distance at which the observer of Ex. 13 could see an object known to be 150 feet above water level?

15. The apothem of a regular hexagon is k feet in length. Find the length of a side.

ANS. $\frac{2k\sqrt{3}}{3}$ ft.

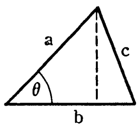
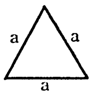
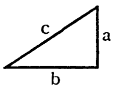
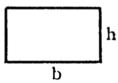
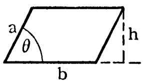
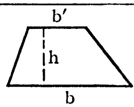
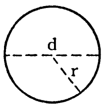
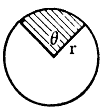
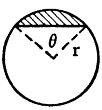
5. LIMITATIONS OF THE METHODS OF PLANE GEOMETRY

In many cases unknown distances can be determined by applications of plane geometry. In a few special cases, angles can also be determined, e.g., if a triangle is proven equilateral, each angle is 60°, etc. However, by the means thus far at hand there is no way to determine unknown sides of many triangles or unknown angles of most triangles, *even when these magnitudes are definitely fixed.* Simple examples of such problems are (**a**) to find the apothem of a regular pentagon whose side is given, (**b**) to find the acute angles of a right triangle whose sides are known, (**c**) to find the altitude of a triangle when a side and an angle adjacent to it are given, (**d**) to find the legs of a right triangle when the hypotenuse and an acute angle are given.

At this point the applications of **Trigonometry** come to hand and presently it will be seen that all the unknown sides and angles of any triangle can be found when sufficient data to fix the triangle are given.

The question of areas of plane figures will be treated in Chapter III where the combined methods of geometry and trigonometry supply a way to handle many problems relating to areas of polygons and circles.

Symbols not illustrated: p = perimeter, s = half-perimeter, $\pi = 3.1416$

Name	*Figure*	*Lengths*	*Area*
Triangle			$A=\frac{1}{2}bh$ $=\sqrt{s(s-a)(s-b)(s-c)}$ $=\frac{1}{2}ab\sin\theta$
Equilateral triangle			$A=\frac{a^2\sqrt{3}}{4}$
Right triangle		$a^2+b^2=c^2$	$A=\frac{1}{2}ab$
Rectangle			$A=bh$ ($=b^2$, if a square)
Parallelogram			$A=bh$ $=ab\sin\theta$
Trapezoid			$A=\frac{1}{2}h(b+b')$
Regular Polygon of n sides			$A=\frac{1}{2}pr$ $=\frac{1}{2}nR^2\sin\theta$
Circle		$C=2\pi r$ $=\pi d$	$A=\pi r^2$ $=\frac{\pi d^2}{4}$
Circular sector		arc $=r\theta$, θ in radians	$A=\frac{1}{2}r^2\theta$, θ in radians $=\frac{\pi r^2\theta}{360}$, θ in degrees
Circular segment			$A=\frac{1}{2}r^2(\theta-\sin\theta)$, θ in radians $=\frac{r^2}{2}\left(\frac{\pi\theta}{180}-\sin\theta\right)$, θ in deg.

Chapter II

THE TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES

6. DEFINITION

Trigonometry is that branch of mathematics that treats of the relations among the sides and angles of a triangle. The usefulness of Trigonometry, though, is not confined to the discussion of triangles, but extends into many branches of mathematics. It is widely used in astronomy, electricity, navigation, physics, map construction, and many phases of engineering work.

7. FUNCTIONS OF A VARIABLE

If two variables are so related that when a value is assigned to one of them there corresponds a value of the other, then the latter is said to be a *function* of the former. Thus, the area of a circle is a function of the radius, the distance a ray of light travels is a function of the time, and $3x^2 + 5x - 7$ is a function of x. The statement that y is a function of x is often written in the symbolic form

$$y = f(x), \text{ or } y = F(x).$$

The equation involving one or more mathematical expressions is not the only means of expressing a functional relation between two variables. Mathematical tables of various kinds are so constructed that when a certain value of one variable is specified, a corresponding value of another variable is determined either exactly or approximately. Thus for every positive number n a table of logarithms gives an approximation of the corresponding value of log n. Another means of expressing a functional relation is the graph. In some cases the graph is simply a visual expression of the relation given by an equation or by a table of values. In other cases, the graph is recorded directly. Certain types of thermometers and pressure gauges, electroscopes, lie detectors, and altimeters graphically record the values of their respective variables as functions of the variable *time*.

8. TRIGONOMETRIC FUNCTIONS

Let α be any acute angle with vertex A and sides AC and AB. From any point on either side, as B, draw BC perpendicular to the other side, forming the right triangle ABC with the right angle at C, and sides a, b, c, opposite the angles with vertices A, B, C, respectively (Fig. 9).* There are six ratios of one side of the right triangle to another side; that is,

$$\frac{a}{c}, \frac{b}{c}, \frac{a}{b}, \text{ and their reciprocals, } \frac{c}{a}, \frac{c}{b}, \frac{b}{a}.$$

When values are assigned to the angle α, corresponding values of these six ratios are determined. Hence these six ratios are called trigonometric functions of the angle α. These functions have been named.

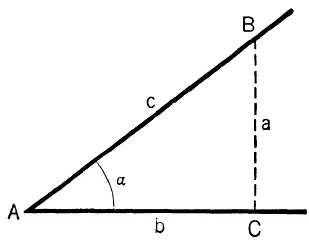


FIGURE 9.

$$\textbf{sine of } \alpha = \frac{a}{c}, \text{ abbreviated as } \textbf{sin } \alpha,$$

$$\textbf{cosine of } \alpha = \frac{b}{c}, \text{ abbreviated as } \textbf{cos } \alpha,$$

* A uniform notation for parts of triangles will be followed in this book: Greek letters for the angles, capital Roman letters for the vertices, and small Roman letters for the corresponding opposite sides. For example, the angles of triangle ABC will be designated by the Greek letters α, β, and γ and the opposite sides by the letters a, b, and c, respectively.

$$\textbf{tangent of } \alpha = \frac{a}{b}, \text{ abbreviated as } \textbf{tan } \alpha,$$

$$\textbf{cotangent of } \alpha = \frac{b}{a}, \text{ abbreviated as } \textbf{cot } \alpha,$$

$$\textbf{secant of } \alpha = \frac{c}{b}, \text{ abbreviated as } \textbf{sec } \alpha,$$

$$\textbf{cosecant of } \alpha = \frac{c}{a}, \text{ abbreviated as } \textbf{csc } \alpha.$$

It is helpful also in remembering the preceding ratios to think of the sides of the right triangle, with respect to angle α, as **opposite side, adjacent side,** and **hypotenuse.** Then, for an acute angle α of a right triangle,

sin α = **opposite side** divided by the **hypotenuse,**
cos α = **adjacent side** divided by the **hypotenuse,**
tan α = **opposite side** divided by the **adjacent side,**
cot α = **adjacent side** divided by the **opposite side,**
sec α = **hypotenuse** divided by the **adjacent side,**
csc α = **hypotenuse** divided by the **opposite side.**

It should also be noted that the values of the trigonometric functions of the angle α are always the same regardless of the position of B on line AB, provided B does not coincide with A. In Fig. 10 it is readily seen that

$$\sin \alpha = \frac{BC}{AB} = \frac{B'C'}{AB'}$$

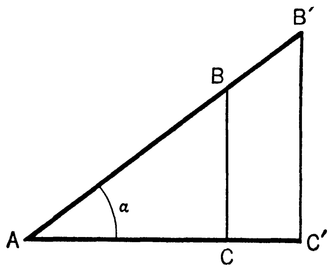


FIGURE 10.

9. VERSED SINE AND COVERSED SINE

In addition to the six trigonometric functions already defined, two others are occasionally used, **versed sine** and **coversed sine,** generally designated by **vers** $\boldsymbol{\alpha}$ and **covers** $\boldsymbol{\alpha}$, and they are defined as follows:

$$\textbf{vers}\,\alpha = 1 - \textbf{cos}\,\alpha;\ \textbf{covers}\,\alpha = 1 - \textbf{sin}\,\alpha.$$

EXERCISES

In each of the following exercises, one of the six trigonometric functions of an acute angle of the right triangle ABC is given. Construct the triangle and find the remaining trigonometric functions of the designated angle and also all the functions of the other acute angle:

1. $\tan \alpha = \frac{3}{4}$.

SOLUTION: Since $\frac{a}{b}$ is given as $\frac{3}{4}$, assume that $a = 3$ and $b = 4$. We know that $a^2 + b^2 = c^2$, therefore $9 + 16 = c^2$. Hence $c = 5$. Now write all the ratios from the definitions of the functions of angles α and β.

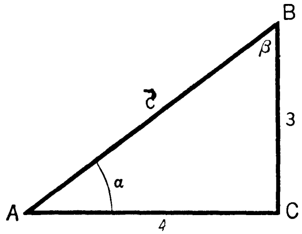


FIGURE 11.

$\sin \alpha = \frac{3}{5}$,	$\sin \beta = \frac{4}{5}$,
$\cos \alpha = \frac{4}{5}$,	$\cos \beta = \frac{3}{5}$,
$\tan \alpha = \frac{3}{4}$,	$\tan \beta = \frac{4}{3}$,
$\cot \alpha = \frac{4}{3}$,	$\cot \beta = \frac{3}{4}$,
$\sec \alpha = \frac{5}{4}$,	$\sec \beta = \frac{5}{3}$,
$\csc \alpha = \frac{5}{3}$,	$\csc \beta = \frac{5}{4}$.

2. $\sin \alpha = \frac{5}{13}$.

3. $\sec \beta = \frac{10}{6}$.

4. $\tan \alpha = \frac{4}{7}$.

5. $\cot \beta = 5$.

6. $\cos \alpha = 0.3$.

7. $\csc \beta = 11$.

8. $\sin \beta = 0.8$.

9. $\cos \beta = \frac{y}{m}$.

10. $\sec \alpha = \frac{6}{\sqrt{7}}$.

11. How many functions of angle α have a in their definitions? b? c?

12. What function is the reciprocal of $\sin \alpha$?

13. Can $\cos \alpha$ be greater than 1? Why?

14. When α is greater than 45°, what is the relation between a and b?

15. Show that for any value of α between 0° and 90°, the values of $\sin \alpha$ and $\cos \alpha$ are proper fractions, the values of $\tan \alpha$ and $\cot \alpha$ may be either proper or improper fractions, and that the values of $\sec \alpha$ and $\csc \alpha$ are improper fractions.

If α is an acute angle, state which is the greater and give reasons:

16. $\sin \alpha$ or $\tan \alpha$.

17. $\cos \alpha$ or $\cot \alpha$.

18. $\sec \alpha$ or $\tan \alpha$.

19. $\csc \alpha$ or $\cot \alpha$.

20. Find the values of the six functions of α if

$$a = 2p,\ b = p^2 - 1, \text{ and } c = p^2 + 1.$$

21. Using the fact that $a^2 + b^2 = c^2$, find the values of the six functions of α when $a = \sqrt{r^2 + s^2}$, $b = \sqrt{2rs}$.

10. TRIGONOMETRIC FUNCTIONS OF 30°, 45°, AND 60°

It has been pointed out that, given the sides of a right triangle, the values of the six trigonometric functions of the acute angles of the triangle are immediately known from the definitions. The question next arises, given the acute angle itself, can its corresponding trigonometric ratios be calculated? The answer is yes, but the method of calculating them is in general beyond the scope of this book. However, there are some special triangles, such as the equilateral triangle and the isosceles right triangle, about which information is easily available both as to angles and to sides. The former enables one to find the trigonometric functions of 30° and of 60°, the latter, those of 45°. The fact that the values of the ratios depend upon the shape and not the size of the right triangle suggests the use of dimensions which prove most convenient.

Let ABD, Fig. 12, be an equilateral triangle with sides equal to 2. Construct AC perpendicular to BD. Then $BC = CD = 1$. Since AC bisects the angle at A, and since each angle of the equilateral triangle is 60°, it is obvious that angle $BAC = 30°$. Then in triangle ABC, the Pythagorean relation gives the equation

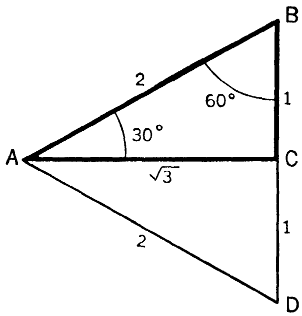


FIGURE 12.

$$\overline{AC}^2 + 1^2 = 2^2,$$

so that $AC = \sqrt{3}$. Hence, from the definitions of the trigonometric functions,

$$\sin 30° = \frac{1}{2}, \qquad \cot 30° = \sqrt{3},$$

$$\cos 30° = \frac{\sqrt{3}}{2}, \qquad \sec 30° = \frac{2}{\sqrt{3}},$$

$$\tan 30° = \frac{1}{\sqrt{3}}, \qquad \csc 30° = 2.$$

Again, from the figure and the definitions,

$$\sin 60° = \frac{\sqrt{3}}{2}, \qquad \cot 60° = \frac{1}{\sqrt{3}},$$

$$\cos 60° = \frac{1}{2}, \qquad \sec 60° = 2,$$

$$\tan 60° = \sqrt{3}, \qquad \csc 60° = \frac{2}{\sqrt{3}}.$$

Let ABC, Fig. 13, be an isosceles right triangle with the equal sides having the value 1. Since $a^2 + b^2 = c^2$, then $1^2 + 1^2 = c^2$. Hence $c = \sqrt{2}$.

From the figure,

$$\sin 45° = \frac{1}{\sqrt{2}},$$
$$\cos 45° = \frac{1}{\sqrt{2}}$$
$$\tan 45° = 1,$$
$$\cot 45° = 1,$$
$$\sec 45° = \sqrt{2},$$
$$\csc 45° = \sqrt{2}.$$

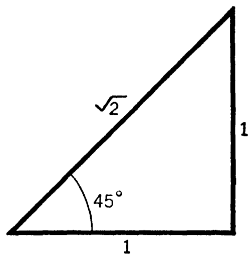


FIGURE 13.

EXAMPLE 1. Given right triangle ABC with $\alpha = 30°$ and $c = 40$ feet. Find sides a and b.

SOLUTION: From the definition of $\sin \alpha$,

$$\frac{a}{c} = \sin \alpha.$$

But $\sin \alpha = \sin 30° = \frac{1}{2}$, and $c = 40$. Hence, by substitution,

$$\frac{a}{40} = \frac{1}{2},$$

and consequently $a = 20$ ft. Likewise, by substituting in

$$\frac{b}{c} = \cos \alpha$$

the values $\cos \alpha = \cos 30° = \frac{\sqrt{3}}{2}$, there results the equation

$$\frac{b}{40} = \frac{\sqrt{3}}{2},$$

which has the solution $\quad b = 20\sqrt{3} = 34.64$ ft.

EXAMPLE 2. Verify that

$$(\sin 60°)^2 + (\tan 60°)^2 + (\cos 60°)^2 = 4.$$

SOLUTION: $\sin 60° = \frac{\sqrt{3}}{2}$, $\tan 60° = \sqrt{3}$, and $\cos 60° = \frac{1}{2}$.

Hence,
$$(\sin 60°)^2 + (\tan 60°)^2 + (\cos 60°)^2 = \tfrac{3}{4} + 3 + \tfrac{1}{4}$$
$$= 4.$$

EXERCISES

Verify the following statements:

1. $\sin 30° - \cos 60° = 0$.
2. $2 \sin 45° \cos 45° = 1$.
3. $\sin 30° \cos 30° \tan 30° + \sin 30° \cos 30° \cot 30° = 1$.
4. $(\sin 45°)^2 + (\cos 45°)^2 = 1$.
5. $\sin 30°(1 + \cos 60°) = (\cos 30°)^2$.
6. $\sin 60° \cos 30° + \cos 60° \sin 30° = 1$.
7. $(\sec 30°)^2 + (\cot 60°)^2 = \frac{5}{3}$.
8. $\tan 45° + \tan 30° \cot 60° = \dfrac{\sec 30°}{\cos 30°}$.
9. Given triangle ABC with $\gamma = 90°$ and $c = 50$ yards, find a and b.
   ANS. $a = 25$ yds., $b = 25\sqrt{3}$ yds.
10. Given triangle ABC with $\gamma = 90°$, and $b = 100$ yards, find a and c and check the results with the Pythagorean relation.
11. The shadow of a vertical radio tower is 120 ft. long when the sun's rays are inclined at an angle of 60° to the horizontal. Find the height of the tower.
    ANS. $120\sqrt{3}$ ft.
12. A certain earthen dam has a vertical cross section in the form of an isosceles trapezoid. The upper base of the trapezoid is 10 yards, its altitude is 10 yards, and the sides are inclined at angles of 30° with the lower base. Find the number of cubic yards in a uniform portion of the dam 100 yards long.

11. TRIGONOMETRIC FUNCTIONS OF COMPLEMENTARY ANGLES. TRIGONOMETRIC CO-FUNCTIONS

The reader will note from the results in the preceding article that $\sin 30° = \cos 60°$, $\tan 30° = \cot 60°$, $\sec 30° = \csc 60°$, $\cos 30° = \sin 60°$, $\cot 30° = \tan 60°$, and that $\csc 30° = \sec 60°$. These relations are plainly true because the side of the triangle which is **adjacent** to the 30° angle is the one which is **opposite** to the 60° angle.

The six equalities just stated depend upon the fact that the angles were acute angles of the same triangle and not upon the number of degrees in each. Any pair of angles whose sum is 90° (i.e., complementary angles) can be made the acute angles of a right triangle and the foregoing six equalities will hold true for their trigonometric functions. Let α and β be two acute angles such that $\alpha + \beta = 90°$. Then there corresponds a right triangle ABC, with the right angle at C, such that α is the angle at A and β the angle at B. Then

$$\sin \alpha = \frac{a}{c} = \cos \beta.$$

But $\beta = 90° - \alpha$, so that

$$\sin \alpha = \cos (90° - \alpha).$$

Similarly,

$$\cos \alpha = \sin (90° - \alpha),$$
$$\tan \alpha = \cot (90° - \alpha),$$
$$\cot \alpha = \tan (90° - \alpha),$$
$$\sec \alpha = \csc (90° - \alpha),$$
$$\csc \alpha = \sec (90° - \alpha).$$

Arrange the six trigonometric functions in pairs as follows: $\sin \alpha$, $\cos \alpha$; $\tan \alpha$, $\cot \alpha$; $\sec \alpha$, $\csc \alpha$. Either function in a pair is called the **co-function** of the other member of the pair. That is, the sine of α is the co-function of the cosine of α; the cosine of α is the co-function of the sine of α; and similarly for the other pairs. The formulas given may be summarized by the statement that *any trigonometric function of an acute angle is equal to the co-function of its complementary angle.*

EXERCISES

1. Express each of the following as a trigonometric function of the complementary angle:

(a) $\sin 52°$. ANS. $\cos 38°$.
(b) $\tan 11°$.
(c) $\cot 31° 42'$. ANS. $\tan 58° 18'$.
(d) $\sec 65° 22'$.
(e) $\cos 89° 5'$. ANS. $\sin 55'$.
(f) $\csc 57° 13'$.
(g) $\cos 2\varphi$. ANS. $\sin (90° - 2\varphi)$.
(h) $\tan (20° - \alpha)$.
(i) $\sin (50° + 2\varphi)$. ANS. $\cos (40° - 2\varphi)$.
(j) $\cot (90° - \theta)$.

2. If it is known that the acute angles θ and φ are such that $\sin \theta = \cos \varphi$, what else is true of θ and φ?

Find an acute angle for which

3. $\sec (\alpha - 15°) = \csc (\alpha + 25°)$. ANS. $\alpha = 40°$.

4. $\tan 2\theta = \cot \theta$.

5. $\cos (30° - \varphi) = \sin (40° + 3\varphi)$. ANS. $\varphi = 10°$.

6. $\cot \left(\frac{3\alpha}{2}\right) = \tan 3\alpha$.

Find the acute angles θ and φ given that

7. $\sin 2\theta = \cos 3\varphi$, and $\theta - \varphi = 20°$. ANS. $\theta = 30°$, $\varphi = 10°$.

8. $\tan (\theta + 35°) = \cot (\varphi - 55°)$, and $2\theta - \varphi = 10°$.

12. FUNDAMENTAL TRIGONOMETRIC RELATIONS

From the fact that $\sin \alpha = \dfrac{a}{c}$ and $\csc \alpha = \dfrac{c}{a}$,

it follows that

$$\sin \alpha = \frac{a}{c} = \frac{1}{\frac{c}{a}} = \frac{1}{\csc \alpha}.$$

In brief, then,

(1) $$\sin \alpha = \frac{1}{\csc \alpha}.$$

It is left to the student to verify in a similar manner that

(2) $$\cos \alpha = \frac{1}{\sec \alpha},$$

(3) $$\tan \alpha = \frac{1}{\cot \alpha},$$

(4) $$\tan \alpha = \frac{\sin \alpha}{\cos \alpha},$$

(5) $$\cot \alpha = \frac{\cos \alpha}{\sin \alpha}.$$

Again, since $a^2 + b^2 = c^2$, dividing both members by c^2 gives the result

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1,$$

which means that

(6) $$\sin^2 \alpha + \cos^2 \alpha = 1.$$

NOTE. $\sin^2 \alpha$ is an abbreviation of the expression $(\sin \alpha)^2$. It must not be confused with the expression $\sin \alpha^2$ For instance, if $\alpha = 8°$, then $\sin^2 \alpha = (\sin 8°)^2$, which is entirely different from $\sin (8^2)°$.

The student should verify that division of the sides of the equation $a^2 + b^2 = c^2$ by b^2 and by c^2 results in the respective relations

(7) $$1 + \tan^2 \alpha = \sec^2 \alpha,$$
(8) $$1 + \cot^2 \alpha = \csc^2 \alpha.$$

These eight relations among the various trigonometric functions are fundamental in further study of trigonometry and should be committed to memory. It should be noted that they hold true for *any* acute angle, that is, they are **identities.**

EXAMPLE. Show that for any acute angle α

$$\cot \alpha \sin \alpha \sec \alpha = 1.$$

PROOF. From the fundamental relations, $\cot \alpha = \frac{\cos \alpha}{\sin \alpha}$ and $\sec \alpha = \frac{1}{\cos \alpha}$.

Hence $$\cot \alpha \sin \alpha \sec \alpha = \left(\frac{\cos \alpha}{\sin \alpha}\right)(\sin \alpha)\left(\frac{1}{\cos \alpha}\right) = 1,$$

which was to be shown.

EXERCISES

1. Verify that each of the relations (1) to (8) are true numerically for $\alpha = 30°$; for $\alpha = 45°$; for $\alpha = 60°$.

After the manner of the example show that the following relations are true for any acute angle:

2. $\dfrac{1 + \sin \alpha}{\cos \alpha} = \sec \alpha + \tan \alpha.$
3. $\dfrac{\sin \theta}{\csc \theta} + \dfrac{\cos \theta}{\sec \theta} = 1.$
4. $(\sin \varphi + \cos \varphi)^2 + (\sin \varphi - \cos \varphi)^2 = 2.$
5. $\tan \beta + \cot \beta = \sec \beta \csc \beta.$
6. $\cos \alpha + \tan \alpha \sin \alpha = \sec \alpha.$
7. Show that if $x = a \sin \theta$, the equation $\dfrac{x}{\sqrt{a^2 - x^2}} = y$ reduces to the form $$\tan \theta = y.$$
8. Show that if $x = a \tan \theta$, the equation $\dfrac{a}{\sqrt{x^2 + a^2}} = s$ reduces to the form $$\cos \theta = s.$$
9. Show that if $z = a \sec \theta$, the equation $\dfrac{a^2}{z\sqrt{z^2 - a^2}} = u$ reduces to the form $$\frac{\cos^2 \theta}{\sin \theta} = u.$$

13. VALUES OF TRIGONOMETRIC RATIOS

In Article 10 the values of the trigonometric ratios for three special angles were computed. Many of the applications of trigonometry depend upon knowing the values of the ratios for *any* angle. Table III in the back of this book gives values, correct to four places, of the sine, cosine, tangent, and cotangent of angles from 0° to 90° by 10′ intervals.

The names of the functions given at the *top* of each section of the table refer to the angle in the *left* column. Thus, sin 24° 40′ = 0.4173, tan 7° 20′ = 0.1287, cot 44° = 1.0355. The reader will at once notice that angles up to 45° can thus be located.

The names of the functions as given at the *bottom* of each section refer to the angles on the *right*. Thus,

$$\cos 67°20' = 0.3854, \quad \sin 55°30' = 0.8241.$$

Note the application of Art. 11, i.e., $\sin \alpha = \cos (90° - \alpha)$, etc.

Suppose now that the function of an acute angle not given in the tables is needed. The process of approximating the value of the desired function is called **interpolation** and is best explained by examples.

EXAMPLE 1. Find sin 38° 25′.

SOLUTION: The value of sin 38° 25′ obviously lies between sin 38° 20′ and sin 38° 30′. It seems reasonable to assume that, since the given angle is halfway between the angles found in the table, its sine should be approximately halfway between sin 38° 20′ and sin 38° 30′. Hence to obtain the desired approximation, add to the value of sin 38° 20′ the product 0.5 times the difference of the two sines. Thus,

$$\begin{aligned}\sin 38^\circ 25' &= \sin 38^\circ 20' + 0.5(\sin 38^\circ 30' - \sin 38^\circ 20') \\ &\qquad \text{(Subtract mentally)} \\ &= 0.6202 + 0.5(0.0023) \\ &= 0.6202 + 0.0012 \text{ (to nearest ten thousandth)} \\ &= 0.6214. \quad \text{Ans.}\end{aligned}$$

NOTE: That the differences between the angles are approximately proportional to the differences between the values of their trigonometric functions can be proved to be exact enough to obtain results to the nearest minute.

EXAMPLE 2. Find tan 73° 7′.

SOLUTION: As above,

$$\begin{aligned}\tan 73^\circ 7' &= \tan 73^\circ 0' + 0.7(\tan 73^\circ 10' - \tan 73^\circ 0') \\ &= 3.271 + 0.7(.034) \\ &= 3.271 + 0.024 \\ &= 3.295. \quad \text{Ans.}\end{aligned}$$

EXAMPLE 3. Find cos 16° 43′.

SOLUTION:

$$\begin{aligned}\cos 16^\circ 43' &= \cos 16^\circ 40' + 0.3\,(\cos 16^\circ 50' - \cos 16^\circ 40') \\ &= 0.9580 + 0.3(0.9572 - 0.9580) \\ &= 0.9580 + 0.3(-\,0.0008) \\ &= 0.9580 - 0.0002 \\ &= 0.9578. \quad \text{Ans.}\end{aligned}$$

If the value of one of the trigonometric functions of an angle is known and the size of the angle to the nearest minute is to be determined, apply an interpolation method to determine an angle whose given function value does not appear in the table. The following examples make the method clear.

EXAMPLE 4. If $\tan \alpha = 0.6148$, find α.

SOLUTION: In Table III, 0.6148 does not appear in the tangent column for any angle. However, it is found to lie between tan 31° 30′ = 0.6128 and tan 31° 40′ = 0.6168; in fact it is readily seen to be exactly *midway* between them. Hence

$$\alpha = 31^\circ 35'.$$

EXAMPLE 5. If $\sin \alpha = 0.5040$, find α.

SOLUTION: From the table note that $\sin 30° 10' = 0.5025$ and that $\sin 30° 20' = 0.5050$. Hence $\alpha = 30° 10' +$ (some fraction of 10′). By inspection, 0.5040 is $\frac{15}{25}$ths or 0.6 of "the way" between 0.5025 and 0.5050.

Hence $\alpha = 30° 10' + 0.6(10') = 30° 16'$. Ans.

EXAMPLE 6. If $\sin \alpha = 0.4419$, find α.

SOLUTION: By inspection,
$$\begin{aligned}\alpha &= 26° 10' + \tfrac{9}{26}(10') \\ &= 26° 10' + 3' \text{ (approximately)} \\ &= 26° 13'. \text{ Ans.}\end{aligned}$$

EXAMPLE 7. If $\cos \beta = 0.4059$, find β.

SOLUTION: By inspection,
$$\begin{aligned}\beta &= 66° 0' + \tfrac{8}{26}(10') \\ &= 66° 3'. \text{ Ans.}\end{aligned}$$

EXERCISES

Verify that the following trigonometric ratios are correct:

1. $\sin 52° 20' = 0.7916$.
2. $\cot 1° 40' = 34.368$.
3. $\tan 68° = 2.4751$.
4. $\cos 34° 30' = 0.8241$.
5. $\sin 44° 45' = 0.7040$.
6. $\cos 32° 14' = 0.8459$.
7. $\tan 67° 18' = 2.3906$.
8. $\sin 55° 57' = 0.8285$.
9. $\cot 33° 18' = 1.5223$.
10. $\tan 65° 44' = 2.2182$.
11. $\cos 45° 51' = 0.6965$.
12. $\sin 71° 45' = 0.9497$.
13. $\tan 11° 16' = 0.1992$.
14. $\cos 27° 8' = 0.8900$.
15. $\cot 84° 52' = 0.08983$.
16. $\sin 19° 19' = 0.3308$.

Find correct to the nearest minute the angles whose trigonometric ratios are given below:

17. $\sin \alpha = 0.6967$. ANS. $\alpha = 44° 10'$.
18. $\cos \alpha = 0.3987$. ANS. $\alpha = 66° 30'$.
19. $\cot \alpha = 21.4704$. ANS. $\alpha = 2° 40'$.
20. $\tan \beta = 2.9600$. ANS. $\beta = 71° 20'$.
21. $\sin \theta = 0.6873$. ANS. $\theta = 43° 25'$.
22. $\cot \delta = 2.0130$. ANS. $\delta = 26° 25'$.
23. $\tan \beta = 0.6060$. ANS. $\beta = 31° 13'$.
24. $\sin \varphi = 0.8854$. ANS. $\varphi = 62° 18'$.
25. $\cos \theta = 0.6520$. ANS. $\theta = 49° 19'$.
26. $\sin \beta = 0.9440$. ANS. $\beta = 70° 44'$.
27. $\cot \alpha = 1.4532$. ANS. $\alpha = 34° 32'$.
28. $\tan \varphi = 0.8905$. ANS. $\varphi = 41° 41'$.
29. $\sin \beta = 0.3001$. ANS. $\beta = 17° 28'$.
30. $\cos \alpha = 0.6698$. ANS. $\alpha = 47° 57'$.

SUMMARY OF FORMULAS, CHAPTER II

(1) $\sin\alpha = \dfrac{1}{\csc\alpha}$.

(2) $\cos\alpha = \dfrac{1}{\sec\alpha}$.

(3) $\tan\alpha = \dfrac{1}{\cot\alpha}$.

(4) $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$.

(5) $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$.

(6) $\sin^2\alpha + \cos^2\alpha = 1$.

(7) $1 + \tan^2\alpha = \sec^2\alpha$.

(8) $1 + \cot^2\alpha = \csc^2\alpha$.

Chapter III

THE RIGHT TRIANGLE

14. SOLVING THE RIGHT TRIANGLE

The sides and angles of a triangle are called the **parts** of a triangle. A right triangle is *determined* if two parts other than the right angle are given, provided that one of the given parts is a side. In plane geometry it is proved that two right triangles are congruent if two sides of one are equal to two sides of the other, or if a side and an acute angle of one are equal to a side and an angle of the other. It is also shown that triangles can be constructed with ruler and compasses when such parts are given.

By means of trigonometry the values of the unknown parts can be computed when known parts sufficient to fix the triangle are given. The process of computing these unknown parts is called **solving** the triangle. The six trigonometric ratios make it possible to form an equation expressing a relation between any two of the sides of a right triangle and either of the acute angles. Hence, if one side of a right triangle and either one of the acute angles or one of the other sides are known, the remaining parts can be calculated. If in the right triangle ABC, β is known, then angle α is found from the relation $\alpha = 90° - \beta°$.

EXAMPLE. Solve right triangle ABC, given $a = 6$ and $\beta = 62° 30'$.

SOLUTION: Write $\cos \beta = \frac{a}{c}$,

or $\cos 62° 30' = \frac{6}{c}$,

and $c = \frac{6}{\cos 62° 30'} = \frac{6}{.4617} = 13.00.$

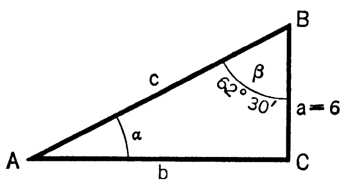


FIGURE 14.

To find b, use the Pythagorean relation $b^2 = c^2 - a^2$, from which

$$b^2 = (c + a)(c - a).$$

Hence

$$b = \sqrt{(13 + 6)(13 - 6)} = \sqrt{(19)(7)} = 11.53.$$

Since $\alpha + \beta = 90°$,

$$\alpha = 90° - 62° 30' = 27° 30'.$$

This completes the *solution* of the triangle, since all unknown parts have been evaluated.

EXERCISES *

In each of the following right triangles, two parts are given. Compute the remaining parts:

1. $a = 12$, $b = 21$. Ans. $\alpha = 29° 45'$, $\beta = 60° 15'$, $c = 24.2$.
2. $b = 11$, $c = 36$.
3. $a = 19$, $c = 40$. Ans. $\alpha = 28° 22'$, $\beta = 61° 38'$, $b = 35.2$.
4. $a = 7$, $\alpha = 44° 12'$.
5. $c = 18$, $\beta = 63° 24'$. Ans. $a = 8.07$, $b = 16.1$, $\alpha = 26° 36'$.
6. $b = 14$, $\alpha = 12'$.
7. Find independently the acute angles of the right triangle whose sides are 5, 12, 13. Check results by showing that their sum is 90°.
8. A person rowing across a river $\frac{1}{2}$ mile wide is carried downstream 300 yds. Find the angle between the course of the boat and the line directly across the river.
9. A ship travels N 35° W at the rate of 14 miles per hour. What is its rate in a westerly direction? Ans. 8.03 mi./hr.
10. Find the length of a side of a regular polygon of 6 sides circumscribed about a circle whose radius is 10 inches.

15. LOGARITHMS OF THE TRIGONOMETRIC FUNCTIONS

The use of logarithms in many of the computations involved in solving triangles is highly desirable. In the Appendix may be found a table of logarithms of the trigonometric functions. Thus, to find the logarithms of the sine of 57° 10′ it is not necessary to look up first sin 57° 10′ and then find the logarithms of this number in the logarithm tables. Instead, simply look up log sin 57° 10′ directly in the table. However, it is well to keep in mind the approximate value of the trigonometric function in order to make certain that the proper characteristic is assigned to the logarithm of the function. The sines and cosines of all angles between 0° and 90° are numbers which are less than one. Hence log sin α and log cos α have negative characteristics for any such angle α. Similarly, log tan α has a negative characteristic for angle α less than 45°, and

* It is suggested that if a slide rule is available, it be used in the computations of these exercises. A 10-inch slide rule usually gives results correct to three significant figures, whereas a 4-place table usually gives results to four significant figures.

log cot α has a negative characteristic for values of α between 45° and 90°. Thus in using the table of logarithms of the trigonometric functions it is necessary to affix minus 10 to the logarithm as found in the table whenever the value of the trigonometric function is less than one.

Interpolations are performed with the logarithms of trigonometric functions in exactly the same manner as explained for the use of the tables of the natural functions in Art. 13.

16. USE OF LOGARITHMS IN SOLVING RIGHT TRIANGLES

The equations determining the unknown parts of a triangle are derived as explained in Art. 14. Logarithms are applied in the solution of the example below in solving a right triangle whose known and unknown parts correspond to those of the example in Art. 14 so that the methods can be compared by the reader.

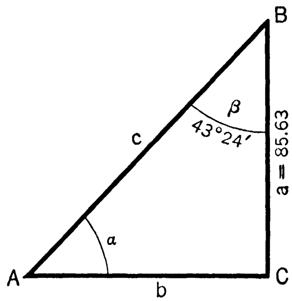


FIGURE 15.

EXAMPLE 1. Solve the right triangle ABC, given $a = 85.63$ and $\beta = 43° 24'$.

SOLUTION: As before, write

$$\cos \beta = \frac{a}{c},$$

or

$$\log \cos \beta = \log a - \log c,$$
$$\log c = \log a - \log \cos \beta$$
$$= 1.9327 - (9.8613 - 10)$$
$$= 2.0714.$$

Hence,

$$c = 117.8.$$

Again, as in the example of Art. 14,

$$b = \sqrt{(c + a)(c - a)},$$

so that

$$\log b = \tfrac{1}{2}[\log (c + a) + \log (c - a)]$$
$$= \tfrac{1}{2}[\log 203.43 + \log 32.17]$$
$$= \tfrac{1}{2}[2.3083 + 1.5074]$$
$$= 1.9078.$$

Hence,

$$b = 80.88.$$

Finally,

$$\alpha = 90° - 43° 24' = 46° 36'.$$

EXAMPLE 2. Solve the right triangle ABC, given $a = 17.2$ and $b = 24.9$.

SOLUTION:
$$\tan \alpha = \frac{a}{b},$$
and
$$\begin{aligned}\log \tan \alpha &= \log a - \log b \\ &= 1.2355 - 1.3962 \\ &= 9.8393 - 10.\end{aligned}$$
From Table II, $\alpha = 34° 38'$.
Hence, $\beta = 90° - 34° 38' = 55° 22'$.

While c is determined from the relation $c^2 = a^2 + b^2$, the right-hand member of this equality cannot be factored and thus it is best to compute c from this equation without logarithms. (Why?) However, c can be readily computed with the aid of logarithms by means of the equation
$$\sin \alpha = \frac{a}{c},$$
from which
$$\log \sin \alpha = \log a - \log c.$$
Hence,
$$\begin{aligned}\log c &= \log a - \log \sin \alpha \\ &= 1.2355 - (9.7546 - 10) \\ &= 1.4809. \\ c &= 30.26.\end{aligned}$$

EXERCISES

Solve the following right triangles:

1. Given $c = 125.7$, $\alpha = 75° 12'$; show that $a = 121.5$, $b = 32.11$.
2. Given $a = 572$, $c = 646$.
3. Given $a = 35.8$, $\alpha = 61° 13'$; show that $c = 40.85$, $b = 19.67$.
4. Given $a = 75$, $b = 82$.
5. Given $a = 2\ 1$, $b = 2$; show that $\alpha = 46° 23'$, $c = 2.9$.
6. Given $b = 68.3$; $\alpha = 24° 30'$.
7. $c = 325$, $\alpha = 27° 34'$; show that $a = 150.4$, $b = 288.1$.
8. $c = .16$, $b = .09$.
9. $\alpha = \theta°$, $c = g$. ANS. $a = g \sin \theta$, $b = g \cos \theta$, $\beta = (90 - \theta)°$.
10. $\alpha = \varphi°$, $c = s$.
11. $\beta = \psi°$, $b = x$. ANS. $a = x \cot \psi$, $c = \dfrac{x}{\sin \psi} = x \csc \psi$, $\alpha = 90° - \psi$.
12. $\alpha = \theta°$, $a = h$.
13. One leg of a right triangle is three times the other. Find the acute angles. ANS. 18° 26′, 71° 34′.
14. Two tangents are drawn from an external point A to a circle. The angle formed by the tangents is 40° and the length of each of the tangents is 50 feet. Find the radius of the circle.
15. Find the radius of the circle inscribed in an equilateral triangle whose side is 8 feet. ANS. 2.31 ft.
16. To measure the distance AB across a river, a line AC is laid off perpendicular to AB. Find the width of the river if angle ACB is 48° and AC is 100 feet.

17. ANGLES OF ELEVATION AND DEPRESSION. PROBLEMS INVOLVING RIGHT TRIANGLES

If an observer is looking at an object *above* his level the angle between the line from the observer's eye to the object and a horizontal line in the same vertical plane is called the **angle of elevation** of the object. If the object is *below* the level of the observer, the corresponding angle is called the **angle of depression.** Thus, in the figure, if the observer is at A, the angle at A is called the angle of elevation of B from the point A. If, on the other hand, the observer is at B looking downward toward A, the angle at B is called the angle of depression of A from the point B.

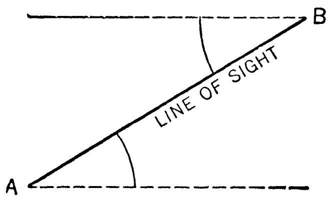


FIGURE 16.

It is obvious that for a given pair of points the angle of elevation of the first from the second is equal to the angle of depression of the second from the first.

In solving problems involving right triangles it is well to draw a figure as nearly to scale as possible and mark the known parts of the triangles involved. Then proceed as in Articles 14 and 16 to determine the required parts. Often more than one triangle must be solved or partly solved before the desired magnitude can be determined, as illustrated in Example 2, below:

EXAMPLE 1. A ship starting at A sailed 12.7 miles to the north and then 8.4 miles east to B. What is the direction of B from A?

SOLUTION: The problem is to find $\angle \alpha$ of Fig. 17. Write

$$\tan \alpha = \frac{8.4}{12.7}.$$

$$\log \tan \alpha = \log 8.4 - \log 12.7.$$

The completion of the problem is left to the student. ANS. North, 33° 29′ east.

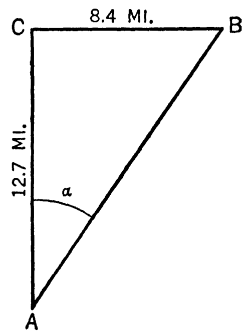


FIGURE 17.

EXAMPLE 2. Find the height of a mountain if when observed from the top of the mountain the angles of depression of two successive mileposts on a level road running directly away from the mountain are 15° 20′ and 10° 45′, respectively.

FIRST SOLUTION: Arrange the information of the problem as shown in the figure. The problem is to determine y. Let $AD = x$ be perpendicular to BC. Note that angle $ACD = 4° 35'$. (Why?) Let $AC = z$, then in triangle AOC,

$$\frac{y}{z} = \sin 15° 20,$$

or

$$y = z \sin 15° 20'.$$

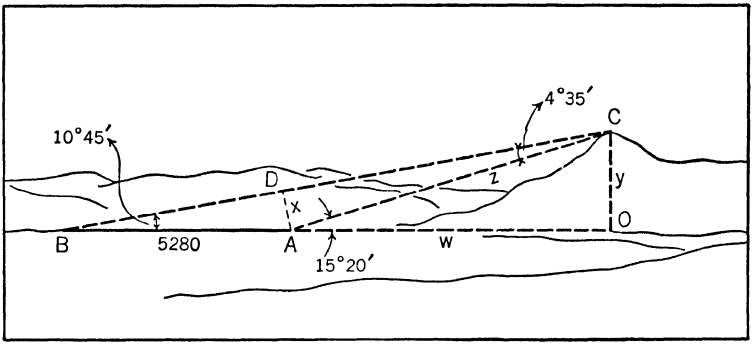


FIGURE 18.

Similarly from rt. triangle ADC,

$$\frac{x}{z} = \sin 4° 35',$$

so that

$$z = \frac{x}{\sin 4° 35'}.$$

Hence, by substitution, $$y = \frac{x \sin 15° 20'}{\sin 4° 35'}.$$

Again in rt. triangle ADB,

$$\frac{x}{5280} = \sin 10° 45',$$

$$x = 5280 \sin 10° 45',$$

from which, finally, $$y = \frac{5280(\sin 10° 45')(\sin 15° 20')}{\sin 4° 35'}.$$

Therefore,

$$\begin{aligned}\log y &= \log 5280 + \log \sin 10° 45' + \log \sin 15° 20' \\ &\quad - \log \sin 4° 35' \\ &= 3.7226 + (9.2707 - 10) + (9.4223 - 10) \\ &\quad - (8.9025 - 10) \\ &= 3.5131.\end{aligned}$$

Hence $$y = 3259 \text{ ft. Ans.}$$

SECOND SOLUTION: In the rt. triangle AOC let

$$OA = w,$$

then

$$\frac{w}{y} = \cot 15° \, 20'.$$

Also in rt. triangle BOC,

$$\frac{w + 5280}{y} = \cot 10° \, 45'.$$

Subtracting the members of the first equation from the corresponding members of the second eliminates w, and gives

$$\frac{5280}{y} = \cot 10° \, 45' - \cot 15° \, 20'.$$

From this,

$$y = \frac{5280}{\cot 10° \, 45' - \cot 15° \, 20'}$$

$$= \frac{5280}{5.267 - 3.647}$$

$$= 3259 \text{ ft.} \quad \text{Ans.}$$

EXERCISES

1. Wishing to know the height of a tree standing on a horizontal plain, I measured from the foot of the tree a horizontal line 150 ft. long, and found at that point the angle of elevation of the top of the tree to be 35° 20′. Required the height. ANS. 106.3 ft.
2. What is the angle of elevation of the sun when a tree 110 ft. high casts a shadow 165 ft. long? ANS. 33° 41′.
3. Find the length of the side of a regular pentagon inscribed in a circle of radius 6.87 in. ANS. 8.08 in.
4. Two stations A and B are 8 miles apart, and a man in a balloon directly above A observes the angle of depression of B to be 10° 14′. Find the height of the balloon. ANS. 1.44 mi.
5. The side of a regular octagon is 23 in. Find the radii of the inscribed and circumscribed circles. ANS. 27.77 in., 30.05 in.
6. The foot of a 30 ft. ladder leaning against a vertical wall rests on level ground 18 ft. from the base of the wall. Find the angle of inclination of the ladder. ANS. 53° 8′.
7. A railway is inclined at an angle of 3° 50′ to the horizontal. How many feet does it rise per mile of track? per mile of horizontal distance? ANS. 352.9 ft.; 353.8 ft.
8. The cross section of a levee is a trapezoid 30 ft. high. Its top is 12 ft. wide and the sides slope downward at an angle of 22° with the horizontal. Find the width of the base. ANS. 160.5 ft.
9. An observer in the top of a lighthouse 95 ft. above sea level finds that the angle of depression of a ship is 6° 15′. Find the horizontal distance of the ship from the lighthouse. ANS. 867.5 ft.
10. To find the width of a river a line AB = 420 ft. is measured parallel to one bank. A stump C on the other bank and straight across from B is sighted and angle CAB is found to be 21° 48′. Find BC. ANS. 168 ft.
11. Assume A to be the top of a flag pole, B its base, and C on a perpendicular to AB from B; what is the height of the pole if BC is 100 ft. and the angle ACB = 72°? ANS. 307.8 ft.

12. A boy is flying a kite and has let out 300 ft. of string. Assume a second boy 200 feet away is right under the kite. What is the height of the kite above the second boy? ANS. 223.6 ft.

13. Taking the radius of the earth as 3960 mi. and regarding the earth as a true sphere, find the radius of a parallel of latitude 30° north or south of the equator. ANS. 3429 mi.

14. A fly with clipped wings goes from one lower corner of a room to the opposite upper corner by the shortest path, necessarily crawling by floor and wall. The room is 14 ft. long, 10 ft. high, and 12 ft. wide. At what point will the fly start going up the wall? At what angle with the long side of the room must the fly start? At what angle with the floor must he start up the wall? Will it make any difference in the distance traveled whether he goes up a side wall or an end wall? How much?

Suggestion: Fold a side wall of the room down and outward to the level of the floor. Connect the starting point with the final point with a straight line.

15. A straight horizontal road leads to the foot of a wall 160 ft. high, and from the top of the wall the angles of depression of two men on the road are 10° and 8°. Find the distance between the men. ANS. 231 ft.

16. Upon the top of a shaft 125 ft. high stands a statue which, at a horizontal distance of 200 ft. from the foot of the shaft, subtends an angle of 3°. How tall is the statue? ANS. 15.04 ft.

17. The horizontal distance between the two extreme positions of the end of a pendulum 36 in. long is 5 in. Through what angle does it swing? ANS. 7° 58′.

18. A man walks up a hill 100 feet from its base. The angle made by the hill with the horizontal level is 15°. Assume the man to be 6 feet in height. From his position the angle of elevation of the top of the hill is 3°. What is the height of the hill? What distance will he have to travel to get to the top? ANS. 33.33 ft., 28.8 ft.

19. Find the length of one side of a regular pentagon if the radius of the circumscribed circle is 10 inches. ANS. 11.75 in.

20. An element of the lateral surface of a right circular cone is inclined at an angle of 70° to the plane of the base. The radius of the base is 20 feet. Find the volume and lateral surface of the cone.

21. Find the radius of a circle if the subtended chord is 20 ft. and the central angle is 42°. ANS. 27.90.

18. VECTORS. COMPONENTS AND RESULTANTS

Any quantity which has associated with it both magnitude and direction can be represented pictorially by an arrow whose length (drawn to some suitable scale) represents the magnitude of the quantity and whose direction represents the direction of the quantity. The arrow is called a **vector** and the quantity capable of such representation, a **vector quantity.** Velocities, accelerations, and forces are well known examples of vector quantities. Thus in Fig. 19, vector ***a*** might represent the velocity of the wind when blowing in an eastward direction and vector ***b*** might be the velocity (with respect to the air) of an airplane headed in a northeastward direction.

The length of the vectors indicates that the speed* of the plane is about twice that of the wind. Again, vector a might represent a force acting upon an object at O tending to move it to the right,

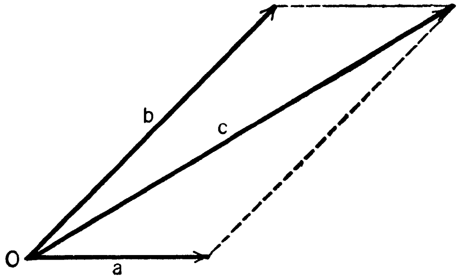


FIGURE 19.

and vector **b** might represent a force twice as great as a acting upward and at a 45° angle with a upon the same object.

The total effect of two vector quantities acting upon an object may be conveniently represented by that vector quantity which is given pictorially by the **diagonal of the parallelogram** of which the two given vectors form adjacent sides. This is known as the **parallelogram law:** The vector given by the diagonal is called the **resultant** of the other two vectors, and they in turn are called its **components** in the given directions. If vectors a and b are the velocity vectors above described, then the resultant c represents the *actual* velocity of the airplane. If a and b are forces acting upon an object at O, then c is the resultant force exactly equal in effect to a and b combined.

The finding of a resultant by the parallelogram law is called **vector addition.** If the resultant vector and one of its components is given, then the other component can be found by completing the parallelogram. This process is called **vector subtraction.** If a resultant vector is given and the angles which its components make with it are also given, the parallelogram may again be completed, thus **resolving** the resultant vector into components. Vector addition, vector subtraction, and the resolving of a vector into components are illustrated for rectilinear motion in Examples 1, 2, and 3 below, respectively. Some cases where right triangles are not involved will be considered in Chapter VIII.

* The word *speed* is used for the magnitude of the velocity. The word *velocity* is reserved for the vector quantity consisting of the speed and the direction.

EXAMPLE 1. A swimmer is swimming with a speed of 2 miles per hour and heads straight across a river having a current velocity of 4 miles per hour. In what direction is the swimmer actually traveling and with what speed?

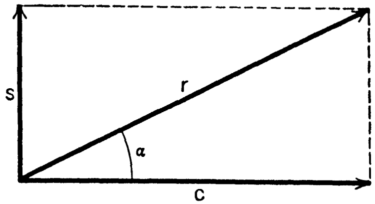


FIGURE 20.

SOLUTION: In Fig. 20, vector s represents the swimmer's velocity, and vector c the current velocity. It is required to find the resultant r in magnitude and direction. From the figure, $r^2 = 2^2 + 4^2 = 20$. Hence the actual speed of the swimmer is $\sqrt{20}$ miles per hour. Since $\tan \alpha = \frac{2}{4} = 0.5000$, from the tables, $\alpha = 26° 34'$. Hence the swimmer actually moves in a line which makes an angle of 26° 34′ with the bank and with a speed of $\sqrt{20} = 4.47$ miles per hour.

EXAMPLE 2. A force of 498.3 lbs. acts vertically upward upon an object. What horizontal force must also act upon the object if the resultant force is to be 625 lbs. acting upward and to the right?

SOLUTION: The two component forces are at right angles to each other. The resultant force of 625 lbs. must be the diagonal of a rectangle as shown in Fig. 21. Hence the horizontal component is the vector h of the figure, and at once,

$$
\begin{aligned}
h^2 &= (625)^2 - (498.3)^2 \\
&= (625 + 498.3)(625 - 498.3) \\
&= (1123.3)(126.7). \\
\therefore \quad 2 \log h &= \log 1123.3 + \log 126.7; \\
\log h &= \tfrac{1}{2}(3.0504 + 2.1028) \\
&= 2.5766. \\
h &= 377.2. \quad \text{Ans.}
\end{aligned}
$$

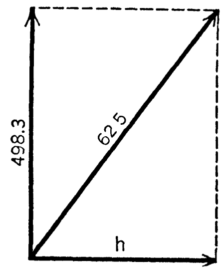


FIGURE 21.

EXAMPLE 3. The acceleration due to gravity of a falling body is 32.2 ft. per second per second. This means that the velocity of the falling body increases 32.2 ft. per second every second. What is the acceleration of a body sliding down a smooth (frictionless) plane inclined at an angle of 52° with the horizontal?

SOLUTION: The gravity vector acting downward must be resolved into two components, one of them parallel to the surface of the plane (vector t of Fig. 22), the other perpendicular to the surface (vector n). The magnitude of vector t is required. Obviously

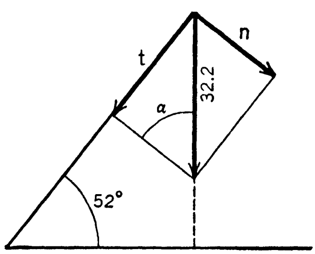


FIGURE 22.

$$\angle \alpha = 52°.$$

Hence, $$\frac{t}{32.2} = \sin \alpha$$

and $$t = 32.2(.7880) = 25.37.$$

Hence the object slides down the plane with an acceleration of 25.37 ft. per second per second (25.37 ft./sec^2.).

EXERCISES

1. A man rides eastward with a speed of 17.32 miles per hour. A breeze is blowing from the north with a speed of 10 miles per hour. From what direction does the wind apparently blow on the face of the man? What is the speed of the wind relative to the man?
   ANS. From a direction 60° east of north; 20 mi./hr.
2. A passenger on a train moving with a speed of 60 ft. per second tosses a stone with a speed of 30 ft. per second out the window of the train at right angles to the track. Considering only the horizontal motion of the stone, in what direction and with what velocity does the stone leave the train?
3. A weight of 15 tons rests on a plane inclined at an angle of 12° with the horizontal. With what force is the weight pressing perpendicularly against the plane? ANS. 14.67 tons.
4. A wire is stretched above the ground at an angle of 18° 40′ with the horizontal. A man pushes a pulley up the wire by walking along and holding a vertical pole against the pulley. If the man walks at the rate of 8 ft. per second, how fast does the pulley move up the wire?
5. Two forces, a 560-lb. horizontal force and a 660-lb. vertical force, act upon an object. Find the resultant force in magnitude and direction.
6. A car rolls down an inclined plane without friction. What is its acceleration if the plane is inclined at an angle of 34° with the horizontal? (g, the vertical acceleration due to gravity = 32.2 ft./sec^2.)
7. A car rolls without friction down an inclined plane. Its acceleration is 16.7 ft. per second per second. Find the angle of inclination of the plane. ANS. 31° 14′.
8. A projectile is fired from a cannon raised to an angle of 40° with the horizontal. The muzzle velocity is 1000 ft. per second. Find the horizontal and vertical components of this velocity.
9. A 200-lb. weight hangs at the end of a rope, the upper end of which is fixed. What horizontal force is required to hold the weight in such a position that the rope makes an angle of 21° with the vertical?
10. Two boys can paddle a canoe with a speed of 5 mi./hr. in still water. In what direction must they point their canoe to paddle directly across a river having a current of 3 mi./hr.?
    ANS. Upstream at angle of 53° 8′ with bank.
11. A car is rolling down an incline making an angle of 18° with the horizontal. At the end of 1st second starting from rest, how fast is the car moving horizontally? vertically downward?
    ANS. 9 46 ft. per sec.; 3.07 ft. per sec.
12. An army aviator flies a compass bearing due north at 100 miles per hour airspeed. What is his position at the end of two hours if an east wind is blowing at 30 miles per hour?
13. Town A is due west of town B at a distance of 68 miles. What time is required for an aviator to fly from town B to town A and return if his cruising airspeed is 120 miles per hour and a 20-mile north wind is blowing? What is his average speed for the round trip?
    ANS. 1 hr. 9 min.; 118.3 mi. per hr.

14. The crew of an anti-aircraft gun spot an enemy plane flying in their general direction. The plane's speed and elevation is determined as 90 miles per hour and 800 ft. respectively. The muzzle velocity of the gun is 2000 feet per second. Assuming that the plane will fly directly across their front and over a land point 200 yards away, compute the angle between the direction of fire and the direction of the plane at the time the gun is fired, if the gunner expects to hit the plane immediately as it crosses his front.

Chapter IV

DIRECTED ANGLES AND UNITS OF MEASUREMENT

19. ANGLES AS MEASURES OF ROTATION

An angle has been considered as the figure formed by two half lines drawn from the same point (Art. 2). On the other hand, the magnitude of any angle can be interpreted as a measure of the rotation necessary to bring one side to its given position from a position of coincidence with the other side. The angle generated by a rotating clock hand or by the spoke of a turning wheel is an illustration. This idea of an angle as a measure of rotation is a more general conception that admits of angles of any size.

The thought now naturally arises that the *direction of rotation* must also enter into the consideration of this more general conception of angles. It is customary to call the side of the angle which marks the beginning of the rotation the **initial side,** and the one representing the ending of the rotation the **terminal side** of the angle. An

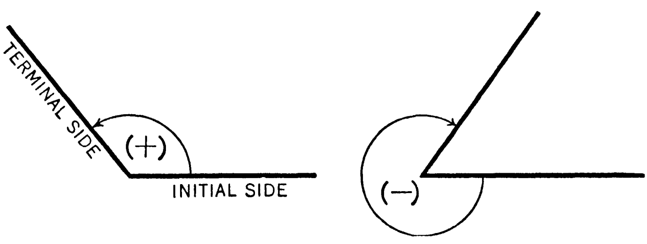


FIGURE 23.

angle is said to be **positive** or **negative** according as the position of the terminal side is reached by a **counter-clockwise** or by a **clockwise** rotation. A curved arrow (Fig. 23) serves to identify the direction of rotation in a given angle.

20. UNITS OF MEASUREMENT

There are several units used in the measurement of angles, the best known being the **degree** (°), the **radian** (rdn.), and the **mil.**

The **degree** is the unit of measurement of angles which is most commonly used in practical measurements. It was defined in Art. 2 as the angle equal to $\frac{1}{360}$ of the angular space about a point. The division of the degree into 60 equal parts called **minutes** and the similar division of the latter into **seconds** is familiar to the student. This system of angular measure is often called the **sexagesimal system.**

Radian or **circular** measure of angles is of great importance in mathematical theory. A **radian** is an angle which intercepts a circular arc whose length is equal to the length of the radius with which the arc is described. Thus, if in Fig. 24 the length of the circular arc s is equal to its radius r, then θ is an angle of one radian. This simple relation of arc to angle when the angle is measured in radians is one of the basic reasons for the usefulness of this unit of measure. If no unit of measure is specifically indicated, it will be understood that the unit is the radian. Thus the expression $\theta = 3$ will be understood to mean $\theta = 3$ *radians.*

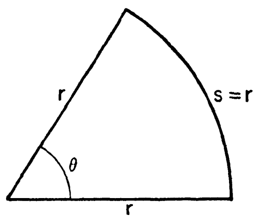


FIGURE 24.

The **mil** is a unit of angular measure generally used by the United States armed forces. It is defined as an angle equal to $\frac{1}{6400}$ of the angular space about a point. An object 1 yard wide subtends an angle of approximately 1 mil at a distance of 1000 yards.

21. RELATIONS BETWEEN THE UNITS

In plane geometry it is shown that two radii intercept an arc on a circle which is proportional to the central angle formed by the radii. This makes it obvious that, since a central angle of *one* radian intercepts an arc on a circle that is one radius in length, then a central angle of two radians intercepts an arc two radii in length, etc. In general, the number of radians in the angle is equal to the number of times the radius is contained in the intercepted circular arc. This means that an angle of 2π radians intercepts an arc $2\pi r$ in length. But $2\pi r = c$, the circumference of the circle. From this comes the important relation

$$2\pi \text{ radians} = 360°,$$

or

(9) $$\boldsymbol{\pi} \textbf{ radians} = \mathbf{180°}.$$

Equation (9) affords a simple method of changing the measure of a given angle from degrees to radians or vice versa. This can be best illustrated with examples.

EXAMPLE 1. Express 60° in radian measure.

SOLUTION: Suppose $x \text{ rdns.} = 60°.$
Now by (9), $\pi \text{ rdns.} = 180°.$
Hence, the proportion $\frac{x}{\pi} = \frac{60}{180},$
from which $x = \frac{\pi}{3}.$ Ans.

EXAMPLE 2. Express 2 radians in sexagesimal measure.

SOLUTION: Given $180° = \pi \text{ rdns.},$
$x° = 2 \text{ rdns.},$
it follows that $\frac{x}{180} = \frac{2}{\pi};$
whence $x = \left(\frac{360}{\pi}\right)^\circ = 114° \, 35.4'.$ Ans.

After division of both sides of equation (9) by π, there results the relation

$$1 \text{ rdn.} = \left(\frac{180}{\pi}\right)^\circ = 57° \, 17.7'.$$

Or, after division by 180,

$$1° = \frac{\pi}{180} \text{ rdn.} = 0.017453 \text{ rdn.}$$

To obtain relations between degree and mil measure, note that, from the definitions of degree and the mil,

$$360° = 6400 \text{ mils.}$$

From this come the two results,

$$1^\circ = \tfrac{6400}{360} \text{ mils} = 17.778 \text{ mils},$$

and

$$1 \text{ mil} = \left(\tfrac{360}{6400}\right)^\circ = 0.05625^\circ.$$

To derive relations between number of radians and number of mils, note that

$$\pi \text{ rdns.} = 3200 \text{ mils},$$

so that

$$1 \text{ rdn.} = \frac{3200}{\pi} \text{ mils} = 1018.6 \text{ mils},$$

and that

$$1 \text{ mil} = \frac{\pi}{3200} \text{ rdn.} = 0.0009817 \text{ rdn.}$$

These relations are summarized in the following set of equations:

(10) $\mathbf{1^\circ = 0.017453 \text{ rdn.} = 17.778 \text{ mils.}}$

(11) $\mathbf{1 \text{ rdn.} = 57.296^\circ = 1018.6 \text{ mils.}}$

(12) $\mathbf{1 \text{ mil} = 0.05625^\circ = 0.0009817 \text{ rdn.}}$

These relations must be understood as being approximations which are accurate to the figures given, with the exception of the relation 1 mil = 0.05625°, which is exact.

EXERCISES

Use the fundamental method employed in the illustrative examples (rather than the conversion relations just given) to obtain the solution of the first sixteen of the following exercises:

Express in radian measure:

1. 60°. Ans. $\frac{\pi}{3}$.

2. 90°.

3. 120°. Ans. $\frac{2\pi}{3}$.

4. 75°.

5. 420°. Ans. $\frac{7\pi}{3}$.

6. 300°.

Express in degree measure:

7. $\frac{\pi}{4}$. Ans. 45°.

8. 2π.

9. $\frac{7\pi}{6}$. Ans. 210°.

10. $\frac{3\pi}{5}$.

11. $\frac{3\pi}{4}$. Ans. 135°.

12. $\frac{5\pi}{3}$.

13. 1600 mils. Ans. 90°.

14. 200 mils.

15. Draw a circle and eight radii which divide it into eight equal sectors. Let one of these radii (preferably one extending to the right of the center) be called r_0, and let the remaining radii be successively r_1, r_2, r_3, etc., named in counterclockwise order from r_0. Finally, let θ_1, θ_2, θ_3, etc., be the angles formed by rotating a radius from the position of r_0 to that of each of the radii r_1, r_2, r_3, . . . , respectively. Express the measure of θ_1, θ_2, θ_3, etc. (**a**) in degrees, and (**b**) in radians. These can be indicated on the figure at the extremities of the corresponding radii.

16. Repeat Exercise 15, dividing the circle into twelve equal parts instead of eight.

Use the conversion relations (10), (11), and (12) to work out the following three exercises:

17. Express 10° as radians; as mils. ANS. 0.17453 rdn.; 177.78 mils.

18. Express 32 mils as degrees; as radians.

19. Express 0.55 radian in degrees; in mils. ANS. 31.51°; 560.2 mils.

20. Find $\cos\frac{\pi}{3}$. ANS. $\frac{1}{2}$.

21. Find $\tan\frac{\pi}{4}$.

22. Find $\sin\frac{\pi}{8}$. ANS. .3827.

23. Find $\cot\frac{\pi}{9}$.

24. Determine in radian measure an acute angle α such that

(a) $\sin\alpha = \frac{1}{2}$. ANS. $\frac{\pi}{6}$.

(b) $\sec\alpha = 2$.

(c) $\tan\alpha = \frac{1}{\sqrt{3}}$. ANS. $\frac{\pi}{6}$.

(d) $\cot\alpha = 1$.

22. THE CENTRAL ANGLE AND THE INTERCEPTED ARC

From the definition of a radian it follows that in a given circle, a central angle of one radian intercepts an arc equal in length to the radius. Hence, by proportion, a central angle of θ radians intercepts an arc equal in length to θ times the radius. Otherwise stated,

(length of arc) = (subtended angle in radians) × (radius);

or symbolically,

$$s = \theta r. \qquad (13)$$

(θ expressed in radians)

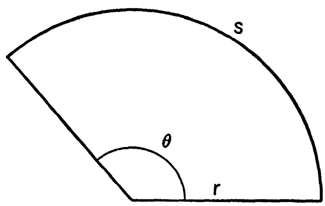


FIGURE 25.

According to (12), 1 mil = 0.0009817 rdn. For four-figure accuracy, round off accordingly and obtain

$$1 \text{ mil} = .0010 \text{ radian}.$$

Hence, the relation $\theta \text{ mils} = \frac{\theta}{1000} \text{ radian}$

is a close approximation. Applied to (13), this gives the approximation formula

$$s = \frac{\theta r}{1000}. \qquad (\theta \text{ expressed in mils})$$

In actual practice, the arc s is replaced by its chord c (nearly equal for *small* angles) and r is considered as the distance to the chord. Thus the approximation formula becomes

(14) $$c = \frac{\theta r}{1000}. \qquad (\theta, \text{ in mils})$$

This can be written in the optional form

(15) $$r = \frac{1000c}{\theta}. \qquad (\theta, \text{ in mils})$$

By means of instruments from which angles in mils can be read directly, it is easy to calculate rapidly the distance r to an object of known height or width c, or, if desired, to calculate the height or width of an object of known distance.

EXAMPLE 1. Find the length of the arc intercepted on a circle of radius 10 inches by a central angle of 40°.

SOLUTION: From (9) Art. 21, $40° = \frac{2\pi}{9}$ rdn.

Now, by (13), $s = r\theta$ (θ in radians).

Hence, $s = 10\frac{(2\pi)}{9} = \frac{20\pi}{9}$ in. Ans.

EXAMPLE 2. What central angle will subtend an arc whose length is 12 inches on a circle with diameter of 5 feet?

SOLUTION: Relation (13) can be written in the form

$$\theta = \frac{s}{r}.$$

Hence, $\theta = \frac{1}{2.5}$ rdn. $= 0.4$ rdn. $= 22° 55'$. Ans.

EXAMPLE 3. A building known to be 140 feet high subtends an angle of 45 mils at a given point of observation. Find the distance of the observer from the building.

SOLUTION: Employ (15) with $c = 140$ and $\theta = 45$.

Hence, distance $= r = \frac{1000(140)}{45} = 3110$ ft. Ans.

EXAMPLE 4. What is the length of a hangar which subtends an angle of 42 mils at a distance of 2400 yards?

SOLUTION: By (14)

$$c = \frac{\theta r}{1000}.$$

Substituting $\theta = 42$ and $r = 2400$ gives

$$c = \frac{42(2400)}{1000} = 101 \text{ yds. Ans.}$$

EXERCISES

1. Using the definition that one mil is an angle equal to $\frac{1}{6400}$ of the angular space about a point, verify the statement that an angle of one mil intercepts an arc of one yard (very nearly) at a distance of 1000 yards.
2. In a circle of 10″ radius, find mentally the length of the arc intercepted by a central angle of (a) 1 rdn., (b) 2 rdns., (c) 5 rdns., (d) π rdns., (e) n rdns.
3. What is the radius of the circle in which the number of radians in a central angle is equal to the length of the intercepted arc? ANS. unity.
4. Find the linear distance traveled by the tip of a 12-inch hour hand of a clock between 6 A.M. and 2 : 30 P.M.
5. The end of a 40-inch pendulum describes an arc of 5 inches. Find the angle through which the pendulum swings. ANS. $\frac{1}{8}$ rdn.
6. Through what angle will a pulley 8 inches in diameter turn while 4 feet of rope pass over it?
7. The front wheel of a buggy is 40 inches in diameter and the rear wheel is 60 inches in diameter. Through how many radians will the front wheel turn during a complete revolution of the rear wheel? ANS. 3π rdns.
8. An observer notes that two points A and B on an enemy front 10,000 feet away subtend an angle of 20 mils. What is the distance AB?
9. A tank known to be 26 feet long subtends an angle of 6 mils. How far is it from the observer? ANS. 4333 ft.
10. Find the angle in mils subtended by a circular target 30 feet in diameter at a distance of 1000 yards.
11. It was remarked that the formula
$$c = \frac{\theta r}{1000}$$
is quite accurate for small angles. Find the percent of error resulting from using this formula when the angle is 200 mils as compared with the true results obtained by converting the measure of the angle to degrees and computing the chord by trigonometric functions. (*Hint.* for convenience, use $r = 1000$.) ANS. 1.94%.

23. AREAS OF THE CIRCULAR SECTOR AND CIRCULAR SEGMENT

It is a well-known theorem of plane geometry that in a given circle the areas of sectors are proportional to their central angles. Hence the proportion

$$\frac{\text{area of sector}}{\text{angle of sector}} = \frac{\text{area of the circle}}{\text{four right angles}}.$$

If the radius of the circle is r, the central angle of the sector is θ radians, and the area of the sector is K, the proportion is

$$\frac{K}{\theta} = \frac{\pi r^2}{2\pi}.$$

This readily is simplified to the form

(16) $K = \frac{1}{2}r^2\theta$. (θ in radians)

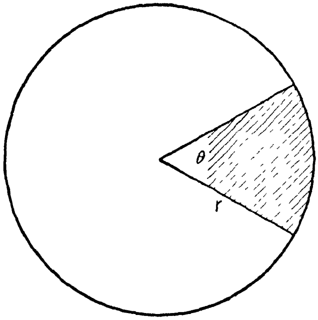


FIGURE 26.

EXAMPLE 1. If the diameter of a certain circle is 16 feet, what is the area of a sector having a central angle of 60°?

SOLUTION: $60° = \frac{\pi}{3} = 1.0472$ radians.

$\therefore \quad K = \frac{1}{2}(8)^2(1.0472) = 33.51$ sq. ft. Ans.

To determine a formula for the area of a circular segment note that the area K' of a segment determined by a chord AB will be given by the equation

K' = area of sector AOB − area of triangle AOB,

or, from (16)

$K' = \frac{1}{2}r^2\theta$ − area of triangle AOB.

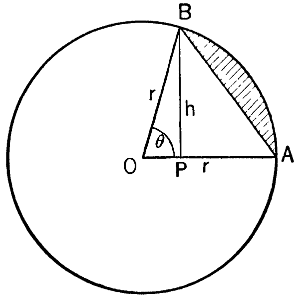


FIGURE 27.

To obtain a formula for the area of triangle AOB, let h be the perpendicular let fall from B to the radius OA. The area of any triangle is equal to half the product of its base and altitude. Hence the

$$\text{area of triangle } AOB = \tfrac{1}{2}bh = \tfrac{1}{2}rh.$$

But from right triangle BPO,

$$\frac{h}{r} = \sin\theta$$

so that

$$h = r\sin\theta.$$

Substitute this value of h in the area formula. Thus the

$$\text{area of triangle } AOB = \tfrac{1}{2}r^2\sin\theta.$$

Finally, the formula for K' becomes

$$K' = \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta$$

or

(17) $$\boldsymbol{K' = \tfrac{1}{2}r^2(\theta - \sin\theta).} \qquad (\theta, \text{ in radians})$$

EXAMPLE 2. Find the area of a circular segment having a central angle of $\frac{\pi}{6}$ radians and a radius of 6 feet.

SOLUTION: By substitution in (17),

$$K' = \tfrac{1}{2}(6^2)\left(\frac{\pi}{6} - \sin\frac{\pi}{6}\right)$$
$$= 18(0.5236 - 0.5000) = 0.4248 \text{ sq. ft. Ans.}$$

24. LINEAR SPEED AND ANGULAR VELOCITY OF ROTATION

Linear speed is usually expressed in feet per second or miles per hour, whereas angular velocity is given in revolutions or radians per second, per minute, or per hour. If an object has a uniform linear speed of v units during a time t, the distance is

$$d = vt.$$

If an object has an angular velocity of ω radians per unit of time, the angle through which the object passes in time t is

$$\theta = \omega t.$$

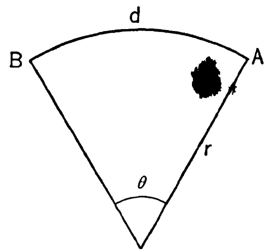


FIGURE 28.

In circular motion

$$d = \theta r, \text{ by (13).}$$

Substituting for d and θ, gives

$$vt = \omega tr.$$

Divided by t, this becomes

$$(18) \qquad \boldsymbol{v = \omega r}.$$

"The linear speed of an object moving in a circular path is equal to the magnitude of the angular velocity of the object multiplied by the radius of the circular path."

EXAMPLE 1. The linear speed of a wheel of radius $\frac{10}{\pi}$ feet is 20 miles per hour. What is its angular velocity in radians per second?

SOLUTION: $v = \omega r.$

$$20 \text{ miles per hour} = \frac{(20)(5280)}{(3600)} \text{ ft. per sec.}$$

$$\therefore \qquad \omega = \frac{v}{r} = \frac{(20)(5280)}{(3600)}\left(\frac{\pi}{10}\right) \text{ rdns. per sec.}$$

$$= 2.93\pi \text{ rdns. per second.}$$

EXAMPLE 2. A pulley $\frac{30}{\pi}$ inches in diameter makes 100 revolutions per minute. What is the linear velocity, in feet per second, of the belt that drives it?

SOLUTION: 100 revolutions per minute $= 2\pi\left(\frac{100}{60}\right)$ radians per sec.

$$v = \omega r = 2\pi\left(\frac{100}{60}\right)\left(\frac{15}{\pi}\right) = 50 \text{ in. per sec.}$$

$$v = 4.17 \text{ ft. per sec.} \quad \text{Ans.}$$

EXERCISES

1. Find the area of the sector of a circle of radius 8 inches if the central angle is 40°. Ans. 22.34 sq. in.
2. A regular hexagon is inscribed in a circle of radius 5 feet. What is the area of the segment formed by one side of the hexagon?
3. A train turns a curve of half a mile radius at 20 miles per hour. What is its angular velocity in radians per minute? Ans. $\frac{2}{3}$ rdn./min.
4. What is the angular velocity of the hour hand of a clock?
5. What is the linear speed of the tip of a 10-inch minute hand? Ans. 1.047 in./min.
6. The earth moves once around the sun in 365 days. If the radius of the earth's orbit about the sun is 93,000,000 miles, what is its linear speed in miles per second? (Assume the orbit to be circular.)
7. An automobile wheel 3 feet in diameter moves along a road at 35 miles per hour. Find the angular velocity in radians per second of a point on the tread of the tire. Ans. 34.2 rdns./sec.
8. A cogwheel is driven by a chain which travels 240 feet per minute. What is the radius of the wheel if it has an angular velocity of 2 radians per second?
9. What is the angular velocity in radians per hour of the earth about its axis? Ans. $\frac{\pi}{12}$ or 0.2618 rdn./hr.
10. A chord of length c cuts a circle of radius c. Find the area of the sector made by drawing radii to the ends of the chord.
11. If the area of a sector of a circle of radius 10 inches is 25π square inches, find the arc length of the sector. Ans. 5π in.

25. RELATIONS OF θ, SIN θ, AND TAN θ FOR SMALL VALUES OF θ

Consider Fig. 29 showing an acute angle of θ radians at the center of a circle of unit radius, with segments whose lengths are respectively y and z perpendicular to one side of θ, one being drawn from the intersection of the other side of the angle and the circle and the other tangent to the circle. Then obviously y is less than the arc s, and it can be proved (See Exercise 1 below.) that s is less than z. In symbols,

$$y < s < z.$$

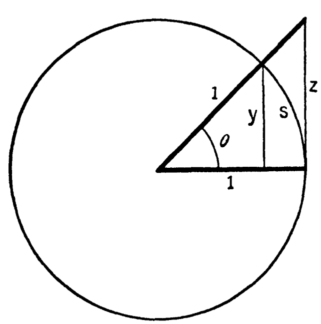


FIGURE 29.

But $\frac{y}{1} = \sin\theta$, so that

$$y = \sin\theta,$$

and similarly

$$z = \tan\theta.$$

Furthermore, since θ is in radian measure, from (13)

$$s = r\theta = \theta.$$

Hence, the foregoing inequality becomes

(a) $\sin\theta < \theta < \tan\theta.$ (θ in radians and $\theta > 0$)

Divide the members of the inequality (a) by $\sin\theta$ and note that $\frac{\tan\theta}{\sin\theta} = \frac{1}{\cos\theta}$, so that

(b) $$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Now suppose that θ approaches zero. The fraction $\frac{1}{\cos\theta}$ will then approach the value 1. (Why?) Because of (b), this necessarily compels the fraction $\frac{\theta}{\sin\theta}$ to approach the value 1. In other words, *$\sin\theta$ is approximately equal to θ for small values of θ.*

Similarly, from division of (a) by $\tan\theta$ and the fact that $\frac{\sin\theta}{\tan\theta} = \cos\theta$,

(c) $$\cos\theta < \frac{\theta}{\tan\theta} < 1;$$

which can be interpreted to mean that *$\tan\theta$ is approximately equal to θ for small values of θ.*

The conclusions stated here in italics justify the substitution of θ for $\sin\theta$ and $\tan\theta$, when approximate results are desired, in computations involving small values of θ. It must be understood that the closeness of the approximation depends upon the smallness of θ.

EXAMPLE 1. Find the arc intercepted by an angle of $1°\,30'$ at a distance of 100 feet from the vertex. Test the accuracy of the result.

SOLUTION: By (13), $s = r\theta$. Replace θ by $\sin\theta$. The approximate result is

$$\begin{aligned} s &= r\sin\theta \\ &= 100(0.0262) \\ &= 2.62 \text{ ft. Ans.} \end{aligned}$$

But by (10), $1° = 0.017453$ rdn., so that $1°\,30' = 0.026179$ rdn. Then

$$\begin{aligned} s &= r\theta \\ &= 100(0.026179) \\ &= 2.6179. \end{aligned}$$

A comparison of the two results shows that the one obtained by using $\sin\theta$ for θ is correct to the nearest one hundredth of a foot.

EXAMPLE 2. Convert $1° 10'$ to radians.

SOLUTION: Let $1° 10' = \theta$ rdns. Then $\theta = \tan \theta$, approximately, so that

$$\theta = \tan \theta = \tan 1° 10' = 0.0204. \quad \text{Ans.}$$

EXERCISES

1. Find to two decimal places the percent of error resulting in replacing θ by $\sin \theta$, if θ is the radian equivalent of (a) 2°, (b) 5°, (c) 10°, (d) 20°.
ANS. (a) 0%, (b) 0.11%, (c) 0.52%, (d) 2.03%.

2. In Fig. 29 prove that arc s is less than z by comparing the expressions for the area of the sector of angle θ and of the right triangle of base 1 and altitude z.

3. Show that the acceleration a of an object sliding down a frictionless incline of angle θ radians is given approximately for small values of θ by the formula

$$a = 32.2\theta.$$

4. Show that the procedure of replacing $\sin \theta$ by θ fails when an approximation formula for the area of circular segments is desired.

5. Following up the idea of the preceding exercise, show that the approximation obtained by replacing θ by $\sin \theta$ is of the same accuracy as the one obtained by replacing the area of circular sector by that of the triangle formed by the radii and chord of the sector.

6. An observer on level ground notes that the angle of elevation of the top of a tower 10,000 feet away is 0.0122 radian. Find the height of the tower by using the $\tan \theta = \theta$ approximation.

MISCELLANEOUS EXERCISES

1. Verify the statement made in Art. 2 that "at a distance of one mile from the vertex of an angle of one second, the sides are separated by a distance approximately equal to the thickness of a lead pencil."

2. Construct a table giving the radian and mil measure for 5°, 10°, 15°, 20°, 25°, 30°, expressing each correct to three decimals.

3. Express 1 radian in degrees, minutes, and seconds to the nearest second.
ANS. $57° 17' 45''$.

4. Express in radian measure (as multiples of π) each of the angles from 0° to 360° which is a multiple of 15°.

5. A railroad track follows a circular arc in making a turn from a direction of northeast to a direction of east. If the radius of the curve is 2000 feet, what is the length of the curve? ANS. 500π ft.

6. A military observer finds that the angle subtended by the highest and lowest points of a bridge is 22 mils. If the bridge is known to be 90 feet in height, what is its distance from the observer?

7. A train moves over the curved track described in Ex. 5 with a speed of 40 miles per hour. What is its angular velocity in radians per minute?
ANS. 1.76 rdn./min.

8. A plane follows the 42nd parallel of latitude through 15 degrees of longitude. What is the distance flown?

9. A cylindrical tank 20 ft. long and 10 ft. in diameter lies in a horizontal position. How many cubic feet of oil does it contain when the greatest depth of the oil is 7 ft.? ANS. 1180 cu. ft.

10. A searchlight which gives satisfactory visibility for a distance of 10 miles sweeps horizontally through an angle of 40°. How many square miles of territory does it thus cover?

11. What fraction of the sun's illumination is in effect when one third of its diameter is showing above the horizon? ANS. 0.292.

12. An observer 110 feet above water level finds that the angle of depression of a distant object floating in the water is 40′. Find the horizontal distance to the object without using the table of trigonometric functions.

13. A belt is stretched around two wheels whose radii are 5 in. and 17 in., respectively, and whose centers are 24 in. apart. Verify that the length of the belt is $24\sqrt{3} + 26\pi$ in.

14. Solve Ex. 13 if the belt is crossed.

15. A ferris wheel of 30 ft. radius is turning so that a point on its circumference is moving with a linear speed of 10 ft./sec. Measured from the instant the point reaches the top of the circle, how far will the projection of the point move along the ground in 3 seconds? ANS. 25.24 ft.

SUMMARY OF FORMULAS, CHAPTER IV

On units of measure.

(9) π radians = 180°.

(10) 1° = 0.017453 rdn. = 17.778 mils.

(11) 1 rdn. = 57.2958° = 1018.6 mils.

(12) 1 mil = 0.05625° = 0.0009817 rdn.

On length of arc, s.

(13) $s = \theta r.$ (θ, in radians)

On approximations to length of chord, c, and its distance from center of circle, r.

(14) $c = \dfrac{\theta r}{1000}.$ (θ, in mils)

(15) $r = \dfrac{1000c}{\theta}.$ (θ, in mils)

On area of sector, K, and area of segment, K'.

(16) $K = \frac{1}{2} r^2\theta.$ (θ, in radians)

(17) $K' = \frac{1}{2} r^2(\theta - \sin\theta).$ (θ, in radians)

On circular motion, giving the linear speed, v, in terms of the angular velocity, ω.

(18) $v = \omega r.$

TRIGONOMETRIC FUNCTIONS OF THE GENERAL ANGLE

26. INTRODUCTION

It will be the object of this chapter to extend the definitions of the six trigonometric functions so as to include the functions of any angle, at the same time keeping the extended definitions in agreement with the original definitions of the functions of an acute angle.

27. A RECTANGULAR COORDINATE SYSTEM

The student is already familiar with the rectangular coordinate system as used in algebra for plotting points and tracing loci in a plane. Such a system will be used in the further discussion of angles in general and their trigonometric functions. The essential features are reviewed here.

Construct two lines perpendicular to each other, one horizontal, the other vertical. Their point of intersection is called the **origin,** the horizontal line the ***X*-axis,** and the vertical line the ***Y*-axis.** By choosing a proper scale any positive or negative number can be represented by fixing a corresponding distance along either of the axes. Conventionally, positive numbers are measured on the X-axis to the right of the origin and up from the origin along the Y-axis, and negative numbers in the opposite directions. It follows that any point in the plane of the two axes can be located by two signed numbers representing the positive or negative directed distance along the X-axis and along the Y-axis, respectively. The first of these numbers is called the **abscissa** of the point, and the second, the **ordinate** of the point. Together they are called the **coordinates** of the point. A point whose abscissa is x and whose ordinate is y is usually denoted by the symbol (x, y).

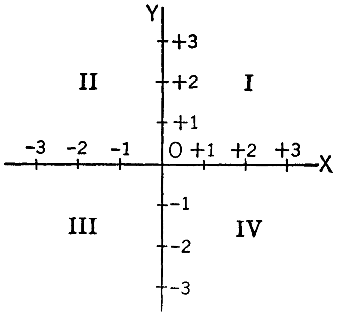


FIGURE 30.

The plane is divided into four quadrants by the coordinate axes, and they are numbered in counterclockwise fashion with number one in the upper right-hand quadrant.

EXERCISES

1. Construct a set of rectangular axes with a number scale on each and locate the following points: A, $(2, 3)$; B, $(-4, 2)$; C, $(0, 4)$; D, $(-2, 0)$; E, $(-3, -5)$; F, $(-1, \sqrt{3})$; G, $(\sqrt{3}, -1)$; O, $(0, 0)$.
2. If (x, y) represents a variable point, state
   (a) the sign of x in each quadrant,
   (b) the sign of y in each quadrant,
   (c) the sign of $\frac{y}{x}$ in each quadrant,
   (d) the sign of $\frac{x}{y}$ in each quadrant.
3. Plot the points $(0, 0)$, $(0, 4)$, $(4, 4)$, and $(4, 0)$. Draw line segments connecting each point with the one immediately following it and the last point with the first. What is the figure?
4. Plot the points $(1, 2)$, $(3, 2 + 2\sqrt{3})$, and $(5, 2)$ and connect them by line segments. What is the figure?

28. DISTANCE BETWEEN TWO POINTS

The distance between two points P_1 and P_2 can be found by using the theorem of Pythagoras. Let the coordinates of P_1 be (x_1, y_1) and of P_2, (x_2, y_2). In Fig. 31,

$$d^2 = a^2 + b^2$$

but $a = x_2 - x_1$, and $b = y_2 - y_1$.

When these values are substituted for a and b and the square root of both sides taken,

$$\textbf{(19)}\quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

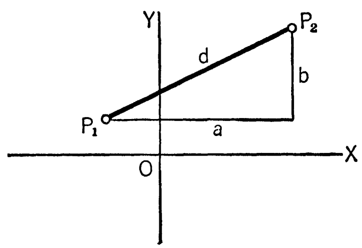


FIGURE 31.

By constructing figures showing P_1 and P_2 in various positions, the student can readily show that this formula for the distance between two points is true no matter where the points P_1 and P_2 are located.

EXAMPLE. Find the distance between the points (3, 4) and (– 5, 1).

SOLUTION: By substitution in formula (19),

$$\begin{aligned} d &= \sqrt{(3 - [-5])^2 + (4 - 1)^2} \\ &= \sqrt{(3 + 5)^2 + (4 - 1)^2} \\ &= \sqrt{73} \text{ units.} \quad \text{Ans.} \end{aligned}$$

EXERCISES

1. Find the distance between the points (4, 1) and (6, – 2). ANS. $\sqrt{13}$.
2. Find the distance from the origin to (– 5, 3).
3. Show that the points (0, – 5), (– 2, 1), (10, 5) are the vertices of a right triangle.
4. Show that the points (5, – 4), (10, 8), (– 7, 1), (– 2, 13) are the vertices of a square.
5. Show that the points (3, 2), (6, 1), (7, – 2) are the vertices of an isosceles triangle.
6. Draw a circle with center at (3, 5) tangent to the x-axis. Does this circle pass through (6, 1)? Through (6, 9)? Through (– 1, 2)?
7. Find the sine and the cosine of the angle with vertex at A of the triangle determined by the three points $A(-8, -2)$, $B(4, 3)$, and $C(4, -2)$.
ANS. $\sin \alpha = \frac{5}{13}$, $\cos \alpha = \frac{12}{13}$.

29. STANDARD POSITION OF AN ANGLE

In the study of the general angle to follow, it is essential that a uniform method of viewing an angle be used. Consider an angle as formed by rotating one side from a position of coincidence with the other in a fixed position. The fixed side of the angle is called the **initial side** and the rotating side in its final position is called the **terminal side.** An angle is said to be in **standard position** with reference to a rectangular coordinate system when the origin of the coordinate system is at the vertex of the angle and the positive half of the X-axis coincides with the initial side of the angle. The terminal side of any **positive** angle is then obtained by a counterclockwise rotation from the x-axis about the origin, and the terminal side of any **negative** angle by a clockwise rotation. Thus, in Fig. 32, angles of + 150° and – 70° are illustrated in standard position. It is obvious that angles of any magnitude and sign can be constructed in this manner.

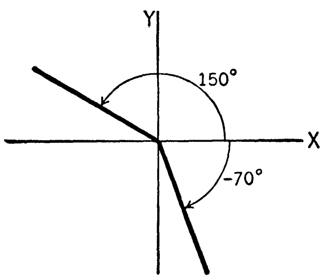


FIGURE 32.

If two or more angles in standard position have the same terminal side, they are said to be **co-terminal.** Thus, angles of 225°, − 135°, and 585° are co-terminal.

It should be noted that the curved arrow denotes the direction of rotation and indicates whether the angle is positive or negative.

EXERCISES

1. Construct the following angles in standard position, give the quadrant in which the terminal side of each angle falls, and name one angle which is co-terminal with each:
   (a) 45°, (b) 210°, (c) − 130°, (d) 750°, (e) − 65°, (f) 270°, (g) − 180°.
2. Specify a negative angle which is co-terminal with each of the following: (a) 90°, (b) 180°, (c) 270°, (d) 360°.

30. THE TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

Let XOT be any angle θ in standard position (see illustrations of θ in different quadrants), and let (x, y) be any point P on the ter-

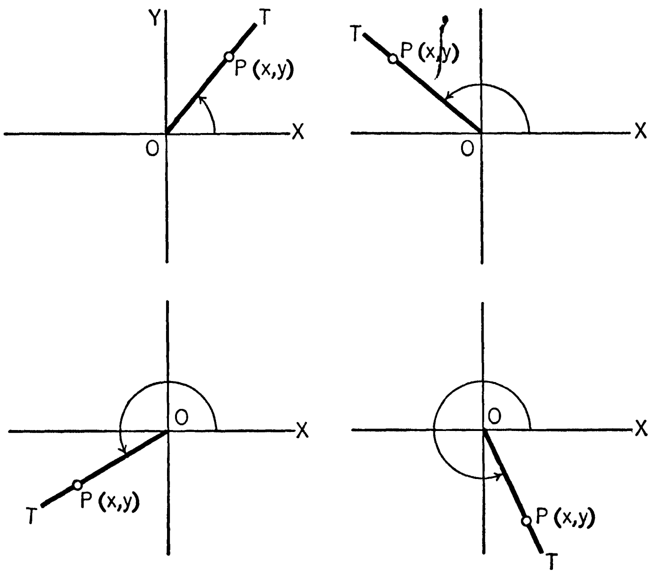


FIGURE 33.

minal side different from (0, 0). Finally let r represent the absolute value of the distance from (0, 0) to (x, y), noting that r is always positive and cannot be negative as can x and y.

The trigonometric ratios of θ are defined as follows:

$$\sin\theta = \frac{y}{r} = \frac{\text{ordinate of } P}{\text{distance of } P \text{ from origin}},$$

$$\cos\theta = \frac{x}{r} = \frac{\text{abscissa of } P}{\text{distance of } P \text{ from origin}},$$

$$\tan\theta = \frac{y}{x} = \frac{\text{ordinate of } P}{\text{abscissa of } P},$$

$$\cot\theta = \frac{x}{y} = \frac{\text{abscissa of } P}{\text{ordinate of } P},$$

$$\sec\theta = \frac{r}{x} = \frac{\text{distance of } P \text{ from origin}}{\text{abscissa of } P},$$

$$\csc\theta = \frac{r}{y} = \frac{\text{distance of } P \text{ from origin}}{\text{ordinate of } P}.$$

To illustrate the application of these definitions, consider a graphical method of determining approximately the values of the trigonometric functions of any given angle. A pair of rectangular axes are set up as shown, the unit of distance being chosen large enough to locate tenths of a unit readily. To facilitate the location of any

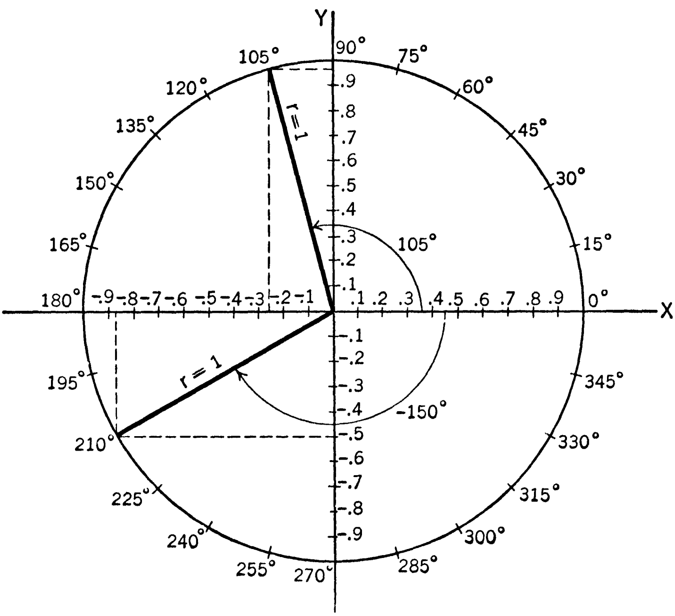


FIGURE 34.

angle in standard position, arcs of 15° intervals from 0° to 360° are marked on a circle of radius 1. The trigonometric functions of any angle, say 105°, are found by locating it on the drawing, and choosing the point (x, y) so that the distance $r = 1$, that is, (x, y) is on the unit circle. By inspection it is seen that

$$x = -0.26 \quad \text{and} \quad y = 0.97. \quad \text{(Approx.)}$$

Hence, from the previous definitions, approximate values of the trigonometric functions of 105° are

$$\sin 105° = \frac{y}{r} = \frac{0.97}{1} = 0.97,$$

$$\cos 105° = \frac{x}{r} = \frac{-0.26}{1} = -0.26,$$

$$\tan 105° = \frac{y}{x} = \frac{0.97}{-0.26} = -3.7,$$

$$\cot 105° = \frac{x}{y} = \frac{-0.26}{0.97} = -0.27,$$

$$\sec 105° = \frac{r}{x} = \frac{1}{-0.26} = -3.8,$$

$$\csc 105° = \frac{r}{y} = \frac{1}{0.97} = 1.03.$$

The student is reminded that the point (x, y) could have been chosen at *any* point on the terminal side of the angle, the distance $r = 1$ being chosen here simply for convenience in computing ratios.

31. SOME OBSERVATIONS REGARDING FUNCTIONS OF THE GENERAL ANGLE

The student's attention is called to several points resulting from the definitions given in the preceding article:

(a) If θ is an acute angle, the definitions of the trigonometric functions are in no way different from those originally given in Art. 8, for in this case x and y are both positive numbers and may be considered the base and perpendicular sides, respectively, of a right triangle of which r is the hypotenuse and of which θ is an acute angle.

(b) In general, the six trigonometric functions of an angle are signed numbers, being positive or negative according to the signs of x and y in the quadrant in which the terminal side of the angle lies when in standard position.

(c) The fact that an angle is negative does not determine the sign of its trigonometric functions. Negative angles are placed in standard position with a clockwise rotation of the terminal side. The signs of the trigonometric functions are determined solely by the signs of x and y, as noted in **(b)**.

(d) Any trigonometric function of any angle, with certain exceptions noted in Article 33, can be determined in some cases exactly and in others approximately by the graphical method.

(e) The Fundamental Relations, formulas (1) to (8), hold for the general angle as well as for the acute angle, except that two of the six functions are not defined for each angle co-terminal with 0°, 90°, 180°, and 270° (See Art. 33). This is easily verified by replacing a, b, c with x, y, r, respectively, in the developments of Art. 12.

EXERCISES

1. Construct four figures, each showing a positive angle with terminal side in a different quadrant. Determine the sign of x and y for each case (what is the sign of r?) and state the signs of the six trigonometric ratios for each of the four angles.

2. Repeat Ex. 1, using four negative angles with terminal sides in the various quadrants.

3. Verify the statement: "The sign of $\sin\theta$ will always be the same as the sign of y." Of what other function can the same statement be made?

4. Complete the statement: "The sign of $\cos\theta$ will always be the same as . . ." Can the same be said of any other function? If so, which one?

5. An angle of $-150°$ has been constructed in standard position in Fig. 34. Taking $r = 1$ as shown, find the values of x and y and verify that $\sin(-150°) = -0.5$, and that $\tan(-150°) = 0.58$. Find also the values of the four remaining functions.

6. Verify the following statement: "If (x, y) is chosen on the terminal side of angle θ so that $r = 1$, then $\sin\theta = y$ and $\cos\theta = x$."

7. Using Fig. 34 and the results of Ex. 6, find correct to two figures the following functions:

(a) $\cos 315°$.	Ans. 0.71.		(e) $\sin 120°$.
(b) $\sin 240°$.	Ans. -0.87.		(f) $\cos 135°$.
(c) $\sin(-30°)$.	Ans. -0.50.		(g) $\cos(-225°)$.
(d) $\cos(-75°)$.	Ans. 0.26.		(h) $\sin(-210°)$.

8. In which quadrant will the terminal side of the following angles lie?

(a) $\sin\alpha > 0$, $\cos\alpha < 0$.
(b) $\sin\beta < 0$, $\tan\beta > 0$.
(c) $\cos\theta > 0$, $\tan\theta < 0$.
(d) $\tan\varphi < 0$. $\cos\varphi < 0$.

9. Prove that the eight Fundamental Relations for trigonometric functions of an acute angle hold for any angle.
10. Find the value of the six trigonometric functions of the angle which when placed in standard position will have its terminal side passing through the point:
    (a) $(-3, 4)$. (c) $(-4, -3)$.
    (b) $(12, -5)$. (d) $(5, -7)$.

32. FINDING THE VALUES OF THE FUNCTIONS OF ANY ANGLE. REDUCTION TO FUNCTIONS OF AN ACUTE ANGLE

The tables of trigonometric functions of angles from 0° to 90° can be used to determine the value of the function of *any* angle. Assuming that h and k are positive numbers, the points $P(h, k)$,

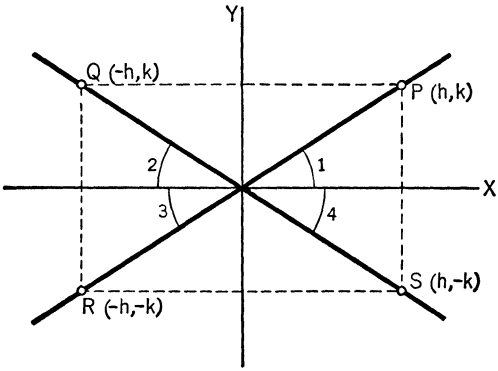


FIGURE 35.

$Q(-h, k)$, $R(-h, -k)$, and $S(h, -k)$ can be plotted as shown in the accompanying figure. The *numerical values** of the coordinates of the four points are equal, and the absolute distance r of each point from the origin is $\sqrt{h^2 + k^2}$. From this, it follows that angles in standard position whose terminal sides pass through P, Q, R, and S, respectively, have *trigonometric functions of the same numerical value.* Of course, the signs of the functions of the various angles depend upon the quadrant in which the terminal side lies, as was explained in Art. 31. Since angles 1, 2, 3, and 4 are equal, it can be stated that *any function of any angle* may be reduced to the same function of a *positive acute angle* by making use of the following three suggestions:

* By the numerical or absolute value of a quantity is meant its value independent of sign. Thus the numerical value of -3 is 3. The numerical value of a quantity is often represented by vertical bars; thus $|-3| = 3$.

(a) *Sketch the angle in standard position.*

(b) *Determine the sign of the desired trigonometric function of the angle from the signs of x and y at a point on its terminal side.*

(c) *The numerical value of the desired function is equal to that of the same function of the acute angle formed by the X-axis and the terminal side of the given angle.*

EXAMPLE 1. Find cos 217° 24′.

SOLUTION: Sketching the angle in standard position shows that its terminal side lies in the third quadrant. Hence the sign of the cosine, $\frac{x}{r}$, is negative. From the third suggestion made previously, the numerical value of cos 217° 24′ is found to be the same as that of cos 37° 24′. Hence

$$\cos 217° 24' = -\cos 37° 24' = -0.7944.$$

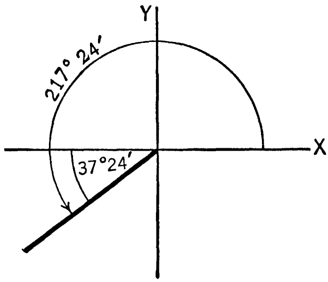


FIGURE 36.

EXAMPLE 2. Find tan 100° 30′.

SOLUTION: The angle is in the second quadrant. The sign of $\frac{y}{x}$, the tangent of the angle, is obviously negative in this quadrant. Why? Thus

$$\tan 100° 30' = -\tan (180° - 100° 30') = -\tan 79° 30' = -5.396.$$

EXAMPLE 3. Given $\sin \alpha = -0.4067$ and $\cos \alpha = 0.9135$, find α.

SOLUTION: The terminal side of α lies in the fourth quadrant because the sign of the sine is negative and the sign of the cosine is positive. The tables give sin 24° = 0.4067, and cos 24° = 0.9135.

Hence $$\alpha = 360° - 24° = 336°.$$

EXERCISES

1. Express the following functions in terms of functions of acute angles:

(a) cos 160°. ANS. − cos 20°.

(b) sin 205°. ANS. − sin 25°.

(c) tan 260°. ANS. tan 80°.

(d) cos (− 85°). ANS. cos 85°.

(e) $\sin \frac{5\pi}{7}$. ANS. $\sin \frac{2\pi}{7}$.

(f) $\sec \frac{-8\pi}{5}$.

(g) $\cos \frac{7\pi}{8}$.

(h) tan 140°.

(i) sin 92°.

(j) $\cos \frac{9\pi}{5}$.

2. Recalling the values of the trigonometric functions of angles of 30°, 45°, and 60°, find without tables the values of the following functions:

(a) $\sin \frac{2\pi}{3}$. ANS. $\frac{\sqrt{3}}{2}$.

(b) $\cos \frac{5\pi}{6}$. ANS. $-\frac{\sqrt{3}}{2}$.

(c) $\tan \frac{5\pi}{4}$. ANS. 1.

(d) $\sin \frac{11\pi}{6}$. ANS. -0.5.

(e) $\csc \frac{-5\pi}{4}$. ANS. $\sqrt{2}$.

(f) $\cos \frac{2\pi}{3}$.

(g) $\tan \frac{3\pi}{4}$.

(h) $\sin \frac{3\pi}{4}$.

(i) $\cos \frac{-\pi}{4}$.

(j) $\sin \frac{-2\pi}{3}$.

3. Find the angle x determined by the conditions given in each instance in the following:

(a) $\sin x = 0.5$, $\cos x = -\frac{\sqrt{3}}{2}$. ANS. $x = \frac{5\pi}{6}$.

(b) $\sin x = -\frac{1}{\sqrt{2}}$, $\cos x$ is positive.

(c) $\sin x = \cos x$, both negative. ANS. $x = 225°$.

(d) $\sin x = -0.5736$, $\cos x = 0.8192$.

(e) $\cos x = -\frac{1}{\sqrt{2}}$, $\tan x = -1$. ANS. $x = \frac{3\pi}{4}$.

(f) $\sin x = 0.5$, $\cot x = \sqrt{3}$.

33. VALUE OF THE FUNCTIONS OF THE QUADRANTAL ANGLES

An angle in standard position whose terminal side coincides with one of the perpendicular axes is called a **quadrantal angle.** Angles of 0°, 90° (or $\frac{\pi}{2}$ radians), 180° (or π), 270° $\left(\text{or } \frac{3\pi}{2}\right)$, 360° (or 2π), and their negatives are those usually considered. (What others are there?) The values of the trigonometric functions of these angles can readily be determined from the definitions of the functions. As before, let (x, y) be a point on the terminal side of the angle, and let r be the distance from the point (x, y) to the origin. It is obvious (see Fig. 37) that for an angle of 0°, the point (x, y) has the coordinates $x = r$ and $y = 0$. Hence,

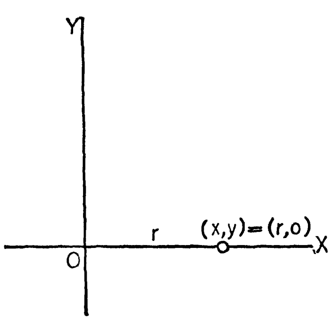


FIGURE 37.

$$\sin 0° = \frac{y}{r} = \frac{0}{r} = 0,$$

$$\cos 0° = \frac{x}{r} = \frac{r}{r} = 1,$$

$$\tan 0° = \frac{y}{x} = \frac{0}{r} = 0,$$

$$\cot 0° = \frac{x}{y} = \frac{r}{0}, \text{ not defined,*}$$

$$\sec 0° = \frac{r}{x} = \frac{r}{r} = 1,$$

$$\csc 0° = \frac{r}{y} = \frac{r}{0}, \text{ not defined.}$$

Again, for an angle of 90° (or $\frac{\pi}{2}$ radians) the point (x, y) on the terminal side makes $x = 0$, and $y = r$. Similar observations can be made for the other quadrantal angles. The student should work the following exercise with care, not with the view of memorizing the results, but with the purpose of mastering the idea and method so that the values of the functions of these angles can readily be determined mentally when needed.

EXERCISE

Copy and complete the following table of functions:

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin \theta$	0	1			
$\cos \theta$	1	0			
$\tan \theta$	[illegible]	∞			
$\cot \theta$	∞	0			
$\sec \theta$	1	∞			
$\csc \theta$	∞	1			

* In mathematics, division by zero is not a permissible operation, so that the fraction $\frac{r}{0}$ is not defined. However, as a variable positive angle approaches zero, the ratio $\frac{x}{y}$ becomes infinitely large as y approaches zero. The symbol ∞ may be used as an abbreviation for this state of affairs. Note that ∞ cannot be considered in this sense as a number. On the other hand, as a variable negative angle approaches zero, its cotangent becomes negatively infinite. This emphasizes the fact that cot 0° is not defined. The same is true for csc 0°, tan $\frac{\pi}{2}$, and others.

34. FUNCTIONS OF NEGATIVE ANGLES

Let θ be any positive angle, and $-\theta$ its negative. Consider them in standard position as in Fig. 38. Let P and P_1 be the respective points of intersection of the terminal sides of θ and $-\theta$ with the unit circle. Let their rectangular coordinates be denoted by (x, y) and (x', y'), respectively.

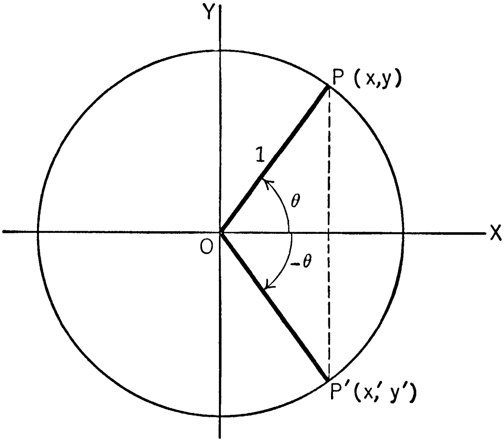


FIGURE 38.

Then for every value of θ,

$$x' = x, \qquad y' = -y.$$

But

$$x' = \cos(-\theta),\ y' = \sin(-\theta),$$
$$x = \cos\theta, \qquad y = \sin\theta.$$

Hence, by substitution,

(20) $$\sin(-\theta) = -\sin\theta,$$

(21) $$\cos(-\theta) = \cos\theta.$$

Moreover from (20), (21), and the fact that $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$, for all values of α for which the functions are defined,

(22) $$\tan(-\theta) = -\tan\theta.$$

The student can establish corresponding formulas for $\cot(-\theta)$, $\sec(-\theta)$, and $\csc(-\theta)$.

EXERCISES

Express the following in terms of functions of positive angles and find their values:

1. $\sin(-45°)$. Ans. $-\sin 45° = -\dfrac{1}{\sqrt{2}}$.

2. $\sec(-510°)$.

3. $\cos(-60°)$. Ans. $\cos 60° = 0.5$.

4. $\csc(-330°)$.

5. tan (– 225°). Ans. – tan 45° = – 1.

6. sin (– 210°).

7. cot (– 120°). Ans. – cot 120° = $\frac{1}{\sqrt{3}}$.

8. cos (– 135°).

35. LOCI IN POLAR COORDINATES

As a variable angle θ in standard position varies from 0 to 2π, its terminal side sweeps over the entire plane so that the location of a given point in the plane can be precisely expressed by giving (1) the value of θ for which the terminal side passes through the point, and (2) the directed distance of the point from the origin, denoted by the symbol ρ. The numbers ρ and θ which locate a point P are called the **polar coordinates** of P. Corresponding to (x, y) for the rectangular coordinates of P, the polar coordinates are given by (ρ, θ).

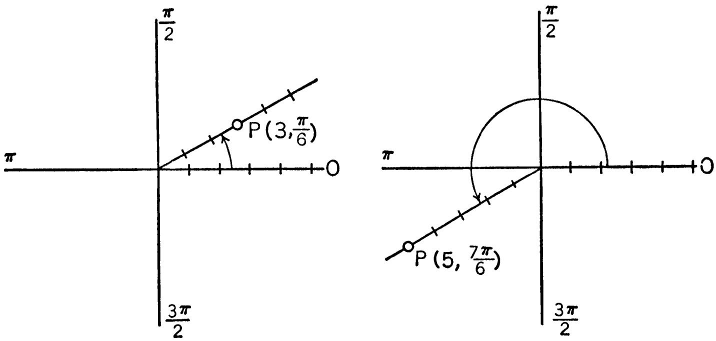


FIGURE 39.

The quantity ρ differs from the distance r, used in the definitions of the trigonometric functions, in that it is a signed number. Positive values of ρ are defined as distances from the origin measured along the terminal side of θ. Negative values of ρ are defined as distances from the origin measured along the *extension of the terminal side of θ through the origin.*

Thus the point $\rho = -4$, $\theta = \frac{\pi}{6}$, or $\left(-4, \frac{\pi}{6}\right)$ is shown in Fig. 40.

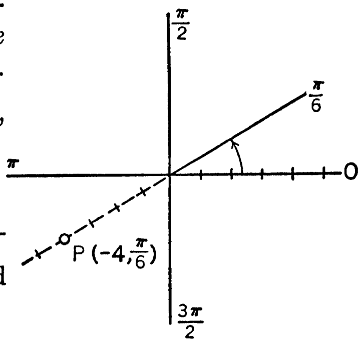


FIGURE 40.

Note that the points whose coordinates are $\left(-4, \frac{\pi}{6}\right)$, $\left(4, \frac{7\pi}{6}\right)$, and $\left(4, -\frac{5\pi}{6}\right)$ are identical in position.

Any equation expressing ρ as a function of θ restricts the variable point (ρ, θ) to occupying only those positions for which the values of ρ and θ satisfy the equation. Hence, by substitution, the locus can be determined by a sufficient number of points, connected in order of value of θ.

The problem of graphing equations in polar coordinates is not essentially different from the problem of graphing equations in cartesian coordinates. The polar equation may, if necessary, usually be solved for ρ explicitly in terms θ and a table of values of ρ corresponding to selected values of θ be computed. In general, the quadrantal values of θ should be included. In most cases a limited range of values of θ will give the complete graph.

EXAMPLE. Trace the locus of the equation $\rho = 4 \sin \theta$.

SOLUTION: The loci of simple equations can usually be satisfactorily traced by locating the points corresponding to $\theta = 0$, $\theta = \frac{\pi}{6}$, $\theta = \frac{\pi}{3}$, etc., $\left(\text{the multiples of } \frac{\pi}{6}\right)$. Other values of θ may be chosen as it seems helpful to do so. It is suggested that the values of ρ and θ be tabulated for the relation $\rho = 4 \sin \theta$ as follows:

θ	$\sin \theta$	$\rho (= 4 \sin \theta)$
0	0	0
$\frac{\pi}{6}$	0.5	2.0
$\frac{\pi}{3}$	0.87	3.48
$\frac{\pi}{2}$	1	4
$\frac{2\pi}{3}$	0.87	3.48
$\frac{5\pi}{6}$	0.5	2
π	0	0
$\frac{7\pi}{6}$	− 0.5	− 2
$\frac{4\pi}{3}$	− 0.87	− 3.48
$\frac{3\pi}{2}$	− 1	− 4
$\frac{5\pi}{3}$	− 0.87	− 3.48
$\frac{11\pi}{6}$	− 0.5	− 4

It will be observed that when θ runs through the third and fourth quadrants, ρ is negative, thus locating P in the first and second quadrants, respectively. In fact, for this particular equation the point P *retraces* the path followed as θ varied from 0 to π. By plotting these points and joining them in order with a smooth curve, the locus shown in Fig. 41 is obtained. This seems to be a circle. This can be further checked by trying other values of θ. (The fact that it *is* a circle is proved in courses in analytic geometry.)

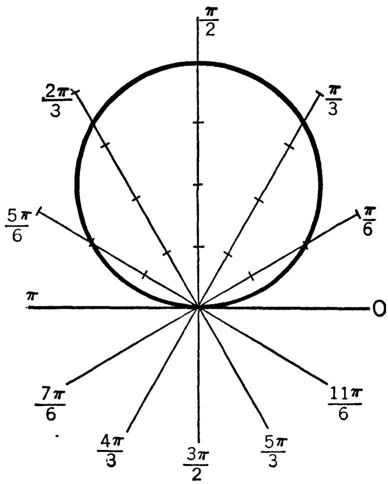


FIGURE 41.

EXERCISES

1. Trace the following loci in polar coordinates:

(a) $\rho = 6 \cos \theta$.

(b) $\rho = 4(1 - \cos \theta)$.

(c) $\rho = 2 - 4 \sin \theta$.

(d) $\rho = \dfrac{8}{2 - \sin \theta}$.

(e) $\rho = 6 \cos 2\theta$.

(f) $\rho = 6 \sin 3\theta$.

(g) $\rho = 3 \cos \theta + 4 \sin \theta$.

(h) $\rho = 4 \sin^2 \theta$.

(i) $\rho \cos \theta - 4 = 0$.

(j) $\rho^2 = 4 \cos 2\theta$.

(k) $\rho = 6 \tan^2 \theta \sec \theta$.

(l) $\theta^2 = 1$.

(m) $\rho = 2\theta$.

2. Verify that the formulas $x = \rho \cos \theta$, $y = \rho \sin \theta$ are relations which can be used to change the equation of a locus from rectangular coordinates to polar coordinates.

Chapter VI

VARIATION AND GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

36. THE SINE AND THE COSINE

Let θ be a variable angle placed in standard position. Let $P(x, y)$ be the point at which the terminal side of θ intersects the circle of unit radius whose center is the origin. Let MP and OM be the ordinate and abscissa of P. Then, for any quadrant,

$$\sin\theta = \frac{y}{r} = \frac{MP}{1} = MP,$$

$$\cos\theta = \frac{x}{r} = \frac{OM}{1} = OM.$$

The student's attention is called to the fact that MP and OM are directed distances, MP being positive or negative as the direction from M to P is up or down and OM being positive or negative as the direction from O to M is right or left.

Although the trigonometric functions are *ratios,* it is thus possible to construct a line segment whose measure is equal to the sine or to the cosine of any angle that is desired. It should be noted that when MP extends above the horizontal axis, its sign is positive, and when below this axis the sign is negative. Similarly the sign of OM is positive or negative according as it extends to the right or to the left of the vertical axis. These directed segments are known as the **line values** of the functions.

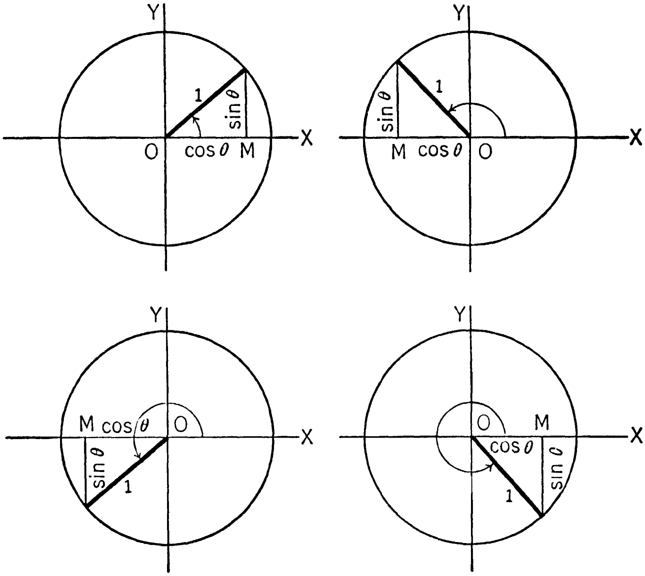


FIGURE 42.

The figure gives four isolated views of θ, but a motion picture in in which θ varies continuously from 0° to 360° would give an excellent illustration of the variation of sin θ and cos θ as functions of a varying angle. As θ increases from 0° to 90°, sin θ increases from 0 to 1; as θ increases from 90° to 270°, sin θ decreases from + 1 through 0 to − 1; as θ continues to increase from 270° to 360°, sin θ increases from − 1 to 0. If now θ continues to increase from 360° to 720°, sin θ repeats the set of values above described. If θ continues to increase indefinitely, sin θ periodically repeats the same set of values for every 360° added to θ. The same sort of thing happens if θ is made to vary in a negative sense. The variation of sin θ can be shown graphically with the aid of a rectangular coordinate system, using values of θ * as abscissas, and the corresponding values of sin θ as ordinates. The results appear in Fig. 43.

A function which repeats itself in this fashion is called a **periodic** function. The interval over which the independent variable runs before the function begins its repetition is called the **period** of the function. Thus the period of the sine is 360° or 2π.

*** θ is here expressed in radians because this enables the use of the same unit on both axes. Degrees could have been used equally well by choosing a proper scale on the horizontal axis.**

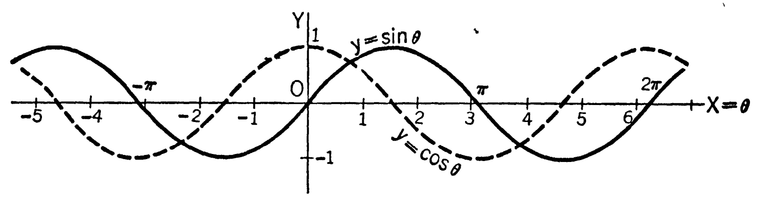


FIGURE 43.

A similar study of the variation of cos θ discloses that cos θ is also a periodic function of period 2π. Its graph as shown in Fig. 43 is seen to differ from that of sine only in its position along the θ axis. The graphical relation can be stated algebraically by the equation

$$\sin(\theta + 90°) = \cos\theta.$$

This relation will be studied more precisely later on.

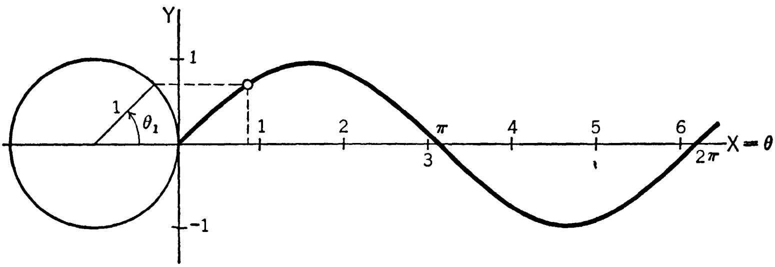


FIGURE 44.

Using the unit circle for obtaining line values of sin θ, the graph of sin θ can be traced accurately and quickly. Draw a unit circle with its center on the X-axis as shown. Choose a value of θ, say θ_1, for which the corresponding point on the graph is desired. Construct this angle as a central angle in the circle, and also measure off on the X-axis the number of radians it contains. Through the point at which the terminal side of θ intersects the circle draw a horizontal line, and through the point corresponding to the chosen value of θ on the X-axis draw a vertical line. The intersection point of these two lines is obviously the required point of the graph, since its ordinate is equal to the line value of sin θ_1. This can be repeated for as many values of θ as are needed to trace the curve.

If the circle is rotated counter-clockwise through 90°, the ordinate of the point located by the intersection of the corresponding vertical and horizontal lines is the line value of cos θ_1. Hence the locus of the points so obtained for all values of θ is the graph of cos θ. This rotation of the circle through 90° again calls attention to the relation

$$\sin (90° + \theta) = \cos \theta.$$

It can be remarked here that graphs of sin θ and cos θ are extensively used in describing various types of wave motion, as for example in sound waves, and certain other periodic phenomena, such as alternating electric currents.

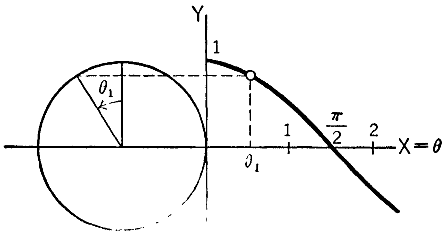


FIGURE 45.

If $f(x)$ represents the deviation of any periodic function of the variable x from its mean value, the greatest value of $|f(x)|$ is called the **amplitude** of the function $f(x)$. Apply this definition to the periodic functions sin θ and cos θ. The amplitude of each is 1, since the greatest value of $|\sin \theta| = 1$ and of $|\cos \theta| = 1$. The student may already be acquainted with the term amplitude as the measure of the deviation from the middle position, for instance the amplitude of the swing of a pendulum bob is the distance traversed by the bob in swinging from the rest position to an extreme point of its arc.

Every ordinate of the graph of the function $f(\theta) = a \sin \theta$, where a is any constant, is seen to be a times the magnitude of the corresponding ordinate of the function $f(\theta) = \sin \theta$. Hence the amplitude of the former function is a. The function $a \cos \theta$ obviously has an amplitude of a also. The graph of 3 *sin* θ, with amplitude 3 is shown in Fig. 46, which includes the graph of the function sin θ for purposes of comparison. Note that the *period* is the same for both functions, namely, 2π.

The function $f(\theta) = 3 \sin 2\theta$ is shown graphically in Fig. 47. Note that its period is π and its amplitude is 3. In general it is not difficult to verify that the functions $a \sin b\theta$ and $a \cos b\theta$ both have periods of $\frac{2\pi}{b}$ and amplitudes of a.

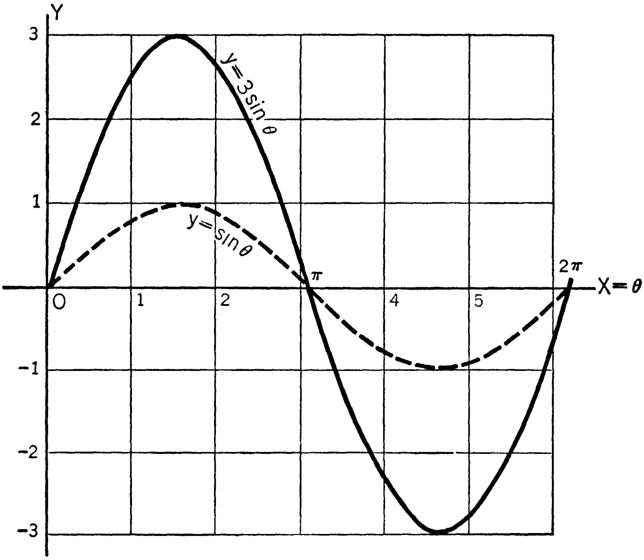


FIGURE 46.

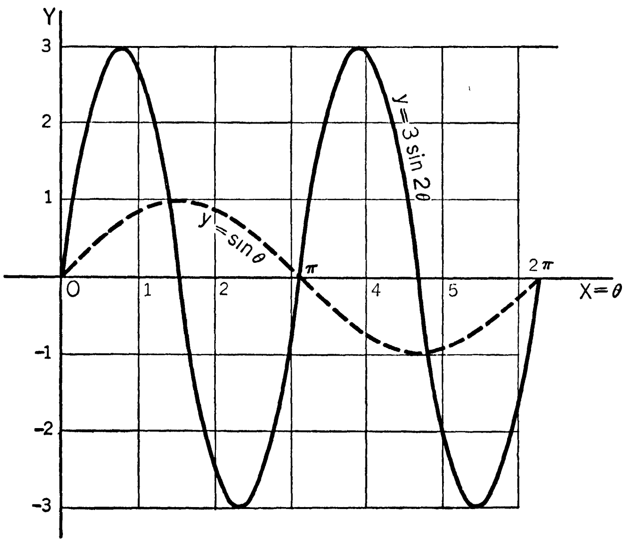


FIGURE 47.

37. THE TANGENT AND COTANGENT

Line values of $\tan \theta$ and $\cot \theta$ can be easily obtained. The idea, as with the sine and cosine functions, is to express each of them as the ratio of the directed lengths of two directed line segments, the segment appearing in the denominator being chosen of length unity, thus making the directed length of the other segment equal to the ratio itself. As before, draw a circle of radius unity and let θ be the central angle in standard position. Construct tangent lines to the circle as shown in the figure.

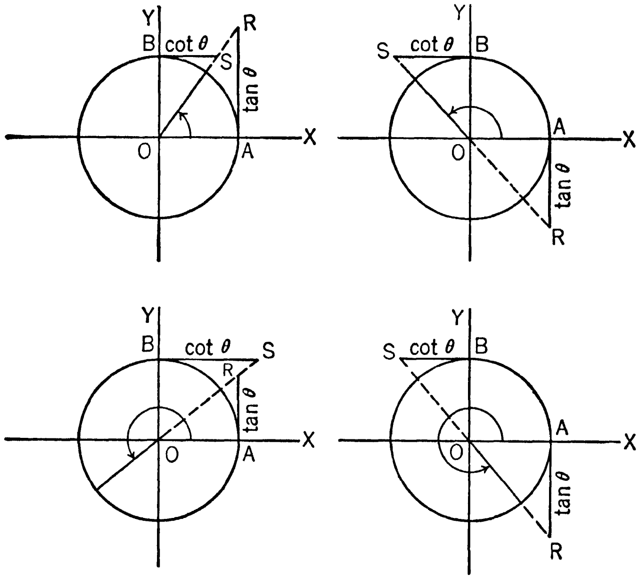


FIGURE 48.

Then in any quadrant, $\tan \theta = \dfrac{AR}{OA} = \dfrac{AR}{1} = AR,$

and $\cot \theta = \dfrac{BS}{OB} = \dfrac{BS}{1} = BS.$

In order to obtain the proper interpretation of the sign of the function, AR is considered positive when extending upward and negative when extending downward. Similarly, BS is considered positive or negative according as it extends to the right or the left.

Using the line values of $\tan \theta$, the graph is obtained as shown in Fig. 49.

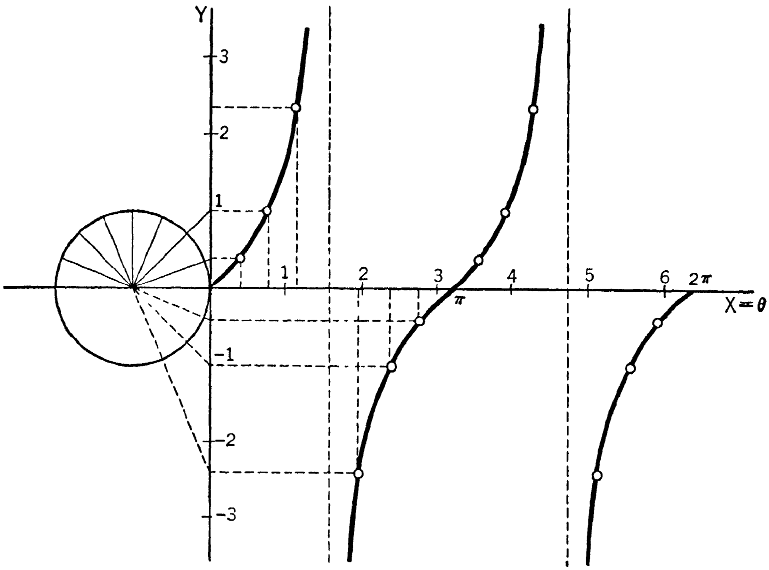


FIGURE 49.

For clearness, construction lines are shown only for values of θ in the first and second quadrants. Observe the variation of $\tan \theta$ as θ varies continuously from 0 to 2π. Tan θ increases from 0 to $+\infty$ as θ varies from 0 to $\frac{\pi}{2}$, increases from $-\infty$ to 0 as θ increases from $\frac{\pi}{2}$ to π, increases from 0 to $+\infty$ as θ increases from π to $\frac{3\pi}{2}$, and increases from $-\infty$ to 0 as θ increases from $\frac{3\pi}{2}$ to 2π. Note that $\tan \theta$ repeats the set of values in the interval from π to 2π that it has from 0 to π. Hence $\tan \theta$ is a periodic function of period π.

38. COMPOSITE FUNCTIONS. ADDITION OF ORDINATES

Occasions arise in mathematics when it is necessary to consider functions which involve sums of trigonometric functions. For example, the equation

(a) $$y = 2 \sin x + \sin (2x) \; *$$

can represent a certain musical sound in which an overtone is present. Other examples of composite functions are:

(b) $$y = \frac{x}{3} + \sin x,$$

(c) $$y = \sin x - \cos x, \text{ etc.}$$

The graph of such functions can be obtained in the usual manner by substituting values of x to obtain the values of y and plotting the corresponding points. However, considerable labor can be saved if the loci of the separate functions are first traced, and then are combined into the composite graph by the method of addition of ordinates. The method is best shown by an example. Consider equation (*b*) above. Trace the loci of the two functions, $y = \frac{x}{3}$ and $y = \sin x$, upon a single set of axes. The first is a straight line and its graph is familiar to the student of algebra.

Since the original function, $y = \frac{x}{3} + \sin x$, is the sum of two functions, the ordinate of its graph corresponding to a chosen value of x is obviously the *algebraic* sum of the ordinates of the graphs of the separate functions corresponding to the same value of x. The

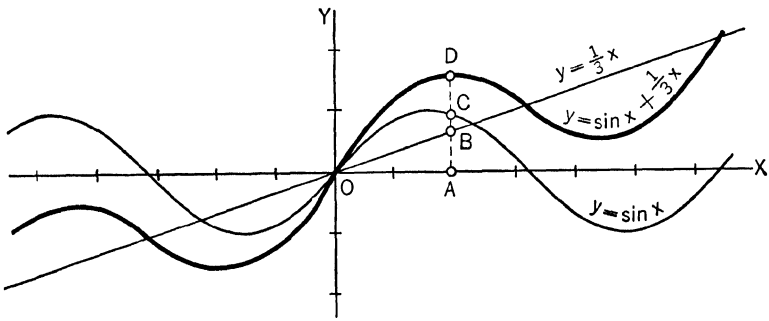


FIGURE 50.

* Heretofore the use of the symbol x to represent the variable angle has been avoided in order that the student might not confuse that symbol with the x and y used in representing rectangular coordinates of a point. However, the use of x to represent the variable angle is common in mathematical works and should become familiar to the student. In fact, since x can assume any real value, $\sin x$ can be looked upon as a trigonometric function of the real variable x, with no thought of angles being involved.

word *algebraic* is used because due regard must be given to the sign of the separate ordinates. Thus for $x = 0$, both $\frac{x}{3}$ and $\sin x$ are 0, hence (0, 0) is a point on the curve. In general when $x = a$, $\frac{x}{3} = AB$ and $\sin x = AC$ (see Fig. 50). Hence the ordinate AD of the composite function is

$$AD = AB + AC.$$

These ordinates can be added graphically, without computing them, by means of dividers, or by marking the edge of a strip of paper to record the lengths.

The graph of the equation (a), $y = 2 \sin x + \sin 2x$ is shown in Figure 51.

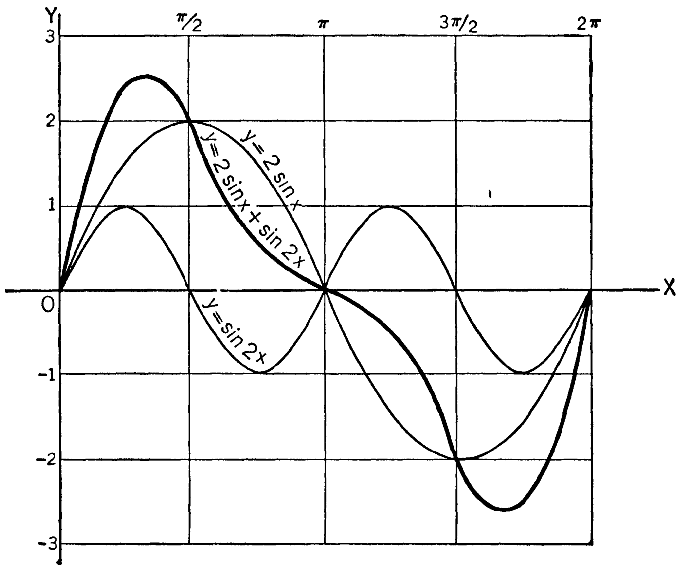


FIGURE 51.

The graphs of $y = 2 \sin x$ and $y = \sin 2x$ are first constructed and then the ordinates of the two graphs for the same value of x are added to obtain the corresponding ordinate for the composite graph. It should be noted that the sum of the two trigonometric functions is, in this case, a periodic function whose period is 2π and consequently the graph may be continued indefinitely to the left and to the right.

EXERCISES

1. Complete the following table on variation of the functions: (The abbreviation I means "increases from"; D means "decreases from.")

Quadrant	*I*	*II*	*III*	*IV*
θ	I 0 to $\frac{\pi}{2}$	I $\frac{\pi}{2}$ to π	I π to $\frac{3\pi}{2}$	I $\frac{3\pi}{2}$ to 2π
$\sin\theta$	I 0 to 1	D 1 to 0	D 0 to -1	I -1 to 0
$\cos\theta$				
$\tan\theta$				
$\cot\theta$				
$\sec\theta$				
$\csc\theta$				

2. Plot the graph of $\cos\theta$, using the unit circle and the line values as suggested previously.

3. Plot the graph of $\cot\theta$.

4. The statement that a variable x ranges over all values from a to b, inclusive, is abbreviated mathematically by the expression: $a \leqq x \leqq b$. Express in this form the range of variation of $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

5. Using the values of the functions of angles of 0°, 30°, 45°, and their multiples, plot the graphs of:

(a) $y = \sec x$, (b) $y = \csc x$.

6. State the period and amplitude of each of the following functions and sketch their graphs:

(a) $y = 2\sin 3x$, (b) $y = 4\cos 2x$.

7. By the method of addition of ordinates, trace the loci represented by the following equations:

(a) $y = \sin x + \cos x$, (c) $y = 2\sin x - \sin 2x$,

(b) $y = x + \cos x$, (d) $y = \frac{x^2}{4} + \sin x$.

8. What may be said concerning the amplitude of the function $f(x) = \tan x$?

Chapter VII

FUNCTIONS INVOLVING MORE THAN ONE ANGLE

39. INTRODUCTION

In addition to the eight fundamental relations of the trigonometric functions considered previously, there is another important group of formulas expressing such trigonometric expressions as $\sin(\theta + \varphi)$, $\tan(\theta - \varphi)$, $\cos 2\theta$, etc., in terms of the proper functions of the single angles. These formulas find frequent use in the applications of trigonometry.

The student is warned to avoid such errors as that of assuming $\cos(\theta + \varphi)$ to be equivalent to $\cos\theta + \cos\varphi$. This fallacy should not tempt anyone who remembers that $\cos(\theta + \varphi)$ is read "the cosine *of* $(\theta + \varphi)$" and not "the cosine *times* $(\theta + \varphi)$." The fact that $\cos(\theta + \varphi)$ is not, in general, equal to $\cos\theta + \cos\varphi$ is readily shown by substituting particular values of θ and φ, say $\theta = 60°$, $\varphi = 30°$, in the two expressions.

Thus, $$\cos(60° + 30°) = \cos 90° = 0.$$

But $$\cos 60° + \cos 30° = \tfrac{1}{2} + \frac{\sqrt{3}}{2}.$$

Since 0 is not equal to $\frac{1}{2} + \frac{\sqrt{3}}{2}$, it is evident that

$$\cos(60° + 30°) \neq \cos 60° + \cos 30°.$$

The following articles state and verify the correct formulas for expressing $\cos(\theta + \varphi)$, $\cos(\theta - \varphi)$, $\sin(\theta + \varphi)$, $\sin(\theta - \varphi)$, $\tan(\theta + \varphi)$, and $\tan(\theta - \varphi)$ in terms of functions of θ and φ alone.

The formulas are frequently called the **addition formulas** of trigonometry. The **double angle formulas** for $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, and the **half angle formulas** for $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$, and $\tan \frac{\theta}{2}$ will also be developed.

40. THE COSINE ADDITION FORMULAS

Given any two angles θ and φ, the following formulas are true:

(23) $\qquad \cos(\theta + \varphi) = \cos\theta \cos\varphi - \sin\theta \sin\varphi,$

(24) $\qquad \cos(\theta - \varphi) = \cos\theta \cos\varphi + \sin\theta \sin\varphi.$

PROOF OF (23). Place angles θ, $\theta + \varphi$, and $-\varphi$ in standard position, let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ be the respective points of intersection of their terminal sides with the unit circle, and also let $P_4(x_4, y_4)$ be the point $(1, 0)$. Then $\overline{P_2P_4} = \overline{P_1P_3}$, because they are chords of the unit circle determined by central angles, both of measure $|\theta+\varphi|$. Therefore, by (19) the formula for the distance between two points,

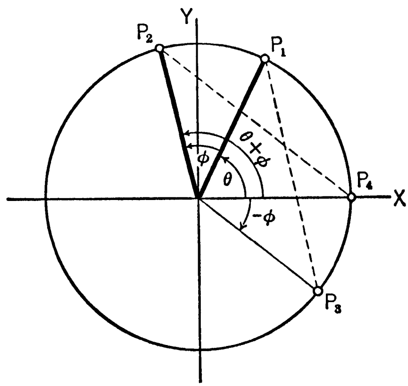


FIGURE 52.

$$\sqrt{(x_2 - x_4)^2 + (y_2 - y_4)^2} = \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2}.$$

But
$$\begin{aligned} x_1 &= \cos\theta, & y_1 &= \sin\theta, \\ x_2 &= \cos(\theta + \varphi), & y_2 &= \sin(\theta + \varphi), \\ x_3 &= \cos(-\varphi) = \cos\varphi, & y_3 &= \sin(-\varphi) = -\sin\varphi, \\ x_4 &= 1, & y_4 &= 0. \end{aligned}$$

Hence the equation of distances becomes

$$\sqrt{[\cos(\theta+\varphi) - 1]^2 + \sin^2(\theta+\varphi)} = \sqrt{(\cos\theta - \cos\varphi)^2 + (\sin\theta + \sin\varphi)^2}.$$

With both sides squared and terms collected, this becomes with the aid of (6),

$$2 - 2\cos(\theta + \varphi) = 2 - 2\cos\theta\cos\varphi + 2\sin\theta\sin\varphi.$$

This simplifies to the required formula,

$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi.$$

PROOF OF (24). Formula (23) has been verified for any angles θ and φ, positive or negative. Hence by means of (23),

$$\cos(\theta - \varphi) = \cos(\theta + [-\varphi]) = \cos\theta\cos[-\varphi] - \sin\theta\sin[-\varphi].$$

But $\cos[-\varphi] = \cos\varphi$, and $\sin[-\varphi] = -\sin\varphi$.

Hence $\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$.

EXAMPLE 1. Find the exact value of cos 75°. (The value found in the table is only *approximately* correct.)

SOLUTION: Express 75° in terms of angles whose trigonometric functions are known exactly. Thus

$$75^\circ = 45^\circ + 30^\circ.$$

$$\begin{aligned} \therefore \qquad \cos 75^\circ &= \cos(45^\circ + 30^\circ) \\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\ &= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}. \quad \text{Ans.} \end{aligned}$$

NOTE: This exact answer must be left in radical form. Why?

EXAMPLE 2. Simplify the expression $\cos(\pi - \theta)$.

SOLUTION: From (24),

$$\begin{aligned} \cos(\pi - \theta) &= \cos\pi\cos\theta + \sin\pi\sin\theta \\ &= (-1)\cos\theta + (0)\sin\theta \\ &= -\cos\theta. \quad \text{Ans.} \end{aligned}$$

EXAMPLE 3. Express $\cos 4\alpha - \cos 6\alpha$ in the form of a product.

SOLUTION: $4\alpha = 5\alpha - \alpha$, and $6\alpha = 5\alpha + \alpha$. Hence

$$\begin{aligned} \cos 4\alpha - \cos 6\alpha &= \cos(5\alpha - \alpha) - \cos(5\alpha + \alpha) \\ &= \cos 5\alpha\cos\alpha + \sin 5\alpha\sin\alpha - (\cos 5\alpha\cos\alpha - \sin 5\alpha\sin\alpha) \\ &= 2\sin 5\alpha\sin\alpha. \quad \text{Ans.} \end{aligned}$$

EXERCISES

1. By using 105° = 45° + 60°, prove that

$$\cos 105^\circ = \frac{\sqrt{2} - \sqrt{6}}{4}.$$

2. Find the exact value of cos 15°.
3. Simplify $\cos 5x \cos 2x + \sin 5x \sin 2x$. ANS. $\cos 3x$.
4. Simplify cos 70° cos 50° − sin 70° sin 50°.
5. After the manner of Example 2, verify the following simplification or *reduction* formulas:

(a) $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$.

(b) $\cos(\pi + \theta) = -\cos\theta$.

(c) $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$.

(d) $\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$.

(e) $\cos(2\pi - \theta) = \cos\theta$.

(f) $\cos(2\pi + \theta) = \cos\theta$.

6. State formulas (23) and (24) in words.
7. Express $\cos 50° + \cos 40°$ as a product by using $50° = 45° + 5°$ and $40° = 45° - 5°$, and simplifying the result. ANS. $\sqrt{2} \cos 5°$.
8. Simplify $\cos(x + y) + \cos(x - y)$.
9. Simplify $\cos 100° - \cos 80°$. ANS. $-2 \sin 10°$.

41. GENERALIZATION OF THE RELATIONS $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$, AND $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

In Art. 11 it was shown that for any acute angle θ,

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \tag{25}$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta. \tag{26}$$

It is now possible to show that these relations hold for any value of θ.

By (24),
$$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta,$$
$$= \sin\theta,$$

which proves (25) for the general case.

Since (25) is now known to be true for any value of θ, by substituting
$$\theta = \frac{\pi}{2} - \alpha$$
in (25), there results
$$\cos\left(\frac{\pi}{2} - \left[\frac{\pi}{2} - \alpha\right]\right) = \sin\left(\frac{\pi}{2} - \alpha\right),$$
or
$$\cos\alpha = \sin\left(\frac{\pi}{2} - \alpha\right).$$
This holds for any value of α.

42. THE SINE ADDITION FORMULAS

The following two formulas are true for any values of the angles θ and φ:

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi, \tag{27}$$
$$\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi. \tag{28}$$

PROOF OF (27). From (25), followed by (24), (25), and (26),

$$\begin{aligned}\sin(\theta + \varphi) &= \cos\left(\frac{\pi}{2} - [\theta + \varphi]\right)\\ &= \cos\left(\left[\frac{\pi}{2} - \theta\right] - \varphi\right)\\ &= \cos\left(\frac{\pi}{2} - \theta\right)\cos\varphi + \sin\left(\frac{\pi}{2} - \theta\right)\sin\varphi\\ &= \sin\theta\cos\varphi + \cos\theta\sin\varphi.\end{aligned}$$

PROOF OF (28). It is obvious that

$$\sin(\theta - \varphi) = \sin(\theta + [-\varphi]).$$

Hence, from (27), (20), and (21),

$$\begin{aligned}\sin(\theta - \varphi) &= \sin\theta\cos(-\varphi) + \cos\theta\sin(-\varphi)\\ &= \sin\theta\cos\varphi - \cos\theta\sin\varphi.\end{aligned}$$

EXERCISES

1. Expand $\sin(2\theta \pm 3\varphi)$. ANS. $\sin 2\theta \cos 3\varphi \pm \cos 2\theta \sin 3\varphi$.
2. Show that $\sin(\alpha + \beta + \gamma) = \sin\alpha\cos\beta\cos\gamma - \sin\alpha\sin\beta\sin\gamma + \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma$.
3. Find the exact value of $\sin 15°$ by using $15° = 45° - 30°$. ANS. $\dfrac{\sqrt{6} - \sqrt{2}}{4}$.
4. Find the exact value of $\sin 75°$.
5. Find the exact value of $\sin 105°$. ANS. $\dfrac{\sqrt{6} + \sqrt{2}}{4}$.
6. Verify the following *reduction formulas:*
   - (a) $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$.
   - (b) $\sin(\pi - \theta) = \sin\theta$.
   - (c) $\sin(\pi + \theta) = -\sin\theta$.
   - (d) $\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$.
   - (e) $\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$.
   - (f) $\sin(2\pi - \theta) = -\sin\theta$.
   - (g) $\sin(2\pi + \theta) = \sin\theta$.
7. Simplify $\sin(3x + y) - \sin(3x - y)$. ANS. $2\cos 3x \sin y$.
8. Simplify $\sin(\theta + 12°) + \sin(\theta - 12°)$.
9. Simplify $\sin x^2 \cos 2x^2 + \cos x^2 \sin 2x^2$. ANS. $\sin 3x^2$.
10. Simplify $\sin(x + y)\cos(x - y) + \cos(x + y)\sin(x - y)$.
11. Simplify $\sin 55° + \sin 35°$. ANS. $\sqrt{2}\cos 10°$.
12. If α is a 1st quadrant angle with $\sin\alpha = \frac{5}{13}$, and if δ is a second quadrant angle with $\sin\delta = \frac{3}{5}$, find the value of $\sin(\alpha - \delta)$; $\cos(\alpha + \delta)$.
13. If $\sin\theta = m$, and $\sin\varphi = n$, m and n positive, verify that
$$\sin(\theta + \varphi) = m\sqrt{1 - n^2} + n\sqrt{1 - m^2}.$$
14. Find $\sin 120°$ by using $120° = 60° + 60°$, and check the result.
15. Express $\sin 4x + \sin 6x$ as a product. ANS. $2\sin 5x \cos x$.
16. State formulas (27) and (28) in words.

43. THE TANGENT ADDITION FORMULAS

The two following formulas are valid for all values of θ and φ, except when θ, φ, $\theta + \varphi$, and $\theta - \varphi$ are odd multiples of $\frac{\pi}{2}$:

$$(29) \qquad \tan(\theta + \varphi) = \frac{\tan\theta + \tan\varphi}{1 - \tan\theta\tan\varphi},$$

$$(30) \qquad \tan(\theta - \varphi) = \frac{\tan\theta - \tan\varphi}{1 + \tan\theta\tan\varphi}.$$

PROOF OF (29). From (4), (27) and (23),

$$\tan(\theta + \varphi) = \frac{\sin(\theta + \varphi)}{\cos(\theta + \varphi)}$$

$$= \frac{\sin\theta\cos\varphi + \cos\theta\sin\varphi}{\cos\theta\cos\varphi - \sin\theta\sin\varphi}.$$

Dividing the numerator and denominator of this fraction by $\cos\theta\cos\varphi$,

$$\tan(\theta + \varphi) = \frac{\dfrac{\sin\theta\cos\varphi}{\cos\theta\cos\varphi} + \dfrac{\cos\theta\sin\varphi}{\cos\theta\cos\varphi}}{\dfrac{\cos\theta\cos\varphi}{\cos\theta\cos\varphi} - \dfrac{\sin\theta\sin\varphi}{\cos\theta\cos\varphi}}.$$

Simplifying, $\tan(\theta + \varphi) = \dfrac{\tan\theta + \tan\varphi}{1 - \tan\theta\tan\varphi}$.

The student should verify (30).

EXERCISES

1. Given $\tan\alpha = \frac{1}{2}$ and $\tan\delta = -\frac{3}{2}$. Find $\tan(\alpha + \delta)$, $\tan(\alpha - \delta)$. ANS. $-\frac{4}{7}$, 8.
2. Find tan 75° by using $75° = 45° + 30°$.
3. Given $\tan\alpha = m$ and $\tan\delta = -\frac{1}{m}$. Find the value of $\alpha - \delta$. ANS. 90°.
4. Find tan 150° by using $150° = 75° + 75°$. Why not use $90° + 60°$?
5. Prove that $\tan(45° + \varphi)\tan(45° - \varphi) = 1$.
6. Simplify $\dfrac{\tan(\alpha + \delta) - \tan\alpha}{1 + \tan(\alpha + \delta)\tan\alpha}$. ANS. $\tan\delta$.
7. Show that $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$.
8. Verify the following reduction formulas:
   (a) $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$. $\left(\textit{Hint:}\ \text{Use } \frac{\pi}{2} = \frac{\pi}{4} + \frac{\pi}{4}\right)$
   (b) $\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$.
   (c) $\tan(\pi - \theta) = -\tan\theta$.
   (d) $\tan(\pi + \theta) = \tan\theta$.

44. A NOTE ON THE PERIODICITY OF THE FUNCTIONS SIN α, COS α, AND TAN α

It is now convenient to give a more precise definition of a *periodic function* than was given in Art. 36.

If y is a function of x, it can be expressed by the following symbolism,

$$y = f(x),$$

and y is called a *periodic function* of x of *period* p, if

$$f(x + np) = f(x)$$

where n is any integer. This means that as x varies from 0 to p, $f(x)$ assumes values which are identical with the corresponding values of $f(x \pm p), f(x \pm 2\,p), f(x \pm 3\,p)$, etc., so that $f(x)$ repeats its values over every interval of magnitude p.

Consider the periodicity of the functions $y = \sin \alpha$, $y = \cos \alpha$, and $y = \tan \alpha$ from the above definition.

First, the function $\qquad f(\alpha) = \sin \alpha$
is investigated.

In this case

$$\begin{aligned} f(\alpha + np) &= \sin (\alpha + np) \\ &= \sin \alpha \cos np + \cos \alpha \sin np. \end{aligned}$$

By definition $\sin \alpha$ is a periodic function, of period p, if

$$\sin (\alpha + np) = \sin \alpha.$$

The right-hand member of the preceding equation will equal $\sin \alpha$ provided $\cos np = 1$ and $\sin np = 0$. These conditions are met when and only when np is a multiple of 2π, i.e., when $p = 2\pi$.

Hence

$$\sin (\alpha + n \cdot 2\,\pi) = \sin \alpha,$$

and $\sin \alpha$ is a periodic function of period 2π.

In the same manner,

$$\begin{aligned} \cos (\alpha + n \cdot 2\pi) &= \cos \alpha \cos (2n\pi) - \sin \alpha \sin (2n\pi) \\ &= \cos \alpha(1) - \sin \alpha(0) \\ &= \cos \alpha. \end{aligned}$$

Hence $\cos \alpha$ is a periodic function of period 2π.

Now consider the function $\tan (\alpha + np)$. From (30)

$$\tan (\alpha + np) = \frac{\tan \alpha + \tan np}{1 - \tan \alpha \tan np}.$$

The right-hand member of this equation will reduce to $\tan \alpha$ when $\tan np = 0$, i.e., when $np = n\pi$, or $p = \pi$. Hence

$$\tan (\alpha + n\pi) = \tan \alpha.$$

Thus $\tan \alpha$ is a periodic function of period π.

EXERCISES

1. Show that $\sin \alpha$ cannot have a period π.
2. Show that $\cos \alpha$ cannot have a period π.
3. Find the period of the function $y = \sin 3x$.
   (*Hint:* Let $\sin [3(x + np)] = \sin 3x$, and determine p.) Ans. $\frac{2\pi}{3}$
4. Find the period of the function

$$y = \cos \left(\frac{3x}{2}\right).$$

45. THE DOUBLE ANGLE FORMULAS

If $\varphi = \theta$, the trigonometric functions of $\theta + \varphi$ become the functions of 2θ.

Thus, from (27)

$$\sin 2\theta = \sin (\theta + \theta) = \sin \theta \cos \theta + \cos \theta \sin \theta.$$

Hence

(31) $\quad \mathbf{\sin 2\theta = 2 \sin \theta \cos \theta.}$

Similarly,

(32) $\quad \mathbf{\cos 2\theta = \cos^2 \theta - \sin^2 \theta.}$

Substituting $\cos^2 \theta = 1 - \sin^2 \theta$ in (32) gives

(33) $\quad \mathbf{\cos 2\theta = 1 - 2 \sin^2 \theta.}$

Substituting $\sin^2 \theta = 1 - \cos^2 \theta$ in (32) gives

(34) $\quad \mathbf{\cos 2\theta = 2 \cos^2 \theta - 1.}$

It is left to the student to verify that

(35) $\quad \mathbf{\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}}. \quad (\tan^2 \theta \neq 1)$

EXERCISES

1. Given $\sin \alpha = \frac{4}{5}$, α in the first quadrant; find $\sin 2\alpha$; $\cos 2\alpha$; $\tan 2\alpha$.
   Ans. $\frac{24}{25}$; $-\frac{7}{25}$; $-\frac{24}{7}$.
2. Find $\sin 60°$, $\cos 60°$, and $\tan 60°$, using $60° = 2(30°)$.
3. Given α in the first quadrant and $\tan \alpha = \frac{1}{2}$, find $\sin 2\alpha$; $\cos 2\alpha$; $\tan 2\alpha$. Find 2α in degrees. Ans. $\frac{4}{5}$; $\frac{3}{5}$; $\frac{4}{3}$; $53°8'$.
4. Prove that the area of a triangle ABC in which $C = 90°$ is $\frac{1}{4}c^2 \sin 2\alpha$.

5. Find the area of a right triangle with hypotenuse 200 feet and with one acute angle equal to 32° 15′. ANS. 9026 sq. ft.
6. Verify the following identities:

(a) $\dfrac{1 + \cos 2\alpha}{\sin 2\alpha} = \cot \alpha.$

(b) $\cot \delta + \tan \delta = \dfrac{2}{\sin 2\delta}.$

(c) $\cos 2u \cos 2v = 2 \sin^2 u \sin^2 v + 2 \cos^2 u \cos^2 v - 1.$

(d) $\cos 4\alpha = 1 - 8 \sin^2 \alpha + 8 \sin^4 \alpha.$

(e) $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}.$

(f) $\sin 3y = 2 \sin \dfrac{3y}{2} \cos \dfrac{3y}{2}.$

7. In solving certain examples in calculus it is necessary to express the second degree functions $\sin^2 x$ and $\cos^2 x$ in terms of *first* degree functions. With the aid of (33) and (34), verify the following relations:

$$\sin^2 x = \frac{1 - \cos 2x}{2},$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}.$$

8. Without tables find the value of 2 sin 15° cos 15°.
9. Express $\cos^2 10\alpha - \sin^2 10\alpha$ as a single function. ANS. $\cos 20\alpha$.
10. Show that in any right triangle in which α and β are the acute angles

$$\sin 2\alpha = \sin 2\beta.$$

46. THE HALF ANGLE FORMULAS

If (33) is solved for $\sin \theta$, there results,

$$\sin \theta = \pm \sqrt{\frac{1 - \cos 2\theta}{2}},$$

where the sign before the radical is determined by the quadrant in which angle θ terminates. Since this result is true for any value of θ, it is true for $\theta = \frac{\varphi}{2}.$ Hence, by substitution, there results,

$$\textbf{(36)} \qquad \sin \frac{\varphi}{2} = \pm \sqrt{\frac{1 - \cos \varphi}{2}},$$

(the sign to be determined by the quadrant in which $\frac{\varphi}{2}$ terminates).

If (34) is treated in a similar manner,

$$\textbf{(37)} \qquad \cos \frac{\varphi}{2} = \pm \sqrt{\frac{1 + \cos \varphi}{2}}.$$

Also,

$$\textbf{(38)} \qquad \tan \frac{\varphi}{2} = \pm \sqrt{\frac{1 - \cos \varphi}{1 + \cos \varphi}} = \frac{\sin \varphi}{1 + \cos \varphi} = \frac{1 - \cos \varphi}{\sin \varphi}.$$

Since $\tan \frac{\varphi}{2}$ and $\sin \varphi$ have the same sign for all values of φ and $1 - \cos \varphi$ is always positive the use of the $\pm$ sign is not needed in the last two expressions of this identity.

It is suggested that the student derive the formula for $\cot \frac{\varphi}{2}$.

EXAMPLE. Find the exact value of tan 22° 30′.

SOLUTION: Since $22° 30' = (\frac{45}{2})°$,

$$\tan 22° 30' = \tan (\tfrac{45}{2})°$$
$$= \sqrt{\frac{1 - \cos 45°}{1 + \cos 45°}}$$
$$= \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}}}$$
$$= \sqrt{2} - 1.$$

EXERCISES

1. Find the exact value of sin 15° by using $15° = \frac{30}{2}°$. ANS. $\frac{\sqrt{2 - \sqrt{3}}}{2}$.
2. Reduce the answer of the preceding exercise to the form given in the answer to Ex. 3, Art. 42.
3. Find tan 30° by using $30° = \frac{60}{2}°$.
4. Show that $\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x}$.
5. If $\sin x = \frac{5}{13}$ and x is an acute angle, find $\sin \frac{x}{2}$. ANS. $\frac{\sqrt{26}}{26}$.
6. Certain problems in Analytical Geometry involve finding the value of $\sin \alpha$ and $\cos \alpha$ when the value of $\tan 2\alpha$ is given, α being an acute angle. Find $\sin \alpha$ and $\cos \alpha$, if $\tan 2\alpha = \frac{24}{7}$. (*Hint:* First find $\cos 2\alpha$.) ANS. $\frac{3}{5}$; $\frac{4}{5}$.
7. Show that the following may be obtained from the *half angle formulas:*

(a) $$\cos 4x = \pm \sqrt{\frac{1 + \cos 8x}{2}}.$$

(b) $$\sin 200° = - \sqrt{\frac{1 - \cos 400°}{2}}.$$

(c) $$\tan \frac{7\delta}{4} = \pm \sqrt{\frac{1 - \cos \frac{7\delta}{2}}{1 + \cos \frac{7\delta}{2}}}.$$

(d) $$\sin (x - y) = \pm \sqrt{\frac{1 - \cos (2x - 2y)}{2}}.$$

47. THE SUM AND DIFFERENCE OF SINES AND COSINES OF TWO ANGLES

The student has already learned how to express the sum of sines and cosines in the form of products for special cases. For example,

$$\begin{aligned}\sin 5x+\sin 3x &= \sin (4x+x)+\sin (4x-x)\\ &= \sin 4x \cos x+\cos 4x \sin x+\sin 4x \cos x-\cos 4x \sin x\\ &= 2 \sin 4x \cos x.\end{aligned}$$

There are formulas for the general cases of $\sin \theta \pm \sin \varphi$ and $\cos \theta \pm \cos \varphi$. After the manner of the previous example write

$$\begin{aligned}\sin \theta+\sin \varphi &= \sin\left(\frac{\theta+\varphi}{2}+\frac{\theta-\varphi}{2}\right)+\sin\left(\frac{\theta+\varphi}{2}-\frac{\theta-\varphi}{2}\right),\\ &= \sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)+\cos\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)+\\ &\quad \sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)-\cos\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right).\end{aligned}$$

Hence

$$\text{(39)} \qquad \sin \theta + \sin \varphi = 2 \sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right).$$

It is left to the student to verify in the same manner that

$$\text{(40)} \qquad \sin \theta - \sin \varphi = 2 \cos\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right),$$

$$\text{(41)} \qquad \cos \theta + \cos \varphi = 2 \cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right),$$

$$\text{(42)} \qquad \cos \theta - \cos \varphi = -2 \sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right).$$

EXERCISES

1. Show that $\cos 75° - \cos 15° = -\frac{\sqrt{2}}{2}$.
2. Show that $\cos 10° + \cos 70° = \sqrt{3} \cos 40°$.
3. Show that $\frac{\sin \alpha + \sin \delta}{\sin \alpha - \sin \delta} = \frac{\tan \frac{1}{2}(\alpha + \delta)}{\tan \frac{1}{2}(\alpha - \delta)}$.
4. Express as the product of two trigonometric functions:
   (a) $\sin 4x + \sin 2x$.
   (b) $\cos 6x + \cos 2x$.
   (c) $\cos 40° - \sin 60°$.
   (d) $\sin 50° + \cos 30°$.
   (e) $\sin 75° + \cos 75°$.
   (f) $\cos 15° - \sin 15°$.
5. Show that $\sin \alpha + \sin \beta + \sin \gamma = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$, provided that $\alpha + \beta + \gamma = 180°$.
   Hint: $\sin \alpha + \sin \beta = 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}$. Also $\gamma = 180° - (\alpha + \beta)$, so that $\sin \gamma = \sin (\alpha + \beta) = 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha+\beta}{2}$.
6. Show that $\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$, provided that $\alpha + \beta + \gamma = 180°$.
7. Show that $\frac{1 + \sin x - \cos x}{1 + \sin x + \cos x} = \tan \frac{x}{2}$.
8. Prove that $\sin 80° = \sin 40° + \cos 70°$.

SUMMARY OF FORMULAS ON RELATIONS OF THE TRIGONOMETRIC FUNCTIONS

(1) $$\sin\theta = \frac{1}{\csc\theta}.$$

(2) $$\cos\theta = \frac{1}{\sec\theta}.$$

(3) $$\tan\theta = \frac{1}{\cot\theta}.$$

(4) $$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

(5) $$\cot\theta = \frac{\cos\theta}{\sin\theta}.$$

(6) $$\sin^2\theta + \cos^2\theta = 1.$$

(7) $$1 + \tan^2\theta = \sec^2\theta.$$

(8) $$1 + \cot^2\theta = \csc^2\theta.$$

(20) $$\sin(-\theta) = -\sin\theta.$$

(21) $$\cos(-\theta) = \cos\theta.$$

(22) $$\tan(-\theta) = -\tan\theta.$$

(23) $$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi.$$

(24) $$\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi.$$

(25) $$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta.$$

(26) $$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta.$$

(27) $$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi.$$

(28) $$\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi.$$

(29) $$\tan(\theta + \varphi) = \frac{\tan\theta + \tan\varphi}{1 - \tan\theta\tan\varphi}.$$

(30) $$\tan(\theta - \varphi) = \frac{\tan\theta - \tan\varphi}{1 + \tan\theta\tan\varphi}.$$

(31) $$\sin 2\theta = 2\sin\theta\cos\theta.$$

(32) $$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$

(33) $$\cos 2\theta = 1 - 2\sin^2\theta.$$

(34) $$\cos 2\theta = 2\cos^2\theta - 1.$$

(35) $$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

(36) $$\sin\tfrac{1}{2}\theta = \pm\sqrt{\frac{1 - \cos\theta}{2}}.$$

(37) $$\cos\tfrac{1}{2}\theta = \pm\sqrt{\frac{1 + \cos\theta}{2}}.$$

(38) $$\tan \tfrac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}.$$

(39) $$\sin\theta + \sin\varphi = 2\sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right).$$

(40) $$\sin\theta - \sin\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right).$$

(41) $$\cos\theta + \cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right).$$

(42) $$\cos\theta - \cos\varphi = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right).$$

MISCELLANEOUS EXERCISES

Prove the following thirty relations for all values of the angles for which the functions are defined:

1. $\tan(45° + \alpha) = \dfrac{1+\tan\alpha}{1-\tan\alpha}.$
2. $\cos(\theta - 270°) = -\sin\theta.$
3. $\sin(60° + v) = (\tfrac{1}{2})(\sqrt{3}\cos v + \sin v).$
4. $\cos(60° - v) = (\tfrac{1}{2})(\cos v + \sqrt{3}\sin v).$
5. $\sin(30° + \beta) = \cos(60° - \beta).$
6. $\tan(45° + \delta) - \tan(45° - \delta) = 2\tan 2\delta.$
7. $\sin(30° + \alpha) + \sin(30° - \alpha) = \cos\alpha.$
8. $\cos^2 x - \sin(30° + x)\sin(30° - x) = \tfrac{3}{4}.$
9. $\tan 15° = 2 - \sqrt{3}.$
10. $\dfrac{\sin 7A + \sin 3A}{\cos 7A + \cos 3A} = \tan 5A.$
11. $\dfrac{\sin 13° + \sin 7°}{\sin 20°} = \dfrac{\cos 3°}{\cos 10°}.$
12. $\dfrac{\sin 17\theta + \sin 11\theta}{\cos 17\theta - \cos 11\theta} = -\cot 3\theta.$
13. $\cos 22° 30' = \tfrac{1}{2}\sqrt{2+\sqrt{2}}.$
14. $\tan\dfrac{\theta}{2} = \csc\theta - \cot\theta.$
15. $\cot(\theta + \varphi) = \dfrac{\cot\theta\cot\varphi - 1}{\cot\theta + \cot\varphi}.$
16. $\cot(\theta - \varphi) = \dfrac{\cot\theta\cot\varphi + 1}{\cot\varphi - \cot\theta}.$
17. $\cot 2\theta = \dfrac{\cot^2\theta - 1}{2\cot\theta}.$
18. $\cot\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}.$
19. $2\csc\alpha\cos\dfrac{\alpha}{2} = \csc\dfrac{\alpha}{2}.$
20. $\csc 2\alpha = \tfrac{1}{2}\sec\alpha\csc\alpha.$
21. $\cos x + \cos 3x + \cos 5x + \cos 7x = 4\cos x\cos 2x\cos 4x.$
22. $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x\cos 2x\sin 4x.$
23. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$

24. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$

25. $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$

26. $\sin 4\theta = 4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta.$

27. $\cos 4\theta = 1 - 8\sin^2\theta + 8\sin^4\theta.$

28. $(\sin\alpha - \cos\alpha)^2 = 1 - \cos\left(\dfrac{\pi}{2} - 2\alpha\right).$

29. $\tan(x + y + z) = \dfrac{\tan x + \tan y + \tan z - \tan x\tan y\tan z}{1 - \tan x\tan y - \tan x\tan z - \tan y\tan z}.$

30. $\dfrac{2}{1 - \cos 2x} = \csc^2 x.$

31. Given $\sin\alpha = -\frac{3}{5}$, $\tan\beta = -\frac{1}{2}$, α terminates in the third quadrant, β terminates in the second quadrant. Find $\cos(2\beta - \alpha)$. ANS. 0.

32. Given α and β as in Ex. 31, find $\sin\left(2\beta - \dfrac{\alpha}{2}\right).$

In the next eight exercises, apply the formulas of this chapter to reduce by inspection each given expression to a form involving a single trigonometric function without radicals or exponents.

33. $\cos\dfrac{x}{2}\cos\dfrac{y}{2} - \sin\dfrac{x}{2}\sin\dfrac{y}{2}.$ ANS. $\cos\dfrac{x + y}{2}.$

34. $\sin\alpha\cos(\alpha - 10°) - \cos\alpha\sin(\alpha - 10°).$

35. $\dfrac{\tan\alpha - \tan(\alpha - 30°)}{1 + \tan\alpha\tan(\alpha - 30°)}.$ ANS. $\tan 30°.$

36. $\cos^2\left(\dfrac{3\alpha}{2}\right) - \sin^2\left(\dfrac{3\alpha}{2}\right).$

37. $\sin^2 6x.$ ANS. $\dfrac{1 - \cos 12x}{2}.$

38. $\cos^2 5x.$

39. $\cos x\cos(x - y) + \sin x\sin(x - y).$ ANS. $\cos y.$

40. $2\sin(a - x)\cos(a - x).$

41. An observer notes that the angle of elevation of a stationary observation balloon is θ. He moves x feet nearer to a point directly under the balloon and finds that the angle of elevation is now φ. Show that the height y of the balloon above the level ground is expressible by the formula

$$y = \frac{x\sin\theta\sin\varphi}{\sin(\varphi - \theta)},$$

a form which is suitable for logarithmic computation.

Chapter VIII

SOLUTION OF OBLIQUE TRIANGLES

48. OBLIQUE TRIANGLES

Triangles which are not right triangles are called *oblique triangles.* One of the angles of such a triangle may be obtuse. In a preceding chapter the trigonometric functions of obtuse angles were evaluated, so that the solution of an oblique triangle, where the word *solution* means the finding of the unknown parts, may be studied with complete generality.

There are four cases as follows:

Case 1. Given two angles and any side.

Case 2. Given two sides and an angle opposite one of them.

Case 3. Given two sides and the included angle.

Case 4. Given the three sides.

Case 2 involves more than one possibility, as will be explained in Art. 50.

It should be noted that the methods of this chapter, although derived for the solution of oblique triangles, are applicable to the solution of right triangles. In the right triangle the given data include the right angle.

49. THE LAW OF SINES

This is a relation between the sides of a triangle and the angles opposite them. Figure 53 illustrates oblique triangles, one with three acute angles, and the other with one obtuse and two acute angles. The conventional notation is applied in the figures, that is, the vertices are A, B, C; the angles are α, β, γ; and the sides opposite them are a, b, and c, respectively. A perpendicular h is constructed from the vertex C to meet the side AB (or AB extended) at D. Then, in the right triangle ADC, $\frac{h}{b} = \sin\alpha.$

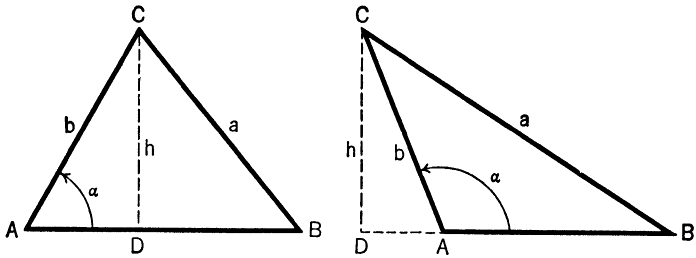


FIGURE 53.

In right triangle DBC, $\frac{h}{a} = \sin\beta.$

If the members of the first equation are divided by the corresponding members of the second equation, h is eliminated and the result is

$$\frac{a}{b} = \frac{\sin\alpha}{\sin\beta},$$

which can be written, $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}.$

By drawing the altitude from vertex A instead of from C, it is found in a similar fashion that

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

From these two equalities it follows that

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}. \tag{43}$$

This result is known as the **Law of Sines.** It can be stated in words as follows:

In any triangle the sides are proportional to the sines of the angles opposite them. (Note that the Law of Sines applies to right as well as oblique triangles.)

50. PROBLEMS INVOLVING THE LAW OF SINES

The equation form (48) of the Law of Sines actually is three equations, each of which is based on the proportionality of two sides of a triangle to the sines of the angles opposite them. A study of these equations shows that cases 1 and 2 of solvable triangles in Article 48 can be solved by means of the Law of Sines. The method for the two cases is illustrated in the following examples.

EXAMPLE OF CASE 1. A side of a parallelogram 598 feet long makes angles of 27° and 42° with the diagonals. Find the lengths of the diagonals.

SOLUTION: Since the diagonals of the parallelogram bisect each other, OA and OB (Fig. 54) can be calculated and respectively doubled. To apply the Law of Sines in solving triangle AOB, the angle opposite AB is needed. It may be computed at once by subtracting the sum of the two known angles from 180° and is found to be 111°. Then from the Law of Sines

$$\frac{OB}{\sin 27^\circ} = \frac{598}{\sin 111^\circ}.$$

But $\sin 111^\circ = \sin 69^\circ$. Hence

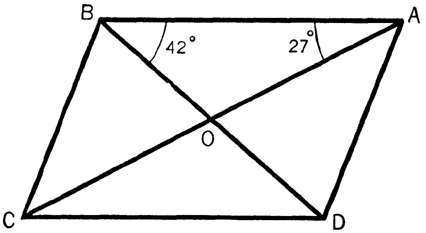


FIGURE 54.

$$\frac{OB}{\sin 27^\circ} = \frac{598}{\sin 69^\circ},$$

or
$$\begin{aligned}\log OB &= \log \sin 27^\circ + \log 598 - \log \sin 69^\circ\\ &= (9.6570 - 10) + 2.7767 - (9.9702 - 10)\\ &= 2.4635.\end{aligned}$$

$\therefore$ $OB = 290.7,$

and $BD = 581.4.$

Similarly, from the equation

$$\frac{OA}{\sin 42^\circ} = \frac{598}{\sin 111^\circ},$$

it follows that

$$AC = 857.0.$$

EXAMPLE OF CASE 2. In a recreation park a children's slide is 27 feet long and makes an angle of 39° with the ground. Its top is reached by a ladder 18 feet long. What is the angle of inclination of the ladder?

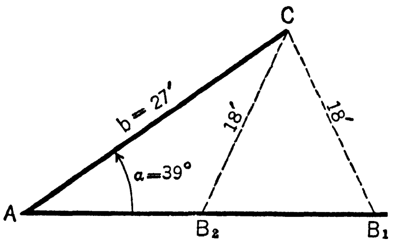


FIGURE 55.

SOLUTION: The inclination of the ladder is given by angle β_1 of Fig. 55. In the triangle AB_1C by the Law of Sines,

$$\frac{\sin \beta_1}{27} = \frac{\sin 39°}{18}.$$

Therefore,
$$\log \sin \beta_1 = \log 27 + \log \sin 39° - \log 18$$
$$= 9.9750 - 10.$$
$$\beta_1 = 70° 45'.$$

NOTE: The angle $\beta_2 = 180° - 70° 45' = 109° 15'$ has the same sine and hence the same log sine as $70° 45'$. Hence from the Law of Sines alone, the angle could be either $70° 45'$ or $109° 15'$. However, the nature of this problem requires the rejection of the obtuse angle. Usually, a figure carefully drawn to represent the conditions of the problem will reveal the correct choice of angle.

Case 2. Given two sides and an angle opposite one of them. This is called the *ambiguous case,* because the information given sometimes is insufficient to determine which solution of the triangle is wanted. The various sub-cases are discussed by means of the following problem:

Given sides a and b and angle α, to construct triangle ABC.

Construct angle α with sides AC and AD and along one side lay off length $b = AC$. With C as center draw an arc of radius a. The third vertex of the triangle is found where the arc intersects the line AD. *If a is less than b,* there are three possibilities:

(1) If a is equal to the perpendicular from C to AD (Fig. 56), then B is at the foot of the perpendicular, and has only one position. The triangle is a right triangle, and $\sin \beta = 1$. Here the Law of Sines gives

$$\sin \beta = \frac{b \sin \alpha}{a} = 1.$$

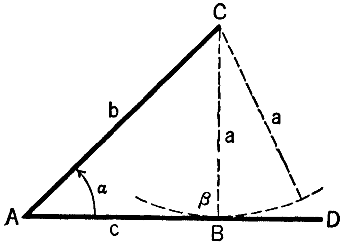


FIGURE 56.

(2) If a is greater than the perpendicular from C to AD (Fig. 57), the arc intersects AD in two points, B_1 and B_2, each of which is the vertex of a triangle which satisfies all the conditions of the problem. It is evident that $\sin \beta = \dfrac{b \sin \alpha}{a} < 1$, for a is greater in this instance than it was in the first one.

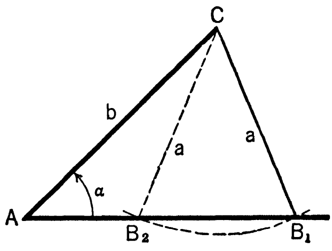


FIGURE 57.

(3) If a is less than the perpendicular from C to AD (Fig. 58), then there is no triangle satisfying the given conditions. Again from the Law of Sines

$$\sin \beta = \frac{b \sin \alpha}{a},$$

and since a here is less than $b \sin \alpha$ it follows that

$$\frac{b \sin \alpha}{a} > 1.$$

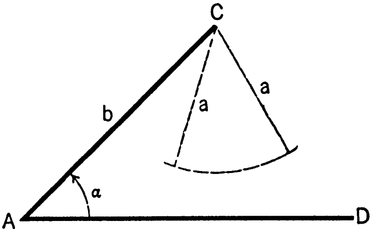


FIGURE 58.

This is impossible since there is no value of β such that $\sin \beta$ is greater than 1.

If a is equal to or greater than b, there are two possibilities:

1. If a is equal to b, then the triangle is isosceles and only one solution is possible, provided that α is an acute angle.

2. If a is greater than b (Fig. 59), the arc with radius a and center at C intersects AD in two points B_1 and B_2 which lie on opposite sides of A. However the triangle AB_1C contains angle α, and the triangle AB_2C does not. Hence only one solution is possible. (It should be noted that this is the only possibility in which α can be obtuse and a solution found.)

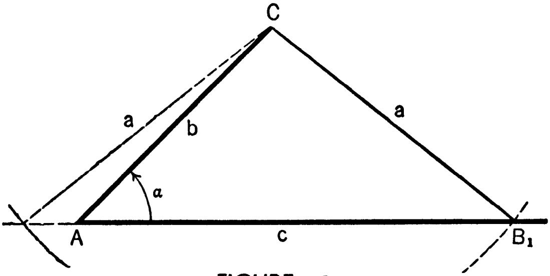


FIGURE 59.

Summary of the Ambiguous Case, given α, b, and a:

I. $a < b$, *so that α is acute. Apply Law of Sines.*

(1) *If* $\sin \beta = 1$, *there is one solution, a right triangle.*

(2) *If* $\sin \beta < 1$, *there are two solutions.*

(3) *If* $\sin \beta > 1$, *there is no solution.*

II. $a = b$.

(1) *If* $\alpha < 90°$, *there is one solution.*

(2) *If* $\alpha \geqq 90°$, *there is no solution.*

III. $a > b$.

There is one solution for every α, acute, right, or obtuse.

EXAMPLE OF I (1). Solve the triangle ABC if $a = 6$, $b = 12$, $\alpha = 30°$.

SOLUTION: Here

$$\sin \beta = \frac{b \sin \alpha}{a} = 1$$

Therefore, $\beta = 90°$, $\gamma = 60°$, and $c = \sqrt{12^2 - 6^2} = \sqrt{108} = 10.39$.

EXAMPLE OF I (2). Solve the triangle if $a = 8$, $b = 12$, $\alpha = 30°$.

SOLUTION: Here

$$\sin \beta = \frac{b \sin \alpha}{a} = \frac{(12)(0.5)}{8} = 0.7500 < 1.$$

Hence there are two solutions so that

$$\beta = 48° \, 35' \text{ or } 131° \, 25'.$$

If $\beta = 48° \, 35'$, then

$$\gamma = 101° \, 25',$$

and
$$c = \frac{b \sin \gamma}{\sin \beta} = \frac{(12)(0.9802)}{0.7500} = 15.68.$$

If $\beta = 131° \, 25'$, then

$$\gamma = 18° \, 35',$$

and
$$c = \frac{(12)(0.3187)}{0.7500} = 5.099.$$

EXAMPLE OF I (3). Solve the triangle if $a = 5$, $b = 12$, $\alpha = 30°$.

SOLUTION: Here

$$\sin \beta = \frac{b \sin \alpha}{a} = \frac{(12)(0.5)}{5} = 1.2 > 1.$$

Hence there is no solution.

EXAMPLE OF II (1). Solve the triangle if $a = 12$, $b = 12$, and $\alpha = 30°$.

SOLUTION: Here

$$\sin \beta = \frac{b \sin \alpha}{a} = \frac{(12)(0.5)}{12} = 0.5000,$$

so $\quad \beta = 30°$ or $150°$.

If $\beta = 150°$, there is no triangle, hence no solution.

Therefore the only solution is

$$\beta = 30°, \ \gamma = 120°, \text{ and } c = \frac{b \sin \gamma}{\sin \beta} = \frac{(12)(0.8660)}{0.5000} = 20.78.$$

EXAMPLE OF III. Solve the triangle if $a = 12$, $b = 8$, and $\alpha = 30°$.

SOLUTION: Here

$$\sin \beta = \frac{b \sin \alpha}{a} = \frac{(8)(0.5)}{12} = 0.3333,$$

so $\beta = 19° 28'$ or $160° 32'$.

If $\beta = 160° 32'$, then the triangle does not contain α. Hence there is only one solution, namely,

$$\beta = 19° 28', \gamma = 129° 32', c = \frac{b \sin \gamma}{\sin \beta} = \frac{(8)(0.7600)}{0.3333} = 18.24.$$

EXERCISES

Solve for the parts called for in each of the triangles ABC in the following exercises:

1. Given $b = 30.36$, $\alpha = 103° 36'$, $\beta = 19° 21'$; show that $\gamma = 57° 3'$, $a = 89.04$, and $c = 76.88$.
2. Given $c = 998$, $\alpha = 37° 58'$, $\beta = 65° 2'$; find γ, a, b.
3. Given $b = 67.85$, $\beta = 13° 57'$, $\gamma = 57° 13'$; show that $a = 266.4$, $c = 236.6$.
4. Given $a = 4.37$, $\beta = 48° 30'$, $\gamma = 72° 8'$; find α, b, c.
5. Given $a = 168$, $b = 97$, $\alpha = 21° 31'$; show that $\beta = 12° 14'$, $c = 254.5$.
6. Given $a = 186.8$, $b = 394.2$, $\beta = 114° 30'$; find α, c.
7. Given $a = 51$, $b = 33$, $\beta = 30° 20'$; find $\alpha = 51° 19'$ or $128° 41'$, and $c = 64.7$ or 23.4 respectively.
8. Given $a = 19$, $b = 20$, and $\alpha = 70° 40'$; solve the triangle.
9. Given $a = 84$, $b = 69$, $\beta = 61° 47'$; show that there is no triangle.
10. Given $c = 8$, $b = 5$, and $\beta = 34° 20'$; solve the triangle.
11. Two men 1000 yds. apart on a plain, and facing each other, find that the angles of elevation of a balloon in the same vertical plane with themselves are 53° and 79° 12′ respectively. Find the distance from the balloon to each observer, and also the height of the balloon above the plain. ANS. height, 1059 yds.
12. A hill rises from a horizontal plain, and wishing to know the airline distance of a point A on the plain to the top B of the hill, an observer selects a point C at the foot of the hill, in the same vertical plane with B and A, and finds that $BC = 910.6$ yds., $AC = 770.4$ yds., angle $CAB = 51° 9'$. Required, the distance AB.

51. THE LAW OF COSINES

In Fig. 60 below are illustrated a right triangle, an acute triangle, and an obtuse triangle. To emphasize the variation of value in a, triangles are chosen having the same values of b and of c in each case. In the right triangle, $a^2 = b^2 + c^2$. Furthermore, it is obvious that in the acute triangle $a^2 = b^2 + c^2 -$ (*a term to be determined*) and that in the obtuse triangle $a^2 = b^2 + c^2 +$ (*a term to be determined*). A single formula is developed which will give the correct value of a^2 for each of the three cases.

In the two oblique triangles let fall a perpendicular h from C to meet AB (or AB extended) at D. In the obtuse case, $\cos \angle DAC = -\cos \alpha$. Hence in that figure, from the relation

$$\frac{DA}{b} = \cos \angle DAC,$$

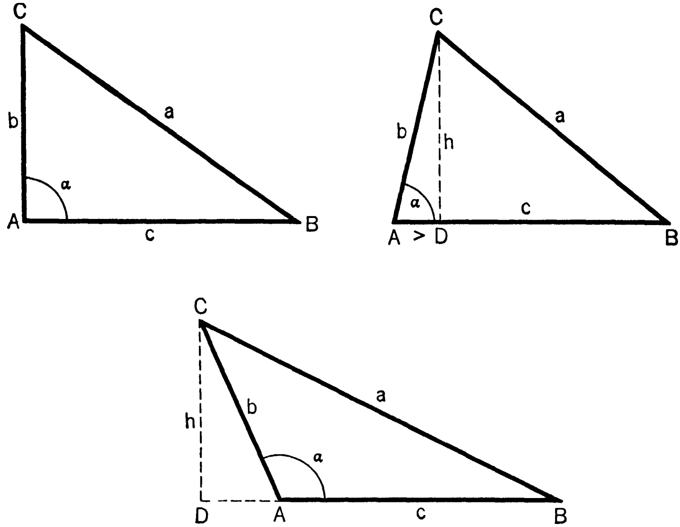


FIGURE 60.

$DA = -b \cos \alpha$. Thus in both figures it follows that

$$DB = c - b \cos \alpha.$$

Also, $$h = b \sin \alpha.$$

But in both figures, $$a^2 = h^2 + \overline{DB}^2.$$

Hence, by substitution,

$$\begin{aligned} a^2 &= b^2 \sin^2 \alpha + (c - b \cos \alpha)^2 \\ &= b^2 \sin^2 \alpha + c^2 - 2bc \cos \alpha + b^2 \cos^2 \alpha \\ &= b^2 (\sin^2 \alpha + \cos^2 \alpha) + c^2 - 2bc \cos \alpha. \end{aligned}$$

Therefore,

(44) $$\mathbf{a^2 = b^2 + c^2 - 2bc \cos \alpha.}$$

It is evident that similar expressions may be obtained for each of the three sides (see Art. 57).

This relation is known as the **Law of Cosines** and may be stated in words as follows:

In any triangle, the square of any side is equal to the sum of the squares of the other two sides diminished by twice the product of these sides times the cosine of their included angle.

Note that the term, $2bc \cos \alpha$, of the right-hand side of (44), meets the requirements of the opening statement; namely, that for the right triangle it becomes zero (reducing the equation to the Pythagorean relation); for the acute triangle it properly diminishes the value of $a^2 + b^2$; and for the obtuse triangle, due to the negative value of $\cos \alpha$, it increases the value of $a^2 + b^2$.

When solved for $\cos \alpha$, (44) gives the *alternate form of the Law of Cosines.*

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}. \tag{45}$$

52. PROBLEMS INVOLVING THE LAW OF COSINES

Forms (44) and (45) of the Law of Cosines provide relations which solve triangles coming under cases (3) and (4) of Art. 48. If two sides and the included angle are given, (44) gives the value of the third side. The remaining angles may then be found by means of the Law of Sines. If three sides are given, (45) gives the means of finding any one of the three angles.

It will be noted that the Law of Cosines is not well adapted to logarithmic computation since logarithmic methods are not suitable for carrying out addition and subtraction.

The application of the Law of Cosines is next illustrated with two examples.

EXAMPLE OF CASE 3. A plane with a cruising speed of 120 mi./hr. heads in the direction north, 52° 30′ east. A 45-mi./hr. wind is blowing toward south, 70° east. Find the actual speed and direction of the plane.

SOLUTION: A figure is constructed using vectors OA and OB to denote the air velocity of the plane and the velocity of the wind, respectively. It is required to find the magnitude of the resultant OC and the size of angle θ which will determine the direction of the resultant. Since the adjacent angles of a parallelogram are supplementary, angle OBC contains 122° 30′. From form (44) of the Law of Cosines as applied to triangle OBC,

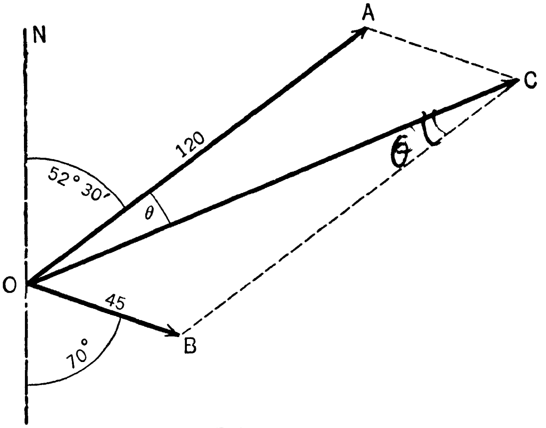


FIGURE 61.

$$\begin{aligned}\overline{OC}^2 &= \overline{BC}^2 + \overline{OB}^2 - 2\,\overline{BC}\cdot\overline{OB}\cdot\cos\beta\\ &= 120^2 + 45^2 - 2(120)(45)\cos 122°\,30'\\ &= 14{,}400 + 2025 - (10{,}800)(-\cos 57°\,30')\\ &= 14{,}400 + 2025 + (10{,}800)(0.5373)\\ &= 14{,}400 + 2025 + 5802.84\\ &= 22{,}227.84;\\ \overline{OC} &= \sqrt{22{,}227.84} = 149.1.\end{aligned}$$

From the Law of Sines, in triangle OBC,

$$\frac{\sin\theta}{45} = \frac{\sin 122°\,30'}{149.1}$$

$$\begin{aligned}\log\sin\theta &= \log 45 + \log\sin 57°\,30' - \log 149.1\\ &= 9.4057 - 10.\\ \therefore\quad \theta &= 14°\,45'.\end{aligned}$$

Thus $\angle NOC = 52°\,30' + 14°\,45' = 67°\,15'$.

The conclusion is that the plane actually flies in a direction 67° 15′ east of north with a speed of 149.1 miles per hour.

Caution: In evaluating $b^2 + c^2 - 2bc\cos\alpha$ the student should avoid the common error of subtracting $2bc$ from $b^2 + c^2$ before multiplying by $\cos\alpha$. The computation is given in detail in the above example to make the proper procedure clear.

EXAMPLE OF CASE 4. Two sides of a triangle are 51 and 65. What must their included angle be in order that the third side shall be 20?

SOLUTION: Let α be the included angle. Then by (45),

$$\cos\alpha = \frac{51^2 + 65^2 - 20^2}{2(51)(65)} = 0.9692,$$

$$\alpha = 14°\,16'.$$

EXERCISES

Solve the triangle ABC as indicated in each of the following:

1. Given $a = 5$, $b = 6$, $\gamma = 15° 28'$; show that $\alpha = 48° 28'$, $\beta = 116° 4'$.
2. Given $b = 5$, $c = 6$, $\alpha = 130°$; find a, γ, β.
3. Given $a = 7$, $b = 8$, $c = 9$; show that $\alpha = 48° 11'$, $\gamma = 73° 24'$.
4. Given $a = 19$, $b = 34$, $c = 49$, find α, β, γ.
5. Given $a = 12$, $b = 16$, $c = 20$; show that $\alpha = 36° 52'$, $\gamma = 90°$.
6. Given triangle PQR, use the two forms of the Law of Cosines to express each part of the triangle in terms of other parts.
7. Two forces of 50 lb. and 80 lb. respectively act upon an object at an angle of 65°. Find the magnitude and direction of the resultant force. ANS. 110.8 lb. and 24° 9′ with 80 lb. force.
8. Show that the Law of Cosines is equivalent to the following theorem of plane geometry: "In an oblique triangle, the square of the side opposite an acute (or obtuse) angle is equal to the sum of the squares of the other sides diminished (or increased) by twice the product of one of these sides times the projection of the other side on it."
9. Two sides of a triangle are 10 and 11, and the included angle is 50°. Find the third side. ANS. 8.92.
10. In surveying a tract of land an engineer found it impracticable to measure the side AB on account of a thick brushwood lying between A and B. He therefore measured AE, 9.17 chains, and EB, 3.12 chains, and found the angle at $E = 73° 7'$. Find the distance from A to B.

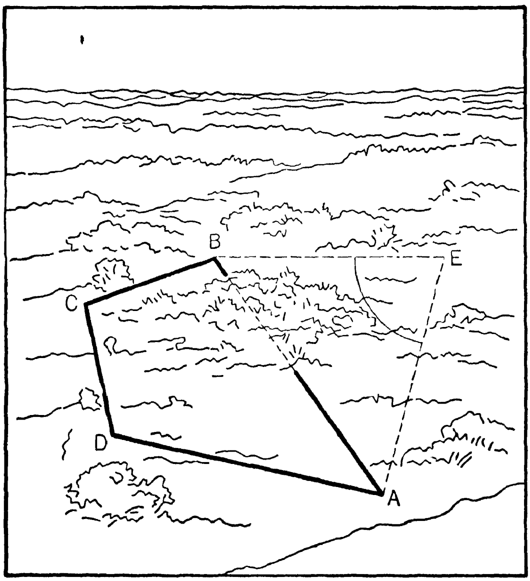


FIGURE 62.

53. THE LAW OF TANGENTS

If two sides and the included angle of a triangle (Case 3, Art. 48) are given, and *only* the other two angles are required, the Law of

Cosines could be used. In this case it would be necessary to first find the third side, then the two angles. Here the solution is a two-step problem, and logarithms may be used only to a limited extent. It is desirable to find a way in which these angles may be found in one step and in which logarithmic computation might be used. Hence the *Law of Tangents.*

In any triangle ABC, $$\frac{\sin \alpha}{\sin \beta} = \frac{a}{b}.$$

By composition and division, $$\frac{\sin \alpha - \sin \beta}{\sin \alpha + \sin \beta} = \frac{a - b}{a + b}.$$

But applying (40) and (39) to the numerator and denominator, respectively, of the left-hand fraction, there results,

$$\frac{2 \cos \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta)}{2 \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta)} = \frac{a - b}{a + b},$$

which readily reduces to

(46) $$\frac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)} = \frac{a - b}{a + b}.$$

This relation is known as the **Law of Tangents.**

Other forms for the Law of Tangents can be obtained by inverting both sides (in case $a \neq b$) or by cyclical permutations as explained in Art. 57.

54. APPLYING THE LAW OF TANGENTS

If two sides, a and b, and the included angle γ of a triangle ABC are given, $\alpha + \beta$ is easily found from the relation $\alpha + \beta + \gamma = 180°$. Hence the value of $\tan \frac{1}{2}(\alpha + \beta)$ may be obtained, or its logarithm if logarithmic computation is to be used. This leaves $\tan \frac{1}{2}(\alpha - \beta)$ as the only unknown in (46). The value of $(\alpha - \beta)$ thus obtained can then readily be combined with the value of $(\alpha + \beta)$ already determined, and values of α and β separately can be found. The method is illustrated by an example.

EXAMPLE. Two sides of a triangular field are 236.7 yds. and 341.3 yds., respectively. They form with each other an angle of 67° 40′. Find the other angles of the field.

SOLUTION: *To avoid negative signs,* let the longer of the two sides correspond to the side a of the law of tangents. If $\gamma = 67° 40'$, $\alpha + \beta = 180° - 67° 40' = 112° 20'$ or $\frac{1}{2}(\alpha + \beta) = 56° 10'$. Now $a - b = 104.6$, and $a + b = 578.0$.

Substituting these values in (46),

$$\frac{\tan \frac{1}{2}(\alpha - \beta)}{\tan 56^\circ\ 10'} = \frac{104.6}{578.0}.$$

Hence, $\log \tan \frac{1}{2}(\alpha - \beta) = \log \tan 56^\circ\ 10' + \log 104.6 - \log 578.0$

$= 0.1737 + 2.0195 - 2.7619$

$= 9.4313 - 10.$

Therefore, $\frac{1}{2}(\alpha - \beta) = 15^\circ\ 6'.$

But $\frac{1}{2}(\alpha + \beta) = 56^\circ\ 10'.$

If these two equalities are added,

$$\alpha = 71^\circ\ 16'.$$

If the first is subtracted from the second,

$$\beta = 41^\circ\ 4'.$$

The solution of the triangle for the third side may now be carried out by using the Law of Sines as in case 1.

NOTE: While it is true that the sum of α, β, and γ should equal 180°, this does not give a complete check on the accuracy of the work. If $\frac{1}{2}(\alpha + \beta)$ is found correctly and this value is combined correctly with the value found for $\frac{1}{2}(\alpha - \beta)$, the resulting three angles will total 180°, regardless of the correctness of the value found for $\frac{1}{2}(\alpha - \beta)$. Hence this check should be used only to discover errors in finding $\frac{1}{2}(\alpha + \beta)$ and in adding and subtracting the last pair of equations.

EXERCISES

Use the Law of Tangents to find the unknown angles in each of the triangles ABC described below:

1. $a = 342.8$, $b = 301.8$, $\gamma = 46^\circ\ 27'$. ANS. $\alpha = 75^\circ\ 12'$, $\beta = 58^\circ\ 21'$.
2. $a = 54.7$, $b = 44.8$, $\gamma = 66^\circ\ 20'$.
3. $b = 4016$, $c = 6104$, $\alpha = 107^\circ\ 13'$. ANS. $\beta = 27^\circ\ 45'$, $\gamma = 45^\circ\ 3'$.
4. $a = 28.56$, $c = 32.48$, $\beta = 57^\circ\ 40'$.
5. $b = 3625$, $c = 3982$, $\alpha = 68^\circ\ 25'$. ANS. $\beta = 51^\circ\ 51'$, $\gamma = 59^\circ\ 45'$.

55. RELATION OF THE HALF ANGLE TO THE THREE SIDES

Let s represent one-half the perimeter (semi-perimeter) of triangle ABC. Then $a + b + c = 2s$. Also, $a + b - c = 2(s - c)$, $a - b + c = 2(s - b)$, $-a + b + c = 2(s - a)$.

When the members of (36) and (37) are squared and cleared of fractions, there results

$$2 \sin^2 \tfrac{1}{2}\alpha = 1 - \cos \alpha, \text{ and } 2 \cos^2 \tfrac{1}{2}\alpha = 1 + \cos \alpha.$$

Replacing $\cos \alpha$ by its value in (45), the first equation becomes

$$2 \sin^2 \tfrac{1}{2}\alpha = \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b - c)^2}{2bc}$$

$$= \frac{(a + b - c)(a - b + c)}{2bc} = \frac{2(s - b)(s - c)}{bc},$$

and from the second equation

$$2\cos^2 \tfrac{1}{2}\alpha = \frac{2bc + b^2 + c^2 - a^2}{2bc} = \frac{(b+c)^2 - a^2}{2bc}$$
$$= \frac{(b+c+a)(b+c-a)}{2bc} = \frac{2s(s-a)}{bc}.$$

From these results there follow

(47) $$\sin \tfrac{1}{2}\alpha = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

(48) $$\cos \tfrac{1}{2}\alpha = \sqrt{\frac{s(s-a)}{bc}}.$$

From (47), and (48),

(49) $$\tan \tfrac{1}{2}\alpha = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

With these three formulas logarithms may be used in determining an angle of a triangle when three sides are given. (It is understood, of course, that the additions and subtractions must be performed first.)

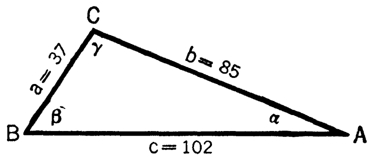


FIGURE 63.

EXAMPLE. The three sides of a triangular field are 37 rds., 102 rds., and 85 rds., respectively. Solve the triangle.

SOLUTION: Let $a = 37$, $b = 85$, and $c = 102$.

Then
$$2s = 37 + 85 + 102 = 224,$$
$$s = 112,\ s - a = 75,\ s - b = 27,\ s - c = 10.$$
$$\sin \tfrac{1}{2}\alpha = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{(27)(10)}{(85)(102)}};$$
$$\log \sin \tfrac{1}{2}\alpha = \tfrac{1}{2}(\log 27 + \log 10 - \log 85 - \log 102).$$

log 27 =	1.4314	log 85 =	1.9294
log 10 =	1.0000	log 102 =	2.0086
	2.4314		3.9380

$$\log \sin \tfrac{1}{2}\alpha = \tfrac{1}{2}(2.4314 - 3.9380)$$
$$= 9.2467 - 10.$$

From the tables,

$$\tfrac{1}{2}\alpha = 10^\circ\, 10',$$
$$\alpha = 20^\circ\, 20'.$$

Similarly, $\sin \frac{1}{2}\beta = \sqrt{\frac{(s-a)(s-c)}{ac}}$

$$= \sqrt{\frac{(75)(10)}{(37)(102)}}.$$

Again by logarithms,

$$\log \sin \tfrac{1}{2}\beta = \tfrac{1}{2}(\log 75 + \log 10 - \log 37 - \log 102).$$

$\log 75 = 1.8751$	$\log 37 = 1.5682$
$\log 10 = \underline{1.0000}$	$\log 102 = \underline{2.0086}$
2.8751	3.5768

$$\log \sin \tfrac{1}{2}\beta = \tfrac{1}{2}(2.8751 - 3.5768)$$
$$= 9.6491 - 10.$$

From the tables,

$$\tfrac{1}{2}\beta = 26^\circ\, 28',$$
$$\beta = 52^\circ\, 56'.$$
$$\gamma = 180^\circ - (20^\circ\, 20' + 52^\circ\, 56') = 106^\circ\, 44'.$$

CHECK: $\sin \frac{1}{2}\gamma = \sqrt{\frac{(s-a)(s-b)}{ab}}$

$$= \sqrt{\frac{(75)(27)}{(37)(85)}},$$

$$\log \sin \tfrac{1}{2}\gamma = \tfrac{1}{2}(\log 75 + \log 27 - \log 37 - \log 85).$$

$\log 75 = 1.8751$	$\log 37 = 1.5682$
$\log 27 = \underline{1.4314}$	$\log 85 = \underline{1.9294}$
3.3065	3.4976

$$\log \sin \tfrac{1}{2}\gamma = \tfrac{1}{2}(3.3065 - 3.4976)$$
$$= 9.9044 - 10.$$

From the tables,

$$\tfrac{1}{2}\gamma = 53^\circ\, 22',$$
$$\gamma = 106^\circ\, 44'.$$

EXERCISES

1. Given $a = 7$, $b = 8$, $c = 9$, verify that $\alpha = 48^\circ\, 12'$, $\gamma = 73^\circ\, 24'$.
2. Given $a = 17$, $b = 21$, $c = 26$, find α, β, γ, and check.
3. Given $a = 6$, $b = 8$, $c = 10$, verify that $\beta = 53^\circ\, 8'$, $\gamma = 90^\circ$.
4. Given $a = b = c = 5$, find α, β, γ, and check.
5. Given $a = 3$, $b = 8$, $c = 9$, verify that $\beta = 61^\circ\, 12'$, $\gamma = 99^\circ\, 36'$.
6. Given $a = 271$, $b = 236$, $c = 185$, find α, β, γ.

56. AREAS OF TRIANGLES

Denote the area of any plane triangle by K. The formulas for K in three cases are given below.

1. *When the base and altitude are given.* If b is the base and h is the altitude, from plane geometry,

$$K = \tfrac{1}{2}bh. \tag{50}$$

In the case of the right triangle either leg may be taken as the base and the other leg as the altitude.

2. *When two sides and the included angle are given.* Let b, c be the sides and α the included angle. (See figures of Art. 49.) The altitude of the triangle from C is h. Hence by (50), $K = \frac{1}{2}ch$. But from the relation $\frac{h}{b} = \sin\alpha$, $h = b\sin\alpha$. Hence by substitution,

$$K = \tfrac{1}{2}bc\sin\alpha. \tag{51}$$

In words, *the area of any triangle is equal to one-half the product of any two sides by the sine of the included angle.*

3. *When three sides are given.* In (31) $\sin 2\theta = 2\sin\theta\cos\theta$. Hence $\sin\alpha = 2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha$. Substituting this value in (51),

$$K = bc\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha.$$

Replacing $\sin\frac{1}{2}\alpha$ and $\cos\frac{1}{2}\alpha$ with their value given in (47) and (48), there results after the factor bc is cancelled in the numerator and the denominator,

$$K = \sqrt{s(s-a)(s-b)(s-c)}. \tag{52}$$

This formula is well adapted to logarithmic computation.

EXAMPLE. Find the area of the triangle with sides $a = 97$, $b = 83$, $c = 71$.

SOLUTION: From (52), $K = \sqrt{s(s-a)(s-b)(s-c)}$, where $2s = 97 + 83 + 71 = 251$, $s = 125.5$, $s - a = 28.5$, $s - b = 42.5$, $s - c = 54.5$; so

$$\begin{aligned} K &= \sqrt{(125.5)(28.5)(42.5)(54.5)},\\ \log K &= \tfrac{1}{2}(\log 125.5 + \log 28.5 + \log 42.5 + \log 54.5).\\ \log 125.5 &= 2.0986\\ \log 28.5 &= 1.4548\\ \log 42.5 &= 1.6284\\ \log 54.5 &= \underline{1.7364}\\ & 6.9182\\ \log K &= \tfrac{1}{2}(6.9182) = 3.4591,\\ K &= 2878.0 \text{ sq. units.} \end{aligned}$$

EXERCISES

1. Given $a = 10, b = 12, c = 16$, find K.
2. Given $a = 46.7, b = 84.5, c = 75.6$, verify that $K = 1755$ sq. units.
3. Two sides of a triangle are 165.4 yds., and 97.3 yds., and the included angle is 60° 40′. Find the area in acres.
4. Show that the area of any quadrilateral equals one-half the product of its diagonals by the sine of the included angle.
5. Given $a = 49, b = 64$, and $K = 1568$. Solve the triangle.

57. CYCLICAL PERMUTATIONS IN THE TRIANGLE FORMULAS

Any of the triangle formulas (44) to (52) can be changed to a similar formula stated in terms of another set of parts of the triangle by replacing each letter of the formula by the letter next in succession to it in such a circular arrangement as shown in Fig. 64. This circular arrangement of letters is called a **cyclical permutation.**

Thus the area formula (51),

$$K = \tfrac{1}{2}bc \sin \alpha,$$

is changed by a cyclical permutation into the formula

$$K = \tfrac{1}{2}ca \sin \beta.$$

Another cyclical change yields the formula

$$K = \tfrac{1}{2}ab \sin \gamma.$$

A third such change results in the initial form.

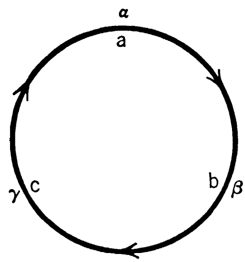


FIGURE 64.

It is recommended, however, that the student depend largely upon the statement form of each of the relations, particularly numbers (43), (44), and (51), thus being able to write the desired relation at once for any triangle in any position.

SUMMARY OF FORMULAS FOR SOLUTION OF TRIANGLES

(43) $$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma},$$ Law of Sines.

(44) $$a^2 = b^2 + c^2 - 2bc \cos \alpha,$$ Law of Cosines.

(45) $$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc},$$ Law of Cosines.

(46) $$\frac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)} = \frac{a - b}{a + b},$$ Law of Tangents.

(47) $$\sin \tfrac{1}{2}\alpha = \sqrt{\frac{(s-b)(s-c)}{bc}}, \text{ where } s = \frac{a+b+c}{2}.$$

(48) $$\cos \tfrac{1}{2}\alpha = \sqrt{\frac{s(s-a)}{bc}}.$$

(49) $$\tan \tfrac{1}{2}\alpha = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

For area of triangle, K.

(50) $K = \tfrac{1}{2}bh.$

(51) $K = \tfrac{1}{2}bc \sin \alpha.$

(52) $K = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \dfrac{a+b+c}{2}$.

MISCELLANEOUS EXERCISES

1. To find the distance AB across a pond, a point C was located 530 ft. from A and 427 ft. from B. Angle ACB was found by measurement to be 37° 20′. Find AB. Ans. 321.4 ft.
2. An airplane search for a missing vessel covered a triangular area with sides 340, 580, and 430 miles respectively. Considering the triangle as a plane figure, how many square miles were searched?
3. Two radio stations A and B are 85 miles apart. Their direction finders reveal that a ship C is so located that angle CAB is 67° 24′ and angle CBA is 52° 50′. Find the distance of the ship from each of the stations. Ans. 78.4 mi. and 90.7 mi.
4. Find the resultant of two velocities of 45 m.p.h. and 32 m.p.h. at an angle of 65°.
5. Find the other two sides of a triangular field in which two angles are 75° and 35° and the side between them is 125 yds. Ans. 76.3 yds., 128.5 yds.
6. In a certain airplane race the course was a triangle with sides 10, 12, and 15 miles. What were the angles at the turns?
7. Two railroads cross at an angle of 52° 24′. A station on one track is 5 miles from the crossing and one on the other track is 7 miles from the crossing. Assuming that the tracks are straight, how far apart are the stations? Ans. Two possibilities, 5.59 mi. or 10.80 mi.
8. A person goes 84 yds. up a slope of 17° 10′ from the edge of a river, and observes that the angle of depression of the edge of the water on the opposite shore is 2° 25′. Find the width of the river.
9. To find the height of a cliff the following measurements were made: At A on level ground the angle of elevation of the top of the cliff was 38° 50′; at B, 2156 ft. nearer the cliff, the angle of elevation of the top of the cliff was 59° 20′. What is the height of the cliff? Ans. 3322 ft.
10. Two sides of a triangle are 784.6 yds. and 688 yds., and their included angle is 43° 16′. Find the area.
11. A triangle has sides 381, 946, and 1053. Find its angles and its area. Ans. 21° 6′, 95° 28′, 63° 26′, K = 179,400.
12. A yacht cruising in an easterly direction with a speed of 12 mi./hr. moves into an ocean current flowing in a direction of north, 35° east, with a speed of 3 mi./hr. Find the resultant speed and direction of the yacht while in the current.

13. The area of a triangle is 54.24 sq. ft. and two of its sides are 10.6 ft. and 13.8 ft. Find the angle between these sides. ANS. 47° 52′.

14. What is the third side of a triangular field in which two sides are 240 yds. and 300 yds. and the angle between them is 75°?

15. A sun dial is part of the surface of a certain concrete garden walk. A wall of the garden is 40 ft. from the sun dial. The ground continues level on the outside of the wall for a distance of 12 ft., then slopes upward at an angle of 24°. 45 ft. up this slope the sun dial is just visible over the wall. How high is the wall? ANS. 7.86 ft.

16. Observers A and B are stationed 4000 ft. apart. They both see the flash of a gun and time the interval until the sound is heard. A hears the sound in 4.2 sec. and B in 6.1 sec. Assuming that light travels instantaneously and that sound travels 1080 ft./sec. in air, find the direction of the gun from each observer in terms of the angles formed with the line AB.

17. A swimmer is at A on one bank of a river, the current rate of which is 2 mi./hr. He can swim 3 mi./hr. in still water and wishes to reach the point B downstream on the opposite bank so located that the line AB makes an angle of 70° with the bank. In what direction must he head in order to swim straight to B?
ANS. Upstream at an angle of 71° 12′ with the bank.

18. One side of a parallelogram is 40, the longer diagonal is 72, and the angle between the two diagonals is 158° 23′. Find the length of the other diagonal.

19. A line 538 ft. long is drawn along one bank of a river. The angles which it makes with lines sighted to an intermediate point on the other bank are 45° 40′ and 57° 30′, respectively. How wide is the river?
ANS. 333.4 ft.

20. A tree stands on a slope of 20°. From a point 180 ft. directly up the slope from the tree, the angle of depression of the top of the tree is 12° 20′. Find the height of the tree.

21. Find the angle of elevation of the sun when the longest shadow that a straight rod can cast is five times the length of the rod itself.
ANS. 11° 32′.

22. A pilot is assigned the task of flying a bombing plane to a target 600 miles due east of his own base. If the plane cruises in still air at 200 mi./hr. and a 30-mi./hr. wind is blowing continuously from the northwest, find the flying time required for the round trip flight.

23. When the angle of elevation of the sun is 48°, a pole standing vertically on a slope whose angle with the horizontal is 15° casts a shadow directly down the slope 44.3 ft. long. How tall is the pole? ANS. 36.06 ft.

24. From a ship the horizontal angle subtended by a cylindrical fort is 8° 20′. 680 ft. nearer to the fort the angle becomes 10° 12′. Find the distance of the ship in the first position from the center of the fort.

25. Three circles of radii 2, 3, and 5 are tangent externally. Find the area inclosed between them. ANS. 1.42 sq. units.

26. A triangular garden ABC has sides $c = 60$ ft., $b = 50$ ft., and $a = 30$ ft. A horse is tied at A outside the garden fence by a rope 90 ft. long. Find the area over which he can graze.

Chapter IX

INVERSE TRIGONOMETRIC FUNCTIONS. TRIGONOMETRIC EQUATIONS.

58. INVERSE FUNCTIONS

If the equation $$y = x^2 - 3$$
is solved for x in terms of y, there results the equation
$$x = \pm\sqrt{y + 3}.$$
The functions $x^2 - 3$ and $\pm\sqrt{y+3}$ are said to be *inversely related functions.* Similarly, the equation
$$y = \log_a x$$
yields the equation $$x = a^y,$$
and the functions $\log_a x$ and a^y are inversely related.

In general, given an equation
$$y = f(x),$$
it is customary to express the results obtained by solving the equation for x in terms of y in the symbolic form
$$x = f^{-1}(y).$$
Note that minus 1 as here used is *not an exponent,* but simply a part of the inverse function symbol.

59. INVERSE TRIGONOMETRIC FUNCTIONS

Let x be a number such that $-1 \leqq x \leqq +1$, and consider the equation
$$\sin\theta = x.$$

The inverse relation can be stated in the form θ *is an angle whose sine is* x. This statement is written in the symbolic form

$$\theta = \sin^{-1} x$$

or in the optional form* $\theta = \text{arc}\sin x$.

In a similar manner the inverses of the other trigonometric functions are indicated by $\cos^{-1} x$, $\tan^{-1} x$, etc., or by *arc cos x*, *arc tan x*, etc. (Note that these are angles, not ratios.)

The inverse of a trigonometric function is a *multiple valued* function. For instance, $\theta = \sin^{-1}\frac{1}{2}$ has two values between 0 and 2π, $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, and the general values $\frac{\pi}{6} + 2n\pi$ and $\frac{5\pi}{6} + 2n\pi$, for all integral values of n. Of the infinite number of values of θ thus given, that value of θ which is *numerically* the smallest is called the *principal value* of $\sin^{-1}\frac{1}{2}$ Hence the principal value of $\sin^{-1}\frac{1}{2}$ is $\frac{\pi}{6}$.

In general, the *principal value* of any inverse trigonometric function is that value which is numerically the smallest, if a smallest exists. If a positive and negative value are numerically equal, the positive value is chosen as the principal one. For instance, $\cos^{-1}\frac{1}{2}$ has the values $\frac{\pi}{3}$ and $-\frac{\pi}{3}$, both numerically equal and smaller than any other values. $\frac{\pi}{3}$ is therefore the principal value of $\cos^{-1}\frac{1}{2}$, according to the statement just made.

Let it be agreed that a capital letter will be used to denote the principal value of an inverse trigonometric function, i.e., $\text{Sin}^{-1} x$, $\text{Arc tan}\, x$, $\text{Sec}^{-1} x$, etc. Then, it can easily be verified that

$$-\frac{\pi}{2} \leqq \text{Sin}^{-1} x \leqq +\frac{\pi}{2},$$
$$0 \leqq \text{Cos}^{-1} x \leqq \pi,$$
$$-\frac{\pi}{2} \leqq \text{Tan}^{-1} x \leqq +\frac{\pi}{2},$$
$$-\frac{\pi}{2} \leqq \text{Cot}^{-1} x \leqq +\frac{\pi}{2},$$
$$0 \leqq \text{Sec}^{-1} x \leqq \pi,$$
$$-\frac{\pi}{2} \leqq \text{Csc}^{-1} x \leqq +\frac{\pi}{2}.$$

* The alternate form $\theta = \text{arc}\sin x$ is derived from the fact that both arc and its central angle are measured in degrees, or in the case of circular measure, that the number of radii in the arc equals the number of radians in the central angle.

EXAMPLE 1. Find tan $(\text{Sin}^{-1} \frac{1}{2})$ and sin $(\text{Sin}^{-1} \frac{1}{2})$.

SOLUTION: Construct $\text{Sin}^{-1} \frac{1}{2}$ as an acute angle of a right triangle with the opposite side 1 and the hypotenuse 2. The adjacent side will then be $\sqrt{2^2 - 1} = \sqrt{3}$.

From the figure

$$\alpha = \text{Sin}^{-1} \tfrac{1}{2} = \text{Tan}^{-1} \frac{1}{\sqrt{3}}, \text{ etc.}$$

Hence by inspection,

$$\tan (\text{Sin}^{-1} \tfrac{1}{2}) = \frac{1}{\sqrt{3}},$$

and* $\sin (\text{Sin}^{-1} \frac{1}{2}) = \frac{1}{2}$.

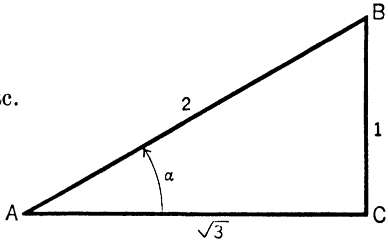


FIGURE 65

EXAMPLE 2. Evaluate cos $[\text{Tan}^{-1} (-\frac{4}{3})]$.

SOLUTION: Construct the figure with the principal value of $\tan^{-1} (-\frac{4}{3})$, as shown.

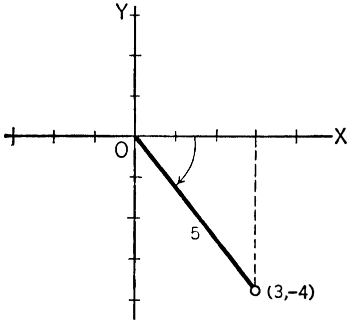


FIGURE 66.

From the figure $\cos [\text{Tan}^{-1} (-\frac{4}{3})] = \frac{3}{5}$.

EXAMPLE 3. Verify the identity $\text{Arc sin} \frac{1}{2} + \text{Arc cos} \frac{1}{2} = \frac{\pi}{2}$.)

SOLUTION: Construct the angles

$$\alpha = \text{Arc sin } \tfrac{1}{2} \text{ and } \beta = \text{Arc cos } \tfrac{1}{2}.$$

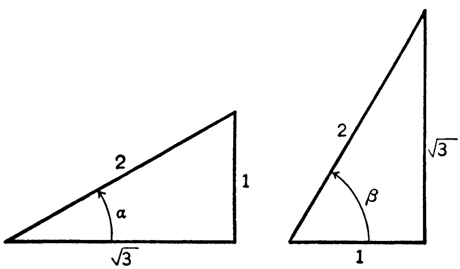


FIGURE 67.

* sin (Arc sin α) $= \alpha$. The name of a man whose name is John—is John.

Let $$\alpha + \beta = K.$$

Taking the sine of both sides of the equation,

$$\sin(\alpha + \beta) = \sin K,$$
$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin K.$$

But from the figure,

$$\sin\alpha = \tfrac{1}{2},\ \cos\alpha = \frac{\sqrt{3}}{2},\ \sin\beta = \frac{\sqrt{3}}{2},\ \cos\beta = \tfrac{1}{2}.$$

Therefore, $(\frac{1}{2})(\frac{1}{2}) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{4} + \frac{3}{4} = 1 = \sin K$

since α and β are both positive angles it is evident that

$$K = \frac{\pi}{2}.$$

EXAMPLE 4. Find sec (Arc tan y).

SOLUTION: Let α = Arc tan y, then $\tan\alpha = y$, and

$$\sec(\text{Arc}\tan y) = \sec\alpha = \sqrt{1 + \tan^2\alpha} = \sqrt{1 + y^2}.$$

The positive sign of the radical must be chosen since α is in the first or fourth quadrants.

EXAMPLE 5. Find or evaluate cos [Arc tan $(-\frac{4}{3})$].

SOLUTION: Let α = Arc tan $(-\frac{4}{3})$, then $\tan\alpha = -\frac{4}{3}$, and

$$\cos[\text{Arc}\tan(-\tfrac{4}{3})] = \cos\alpha = \frac{1}{\sec\alpha} = \frac{1}{\sqrt{1 + \tan^2\alpha}}$$
$$= \frac{1}{\sqrt{(-\frac{4}{3})^2 + 1}}$$
$$= \tfrac{3}{5}.$$

EXERCISES

Find the values of the following:

1. Arc sin 1. ANS. $\frac{\pi}{2}$.
2. Arc cos 0.
3. $\text{Tan}^{-1}\sqrt{3}$. ANS. $\frac{\pi}{3}$.
4. $\text{Cot}^{-1}\,1$.
5. Arc sec 2. ANS. $\frac{\pi}{3}$.
6. Arc csc 2.
7. sec $(\text{Sec}^{-1}\,y)$. ANS. y.
8. cos (Arc sin y).
9. tan (Arc sec y). ANS. $\sqrt{y^2 - 1}$.
10. cot (Arc tan y).

Find the following, using always the principal values of the functions:

11. $\text{Cos}^{-1}\,0 + \text{Sin}^{-1}\,\frac{1}{2}$. ANS. $\frac{2\pi}{3}$.
12. $\text{Tan}^{-1}\sqrt{3} + \text{Sec}^{-1}2$.
13. $\text{Cos}^{-1}(\cos 112°)$. ANS. 112°.
14. $\text{Sin}^{-1}(\sin 90°)$.
15. $\text{Tan}^{-1}(\sin 270°)$. ANS. $-\frac{\pi}{4}$.
16. Arc cot (cos 100°).

17. $\text{Arc}\cos\sqrt{\frac{(1+\cos x)}{2}}$. ANS. $\frac{x}{2}$.

18. $\text{Sin}^{-1}(\tan - \frac{1}{2})$.

19. $\text{Tan}^{-1}(0.3346)$. ANS. $18°\ 30'$.

20. $\text{Cos}^{-1}(-0.4899)$.

21. $\text{Sin}^{-1}(-0.7451)$. ANS. $-48°\ 10'$.

22. $\text{Cot}^{-1}(1.580)$.

Find the value without the use of tables:

23. $\sin(2\ \text{Sin}^{-1}\frac{4}{5})$. ANS. $\frac{24}{25}$.

24. $\tan(2\ \text{Arc}\tan 3)$.

25. $\tan(\frac{1}{2}\ \text{Cos}^{-1}\frac{3}{5})$. ANS. $\frac{1}{2}$.

26. $\cos(2\ \text{Sin}^{-1}\frac{3}{5})$.

27. $\sin(\text{Cos}^{-1}x + \text{Cos}^{-1}2y)$. ANS. $2y\sqrt{1-x^2} + x\sqrt{1-4y^2}$.

28. $\cos(\text{Sin}^{-1}2x - \text{Sin}^{-1}3y)$.

29. $\tan(\text{Tan}^{-1}x + \text{Cot}^{-1}2y)$. ANS. $\frac{2xy+1}{2y-x}$.

30. $\sin^2(\frac{1}{2}\ \text{Cos}^{-1}2x)$.

60. TRIGONOMETRIC EQUATIONS

An equation involving trigonometric functions of the unknown (or unknowns) is called a trigonometric equation. If the equation is true for every value of the unknown(s) for which the equation is defined, it is called an **identity.** Thus the equations,

$$\sin^2 x + \cos^2 x = 1,$$
$$\sin(x+y) = \sin x\cos y + \cos x \sin y,$$

and

$$x^2 - a^2 = (x+a)(x-a)$$

are identities, the first two being true for all values of the variable angles x and y, and the last being true for any value of the variable x. The first two are called trigonometric identities, the last, an algebraic identity.

On the other hand, a **conditional equation,** usually called simply an equation, is true only for certain values of the variables. An example of a conditional trigonometric equation is

$$\sin x = \tfrac{1}{2},$$

which is obviously satisfied only by certain values of the angle x. In the same manner the conditional equation

$$3x = 15$$

is satisfied by the single value of $x = 5$.

In the following article the general solutions of certain simple trigonometric equations will be given. This will be followed in the next article with suggestions for reducing the solution of more complicated types of trigonometric equations to these simple forms.

61. SOLUTIONS OF THE EQUATIONS SIN $\theta = a$, COS $\theta = a$, TAN $\theta = a$

First consider the special cases for $a = 0$. From Art. 33, the solutions are obtained by inspection. Obviously $\sin\theta = 0$ for the values $\theta = 0$, $\theta = \pm\pi$, $\theta = \pm 2\pi$, etc., so that, if n is any integer (positive, negative, or zero), the equation

(53) **$\sin\theta = 0$ has solutions given by the equation $\theta = n\pi$.**

Similarly, the following results can be obtained:

(54) **$\cos\theta = 0$ has solutions given by the equation $\theta = \frac{\pi}{2} + n\pi$,**

(55) **$\tan\theta = 0$ has solutions given by the equation $\theta = n\pi$,**

(56) **$\cot\theta = 0$ has solutions given by the equation $\theta = \frac{\pi}{2} + n\pi$.**

Evidently the equations $\sec\theta = 0$ and $\csc\theta = 0$ have no real solutions.

Now consider the equation

$$\sin x = a.$$

Obviously, one solution is

$$x = \mathrm{Sin}^{-1} a,$$

and since $\sin x = \sin(\pi - x)$, another solution is

$$x = \pi - \mathrm{Sin}^{-1} a.$$

These solutions will be generalized by the addition of $2n\pi$ (i.e., any multiple of 2π) to them, the first giving the set of solutions given by the equation

$$x = \mathrm{Sin}^{-1} a + 2n\pi,$$

and from the second,

$$\begin{aligned} x &= \pi - \mathrm{Sin}^{-1} a + 2n\pi \\ &= (2n + 1)\pi - \mathrm{Sin}^{-1} a. \end{aligned}$$

These two generalized solutions can be combined into the *complete* set of solutions

$$x = n\pi + (-1)^n \mathrm{Sin}^{-1} a,$$

for the even values of n will give the first set of solutions and the odd values will give the second set. Hence the equation

(57) **$\sin x = a$ has solutions given by the equation**
$x = n\pi + (-1)^n \operatorname{Sin}^{-1} a.$

Next consider the equation

$$\cos x = a.$$

As before, one solution is

$$x = \operatorname{Cos}^{-1} a.$$

But by (21), $\cos x = \cos(-x)$. Hence another solution is

$$x = -\operatorname{Cos}^{-1} a.$$

Thus the equation

(58) **$\cos x = a$ has solutions given by the equation**
$x = 2n\pi \pm \operatorname{Cos}^{-1} a.$

Again, the identity $\tan x = \tan(\pi + x)$ leads by reasoning similar to that used in solving the equation $\sin x = a$ to the result that the equation

(59) **$\tan x = a$ has solutions given by the equation**
$x = n\pi + \operatorname{Tan}^{-1} a.$

Solutions could be given for the equations $\cot x = a$, $\sec x = a$, and $\csc x = a$ but are omitted from this book because of the ease with which the functions here involved can be eliminated by application of the reciprocal relations

$$\cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

EXAMPLE. Find the general solution of the equation $\sin x = \frac{\sqrt{2}}{2}$ and give the values of n which determine solutions which could be angles of a triangle.

SOLUTION: By (57) the general solution is

$$x = n\pi + (-1)^n \operatorname{Sin}^{-1}\left(\frac{\sqrt{2}}{2}\right)$$
$$= n\pi + (-1)^n \left(\frac{\pi}{4}\right).$$

Now arrange the particular solutions corresponding to the various values of n in tabular form as follows:

$n =$	0,	1,	-1,	2,	-2,	3,	-3,	$\cdots$
$x =$	$\frac{\pi}{4}$,	$\frac{3\pi}{4}$,	$\frac{-5\pi}{4}$,	$\frac{9\pi}{4}$,	$\frac{-7\pi}{4}$,	$\frac{11\pi}{4}$,	$\frac{-13\pi}{4}$,	$\cdots$

It is obvious that $\frac{\pi}{4}$, or 45°, and $\frac{3\pi}{4}$, or 135°, are the only solutions which are possible angles of a triangle.

NOTE: In many cases particular solutions of equations of the form treated above can be found by inspection without applying the formulas for the general solutions. In the example just given, the two particular solutions $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ could have been readily found by inspection. The student should be familiar with both methods.

EXERCISES

1. Find the general solution and all positive particular solutions between 0 and 2π for the equation

$$\tan x = \sqrt{3}. \qquad \text{ANS. } n\pi + \frac{\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}.$$

2. Find the general solution and all particular solutions between -2π and $+2\pi$ for the equation

$$\cos x = -0.5.$$

3. Find the general solution of the equation

$$\sin x = -0.5. \qquad \text{ANS. } n\pi + (-1)^{n+1}\frac{\pi}{6}.$$

4. Find the general solution of the equation

$$\cos x = -1.$$

62. SOLUTION OF OTHER FORMS OF TRIGONOMETRIC EQUATIONS

In general, the solutions of trigonometric equations of more complicated form are effected by reducing them to systems of equations of the form treated in the preceding article. No one method of reduction will cover all types, and frequently the discovery of a method of reduction is a matter of ingenuity. The following examples will illustrate the method of solving some of the frequently encountered types.

EXAMPLE 1. *Reducible by algebraic factoring.* Solve the equation

$$2 \sin x \cos x + 2 \sin x - \cos x - 1 = 0.$$

SOLUTION: The equation can be written in the factored form

$$(\cos x + 1)(2 \sin x - 1) = 0,$$

from which, $\cos x + 1 = 0$ or $2 \sin x - 1 = 0.$

The first of these resulting equations can be written

$$\cos x = -1,$$

which, by (58), has the general solution

$$x = 2n\pi \pm \operatorname{Cos}^{-1}(-1)$$
$$= 2n\pi \pm \pi.$$

The second of the resulting equations can be written

$$\sin x = \tfrac{1}{2},$$

which, by (57) has the general solution

$$x = n\pi + (-1)^n \operatorname{Sin}^{-1}(\tfrac{1}{2})$$
$$= n\pi + (-1)^n \frac{\pi}{6}.$$

Hence the solutions of the given equation are

$$x = 2n\pi \pm \pi, \text{ and } x = n\pi + (-1)^n \frac{\pi}{6}.$$

EXAMPLE 2. *Reducible with the aid of the fundamental identities.* Solve the equation for values of x between 0° and 360°.

$$3 \cos x + \sin x = 1.$$

SOLUTION: Substitute for $\sin x$ its value in terms of $\cos x$, and the equation becomes $\quad 3 \cos x + \sqrt{1 - \cos^2 x} = 1,$

or $\quad \sqrt{1 - \cos^2 x} = 1 - 3 \cos x.$

Squaring both sides of the equation,

$$1 - \cos^2 x = 1 - 6 \cos x + 9 \cos^2 x,$$

which reduces to $\quad 5 \cos^2 x - 3 \cos x = 0,$

or, in factored form, $\quad \cos x(5 \cos x - 3) = 0.$

Hence $\quad \cos x = 0, \text{ or } \cos x = \tfrac{3}{5}.$

From (58) $\quad x = 2n\pi \pm \frac{\pi}{2}, \text{ or } x = 2n\pi \pm \operatorname{Cos}^{-1} \tfrac{3}{5}.$

NOTE: The student will recall that solutions of equations obtained by squaring both members or by multiplying both members by a factor containing the unknown may introduce extraneous values. Hence all such solutions should be checked carefully.

In this problem the values that will check in the original equation are

$$x = 2n\pi + \frac{\pi}{2} \text{ and } x = 2n\pi - \operatorname{Cos}^{-1} \tfrac{3}{5}.$$

Hence, $x = \frac{\pi}{2}$, $x = -53° 8'$, and $x = 360° - 53° 8' = 306° 52'$ are solutions between 0° and 360°.

$x = 2n\pi - \frac{\pi}{2}$, and $x = 2n\pi + \operatorname{Cos}^{-1} \tfrac{3}{5}$ are extraneous values.

ALTERNATE SOLUTION OF THIS EXAMPLE.

SOLUTION: Given any two real numbers a and b not both zero, there exists an angle α such that $\sin \alpha = \frac{a}{\sqrt{a^2 + b^2}}$, and $\cos \alpha = \frac{b}{\sqrt{a^2 + b^2}}$, since $\sin^2 \alpha + \cos^2 \alpha = 1$ for every value of a and b not both zero.

Let $a = 3$ and $b = 1$ in the equation

$$3 \cos x + \sin x = 1.$$

Divide both sides of the equation by the $\sqrt{10}$. The resulting equation,

$$\frac{3}{\sqrt{10}}\cos x + \frac{1}{\sqrt{10}}\sin x = \frac{1}{\sqrt{10}},$$

can be written in the form

$$\sin x \cos \alpha + \cos x \sin \alpha = \frac{1}{\sqrt{10}},$$

where $\alpha = \operatorname{Sin}^{-1}\frac{3}{\sqrt{10}}$. This equation reduces to the form

$$\sin(x+\alpha) = \frac{1}{\sqrt{10}}.$$

By (57), this has the general solution

$$x + \alpha = n\pi + (-1)^n \operatorname{Sin}^{-1}\frac{1}{\sqrt{10}},$$

or

$$x = n\pi + (-1)^n \operatorname{Sin}^{-1}\frac{1}{\sqrt{10}} - \operatorname{Sin}^{-1}\frac{3}{\sqrt{10}}.$$

The reduction of this solution to degree measure is left to the student. (It should be noted that any equation of the form $a \cos x + b \sin x = c$ can be solved by this method.)

EXAMPLE 3. *Reducible by the factor formulas.* Solve the equation

$$\sin 5x + \sin 3x + \cos x = 0.$$

SOLUTION: By applying (39), or by using $5x = 4x + x$ and $3x = 4x - x$ and applying the sine addition formulas, the left side of the given equation can be reduced to the form

$$2 \sin 4x \cos x + \cos x = 0,$$

which in factored form is

$$\cos x(2 \sin 4x + 1) = 0.$$

Hence,

$$\cos x = 0, \text{ or } \sin 4x = -\tfrac{1}{2}.$$

The solution of the first equation is $x = n\pi + \frac{\pi}{2}$, and that of the second,

$$x = \tfrac{1}{4}[n\pi + (-1)^n \operatorname{Sin}^{-1}(-\tfrac{1}{2})] = \tfrac{1}{4}\left[n\pi - (-1)^n\left(\frac{\pi}{6}\right)\right].$$

The two solutions together are the solutions of the given equation.

EXERCISES

For each of the following equations, find the general solution and check particular solutions from 0 to 2π.

1. $4 \sin^2 x - 4 \sin x + 1 = 0$. ANS. $x = n\pi + (-1)^n \frac{\pi}{6}$.
2. $\sin x - \cos x = 0$.
3. $3 \tan x - \cot x = 0$. ANS. $x = n\pi \pm \frac{\pi}{6}$.
4. $2 \cos^2 2\alpha + \cos 2\alpha - 1 = 0$.

5. $\cos 2\theta + \sin \theta = 0$. ANS. $\theta = n\pi - (-1)^n \frac{\pi}{6}$; $\theta = 2n\pi + \frac{\pi}{2}$.

6. $4 \csc^2 \varphi - 7 \cot^2 \varphi - 2 = 0$.

7. $\sin 3x + \sin 2x + \sin x = 0$. (*Hint:* Group the first and last terms together.) ANS. $x = \frac{n\pi}{2}$; $x = 2n\pi \pm \frac{2}{3}\pi$.

8. $2 \tan \alpha = 2 - \sec^2 \alpha$.

9. $\cos 4x + \sin 2x = 0$. ANS. $x = \frac{n\pi}{2} + (-1)^{n+1} \cdot \frac{\pi}{12}$; $x = n\pi + \frac{\pi}{4}$.

10. $\sin \frac{x}{2} - \cos x = 1$.

11. $\sin 2\theta = \cos^2 \theta - \sin^2 \theta$. ANS. $\theta = \frac{\pi}{2}(n + \frac{1}{4})$.

12. $\sin 3x = \sin 5x$.

13. $2 \sin x \cos x + \cos x - 2 \sin x = 1$.

ANS. $x = 2n\pi$; $x = \frac{\pi}{6}[6n + (-1)^{n+1}]$.

14. $3 \cos x + 4 \sin x = 5$.

15. $\cos x - \sin x = 1$. ANS. $x = 2n\pi$; $x = \pi(2n - \frac{1}{2})$.

16. $2 \cos x - \sin x = \frac{1}{2}\sqrt{10}$.

17. $\text{Sin}^{-1} x + \text{Sin}^{-1} 2x = \frac{\pi}{2}$. (*Hint:* Take cosine of both sides of the equation.) ANS. $x = \frac{\sqrt{5}}{5}$.

18. $\text{Arc} \cos 2x + \text{Arc} \sin x = \frac{\pi}{4}$.

19. $\sin 6x + \sin 5x + \sin 4x = 0$. ANS. $x = \frac{n\pi}{5}$; $x = 2\pi(n \pm \frac{1}{3})$.

20. $3 \sin x \tan x - 9 \sin x - \tan x + 3 = 0$.

21. Given the pair of equations $\rho \sin \theta = 4$ and $\rho \cos \theta = 3$, find values of ρ and θ which are a simultaneous solution of the equations. (*Hint:* Square both sides of each equation, and add.) ANS. $\rho = 5$, $\theta = \text{Sin}^{-1} \frac{4}{5}$.

22. Solve the pair $\rho \cos \theta = 1$, $\rho \sin \theta = \sqrt{3}$.

23. Solve the pair $\rho \cos \theta = 8$, $\rho \sin \theta = -6$. ANS. $\rho = 10$, $\theta = -\text{Sin}^{-1} \frac{6}{10}$.

24. If the end view of a cylindrical tank is as shown in the figure, and if the length of the tank is L ft., find a formula for the volume of water in the tank in terms of a, h, and L. From this find the volume for $h = 1$, $a = 3$, and $L = 10$.
ANS. $L(a^2 \sin^{-1} \frac{h}{a} + \frac{\pi a^2}{2} + h\sqrt{a^2 - h^2})$; 200 cu. ft.

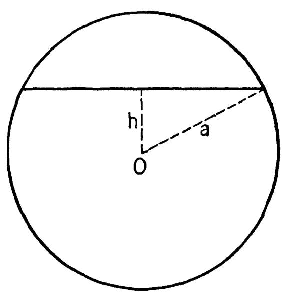


FIGURE 68.

25. *For those who want a mettle tester.* A vertical tower of height 100 feet stands on a hillside. Two stones in a line directly downhill from the tower are 50 feet apart. The angles of depression of these two stones from the top of the tower are 42° and 48°, respectively. Find the angle of inclination of the hill. ANS. 24° 5′.

Chapter X

COMPLEX NUMBERS AND DE MOIVRE'S THEOREM

63. COMPLEX NUMBERS

It is the usual practice to represent the imaginary unit $\sqrt{-1}$ by the symbol i.

Note that if $\quad i = \sqrt{-1},$

then, $\quad i^2 = -1,$

$i^3 = -i,$

$i^4 = +1,$

$i^5 = i,$ and so on.

It is readily seen that odd powers of i will equal either $+i$ or $-i$, and even powers of i will equal either 1 or -1. A **complex number** is a number of the form $a + bi$, where a and b are real. The number a is called the **real part** and bi the **imaginary part.** Any real number may be expressed as a complex number in which b is 0. If a is 0, the number is called a **pure imaginary number.** Any complex number, not real, is imaginary, except that 0 may be considered as either real or as a pure imaginary. The number 5 can be expressed as $5 + 0i$; $\sqrt{-5}$ may be expressed as $0 + i\sqrt{5}$.

Two complex numbers $a + bi$ and $c + di$ are equal if $a = c$ and $b = d$.

In the four fundamental operations involving complex numbers, i is subject to all the operational rules of real numbers, as $(5 - 3i)^2 = 25 - 30i + 9i^2 = 16 - 30i$.

If two complex numbers are of the form $a + bi$ and $a - bi$, each is called the *conjugate* of the other. For example, the conjugate of $2 - 3i$ is $2 + 3i$.

64. GRAPHICAL REPRESENTATION OF COMPLEX NUMBERS

Using the co-ordinate axes as illustrated in Fig. 69, the number $a + bi$ is represented by the point P with directed distance a from the imaginary axis, and with distance b from the real axis. These co-ordinate axes differ from the usual rectangular system in that the X-axis becomes the axis of real numbers, and the Y-axis, the axis of pure imaginary numbers. The plane which these axes determine is called the *complex plane.*

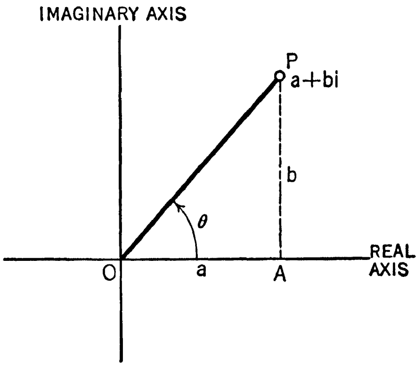


FIGURE 69.

The length r of the **radius vector** OP is called the **modulus** of the complex number $a + bi$, and θ is called its **amplitude** or **argument.**

Let $\quad OP = r,$

then,

$$a = r\cos\theta,$$
$$b = r\sin\theta,$$
$$a + bi = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta).$$

This is called the **polar form** of the complex number.

EXERCISES

Plot each number and find its modulus and smallest positive amplitude:

1. $3 + 4i$. Ans. $r = 5, \theta = \text{Sin}^{-1}\frac{4}{5}$.
2. $3\cos 30° + 3i\sin 30°$.
3. $6 - 5i$. Ans. $r = \sqrt{61}, \theta = 2\pi - \text{Sin}^{-1}\left(\frac{-5}{\sqrt{61}}\right)$.
4. $2\cos(-60°) + 2i\sin(-60°)$.
5. 7. Ans. $r = 7, \theta = 0$.
6. $\cos 180° + i\sin 180°$.
7. $4i$. Ans. $r = 4, \theta = \frac{\pi}{2}$.
8. $\cos 270° + i\sin 270°$.

65. ADDITION AND SUBTRACTION OF COMPLEX NUMBERS

The sum of two complex numbers $a + bi$ and $c + di$ is a complex number defined by the equation

$$(a + bi) + (c + di) = (a + c) + i(b + d).$$

The geometric sum of two complex numbers is obtained as follows: Draw the radius vector of each of the two complex numbers $a + bi$ and $c + di$; with these as adjacent sides of a parallelogram complete the figure, as in Fig. 70.

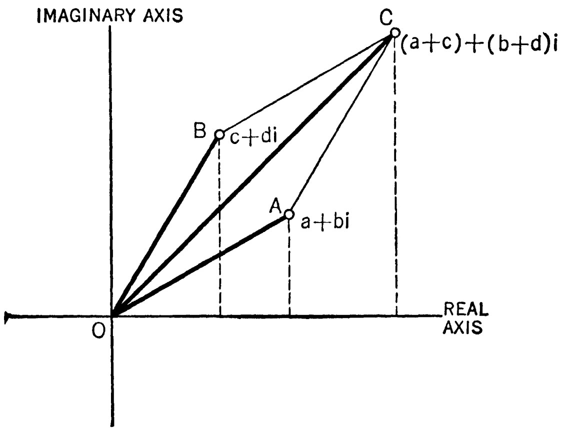


FIGURE 70.

From the geometry of the figure, $(a + bi) + (c + di) = (a + c) + i(b + d)$. The modulus of the new complex number is

$$| OC | = \sqrt{(a + c)^2 + (b + d)^2}.$$

Now consider $(a + bi) - (c + di) = (a - c) + i(b - d)$. The geometric subtraction of two complex numbers is obtained as follows: Draw the radius vector of each of the two complex numbers $a + bi$ and $- c - di$, and proceed as in addition. The length of the diagonal drawn from O of the resulting parallelogram is the modulus of the difference of the two complex numbers. (See Fig. 71.) In the figure the modulus of the difference is

$$| OC | = \sqrt{(a - c)^2 + (b - d)^2}.$$

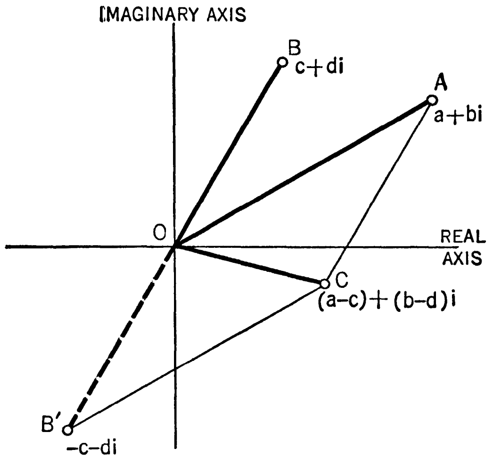


FIGURE 71.

EXERCISES

Perform the indicated operation algebraically and check graphically:

1. $(4 + 2i) + (3 - 2i)$.
2. $(6 + 0i) + (0 - 2i)$.
3. $(2 - i) + (3 + 4i)$.
4. $(3 + 2i) - (4 + 7i)$.
5. $(1 + 2i) + (-2 - 3i)$.
6. $(4 - 5i) - (2i + 1)$.
7. $4 + (2i - 3)$.
8. $3 - 2i$.
9. $2i + (4 - 6i)$.
10. $4i - 5$.
11. Find graphically: $(2 - 3i) + (3 + 2i) - (1 - 4i)$.

66. MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS

Given the complex numbers $a + bi$ and $c + di$. To find the product, proceed algebraically, that is

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2\\ &= ac - bd + i(ad + bc).\end{aligned}$$

The modulus of the product is $\sqrt{(ac - bd)^2 + (ad + bc)^2}$; the amplitude is $\theta = \tan^{-1}\left(\frac{ad + bc}{ac - bd}\right)$.

EXAMPLE. Find the product of $3 + 4i$ and $2 - 3i$.

SOLUTION: $(3 + 4i)(2 - 3i) = 6 - i - 12i^2 = 18 - i$.

The quotient of two complex numbers can be reduced to the standard form of complex numbers by expressing the quotient in the fraction form and then multiplying numerator and denominator of the fraction by the conjugate of the denominator. Thus

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di} \quad (c \text{ and } d \text{ not both } 0)$$
$$= \frac{ac+bci-adi-bdi^2}{c^2+d^2}$$
$$= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i.$$

EXAMPLE. Divide $3+4i$ by $2-3i$.

SOLUTION:
$$\frac{3+4i}{2-3i} = \frac{3+4i}{2-3i}\cdot\frac{2+3i}{2+3i}$$
$$= \frac{-6+17i}{13}$$
$$= -\tfrac{6}{13} + \tfrac{17}{13}i.$$

EXERCISES

Perform the indicated operation in the following:

1. $(2+3i)(2-3i)$. ANS. 13.
2. $(2+3i)(3i-2)$.
3. $(4i)(2i)$. ANS. -8.
4. $\dfrac{2}{i}$
5. $(\sqrt{-3})(\sqrt{-7})$. ANS. $-\sqrt{21}$.
6. $\dfrac{3i}{(2-i)}$.
7. $(\sqrt{-2}+7)(3+\sqrt{-5})$.
   ANS. $21-\sqrt{10}+(7\sqrt{5}+3\sqrt{2})i$.
8. $\dfrac{(\sqrt{-3}+5)}{(4-\sqrt{-2})}$.

67. THE PRODUCT OF TWO OR MORE COMPLEX NUMBERS IN THE POLAR FORM. DE MOIVRE'S THEOREM

If two complex numbers

$$z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2$$

are expressed in *polar form,* their product is

$$z_1z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2).$$

By actual multiplication and the use of the fact that $i^2 = -1$, this product can be written in the form

$$z_1z_2 = r_1r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$
$$= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)].$$

Using this result, the product of three complex numbers is easily obtained, as

$$z_1z_2z_3 = r_1r_2r_3[\cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)].$$

When the process is continued, the general result is

$$z_1z_2\cdots z_n = r_1r_2\cdots r_n\,[\cos(\theta_1+\theta_2+\cdots+\theta_n)+i\sin(\theta_1+\theta_2+\cdots+\theta_n)].$$

If now each of the factors in this formula is set equal to

$r(\cos\theta + i\sin\theta)$, there results

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

In case $r = 1$, the complex number is located graphically on a unit circle with center at the origin, and the formula becomes *De Moivre's Theorem:*

(60) $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

(NOTE: De Moivre's Theorem is proved here for the case when n is a positive integer. It can be shown that it is also true if n is any rational number.)

EXAMPLE 1. Express $(\sqrt{3} + i)^{10}$ as a complex number.

SOLUTION: First express $\sqrt{3} + i$ in the form

$$r(\cos\theta + i\sin\theta)$$

by remembering that $r = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2$,

and that $\theta = \text{Tan}^{-1}\left(\frac{y}{x}\right) = \text{Tan}^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

Hence $\theta = \frac{\pi}{6}$, and $\sin\theta = \frac{1}{2}$, and $\cos\theta = \frac{\sqrt{3}}{2}$.

Therefore $\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

By De Moivre's Theorem,

$$\begin{aligned}(\sqrt{3} + i)^{10} &= 2^{10}\left(\cos\frac{10\pi}{6} + i\sin\frac{10\pi}{6}\right)\\ &= 1024\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\\ &= 512 - 512\sqrt{3}\,i.\end{aligned}$$

EXAMPLE 2. Find the three cube roots of unity (i.e., of 1).

SOLUTION: Here $r(\cos\theta + i\sin\theta) = 1$.

Hence, $r = 1$, and $\theta = 0, 2\pi, 4\pi$, etc.
When $\theta = 0$,

$$\begin{aligned}\sqrt[3]{1} &= (\cos 0 + i\sin 0)^{\frac{1}{3}}\\ &= (\cos\tfrac{0}{3} + i\sin\tfrac{0}{3})\\ &= 1, \text{ the first cube root of unity.}\end{aligned}$$

When $\theta = 2\pi$,

$$\begin{aligned}\sqrt[3]{1} &= (\cos 2\pi + i \sin 2\pi)^{\frac{1}{3}}\\ &= \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)\\ &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \text{ the second cube root of unity.}\end{aligned}$$

When $\theta = 4\pi$,

$$\sqrt[3]{1} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}.$$

The values 6π, 8π, etc., for θ will be found to yield repetitions of the above values.

(NOTE. In general there will be found n distinct nth roots of unity.)

EXERCISES

1. Express the following complex numbers in the form $r(\cos \theta + i \sin \theta)$:
   (a) $2\sqrt{3} + 2i$. ANS. $4\left[\cos\left(\frac{\pi}{6} + 2n\pi\right) + i \sin\left(\frac{\pi}{6} + 2n\pi\right)\right]$.
   (b) $1 + i\sqrt{3}$.
   (c) i. ANS. $\cos\left(\frac{\pi}{2} + 2n\pi\right) + i \sin\left(\frac{\pi}{2} + 2n\pi\right)$.
   (d) $-1 - i$.
2. Reduce the following to the form of a complex number by use of De Moivre's Theorem. Check by algebraic multiplication:
   (a) $(1 + i)^4$. (c) $(-2i)^5$.
   (b) $(1 + i\sqrt{3})^3$. (d) $(-3 + 2i)^3$.
3. By De Moivre's Theorem reduce each of the following to the form $a + bi$:
   (a) $\pm\sqrt{i}$. ANS. $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$; $-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$.
   (b) $\pm\sqrt{-1 + i}$.
   (c) $[32(\cos 60° + i \sin 60°)]^{\frac{1}{5}}$. ANS. $1.956 + 0.416i$.
   (d) $\pm(2\sqrt{3} + 2i)^{\frac{1}{2}}$.
   (e) $\sqrt[3]{5}$. ANS. $\sqrt[3]{5}$ times each of the cube roots of unity.
   (f) $\sqrt[3]{8}$.
   (g) $\pm\sqrt{1}$. ANS. 1; -1.
   (h) $\sqrt[3]{-1 - i}$.
4. Solve the equation $x^3 - 1 - i = 0$.
   ANS. $1.08 + 0.29i$; $-0.79 + 0.79i$; $-0.29 - 1.08i$.
5. Solve the equation $x^3 - 4\sqrt{3} + 4i = 0$.

68. THE QUOTIENT OF TWO COMPLEX NUMBERS IN THE POLAR FORM

Given the two complex numbers

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1),$$
$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2).$$

Then,
$$\frac{z_1}{z_2} = \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)}. \quad (r_2 \neq 0)$$

Multiply the numerator and denominator of the right-hand member by $\cos\theta_2 - i\sin\theta_2$, and there results

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2}(\cos\overline{\theta_1 - \theta_2} + i\sin\overline{\theta_1 - \theta_2}).$$

This result can be expressed in words as follows: "The modulus of the quotient of two complex numbers is the quotient of their moduli, and the amplitude of the quotient is the amplitude of the numerator less that of the denominator (denominator not zero)."

EXAMPLE. Divide $3 + 4i$ by $2 - 3i$.

SOLUTION: Here $r_1 = \sqrt{3^2 + 4^2} = 5$, $\theta_1 = \text{Tan}^{-1}\frac{4}{3}$, $r_2 = \sqrt{2^2 + 3^2}, = \sqrt{13}$, and $\theta_2 = \text{Tan}^{-1}(-\frac{3}{2})$. From the tables it is found that $\theta_1 = 53°\ 8'$, and $\theta_2 = -\ 56°\ 19'$.

Therefore, $$\frac{3 + 4i}{2 - 3i} = \left(\frac{5}{\sqrt{13}}\right)(\cos 109°\ 27' + i\sin 109°\ 27')$$

$$= \left(\frac{5}{\sqrt{13}}\right)(-\ 0.3330 + 0.9429\,i).$$

This checks with the example given under Art. 66.

EXERCISES

Reduce the following complex numbers to polar form, find the quotient and plot all points:

1. Divide $3 + 2i$ by $1 - i$.
2. Divide $3i + 2$ by $3 + 2i$.
3. Divide i by $3 + 4i$.
4. Divide 5 by $i - 1$.
5. Divide $3i$ by $4i$.
6. Divide 4 by 2.
7. Divide $\sqrt{3} - i$ by $1 + i$.
8. Divide $i + \sqrt{3}$ by $1 - i$.

69. REVIEW EXERCISES IN PLANE TRIGONOMETRY

1. Express the following angles in terms of radians, if degrees are given, and in terms of degrees, if radians are given:

   (a) $\frac{\pi}{2}$. (c) $\frac{3\pi}{2}$. (e) $\frac{4\pi}{3}$. (g) 120°. (i) 135°.

   (b) $\frac{2\pi}{3}$. (d) $\frac{5\pi}{6}$. (f) 180°. (h) 315°. (j) 225°.

2. Prove the eight fundamental trigonometric identities.
3. Find the value of each of the following:

   (a) $\cos 45° \cos 135° - \sin 45° \sin 135°$.

   (b) $\dfrac{\tan 240° + \tan 60°}{1 - \tan 240° \tan 60°}$.

4. Find the general values of the following and construct the values less than 360°:

   (a) arc sin $\frac{3}{5}$. (c) arc tan $\frac{3}{4}$. (e) arc cot $(-\ 1)$.

   (b) arc cos $\frac{4}{5}$. (d) arc csc 2. (f) arc sec $(-\ 2)$.

5. Prove the following identities:

(a) $\dfrac{\cot A \cos A}{\cot A + \cos A} = \dfrac{\cot A - \cos A}{\cot A \cos A}$.

(b) $\dfrac{1 + \csc x}{\csc x - 1} = \dfrac{1 + \sin x}{1 - \sin x}$.

(c) $\dfrac{\cos B}{1 - \sin B} = 2 \tan B + \dfrac{1 - \sin B}{\cos B}$.

(d) $\dfrac{(1 + \sin x)}{2 \cos x}\left[\sqrt{\dfrac{1 - \sin x}{1 + \sin x}} + \sqrt{\dfrac{1 + \sin x}{1 - \sin x}}\right]\sqrt{\dfrac{1 - \sin x}{1 + \sin x}} = \sec x$.

(e) $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{\csc B + \csc A}{\csc B - \csc A}$.

6. Find the values of θ between 0 and 360° which satisfy each of the following equations:

(a) $3 \tan^2 \theta - 2\sqrt{3} \tan \theta + 1 = 0$.
(b) $2 \cos^2 2\theta + \cos 2\theta - 1 = 0$.
(c) $\sin 2\theta = 2 \sin \theta$.
(d) $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$.

7. A regular polygon of twelve sides is inscribed in a circle of radius 20.6. Find the length of one side of the polygon.

8. Express the following functions as functions of acute positive angles:

(a) sin 160°.
(b) cot 265°.
(c) sec 275°.
(d) cos 281° 30′.
(e) tan (− 68° 30′).
(f) cot 1050°.
(g) csc 840°.
(h) sin 700°.
(i) cot (− 425°).
(j) sin (− 450°).

9. Show that

(a) $\tan (225° - A) = \tan (45° - A)$.
(b) $\sin (135° + A) = \sin (45° - A)$.

10. Find a value of B for which

(a) $\sin 3B = \cos 2B$.
(b) $\tan B = \cot (2B - 15°)$.

11. Make a diagram representing by line segments the values of the six trigonometric functions of (1) 135°, (2) 120°, (3) 225°, (4) 315°, (5) − 150°.

12. Graph the curves represented by the following equations for values of x between 0 and 2π:

(a) $y = \sin x$.
(b) $y = \cos x$.
(c) $y = \tan x$.
(d) $y = \cot x$.
(e) $y = \sec x$.
(f) $y = \csc x$.
(g) $y = \sin 2x$.
(h) $y = 2 \sin 2x$.
(i) $y = 2 \sin \left(2x - \dfrac{\pi}{6}\right)$.
(j) $y = \cos 2x$.
(k) $y = 3 \cos 2x$.
(l) $y = 3 \cos \left(2x - \dfrac{\pi}{6}\right)$.
(m) $y = \sin x + \cos x$.
(n) $y = x^2 + \sin x$.
(o) $y = x + \tan x$.
(p) $\rho = 2 \sin 2x$.
(q) $\rho = 3 \cos 3x$.

13. Find the values of each of the following:

(a) $\tan (\text{Arc cot } \sqrt{3})$.
(b) $\cos (\text{Sin}^{-1} \tfrac{1}{2})$.
(c) $\sin [\text{Arc sin } (-\tfrac{1}{2})]$.
(d) $\sin [\text{Arc cot } (-\sqrt{3})]$.

14. Find a general expression for the general values of A which satisfy each of the following equations, and write all positive values of A less than 360°:

(a) $3 \tan A = 2 \cos A$.
(b) $3 \tan^2 A - 2\sqrt{3} \tan A + 1 = 0$.
(c) $2 \cos A - \cot A = 0$.
(d) $\sin^3 A - \cos^3 A = 0$

15. Find the principal value of each of the following:

(a) $\cos^{-1} \frac{\sqrt{2}}{2}$.
(b) $\sin^{-1} (-\frac{1}{2})$.
(c) $\tan^{-1} (\sqrt{3})$.
(d) $\cot^{-1} \left(-\frac{1}{\sqrt{3}}\right)$.
(e) $\sec^{-1} (2)$.
(f) $\csc^{-1} (-2)$.
(g) $\cos^{-1} (2)$.
(h) $\tan^{-1} (-1)$.
(i) $\cot^{-1} (-\sqrt{3})$.
(j) $\sin^{-1} \left(\frac{\sqrt{3}}{2}\right)$.

16. Given positive angles A and B, whose sum is less than 90°, prove

(a) $\sin (A + B) = \sin A \cos B + \cos A \sin B$.
(b) $\cos (A + B) = \cos A \cos B - \sin A \sin B$.
(c) $\tan (A + B) = \frac{(\tan A + \tan B)}{(1 - \tan A \tan B)}$.

17. Using the results of problem 16, prove the double angle and half angle formulas for a positive angle less than 45°.

18. Find the values of the six trigonometric functions of

(a) 15°.
(b) 105°.
(c) 75°.
(d) $22\frac{1}{2}$°.
(e) 195°.
(f) $52\frac{1}{2}$°.

19. Complete the following formulas:

(a) $\sin A + \sin B =$
(b) $\sin A - \sin B =$
(c) $\cos A + \cos B =$
(d) $\cos A - \cos B =$

20. Express $\sin 3A$ in terms of $\sin A$.

21. Express $\cos 3A$ in terms of $\cos A$.

22. Simplify $\frac{\sin 70° + \sin 50°}{\cos 70° - \cos 50°}$.

23. Prove the following identities:

(a) $\frac{\tan x + \tan y}{\tan x - \tan y} = \frac{\sin (x + y)}{\sin (x - y)}$.

(b) $\frac{1 + \sin B - \cos B}{1 + \sin B + \cos B} = \tan \frac{B}{2}$.

(c) $\frac{2 \sin B - \sin 2B}{2 \sin B + \sin 2B} = \tan^2 \frac{B}{2}$.

24. Give the complete proof that the Law of Sines holds for the triangle shown in the figure.

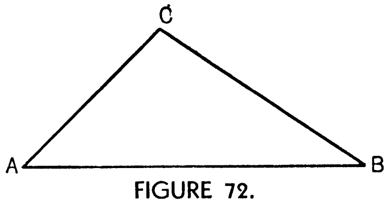


FIGURE 72.

25. Give the complete proof that the Law of Cosines holds for the triangle of problem 24.

26. In each of the following cases, three parts of an oblique triangle are given. Write out the formulas which should be used to find each of the missing parts:

(a) b, c, α.
(b) a, c, α.
(c) a, b, γ.
(d) α, β, b.
(e) a, b, c.
(f) α, β, γ.
(g) α, γ, c.
(h) a, c, β.
(i) α, γ, b.
(j) α, β, c.

27. Express each of the following complex numbers in polar form:

(a) $2 + 2i$.
(b) $1 + \sqrt{3}\, i$.
(c) $1 - i$.
(d) $-3 - 3i$.

28. Express each of the following complex numbers in the form $a + bi$:
(a) $(1 - i)^3$. (b) $(1 + i)^5$. (c) $(1 + \sqrt{3}\,i)^{50}$.

29. Find the (a) three cube roots of 1; (b) five 5th roots of 32; (c) six 6th roots of 64; (d) three cube roots of $(1 + i)$.

30. Given the accompanying figure, prove that

(1) (a) $x = \dfrac{a \sin \alpha \sin \beta}{\sin (\alpha - \beta)}$.

(b) $y = \dfrac{a \cos \alpha \sin \beta}{\sin (\alpha - \beta)}$.

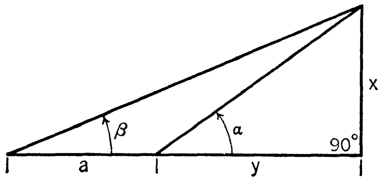


FIGURE 73.

Part Two

SPHERICAL TRIGONOMETRY

Chapter XI

USEFUL FACTS ABOUT SOLID FIGURES

70. FUNDAMENTAL DEFINITIONS AND RELATIONS

A figure is a **three-dimensional figure** if not all of it lies in a single plane. Thus a plane and an intersecting straight line form a three-dimensional figure.

A line is **perpendicular to a plane** if it is perpendicular to any (i.e., every) line in the plane drawn through the point of intersection of the line and plane. By the **distance from a point to a plane** is meant the distance from the point to the plane along a line perpendicular to the plane. A line is parallel to a given plane if it has no point in common with the plane.

Two planes are parallel if they have no points in common. The distance between two parallel planes is the length of the perpendicular line segment joining them.

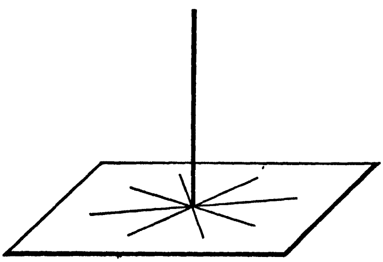


FIGURE 74.

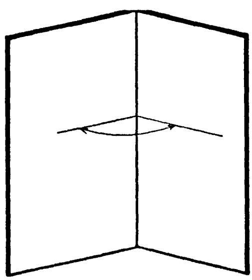


FIGURE 75.

A line cutting two or more planes is called a **transversal.** If three or more planes are parallel, they intercept proportional segments on two transversals. From this it is seen that any line segment joining two parallel planes is bisected by a third plane parallel to the given planes and midway between them.

When two planes intersect they form a **dihedral angle.** More properly, they form four dihedral angles. A dihedral angle may be defined as the figure formed by two **half-planes** with a common **edge.** The half-planes are called the **faces** of the dihedral angle. A dihedral angle is measured in degrees or in radians, the measure being that of the **plane angle of the dihedral angle,** which is the angle formed by the intersections of the faces of the dihedral angle with a third plane perpendicular to the edge of the dihedral angle, as shown in Fig. 75. If two planes form a right dihedral angle, they are said to be perpendicular.

If three or more planes meet in a unique point, the resulting figure will be of the type illustrated in Fig. 76. This is called a **polyhedral angle.** The common point is the **vertex** and the lines of intersections of the planes are the **edges** of the polyhedral angle. The plane angles at the vertex are called its **face angles.** It can be shown that the sum of the face angles of a convex polyhedral angle (one such that the polygon formed by the intersection by a plane with all of the faces is a convex polygon) is less than 360°. A trihedral angle is a polyhedral angle with three faces.

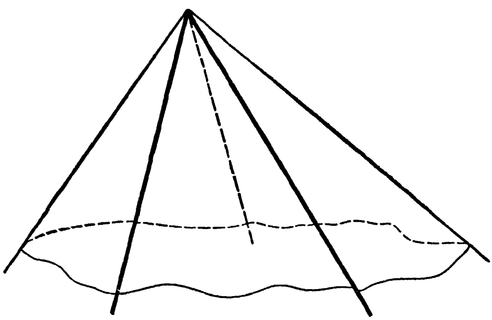


FIGURE 76.

A **polyhedron** is a closed surface formed by plane faces. The line segments which are the intersections of the faces are the **edges** of the polyhedron. A **regular** polyhedron is one whose faces are all congruent regular polygons and whose polyhedral angles are congruent. There are only five regular polyhedrons when classified as to number of faces. A regular polyhedron of four, six, eight, twelve, or twenty faces is called a regular tetrahedron, hexahedron (cube), octahedron, dodecahedron, or icosahedron, respectively.

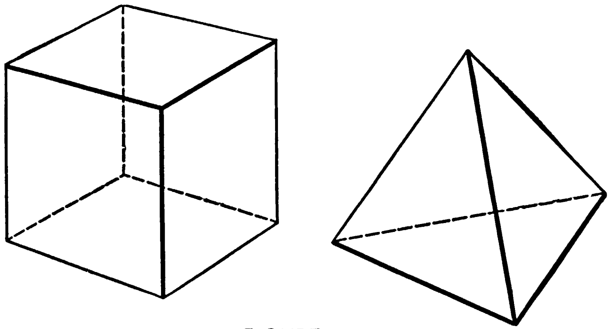


FIGURE 77.

71. SIMILAR FIGURES

Informally speaking, two figures are **similar** if they have the same shape. More precisely, two figures are similar if to every point of one there corresponds a single point of the other in such a manner that the distances between corresponding pairs of points of one of them are proportional to the corresponding distances of the other. Unless two figures are congruent as well as similar, one of them may be looked upon as an undistorted magnification of the other.

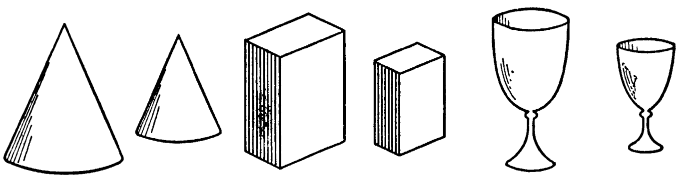


FIGURE 78.

Formulas for surface area and volume for certain special three-dimensional figures will presently be stated. However, corresponding distances, areas, and volumes of similar figures bear important proportional relations which enable useful results to be obtained, even when no formulas exist. The three relations, the first of which is obvious from the definition of similar figures, follow:

Corresponding distances on (or within) similar figures are proportional.

Corresponding areas of similar figures are proportional to the *squares* of corresponding distances.

Corresponding volumes of similar figures are proportional to the *cubes* of corresponding distances.

EXERCISES

1. How many different planes can be passed through a given point? a given line? a line and a point not on the line? two intersecting lines?
2. How many lines can be drawn through a given point parallel to a given plane which does not contain the point? How many lines can be drawn through a given point parallel to each of two intersecting planes?
3. Two lines are said to be *parallel* if they lie in the same plane and have no point in common. They are said to be *skew lines* if they do not lie in a plane and have no point in common. How many lines can be drawn perpendicular to each of two parallel lines? How many can be drawn perpendicular to each of two skew lines?
4. If a straight line is perpendicular to a given plane, what can be said of every plane passing through this line?
5. In plane geometry it is proved that two lines which are perpendicular to a given line are parallel to each other. Does this make a true statement if the words *lines* and *line* are replaced by the words *planes* and *plane?* Illustrate your answer.
6. Give illustrations seen daily of: parallel planes, perpendicular planes, dihedral angle, trihedral angle, skew lines, similar solids.
7. Assuming that weight is proportional to volume in the similar solids considered in these exercises, what is the weight of a man 5 feet 5 inches tall if a man of similar build who is 5 feet 10 inches tall weighs 180 pounds? ANS. 144 lbs.
8. From the vertices of a square, line segments are drawn to a point not in the plane of the square. A second plane is passed through these segments so as to be parallel to the plane of the square and midway between it and the given point. Find the relation of the area of the given square to the area of the square whose vertices are the points of intersection of the second plane with the four segments.
9. A certain athletic trophy is 26 inches tall, rests upon a base covering 50 square inches, and weighs 12 pounds. What is the area of the base and the weight of a similar trophy 13 inches tall? ANS. 12.5 sq. in.; 1.5 lbs.
10. A certain bottle 5 inches tall holds 60 c.c. of liquid. What is the capacity of a similar bottle 7 inches tall?
11. If the air space in a room is twice that in a room of similar shape, compare the cost of papering the two rooms, assuming that the cost is proportional to the area.

    ANS. The cost for the larger room is $(2)^{\frac{2}{3}}$, or approximately 1.57, times that for the smaller room.
12. Construct models of the five regular polyhedrons by cutting out stiff paper forms similar to the patterns in Fig. 79. After properly folding on the dotted lines, the unjoined edges can be fastened with Scotch tape or other adhesive.
13. Prove that there are no more than five regular polyhedrons, using as a basis for the proof the fact that the sum of the face angles of a polyhedral angle is less than 360°.

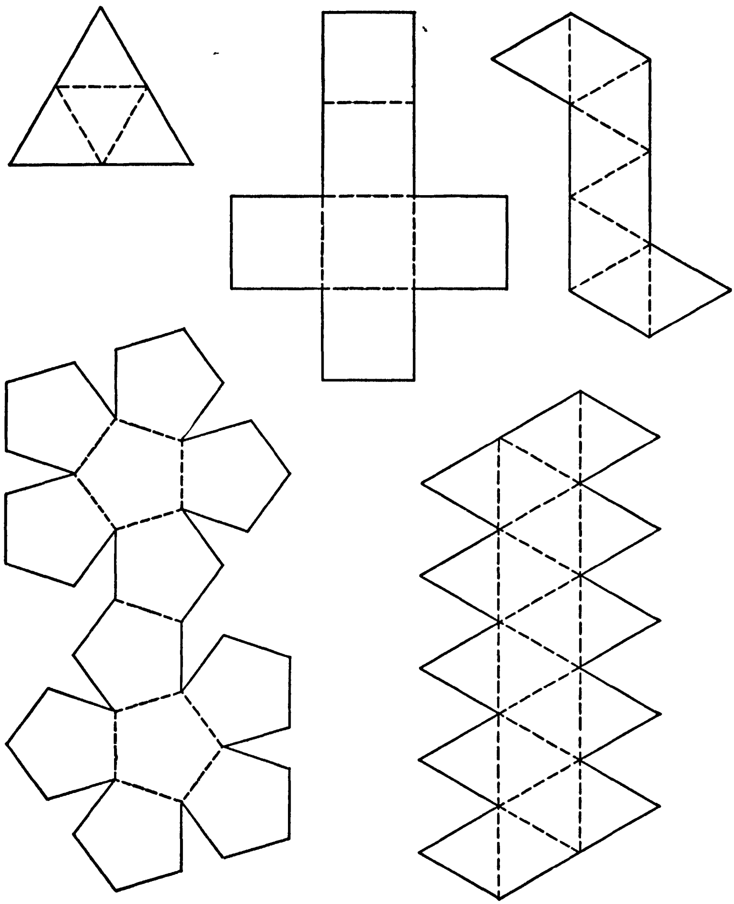


FIGURE 79.

72. PRISMS. PARALLELEPIPEDS

A **prism** is a polyhedron two faces of which are congruent polygons in parallel planes and the other two faces of which are parallelograms. The congruent polygons are called the **bases** of the prism and the distance between their planes, the **altitude** of the prism. Prisms are classified as triangular, hexagonal, etc., according as their bases are triangles, hexagons, etc., respectively. The parallel line segments joining the corresponding vertices of the bases are called the **lateral edges** of the prism. A **right prism** is one whose lateral edges are perpendicular to the bases. A **regular prism** is a right prism having regular polygons for its bases. If a plane is passed through a prism perpendicular to its lateral edges and cutting them all, the polygon thus formed is called a **right section** of the prism.

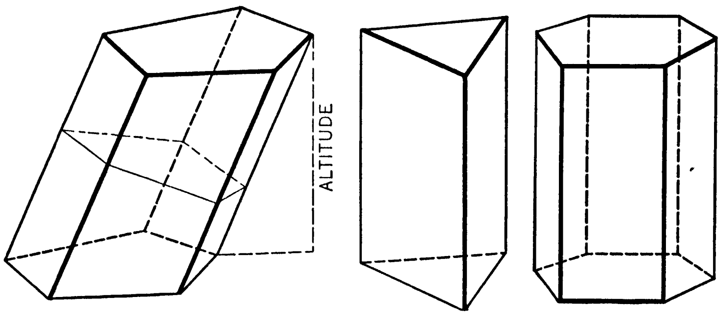


FIGURE 80.

A **parallelepiped** is a prism all of whose faces are parallelograms. If the six faces of a parallelepiped are all rectangles, it is called a **rectangular parallelepiped.** If all of the faces of a parallelepiped are squares, it is called a **cube.**

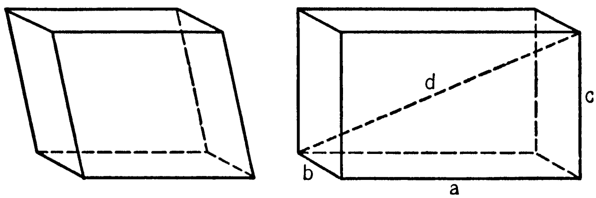


FIGURE 81.

The **volume** of any prism is equal to the product of the area of its base and its altitude. The volume is also equal to the product of its lateral edge and the area of its right section. The **lateral area** of a prism is equal to the product of its lateral edge and the perimeter of its right section. In formulas:

$$V = Bh = eK, \qquad S = ep.$$

Of course the volume and area formulas of prisms hold for the parallelepipeds, but they can be considerably simplified for the rectangular parallelepiped and the cube. Denoting the total surface area by T and using the notation of the figures, for the rectangular parallelepiped,

$$V = abc, \qquad T = 2(ab + ac + bc),$$

and for the cube,

$$V = a^3, \qquad T = 6a^2.$$

The diagonal of a rectangular parallelepiped is given by the formula

$$d = \sqrt{a^2 + b^2 + c^2}.$$

EXERCISES

1. If iron weighs about 450 lbs. per cubic foot, find the weight of an iron bar in the form of a regular hexagonal prism 6 ft. long if each edge of the hexagonal base is 1 in. long. ANS. 48.71 lbs.
2. Find the volume in cubic yards of a concrete foundation wall 3 ft. high, 1 ft. thick, and supporting a rectangular building 30 ft. by 60 ft.
3. Find the length of the diagonal of a cube the volume of which is 3375 cu. in. ANS. $15\sqrt{3}$ in.
4. A certain stretch of Mississippi river levee is uniform enough for a half mile of its length to be considered as a good approximation to a prism. Its right section is a trapezoid with bases 120′ and 30′ and height 40′. Find the number of cubic yards of earth in the half mile section of the levee.
5. The pedestal for a certain statue is in the form of a regular octagonal prism 6 ft. high with the sides of the base each 6 in. long. Find its volume. ANS. 7.24 cu. ft.
6. Find the volume of the largest beam with square cross section that can be sawed from a uniform, round log 2 ft. in diameter and 20 ft. long.
7. If it costs $6.00 to line a certain packing case with copper preparatory to an Arctic expedition, what is the cost of lining a case of similar shape but having 8 times the capacity? ANS. $24.00.
8. As a locomotive traveling 50 ft. per sec. passes over a bridge 60 ft. above the water level, a boat passes directly under the locomotive at right angles to the bridge with a speed of 20 ft. per sec. If both objects continue along straight paths, by what distance will they be separated at the end of 10 seconds?
9. Express in terms of the volume, v, the edge, area, and the length of the diagonal of a cube.

73. CYLINDRICAL SURFACES. CYLINDERS

If a straight line is moved in such a manner as to be parallel to a given line and at the same time to completely traverse a closed plane curve, the line is said to generate a **closed cylindrical surface.** The moving line is called the **generator** and any one of its positions is an **element** of the cylindrical surface. The surface is of course considered as of indefinite extent along its elements.

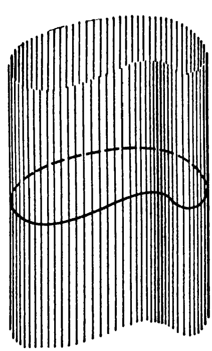


FIGURE 82.

A solid bounded by a cylindrical surface and two parallel planes is called a cylinder. The two parallel portions of the planes which partly bound the cylinder are its **bases.** Its **altitude** is the perpendicular distance between its bases, and the curved surface is called its **lateral sur-**

face. A cylinder whose elements are perpendicular to its bases is called a **right cylinder.** A right cylinder with circular bases is a **right circular cylinder.**

Volume and Area of the Cylinder. Since the lateral area and the volume of a cylinder can be approximated as closely as desired by the lateral area and volume of an inscribed or circumscribed prism,

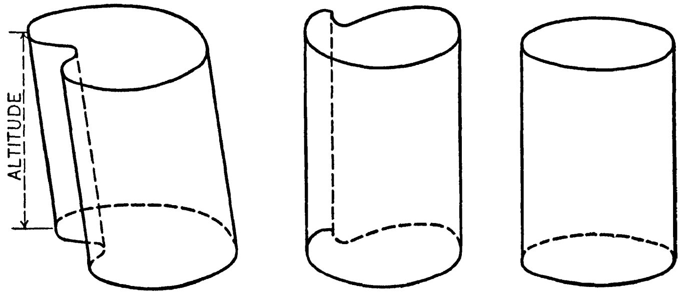


FIGURE 83.

the lateral area and volume formulas for the cylinder are the same as the corresponding formulas for the prism. If e = length of an element of the cylinder, B the area of the base, h the height, and p_r the perimeter of the right section,

$$S = ep_r, \qquad V = Bh.$$

In the case of the right circular cylinder, $e = h$, and $B = \pi r^2$, $p_r = 2\pi r$, where r is the radius of the base. Hence for the *right circular cylinder,*

$$V = \pi r^2 h, \qquad S = 2\pi rh, \qquad T = 2\pi r(h + r).$$

EXERCISES

1. If an oblique (not right) cylinder has a circular base, is its right section a circle?

2. Find the volume and lateral area of a right circular cylinder if the radius of the base is 6 in. and the altitude is 10 in.

3. Find the volume of an oblique cylinder whose base is a circle of radius 4 and whose altitude is 8. Can the lateral area be found?

ANS. 128π; not by elementary methods.

4. Find the number of square inches of material required to make a can in the form of a right circular cylinder 5 in. tall and 4 in. in diameter. Neglect seams and waste.

5. The diameter of a right circular cylinder is 8, and the diagonal of the largest rectangle made by a vertical section is 16. Find the altitude.

ANS. 13.856.

6. Find the altitude of a right circular cylinder whose volume is 90π cu. in. and the radius of whose base is 3 in.

7. If the altitude of a right circular cylinder is equal to its diameter, find the ratio of the numerical value of its total area to that of its volume.

Ans. $\frac{T}{V} = \frac{3}{r}$.

8. A regular hexagonal prism is inscribed in a right circular cylinder the altitude of which is equal to the diameter. Find the difference between the volumes of the cylinder and of the prism if the side of the hexagonal base of the prism is 4 in. long.

9. A cylindrical tank 8 ft. in diameter, partly filled with water, is lying on its side. If the greatest depth of the water is 6 ft., and if the tank is 10 ft. long, what is the number of gallons of water in the tank? (1 cu. ft. = 7.5 gal., approx.)

Ans. $300\left(\sqrt{3} + \frac{8\pi}{3}\right)$ gal.

10. A rectangle whose sides are b and h is turned about the side h as an axis and then about the side b. Find the volumes and total areas of the two right circular cylinders thus generated.

74. PYRAMIDS

A **pyramidal surface** is a surface generated by a line through a fixed point which moves so as to completely traverse a polygon whose plane does not contain the given point. The fixed point is called the **vertex** of the surface, and the generating line in any of its positions is called an **element** of it. The solid bounded by a pyramidal surface and a plane section cutting all the elements is a **pyramid.** The **lateral surface** of a pyramid is composed of the triangular **faces** which meet at the vertex of the pyramid. The sides common to two such triangles are the **lateral edges** of the pyramid. Pyramids are classified according to the shape of the base, as **triangular, quadrangular, pentagonal,** etc. A triangular pyramid is often known also as a **tetrahedron.** The **altitude** of a pyramid is the perpendicular distance from the vertex to the plane of the base.

A **regular pyramid** is one whose base is a regular polygon and is such that the perpendicular from the vertex meets the base at its center. The altitude of any one of the triangular faces of a regular pyramid is called its **slant height.**

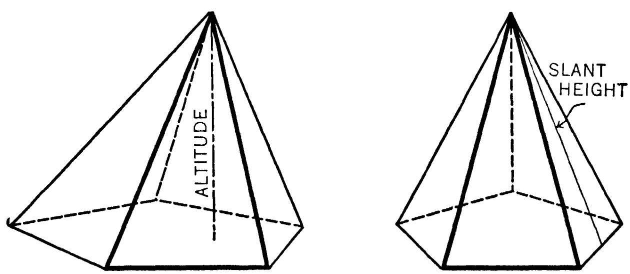


FIGURE 84.

The volume of any pyramid is equal to one-third of the product of the area of its base and altitude. In formula,

$$V = \tfrac{1}{3}Bh,$$

where B is the area of the base and h the altitude.

The lateral area of a *regular* pyramid is equal to one-half the product of the perimeter of its base by its slant height. Or, if p represents the perimeter of the base and l the slant height,

$$S = \tfrac{1}{2}pl.$$

EXERCISES

1. A flower bed is in the form of a regular pyramid, with a square base 5 ft. on each side. The altitude is 3 ft. Find the number of cubic feet of soil used in its construction. ANS. 25 cu. ft.
2. Find the lateral area of a regular pyramid if its slant height is 10 in. and the perimeter of its base is 25 in.
3. Find the lateral area of a regular pyramid with altitude 6 ft. and base 16 ft. square. ANS. 320 sq. ft.
4. The lateral area of a regular hexagonal pyramid is 72 sq. ft. and the slant height is 12 ft. Find the volume.
5. Find the volume of a regular triangular pyramid, the sides of the base being 5 in. long and the altitude being 6 in. ANS. $\frac{25\sqrt{3}}{2}$ cu. in.
6. A pyramid is cut into two parts by a plane parallel to its base. If the plane bisects the altitude of the pyramid, compare the volumes of the two portions of the pyramid.
7. Find the volume and total surface of a regular tetrahedron whose edges are e inches. ANS. $\frac{e^3\sqrt{2}}{12}$ cu. in.; $e^2\sqrt{3}$ sq. in.

   (*Hint:* The vertex of a regular tetrahedron is directly "over" the point of intersection of the medians of the base.)
8. A tent is to be made in the form of a regular hexagonal pyramid. If the height is to be 12 ft., what must each side of the base be in order that the tent may inclose 400 cu. ft. of space?
9. A mound of earth of the shape shown in the figure has a rectangular base 16 yds. long and 8 yds. wide. Its perpendicular height is 5 yds. and the length of the top edge is 8 yds. Find the number of cubic yards of earth in the mound. ANS. 266.67 cu. yds.

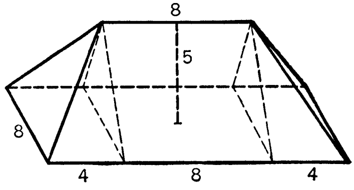


FIGURE 85.

75. CONICAL SURFACES. CONES

If a line through a fixed point is made to completely traverse a closed plane curve not a polygon, it generates a **conical surface.** (Compare this with the definition of a pyramidal surface in Article

74.) A **cone** is a solid bounded by a conical surface and a plane cutting all its elements, where the word **element** refers to any position of the generating line. The plane part of the surface of a cone is its

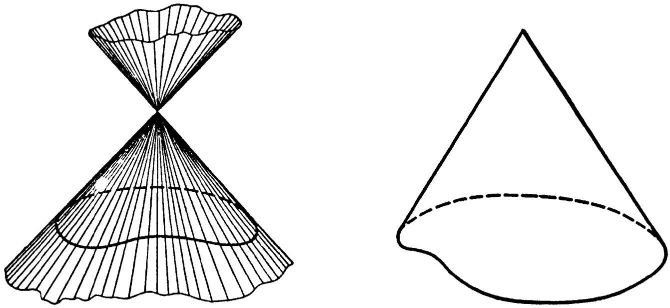


FIGURE 86.

base, and the curved part its **lateral** surface. A cone which has a circular cross section such that a line through the center of the circle and perpendicular to its plane passes through the **vertex** of the cone is called a **circular cone.** If the base is such a circle, the cone is called a **right circular cone.** Otherwise, it is called an **oblique cone.** The **axis** of a circular cone is the line through the centers of all its circular cross sections.

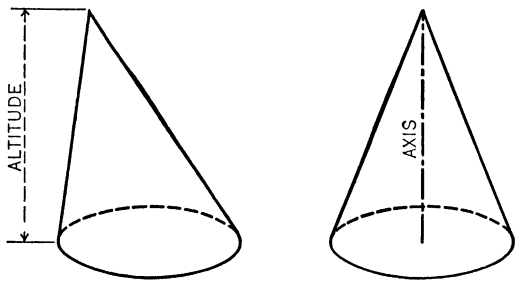


FIGURE 87.

The cone is related to the pyramid in precisely the same manner that the cylinder is related to the prism. The fundamental formulas for surface and volume of the pyramid hold for the cones. The volume of any cone is equal to one-third the product of the area of its base and its altitude. The lateral area of oblique cones, even those with circular bases, cannot be expressed in a simple formula, but the lateral area of a **right circular cone** is one-half the product of the length of an element and the perimeter of the base.

If r = the radius of the base of a right circular cone, h = its altitude, and l, the length of an element,

$$V = \tfrac{1}{3}\pi r^2 h,$$

$$S = \pi r l, \qquad T = \pi r(r + l).$$

EXERCISES

1. Find the lateral area of a right circular cone with radius of base 6 in. and slant height (length of element) 10 in. Find also its volume.
   ANS. 60π sq. in.; 96π cu. in.
2. Find the lateral area of a right circular cone with radius of base 8 and altitude 15.
3. The lateral area of a right circular cone is 120π sq. in. and the radius of its base is 4 in. Find its slant height; its altitude; its volume.
   ANS. 30 in.; $2\sqrt{221}$ in.; $\frac{32\pi}{3}\sqrt{221}$ cu. in.
4. Find the volume of an oblique cone with altitude 8 ft. and with circular base of radius 6 ft.
5. If the radius of the base of a right circular cone is r and the lateral area is S, express the value of the slant height in terms of r and S. Express also the height in terms of r and S.
   SECOND ANSWER: $h = \frac{\sqrt{S^2 - \pi^2 r^4}}{\pi r}$.
6. The lateral area of a right circular cone is S and the slant height is l. Express the radius of the base and the altitude in terms of S and l.
7. Find the volume of a right circular cone with slant height 10 in. and circumference of base 12π in.
   ANS. 96π cu. in.
8. What is the difficulty in finding the lateral area of an oblique cone or of a cone whose base is not circular? Does the same difficulty exist in finding the volume of such a cone?
9. Find the area and the volume of the figure developed by an equilateral triangle with sides b if it is revolved about one of its sides.
   ANS. $S = \pi b^2\sqrt{3}$; $V = \frac{1}{4}\pi b^3$.
10. Find the area and volume of the solid developed by revolving a square about one of its diagonals.
11. In a right circular cone with a radius of base 6 in. and altitude 18 in. a right circular cylinder of height 12 in. is inscribed as shown in the figure. Find the volume of the cylinder.
    ANS. 48π cu. in.
12. Through one vertex of an equilateral triangle with sides a, draw a line l parallel to the opposite side. Find the volume and area of the figure developed by revolving the triangle about the line l.

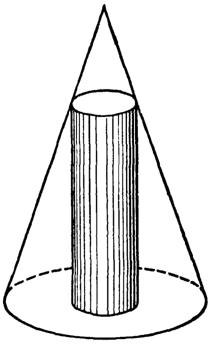


FIGURE 88.

76. THE FRUSTUM

Most of the statements to be made in the following discussion apply equally to pyramids and to cones. Hence, to emphasize the parallelism of the two cases, as well as for brevity, we shall discuss the frustum of the pyramid and of the cone simultaneously.

A **frustum of a pyramid** (or **cone**) is a portion of a pyramid (or cone) included between the base and a section made by a plane parallel to the base. This section is also called a *base* of the frustum. The bases of a frustum of a pyramid are similar polygons, while those of a frustum of a right circular cone are both circles. The **altitude** of a frustum of a pyramid or of a cone is the perpendicular distance between its bases.

The **lateral faces** of a frustum of a **regular** pyramid are congruent trapezoids. Their common altitude is called the **slant height** of the frustum. Obviously, only frustums of *regular* pyramids have a slant height. The slant height of a frustum of a right circular cone is the length of a segment of an element of the cone included between the bases.

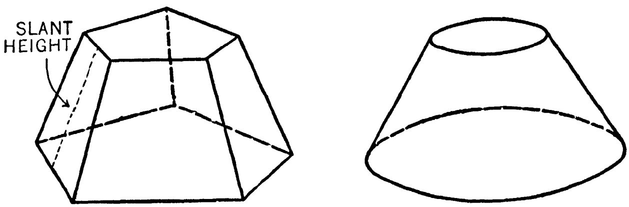


FIGURE 89.

The volume of any frustum of a cone or pyramid is equal to the sum of three cones (or pyramids) whose common altitude is the altitude of the frustum and whose bases are the two bases and a mean proportional between them. Or, if B and B' are the bases and h the altitude of a frustum, the volume is given by the formula

$$V = \frac{h}{3}(B + B' + \sqrt{BB'}).$$

If p and p' are the perimeters of the bases of a frustum and if l is its slant height, then the lateral area of the frustum is given by the formula

$$S = \frac{l}{2}(p + p').$$

EXERCISES

1. If r and r' are the radii of the bases of a frustum of a right circular cone, show that the volume V, the lateral surface S, and the total surface T are given by the formulas

$$V = \frac{\pi h}{3}(r^2 + r'^2 + rr'),$$
$$S = \pi l(r + r'),$$
$$T = \pi[r(r + l) + r'(r' + l)].$$

2. Using the results of Ex. 1, find V, S, and T for the frustum of a right circular cone having bases whose radii are 6 and 11, respectively, and having an altitude of 12.

3. The radii of the bases of a frustum of a right circular cone are 9′ and 6′, respectively. Its slant height is 5′. Find its volume and the volume of the original cone. ANS. 228π cu. ft.; 324π cu. ft.

4. A certain type of concrete building pier is in the form of a frustum of a regular pyramid. It is 3 ft. high and its bases are squares with sides 2 ft. and 1 ft., respectively. Find the number of cubic yards of material in 1000 such piers.
5. A factory chimney is in the form of a frustum of a regular square pyramid. The chimney is 140 ft. high and the edges of its bases are 14 ft. and 8 ft., respectively. All cross sections of the flue are 6 ft. square. How many cubic feet of material does the chimney contain? ANS. 12,320 cu. ft.
6. The radii of the bases of a frustum of a right circular cone are 4 in. and 6 in., respectively. Its altitude is to be chosen so as to make the volume 228π cu. in. Find the altitude.
7. The measurements of a round cake pan were found to be as shown in the cross-section view below at the right. Find the number of square inches of material in the pan, it being of aluminum and having no seams. ANS. 136.4 sq. in.

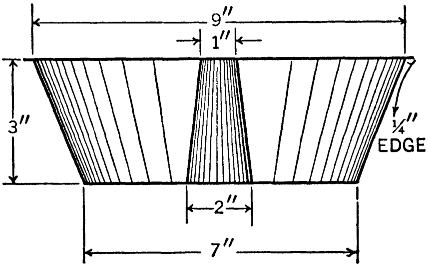


FIGURE 90.

8. A certain storage bin is to hold 158 cubic feet of grain and is to be made in the form of an inverted frustum of a regular square pyramid. If the upper base is 7 ft. square and the depth is 6 ft., find the sides of the lower base.

77. THE PRISMATOID AND ITS MENSURATION

Any polyhedron all of whose vertices lie in two parallel planes is called a **prismatoid.** The faces of the prismatoid which lie in the parallel planes are called its bases. In the case in which only one vertex of the prismatoid lies in one of the planes, or if all of the vertices of that plane are on a straight line, the area of the base of the prismatoid in that plane is of course zero. From the definition, it is clear that prisms, pyramids, and frustums of pyramids are also prismatoids.

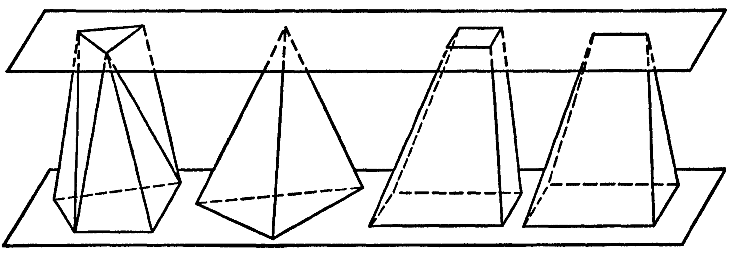


FIGURE 91.

The plane section of a prismatoid made by a plane parallel to the planes of the bases and midway between them is called the **mid-**

section of the prismatoid. The lateral faces of a prismatoid are either triangles, parallelograms or trapezoids.

If B and B' are the areas of the bases of a prismatoid, M the area of its midsection, and h its altitude, its volume V is given by the following formula which is known as the *Prismatoidal Formula:*

$$V = \frac{h}{6}(B + B' + 4M).$$

In finding the value of M for a given prismatoid, it is frequently necessary to remember that each side of M bisects two of the lateral edges of the prismatoid and hence its length is the **average** of two of the sides of the bases. Thus in the figure, $RS = \frac{1}{2}(0 + 8) = 4$ and $ST = \frac{1}{2}(3 + 6) = 4.5$. Thus if the sides of the bases of a prismatoid are known, the sides of the mid-section can be determined, which will in many cases lead to the value of M.

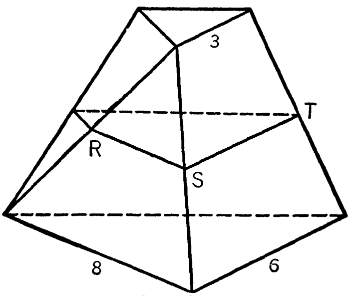


FIGURE 92.

NOTE: The Prismatoidal Formula is remarkable in that it gives the correct volume for many solids other than prismatoids. In addition, it gives a very good approximation to the correct volume of many irregular solids, such as earth fills, ditches, etc., so that it is much used by engineers both for exact and for approximate calculations.

78. THE THEOREM OF PAPPUS

If any closed plane figure is revolved about a line in its plane but not intersecting the plane figure, a solid figure is generated. The surface area and volume of such a solid were first determined by Pappus of Alexandria (born about 340 A.D.). His theorem states that the area of such a solid is equal to the product of the perimeter of the given figure and the distance through which its center of area moves. The volume of the solid is equal to the product of the area of the given figure and the distance through which its center of area moves. Thus if

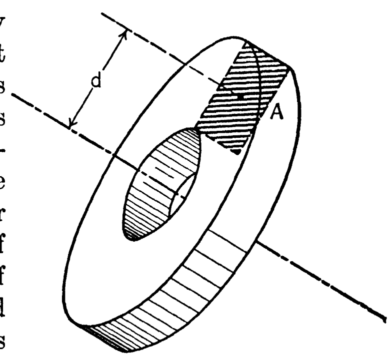


FIGURE 93.

A is the area of the generating plane figure, P is its perimeter, and d, the distance from its center of area to the axis of revolution, the surface area S and the volume V of the solid generated by a complete revolution,

$$S = 2\pi dP; \qquad V = 2\pi dA.$$

For a circle or regular polygon the center of area is the center of the circle or the polygon.

EXERCISES

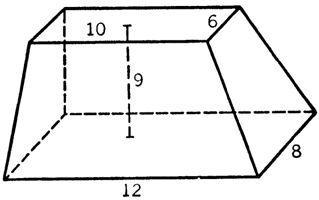


FIGURE 94.

1. Find the volume of the solid pictured in Fig 94 if its bases are rectangles with their corresponding sides parallel. ANS. 696 cu. ft.
2. Is the solid of Ex. 1 a frustum of a pyramid? Prove your conclusion.
3. The inner radius of an inflated inner tube is 10 in. and the cross section of the tube is a circle 5 in. in diameter. Find the surface area and the total volume of the tube and air contained in it. ANS. 1234 sq. in.; 1542 cu. in.
4. Prove that the Prismatoidal Formula correctly gives the volume of a cone.

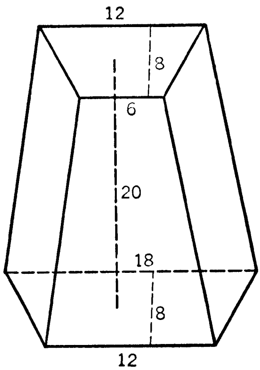


FIGURE 95.

5. A gasoline tank was made in the form shown in Fig 95 so as to fit into the prow of a motor-boat. If the bases of the figure are trapezoids, find the number of gallons capacity of the tank. (1 gal. = 231 cu. in.)
6. An equilateral triangle with sides 10 inches is revolved about a line in its plane parallel to one of its sides and at a distance of 20 inches from this side. Find the surface area and the volume of the figure thus generated. Two answers. (Suggestion: The center of area of a triangle is the intersection point of its medians.)
7. A circle of radius r revolves about one of its tangents. Find the area and volume of the solid generated. ANS. $S = (2\pi r)^2$; $V = 2\pi^2 r^3$.
8. A rectangle whose length is y and whose width is Δx completely revolves about a line parallel to its long side and at a distance of x from the center of area of the rectangle. Give a formula for the volume of the cylindrical shell thus generated.

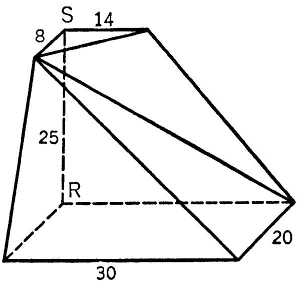


FIGURE 96.

9. Find the volume of the prismatoid illustrated in Fig. 96. The upper base is a right triangle, the lower base is a rectangle. RS is an altitude of the solid and is 25 ft. long. ANS. 6467 cu. ft.
10. An earthen embankment across a valley has bases which are approximate rectangles. The length and width of the top are 185 ft. and 20 ft., respectively,

while the corresponding dimensions of the base are 130 ft. and 58 ft. Its height is 22 ft. Find the number of cubic yards of earth in it.

11. A pile of coal 18 ft. high is in the form shown in Fig 97. Its base is a rectangle 50 ft. long and 28 ft. wide. The top ridge is parallel to the plane of the base and is 32 ft. long. Find the number of loads of 96 cu. ft. each in the pile. ANS. 115.5.

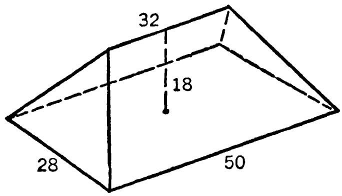


FIGURE 97.

12. A model hexagonal prism (regular) is made with two regular hexagons of cardboard as bases. The bases are connected by parallel threads which form the lateral edges of the prism. The sides of the bases are 2 in. long and the altitude of the prism (the length of the threads) is 6 in. While the threads are kept taut, the lower base is now turned through an angle of 120°. The resulting position of the model may be considered as a solid such that every plane section parallel to the bases is a regular hexagon. Find its volume.

ANS. $18\sqrt{2}$ cu. in.

(*Hint:* Project all lines of the figure upon one of the bases.)

79. THE SPHERE. FIGURES ON THE SPHERE

A **sphere** is a closed surface every point of which is equidistant from a point called its **center.** A line segment joining the center to a point on the sphere is a **radius** of the sphere.

The intersection of a plane with a sphere is a circle. If the plane of the circle passes through the center of the sphere, the circle is a **great circle** of the sphere. Otherwise, it is a **small circle.** The diameter of the sphere which is perpendicular to the plane of a circle is the *axis* of the circle, and its extremities are the **poles** of the circle. In Fig. 98, P and P' are poles of the circles shown.

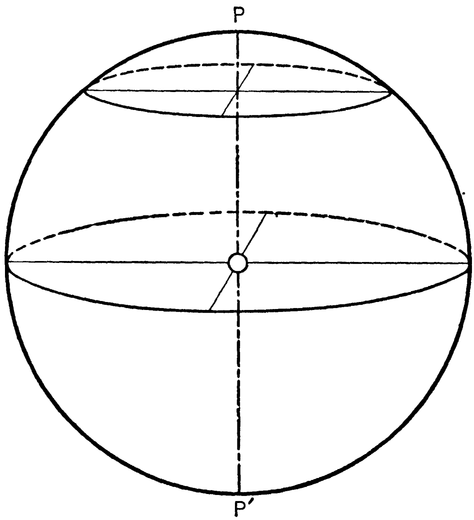


FIGURE 98.

The length of the arc of a great circle joining a point of a given circle to its pole is called the **polar distance** of the given circle.

The **distance** between two points of a sphere is the length of the minor arc (arc less than a semicircle) of the great circle passing through the two points. Arcs of great circles are usually measured in degrees (or radians) of arc.

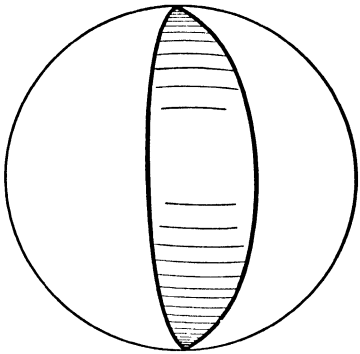


FIGURE 99.

A **lune** is the figure on a sphere formed by two halves of great circles with common end points. A **zone** is the portion of the surface of a sphere included between two parallel planes, one of which may be tangent to the sphere.

The angle between two arcs is measured by the angle formed by their tangents at a point of intersection. Thus the angle between the arcs of great circles is equal in measure to the dihedral angle formed by the planes of the arcs (Fig. 104).

If A, B, and C are points on a sphere which lie in a plane not containing the center of the sphere, **the spherical triangle** ABC is defined to be the figure formed by the minor arcs of great circles which connect A, B, and C in pairs. Since the sides of such a triangle are arcs of great circles, they are measured in degrees. This definition of a spherical triangle requires that the sides of the spherical triangle determined by three points on a sphere shall each be less than 180°. It can be shown that each of the angles of the triangle must be less than 180°. (It is possible to define spherical triangles with sides greater than 180°, but such triangles are seldom used in the

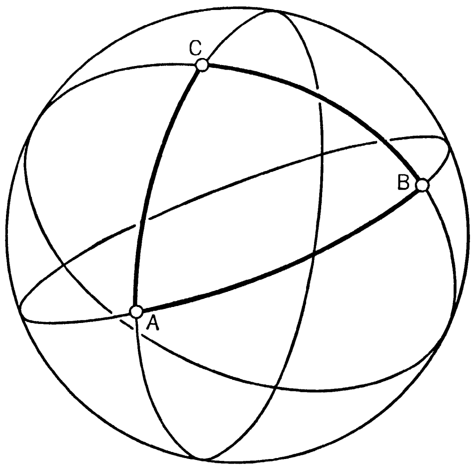


FIGURE 100.

applications to geometry and spherical trigonometry and are a source of complications in many theorems.)

The sum of the sides of a spherical triangle is less than 360°.

The sum of the angles of a spherical triangle is greater than 180° and less than 540°.

Let ABC be a spherical triangle with a, b, and c, the opposite sides. With A and B as poles, describe great circles. These will intersect in two points, one of which will be on the same side of c as C. Call this point C'. In the same way let great circles whose poles are A and C intersect at B', B' being on the same side of b as B. Finally locate A' in corresponding manner. The spherical triangle $A'B'C'$ is called the **polar triangle** of triangle ABC.

Figure 101 shows triangle ABC and its polar triangle $A'B'C'$. It is obvious that A' is at a quadrant's distance (90° of arc) from B and C, so that A' is the pole of arc a. Similarly B' is the *pole* of b and C' is the pole of c. This means that:

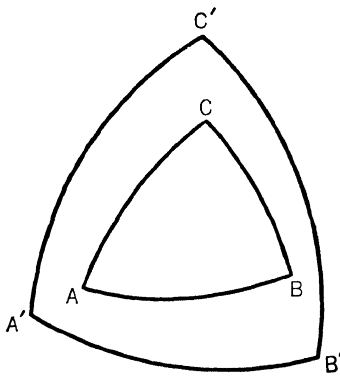


FIGURE 101.

If $A'B'C'$ is the polar triangle of triangle ABC, then ABC is the polar triangle of triangle $A'B'C'$.

This takes on great importance when combined with another readily proved fact that:

Any angle of a spherical triangle is equal to the supplement of the opposite side of its polar triangle, or symbolically,

$$\alpha = 180° - a', \qquad \alpha' = 180° - a,$$
$$\beta = 180° - b', \qquad \beta' = 180° - b,$$
$$\gamma = 180° - c', \qquad \gamma' = 180° - c.$$

These relations establish a duality between sides and angles which enables one to exchange the words "side" and "angle" in many theorems in Trigonometry and obtain a *dual* theorem, proper allowances for certain changes in sign being made.

EXERCISES

1. Airport B is due west of airport A, both being in the north temperate zone. Show by means of a diagram that an aviator wishing to fly the great circle route (shortest route) from A to B would start in a direction which is north of due west.

2. Show that to every spherical triangle there corresponds a trihedral angle at the center of the sphere whose face angles are equal (are of the same measure) to the corresponding sides of the triangle and whose dihedral angles formed by the faces are equal to the corresponding angles of the triangle.

3. A plane leaves Point Barrow, Alaska, and flies directly over the North Pole to Moscow, Russia. If the latitude of Point Barrow is 72°, north, and that of Moscow is 56°, north, what is the distance flown? ANS. 3600 miles.

4. In a sphere of radius 10 inches a right circular cylinder with radius 6 inches is inscribed. Find the volume of the cylinder.

5. A right circular cone of altitude 18 inches is inscribed in a sphere of radius 13 inches. Find the volume of the cone. ANS. 864π cu. in.

6. A right circular cylinder of radius r is inscribed in a sphere of radius a. Express the volume and the total surface area of the cylinder in terms of r and a.

7. A right circular cone of altitude h is inscribed in a sphere of radius r. Express the radius of the cone in terms of h and r. ANS. $\sqrt{h(2r - h)}$.

8. What is the polar distance of a great circle?

9. Compare the circumference of a great circle with that of a small circle parallel to it and 60° from it. ANS. $c = 2c'$.

10. Each standard time belt covers 15° of longitude. How many miles wide is a time belt at 30° latitude? at 50°? at 90°?

11. What is the *dual* of the statement, "Any angle of a spherical triangle is equal to the supplement of the opposite side of its polar triangle"?

12. The three angles of a spherical triangle are 60°, 70°, and 100°. What is the number of degrees in each of the sides of the polar triangle?

13. Will Ex 12 and its answer remain a possible problem and answer if the words *sides* and *angles* are exchanged throughout?

14. Verify that if two angles of a spherical triangle are right angles (a *birectangular* triangle), the opposite sides are quadrants and the third angle is equal to the third side.

15. Prove that the polar triangle of a birectangular triangle is also birectangular.

16. Show that if the three angles of a spherical triangle are right angles (a *trirectangular* triangle), the sides are all quadrants.

17. Prove that the polar triangle of a trirectangular triangle is the triangle itself.

18. A hunter left his camp at A and walked 10 miles south to B. He then walked 10 miles east to C, where he shot a bear. From C, he walked 10 miles back to A. What color was the bear?

19. Three spheres of radius a are placed on a horizontal plane so that each is tangent to the other two. A sphere of the same size is now placed on top of them and tangent to each of the three. Find the distance from the plane to the top of the upper sphere. ANS. $2a\left(1 + \frac{\sqrt{6}}{3}\right)$.

20. A circular hole 4 inches in diameter is cut in a board, and a ball 4 inches in radius is made to rest in it. At what angle must the board be held with the horizontal so that the ball will just roll out of the hole? (*Hint:* The board must be held so that the center of the ball will be vertically above the edge of the hole.)

80. SPHERICAL AREAS AND VOLUMES

The area of the surface of a sphere is equal to the area of four great circles, or

$$S = 4\pi r^2,$$

where r refers to the radius of the sphere.

The area of a zone of a sphere is equal to the product of the altitude h of the zone and the circumference of a great circle, or $S = 2\pi rh$. (It is interesting to note that the area of a zone is equal to the area of its projection upon a cylinder circumscribed about the sphere and with axis perpendicular to the plane of a base of the zone.)

The **spherical degree** is a frequently used unit of area on a sphere and is defined as $\frac{1}{720}$ of the area of the sphere. Obviously, the area of a lune whose angle is $\frac{1}{2}°$, or of half a lune whose angle is 1°, is equal to one spherical degree. The area (in spherical degrees) which a polyhedral angle at the center of a sphere cuts out on a sphere is taken as the **measure** of the polyhedral angle. Thus the area of a spherical triangle, in spherical degrees, is the measure of the trihedral angle determined by the triangle.

Since the sum of the angles of a spherical triangle is greater than 180°, there will always be a positive excess when 180° is subtracted from the sum of the angles. This excess is called the **spherical excess** of the triangle. Thus the spherical excess of the spherical triangle with angles α, β, and γ is $\alpha + \beta + \gamma - 180°$. It can be shown that:

The area of a spherical triangle, in spherical degrees, is equal to its spherical excess.

A **spherical segment** is a solid bounded by two parallel planes and the portion of a sphere intercepted between them. One of the planes may be tangent to the sphere (Fig. 102).

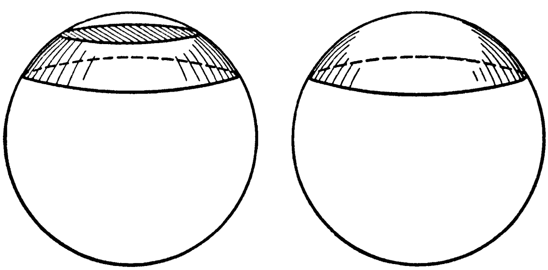


FIGURE 102.

A **spherical sector** is a solid bounded by a zone and the one or two circular cones (with vertex or vertices at the center of the sphere) which intercept the zone. The two types of sectors are illustrated in Fig. 103.

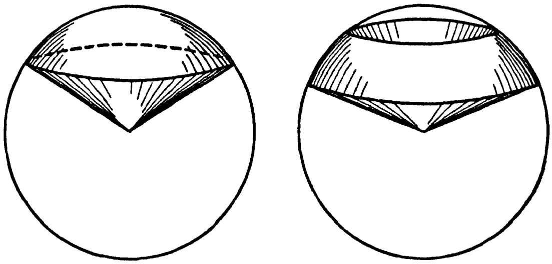


FIGURE 103.

A Fundamental Principle. The volume of any portion of a sphere cut out by a conical or pyramidal surface with vertex at the center of the sphere is equal to one-third the product of the area of the intercepted portion of the surface of the sphere and the radius of the sphere.

Since the above principle applies to the entire sphere, the volume of the sphere is one-third the product of its area and its radius, or

$$V = \tfrac{4}{3}\pi r^3.$$

The principle applies also to the volume of a spherical sector, so that the volume of a spherical sector is one-third the product of the area of the intercepted zone and the radius of the sphere. In formula,

$$V = \tfrac{2}{3}\pi r^2 h,$$

h being the altitude of the zone.

The volume of a spherical segment of *one* base with radius a and altitude h is

$$V = \frac{\pi h^2}{3}(3r - h),$$

or

$$V = \frac{\pi h}{6}(3a^2 + h^2).$$

The volume of a spherical segment having two bases of radii a and b, respectively, is

$$V = \frac{\pi h}{6}(3a^2 + 3b^2 + h^2).$$

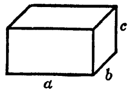
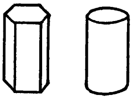
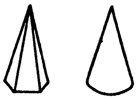
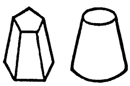
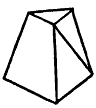
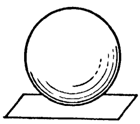
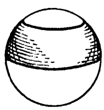
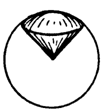
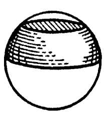
EXERCISES

1. Find the surface area and the volume of a spherical balloon 40 feet in radius. ANS. 20,100 sq. ft.; 268,000 cu. ft.
2. Develop a formula for the volume of a spherical shell whose outer and inner radii are R and r, respectively.
3. Find the area of a spherical triangle on a sphere of 100-foot radius if the angles of the triangle are 70°, 80°, and 90°, respectively. Give the answer in (a) spherical degrees, (b) square feet. ANS. (a) 60; (b) 10,500.
4. If the price of oranges were proportional to the volume and if oranges 2 inches in diameter sell for 20 cents a dozen, how much should be asked for oranges 3 inches in diameter?
5. A certain vat for heating chemicals is in the form of a hemisphere with inner diameter of 7 feet. Find the capacity in gallons and the inner surface area. ANS. 674 gal.; 77 sq. ft.
6. How many spherical shot 0.1 in. in diameter can be made from 25 cubic inches of lead?
7. Show that the area of the zone of a sphere of radius r illuminated by a source of light placed at a distance d from the surface is

$$S = \frac{2\pi d}{r + d}.$$

8. Assuming that a man owns the earth under his property to the center of the earth, how many cubic miles of the earth does a man own who has a ranch covering 240 square miles? Assume the radius of the earth to be 4000 miles.
9. Show that the prismoidal formula holds for the volume of a sphere. (*Hint:* Let the bases each be of area zero and the midsection be a great circle of the sphere.)
10. The area of a certain zone is 72π square feet and its altitude is 4 feet. Find the radius of the sphere of which it is part.
11. A cylindrical water tank 12 feet long at the edges and 5 feet in diameter has ends in the form of zones of spheres 5 feet in radius. Find the cubic capacity of the tank. ANS. 79.3π cu. ft.
12. Find the volume of a lens whose shape is that of a spherical segment, the thickness of the lens being 0.5 inch and the diameter 2 inches.
13. A circular cylinder 5 inches in diameter is filled to a depth of 6 inches with water. A spherical wooden ball 2 inches in diameter is dropped into the water and floats with one-fourth of its vertical diameter above the water level. How high does the water level rise in the vessel? ANS. 0.18 in.
14. A cylindrical hole of radius 5 inches is drilled through the center of a spherical solid of radius 10 inches. Find the volume of the remaining portion of the sphere.
15. Show that half of the earth's surface is included between parallels of latitude 30° N and 30° S.
16. A water tank is built in the form of a right circular cylinder on a hemispherical base and surmounted by a right circular cone the base of which fits the top of the tank. The diameter of the tank is 14 feet and the over-all height is 30 feet. The altitude of the cone is equal to its radius. If the tank is intended to hold water to the top of the cone, find its capacity in cu. ft. Find the surface area of the tank. ANS. $V = 1127\pi$ cu. ft.; $T = 391.3\pi$ sq. ft.

Symbols: S and T = lateral and total surface; p and p', r and r', B and B' = perimeters, radii, and areas of bases, respectively, s = slant height; M = area of mid-section; h = altitude.

Name	*Figure*	*Area*	*Volume*
Rectangular Parallelepiped		$T = 2(ab + bc + ac)$	$V = abc$
Prism and Cylinder		$T = B + B' + ph$ $S = 2\pi rh$ $T = 2\pi r(r+h)$	$V = Bh$ $V = \pi r^2 h$ (rt. cir. cylinder)
Pyramid and Cone		$T = B + sp/2$ $S = \pi rs$ $T = \pi r(r+s)$	$V = \frac{1}{3} Bh$ $V = \frac{1}{3} \pi r^2 h$ (rt. cir. cone)
Frustum of Pyramid or Cone		$S = \frac{1}{2} s(p+p')$ $S = \pi s(r+r')$	$V = \frac{1}{3} h(B + B' + \sqrt{BB'})$ $V = \frac{1}{3} \pi h(r^2 + r'^2 + rr')$ (regular fr.)
Prismatoid			$V = \frac{h}{6}(B + B' + 4M)$
Sphere		$T = 4\pi r^2$	$V = \frac{4}{3} \pi r^3$
Zone		$S = 2\pi rh$	
Spherical Sector			$V = \frac{2}{3} \pi r^2 h$
Spherical Segment			$V = \frac{\pi h^2}{3}(3r - h)$ (for one base) $V = \frac{\pi h}{6}(3a^2 + 3b^2 + h^2)$ (2 bases, radii, a,b)

Chapter XII

SPHERICAL TRIANGLES

81. THE SPHERICAL TRIANGLE

In plane trigonometry, triangles lying on the surface of the earth have been considered as plane triangles. This is because the portions of the earth's surface considered have been sufficiently small to justify the assumption that errors due to the curvature of the earth were less than those resulting from the use of the approximate numbers obtained by measurements and from tables.

Important problems of navigation, long-distance surveying, and astronomy demand a study of the relations of the sides and angles of triangles on a sphere, that is, a study of **spherical trigonometry.**

If A, B, and C are three points on a sphere and are such that their plane does not contain the center of the sphere, then the **spherical triangle** ABC is defined to be the figure on the sphere formed by the minor arcs of great circles connecting A, B, and C in pairs.

The notation used for spherical triangles will correspond to that used for plane triangles. If A, B, and C are the vertices of a spherical triangle, the corresponding opposite sides are a, b, and c, while the spherical angles at the respective vertices are α, β, and γ.

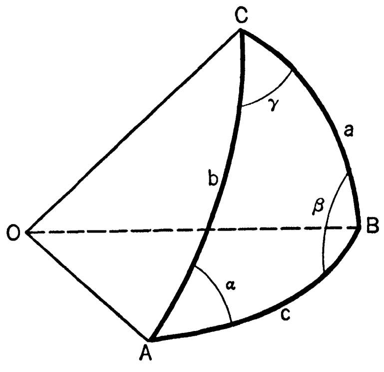


FIGURE 104.

NOTE: The measures of the sides of a spherical triangle ABC are the measures of the face angles of the spherical pyramid O–ABC, where O is the center of the sphere. The angles α, β, and γ have the same measures as the dihedral angles formed by the intersections of the plane faces of the pyramid O–ABC. The sides a, b, and c are usually expressed in degrees. For simplicity, the radius of the sphere will be taken equal to 1, except in applications where linear or square units are desirable in the answers.

82. SPECIAL GEOMETRIC PROPERTIES

(a) The length of a side is the (shortest) distance on the sphere between the two corresponding vertices.

(b) Any side or angle is less than 180°.

(c) Each side is less than the sum of the other two sides.

(d) The sum of the angles of a spherical triangle is greater than 180° and less than 540°.

(e) The sum of the sides is less than 360°.

(f) If $A'B'C'$ is the polar triangle of ABC, then ABC is the polar triangle of $A'B'C'$.

(g) In two mutually polar triangles, each angle of one is the supplement of the opposite side of the other; that is

$$\alpha = 180^\circ - a', \qquad \alpha' = 180^\circ - a,$$
$$\beta = 180^\circ - b', \qquad \beta' = 180^\circ - b,$$
$$\gamma = 180^\circ - c', \qquad \gamma' = 180^\circ - c.$$

(h) In a right spherical triangle, an oblique angle and its opposite side are in the same quadrant.

(i) In a right spherical triangle, either all sides are in the first quadrant or else one side is in the first quadrant and two are in the second quadrant

83. THE LAW OF SINES

In any spherical triangle, the sines of the angles are proportional to the sines of the opposite sides. To prove this, consider the spherical triangle ABC in Fig. 105. From the center O of the sphere draw radii to the three vertices of the triangle A, B, and C. Through any point P on OC pass planes perpendicular to OA and OB. These planes intersect in a line PE perpendicular to the plane OAB.

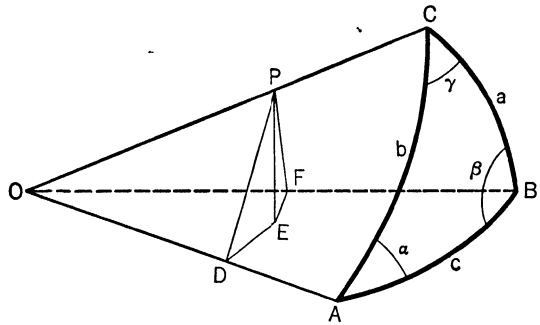


FIGURE 105.

It is evident that angle $\alpha = \angle EDP$, and angle $\beta = \angle EFP$.

In the right triangle PED, $\sin \alpha = \dfrac{PE}{PD}$.

In the right triangle PEF, $\sin \beta = \dfrac{PE}{PF}$.

In the right triangle PDO, $\sin b = \dfrac{PD}{PO}$.

In the right triangle PFO, $\sin a = \dfrac{PF}{PO}$.

Then,
$$\frac{\sin \alpha}{\sin a} = \frac{\frac{PE}{PD}}{\frac{PF}{PO}} = \left(\frac{PE}{PD}\right)\left(\frac{PO}{PF}\right),$$

and
$$\frac{\sin \beta}{\sin b} = \frac{\frac{PE}{PF}}{\frac{PD}{PO}} = \left(\frac{PE}{PF}\right)\left(\frac{PO}{PD}\right).$$

Hence,
$$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b}.$$

It can readily be shown that $\dfrac{\sin \alpha}{\sin a} = \dfrac{\sin \gamma}{\sin c}$.

Therefore,

(61) $$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c},$$ the *Law of Sines.*

The student should note that the proof of the foregoing law has been limited to the case in which all the sides are less than 90°. Other cases are left as an exercise.

84. THE LAW OF COSINES

In a spherical triangle, the cosine of any side is equal to the product of the cosines of the other two sides plus the product of the sines of these two sides and the cosine of the included angle.

PROOF. In Fig. 106, ABC is a spherical triangle. From the center O of the sphere draw radii to the three vertices of the triangle A, B, C. Pass a plane through C perpendicular to OC. This plane intersects the planes COA and COB in the lines CD and CE, respectively.

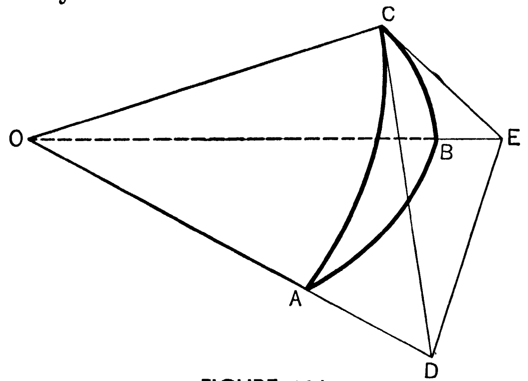


FIGURE 106.

Then $CD = \tan b$, $CE = \tan a$, $OD = \sec b$, $OE = \sec a$, since $r = 1$.

Applying the Law of Cosines from plane trigonometry to the two plane triangles CDE and ODE gives the results

$$(DE)^2 = \tan^2 b + \tan^2 a - 2(\tan b)(\tan a) \cos \gamma,$$
$$(DE)^2 = \sec^2 b + \sec^2 a - 2(\sec b)(\sec a) \cos c.$$

Equating the right sides of the two equations results, after simplifying, in

(62) $\mathbf{\cos c = \cos a \cos b + \sin a \sin b \cos \gamma}$. *The Law of Cosines.*

The Law of Cosines applied to the polar triangle $A'B'C'$ gives

$$\cos c' = \cos a' \cos b' + \sin a' \sin b' \cos \gamma'.$$

Since $c' = 180° - \gamma$, $a' = 180° - \alpha$, $b' = 180° - \beta$, $\gamma' = 180° - c$, then $-\cos \gamma = (-\cos \alpha)(-\cos \beta) + \sin \alpha \sin \beta(-\cos c)$.

(63) $\mathbf{\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c.}$

The foregoing proof has been based on the assumption that the sides of the triangle are less than 90°. The student may consider the more general case as an exercise. Additional formulas may be obtained by a cyclical permutation of the letters.

The cosine of any angle is equal to the product of the sines of the other two angles and the cosine of the opposite side less the product of the cosines of the opposite angles. The Law of Angles.

85. SPHERICAL RIGHT TRIANGLES

A spherical right triangle is a triangle in which one of the spherical angles is 90°. Assuming the right angle is γ, then from the Law of Sines and the Law of Cosines, there result,

(64) $$\sin a = \sin \alpha \sin c,$$
(64′) $$\sin b = \sin \beta \sin c,$$
(65) $$\cos c = \cos a \cos b,$$
(66) $$\cos \alpha = \sin \beta \cos a,$$
(66′) $$\cos \beta = \sin \alpha \cos b.$$

Also by combining certain of the above equations, there result,

(67) $$\cos \alpha = \tan b \cot c,$$
(67′) $$\cos \beta = \tan a \cot c,$$
(68) $$\sin a = \cot \beta \tan b,$$
(68′) $$\sin b = \cot \alpha \tan a,$$
(69) $$\cos c = \cot \alpha \cot \beta.$$

86. NAPIER'S RULES

Formulas (64)–(69) may be obtained by means of two rules, known as *Napier's Rules of Circular Parts.* These parts are a, b, $(90° - \alpha)$, $(90° - \beta)$, and $(90° - c)$, and are designated as a, b, co-α, co-β, and co-c. These circular parts are placed in a circle in the same order in which a, b, α, c, and β occur in the spherical right triangle.

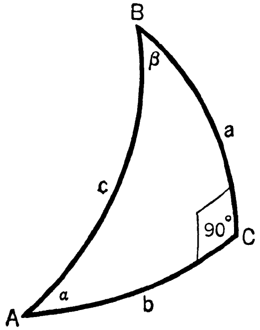


FIGURE 107.

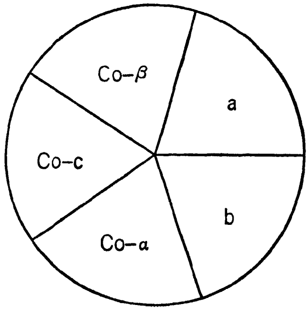


FIGURE 108.

If any three of the parts are considered, it is always possible to select one of them as the middle part, and the other two as either adjacent or opposite parts. For example, if a, b, and co-α are considered, then b is the middle part, while a and co-α are adjacent parts. If co-c, a, and b are considered, then co-c is the middle part, and a and b are the opposite parts.

Napier's Rules are as follows:

(a) The sine of the middle part is equal to the product of the tangents of the adjacent parts, e.g., $\sin a = \tan b \cot \beta$, etc.

(b) The sine of the middle part is equal to the product of the cosines of the opposite parts, e.g., $\sin a = \sin c \sin \alpha$, etc.

87. CASES THAT ARE SOLVABLE IN A RIGHT SPHERICAL TRIANGLE

A right spherical triangle can be solved if there is given in addition to the right angle:

- **(a)** Two sides.
- **(b)** One side and the hypotenuse.
- **(c)** One side and the opposite angle — an ambiguous case.
- **(d)** One side and the adjacent angle.
- **(e)** The hypotenuse and one angle.
- **(f)** The two angles.

EXAMPLE 1. Given $a = 120°$, $b = 45°$; find α, β, c.

SOLUTION: By (65)

$$\begin{aligned} \cos c &= \cos a \cos b \\ &= \cos 120° \cos 45° \\ &= \left(-\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) \\ &= -\frac{\sqrt{2}}{4} \\ &= -0.3536. \end{aligned}$$

Therefore, $$c = 180° - 69° 18' = 110° 42'.$$

By (68′)

$$\begin{aligned} \cot \alpha &= \frac{\sin b}{\tan a} \\ &= \frac{0.7071}{-1.7321} \\ &= -0.4082. \end{aligned}$$

Therefore, $$\alpha = 180° - 67° 48' = 112° 12'.$$

By (68) $\cot\beta = \frac{\sin a}{\tan b}$

$= \frac{0.8661}{1.0000}$

$= 0.8661.$

Therefore, $\beta = 49° 6'.$

CHECK: $\cos c = \cot\alpha \cot\beta$ by (69). Substituting the values found for c, α, and β reduces each member to the value 0.3536, which verifies the result.

EXAMPLE 2. Given $\beta = 75°$, $b = 60°$; find α, a, c.

SOLUTION: From Fig. 107 it is shown that if a is the middle part then b and co-β are adjacent parts. Hence, by Napier's Rules,

$$\sin a = \tan b \cot\beta$$
$$= (1.7321)(0.2680)$$
$$= 0.4642.$$

Therefore, $a = 27° 40'.$

By (64′) $\sin c = \frac{\sin b}{\sin\beta}$

$= \frac{0.8660}{0.9659}$

$= 0.8966.$

Therefore, $c = 63° 43'.$

By (66′) $\sin\alpha = \frac{\cos\beta}{\cos b}$

$= \frac{0.2588}{0.5000}$

$= 0.5176.$

Therefore, $\alpha = 31° 10'.$

CHECK: By (64)

$$\sin a = \sin\alpha \sin c$$
$$= (0.5176)(0.8967)$$
$$= 0.4641$$
$$= \sin a, \text{ approximately.}$$

NOTE: In this case in which a side and its opposite angle are given, the other side must be determined by its sine, and therefore may have two values, each being the supplement of the other. Thus in Fig. 109, if ABC is the given triangle, and $BAB'C$ is a lune whose angle $B' = B = 75°$, then each of the two triangles ABC and $AB'C$ satisfy all of the conditions. Hence,

$a = 27° 40'$,
or $a = 152° 20'$.
$c = 63° 44'$,
or $c = 116° 16'$.
$\alpha = 31° 10'$,
or $\alpha = 148° 50'$.

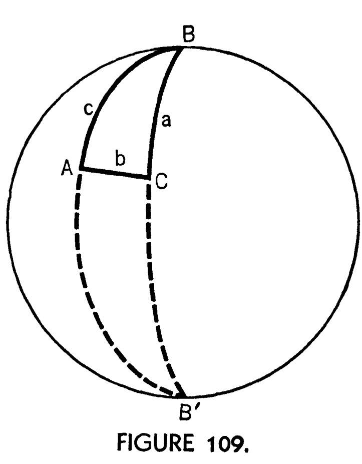


FIGURE 109.

EXAMPLE 3. Given $a = 165°$, $\beta = 120°$; find α, b, c.

SOLUTION: In Fig. 107 co-β is the middle part to the two adjacent parts co-c and a. Hence by Napier's Rules,

$$\cos\beta = \cot c \tan a,$$

or

$$\begin{aligned} \cot c &= \cos\beta \cot a \\ &= (-0.5000)(-3.732) \\ &= 1.8660. \end{aligned}$$

Therefore, $c = 28°\ 11'$.

By (66)

$$\begin{aligned} \cos\alpha &= \sin\beta \cos a \\ &= (0.8660)(-0.9659) \\ &= -0.8365. \end{aligned}$$

Therefore, $\alpha = 180° - 33°\ 14' = 146°\ 46'$.

By (68)

$$\begin{aligned} \tan b &= \sin a \tan\beta, \\ &= (0.2588)(-1.732) \\ &= -0.4482. \end{aligned}$$

Therefore, $b = 180° - 24°\ 9' = 155°\ 51'$.

CHECK: By (67)

$$\begin{aligned} \cos\alpha &= \tan b \cot c \\ &= (0.4482)(1.8660) \\ &= -0.8364, \text{ which is approximately equal to } \cos\alpha. \end{aligned}$$

EXERCISES

Assume ABC to be a right spherical triangle with $\gamma = 90°$ in the following:

1. Given $a = 50°\ 45'$, $c = 110°\ 30'$; find $\alpha = 55°\ 46'$, $\beta = 117°\ 14'$, $b = 123°\ 37'$.
2. Given $b = 67°$, $a = 48°$; find α, β, c.
3. Given $\beta = 12°\ 19'$, $a = 118°\ 54'$; find $\alpha = 95°\ 55'$, $b = 10°\ 49'$, $c = 118°\ 20'$.
4. Given $c = 93°\ 15'$, $\beta = 70°\ 20'$; find α, a, b.
5. Given $\alpha = 113°\ 9'$, $\beta = 130°\ 18'$; find $a = 121°\ 2'$, $b = 134°\ 43'$, $c = 68°\ 44'$.
6. Given $\alpha = 135°$, $a = 150°$; find β, b, c. (NOTE: Two values for β, b, c.)

88. THE AREA OF A SPHERICAL TRIANGLE

The area, in spherical degrees, of a spherical triangle is equal to the difference between the sum of the angles and 180°. This difference is also called the **spherical excess** of the triangle. The surface of a sphere contains 720 spherical degrees. If the spherical excess of a triangle is 120°, then the area of the triangle is 120 spherical degrees, or $\frac{120}{720} = \frac{1}{6}$ of the area of the sphere. It is necessary to know the radius of the sphere in order to find the area of the spherical triangle in square units. Let E be the spherical excess of a triangle. Then by definition,

$$E = \alpha + \beta + \gamma - \pi.$$

If K is the area in square units and r the radius of a sphere, then

$$K = \left(\frac{E}{720}\right)(4\pi r^2),$$

or

(70) $$K = \left(\frac{E}{180}\right)(\pi r^2).$$

EXAMPLE. Find the area of a spherical triangle whose angles are 90°, 100°, 120°, and whose radius is 12 inches.

SOLUTION:
$$\begin{aligned} E &= 90 + 100 + 120 - 180 \\ &= 130. \\ K &= (\tfrac{130}{180})(\pi)(12)^2 \\ &= 104\pi \text{ sq. in.} \end{aligned}$$

EXERCISES

Find the area in spherical degrees and in square units (in terms of r) of the following spherical triangles:

1. $\alpha = 70°$, $\beta = 100°$, $\gamma = 85°$. ANS. 85 sp. deg.; $\frac{17\pi r^2}{36}$.
2. $\alpha = 80°$, $\beta = 60°$, $\gamma = 140°$.
3. $a = 118° 54'$, $\beta = 12° 19'$, $\gamma = 90°$. ANS. 18.23 sp. deg.; $0.101(\pi r^2)$
4. $c = 98° 10'$; $\beta = 68°$; $\gamma = 90°$.

89. SOLUTION OF THE OBLIQUE SPHERICAL TRIANGLE

There are six possible cases that may arise in the solution of oblique spherical triangles. When any three of the six parts are given, the remaining three can be found. The fact that six parts may be arranged into exactly six essentially different combinations of three each assures that the following represent all possible cases, listed according to given parts:

1. Two sides and the included angle.
2. Two angles and the included side.
3. Two sides and the angle opposite one of them — an ambiguous case.
4. Two angles and the side opposite one of them — an ambiguous case.
5. The three sides.
6. The three angles.

The spherical triangles in cases 2, 4, and 6 may be reduced to those of cases 1, 3, and 5 by "passing to the polar triangle." For example, if in case 2 the given parts are α, β, and c, the polar triangle relations are $a' = 180° - \alpha$, $b' = 180° - \beta$, $\gamma' = 180° - c$. The polar triangle can now be solved in a manner similar to 1. This illustration shows how we are able to solve the six cases on the basis of three only. It is possible to accomplish the solutions with the Law of Sines and the Law of Cosines which can be used to express a relation between the three knowns and one unknown. However, two handicaps arise when every solution is attempted by these means; first, logarithmic methods cannot be used throughout; second, when the sine of an angle is found, there arises a question as to whether the angle is an acute angle or its obtuse supplement, making laborious checking necessary. In order to simplify the computations, the following additional formulas and others obtained from them by cyclical purmutation of α, β, γ and a, b, c, are generally used, two of which will be proved and a suggestion of the development of the others made in the form of a note.

$$(71)\quad \sin\tfrac{1}{2}\alpha = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}, \text{ where } s = \frac{a+b+c}{2}.$$

$$(72)\quad \cos\tfrac{1}{2}\alpha = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}.$$

$$(73)\quad \tan\tfrac{1}{2}\alpha = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}.$$

$$(74)\quad \sin\tfrac{1}{2}a = \sqrt{\frac{-\cos S\cos(S-\alpha)}{\sin\beta\sin\gamma}}, \text{ where } S = \frac{\alpha+\beta+\gamma}{2}.$$

$$(75)\quad \cos\tfrac{1}{2}a = \sqrt{\frac{\cos(S-\beta)\cos(S-\gamma)}{\sin\beta\sin\gamma}}.$$

$$(76)\quad \tan\tfrac{1}{2}a = \sqrt{\frac{-\cos S\cos(S-\alpha)}{\cos(S-\beta)\cos(S-\gamma)}}.$$

$$(77)\quad \tan\tfrac{1}{2}a = \frac{\sin\tfrac{1}{2}(\beta+\gamma)\tan\tfrac{1}{2}(b-c)}{\sin\tfrac{1}{2}(\beta-\gamma)}.$$

$$(78)\quad \cot\tfrac{1}{2}\alpha = \frac{\sin\tfrac{1}{2}(b+c)\tan\tfrac{1}{2}(\beta-\gamma)}{\sin\tfrac{1}{2}(b-c)}.$$

Proof of (71). Let $2s = a + b + c$, then $a + b - c = 2(s - c)$, $a + c - b = 2(s - b)$, $b + c - a = 2(s - a)$.

Using the Law of Cosines,

$$\cos\alpha = \frac{\cos a - \cos b\cos c}{\sin b \sin c}.$$

But $$2\sin^2 \tfrac{1}{2}\alpha = 1 - \cos\alpha$$
$$= 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$
$$= \frac{\sin b \sin c - \cos a + \cos b \cos c}{\sin b \sin c}$$
$$= \frac{\cos(b - c) - \cos a}{\sin b \sin c}$$
$$= \frac{2 \sin \tfrac{1}{2}(a + b - c) \sin \tfrac{1}{2}(a - b + c)}{\sin b \sin c}.$$

Therefore, $$\sin \tfrac{1}{2}\alpha = \sqrt{\frac{\sin \tfrac{1}{2}(2)(s - c)(\sin \tfrac{1}{2}(2)(s - b)}{\sin b \sin c}},$$

which gives the desired formula,

$$\sin \tfrac{1}{2}\alpha = \sqrt{\frac{\sin(s - b)\sin(s - c)}{\sin b \sin c}}.$$

Proof of (77). Using the Law of Sines.

$$\frac{\sin a}{\sin b} = \frac{\sin\alpha}{\sin\beta}$$

By division $$\frac{\sin a - \sin b}{\sin b} = \frac{\sin\alpha - \sin\beta}{\sin\beta},$$

and $$\sin a - \sin b = \frac{\sin b}{\sin\beta}(\sin\alpha - \sin\beta).$$

From the Law of Cosines,

$$\cos a = \cos b \cos c + \sin b \sin c \cos\alpha,$$
$$\cos b = \cos a \cos c + \sin a \sin c \cos\beta.$$

Adding, $$\cos a + \cos b = \cos b \cos c + \cos a \cos c + \sin b \sin c \cos\alpha + \sin a \sin c \cos\beta.$$

Hence,

$$(\cos a + \cos b)(1 - \cos c) = \sin c(\sin a \cos\beta + \sin b \cos\alpha),$$

and $$(\cos a + \cos b)\tan \tfrac{1}{2}c = \sin a \cos\beta + \sin b \cos\alpha.$$

Substituting, $$\sin a = \frac{\sin b \sin\alpha}{\sin\beta},$$

$$(\cos a + \cos b)\tan \tfrac{1}{2}c = \frac{\sin b(\sin\alpha\cos\beta + \cos\alpha\sin\beta)}{\sin\beta}$$
$$= \frac{\sin b}{\sin\beta}\sin(\alpha + \beta).$$

Then $$\frac{\sin a - \sin b}{(\cos a + \cos b)\tan \tfrac{1}{2}c} = \frac{\sin\alpha - \sin\beta}{\sin(\alpha + \beta)}.$$

But $$\frac{\sin a - \sin b}{\cos a + \cos b} = \tan \tfrac{1}{2}(a - b),$$

and $$\frac{\sin \alpha - \sin \beta}{\sin (\alpha + \beta)} = \frac{\sin \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}(\alpha + \beta)}.$$

Substituting and simplifying,

$$\tan \tfrac{1}{2}c = \frac{\sin \frac{1}{2}(\alpha + \beta) \tan \frac{1}{2}(a - b)}{\sin \frac{1}{2}(\alpha - \beta)}.$$

Replace c by a, α by β, β by γ, a by b, and b by c to obtain (77).

NOTE: The derivation of the formula for $\cos \frac{1}{2}\alpha$ in terms of the sides, formula (72), is readily obtained by changing the proof of (71) to make use of the relation

$$2 \cos^2 \tfrac{1}{2}\alpha = 1 + \cos \alpha$$

instead of the relation

$$2 \sin^2 \tfrac{1}{2}\alpha = 1 - \cos \alpha.$$

Division of (71) by (72) gives (73), the formula for $\tan \frac{1}{2}\alpha$ in terms of the sides.

Formulas (74), (75), and (76) can be obtained from (72), (71), and (73), respectively, by using the polar triangle relations. Thus

$$s = \frac{a + b + c}{2} = \frac{540° - (\alpha' + \beta' + \gamma')}{2} = 270° - S',$$

where $$S' = \frac{\alpha' + \beta' + \gamma'}{2}.$$

Hence, for example,

$$\begin{aligned} \sin (s - b) &= \sin (270° - S' - [180° - \beta']) \\ &= \sin (90° - [S' - \beta']) \\ &= \cos (S' - \beta'). \end{aligned}$$

Thus (71),

$$\sin \tfrac{1}{2}\alpha = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}},$$

becomes

$$\cos \tfrac{1}{2}a' = \sqrt{\frac{\cos (S' - \beta') \cos (S' - \gamma')}{\sin \beta' \sin \gamma'}},$$

which, with primes dropped, is (75). The developments of (74) from (72), (76) from (73), and (78) from (77) are similarly made.

EXAMPLE 1. CASE 1. Given $a = 100°$, $c = 60°$, $\beta = 105°$; find α, γ, b.

SOLUTION: Using the Law of Cosines,

$$\begin{aligned} \cos b &= \cos a \cos c + \sin a \sin c \cos \beta, \text{ and substituting,} \\ \cos b &= (-0.1737)(0.5000) + (0.9848)(0.8660)(-0.2588) \\ &= -0.3076. \end{aligned}$$

Therefore, $$b = 180° - 72° 5' = 107° 55'.$$

Again using the Law of Cosines,

$$\begin{aligned} \cos \alpha &= \frac{\cos a - \cos b \cos c}{\sin b \sin c} \\ &= \frac{-0.1737 - (0.5000)(-0.3076)}{(0.9515)(0.8660)} \\ &= -.0242. \end{aligned}$$

Therefore, $$\alpha = 180° - 88° 37' = 91° 23'.$$

Again using the Law of Cosines,

$$\cos \gamma = \frac{\cos c - \cos a \cos b}{\sin a \sin b}$$
$$= \frac{0.5000 - (-0.1737)(-0.3076)}{(0.9848)(0.9515)}$$
$$= -0.4765.$$

Hence, $\gamma = 180° - 61° 33' = 118° 27'.$

As a check, use the Law of Sines,

$$\frac{\sin \alpha}{\sin a} = \frac{\sin \gamma}{\sin c}.$$

When values are substituted, each member of the equation is found to have approximately the value 1.015.

EXAMPLE 2. CASE 2. Given $\alpha = 48° 30'$, $\beta = 62° 54'$, $c = 114° 30'$; find a, b, γ.

SOLUTION: By changing to the polar triangle the solution can be found with the aid of the Law of Cosines. Another method is to apply the law of angles (63),

$$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c$$
$$= -\cos 48° 30' \cos 62° 54' + \sin 48° 30' \sin 62° 54' \cos 114° 30'$$
$$= -(0.6626)(0.4555) + (0.7490)(0.8902)(-0.4147)$$
$$= -0.5783.$$

Therefore,

$$\gamma = 180° - 54° 40' = 125° 20'.$$

Application of the Law of Sines gives

$$\sin a = \frac{\sin \alpha \sin c}{\sin \gamma},$$

from which it is readily found that

$$\log \sin a = 0.9219 - 1.$$

Thus $a = 56° 40'$ or $180° - 56° 40' = 123° 20'.$

$a - c$ and $\alpha - \gamma$ agree in sign for $a = 56° 40'$, but disagree for $a = 123° 20'$. The latter value is discarded (see note in Example 3). Note that Case 2 is not an ambiguous case, but that the use of the Law of Sines introduces an ambiguity. This suggests that the Law of Sines should be avoided in the unambiguous cases. The unique value of a in this problem could have been found by means of the Law of Angles (62), or with the aid of logarithms by means of a choice of (74), (75), or (76).

To find b, use (77),

$$\tan \tfrac{1}{2}b = \frac{\sin \frac{1}{2}(\gamma + \alpha) \tan \frac{1}{2}(c - a)}{\sin \frac{1}{2}(\gamma - \alpha)}$$
$$= \frac{\sin 86^\circ 55' \tan 28^\circ 55'}{\sin 38^\circ 25'}.$$
$$\log \tan \tfrac{1}{2}b = (0.9994 - 1) + (0.7423 - 1) - (0.7934 - 1)$$
$$= 0.9483 - 1.$$

Hence, $b = 83^\circ 12'$.

A good check would be to show that a, b, and γ obey the Law of Cosines, i.e., that $\cos a \cos b + \sin a \sin b \cos \gamma$ is equal to $\cos c$.

EXAMPLE 3. CASE 3. Given $a = 42^\circ 45'$, $c = 52^\circ 35'$, $\gamma = 77^\circ 21'$; find α, β, and b.

SOLUTION: By the Law of Sines

$$\sin \alpha = \frac{\sin a \sin \gamma}{\sin c}.$$

Taking the logarithm of both sides and substituting values,

$$\log \sin \alpha = (0.8317 - 1) + (0.9893 - 1) - (0.9000 - 1)$$
$$= 0.9210 - 1,$$

so that $\alpha = 56^\circ 29'$, or $\alpha = 180^\circ - 56^\circ 29' = 123^\circ 31'$.

NOTE: In order that each of the values of α be admissible, it is necessary that it lead to a positive value of $\tan \frac{1}{2}b$ through formula (82),

$$\tan \tfrac{1}{2}b = \frac{\sin \frac{1}{2}(\alpha + \gamma) \tan \frac{1}{2}(a - \ )}{\sin \frac{1}{2}(\alpha - \gamma)}$$

It is fairly obvious that the right-hand member of this equation will be positive when $a - c$ and $\alpha - \gamma$ have the same sign. Otherwise, it will be negative. $\alpha = 56^\circ 29'$ passes this test, but $\alpha = 123^\circ 31'$ fails to pass it and must be ruled inadmissible.

Discarding the larger value of α as inadmissible, the value of b can now be determined by formula (82),

$$\tan \tfrac{1}{2}b = \frac{\sin \frac{1}{2}(\alpha + \gamma) \tan \frac{1}{2}(a - c)}{\sin \frac{1}{2}(\alpha - \gamma)}.$$

Taking the logarithm of each member and substituting the known values,

$$\log \tan \tfrac{1}{2}b = \log \sin 66^\circ 55' + \log \tan 4^\circ 55' - \log \sin 10^\circ 26'$$
$$= 0.6405 - 1.$$

Hence, $b = 47^\circ 12'$.

To find β, use the Law of Sines,

$$\sin \beta = \frac{\sin b \sin \gamma}{\sin c},$$

from which

$$\log \sin \beta = \log \sin 47° 12' + \log \sin 77° 21' - \log \sin 52° 35'$$
$$= 0.9548 - 1.$$

Thus $\beta = 64° 19'$, or $\beta = 115° 41'$.

The obtuse value of β is not admissible here, for, writing (82) in the form

$$\tan \tfrac{1}{2}a = \frac{\sin \frac{1}{2}(\gamma + \beta) \tan \frac{1}{2}(c - b)}{\sin \frac{1}{2}(\gamma - \beta)},$$

the right-hand member becomes positive when evaluated with $\beta = 64° 19'$, but negative with $\beta = 115° 41'$. In short, $\gamma - \beta$ differs in sign from $c - b$ when $\beta = 115° 41'$, which eliminates this value of β.

EXAMPLE 4. CASE 4. Given $\alpha = 80°$, $\gamma = 116°$, $a = 84°$; find β, b, and c.

SOLUTION: From the Law of Sines,

$$\sin c = \frac{\sin \gamma \sin a}{\sin \alpha}$$
$$= \frac{\sin 116° \sin 84°}{\sin 80°}$$
$$= 0.9076.$$

Hence, $c = 65° 11'$ or $114° 49'$.

Since $\alpha - \gamma$ and $a - c$ are different in sign for $c = 65° 11'$, but have the same sign for $c = 114° 49'$, the solution must be the single value

$$c = 114° 49'.$$

To find β, make use of (78),

$$\cot \tfrac{1}{2}\beta = \frac{\sin \frac{1}{2}(c + a) \tan \frac{1}{2}(\gamma - \alpha)}{\sin \frac{1}{2}(c - a)}$$
$$= \frac{\sin 99° 25' \tan 18°}{\sin 15° 25'}.$$

From this,

$$\beta = 79° 20'.$$

Finally, b can be found from (77),

$$\tan \tfrac{1}{2}b = \frac{\sin \frac{1}{2}(\gamma + \alpha) \tan \frac{1}{2}(c - a)}{\sin \frac{1}{2}(\gamma - \alpha)}$$
$$= \frac{\sin 98' \tan 15° 25'}{\sin 18°},$$

from which,

$$b = 82° 56'.$$

As a check, it can be shown that $\dfrac{\sin \beta \sin c}{\sin b} = \sin \gamma$, thus satisfying the Law of Sines with the three answers.

EXAMPLE 5. CASE 5. Given $a = 38°$, $b = 42°$, $c = 51°$; find α, β, γ.

SOLUTION: By (76),

$$\sin \tfrac{1}{2}\alpha = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}.$$

In logarithmic form, with values substituted,

$$\begin{aligned}\log \sin \tfrac{1}{2}\alpha &= \tfrac{1}{2}[\log \sin 23° 30' + \log \sin 14° 30' - \log \sin 42° - \log \sin 51°] \\ &= 0.6416 - 1.\end{aligned}$$

Therefore,

$$\alpha = 51° 58'.$$

To find β, use (72) in the form

$$\cos \tfrac{1}{2}\beta = \sqrt{\frac{\sin s \sin (s - b)}{\sin c \sin a}}.$$

From this,

$$\begin{aligned}\log \cos \tfrac{1}{2}\beta &= \tfrac{1}{2}[\log \sin 65° 30' + \log \sin 23° 30' - \log \sin 51° - \log \sin 38°] \\ &= \tfrac{1}{2}[0.9590 - 1 + 0.6007 - 1 - (0.8905 - 1 + 0.7893 - 1)] \\ &= 0.9400 - 1,\end{aligned}$$

from which

$$\beta = 58° 52'.$$

Again using (72) to find γ,

$$\cos \tfrac{1}{2}\gamma = \sqrt{\frac{\sin s \sin (s - c)}{\sin a \sin b}}.$$

Thus,

$$\begin{aligned}\log \cos \tfrac{1}{2}\gamma &= \tfrac{1}{2}[\log \sin 65° 30' + \log \sin 14° 30' - \log \sin 38° - \log \sin 42°] \\ &= \tfrac{1}{2}[0.9590 - 1 + 0.3989 - 1 - (0.7893 - 1 + 0.8255 - 1)] \\ &= 0.8716 - 1.\end{aligned}$$

Hence, $\gamma = 83° 50'$.

Formula (74) can be used for a simultaneous check on all three answers.

EXAMPLE 6. CASE 6. Given $\alpha = 133° 14'$, $\beta = 159° 16'$, $\gamma = 142° 24'$; find a, b, and c.

SOLUTION: The solution can be made by solving the polar triangle with the Law of Cosines, by solving with the Law of Angles, or, with logarithms, by means of (74). Using the latter method, by (74)

$$\begin{aligned}\sin \tfrac{1}{2}a &= \sqrt{\frac{- \cos S \cos (S - \alpha)}{\sin \beta \sin \gamma}} \\ &= \sqrt{\frac{- \cos 217° 27' \cos 84° 13'}{\sin 159° 16' \sin 142° 24'}} \\ &= \sqrt{\frac{\cos 37° 27' \cos 84° 13'}{\sin 20° 44' \sin 37° 36'}}.\end{aligned}$$

From this,

$$\log \sin \tfrac{1}{2}a = 0.7844 - 1.$$

Therefore, $a = 75°.$

Similarly, by (74),

$$\sin \tfrac{1}{2}b = \sqrt{\frac{-\cos S \cos (S - \beta)}{\sin \alpha \sin \gamma}}$$

$$= \sqrt{\frac{\cos 37° 27' \cos 58° 11'}{\sin 46° 46' \sin 37° 36'}}.$$

$$\log \sin \tfrac{1}{2}b = 0.9870 - 1.$$

$$b = 152° 6'.$$

Again, by (74),

$$\sin \tfrac{1}{2}c = \sqrt{\frac{-\cos S \cos (S - \gamma)}{\sin \alpha \sin \beta}}$$

$$= \sqrt{\frac{\cos 37° 27' \cos 75° 3'}{\sin 46° 46' \sin 20° 44'}}.$$

From this, $c = 126° 4'.$

Formula (71) can be used as a check on the three results.

EXERCISES

1. Given $a = 102°$, $b = 139°$, $\gamma = 72°$; find $\alpha = 84° 42'$, $\beta = 138° 8'$, $c = 69° 12'$.
2. Given $b = 47°$, $c = 98°$, $\alpha = 112°$; find a, β, γ.
3. Given $a = 117° 6'$, $c = 54° 40'$, $\beta = 123° 20'$; find $\alpha = 96° 48'$, $\gamma = 65° 30'$, $b = 131° 30'$.
4. Given $\alpha = 78° 12'$, $\beta = 86° 18'$, $c = 112° 43'$; find a, b, γ.
5. Given $\beta = 101° 45'$, $\gamma = 123° 40'$, $a = 60°$; find $\alpha = 72° 52'$, $b = 117° 28'$, $c = 131° 3'$.
6. Given $\alpha = 73° 30'$, $\gamma = 114° 25'$, $b = 28° 15'$; find a, β, c.
7. Given $a = 118°$, $c = 78° 2'$, $\alpha = 115° 30'$; find $\gamma = 90°$, $b = 116° 13'$, $\beta = 113° 30'$.
8. Given $b = 98° 40'$, $c = 57° 10'$, $\beta = 75° 50'$; find α, a, γ.
9. Given $a = 62° 6'$, $b = 134° 51'$, $\alpha = 103° 22'$; find $\beta = 128° 42'$, $\gamma = 138° 55'$, $c = 143° 21'$.
10. Given $\alpha = 102° 35'$, $\beta = 67° 40'$, $a = 35° 18$; find b, c, γ.
11. Given $\alpha = 70° 40'$, $\gamma = 97° 58'$, $a = 72° 20'$; find $c = 90°$, $b = 66° 27'$, $\beta = 65° 12'$.
12. Given $\beta = 70° 25'$, $\gamma = 49° 15'$, $b = 103° 50'$; find a, α, c.
13. Given $a = 30° 3'$, $b = 61° 50'$, $c = 49° 58'$; find $\alpha = 33° 38'$, $\beta = 102° 46'$, $\gamma = 57° 54'$.
14. Given $a = 120°$, $b = 105°$, $c = 60°$; find α, β, γ.
15. Given $\alpha = 29° 40'$, $\beta = 42° 40'$, $\gamma = 159° 55'$; find $a = 40°$, $b = 118° 20'$, $c = 153° 30'$.
16. Given $\alpha = 140° 40'$, $\beta = 105° 45'$, $\gamma = 135° 20'$; find a, b, c.

90. THE TERRESTRIAL SPHERE

The earth may be regarded as a true sphere having a radius of 3960 miles, which is approximately the mean of the equatorial radius and the polar radius. The minor arc of a great circle joining two points on the earth is the shortest distance between the points, and, in fact, is called *the distance* between them. Hence the distance between two points can be expressed in degrees of arc between them. An arc of 1 degree has a length which is $\frac{1}{360}$ of the circumference of the circle. Hence on a great circle

$$1° = \frac{2\pi(3960)}{360} = 69.12 \text{ miles, approximately.}$$

A **nautical mile** is the length of a great circle arc of 1′. Hence 1° = 60 nautical miles and 1 nautical mile = 1.152 statute miles (approx.). A speed of 1 nautical mile per hour is called a *knot.* Thus a speed of 30 knots means a speed of 30 nautical miles per hour. (Note that the correct expression for the speed is *30 knots,* not *30 knots per hour.*)

Meridians of **longitude** are great semicircles joining the north and south poles of the earth. The meridian of reference passes through Greenwich, England. The longitude of any point on the earth is the angle at either pole formed by the meridian through the point and the reference meridian, measured east or west from the latter. The **latitude** of a point is the number of degrees it lies north or south of the equator along the meridian through the point. The locus of all points on the earth having the same latitude is a small circle lying in a plane parallel to that of the equator. Such a circle is called a **circle of latitude.**

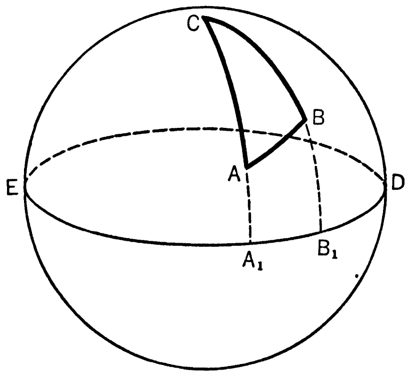


FIGURE 110.

In Fig. 110 let A and B be two points on the surface of the terrestrial sphere. Let C be the north pole, and let A_1B_1D be the equator, with CAA_1 and CBB_1 the meridians through A and B, respectively. Then AA_1 is the latitude of A and BB_1 the latitude of B. Hence in the spherical triangle ABC,

$$AC = 90° - AA_1,$$
$$BC = 90° - BB_1,$$

where the subtraction is algebraic, so that AC is greater than 90° if A is in the southern hemisphere. Furthermore, the angle ACB (or arc A_1B_1) is the difference between the longitudes of A and B. This means that if the latitude and longitude of each of two points are known, the spherical triangle determined by the points and the pole can be solved as belonging to Case 1, having two sides and the included angle given. The solution of the triangle establishes the distance between the points (the arc AB), and the **bearing,** or initial direction, of each point from the other. More precisely the angles CAB and CBA, respectively, give the bearing of B from A and of A from B. In general, the actual direction changes constantly as one follows a great circle course from one point to another.

EXAMPLE 1. Find the distance and bearing of New Orleans (lat. 29° 58′ N, long. 90° 3′ W) from New York (lat. 40° 44′ N, long. 73° 58′ W).

SOLUTION: In Fig. 110, let the points A and B be New Orleans and New York, respectively. Then in the spherical triangle ACB, the angle at C is $90° 3' - 73° 58' = 16° 5'$, arc $CB = 90° - 40° 44'$, and arc $CA = 90° - 29° 58'$. Hence, in the spherical triangle ABC, there is given

$$\gamma = 16° 5',$$
$$a = 49° 16',$$
$$b = 60° 2'.$$

Using the Law of Cosines,

$$\begin{aligned}\cos c &= \cos a \cos b + \sin a \sin b \cos \gamma \\ &= (0.6525)(0.4995) + (0.7578)(0.8663)(0.9609) \\ &= 0.9568.\end{aligned}$$

Therefore, $c = 16° 54' = 16.9°$.

Since $1° = 69.12$ miles, the distance from New Orleans to New York is $(16.9)(69.12) = 1168$ statute miles.

The bearing of New Orleans from New York is given by the angle β of the triangle ABC. This can be found by (72). Thus

$$\cos \tfrac{1}{2}\beta = \sqrt{\frac{\sin s \sin (s - b)}{\sin a \sin c}}$$
$$= \sqrt{\frac{\sin 63^\circ 6' \sin 3^\circ 4'}{\sin 49^\circ 16' \sin 16^\circ 54'}}.$$

Logarithmic solution of this equation gives the result

$$\beta = 124^\circ 32'.$$

EXAMPLE 2. A person travels due east from New York for 100 miles. What is his new position? What is the shortest distance that could have been covered to reach the position?

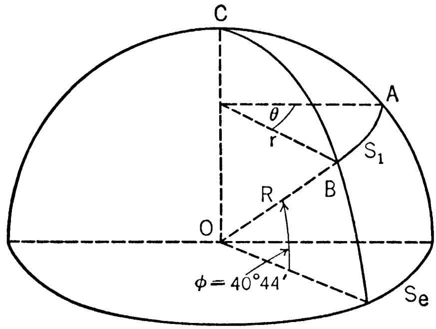


FIGURE 111.

SOLUTION: The position of New York was given in Example 1, lat. 40° 44′ N, long. 73° 58′ W. In Fig. 111, let B represent New York, and A the point 100 miles due east. Obviously, its latitude is the same as that of New York. Let r be the radius of the circle of latitude through A and B let R, be the radius of the earth, and let s_1 and s_e be the arcs on the respective circles, each determined by the central angle θ. Now from (13),

$$s_1 = r\theta \text{ and } s_e = R\theta.$$

Elimination of θ between these equations gives the proportion

$$\frac{s_1}{s_e} = \frac{r}{R}.$$

But, if φ is the latitude of A and of B, $\frac{r}{R} = \cos \varphi$. Hence

$$\frac{s_1}{s_e} = \cos \varphi.$$

From this,

$$s_e = \frac{s_1}{\cos \varphi} = \frac{100}{\cos 40° 44'} \doteq 132.0 \text{ miles.}$$

In degrees, $s_e = \dfrac{132.0}{69.12} = 1.909° = 1° 54'$, so that $\theta = 1° 54'$. Hence the longitude of A is $73° 58' - 1° 54' = 72° 4'$ W.

To find c, the great circle arc between A and B, apply the law of cosines to the spherical triangle ABC, obtaining

$$\begin{aligned} \cos c &= (\cos 49° 16')^2 + (\sin 49° 16')^2(\cos 1° 54') \\ &= 0.9997. \\ c &= 1° 24', \end{aligned}$$

a somewhat doubtful value as a result of the use of the cosine of the small angle $1° 54'$. Since $1° 24' = 1.4°$, in miles,

$$c = (1.4)(69.12) = 96.8 \text{ miles.}$$

NOTES: (1) The doubtfulness of the result found as the great circle distance from A to B illustrates the opening statement of the chapter concerning short distances on the earth's surface.

(2) The formula $\dfrac{s_1}{s_e} = \cos \varphi$ is a general conclusion which can be given by the statement, *the length of an arc of a circle of latitude between two meridians of longitude is equal to the product of the cosine of the latitude and the length of the arc of the equator intercepted by the meridians.*

EXERCISES

1. Find the distance from New York (lat. 40° 44′ N, long. 73° 58′ W) to San Francisco (lat. 37° 47′ N, long. 122° 26′ W). ANS. 2569 miles.
2. Find the distance from Los Angeles (lat. 34° 3′ N, long. 118° 15′ W) to Tokio (lat. 35° 39′ N, long. 139° 45′ E).
3. Find the distance and bearing of Honolulu (lat. 21° 18′ N, long. 157° 52′ W) from San Francisco. ANS. 2396 miles; S 71° 48′ W.
4. Find the distance and bearing of Sydney, Australia (lat. 33° 52′ S, long. 151° 12′ E) from Los Angeles.
5. Find the distance and bearing of London (lat. 51° 30′ N, long. 0°) to Washington (lat. 38° 53′ N, long. 77° 1′ W). ANS. 3672 miles; N 71° 29′ W.
6. What is the position of a battleship that makes 20 knots when it is four days out of San Francisco on its way to Sydney? (1 knot = 1′ on a great circle per hour = 1 nautical mile per hour.)
7. What is the position of a cruiser (latitude and longitude) 500 miles due east of Manila (lat. 14° 35′ N, 120° 59′ E)? ANS. lat. 14° 35′ N; long. 128° 28′ E.
8. Find the area in square miles of a spherical triangle with vertices at San Francisco, Honolulu, and Los Angeles.
9. Find the distance of Panama (lat. 8° 57′ N, long. 79° 32′ W) to Liverpool (lat. 53° 24′ N, long. 3° 4′ W). ANS. 5168 miles.

91. THE CELESTIAL SPHERE

The celestial sphere is an imaginary sphere of indefinite radius on whose surface all celestial objects are assumed to be located. It may be visualized as the dome of the sky.

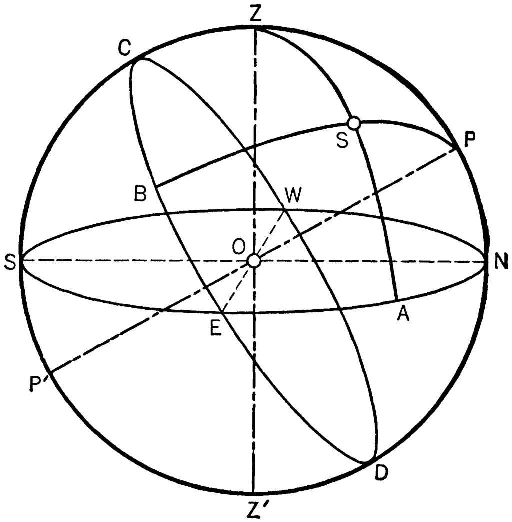


FIGURE 112.

In the figure, the observer is at O. The plane of the observer's terrestrial horizon intersects the celestial sphere in the *horizon*, the great circle $NWSE$. The poles Z and Z' of the horizon are called the *zenith* and *nadir*, respectively. The zenith, Z, appears to be vertically above the observer.

The axis of rotation of the earth intersects the celestial sphere in the **north celestial pole,** P, and the **south celestial pole,** P'. Great circles through the poles are called **celestial meridians,** and that one through Z is the **prime,** or **noon,** meridian. The great circle $CBEDW$ whose plane is perpendicular to PP' is the **celestial equator.**

Due to the rotation of the earth, the sun S, or any heavenly body, appears to move on a small circle having P and P' as poles, the motion being from east to west. The angle between the prime meridian and the meridian through S is called the **hour angle** of S. Celestial meridians are also known as **hour circles.**

The **altitude** of S is its angular distance from the celestial horizon measured along the vertical circle through S and Z. In Fig. 112. the altitude of S is the arc AS. The **declination** of S is its angular distance from the equator measured along a celestial meridian. It is the arc BS in the figure. The **azimuth** of S is the angle PZS, which equals the measure of the arc AN.

It is seen that the latitude of the observer is the arc CZ. Since CZ is equal to PN, the altitude of P, the latitude of an observer can be determined by measuring the angular distance of the pole above the horizon. In the northern hemisphere, the North Star, or Pole Star, is approximately at P.

The spherical triangle ZPS can be solved if any three parts are known. In the triangle,

$$\begin{aligned} PS &= 90° - BS \text{ (declination)}, \\ ZS &= 90° - SA \text{ (altitude)}, \\ ZP &= 90° - PN \text{ (latitude)}, \\ \angle ZPS &= \text{hour angle of } S, \\ \angle PZS &= \text{azimuth of } S. \end{aligned}$$

A heavenly body appears to make one complete revolution about the earth in 24 hours. Hence its hour angle changes by 15° each hour. The altitude of the sun can be found by the observer by the use of some instrument, as the sextant. The declination of the sun varies with the date and is determined from tables and in many almanacs. It varies from $-23° 30'$ to $+23° 30'$.

To find the time of an observation, it is necessary to divide the hour angle by 15. This gives the number of hours before, or after, noon of the observation, according to whether the observation was made in the morning or afternoon.

While the sun is the most important heavenly body for such observations, any result obtained by solar observations can be obtained from stellar observations. The sun has been used as a reference point to keep the idea definite.

EXAMPLE 1. **Given the latitude as 30° N, the altitude of the sun as 35°, and the declination of the sun as 10°, find the time of the observation if it is made before noon.**

Solution: In Fig. 112, let $PN = 30°$, $AS = 35°$, $BS = 10°$, then in triangle ZPS,

$$s = ZP = 90° - 30° = 60°,$$
$$p = ZS = 90° - 35° = 55°,$$
$$z = PS = 90° - 10° = 80°.$$

Using the law of cosines, to find the angle at P,

$$\begin{aligned} \cos \angle P &= \frac{\cos p - \cos s \cos z}{\sin s \sin z} \\ &= \frac{\cos 55° - \cos 60° \cos 80°}{\sin 60° \sin 80°} \\ &= \frac{0.5736 - (0.5000)(0.1737)}{(0.866)(0.9848)} \\ &= 0.5707. \end{aligned}$$

Therefore, $\angle P = 55° \, 12' = 55.2°.$

Since the hour angle changes 15° every hour, the time is $\frac{55.2}{15} = 3$ hours and 41 minutes before noon, or 8 : 19 A.M., local time.

EXAMPLE 2. Find the time of sunrise at New Orleans (lat. 29° 58′ N) on its shortest day (declination − 23° 30′).

Solution: In Fig. 112, $PN = 29° \, 58'$, $AS = 0°$, and $BS = - 23° \, 30'$. Then in triangle ZPS,

$$s = 90° - 29° \, 58' = 60° \, 2',$$
$$p = 90° - 0° = 90°,$$
$$z = 90° + 23° \, 30' = 113° \, 30'. \quad \text{Solve for angle } P.$$

This is a *quadrantal triangle*, i.e., one side of the spherical triangle is equal to 90°. A quadrantal triangle is most readily solved by passing to the polar triangle, which of course is a right spherical triangle. Therefore

$$S' = 180° - 60° \, 2' = 119° \, 58',$$
$$P' = 180° - 90° = 90°,$$
$$Z' = 180° - 113° \, 30' = 66° \, 30',$$
$$p' = 180° - P. \quad \text{Solve for } p'.$$

Using Napier's Rule

$$\begin{aligned} \cos p' &= \cot S' \cot Z', \text{ and substituting} \\ &= \cot 119° \, 58' \cot 66° \, 30' \\ &= (-0.5766)(0.4348) \\ &= -0.2507. \end{aligned}$$

Therefore, $p' = 180° - 75° \, 56' = 104° \, 4'.$

Hence, $P = 180° - 104° \, 4' = 75° \, 56' = 75.93°.$

Since the hour angle changes 15° every hour, the time is $\frac{75.93}{15} = 5$ hours and 4 minutes before noon, or 6 : 56 A.M.

EXERCISES

1. The altitude of the sun at latitude 47° 30′ N is 35°. Find the time if the declination is 12° 15′, and the observation was made in the afternoon.
   ANS. 3 : 43 P.M.
2. At latitude 40° N the altitude of a star is 15°. Find the declination if its azimuth is 68° 50′.
3. Find the declination of a star that sets in the southeast at latitude 35° S.
   ANS. 53° 54′.
4. Find the time of sunset at Washington (lat. 38° 53′ N) when its declination is − 10° 45′.
5. The altitude of a star is 47°, its declination is 53°, and its hour angle is 75°. Find the azimuth. ANS. 61° 30′.
6. The declination of the sun is 15°, its altitude 75°, and the time is 10 A.M. Find the latitude of the observer.
7. An observer finds that the altitude of the sun is 14° 18′. Its declination is 18° 36′. If the latitude is 50° 13′ N, and if the observation is made in the morning, find the hour of the day. If the observation is made at 9 A.M., Greenwich time, what is the longitude? ANS. long. 44° 49′ W.
8. Find the longitude of the position of the observation in Ex. 1 if the Greenwich time was 8 : 23 P.M.

SUMMARY OF FORMULAS IN SPHERICAL TRIGONOMETRY*

The Law of Sines.

(61) $$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}.$$

The Law of Cosines,

(62) $$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma.$$

The Law of Angles,

(63) $$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c.$$

Special formulas for the spherical right triangle:

(64) $\sin a = \sin \alpha \sin c.$

(64′) $\sin b = \sin \beta \sin c.$

(65) $\cos c = \cos a \cos b.$

(66) $\cos \alpha = \sin \beta \cos a.$

(66′) $\cos \beta = \sin \alpha \cos b.$

(67) $\cos \alpha = \tan b \cot c.$

(67′) $\cos \beta = \tan a \cot c.$

(68) $\sin a = \cot \beta \tan b.$

(68′) $\sin b = \cot \alpha \tan a.$

(69) $\cos c = \cot \alpha \cot \beta.$

*Each of the formulas for functions of sides or angles of the general spherical triangle can be used to derive two companion formulas by cyclic rearrangement of letters (cf. page 104).

Area of spherical triangle,

(70) $$K = \frac{\pi r^2 E}{180}, \text{ where } E = \alpha + \beta + \gamma - 180.$$

Formulas to use when 3 sides are given:

(71) $$\sin \tfrac{1}{2}\alpha = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}, \text{ where } s = \frac{a + b + c}{2}$$

(72) $$\cos \tfrac{1}{2}\alpha = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}}.$$

(73) $$\tan \tfrac{1}{2}\alpha = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}}.$$

Formulas to use when 3 angles are given:

(74) $$\sin \tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos (S - \alpha)}{\sin \beta \sin \gamma}}, \text{ where } S = \frac{\alpha + \beta + \gamma}{2}.$$

(75) $$\cos \tfrac{1}{2}a = \sqrt{\frac{\cos (S - \beta) \cos (S - \gamma)}{\sin \beta \sin \gamma}}.$$

(76) $$\tan \tfrac{1}{2}a = \sqrt{\frac{-\cos S \cos (S - \alpha)}{\cos (S - \beta) \cos (S - \gamma)}}.$$

Formulas to use when two sides and the angles opposite them are given:

(77) $$\tan \tfrac{1}{2}a = \frac{\sin \frac{1}{2}(\beta + \gamma) \tan \frac{1}{2}(b - c)}{\sin \frac{1}{2}(\beta - \gamma)}.$$

(78) $$\cot \tfrac{1}{2}\alpha = \frac{\sin \frac{1}{2}(b + c) \tan \frac{1}{2}(\beta - \gamma)}{\sin \frac{1}{2}(b - c)}.$$

APPENDIX

I. ACCURACY OF COMPUTED RESULTS

The most important consideration connected with accuracy of answers is the question of the accuracy of the numbers used to obtain the answer. Some of the numbers will be *exact,* as for example the 2 in the formula $c = 2\pi r$, others will be *approximate,* as are all numbers obtained by measurement and most of the numbers obtained from tables. (Exceptions are such numbers as $\sin 30° = 0.5$, $\log 100 = 2$, which are exact.) Irrational numbers such as π, $\sqrt{2}$, $\sqrt[3]{5}$, etc., are usually represented by approximate decimal expressions.

One way of expressing the degree of accuracy of an approximate number is to state the number of *significant figures* (or digits) to which it is accurate. Thus if the number π is equal to $3.14159265358979 \cdots$, where the dots mean that the digits may be continued indefinitely, it is customary to say that 3.14, 3.142, 3.1416, and 3.141593 are approximations of π correct to 3, 4, 5, and 7 significant figures, respectively. Significant figures are digits which give meaning to a number regardless of the location of the decimal point. The digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are always significant when present in a number. Zero is not significant if it occurs to the left of all the other digits in a number, as in 0.45 or 0.00045; it is significant if it occurs between other digits, as in 205 and in 6.007; its significance is in doubt if it occurs to the right of the other digits, as in 40 or 3300. If the approximate number 40 actually represents a certain magnitude more exactly than 39 or 41 does, then the figure 0 is significant, but if it merely represents a number closer to 40 than to 30 or 50, the 0 is not significant. It is proper to be specific in such cases with a statement. For example, if 3300 is a better approximation to a certain measurement than 3290 or 3310 are, with no other information as to its accuracy, it is said to be correct to 3 significant figures, or correct to the nearest 10.

As a chain is no stronger than its weakest link, so a numerical result obtained by multiplications and divisions with approximate numbers will, as a rule, contain no more significant figures than that one of the given numbers having the least number of significant figures. This is a safe working rule. As an illustration, suppose the radius of a circle is 22 inches to the nearest inch, and it is desired to know the area of the circle from the formula πr^2. If the value of π is taken as the approximate number 3.1416, the value of πr^2 as found from the formula is 1530.5344 square inches, an obvious absurdity — even worse, a delusion of accuracy. Actually, all that can be said is that the area is 1500 sq. in., correct to two significant figures. On the other hand, if the radius is 22.000, correct to five significant figures, the answer is 1530.5 square inches, since both component numbers were correct to 5 significant figures.

No very simple rule can be stated for addition and subtraction of approximate numbers, but when the given numbers are arranged with decimal points in a vertical line, it is a simple matter to see which digits of the answer are determined entirely by accurate figures and which ones should be regarded as approximate.

In rounding off numbers to a prescribed number of significant digits, the choice of whether to leave the last retained digit unchanged or to increase it by one is too familiar to the reader to need repetition except in the case that the discarded digits amount to exactly half a unit of the desired accuracy. It is considered good practice by computers to make the last retained digit an *even* one in such cases. Thus the numbers 0.68645, 0.42375, and 3426500 become 0.6864, 0.4238, and 3426000, respectively, when rounded off to accuracy of four significant figures. Judicious rounding off of numbers will save time without sacrifice of accuracy if done *before* computation is begun. If in the first illustration of area of a circle with radius 22 inches, the value of π had been rounded off to 3.14, the area could have been computed more rapidly and with the same accuracy as when using 3.1416.

Usually, measurements can be made more accurately than to two significant figures when suitable units are used. It has been tacitly assumed that the measurement of the quantities treated in this book are accurate to four significant figures. The tables are also accurate to four significant figures. Hence it is generally to be expected that answers will be accurate to not more than four significant figures.

Since the trigonometric functions are ratios of lengths the accuracy in the determination of the values of the functions corresponds to the accuracy in the determination of the lengths. The accuracy in the determination of angles from their functions depends somewhat on the size of the angle. However, as a general rule, functions expressed with 3, 4, 5, 6 significant figures correspond in accuracy of the angle respectively to the nearest tenth of a degree, minute, tenth of a minute, and to the nearest second.

II. COMMON LOGARITHMS

The following paragraphs are arranged to give in brief compass the essentials of common logarithms. The numbered sentences in italics, separated from their respective explanations, will serve as an outline of the development. It is assumed that the student is familiar with the algebraic laws of exponents, namely, if a is any real number different from zero, x and y positive numbers,

$$(a^x)(a^y) = a^{x+y},$$

$$\frac{a^x}{a^y} = a^{x-y},$$

$$(a^x)^y = a^{xy},$$

$$\sqrt[y]{a^x} = a^{\frac{x}{y}}.$$

The following definitions from algebra will also be assumed:

$$a^{-x} = \frac{1}{a^x},$$

$$a^0 = 1.$$

1. *Every positive real number can be expressed as a power of ten* The proof of this statement is beyond the scope of this book. The student is asked to accept the following statement: If

$$N = 10^x,$$

where N is any given positive real number, it is possible to calculate the value of x to any desired degree of accuracy. This will seem reasonable if the following illustrations, which can be verified by the student, are studied:

$1000 = 10^3$, $\quad 3.162 = \sqrt{10} = 10^{0.5}$, approximately,

$10 = 10^1$, $\quad 2.154 = \sqrt[3]{10} = 10^{0.3333}$, approximately,

$1 = 10^0$, $\quad 31.62 = \sqrt{10^3} = 10^{\frac{3}{2}} = 10^{1.5}$, approximately,

$0.01 = \frac{1}{10^2} = 10^{-2}$, $\quad 0.3162 = \frac{1}{\sqrt{10}} = 10^{-0.5}$, approximately.

2. *If $N = 10^x$, the exponent x is called the common logarithm, or simply the logarithm, of the number N.* Thus the logarithm of 100 is 2 because $10^2 = 100$; the logarithm of 2.154 is 0.3333 because $10^{.3333} = 2.154$; the logarithm of 0.3162 is -0.5 because $10^{-.5} = 0.3162$; the logarithm of 1 is 0 because $10^0 = 1$; etc. The abbreviated form *log N* is used to represent the expression "the logarithm of the number *N*," and it should be so read. Thus we write $\log 100 = 2$, $\log 10 = 1$, $\log 31.62 = 1.5$, etc. Finally, if $N = 10^x$, then $\log N = x$.

3. *The logarithms of numbers which are integral (whole number) powers of ten are obtained by inspection.* This statement has already been put into effect in the preceding discussion and is made here for the sake of emphasis. The student can readily determine such results as

$$\log 100{,}000 = 5,$$
$$\log 0.001 = -3,$$
and
$$\log 100 = 2.$$

4. *The logarithms of numbers of three digits from 1.00 to 9.99, inclusive, are given to four decimal places in Table II.* To find the logarithm of such a number in the table, locate the first two digits in the column headed by n, hold their position, and proceed horizontally to the right until the column headed by the third digit of the given number is reached. The number thus located, with a decimal point placed at its left, is the logarithm of the given number. For example, to find log 5.62, look down the column under N until 56 is located (all decimal points are omitted in the table), then move across the table until the column headed by 2 is reached. The corresponding entry, 0.7497, is the logarithm of 5.62. As other examples, the student should verify that $\log 1.03 = 0.0128$ and that $\log 7.48 = 0.8739$.

5. *Logarithms can be used to simplify computations involving the operations of multiplication, division, and finding powers and roots.* Consider a few illustrative examples.

EXAMPLE 1. Find $\dfrac{(3.23)(2.99)}{2.21}$.

SOLUTION: From the tables,

$$\log 3.23 = 0.5092, \quad \therefore 3.23 = 10^{0.5092};$$
$$\log 2.99 = 0.4757, \quad \therefore 2.99 = 10^{0.4757};$$
$$\log 2.21 = 0.3444, \quad \therefore 2.21 = 10^{0.3444}.$$

$$\text{Hence,}\quad \frac{(3.23)(2.99)}{2.21} = \frac{(10^{0.5092})(10^{0.4757})}{10^{0.3444}},$$
$$= 10^{0.5092+0.4757-0.3444},$$
$$= 10^{0.6405}.$$

Reference to the tables will show that the number whose logarithm is 0.6405 is 4.37. Thus $\frac{(3.23)(2.99)}{2.21} = 4.37.$

NOTE: The reader may not agree that this method "simplifies" the operations, but he certainly must agree that the operations of multiplication and division have been reduced to the *simpler* operations of addition and subtraction. Later, improved methods of manipulating the logarithms will be shown.

EXAMPLE 2. Find $\sqrt[3]{9}$.

SOLUTION: After the manner of Ex. 1, log 9 = 0.9542, so that $9 = 10^{0.9542}$.

$$\text{Hence,}\quad \sqrt[3]{9} = \sqrt[3]{10^{0.9542}} = (10)^{\frac{0.9542}{3}} = 10^{0.3181}.$$

From the table 0.3181 is the logarithm of 2.08.

$$\text{Hence,}\quad \sqrt[3]{9} = 2.08.$$

EXERCISES

Apply the method of the above examples to compute each of the following, correct to the nearest third digit:

1. $(4.33)(1.94)$. ANS. 8.40.
2. $6.5\sqrt{2.88}$.
3. $\frac{8.26}{2.67}$. ANS. 3.09.
4. $\frac{(7.57)(4.9)}{5.18}$.
5. $(1.06)^{20}$. ANS. 3.21.
6. $(7.36)^{\frac{2}{3}}$.
7. $\sqrt{7.51}$. ANS. 2.74.
8. $\sqrt{(6.41)(3.29)}$.
9. $\sqrt[3]{8.92}$. ANS. 2.07.
10. $\frac{9.37}{(2.4)(3.25)}$.

6. *By a process called interpolation, the logarithm of any four digit number from 1.000 to 9.999 can be determined from the table. Conversely, given the logarithm of any such number, the number can be determined correct to four digits.* These statements can best be justified by the use of examples.

EXAMPLE 1. Find log 1.954.

SOLUTION: The value of log 1.954 obviously is between that of log 1.950 and of log 1.960. In fact, it would seem to be four-tenths "of the way" between them. (Experience shows that this assumption of proportionality is accurate enough to permit the use of the tables in finding the logarithms of numbers within the range stated above.) Upon this assumption,

$$\begin{aligned}\log 1.954 &= \log 1.950 + 0.4(\log 1.960 - \log 1.950),\\ &= 0.2900 + 0.4(0.2923 - 0.2900),\\ &= 0.2900 + 0.4(.0023),\\ &= 0.2909. \text{ Ans.}\end{aligned}$$

In actual practice, since all of the logarithms are to ten thousandths, it is customary to make the subtraction mentally, obtaining 23 (ten thousandths), multiply by 0.4, obtaining 9 (ten thousandths) and then add to the first logarithm.

EXAMPLE 2. Find the number whose logarithm is 0.2420.

SOLUTION: Examination of the table shows that 0.2420 lies between the logarithms of 1.740 and 1.750. Thus, if $\log n = 0.2420$,

$$\begin{aligned}\log 1.740 &= 0.2405,\\ \log n &= 0.2420,\\ \log 1.750 &= 0.2430.\end{aligned}$$

Assuming proportionality, it is obvious that n is $\frac{15}{25} = 0.8$ "of the way" between 1.740 and 1.750. Hence,

$$n = 1.748. \quad \text{Ans.}$$

EXERCISES

Find the logarithms of the following numbers:

1. 3.655. ANS. 0.5629. **2.** 8.202.
3. 8.912. ANS. 0.9500. **4.** 3.645.
5. 2.576. ANS. 0.4109. **6.** 1.144.
7. 3.142. ANS. 0.4972. **8.** 5.338.
9. 3.1416. ANS. 0.4972. **10.** 5.337831.
11. 4.004. ANS. 0.6025. **12.** 8.999.

Find correct to four digits the numbers whose logarithms are:

13. 0.4806. ANS. 3.024. **14.** 0.6175.
15. 0.6388. ANS. 4.353. **16.** 0.3434.
17. 0.8303. ANS. 6.766. **18.** 0.3314.
19. 0.4400. ANS. 2.754. **20.** 0.5701.
21. 0.6605. ANS. 4.576. **22.** 0.6301.
23. 0.8245. ANS. 6.676. **24.** 0.3844.

7. *Since logarithms are exponents, indicating powers of ten, they obey the laws of exponents so that laws of logarithms are:*

$$\text{(I)} \quad \log (A \cdot B) = \log A + \log B,$$

$$\text{(II)} \quad \log \left(\frac{A}{B}\right) = \log A - \log B,$$

$$\text{(III)} \quad \log (A)^p = p \log A,$$

$$\text{(IV)} \quad \log \sqrt[r]{A} = \frac{1}{r} \log A.$$

In the same way that $10^2 \times 10^3 = 10^{2+3} = 10^5$, it is seen that

$$A \cdot B = 10^{\log A} \times 10^{\log B} = 10^{\log A + \log B},$$

which is equivalent to (I) above. The other three laws can be verified similarly.

EXAMPLE. Use the laws of logarithms to express the logarithm of N if

(a) $N = x^3y\sqrt{4 + x^2}$,
(b) $N = (3.14)(10^2)$.

SOLUTIONS: (a) Combining the logarithm laws for products, (I), for powers, (III), and for roots, (IV), $\log N$ can be written

$$\log N = 3 \log x + \log y + \tfrac{1}{2} \log (4 + x^2).$$

(b) If the product law is applied, log 3.14 determined from the table, and $\log 10^2$ found by inspection, then

$$\begin{aligned} \log N &= \log 3.14 + \log (10^2) \\ &= 0.4969 + 2 \\ &= 2.4969. \end{aligned}$$

EXERCISES

Express the logarithm of both members of the equation in each of the following and find the value of the logarithm in the numerical cases:

1. $K = \sqrt{s(s - a)(s - b)(s - c)}$.
   ANS. $\log K = \frac{1}{2}[\log s + \log (s - a) + \log (s - b) + \log (s - c)]$.
2. $x = \sqrt{\dfrac{(s - b)(s - c)}{s(s - a)}}$.
3. $V = \frac{4}{3}\pi r^3$. ANS. $\log V = \log 4 + \log \pi + 3 \log r - \log 3$.
4. $l = ar^{k-1}$.
5. $n = ke^x$. ANS. $\log n = \log k + x \log e$.
6. $S = P(1 + i)^n$.
7. $N = 4.65(10^5)$. ANS. $\log N = 5.6675$.
8. $N = 9.23(10^3)$.
9. $q = \dfrac{3.41(10^4)}{7.34}$. ANS. $\log q = 3.6671$.
10. $x = \dfrac{5.8(10^2)}{6.66}$.

8. *The logarithm of any positive number N (rounded off to 4 significant digits) can be determined by first expressing the number in the so-called standard form $N = n(10^k)$, where n is between 1 and 10 and where k is a positive or negative integer or is zero.*

EXAMPLE 1. If $N = 634{,}000$, what is $\log N$?

SOLUTION: In *standard form*

$$N = 6.34(10^5).$$

Hence, by the method of Example (b) of the preceding article,

$$\begin{aligned} \log N &= \log 6.34 + \log (10^5) \\ &= 0.8021 + 5 \\ &= 5.8021. \quad \text{Ans.} \end{aligned}$$

EXAMPLE 2. Find log 0.05628.

SOLUTION: In standard form

$$0.05628 = 5.628(10^{-2}).$$

Hence,

$$\begin{aligned} \log 0.05628 &= \log 5.628 + \log (10^{-2}) \\ &= 0.7504 - 2. \quad \text{Ans.} \end{aligned}$$

It proves convenient to write logarithms such as found in Example 2 with the negative part kept separated from the positive part. In fact, many persons prefer to adjust such logarithms so that the negative part is -10. Thus, since $-2 = 8 - 10$,

$$\log 0.05628 = 8.7504 - 10.$$

In composite logarithms such as found in the examples just given, the decimal part of the logarithm is called the *mantissa* and the integral part, the *characteristic*. Thus in

$$\log 634{,}000 = 5.8021,$$

the mantissa of the logarithm is 0.8021 and the characteristic is 5. Again in

$$\log 0.05628 = 0.7504 - 2 \ (\text{or } 8.7504 - 10),$$

the mantissa is 0.7504 and the characteristic is -2.

From the expression $N = n(10^k)$, the characteristic of $\log N$ is k and the mantissa is $\log n$.

EXERCISES

Making use of the answers given to the odd-numbered exercises from 1 to 11, Paragraph 6, find the logarithms of the numbers below:

1. 365.5. ANS. 2.5629. **2.** 36550.
3. 0.003655. ANS. $0.5629 - 3$. **4.** 0.3655.
5. 89.12. ANS. 1.9500. **6.** $8.912(10^8)$.
7. 0.08912. ANS. $8.9500 - 10$. **8.** 0.00008912.
9. 2576. ANS. 3.4109. **10.** 257600.

With the aid of the tables find the logarithms of the following numbers, performing any intermediate calculations mentally:

11. 5930. ANS. 3.7731. **12.** 47.4.
13. 4.56. ANS. 0.6590. **14.** 21.
15. 0.005. ANS. $7.6990 - 10$. **16.** 8040.
17. 101. ANS. 2.0043. **18.** 0.387
19. 0.0712. ANS. $0.8525 - 2$. **20.** 7000.

9. *If log N is given, N can be determined by reversing the steps suggested in Paragraph 8 for finding the logàrithm of a number.* This will be illustrated with examples.

EXAMPLE 1. If $\log N = 3.8142$, what is N?

SOLUTION: Since $\log N = 0.8142 + 3$, then

$$\begin{aligned} N &= (\text{number whose log is } 0.8142) \cdot (\text{number whose log is } 3) \\ &= 6.52(10^3) \quad [\text{from the tables}], \\ &= 6520. \quad \text{Ans.} \end{aligned}$$

ALTERNATE SOLUTION OF 1. Since $\log N = 3.8142$, then by the definition of a logarithm,

$$\begin{aligned} N &= 10^{3.8142} \\ &= 10^{0.8142} \times 10^3 \\ &= 6.52(10^3) \quad [\text{from the tables}] \\ &= 6520. \quad \text{Ans.} \end{aligned}$$

EXAMPLE 2. If $\log N = 7.7290 - 10$, what is N?

SOLUTION: Since $\log N = 0.7290 - 3$, then

$$\begin{aligned} N &= (\text{number whose log is } 0.7290)(\text{number whose log is } -3) \\ &= 5.358(10^{-3}) \\ &= 0.005358. \end{aligned}$$

EXERCISES

Using the answers to Exercises 13–23, Par. 6, find the value of N, given that:

1. $\log N = 1.4806$. ANS. $N = 30.24$.
2. $\log N = 9.4806 - 10$.
3. $\log N = 3.4806$. ANS. $N = 3024$.
4. $\log N = 2.4806$.
5. $\log N = 0.6388 - 2$. ANS. $N = 0.04353$.
6. $\log N = 1.6388$.
7. $\log N = 3.6388$. ANS. $N = 4353$.
8. $\log N = 7.6388 - 10$.
9. $\log N = 9.8303 - 10$. ANS. $N = 0.6766$.
10. $\log N = 2.8303$.

With the aid of the tables, find x, given that:

11. $\log x = 8.7574 - 10$. ANS. $x = 0.0572$.
12. $\log x = 2.4082$.
13. $\log x = 1.5132$. ANS. $x = 32.6$.
14. $\log x = 9.2504 - 10$.
15. $\log x = 0.9325$. ANS. $x = 8.56$.
16. $\log x = 3.8222$.
17. $\log x = 0.9046 - 3$. ANS. $x = 0.008028$.
18. $\log x = 1.7828$.
19. $\log x = 4.3724$. ANS. $x = 23570$.
20. $\log x = 0.4545$.

10. *Calculations involving multiplication, division, powers, and roots may be performed by means of simpler operations with the logarithms of the given numbers.* In Paragraph 7 it was shown how to express the logarithm of a given expression involving these operations. Paragraphs 8 and 9 provide the information necessary to complete the calculation. Certain devices can be used in order to avoid obtaining *negative fractions* in the computations, as shown in Example 2 below. Negative fractions are avoided if the tables are to be used, since the entries in the tables are all positive.

EXAMPLE 1. With logarithms find $\frac{(65.4)(1.657)^2}{(0.872)(43.6)}$.

SOLUTION: Let $N = \frac{(65.4)(1.657)^2}{(0.872)(43.6)}$.

Then $\log N = \log 65.4 + 2 \log 1.657 - (\log 0.872 + \log 43.6)$.

A suggested arrangement of the details follows:

$$\begin{aligned} \log 1.657 &= 0.2193. \\ 2 \log 1.657 &= 0.4386 \\ \log 65.4 &= \underline{1.8156} \quad (+) \\ \log (\text{numerator}) &= 2.2542 \\ \log (\text{denominator}) &= \underline{1.5800} \quad (-) \\ \log N &= 0.6742, \end{aligned}$$

$$\begin{aligned} \log 0.872 &= 0.9405 - 1 \\ \log 43.6 &= \underline{1.6395} \quad (+) \\ \log (\text{denom.}) &= 1.5800 \end{aligned}$$

and from the tables

$$N = 4.723. \quad \text{Ans.}$$

EXAMPLE 2. Find $\sqrt[3]{\frac{2.304}{87.5}}$.

SOLUTION: Let $x = \sqrt[3]{\frac{2.304}{87.5}}$.

Then $\log x = \frac{1}{3}(\log 2.304 - \log 87.5)$.

Computation:

$$\begin{aligned} \log 2.304 &= 0.3625 \\ \log 87.5 &= \underline{1.9420} \quad (-) \end{aligned}$$

Subtraction in this form results in a negative fraction. To avoid this difficulty, simply add zero in the form 2 − 2 to the first logarithm, thus writing 0.3625 as 2.3625 − 2. Then

$$\begin{aligned} \log 2.304 &= 2.3625 - 2 \\ \log 87.5 &= \underline{1.9420} \quad (-) \\ \log (\text{fraction}) &= 0.4205 - 2 \end{aligned}$$

Now, $\log x = \frac{1}{3}(0.4205 - 2)$.

Again, if the division is carried out in the present form, a negative decimal will result. This difficulty can be avoided by use of the equivalent form, with 0 = 1 − 1 added in the parenthesis,

$$\begin{aligned} \log x &= \tfrac{1}{3}(1.4205 - 3) \\ &= 0.4735 - 1, \end{aligned}$$

and from the tables, $x = 0.2975$. Ans.

EXERCISES

By means of logarithms calculate the values of the unknowns to four significant figures:

1. $N = 371.5 \times 4.27$. ANS. 1586.
2. $x = 34.7 \times 484 \times 0.495$.
3. $t = \dfrac{834}{6.43}$. ANS. 129.7.
4. $n = \dfrac{5600}{0.034 \times 767}$.
5. $x = \sqrt{84.69}$. ANS. 9.202.
6. $r = \sqrt[3]{285}$.
7. $y = (7.452)^3$. ANS. 413.9.
8. $x = (17.5)^{\frac{3}{4}}$.
9. $n = \dfrac{3.14}{88.4}$. ANS. 0.03551.
10. $m = \dfrac{62.82}{0.372}$.
11. $z = \sqrt{0.684}$. ANS. 0.8272.
12. $n = \sqrt[3]{0.612}$.

Solve each of the following equations, applying the following suggestions: (a) Take the logarithm of each side of the equation and simplify by the laws of logarithms, (b) solve the resulting equation for the logarithm of the unknown, and (c) evaluate the unknown as in the first twelve exercises.

13. If n is the number of shot 0 12 in. in radius that can be made from a cylinder of lead 6 in. long and 3 in. in radius, then

$$\frac{4\pi(0.12)^3 n}{3} = \pi(1.5)^2(6).$$

Find n.

14. $62.5\,x^3 = 546{,}000$. ANS. $x = 20.59$.
15. $\dfrac{x}{0.5645} = \dfrac{67.5}{83.7}$.
16. $x^2 = (4630)(781)$. ANS. $x = 1902$.
17. $2.556 = 3.14\sqrt{\dfrac{l}{32.2}}$.
18. $\frac{4}{3}(3.1416)r^3 = 205^3$. ANS. 127.2.
19. $L = 0.492(.001189)(166)(95.4)^2$.

SUMMARY OF LOGARITHMS

(*a*) *Definition.* The common logarithm of a number N is the value of x which satisfies the equation

$$10^x = N.$$

(*b*) *Rules of logarithms.*

$$\text{I.}\quad \log(AB) = \log A + \log B.$$

$$\text{II.}\quad \log\left(\frac{A}{B}\right) = \log A - \log B.$$

$$\text{III.}\quad \log(A)^p = p \log A.$$

$$\text{IV.}\quad \log \sqrt[r]{A} = \frac{1}{r}\log A.$$

(*c*) *Mantissa and characteristic.* Given a number N, express it in the form

$$N = n(10^k),$$

where n is between 1 and 10, and where k is an integer or zero. The mantissa of log n is the mantissa of log N and can be found in the tables. k is the characteristic of log N.

(*d*) *Rule of mantissa.* The mantissa of the logarithm of a number does not depend upon the location of the decimal point in the number.

(*e*) *Pointing off the answer.* The number found in the table corresponding to a given mantissa is between 1 and 10, with decimal point between the first and second significant figures. The characteristic of the logarithm of the number indicates how many places to move the decimal point, to right or left, depending upon the sign.

TABLES

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$		n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$
1	1	1	1	1	**51**	2601	132651	7.141	3.708
2	4	8	1.414	1.260	52	2704	140608	7.211	3.733
3	9	27	1.732	1.442	53	2809	148877	7.280	3.756
4	16	64	2.000	1.587	54	2916	157464	7.348	3.780
5	25	125	2.236	1.710	55	3025	166375	7.416	3.803
6	36	216	2.449	1.817	56	3136	175616	7.483	3.826
7	49	343	2.646	1.913	57	3249	185193	7.550	3.849
8	64	512	2.828	2.000	58	3364	195112	7.616	3.871
9	81	729	3.000	2.080	59	3481	205379	7.681	3.893
10	100	1000	3.162	2.154	60	3600	216000	7.746	3.915
11	121	1331	3.317	2.224	**61**	3721	226981	7.810	3.936
12	144	1728	3.464	2.289	62	3844	238328	7.874	3.958
13	169	2197	3.606	2.351	63	3969	250047	7.937	3.979
14	196	2744	3.742	2.410	64	4096	262144	8.000	4.000
15	225	3375	3.873	2.466	65	4225	274625	8.062	4.021
16	256	4096	4.000	2.520	66	4356	287496	8.124	4.041
17	289	4913	4.123	2.571	67	4489	300763	8.185	4.062
18	324	5832	4.243	2.621	68	4624	314432	8.246	4.082
19	361	6859	4.359	2.668	69	4761	328509	8.307	4.102
20	400	8000	4.472	2.714	70	4900	343000	8.367	4.121
21	441	9261	4.583	2.759	**71**	5041	357911	8.426	4.141
22	484	10648	4.690	2.802	72	5184	373248	8.485	4.160
23	529	12167	4.796	2.844	73	5329	389017	8.544	4.179
24	576	13824	4.899	2.884	74	5476	405224	8.602	4.198
25	625	15625	5.000	2.924	75	5625	421875	8.660	4.217
26	676	17576	5.099	2.962	76	5776	438976	8.718	4.236
27	729	19683	5.196	3.000	77	5929	456533	8.775	4.254
28	784	21952	5.291	3.037	78	6084	474552	8.832	4 273
29	841	24389	5.385	3.072	79	6241	493039	8.888	4.291
30	900	27000	5.477	3.107	80	6400	512000	8.944	4.309
31	961	29791	5.568	3.141	**81**	6561	531441	9.000	4.327
32	1024	32768	5.657	3.175	82	6724	551368	9.055	4.344
33	1089	35937	5.745	3.208	83	6889	571787	9.110	4.362
34	1156	39304	5.831	3.240	84	7056	592704	9.165	4.380
35	1225	42875	5.916	3.271	85	7225	614125	9.220	4.397
36	1296	46656	6.000	3.302	86	7396	636056	9.274	4.414
37	1369	50653	6.083	3.332	87	7569	658503	9.327	4.431
38	1444	54872	6.164	3.362	88	7744	681472	9.381	4.448
39	1521	59319	6.245	3.391	89	7921	704969	9.434	4.465
40	1600	64000	6.325	3.420	90	8100	729000	9.487	4.481
41	1681	68921	6.403	3.448	**91**	8281	753571	9.539	4.498
42	1764	74088	6.481	3.476	92	8464	778688	9.592	4.514
43	1849	79507	6.557	3.503	93	8649	804357	9.644	4.531
44	1936	85184	6.633	3.530	94	8836	830584	9.695	4.547
45	2025	91125	6.708	3.557	95	9025	857375	9.747	4.563
46	2116	97336	6.782	3.583	96	9216	884736	9.798	4.579
47	2209	103823	6.856	3.609	97	9409	912673	9.849	4.595
48	2304	110592	6.928	3.634	98	9604	941192	9.899	4.610
49	2401	117649	7.000	3.659	99	9801	970299	9.950	4.626
50.	2500	125000	7.071	3.684	100	10000	1000000	10.000	4.642
n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	43	42	41	40	39	38	37	36	35	34	33	32
1	4.3	4.2	4.1	4.0	3.9	3.8	3.7	3.6	3.5	3.4	3.3	3.2
2	8.6	8.4	8.2	8.0	7.8	7.6	7.4	7.2	7.0	6.8	6.6	6.4
3	12.9	12.6	12.3	12.0	11.7	11.4	11.1	10.8	10.5	10.2	9.9	9.6
4	17.2	16.8	16.4	16.0	15.6	15.2	14.8	14.4	14.0	13.6	13.2	12.8
5	21.5	21.0	20.5	20.0	19.5	19.0	18.5	18.0	17.5	17.0	16.5	16.0
6	25.8	25.2	24.6	24.0	23.4	22.8	22.2	21.6	21.0	20.4	19.8	19.2
7	30.1	29.4	28.7	28.0	27.3	26.6	25.9	25.2	24.5	23.8	23.1	22.4
8	34.4	33.6	32.8	32.0	31.2	30.4	29.6	28.8	28.0	27.2	26.4	25.6
9	38.7	37.8	36.9	36.0	35.1	34.2	33.3	32.4	31.5	30.6	29.7	28.8

Prop. Parts

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Prop. Parts

	31	**30**	**29**	**28**	**27**	**26**	**25**	**24**	**23**	**22**	**21**
1	3 1	3.0	2.9	2.8	2.7	2 6	2.5	2 4	2 3	2.2	2.1
2	6.2	6.0	5.8	5 6	5.4	5 2	5.0	4.8	4.6	4 4	4.2
3	9.3	9 0	8.7	8 4	8.1	7 8	7 5	7 2	6.9	6.6	6.3
4	12.4	12.0	11.6	11 2	10 8	10 4	10 0	9 6	9.2	8 8	8 4
5	15.5	15 0	14 5	14.0	13 5	13 0	12 5	12 0	11.5	11.0	10 5
6	18.6	18 0	17.4	16.8	16.2	15 6	15 0	14 4	13 8	13.2	12 6
7	21.7	21 0	20.3	19 6	18 9	18 2	17 5	16.8	16.1	15.4	14 7
8	24.8	24.0	23 2	22 4	21 6	20 8	20 0	19 2	18.4	17 6	16 8
9	27.9	27.0	26.1	25 2	24 3	23.4	22.5	21 6	20.7	19 8	18.9

Prop. Parts

x	sin x	cos x	tan x	cot x	sec x	cosec x	
0° 0′	.00000	1.0000	.00000	∞	1.0000	∞	**90° 0′**
10′	.00291	1.0000	.00291	343.77	1.0000	343.78	50′
20′	.00582	1.0000	.00582	171.88	1.0000	171.89	40′
30′	.00873	1.0000	.00873	114.59	1.0000	114.59	30′
40′	.01164	.9999	.01164	85.940	1.0001	85.946	20′
50′	.01454	.9999	.01455	68.750	1.0001	68.757	10′
1° 0′	.01745	.9998	.01746	57.290	1.0002	57.299	**89° 0′**
10′	.02036	.9998	.02036	49.104	1.0002	49.114	50′
20′	.02327	.9997	.02328	42.964	1.0003	42.976	40′
30′	.02618	.9997	.02619	38.188	1.0003	38.202	30′
40′	.02908	.9996	.02910	34.368	1.0004	34 382	20′
50′	.03199	.9995	.03201	31.242	1.0005	31.258	10′
2° 0′	.03490	.9994	.03492	28.6363	1.0006	28.654	**88° 0′**
10′	.03781	.9993	03783	26.4316	1.0007	26.451	50′
20′	.04071	.9992	.04075	24.5418	1.0008	24.562	40′
30′	.04362	.9990	.04366	22.9038	1.0010	22.926	30′
40′	.04653	.9989	.04658	21.4704	1.0011	21.494	20′
50′	.04943	.9988	.04949	20.2056	1.0012	20.230	10′
3° 0′	.05234	.9986	.05241	19.0811	1 0014	19.107	**87° 0′**
10′	.05524	.9985	.05533	18.0750	1 0015	18.103	50′
20′	.05814	.9983	.05824	17.1693	1.0017	17.198	40′
30′	.06105	.9981	.06116	16.3499	1.0019	16 380	30′
40′	.06395	.9980	.06408	15.6048	1 0021	15 637	20′
50′	.06685	.9978	.06700	14.9244	1.0022	14.958	10′
4° 0′	.06976	.9976	.06993	14.3007	1.0024	14 336	**86° 0′**
10′	.07266	.9974	.07285	13.7267	1.0027	13.763	50′
20′	.07556	.9971	.07578	13.1969	1.0029	13.235	40′
30′	.07846	.9969	.07870	12.7062	1.0031	12.746	30′
40′	.08136	.9967	.08163	12.2505	1.0033	12.291	20′
50′	.08426	.9964	.08456	11.8262	1.0036	11.868	10′
5° 0′	.08716	.9962	.08749	11.4301	1.0038	11.474	**85° 0′**
10′	.09005	.9959	.09042	11.0594	1.0041	11.105	50′
20′	.09295	.9957	.09335	10.7119	1.0044	10.758	40′
30′	.09585	.9954	.09629	10.3854	1.0046	10.433	30′
40′	.09874	.9951	.09923	10.0780	1.0049	10.128	20′
50′	.10164	.9948	.10216	9.7882	1.0052	9.839	10′
6° 0′	.10453	.9945	.10510	9.5144	1.0055	9.5668	**84° 0′**
10′	.10742	.9942	.10805	9.2553	1.0058	9.3092	50′
20′	.11031	.9939	.11099	9.0098	1.0061	9.0652	40′
30′	.11320	.9936	.11394	8.7769	1.0065	8.8337	30′
40′	.11609	.9932	.11688	8.5555	1.0068	8.6138	20′
50′	.11898	.9929	.11983	8.3450	1.0072	8.4647	10′
7° 0′	.12187	.9925	.12278	8.1443	1.0075	8.2055	**83° 0′**
10′	.12476	.9922	.12574	7.9530	1.0079	8.0157	50′
20′	.12764	.9918	.12869	7.7704	1.0083	7.8344	40′
30′	.13053	.9914	.13165	7.5958	1.0086	7.6613	30′
	cos x	sin x	cot x	tan x	cosec x	sec x	x

x	sin x	cos x	tan x	cot x	sec x	cosec x	
30′	.1305	.9914	.1317	7.5958	1.0086	7.6613	30′
40′	.1334	.9911	.1346	7.4287	1.0090	7.4957	20′
50′	.1363	.9907	.1376	7.2687	1.0094	7.3372	10′
8° 0′	.1392	.9903	.1405	7.1154	1.0098	7.1853	**82°** 0′
10′	.1421	.9899	.1435	6.9682	1.0102	7.0396	50′
20′	.1449	.9894	.1465	6.8269	1.0107	6.8998	40′
30′	.1478	.9890	.1495	6.6912	1.0111	6.7655	30′
40′	.1507	.9886	.1524	6.5606	1.0116	6.6363	20′
50′	.1536	.9881	.1554	6.4348	1.0120	6.5121	10′
9° 0′	.1564	.9877	.1584	6.3138	1.0125	6.3925	**81°** 0′
10′	.1593	.9872	.1614	6.1970	1.0129	6.2772	50′
20′	.1622	.9868	.1644	6.0844	1.0134	6.1661	40′
30′	.1650	.9863	.1673	5.9758	1.0139	6.0589	30′
40′	.1679	.9858	.1703	5.8708	1.0144	5.9554	20′
50′	.1708	.9853	.1733	5.7694	1.0149	5.8554	10′
10° 0′	.1736	.9848	.1763	5.6713	1.0154	5.7588	**80°** 0′
10′	.1765	.9843	.1793	5.5764	1.0160	5.6653	50′
20′	.1794	.9838	.1823	5.4845	1.0165	5.5749	40′
30′	.1822	.9833	.1853	5.3955	1.0170	5.4874	30′
40′	.1851	.9827	.1883	5.3093	1.0176	5.4026	20′
50′	.1880	.9822	.1914	5 2257	1.0182	5.3205	10′
11° 0′	.1908	.9816	.1944	5.1446	1.0187	5.2408	**79°** 0′
10′	.1937	.9811	.1974	5.0658	1.0193	5.1636	50′
20′	.1965	.9805	.2004	4.9894	1.0199	5.0886	40′
30′	.1994	.9799	.2035	4.9152	1.0205	5.0159	30′
40′	.2022	.9793	.2065	4.8430	1.0211	4.9452	20′
50′	.2051	.9787	.2095	4.7729	1.0217	4.8765	10′
12° 0′	.2079	.9781	.2126	4.7046	1.0223	4.8097	**78°** 0′
10′	.2108	.9775	.2156	4.6382	1.0230	4.7448	50′
20′	.2136	.9769	.2186	4.5736	1.0236	4.6817	40′
30′	.2164	.9763	.2217	4.5107	1.0243	4.6202	30′
40′	.2193	.9757	.2247	4.4494	1.0249	4.5604	20′
50′	.2221	.9750	.2278	4.3897	1.0256	4.5022	10′
13° 0′	.2250	.9744	.2309	4.3315	1.0263	4.4454	**77°** 0′
10′	.2278	.9737	.2339	4 2747	1.0270	4.3901	50′
20′	.2306	.9730	.2370	4.2193	1.0277	4.3362	40′
30′	.2334	.9724	.2401	4.1653	1.0284	4.2837	30′
40′	.2363	.9717	.2432	4.1126	1.0291	4.2324	20′
50′	.2391	.9710	.2462	4.0611	1.0299	4.1824	10′
14° 0′	.2419	.9703	.2493	4.0108	1.0306	4.1336	**76°** 0′
10′	.2447	.9696	.2524	3.9617	1.0314	4.0859	50′
20′	.2476	.9689	.2555	3.9136	1.0321	4.0394	40′
30′	.2504	.9681	.2586	3.8667	1.0329	3.9939	30′
40′	.2532	.9674	.2617	3.8208	1.0337	3.9495	20′
50′	.2560	.9667	.2648	3.7760	1.0345	3.9061	10′
15° 0′	.2588	.9659	.2679	3.7321	1.0353	3.8637	**75°** 0′
	cos x	sin x	cot x	tan x	cosec x	sec x	x

x	sin *x*	cos *x*	tan *x*	cot *x*	sec *x*	cosec *x*	
15° 0′	.2588	.9659	.2679	3.7321	1.0353	3.8637	**75° 0′**
10′	.2616	.9652	.2711	3.6891	1.0361	3.8222	50′
20′	.2644	.9644	.2742	3.6470	1.0369	3.7817	40′
30′	.2672	.9636	.2773	3.6059	1.0377	3.7420	30′
40′	.2700	.9628	.2805	3.5656	1.0386	3.7032	20′
50′	.2728	.9621	.2836	3.5261	1.0394	3.6652	10′
16° 0′	.2756	.9613	.2867	3.4874	1.0403	3.6280	**74° 0′**
10′	.2784	.9605	.2899	3 4495	1 0412	3.5915	50′
20′	.2812	.9596	.2931	3.4124	1.0421	3.5559	40′
30′	.2840	.9588	.2962	3.3759	1.0430	3.5209	30′
40′	.2868	.9580	.2994	3.3402	1 0439	3 4867	20′
50′	.2896	.9572	.3026	3.3052	1.0448	3.4532	10′
17° 0′	.2924	.9563	.3057	3 2709	1.0457	3.4203	**73° 0′**
10′	.2952	.9555	.3089	3.2371	1.0466	3.3881	50′
20′	.2979	.9546	.3121	3.2041	1.0476	3.3565	40′
30′	.3007	.9537	.3153	3 1716	1.0485	3.3255	30′
40′	.3035	.9528	.3185	3.1397	1.0495	3.2951	20′
50′	.3062	.9520	.3217	3.1084	1.0505	3.2653	10′
18° 0′	.3090	.9511	.3249	3 0777	1 0515	3.2361	**72° 0′**
10′	.3118	.9502	.3281	3.0475	1 0525	3.2074	50′
20′	.3145	.9492	.3314	3 0178	1.0535	3.1792	40′
30′	.3173	.9483	.3346	2.9887	1.0545	3.1516	30′
40′	.3201	.9474	.3378	2.9600	1.0555	3.1244	20′
50′	.3228	.9465	.3411	2.9319	1.0566	3.0977	10′
19° 0′	.3256	.9455	.3443	2.9042	1.0576	3.0716	**71° 0′**
10′	.3283	.9446	.3476	2 8770	1.0587	3.0458	50′
20′	.3311	.9436	.3508	2.8502	1.0598	3.0206	40′
30′	.3338	.9426	.3541	2.8239	1.0609	2 9957	30′
40′	.3365	.9417	.3574	2.7980	1.0620	2.9714	20′
50′	.3393	.9407	.3607	2.7725	1.0631	2.9474	10′
20° 0′	.3420	.9397	.3640	2.7475	1.0642	2.9238	**70° 0′**
10′	.3448	.9387	.3673	2.7228	1.0653	2.9006	50′
20′	.3475	.9377	.3706	2.6985	1.0665	2.8779	40′
30′	.3502	.9367	.3739	2.6746	1.0676	2.8555	30′
40′	.3529	.9356	.3772	2.6511	1.0688	2.8334	20′
50′	.3557	.9346	.3805	2.6279	1.0700	2.8118	10′
21° 0′	.3584	.9336	.3839	2.6051	1.0712	2.7904	**69° 0′**
10′	.3611	.9325	.3872	2.5826	1.0724	2.7695	50′
20′	.3638	.9315	.3906	2.5605	1.0736	2.7488	40′
30′	.3665	.9304	.3939	2.5386	1.0748	2.7285	30′
40′	.3692	.9293	.3973	2.5172	1.0760	2.7085	20′
50′	.3719	.9283	.4006	2.4960	1.0773	2.6888	10′
22° 0′	.3746	.9272	.4040	2.4751	1.0785	2.6695	**68° 0′**
10′	.3773	.9261	.4074	2.4545	1.0798	2.6504	50′
20′	.3800	.9250	.4108	2.4342	1.0811	2.6316	40′
30′	.3827	.9239	.4142	2.4142	1.0824	2 6131	30′
	cos *x*	sin *x*	cot *x*	tan *x*	cosec *x*	sec *x*	*x*

x	sin *x*	cos *x*	tan *x*	cot *x*	sec *x*	cosec *x*	
30′	.3827	.9239	.4142	2.4142	1.0824	2.6131	30′
40′	.3854	.9228	.4176	2.3945	1.0837	2.5949	20′
50′	.3881	.9216	.4210	2.3750	1.0850	2.5770	10′
23° 0′	.3907	.9205	.4245	2.3559	1.0864	2.5593	**67° 0′**
10′	.3934	.9194	.4279	2.3369	1.0877	2.5419	50′
20′	.3961	.9182	.4314	2.3183	1.0891	2.5247	40′
30′	.3987	.9171	.4348	2.2998	1.0904	2.5078	30′
40′	.4014	.9159	.4383	2.2817	1.0918	2.4912	20′
50′	.4041	.9147	.4417	2.2637	1.0932	2.4748	10′
24° 0′	.4067	.9135	.4452	2.2460	1.0946	2.4586	**66° 0′**
10′	.4094	.9124	.4487	2.2286	1.0961	2.4426	50′
20′	.4120	.9112	.4522	2.2113	1.0975	2.4269	40′
30′	.4147	.9100	.4557	2.1943	1.0990	2.4114	30′
40′	.4173	.9088	.4592	2.1775	1.1004	2.3961	20′
50′	.4200	.9075	.4628	2.1609	1.1019	2.3811	10′
25° 0′	.4226	.9063	.4663	2.1445	1.1034	2.3662	**65° 0′**
10′	.4253	.9051	.4699	2.1283	1.1049	2.3515	50′
20′	.4279	.9038	.4734	2.1123	1.1064	2.3371	40′
30′	.4305	.9026	.4770	2.0965	1.1079	2.3228	30′
40′	.4331	.9013	.4806	2.0809	1.1095	2.3088	20′
50′	.4358	.9001	.4841	2.0655	1.1110	2.2949	10′
26° 0′	.4384	.8988	.4877	2.0503	1.1126	2.2812	**64° 0′**
10′	.4410	.8975	.4913	2.0353	1.1142	2.2677	50′
20′	.4436	.8962	.4950	2.0204	1.1158	2.2543	40′
30′	.4462	.8949	.4986	2.0057	1.1174	2.2412	30′
40′	.4488	.8936	.5022	1.9912	1.1190	2.2282	20′
50′	.4514	.8923	.5059	1.9768	1.1207	2.2154	10′
27° 0′	.4540	.8910	.5095	1.9626	1.1223	2.2027	**63° 0′**
10′	.4566	.8897	.5132	1.9486	1.1240	2.1902	50′
20′	.4592	.8884	.5169	1.9347	1.1257	2.1779	40′
30′	.4617	.8870	.5206	1.9210	1.1274	2.1657	30′
40′	.4643	.8857	.5243	1.9074	1.1291	2.1537	20′
50′	.4669	.8843	.5280	1.8940	1.1308	2.1418	10′
28° 0′	.4695	.8829	.5317	1.8807	1.1326	2.1301	**62° 0′**
10′	.4720	.8816	.5354	1.8676	1.1343	2.1185	50′
20′	.4746	.8802	.5392	1.8546	1.1361	2.1070	40′
30′	.4772	.8788	.5430	1.8418	1.1379	2.0957	30′
40′	.4797	.8774	.5467	1.8291	1.1397	2.0846	20′
50′	.4823	.8760	.5505	1.8165	1.1415	2.0736	10′
29° 0′	.4848	.8746	.5543	1.8040	1.1434	2.0627	**61° 0′**
10′	.4874	.8732	.5581	1.7917	1.1452	2.0519	50′
20′	.4899	.8718	.5619	1.7796	1.1471	2.0413	40′
30′	.4924	.8704	.5658	1.7675	1.1490	2.0308	30′
40′	.4950	.8689	.5696	1.7556	1.1509	2.0204	20′
50′	.4975	.8675	.5735	1.7437	1.1528	2.0101	10′
30° 0′	.5000	.8660	.5774	1.7321	1.1547	2.0000	**60° 0′**
	cos *x*	sin *x*	cot *x*	tan *x*	cosec *x*	sec *x*	*x*

x	sin *x*	cos *x*	tan *x*	cot *x*	sec *x*	cosec *x*	
30° 0′	.5000	.8660	.5774	1.7321	1.1547	2.0000	**60° 0′**
10′	.5025	.8646	.5812	1.7205	1.1567	1.9900	50′
20′	.5050	.8631	.5851	1.7090	1.1586	1.9801	40′
30′	.5075	.8616	.5890	1.6977	1.1606	1.9703	30′
40′	.5100	.8601	.5930	1.6864	1.1626	1.9606	20′
50′	.5125	.8587	.5969	1.6753	1.1646	1.9511	10′
31° 0′	.5150	.8572	.6009	1.6643	1.1666	1.9416	**59° 0′**
10′	.5175	.8557	.6048	1.6534	1.1687	1.9323	50′
20′	.5200	.8542	.6088	1.6426	1.1708	1.9230	40′
30′	.5225	.8526	.6128	1.6319	1.1728	1.9139	30′
40′	.5250	.8511	.6168	1.6212	1.1749	1.9049	20′
50′	.5275	.8496	.6208	1.6107	1.1770	1.8959	10′
32° 0′	.5299	.8480	.6249	1.6003	1.1792	1.8871	**58° 0′**
10′	.5324	.8465	.6289	1.5900	1.1813	1.8783	50′
20′	.5348	.8450	.6330	1.5798	1.1835	1.8699	40′
30′	.5373	.8434	.6371	1.5697	1.1857	1.8612	30′
40′	.5398	.8418	.6412	1.5597	1.1879	1.8527	20′
50′	.5422	.8403	.6453	1.5497	1.1901	1.8444	10′
33° 0′	.5446	.8387	.6494	1.5399	1.1924	1.8361	**57° 0′**
10′	.5471	.8371	.6536	1.5301	1.1946	1.8279	50′
20′	.5495	.8355	.6577	1.5204	1.1969	1.8198	40′
30′	.5519	.8339	.6619	1.5108	1.1992	1.8118	30′
40′	.5544	.8323	.6661	1.5013	1.2015	1.8039	20′
50′	.5568	.8307	.6703	1.4919	1.2039	1.7960	10′
34° 0′	.5592	.8290	.6745	1.4826	1.2062	1.7883	**56° 0′**
10′	.5616	.8274	.6787	1.4733	1.2086	1.7806	50′
20′	.5640	.8258	.6830	1.4641	1.2110	1.7730	40′
30′	.5664	.8241	.6873	1.4550	1.2134	1.7655	30′
40′	.5688	.8225	.6916	1.4460	1.2158	1.7581	20′
50′	.5712	.8208	.6959	1.4370	1.2183	1.7507	10′
35° 0′	.5736	.8192	.7002	1.4281	1.2208	1.7435	**55° 0′**
10′	.5760	.8175	.7046	1.4193	1.2233	1.7362	50′
20′	.5783	.8158	.7089	1.4106	1.2258	1.7291	40′
30′	.5807	.8141	.7133	1.4019	1.2283	1.7221	30′
40′	.5831	.8124	.7177	1.3934	1.2309	1.7151	20′
50′	.5854	.8107	.7221	1.3848	1.2335	1.7082	10′
36° 0′	.5878	.8090	.7265	1.3764	1.2361	1.7013	**54° 0′**
10′	.5901	.8073	.7310	1.3680	1.2387	1.6945	50′
20′	.5925	.8056	.7355	1.3597	1.2413	1.6878	40′
30′	.5948	.8039	.7400	1.3514	1.2440	1.6812	30′
40′	.5972	.8021	.7445	1.3432	1.2467	1.6746	20′
50′	.5995	.8004	.7490	1.3351	1.2494	1.6681	10′
37° 0′	.6018	.7986	.7536	1.3270	1.2521	1.6616	**53° 0′**
10′	.6041	.7969	.7581	1.3190	1.2549	1.6553	50′
20′	.6065	.7951	.7627	1.3111	1.2577	1.6489	40′
30′	.6088	.7934	.7673	1.3032	1.2605	1.6427	30′
	cos *x*	sin *x*	cot *x*	tan *x*	cosec *x*	sec *x*	*x*

x	sin x	cos x	tan x	cot x	sec x	cosec x	
30′	.6088	.7934	.7673	1.3032	1.2605	1.6427	30′
40′	.6111	.7916	.7720	1.2954	1.2633	1.6365	20′
50′	.6134	.7898	.7766	1.2876	1.2662	1.6304	10′
38° 0′	.6157	.7880	.7813	1.2799	1.2690	1.6243	52° 0′
10′	.6180	.7862	.7860	1.2723	1.2719	1.6183	50′
20′	.6202	.7844	.7907	1.2647	1.2748	1.6123	40′
30′	.6225	.7826	.7954	1.2572	1.2779	1.6064	30′
40′	.6248	.7808	.8002	1.2497	1.2808	1.6005	20′
50′	.6271	.7790	.8050	1.2423	1.2837	1.5948	10′
39° 0′	.6293	.7771	.8098	1.2349	1.2868	1.5890	51° 0′
10′	.6316	.7753	.8146	1.2276	1.2898	1.5833	50′
20′	.6338	.7735	.8195	1.2203	1.2929	1.5777	40′
30′	.6361	.7716	.8243	1.2131	1.2960	1.5721	30′
40′	.6383	.7698	.8292	1.2059	1.2991	1.5666	20′
50′	.6406	.7679	.8342	1.1988	1.3022	1.5611	10′
40° 0′	.6428	.7660	.8391	1.1918	1.3054	1.5557	50° 0′
10′	.6450	.7642	.8441	1.1847	1.3086	1.5504	50′
20′	.6472	.7623	.8491	1.1778	1.3118	1.5450	40′
30′	.6494	.7604	.8541	1.1708	1 3151	1.5398	30′
40′	.6517	.7585	.8591	1.1640	1.3184	1.5346	20′
50′	.6539	.7566	.8642	1.1571	1.3217	1.5294	10′
41° 0′	.6561	.7547	.8693	1.1504	1.3250	1.5243	49° 0′
10′	.6583	.7528	.8744	1.1436	1.3284	1.5192	50′
20′	.6604	.7509	.8796	1.1369	1.3318	1.5142	40′
30′	.6626	.7490	.8847	1.1303	1.3352	1.5092	30′
40′	.6648	.7470	.8899	1.1237	1.3386	1.5042	20′
50′	.6670	.7451	.8952	1.1171	1.3421	1.4993	10′
42° 0′	.6691	.7431	.9004	1.1106	1.3456	1.4945	48° 0′
10′	.6713	.7412	.9057	1.1041	1.3492	1.4897	50′
20′	.6734	.7392	.9110	1.0977	1.3527	1.4849	40′
30′	.6756	.7373	.9163	1.0913	1.3563	1.4802	30′
40′	.6777	.7353	.9217	1.0850	1.3600	1.4755	20′
50′	.6799	.7333	.9271	1.0786	1.3636	1.4709	10′
43° 0′	.6820	.7314	.9325	1.0724	1.3673	1.4663	47° 0′
10′	.6841	.7294	.9380	1.0661	1.3711	1.4617	50′
20′	.6862	.7274	.9435	1.0599	1.3748	1.4572	40′
30′	.6884	.7254	.9490	1.0538	1.3786	1.4527	30′
40′	.6905	.7234	.9545	1.0477	1.3824	1.4483	20′
50′	.6926	.7214	.9601	1.0416	1.3863	1.4439	10′
44° 0′	.6947	.7193	.9657	1.0355	1.3902	1.4396	46° 0′
10′	.6967	.7173	.9713	1.0295	1.3941	1.4352	50′
20′	.6988	.7153	.9770	1.0235	1.3980	1.4310	40′
30′	.7009	.7133	.9827	1.0176	1.4020	1.4267	30′
40′	.7030	.7112	.9884	1.0117	1.4061	1.4225	20′
50′	.7050	.7092	.9942	1.0058	1.4101	1.4184	10′
45° 0′	.7071	.7071	1.0000	1.0000	1.4142	1.4142	45° 0′
	cos x	sin x	cot x	tan x	cosec x	sec x	x

x	log sin	d	log cos	d	log tan	d	log cot	
0° 0′	−∞		10.0000	0	−∞		∞	90° 0′
10′	7.4637	3011	.0000	0	7.4637	3011	2.5363	50′
20′	.7648	1760	.0000	0	.7648	1761	.2352	40′
30′	.9408	1250	.0000	0	.9409	1249	.0591	30′
40′	8.0658	969	.0000	0	8.0658	969	1.9342	20′
50′	.1627	792	.0000	1	.1627	792	.8373	10′
1° 0′	8.2419	669	9.9999	0	8.2419	670	1.7581	89° 0′
10′	.3088	580	.9999	0	.3089	580	.6911	50′
20′	.3668	511	.9999	0	.3669	512	.6331	40′
30′	.4179	458	9.9999	1	.4181	457	.5819	30′
40′	.4637	413	.9998	0	.4638	415	.5362	20′
50′	.5050	378	.9998	1	.5053	378	.4947	10′
2° 0′	8.5428	348	9.9997	0	8.5431	348	1.4569	88° 0′
10′	.5776	321	.9997	1	.5779	322	.4221	50′
20′	.6097	300	.9996	0	.6101	300	.3899	40′
30′	.6397	280	.9996	1	.6401	281	.3599	30′
40′	.6677	263	.9995	0	.6682	263	.3318	20′
50′	.6940	248	.9995	1	.6945	249	.3055	10′
3° 0′	8.7188	235	9.9994	1	8.7194	235	1.2806	87° 0′
10′	.7423	222	.9993	0	.7429	223	.2571	50′
20′	.7645	212	.9993	1	.7652	213	.2348	40′
30′	.7857	202	.9992	1	.7865	202	.2135	30′
40′	.8059	192	.9991	1	.8067	194	.1933	20′
50′	.8251	185	.9990	1	.8261	185	.1739	10′
4° 0′	8.8436	177	9.9989	0	8.8446	178	1.1554	86° 0′
10′	.8613	170	.9989	1	.8624	171	.1376	50′
20′	.8783	163	.9988	1	.8795	165	.1205	40′
30′	.8946	158	.9987	1	.8960	158	.1040	30′
40′	.9104	152	.9986	1	.9118	154	.0882	20′
50′	.9256	147	.9985	2	.9272	148	.0728	10′
5° 0′	8.9403	142	9.9983	1	8.9420	143	1.0580	85° 0′
10′	.9545	137	.9982	1	.9563	138	.0437	50′
20′	.9682	134	.9981	1	.9701	135	.0299	40′
30′	.9816	129	.9980	1	.9836	130	.0164	30′
40′	.9945	125	.9979	2	.9966	127	.0034	20′
50′	9.0070	122	.9977	1	9.0093	123	0.9907	10′
6° 0′	9.0192	119	9.9976	1	9.0216	120	0.9784	84° 0′
10′	.0311	115	.9975	2	.0336	117	.9664	50′
20′	.0426	113	.9973	1	.0453	114	.9547	40′
30′	.0539	109	.9972	1	.0567	111	.9433	30′
40′	.0648	107	.9971	2	.0678	108	.9322	20′
50′	.0755	104	.9969	1	.0786	105	.9214	10′
7° 0′	9.0859	102	9.9968	2	9.0891	104	0.9109	83° 0′
10′	.0961	99	.9966	2	.0995	101	.9005	50′
20′	.1060	97	.9964	1	.1096	98	.8904	40′
30′	.1157		.9963		.1194		.8806	30′
	log cos	d	log sin	d	log cot	d	log tan	x

Small Angles

x	S	T
<1°	6.4637	6.4637
1°	6.4637	6.4638
2°	6.4636	6.4639
3°	6.4635	6.4641
4°	6.4634	6.4644
5°	6.4631	6.4649

Prop. Parts.

	113	111	109
1	11 3	11 1	10.9
2	22 6	22 2	21 8
3	33.9	33 3	32.7
4	45 2	44 4	43.6
5	56 5	55 5	54.5
6	67 8	66 6	65 4
7	79 1	77 7	76 3
8	90 4	88 8	87.2
9	101.7	99.9	98.1

	108	107	105
1	10 8	10 7	10 5
2	21 6	21 4	21 0
3	32 4	32 1	31 5
4	43 2	42 8	42 0
5	54.0	53 5	52.5
6	64 8	64 2	63 0
7	75 6	74 9	73 5
8	86 4	85 6	84.0
9	97.2	96 3	94.5

	104	102	101
1	10.4	10 2	10 1
2	20 8	20 4	20 2
3	31 2	30 6	30 3
4	41 6	40 8	40 4
5	52 0	51.0	50.5
6	62 4	61 2	60 6
7	72 8	71.4	70 7
8	83 2	81.6	80 8
9	93 6	91.8	90 9

	99	98	97	95
1	9 9	9 8	9 7	9 5
2	19 8	19 6	19 4	19 0
3	29 7	29 4	29 1	28 5
4	39 6	39 2	38 8	38 0
5	49 5	49 0	48.5	47 5
6	59 4	58 8	58 2	57 0
7	69 3	68 6	67 9	66 5
8	79 2	78 4	77.6	76.0
9	89.1	88 2	87.3	85.5

	143	142	138	137	135	134	130	129	127	125	123	122	119	117	115	114
1	14 3	14 2	13 8	13 7	13 5	13 4	13 0	12 9	12 7	12 5	12 3	12 2	11 9	11.7	11 5	11 4
2	28 6	28 4	27 6	27 4	27 0	26.8	26 0	25 8	25.4	25.0	24 6	24 4	23 8	23 4	23 0	22 8
3	42 9	42 6	41.4	41.1	40 5	40 2	39 0	38 7	38 1	37 5	36 9	36 6	35 7	35.1	34.5	34 2
4	57.2	56 8	55 2	54 8	54 0	53.6	52 0	51.6	50 8	50 0	49 2	48.8	47.6	46.8	46 0	45 6
5	71.5	71 0	69 0	68.5	67 5	67 0	65.0	64 5	63 5	62 5	61.5	61.0	59 5	58 5	57.5	57 0
6	85 8	85 2	82 8	82 2	81 0	80 4	78 0	77 4	76 2	75 0	73 8	73 2	71 4	70 2	69 0	68 4
7	100.1	99 4	96 6	95 9	94 5	93 8	91 0	90 3	88 9	87 5	86.1	85 4	83 3	81.9	80 5	79.8
8	114.4	113 6	110 4	109 6	108 0	107.2	104.0	103 2	101 6	100 0	98 4	97 6	95 2	93 6	92 0	91 2
9	128 7	127 8	124 2	123 3	121 5	120 6	117.0	116 1	114 3	112 5	110 7	109 8	107.1	105 3	103 5	102 6

x	log sin	d	log cos	d	log tan	d	log cot	
30′	9.1157	95	9.9963	2	9.1194	97	0.8806	30′
40′	.1252	93	.9961	2	.1291	94	.8709	20′
50′	.1345	91	.9959	1	.1385	93	.8615	10′
8° 0′	9.1436	89	9.9958	2	9.1478	91	0.8522	**82° 0′**
10′	.1525	87	.9956	2	.1569	89	.8431	50′
20′	.1612	85	.9954	2	.1658	87	.8342	40′
30′	.1697	84	.9952	2	.1745	86	.8255	30′
40′	.1781	82	.9950	2	.1831	84	.8169	20′
50′	.1863	80	.9948	2	.1915	82	.8085	10′
9° 0′	9.1943	79	9.9946	2	9.1997	81	0.8003	**81° 0′**
10′	.2022	78	.9944	2	.2078	80	.7922	50′
20′	.2100	76	.9942	2	.2158	78	.7842	40′
30′	.2176	75	.9940	2	.2236	77	.7764	30′
40′	.2251	73	.9938	2	.2313	76	.7687	20′
50′	.2324	73	.9936	2	.2389	74	.7611	10′
10° 0′	9.2397	71	9.9934	3	9.2463	73	0.7537	**80° 0′**
10′	.2468	70	.9931	2	.2536	73	.7464	50′
20′	.2538	68	.9929	2	.2609	71	.7391	40′
30′	.2606	68	.9927	3	.2680	70	.7320	30′
40′	.2674	66	.9924	2	.2750	69	.7250	20′
50′	.2740	66	.9922	3	.2819	68	.7181	10′
11° 0′	9.2806	64	9 9919	2	9.2887	66	0.7113	**79° 0′**
10′	2870	64	.9917	3	.2953	67	.7047	50′
20′	.2934	63	.9914	2	.3020	65	.6980	40′
30′	.2997	61	.9912	3	.3085	64	.6915	30′
40′	.3058	61	.9909	2	.3149	63	.6851	20′
50′	.3119	60	.9907	3	.3212	63	.6788	10′
12° 0′	9.3179	59	9.9904	3	9.3275	61	0.6725	**78° 0′**
10′	.3238	58	.9901	2	.3336	61	.6664	50′
20′	.3296	57	9899	3	.3397	61	.6603	40′
30′	.3353	57	.9896	3	.3458	59	.6542	30′
40′	.3410	56	.9893	3	.3517	59	.6483	20′
50′	.3466	55	.9890	3	.3576	58	.6424	10′
13° 0′	9.3521	54	9.9887	3	9.3634	57	0.6366	**77° 0′**
10′	.3575	54	.9884	3	.3691	57	.6309	50′
20′	.3629	53	.9881	3	.3748	56	.6252	40′
30′	.3682	52	.9878	3	.3804	55	.6196	30′
40′	.3734	52	.9875	3	.3859	55	.6141	20′
50′	.3786	51	.9872	3	.3914	54	.6086	10′
14° 0′	9.3837	50	9.9869	3	9.3968	53	0.6032	**76° 0′**
10′	.3887	50	.9866	3	.4021	53	.5979	50′
20′	.3937	49	.9863	4	.4074	53	.5926	40′
30′	.3986	49	.9859	3	.4127	51	.5873	30′
40′	.4035	48	.9856	3	.4178	52	.5822	20′
50′	.4083	47	.9853	4	.4230	51	.5770	10′
15° 0′	9.4130		9.9849		9.4281		0.5719	**75° 0′**
	log cos	d	log sin	d	log cot	d	log tan	x

Prop. Parts

	73	71	70	69	68
1	7.3	7.1	7.0	6 9	6 8
2	14.6	14.2	14.0	13 8	13 6
3	21 9	21.3	21.0	20.7	20 4
4	29 2	28 4	28.0	27 6	27 2
5	36 5	35 5	35 0	34.5	34 0
6	43 8	42.6	42 0	41 4	40 8
7	51 1	49.7	49.0	48 3	47 6
8	58 4	56 8	56 0	55 2	54 4
9	65.7	63.9	63.0	62 1	61 2

	67	66	65	64	63
1	6 7	6.6	6 5	6 4	6 3
2	13 4	13 2	13 0	12 8	12 6
3	20 1	19 8	19 5	19 2	18 9
4	26 8	26,4	26 0	25 6	25 2
5	33 5	33 0	32 5	32 0	31 5
6	40 2	39 6	39 0	38 4	37 8
7	46 9	46 2	45.5	44.8	44 1
8	53.6	52 8	52 0	51.2	50 4
9	60 3	59.4	58.5	57.6	56 7

	61	60	59	58	57
1	6 1	6.0	5 9	5 8	5 7
2	12 2	12 0	11 8	11 6	11 4
3	18 3	18 0	17.7	17 4	17 1
4	24 4	24 0	23 6	23 2	22 8
5	30 5	30 0	29 5	29 0	28 5
6	36 6	36.0	35 4	34 8	34 2
7	42 7	42 0	41 3	40 6	39 9
8	48 8	48 0	47 2	46.4	45 6
9	54 9	54.0	53 1	52.2	51 3

	56	55	54	53	52
1	5 6	5 5	5.4	5 3	5 2
2	11 2	11 0	10.8	10 6	10 4
3	16 8	16.5	16 2	15 9	15 6
4	22 4	22 0	21 6	21 2	20 8
5	28 0	27.5	27 0	26 5	26 0
6	33 6	33 0	32 4	31 8	31 2
7	39 2	38.5	37.8	37 1	36 4
8	44 8	44 0	43 2	42 4	41 6
9	50 4	49 5	48.6	47.7	46 8

	51	50	48	47
1	5 1	5 0	4 8	4 7
2	10 2	10.0	9.6	9 4
3	15 3	15 0	14 4	14 1
4	20 4	20 0	19.2	18 8
5	25 5	25 0	24 0	23.5
6	30 6	30.0	28 8	28 2
7	35 7	35 0	33.6	32 9
8	40 8	40 0	38 4	37 6
9	45.9	45.5	43.2	42.3

	97	94	93	91	89	87	86	85	84	82	81	79	78	77	76	75	74
1	9.7	9.4	9 3	9 1	8 9	8 7	8 6	8 5	8.4	8 2	8 1	7.9	7 8	7.7	7.6	7.5	7 4
2	19.4	18 8	18 6	18.2	17.8	17 4	17 2	17 0	16 8	16.4	16.2	15 8	15 6	15.4	15 2	15.0	14.8
3	29.1	28 2	27 9	27.3	26.7	26.1	25 8	25 5	25 2	24 6	24 3	23.7	23.4	23 1	22 8	22 5	22 2
4	38 8	37.6	37.2	36.4	35 6	34 8	34.4	34 0	33 6	32.8	32 4	31 6	31.2	30 8	30 4	30 0	29 6
5	48.5	47 0	46.5	45.5	44 5	43.5	43.0	42.5	42.0	41.0	40 5	39.5	39.0	38.5	38.0	37.5	37 0
6	58.2	56.4	55.8	54.6	53.4	52.2	51.6	51 0	50.4	49 2	48 6	47.4	46.8	46.2	45.6	45 0	44.4
7	67 9	65.8	65 1	63.7	62.3	60 9	60.2	59 5	58.8	57.4	56.7	55.3	54.6	53.9	53.2	52 5	51.8
8	77.6	75 2	74.4	72.8	71 2	69.6	68 8	68 0	67 2	65 6	64 8	63.2	62.4	61.6	60.8	60 0	59.2
9	87.3	84 6	83 7	81 9	80 1	78.3	77.4	76.5	75.6	73 8	72 9	71.1	70.2	69.3	68.4	67 5	66 6

x	log sin	d	log cos	d	log tan	d	log cot	
15° 0′	9.4130	47	9.9849	3	9.4281	50	0.5719	**75° 0′**
10′	.4177	46	.9846	3	.4331	50	.5669	50′
20′	.4223	46	.9843	4	.4381	49	.5619	40′
30′	.4269	45	.9839	3	.4430	49	.5570	30′
40′	.4314	45	.9836	4	.4479	48	.5521	20′
50′	.4359	44	.9832	4	.4527	48	.5473	10′
16° 0′	9.4403	44	9.9828	3	9.4575	47	0.5425	**74° 0′**
10′	.4447	44	.9825	4	.4622	47	.5378	50′
20′	.4491	42	.9821	4	.4669	47	.5331	40′
30′	.4533	43	.9817	3	.4716	46	.5284	30′
40′	.4576	42	.9814	4	.4762	46	.5238	20′
50′	.4618	41	.9810	4	.4808	45	.5192	10′
17° 0′	9.4659	41	9.9806	4	9.4853	45	0.5147	**73° 0′**
10′	.4700	41	.9802	4	.4898	45	.5102	50′
20′	.4741	40	.9798	4	.4943	44	5057	40′
30′	.4781	40	.9794	4	.4987	44	.5013	30′
40′	.4821	40	.9790	4	5031	44	.4969	20′
50′	.4861	39	.9786	4	5075	43	.4925	10′
18° 0′	9.4900	39	9.9782	4	9.5118	43	0.4882	**72° 0′**
10′	.4939	38	.9778	4	.5161	42	.4839	50′
20′	.4977	38	.9774	4	.5203	42	.4797	40′
30′	.5015	37	.9770	5	.5245	42	.4755	30′
40′	.5052	38	.9765	4	.5287	42	.4713	20′
50′	.5090	36	.9761	4	.5329	41	.4671	10′
19° 0′	9.5126	37	9.9757	5	9.5370	41	0.4630	**71° 0′**
10′	.5163	36	.9752	4	.5411	40	.4589	50′
20′	.5199	36	.9748	5	.5451	40	.4549	40′
30′	.5235	35	.9743	4	.5491	40	.4509	30′
40′	.5270	36	.9739	5	.5531	40	.4469	20′
50′	.5306	35	.9734	4	.5571	40	.4429	10′
20° 0′	9.5341	34	9.9730	5	9 5611	39	0.4389	**70° 0′**
10′	.5375	34	.9725	4	.5650	39	.4350	50′
20′	.5409	34	9721	5	.5689	38	4311	40′
30′	.5443	34	9716	5	.5727	39	.4273	30′
40′	.5477	33	.9711	5	.5766	38	.4234	20′
50′	.5510	33	.9706	4	5804	38	4196	10′
21° 0′	9.5543	33	9.9702	5	9.5842	37	0 4158	**69° 0′**
10′	.5576	33	.9697	5	.5879	38	.4121	50′
20′	.5609	32	9692	5	.5917	37	.4083	40′
30′	.5641	32	.9687	5	.5954	37	.4046	30′
40′	.5673	31	.9682	5	.5991	37	.4009	20′
50′	.5704	32	.9677	5	.6028	36	.3972	10′
22° 0′	9.5736	31	9.9672	5	9.6064	36	0.3936	**68° 0′**
10′	.5767	31	.9667	6	.6100	36	.3900	50′
20′	.5798	30	.9661	5	.6136	36	.3864	40′
30′	.5828		.9656		6172		.3828	30′
	log cos	d	log sin	d	log cot	d	log tan	x

Prop. Parts

	50	49	48	47
1	5 0	4 9	4 8	4 7
2	10 0	9 8	9 6	9 4
3	15 0	14 7	14 4	14 1
4	20 0	19 6	19 2	18 8
5	25 0	24 5	24 0	23 5
6	30 0	29 4	28 8	28 2
7	35 0	34 3	33 6	32 9
8	40 0	39 2	38 4	37 6
9	45 0	44 1	43 2	42 3

	46	45	44	43
1	4 6	4 5	4 4	4 3
2	9 2	9 0	8 8	8 6
3	13 8	13 5	13 2	12 9
4	18 4	18 0	17 6	17 2
5	23 0	22 5	22 0	21 5
6	27 6	27 0	26 4	25 8
7	32 2	31 5	30 8	30 1
8	36 8	36 0	35 2	34 4
9	41 4	40 5	39 6	38.7

	42	41	40	39
1	4 2	4 1	4 0	3 9
2	8 4	8 2	8 0	7 8
3	12 6	12 3	12 0	11 7
4	16 8	16 4	16 0	15 6
5	21 0	20 5	20 0	19 5
6	25 2	24 6	24 0	23 4
7	29 4	28 7	28 0	27 3
8	33 6	32 8	32 0	31 2
9	37 8	36 9	36 0	35 1

	38	37	36	35
1	3 8	3 7	3 6	3 5
2	7 6	7 4	7 2	7 0
3	11 4	11 1	10 8	10 5
4	15 2	14 8	14 4	14 0
5	19 0	18 5	18 0	17 5
6	22 8	22 2	21 6	21 0
7	26 6	25 9	25 2	24 5
8	30 4	29 6	28 8	28 0
9	34 2	33 3	32 4	31 5

	34	33	32	31
1	3 4	3 3	3 2	3 1
2	6 8	6 6	6 4	6 2
3	10 2	9 9	9 6	9 3
4	13 6	13 2	12 8	12 4
5	17 0	16 5	16 0	15 5
6	20 4	19 8	19 2	18 6
7	23 8	23 1	22.4	21.7
8	27 2	26 4	25 6	24 8
9	30 6	29 7	28.8	27.9

Prop. Parts

x	log sin	d	log cos	d	log tan	d	log cot	
30′	9.5828	31	9.9656	5	9.6172	36	0.3828	30′
40′	.5859	30	.9651	5	.6208	35	.3792	20′
50′	.5889	30	.9646	6	.6243	36	.3757	10′
23° 0′	9.5919	29	9.9640	5	9.6279	35	0.3721	**67° 0′**
10′	.5948	30	.9635	6	.6314	34	.3686	50′
20′	.5978	29	.9629	5	.6348	35	.3652	40′
30′	.6007	29	.9624	6	.6383	34	.3617	30′
40′	.6036	29	.9618	5	.6417	35	.3583	20′
50′	.6065	28	.9613	6	.6452	34	.3548	10′
24° 0′	9.6093	28	9.9607	5	9.6486	34	0.3514	**66° 0′**
10′	.6121	28	.9602	6	.6520	33	.3480	50′
20′	.6149	28	.9596	6	.6553	34	.3447	40′
30′	.6177	28	.9590	6	.6587	33	.3413	30′
40′	.6205	27	.9584	5	.6620	34	.3380	20′
50′	.6232	27	.9579	6	.6654	33	.3346	10′
25° 0′	9.6259	27	9.9573	6	9.6687	33	0.3313	**65° 0′**
10′	.6286	27	.9567	6	.6720	32	.3280	50′
20′	.6313	27	.9561	6	.6752	33	.3248	40′
30′	.6340	26	.9555	6	.6785	32	.3215	30′
40′	.6366	26	.9549	6	.6817	33	.3183	20′
50′	.6392	26	.9543	6	.6850	32	.3150	10′
26° 0′	9.6418	26	9.9537	7	9.6882	32	0.3118	**64° 0′**
10′	.6444	26	.9530	6	.6914	32	.3086	50′
20′	.6470	25	.9524	6	.6946	31	.3054	40′
30′	.6495	26	.9518	6	.6977	32	.3023	30′
40′	.6521	25	.9512	7	.7009	31	.2991	20′
50′	.6546	24	.9505	6	.7040	32	.2960	10′
27° 0′	9.6570	25	9.9499	7	9.7072	31	0.2928	**63° 0′**
10′	.6595	25	.9492	6	.7103	31	.2897	50′
20′	.6620	24	.9486	7	.7134	31	.2866	40′
30′	.6644	24	.9479	6	.7165	31	.2835	30′
40′	.6668	24	.9473	7	.7196	30	.2804	20′
50′	.6692	24	.9466	7	.7226	31	.2774	10′
28° 0′	9.6716	24	9.9459	6	9.7257	30	0.2743	**62° 0′**
10′	.6740	23	.9453	7	.7287	30	.2713	50′
20′	.6763	24	.9446	7	.7317	31	.2683	40′
30′	.6787	23	.9439	7	.7348	30	.2652	30′
40′	.6810	23	.9432	7	.7378	30	.2622	20′
50′	.6833	23	.9425	7	.7408	30	.2592	10′
29° 0′	9.6856	22	9.9418	7	9.7438	29	0.2562	**61° 0′**
10′	.6878	23	.9411	7	.7467	30	.2533	50′
20′	.6901	22	.9404	7	.7497	29	.2503	40′
30′	.6923	23	.9397	7	.7526	30	.2474	30′
40′	.6946	22	.9390	7	.7556	29	.2444	20′
50′	.6968	22	.9383	8	.7585	29	.2415	10′
30° 0′	9.6990		9.9375		9.7614		0.2386	**60° 0′**
	log cos	d	log sin	d	log cot	d	log tan	x

Prop. Parts

	36	35	34	33	32	31	30	29	28	27	26	25	24	23	22
1	3.6	3.5	3.4	3.3	3.2	3.1	3.0	2.9	2.8	2.7	2.6	2.5	2.4	2.3	2.2
2	7.2	7.0	6.8	6.6	6.4	6.2	6.0	5.8	5.6	5.4	5.2	5.0	4.8	4.6	4.4
3	10.8	10.5	10.2	9.9	9.6	9.3	9.0	8.7	8.4	8.1	7.8	7.5	7.2	6.9	6.6
4	14.4	14.0	13.6	13.2	12.8	12.4	12.0	11.6	11.2	10.8	10.4	10.0	9.6	9.2	8.8
5	18.0	17.5	17.0	16.5	16.0	15.5	15.0	14.5	14.0	13.5	13.0	12.5	12.0	11.5	11.0
6	21.6	21.0	20.4	19.8	19.2	18.6	18.0	17.4	16.8	16.2	15.6	15.0	14.4	13.8	13.2
7	25.2	24.5	23.8	23.1	22.4	21.7	21.0	20.3	19.6	18.9	18.2	17.5	16.8	16.1	15.4
8	28.8	28.0	27.2	26.4	25.6	24.8	24.0	23.2	22.4	21.6	20.8	20.0	19.2	18.4	17.6
9	32.4	31.5	30.6	29.7	28.8	27.9	27.0	26.1	25.2	24.3	23.4	22.5	21.6	20.7	19.8

Prop. Parts

x	log sin	d	log cos	d	log tan	d	log cot	
30° 0′	9.6990	22	9.9375	7	9.7614	30	0.2386	60° 0′
10′	.7012	21	.9368	7	.7644	29	.2356	50′
20′	.7033	22	.9361	8	.7673	28	.2327	40′
30′	.7055	21	.9353	7	.7701	29	.2299	30′
40′	.7076	21	.9346	8	.7730	29	.2270	20′
50′	.7097	21	.9338	7	.7759	29	.2241	10′
31° 0′	9.7118	21	9.9331	8	9.7788	28	0 2212	59° 0′
10′	.7139	21	.9323	8	.7816	29	2184	50′
20′	.7160	21	.9315	7	.7845	28	2155	40′
30′	.7181	20	.9308	8	.7873	29	2127	30′
40′	.7201	21	.9300	8	.7902	28	.2098	20′
50′	.7222	20	.9292	8	.7930	28	.2070	10′
32° 0′	9.7242	20	9 9284	8	9.7958	28	0 2042	58° 0′
10′	.7262	20	.9276	8	.7986	28	.2014	50′
20′	.7282	20	.9268	8	.8014	28	.1986	40′
30′	.7302	20	.9260	8	8042	28	.1958	30′
40′	.7322	20	.9252	8	.8070	27	.1930	20′
50′	.7342	19	9244	8	.8097	28	.1903	10′
33° 0′	9 7361	19	9 9236	8	9.8125	28	0.1875	57° 0′
10′	.7380	20	.9228	9	.8153	27	.1847	50′
20′	.7400	19	.9219	8	.8180	28	.1820	40′
30′	.7419	19	.9211	8	.8208	27	.1792	30′
40′	.7438	19	.9203	9	.8235	28	.1765	20′
50′	.7457	19	.9194	8	.8263	27	1737	10′
34° 0′	9.7476	18	9 9186	9	9.8290	27	0 1710	56° 0′
10′	.7494	19	.9177	8	.8317	27	.1683	50′
20′	.7513	18	.9169	9	.8344	27	.1656	40′
30′	.7531	19	9160	9	.8371	27	.1629	30′
40′	.7550	18	.9151	9	.8398	27	.1602	20′
50′	.7568	18	.9142	8	.8425	27	.1575	10′
35° 0′	9.7586	18	9 9134	9	9.8452	27	0.1548	55° 0′
10′	.7604	18	9125	9	.8479	27	.1521	50′
20′	.7622	18	.9116	9	.8506	27	.1494	40′
30′	.7640	17	.9107	9	8533	26	.1467	30′
40′	.7657	18	9098	9	.8559	27	.1441	20′
50′	.7675	17	.9089	9	8586	27	.1414	10′
36° 0′	9.7692	18	9 9080	10	9.8613	26	0.1387	54° 0′
10′	.7710	17	.9070	9	8639	27	.1361	50′
20′	.7727	17	.9061	9	.8666	26	.1334	40′
30′	.7744	17	.9052	10	.8692	26	.1308	30′
40′	.7761	17	.9042	9	.8718	27	.1282	20′
50′	.7778	17	.9033	10	.8745	26	.1255	10′
37° 0′	9.7795	16	9 9023	9	9.8771	26	0 1229	53° 0′
10′	.7811	17	.9014	10	8797	27	.1203	50′
20′	.7828	16	.9004	9	8824	26	.1176	40′
30′	.7844		.8995		.8850		.1150	30′
	log cos	d	log sin	d	log cot	d	log tan	x

Prop. Parts

	30	29	28
1	3 0	2 9	2 8
2	6 0	5 8	5.6
3	9 0	8 7	8.4
4	12 0	11 6	11.2
5	15 0	14 5	14 0
6	18 0	17 4	16 8
7	21 0	20 3	19 6
8	24 0	23 2	22 4
9	27 0	26 1	25 2

	27	26	22
1	2 7	2 6	2 2
2	5 4	5 2	4 4
3	8 1	7 8	6 6
4	10 8	10 4	8 8
5	13 5	13 0	11 0
6	16 2	15 6	13 2
7	18 9	18 2	15 4
8	21 6	20 8	17 6
9	24 3	23 4	19 8

	21	20	19
1	2 1	2 0	1 9
2	4 2	4 0	3 8
3	6 3	6 0	5 7
4	8 4	8 0	7 6
5	10 5	10 0	9 5
6	12 6	12 0	11 4
7	14 7	14 0	13 3
8	16 8	16 0	15 2
9	18 9	18 0	17 1

	18	17	16
1	1 8	1 7	1 6
2	3 6	3 4	3 2
3	5 4	5 1	4 8
4	7 2	6 8	6 4
5	9 0	8 5	8 0
6	10 8	10 2	9 6
7	12 6	11 9	11 2
8	14 4	13 6	12 8
9	16 2	15 3	14 4

	9	8	7
1	9	8	7
2	1 8	1 6	1 4
3	2 7	2 4	2 1
4	3 6	3 2	2 8
5	4 5	4 0	3 5
6	5 4	4 8	4 2
7	6 3	5 6	4 9
8	7 2	6 4	5 6
9	8 1	7.2	6 3

Prop. Parts

x	log sin	d	log cos	d	log tan	d	log cot	
30′	9.7844	17	9.8995	10	9.8850	26	0.1150	30′
40′	.7861	16	.8985	10	.8876	26	.1124	20′
50′	.7877	16	.8975	10	.8902	26	.1098	10′
38° 0′	9.7893	17	9.8965	10	9.8928	26	0.1072	**52° 0′**
10′	.7910	16	.8955	10	.8954	26	.1046	50′
20′	.7926	15	.8945	10	.8980	26	.1020	40′
30′	.7941	16	.8935	10	.9006	26	.0994	30′
40′	.7957	16	.8925	10	.9032	26	.0968	20′
50′	.7973	16	.8915	10	.9058	26	.0942	10′
39° 0′	9.7989	15	9.8905	10	9.9084	26	0.0916	**51° 0′**
10′	.8004	16	.8895	11	.9110	25	.0890	50′
20′	.8020	15	.8884	10	.9135	26	.0865	40′
30′	.8035	15	.8874	10	.9161	26	.0839	30′
40′	.8050	16	.8864	11	.9187	25	.0813	20′
50′	.8066	15	.8853	10	.9212	26	.0788	10′
40° 0′	9.8081	15	9.8843	11	9.9238	26	0.0762	**50° 0′**
10′	.8096	15	.8832	11	.9264	25	.0736	50′
20′	.8111	14	.8821	11	.9289	26	.0711	40′
30′	.8125	15	.8810	10	.9315	26	.0685	30′
40′	.8140	15	.8800	11	.9341	25	.0659	20′
50′	.8155	14	.8789	11	.9366	26	.0634	10′
41° 0′	9.8169	15	9.8778	11	9.9392	25	0.0608	**49° 0′**
10′	.8184	14	.8767	11	.9417	26	.0583	50′
20′	.8198	15	.8756	11	.9443	25	.0557	40′
30′	.8213	14	.8745	12	.9468	26	.0532	30′
40′	.8227	14	.8733	11	.9494	25	.0506	20′
50′	.8241	14	.8722	11	.9519	25	.0481	10′
42° 0′	9.8255	14	9.8711	12	9.9544	26	0.0456	**48° 0′**
10′	.8269	14	.8699	11	.9570	25	.0430	50′
20′	.8283	14	.8688	12	.9595	26	.0405	40′
30′	.8297	14	.8676	11	.9621	25	.0379	30′
40′	.8311	13	.8665	12	.9646	25	.0354	20′
50′	.8324	14	.8653	12	.9671	26	.0329	10′
43° 0′	9.8338	13	9.8641	12	9.9697	25	0.0303	**47° 0′**
10′	.8351	14	.8629	11	.9722	25	.0278	50′
20′	.8365	13	.8618	12	.9747	25	.0253	40′
30′	.8378	13	.8606	12	.9772	26	.0228	30′
40′	.8391	14	.8594	12	.9798	25	.0202	20′
50′	.8405	13	.8582	13	.9823	25	.0177	10′
44° 0′	9.8418	13	9.8569	12	9.9848	26	0.0152	**46° 0′**
10′	.8431	13	.8557	12	.9874	25	.0126	50′
20′	.8444	13	.8545	13	.9899	25	.0101	40′
30′	.8457	12	.8532	12	.9924	25	.0076	30′
40′	.8469	13	.8520	13	.9949	26	.0051	20′
50′	.8482	13	.8507	12	.9975	25	.0025	10′
45° 0′	9 8495		9 8495		0.0000		0.0000	**45° 0′**
	log cos	d	log sin	d	log cot	d	log tan	*x*

Prop. Parts

	26	**25**
1	2 6	2 5
2	5.2	5 0
3	7.8	7.5
4	10 4	10 0
5	13 0	12 5
6	15 6	15.0
7	18 2	17.5
8	20 8	20.0
9	23.4	22.5

	17	**16**	**15**
1	1 7	1.6	1 5
2	3.4	3 2	3 0
3	5.1	4.8	4 5
4	6.8	6 4	6 0
5	8 5	8.0	7.5
6	10.2	9.6	9 0
7	11 9	11.2	10.5
8	13 6	12.8	12 0
9	15.3	14 4	13.5

	14	**13**	**12**
1	1 4	1.3	1 2
2	2 8	2 6	2 4
3	4 2	3.9	3 6
4	5 6	5 2	4 8
5	7 0	6 5	6 0
6	8 4	7.8	7 2
7	9 8	9 1	8 4
8	11 2	10 4	9 6
9	12.6	11.7	10 8

	11	**10**
1	1 1	1.0
2	2 2	2 0
3	3 3	3 0
4	4 4	4 0
5	5 5	5 0
6	6 6	6 0
7	7 7	7.0
8	8 8	8 0
9	9.9	9.0

Prop. Parts

INDEX

(Numbers refer to pages.)

Abscissa, 50
Addition formulas, cosine, 75
  sine, 77
  tan, 79
Addition of vectors, 32
Altitude of heavenly body, 176
Ambiguous case, 91
Amplitude, of periodic function, 67
  of complex number, 119
Apothem, 5
Arc, 4
Area, 42, 43, 102, 137, 138, 140, 142, 145, 150
Argument, 119
Azimuth, 176

Bearing, 172

Celestial sphere, 175
Characteristic, 187
Chord, 4
Circle, definition, 4
  circumscribed, 5
  inscribed, 5
  great, 146
  small, 146
Circumference, 4
Co-functions, 17
Complementary angles, trigonometric functions of, 17
Complex numbers, definition, 118
  addition and subtraction of, 120
  multiplication and division of, 121, 122, 124
Components, 32
Composite functions, 70
Conditional equation, 111
Coordinates, 50
Cosine, 12, 52
Cosines, Law of, plane, 94
  spherical, 157
Co-terminal angles, 52
Coversed sine, 13
Cyclical permutations, 104
Cylinders, 136

Decagon, 4
Declination, 176
Degree, plane, 2, 37
  spherical, 150

De Moivre's Theorem, 123
Depression, angle of, 28
Diameter, 4
Dihedral angle, 131
  plane angle of, 131
Distance, between two points, 50
  from point to plane, 130
Double angle formulas, 81

Elevation, angle of, 28
Equations, trigonometric, 111
  solution of, 112
Equator, celestial, 175
Equilateral triangle, 3
Excess, spherical, 150

Frustum, 141
Function, of a variable, 11
  trigonometric, 12

Graphs, of sine and cosine, 66
  of tangent, 70
  of composite functions, 70

Half angle formulas, 82
Half line, 2
Heptagon, 4
Hexagon, 4
Horizon, 175
Hour angle, 175
Hour circle, 175
Hypotenuse, 3

Identities, fundamental trigonometric, 19
  in general, 111
Initial side, 36
Inscribed circle, 5
Interpolation, 20
Inverse functions, 107
  trigonometric, 107
Isosceles triangle, 3

Knot, 171

Latitude, 171
Law, of angles, 158
  of cosines, 94, 157
  of sines, 88, 155
  of tangents, 98
Leg, 3

Line, 3
  segment of, 3
Line values, of sine and cosine, 64
  of tangent and cotangent, 69
Logarithm, common, 182
  of trigonometric functions, 25
Lune, 147

Mantissa, 187
Median, 4
Meridian, of longitude, 175
  celestial, 175
Mil, 37
Minute, 3
Modulus, 119

Napier's Rule, 158
Nautical mile, 171
Negative angle, 51
  functions of, 60
North star, 176

Oblique triangles, 88
Octagon, 4
Ordinate, 50

Pappus, Theorem of, 144
Parallelogram, 4
Parallelepiped, 135
Parts of triangle, 24
Pentagon, 4
Periodic function, 65, 80
  period of, 65, 80
Permutations, cyclical, 104
Perpendicular to a plane, 130
Polar, coordinates, 51
  distance, 146
  triangle, 148
Pole, of circles of sphere, 146
  celestial, 175
  star, 175
Polygon, 3
Polyhedral angle, 131
Polyhedron, 131
Prism, 134
Prismatoid, 143
  formula for volume of, 144
Pure imaginary number, 118
Pythagorean relation, 7
Pyramid, 138

Quadrantal angles, 58
Quadrilateral, 4

Radian, 37
Radius, of circle, 4
  of regular polygon, 5
Radius vector, 119
Ray, 2
Rectangle, 4
Rectangular coordinates, 49
Reduction to functions of acute angles, 56
Regular polygon, 3
Resultant, 32
Right triangle, 3

Scalene triangle, 3
Second, 2
Sector, 4, 42, 150
Segment, 4, 43, 150
Significant figures, 180
Similar, figures, 132
  polygons, 5
Sines, Law of, 88, 155
Solution of triangles, 24
Speed, 44
Sphere, 146
  celestial, 175
  terrestrial, 171
Spherical, degree, 150
  excess, 150
  sector, 150
  triangle, 154
  area, 161
Square, 4
Standard position of angle, 51
Straight line, 2
Subtraction of vectors, 32
Surfaces, conical, 139
  cylindrical, 136

Terminal side, 36, 51
Terrestrial sphere, 171
Three dimensional figures, 130
Transversal, 131
Trapezoid, 4
Triangle, definition, 3
  polar, 148
  spherical, 147, 154
Trigonometric functions, 12, 14, 52
Trigonometry, definition of, 11

Vector, definition, 31
  quantity, 31
Velocity, angular, 44
Versed sine, 13
Vertex, 2
Volume, 137, 139, 140, 142, 145, 150, 151

Zenith, 175
Zone, 147